MATHEMATICS

Trigonometry

1. Write the principal value of $\tan^{-1}(1) + \cos^{-1}(-\frac{1}{2})$.
2. Write the value of $\tan\left(2\tan^{-1}\frac{1}{5}\right)$
3. Find the value of the following: $\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{1-y^2}{1+y^2}\right], |x| < 1, y > 0$ and $xy > 1$.
4. Prove that: $\tan^{-1}\left(\frac{1}{2}\right) + tan^{-1}\left(\frac{1}{5}\right) + tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$

Linear Algebra

5. Find the value of a if

$$\begin{pmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{pmatrix} = \begin{pmatrix} -1 & 5 \\ 0 & 13 \end{pmatrix}$$

6. If

$$\begin{vmatrix} x+1 & x-1 \\ x-3 & x+2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix}$$

then write the value of x.

7. If

$$\begin{pmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{pmatrix} = A + \begin{pmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{pmatrix}$$

then find the matrix A.

8. Using properties of determinants, prove the following:

$$\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2$$

9. If $\mathbf{a} = x\hat{i} + 2\hat{j} - z\hat{k}$ and $\mathbf{b} = 3\hat{i} - y\hat{j} + \hat{k}$ are two equal vectors, then write the value of $x + y + z$.

10. If a unit vector $\mathbf{a}$ makes angles $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find the value of θ.

11. Find the Cartesian equation of the line which passes through the point $(-2, 4, -5)$ and is parallel to the line $\frac{x+3}{3} = \frac{4-y}{5} = \frac{z+8}{6}$.

12. If $\mathbf{a}$ and $\mathbf{b}$ are two vectos such that $|\mathbf{a} + \mathbf{b}| = |\mathbf{a}|$, then prove that $2\mathbf{a} + \mathbf{b}$ is perpendicular to $\mathbf{b}$.

13. Find the coordinates of the point, where the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2}$ intersects the plane $x - y + z - 5 = 0$. Also find the angle between the line and the plane.

14. Find the vector equation of the plane which contains the line of intersection of the planes $\mathbf{r} \cdot \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) - 4 = 0$ and $\mathbf{r} \cdot \left(2\hat{i} + \hat{j} - \hat{k}\right) + 5 = 0$ and which is perpendicular to the plane $\mathbf{r} \cdot \left(5\hat{i} + 3\hat{j} - 6\hat{k}\right) + 8 = 0$.

15. Find the vector equation of the plane passing through three points with position vectors $\hat{i} + \hat{j} - 2\hat{k}, 2\hat{i} - \hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} + \hat{k}$. Also find the coordinates of the point of intersection of this plane and the line $\mathbf{r} = 3\hat{i} - \hat{j} - \hat{k} + \lambda\left(2\hat{i} - 2\hat{j} + \hat{k}\right)$.

16. A school wants to award its students for the values of Honesty, Regularity and Hard work with a total cash award of \$6,000. Three times the award money for Hardwork added to that given for honesty amounts to \$11,000. The award money given for Honesty and Hard work together is double the one given for Regularity. Represent the above situation algebraically and find the award money for each value, using matrix method. Apart from these values, namely, Honesty, Regularity and Hard work, suggest one more value which the school must include for awards.

Differential equation

17. Write the degree of the differential equation $x^3\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^4 = 0$.

18. Show that the differential equation $2ye^{\frac{x}{y}}\,dx + \left(y - 2x\,e^{\frac{x}{y}}\right)dy = 0$ is homogeneous. Find the particular solution of this differential equation, given that $x = 0$ when $y = 1$.

Calculus

19. The amount of pollution content added in air in the city due to x-diesel vehicles is given by $P(x) = 0.005x^3 + 0.02x^2 + 30x$. Find the marginal increase in pollution content when 3 diesel vehicles are added and write which value is indicated in the above question.

20. Show that the function f in $A = \mathbb{R} - \left\{\frac{2}{3}\right\}$ defined as $f(x) = \frac{4x+3}{6x-4}$ is one-one and onto.Hence find f^{-1}.

21. Differentiate the following function with respect to x: $(\log x)^x + x^{\log x}$

22. If $y = \log\left[x + \sqrt{x^2 + a^2}\right]$, show that $\left(x^2 + a^2\right)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$.

23. Show that the function $f(x) = |x - 3|, x \in \mathbb{R}$, is continuous but not differentiable at $x = 3$.

24. If $x = a\sin t$ and $y = a\left(\cos t + \log\tan\frac{t}{2}\right)$, find $\frac{d^2y}{dx^2}$.

25. Evaluate: $\int \frac{\sin(x-a)}{\sin(x+a)}dx$

26. Evaluate: $\int \frac{5x-2}{1+2x+3x^2}dx$

27. Evaluate: $\int \frac{x^2}{(x^2+4)((x^2+9)}dx$

28. Evaluate: $\int_0^4 (|x| + |x-2| + |x-4|)\,dx$

29. Show that the height of the cylinder of maximum volume, that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.

30. Find the equation of the normal at a point on the curve $x^2 = 4y$ which passes through the point $(1, 2)$. Also find the equation of the corresponding tangent.

31. Using integration, find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.

32. Using integration, find the area of the region enclosed between the two circles $x^2 + y^2 = 4$ and $(x-2)^2 + y^2 = 4$.

Probability

33. A speaks truth in 60% of the cases, while B in 90% of the cases. In what percent of cases are they likely to contradict each other in stating the same fact? In the cases of contradiction do you think, the statement of B will carry more weight as he speaks truth in more number of cases than A?

34. Assume that the chances of a patient having a heart attack is 40%. Assuming that a meditation and yoga course reduces the risk of heart attack by 30% and prescription of certain drug reduces its chance by 25%. At a time a patient can choose any one of the two options with equal probabilities. It is given that after going through one of the two options, the patient selected at random suffers a heart attack. Find the probability that the patient followed a course of meditation and yoga. Interpret the result and state which of the above stated methods is more beneficial for

the patient.

Linear programming

35. A cooperative society of farmers has 50 hectares of land to grow two crops A and B. The profits from crops A and B per hectare are estimated as $10,500$ and $9,000$ respectively. To control weeds, a liquid herbicide has to be used for crops A and B at the rate of 20 litres and 10 litres per hectare, respectively. Further not more than 800 litres of herbicide should be used in order to protect fish and wildlife using a pond which collects drainage from this land. Keeping in mind that the protection of fish and other wildlife is more important than earning profit, how much land should be allocated to each crop so as to maximize the total profit? Form an LPP from the above and solve it graphically. Do you agree with the message that the protection of wildlife is utmost necessary to preserve the balance in environment ?