Nonlinear Systems

Sheet 2 — HT22

For Tutors Only — Not For Distribution

Stability, Lyapunov functions, centre manifold, bifurcations

Section A

1. The complex Landau equation
$$\dot{z} = az - b|z|^2 z,$$
arises in nonlinear stability theory. Here $z(t)$ is complex-valued and a, b are complex numbers (assume that $\mathrm{Re}(a) > 0$). Write the equation as a system of two real equations for $r(t)$ and $\theta(t)$ where $z = r(t)\mathrm{e}^{\mathrm{i}\theta(t)}$. Discuss the existence of periodic solutions in terms of the constants a and b.

Solution:

Writing $z = r\mathrm{e}^{\mathrm{i}\theta}$ and $a = a_1 + \mathrm{i}a_2$, $b = b_1 + \mathrm{i}b_2$ the equation is
$$(\dot{r} + \mathrm{i}r\dot{\theta})\mathrm{e}^{\mathrm{i}\theta} = (a_1 + \mathrm{i}a_2)r\mathrm{e}^{\mathrm{i}\theta} - (b_1 + \mathrm{i}b_2)r^3\mathrm{e}^{\mathrm{i}\theta}.$$
Cancelling $\mathrm{e}^{\mathrm{i}\theta}$ the real and imagingary parts give
$$\dot{r} = a_1 r - b_1 r^3,$$
$$\dot{\theta} = a_2 - b_2 r^2.$$
There are equilibria in r at $r = 0$ and $r^2 = a_1/b_1$. Since we are given that $a_1 > 0$ there is a non-zero equilirbium for r if and only if $b_1 > 0$.

In addition, for this to be a periodic orbit and not ring of fixed points we require $a_2 b_1 - a_1 b_2 \neq 0$, since otherwise $\dot{\theta} = 0$.

2. Discuss the stability of the equilibria and limit cycles of
$$\begin{aligned}\dot{x} &= -y + x\sin r,\\ \dot{y} &= x + y\sin r\end{aligned}$$
where $r^2 = x^2 + y^2$.

Solution:

We write the system in polar coordinates
$$r\dot{r} = x\dot{x} + y\dot{y} = -xy + x^2\sin r + yx + y^2\sin r = r^2\sin r \qquad \Rightarrow \qquad \dot{r} = r\sin r.$$
To work out $\dot{\theta}$ you can differentiate $\tan^{-1}(y/x)$, or,
$$\left.\begin{aligned}\dot{x} &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta,\\ \dot{y} &= \dot{r}\sin\theta + r\dot{\theta}\cos\theta,\end{aligned}\right. \quad \Rightarrow \quad \dot{y}r\cos\theta - \dot{x}r\sin\theta = r^2\dot{\theta} = x\dot{y} - y\dot{x}$$

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so

$$\dot{\theta} = \frac{x\dot{y} - \dot{x}y}{x^2 + y^2} = 1.$$

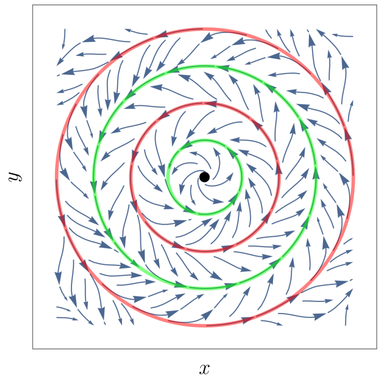
Since $\dot{\theta} \neq 0$ the only fixed point is $r = 0$ corresponding to $(0, 0)$. For small r

$$\dot{r} = r \sin r \sim r^2 \qquad \Rightarrow \text{ the origin is an unstable spiral.}$$

There are peroidic orbits whenever $\sin r = 0$, i.e. $r = n\pi$, $n \in \mathbb{Z}$. Those with radii $r = 2n\pi$, $n \in \mathbb{Z}^+$ are unstable, while those with $r = (2n + 1)\pi$, $n \in \mathbb{Z}^+$ are stable. To see this write $r = n\pi + \rho$ and linearise in ρ to give

$$\dot{\rho} = (n\pi + \rho)\sin(n\pi + \rho) = (n\pi + \rho)\cos(n\pi)\sin\rho \sim n\pi(-1)^n\rho.$$

There is exponential decay when n is odd and exponential growth when n is even.



Section B

3. Consider the equation

$$\ddot{x} = w - 2x + x^2$$

where $w \geq 0$ is a parameter.

(i) Show that the evolution of x conserves a form of the energy and identify the potential function.

(ii) From the potential function, sketch the phase portrait for $w = 0$. Identify important orbits.

(iii) What happens as w increases? Find the critical value w such that the system does not support any periodic orbit.

Solution:

(i) Defining the potential function

$$V = -wx + x^2 - \frac{x^3}{3},$$

the equation is

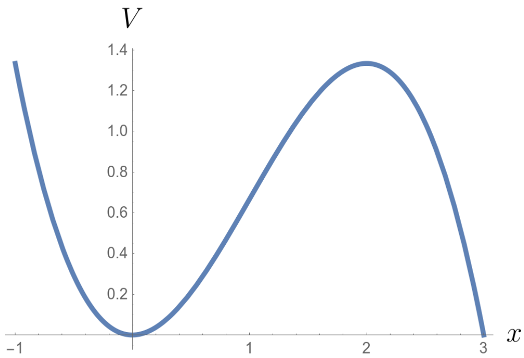
$$\ddot{x} = -\frac{\mathrm{d}V}{\mathrm{d}x}.$$

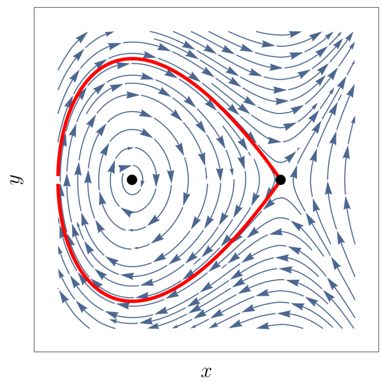
Multiplying through by $\dot{x}$ and integrating with respect to t gives

$$\frac{1}{2}\dot{x}^2 + V(x) = E,$$

where E is constant. Thus E (the energy) is conserved along trajectories.

(ii) Stationary points of the potential are equilibrium points for the system. For $w = 0$, the potential has a minimum ($V = 0$) at $x = 0$ and a maximum ($V = 4/3$) at $x = 2$, which correspond to a center and a saddle point respectively. There is also a homoclinic orbit from the saddle point which circles around the center before returning to the saddle point again. There is a set of trajectories in the neighbourhood of the centre with $0 < V < 4/3$ which are periodic solutions confined by the homoclinic orbit.

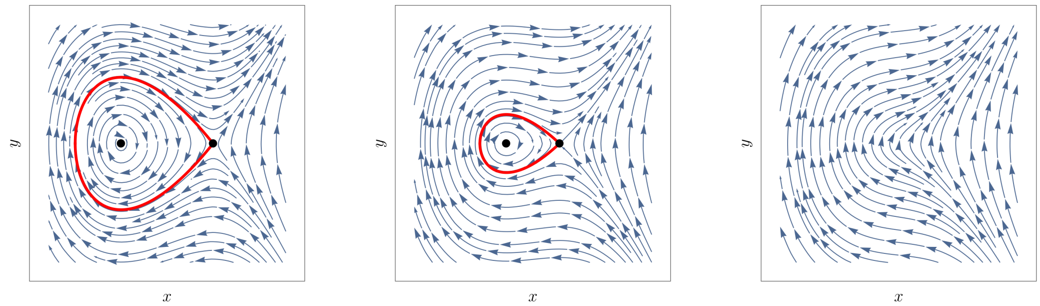




(iii) The potential V has stationary points at

$$x = 1 \pm \sqrt{1-w}.$$

The minimum corresponds to a centre with associated periodic orbits. As w increases from zero, the fixed points move closer together until they merge when $w = 1$. For $w > 1$, there are no longer any fixed points and $x \to \infty$ as $t \to \infty$.



$w = 0.4$ $\qquad$ $w = 0.8$ $\qquad$ $w = 1.2$

4. (i) Consider a vector field $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$. Assume that $H = H(\mathbf{x})$ is a first integral ($\dot{H} = 0$). Let $\mathbf{x}_0$ be a fixed point. Prove that if $\mathbf{x}_0$ is a nondegenerate minimum of H, then $\mathbf{x}_0$ is stable.

   (ii) Let V be a $C^r (r \geq 1)$ function of $\mathbf{x} \in \mathbb{R}^n$. A *gradient vector field* or *gradient flow* is defined by

$$\dot{\mathbf{x}} = -\nabla V(\mathbf{x})$$

   Show a gradient vector field cannot have periodic or homoclinic orbits (Hint: Use $V(x)$ as a Lyapunov function).

Solution:

(i) We are given that $\dot{H}(\mathbf{x}) = 0$ along trajectories, and that $\nabla H(\mathbf{x}_0) = 0$ since $\mathbf{x}_0$ is a minimum of H. If $\mathbf{x}_0$ is a nondegenerate minimum, then there exists an open neighbourhood of $\mathbf{x}_0$ such that $H(\mathbf{x}) > H(\mathbf{x}_0)$ for $\mathbf{x} \in W \setminus \{\mathbf{x}_0\}$. We define $V(\mathbf{x}) = H(\mathbf{x}) - H(\mathbf{x}_0)$. Then

$$\dot{V} = \dot{H} = 0, \qquad V(\mathbf{x}_0) = 0, \qquad V(\mathbf{x}_0) > 0 \text{ for } \mathbf{x} \in W \setminus \{\mathbf{x}_0\}.$$

Thus V is a Lyapunov function and we deduce that $\mathbf{x}_0$ is stable (but not asymptotically stable).

(ii) We use $V(\mathbf{x})$ as a Lyapunov function. A periodic orbit of period T has $\mathbf{x}(0) = \mathbf{x}(T)$ (and therefore $V(\mathbf{x}(0)) = V(\mathbf{x}(T))$), while a homoclinic connection has $\mathbf{x}(-\infty) = \mathbf{x}(\infty)$ (and therefore $V(\mathbf{x}(-\infty)) = V(\mathbf{x}(\infty))$). However

$$V(\mathbf{x}(t_1)) - V(\mathbf{x}(t_0)) = \int_{t_0}^{t_1} \frac{\mathrm{d}V}{\mathrm{d}t}\,\mathrm{d}t = \int_{t_0}^{t_1} \nabla V \cdot \dot{\boldsymbol{x}}\,\mathrm{d}t = -\int_{t_0}^{t_1} |\dot{\mathbf{x}}|^2\,\mathrm{d}t \leq 0,$$

with equality if and only if $\dot{\mathbf{x}} = 0$ for all $t_0 < t < t_1$. Thus the only periodic orbits are fixed points.

5. Show that the origin is a stable point of equilibrium for the nonlinear system

$$\begin{aligned} \dot{x} &= y - x^3, \\ \dot{y} &= -x^3, \end{aligned}$$

but that it is an unstable point of equilibrium for the linearized system there [Hint: Consider Lyapunov functions of the form $V = x^m + cy^n$.]

Solution:

The linearised system is

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= 0. \end{aligned}$$

Eigenvalues are $\lambda = 0$ (twice) which tells us nothing about the stability of the nonlinear system. We can see that the general solution of the linear system is

$$x = At + B, \quad y = A,$$

and therefore that any perturbation in which $y \neq 0$ will leave the neighbourhood of the origin. Thus the origin is an unstable equilibrium point of the linear system.

For the nonlinear system introduce a Lyapunov function V of the form

$$V = x^m + cy^n,$$

which has the required properties $V(0,0) = 0$ and $V(x,y) > 0\ \forall (x,y) \neq (0,0)$ provided m and n are even. Differentiating with respect to t gives

$$\dot{V} = mx^{m-1}\dot{x} + cny^{n-1}\dot{y} = mx^{m-1}y - mx^{m+2} - cnx^3y^{n-1}$$

which suggests trying $m = 4$, $n = 2$, and $c = 2$ to match the powers of x and y and the coefficients to cancel the first and third term (the terms which are not of one sign). This gives

$$\dot{V} = -4x^6 \leq 0.$$

Thus $(0,0)$ is (Lyapunov) stable.

6. Consider the system

$$\begin{aligned}\dot{x} &= xy + ax^3 + xy^2,\\ \dot{y} &= -y + bx^2 + x^2y.\end{aligned}$$

(i) Use an analysis of the dynamics on the centre manifold to show that the origin is asymptotically stable if $a + b < 0$ and unstable if $a + b > 0$.

(ii) What happens if $a + b = 0$? Is the origin stable or unstable?

Solution:

(i) The Jacobian at the origin is

$$D\mathbf{f} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix},$$

which has eigenvalues $\lambda = -1, 0$ and corresponding eigenvectors $(0,1)^T$ and $(1,0)^T$. The centre subspace is spanned by the second of these. We construct the center manifold in the form

$$y = h(x) = c_2x^2 + c_3x^3 + c_4x^4 + O(x^5),$$

where the expansion starts at $O(x^2)$ so that the manifold lies tangent to the center subspace. Substituting the Taylor expansion into the equation $\dot{y} = h_x\dot{x}$ gives

$$(b - c_2)x^2 - c_3x^3 + (c_2 - c_4)x^4 = 2c_2(c_2 + a)x^4 + O(x^5).$$

Equating coefficients gives $c_2 = b$, $c_3 = 0$, and $c_4 = b - 2b(a + b)$, leading to the extended center manifold having the form:

$$y = bx^2 + [b - 2b(a + b)]\, x^4 + O(x^5).$$

Substituting into the equation for $\dot{x}$ gives the dynamics on the centre manifold as

$$\dot{x} = (a+b)x^3 + \left[b^2 + b - 2b(a+b)\right] x^5 + O(x^6).$$

We see that the origin is asymptotically stable for $a+b<0$ and unstable for $a+b>0$, as required.

(ii) If $a+b=0$, then the stability is determined by considering the term of $O(x^5)$. In this case, if $b^2+b>0$ (i.e. $b>0$ or $b<-1$), then the origin is unstable, whilst if $b^2+b<0$, (i.e. $0<b<1$), then the origin is asymptotically stable. However, if $b=-1$ or 0, then we need to go to even higher order in x to determine the stability.

7. Consider the system

$$\begin{aligned} \dot{x} &= y - x - x^2, \\ \dot{y} &= \mu x - y - y^2. \end{aligned}$$

Find the value of μ for which there is a bifurcation at the origin. Find the evolution equation on the extended centre manifold correct to quadratic terms in the Taylor expansion and determine the type of bifurcation.

Solution: The Jacobian at the origin is

$$D\mathbf{f}(0,0) = \begin{bmatrix} -1 & 1 \\ \mu & -1 \end{bmatrix} \qquad \Rightarrow \qquad \lambda = -1 \pm \sqrt{\mu}.$$

There is a change in stability at $\mu = 1$ which suggests a bifurcation. The associated linear subspaces are:

$$E^s = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}, \qquad E^c = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

We therefore change variable by setting $\mu = 1 + \tilde{\mu}$ and

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad \begin{bmatrix} \xi \\ \eta \end{bmatrix} = P^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

giving

$$\xi = \frac{x+y}{2}, \qquad \eta = \frac{x-y}{2},$$

to give the extended system

$$\begin{aligned} \dot{\xi} &= \frac{1}{2}\left(\tilde{\mu}(\xi+\eta) - (\xi+\eta)^2 - (\xi-\eta)^2\right) = \frac{\tilde{\mu}}{2}(\xi+\eta) - \xi^2 - \eta^2, \\ \dot{\eta} &= \frac{1}{2}\left(-(\tilde{\mu}+2)(\xi+\eta) + 2(\xi-\eta) - (\xi+\eta)^2 + (\xi-\eta)^2\right) = -\frac{\tilde{\mu}}{2}\xi - \left(2 + \frac{\tilde{\mu}}{2}\right)\eta - 2\xi\eta, \\ \dot{\tilde{\mu}} &= 0. \end{aligned}$$

Now we look for the extended centre manifold in the form

$$\eta = h(\xi, \tilde{\mu}).$$

Using the dynamical equations in the expression

$$\dot{\eta} = h_\xi \dot{\xi} + h_{\tilde{\mu}} \dot{\tilde{\mu}}$$

gives

$$-\frac{\tilde{\mu}}{2}\xi - \left(2 + \frac{\tilde{\mu}}{2}\right) h - 2\xi h = h_\xi \left(\frac{\tilde{\mu}}{2}(\xi + h) - \xi^2 - h^2\right)$$

Now Taylor expanding

$$h(\xi, \tilde{\mu}) = a_2 \xi^2 + b_2 \tilde{\mu}^2 + c_1 \xi \tilde{\mu} + \cdots ,$$

gives

$$-\left(\frac{1}{2} + 2c_1\right) \tilde{\mu}\xi - 2a_2\xi^2 - 2b_2\tilde{\mu}^2 = O(\{\xi, \tilde{\mu}\}^3)$$

and equating coefficients gives

$$a_2 = 0, \qquad b_2 = 0, \qquad c_1 = -\frac{1}{4}.$$

Substituting the centre manifold into the equation for $\dot{\xi}$ gives

$$\dot{\xi} = \frac{1}{2}\tilde{\mu}\xi - \xi^2 + O(\{\xi, \tilde{\mu}\}^3),$$

which is the canonical form of a transcritical bifurcation.

Note In fact, we don't need to change variable to put the system in standard form when it is this simple. Sticking with x and y we identify the centre subspace as the space $y = x$. Without a change of variables the expansion of the centre manifold must agree with this centre subspace at linear order, i.e. we must write

$$y = h(x, \tilde{\mu}) = x + a_2 x^2 + b_1 \tilde{\mu} + b_2 \tilde{\mu}^2 + c_1 x \tilde{\mu} + \cdots .$$

Then using the dynamical equations in

$$\dot{y} = h_x \dot{x} + h_{\tilde{\mu}} \dot{\tilde{\mu}}$$

gives

$$(1 + \tilde{\mu})x - h - h^2 = h_x(h - x - x^2).$$

Using the Taylor expansion gives

$$\tilde{\mu}x - a_2x^2 - b_1\tilde{\mu} - b_2\tilde{\mu}^2 - c_1x\tilde{\mu} - b_1^2\tilde{\mu}^2 - x^2 - 2b_1x\tilde{\mu} = (a_2x^2 + b_1\tilde{\mu} + b_2\tilde{\mu}^2 + c_1x\tilde{\mu} - x^2) + (2a_2x + c_1\tilde{\mu})b_1\tilde{\mu}.$$

Equating coefficients gives

$$b_1 = 0, \qquad b_2 = 0, \qquad a_2 = 0, \qquad c_1 = \frac{1}{2}.$$

Substituting the center manifold into the equation for $\dot{x}$ then gives

$$\dot{x} = \frac{1}{2}\tilde{\mu}x - x^2 + O(\{x, \tilde{\mu}\}^3),$$

from which we can identify the bifurcation as transcritical. We can check that this is the same as we got before by observing that on the centre manifold

$$x = \xi + \eta = \xi - \frac{\xi\tilde{\mu}}{4} + \cdots$$

so that

$$\begin{aligned}
\dot{\xi}\left(1 - \frac{\tilde{\mu}}{4}\right) &= \frac{1}{2}\tilde{\mu}\xi\left(1 - \frac{\tilde{\mu}}{4}\right) - \xi^2\left(1 - \frac{\tilde{\mu}}{4}\right)^2 + O(\{\xi, \tilde{\mu}\}^3) \\
\Rightarrow \qquad \dot{\xi} &= \frac{1}{2}\tilde{\mu}\xi - \xi^2\left(1 - \frac{\tilde{\mu}}{4}\right) + O(\{\xi, \tilde{\mu}\}^3) \\
&= \frac{1}{2}\tilde{\mu}\xi - \xi^2 + O(\{\xi, \tilde{\mu}\}^3).
\end{aligned}$$

Note that we could equally well have chosen to write x in terms of y to describe the centre manifold.

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Section C

8. By using ideas similar to Lyapunov's method, show that all trajectories of the **Lorenz system**

$$\begin{aligned}\dot{x} &= \sigma(y-x),\\ \dot{y} &= \rho x - xz - y,\\ \dot{z} &= xy - \beta z,\end{aligned}$$

with positive parameters σ, ρ, and β eventually enter and remain inside a large sphere S of the form $x^2 + y^2 + (z-\rho-\sigma)^2 = C$, for C sufficiently large.

Solution: Define $V = x^2 + y^2 + (z - r - \sigma)^2$ and differentiate with respect to t to give

$$\begin{aligned}\dot{V} &= 2x\dot{x} + 2y\dot{y} + 2\left(z - r - \sigma\right)\dot{z},\\ &= 2x\sigma\left(y - x\right) + 2y\left(rx - xz - y\right) + 2\left(z - r - \sigma\right)\left(xy - \beta z\right),\\ &= -2\sigma x^2 - 2y^2 - 2\beta\left(z - \frac{r+\sigma}{2}\right)^2 + \frac{\beta\left(r+\sigma\right)^2}{2}.\end{aligned}$$

We see $\dot{V} = 0$ on the ellipsoid

$$2\sigma x^2 + 2y^2 + 2\beta\left(z - \frac{r+\sigma}{2}\right)^2 = \frac{\beta\left(r+\sigma\right)^2}{2},$$

and that outside this ellipsoid $\dot{V} < 0$. Now choose C to be large enough that the sphere $x^2 + y^2 + (z - r - \sigma)^2 = C$ strictly encloses the ellipsoid. Then there exists μ such that $\dot{V} < \mu < 0$ for any point $\mathbf{x}$ outside the sphere. Thus any trajectory starting outside the sphere satisfies $\dot{V} < \mu$ while it remains outside the sphere, and so will eventually cross $V = C$ and enter the sphere. Since $\dot{V} < 0$ on the surface of the sphere the trajectory can never leave the sphere again.

9. A simple model for the motion of a glider is given by the equations

$$\begin{aligned}\dot{y} &= -\sin\theta - ay^2,\\ \dot{\theta} &= y - \frac{\cos\theta}{y},\end{aligned}$$

where y is the velocity, θ is the angle between the glider and the horizontal, and a is the ratio of the drag coefficient to lift coefficient. For $a = 0$ show that $V = y^3 - 3y\cos\theta$ is a conserved quantity and sketch the phase portrait. Interpret your result (What does the glider do? What is its path?).

[*] For $a > 0$ (positive drag), linearise the system around its fixed points and discuss the stability. Again, interpret the results in terms its motion.

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Solution:

With $a = 0$ we evaluate

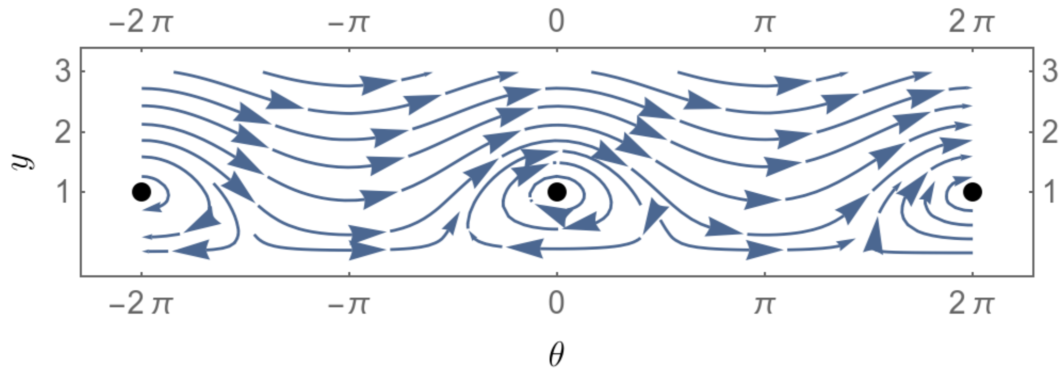
$$\dot{V} = 3y^2\dot{y} - 3\dot{y}\cos\theta + 3y\dot{\theta}\sin\theta = (3y^2 - 3\cos\theta)(-\sin\theta) + 3y\sin\theta\left(y - \frac{\cos\theta}{y}\right) = 0$$

so that V is conserved along trajectories.

There are equilibrium points at $\theta = n\pi$ ($n \in \mathbb{Z}$) and $y^2 = \cos\theta = (-1)^n$. Thus the only fixed points in the range $0 \leq \theta < 2\pi$ are $\theta = 0$, $y = \pm 1$. The linearised system about $(1, 0)$ is

$$\begin{aligned} \dot{y}_1 &= -\theta_1 \\ \dot{\theta} &= 2y_1 \end{aligned} \qquad \Rightarrow \qquad \lambda^2 + 2 = 0$$

so that it is a centre. Close to the centre there are peridic orbits in which the glider bobs up and down in speed and angle of attack. Far from the centre are orbits in which θ continually increases with the glider doing loop-the-loops.



With $a > 0$ the fixed points satisfy $\tan\theta = -a$ and $y^2 = \cos\theta$, so that $y = \pm 1/(1 + a^2)^{1/4}$. Considering the poisitive solution, the Jacobian is given by

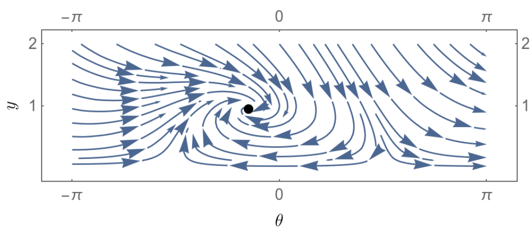
$$D\mathbf{f} = \begin{bmatrix} \frac{-2a}{(1+a^2)^{1/4}} & -\frac{1}{\sqrt{1+a^2}} \\ 2 & -\frac{a}{(1+a^2)^{1/4}} \end{bmatrix}, \tag{1}$$

which has

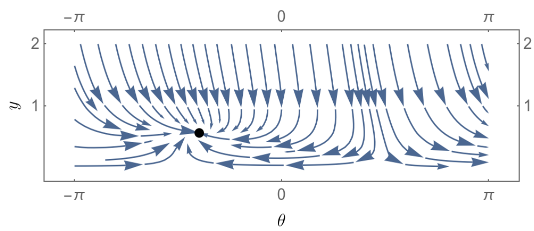
$$\operatorname{tr}(D\mathbf{f}) = -\frac{3a}{(1+a^2)^{1/4}} < 0, \quad \det(D\mathbf{f}) = 2\sqrt{a^2+1} > 0, \quad \operatorname{tr}(D\mathbf{f})^2 - 4\det(D\mathbf{f}) = \frac{a^2 - 8}{\sqrt{1+a^2}}.$$

Thus the fixed points are stable spirals for $a < 2\sqrt{2}$ and stable nodes for $a > 2\sqrt{2}$.

For $a < 2\sqrt{2}$, the glider is lightly damped and bobs up and down before it slows to constant velocity and angle of attack, whilst for $a > 2\sqrt{2}$ it is overdamped and tends to the constant velocity and angle without oscillating.



$a = 0.5$



$a = 3$

10. A bead is free to slide without friction on a circular wire hoop of radius L. The hoop spins about its vertical axis with angular velocity ω. After nondimensionalisation, the equation governing the position $\theta(t)$ (measured from the bottom of the hoop) is

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \sin\theta - \alpha\sin\theta\cos\theta = 0,$$

where $\alpha = \omega^2 L/g$.

(i) Discuss the behaviour of this system, as α increases from zero, from the point of view of bifurcation theory.

(ii) Write down an energy integral for the system. Find the smallest constant $v > 0$ (in terms of the parameters) such that, if initially $\theta = \pi/2, |\dot{\theta}| > v$, then the bead will continually encircle the hoop in one direction.

(iii) [*] What happens if linear damping is added to the system (that is, $-\mu\dot{\theta}$ is added on the equation's RHS with $\mu > 0$)? (NB: There is a real zoo of possible bifurcations in this system. A simple and good starting point is to find the critical rotation speed at which $\theta = 0$ becomes unstable. Describe this bifurcation).

Solution:

(i) We write the equation as a system in the form

$$\begin{aligned}\dot{\theta} &= y,\\ \dot{y} &= -\sin\theta + \alpha\sin\theta\cos\theta,\end{aligned}$$

which has fixed points given by $y = 0$ and

$$\sin\theta = 0, \qquad \text{or} \qquad \cos\theta = \frac{1}{\alpha}.$$

There are always fixed points at $\theta = 0$ and $\theta = \pi$ (plus $2n\pi$, $n \in \mathbb{Z}$). For $0 \le \alpha \le 1$ these are the only fixed points. For $\alpha > 1$ two new fixed points $\pm\theta^*$ (the roots of $\cos\theta^* = 1/\alpha$) appear via a (degenerate) pitchfork bifurcation of the branch $\theta = 0$. The linearisation about $(0, 0)$ is

$$\begin{aligned}\dot{\theta} &= y,\\ \dot{y} &= (\alpha - 1)\theta,\end{aligned} \qquad \Rightarrow \qquad \lambda^2 = \alpha - 1$$

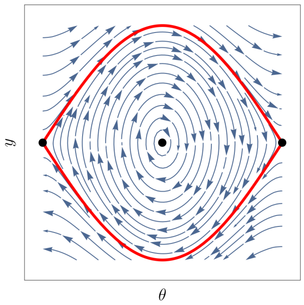
so that the origin is a center (and in fact Lyapunov stable) for $\alpha < 1$ and unstable (a saddle) if $\alpha > 1$. The linearisation about $(0, \pi)$ is

$$\begin{aligned}\dot{\theta}_1 &= y,\\ \dot{y} &= (\alpha + 1)\theta_1,\end{aligned} \qquad \Rightarrow \qquad \lambda^2 = \alpha + 1$$

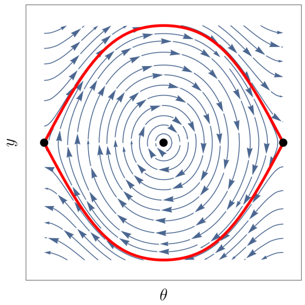
so that $\theta = \pi$ is a saddle. The linearisation about $(0, \theta^*)$ is

$$\begin{aligned}\dot{\theta}_1 &= y,\\ \dot{y} &= -(\alpha\sin^2\theta^*)\,\theta_1,\end{aligned} \qquad \Rightarrow \qquad \lambda^2 = -\alpha\sin^2\theta^*$$

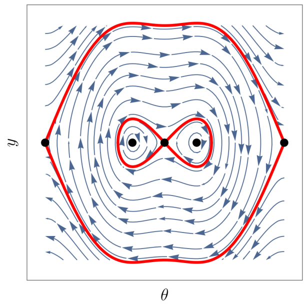
so that these points are centres (and Lyapunov stable). Thus the bifurcation is supercritical.



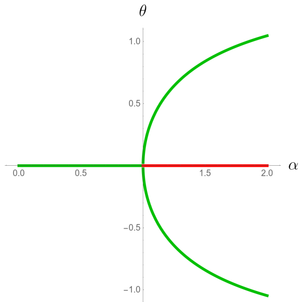
$\alpha = 0$



$\alpha = 0.5$



$\alpha = 1.5$



(ii) We first write the equation as

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \sin\theta - \frac{\alpha}{2}\sin 2\theta = 0.$$

Multiplying by $\dot{\theta}$ and integrating gives

$$\frac{1}{2}\dot{\theta}^2 + V(\theta) = E = \text{ constant},$$

where the potential

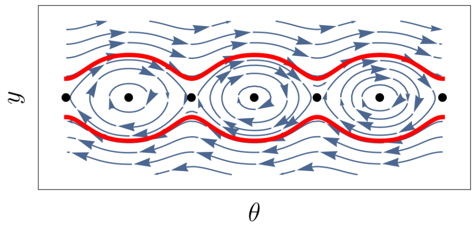
$$V = -\cos\theta + \frac{\alpha}{4}\cos 2\theta.$$

For the bead to continually circle the hoop in one direction, E must be large enough that there is no solution to $V(\theta) = E$ [otherwise this would be the point at which $\dot{\theta} = 0$ and the bead switches direction]. The maximum value of V occurs when $\theta = \pi$ and is $(1+\alpha/4)$. Thus if $E > 1 + \alpha/4$ the bead will continually circle the hoop in one direction. The initial conditions specified give

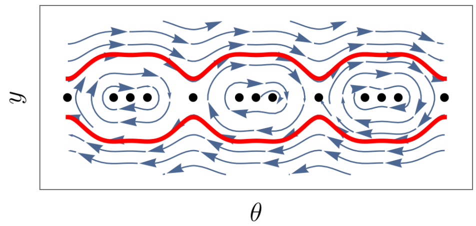
$$E = \frac{v^2}{2} - \frac{\alpha}{4}.$$

Thus we require

$$\frac{v^2}{2} - \frac{\alpha}{4} > 1 + \frac{\alpha}{4} \qquad \Rightarrow \qquad |v| > \sqrt{2+\alpha}.$$



$\alpha = 0.5$



$\alpha = 1.5$

(iii) Adding linear damping gives

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \mu\frac{\mathrm{d}\theta}{\mathrm{d}t} + \sin\theta - \alpha\sin\theta\cos\theta = 0,$$

which is the system

$$\begin{aligned}\dot{\theta} &= y,\\ \dot{y} &= -\mu y - \sin\theta + \alpha\sin\theta\cos\theta,\end{aligned}$$

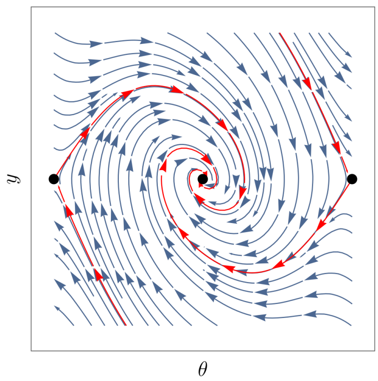
which has exactly the same fixed points as the undamped system. The Jacobian is

$$D\mathbf{f} = \begin{bmatrix} 0 & 1 \\ -\cos\theta + \alpha\cos 2\theta & -\mu \end{bmatrix},$$

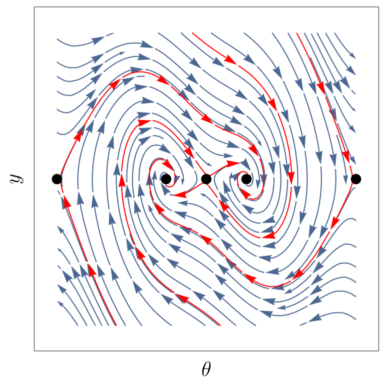
which, for the fixed point $(0,0)$, has eigenvalues given by

$$\lambda = \frac{-\mu \pm \sqrt{\mu^2 + 4(\alpha - 1)}}{2}.$$

For small μ and α the origin is now a stable spiral. When $1 - \mu^2/4 < \alpha < 1$ it switches to a stable node, and for $\alpha > 1$ it switches to an unstable node. The bifurcation is a supercritical (non-degenerate) pitchfork bifurcation.



$\alpha = 0.5$, $\mu = 0.5$



$\alpha = 1.5$, $\mu = 0.5$