Nonlinear Systems

Sheet 1 — HT22

For Tutors Only — Not For Distribution

Linear systems, invariant sets, attracting sets

Section A

1. Consider the system $\dot{\mathbf{x}} = A\mathbf{x}$ where $\mathbf{x} \in \mathbb{R}^3$ and

$$A = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 2 & 0 & 6 \end{bmatrix}.$$

Without solving the system, find the stable, unstable and center subspaces and sketch the phase portrait.

Solution:

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 2 & 0 \\ -2 & -\lambda & 0 \\ 2 & 0 & 6-\lambda \end{bmatrix} = (\lambda^2 + 2)(6 - \lambda) = 0$$

gives the eigenvalues and corresponding eigenvectors as

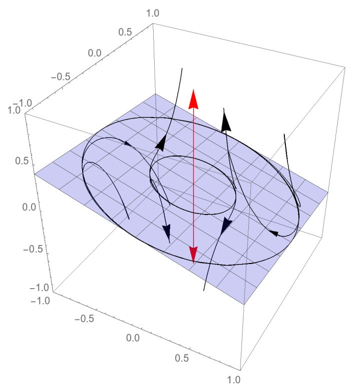
$$\lambda_1 = 6, \quad \mathbf{w}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \lambda_2 = 2\mathrm{i}, \quad \mathbf{w}_2 = \begin{bmatrix} 10\mathrm{i} \\ 10 \\ -1-3\mathrm{i} \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ -1 \end{bmatrix} + \mathrm{i} \begin{bmatrix} 10 \\ 0 \\ -3 \end{bmatrix}$$

$$\lambda_3 = -2\mathrm{i}, \quad \mathbf{w}_3 = \begin{bmatrix} -10\mathrm{i} \\ 10 \\ -1+3\mathrm{i} \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ -1 \end{bmatrix} - \mathrm{i} \begin{bmatrix} 10 \\ 0 \\ -3 \end{bmatrix}.$$

Therefore, the stable E^s, unstable E^u, and center E^c subspaces are

$$E^s = \emptyset, \qquad E^u = \mathrm{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \qquad E^c = \mathrm{span}\left\{ \begin{bmatrix} 0 \\ 10 \\ -1 \end{bmatrix}, \begin{bmatrix} 10 \\ 0 \\ -3 \end{bmatrix} \right\};$$

that is, the empty set, the z-axis, and the plane specified by $z = -3x/10 - y/10$.



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2. The system

$$\dot{x} = -x \tag{1}$$
$$\dot{y} = -y + x^2 \tag{2}$$
$$\dot{z} = z + x^2 \tag{3}$$

defines a flow $\varphi_t : \mathbb{R}^3 \to \mathbb{R}^3$. Show that the set $S = \{(x, y, z) \in \mathbb{R}^3 | z = -x^2/3\}$ is an **invariant set** of this flow. Sketch this set in phase space and identify other interesting orbits (such as fixed points).

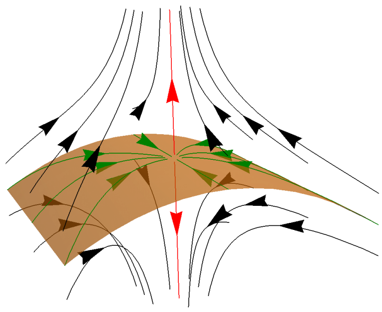
Solution: Let $Q = z + x^2/3$. Differentiating with respect to t along a trajectory using the chain rule gives

$$\dot{Q} = \dot{z} + \frac{2x\dot{x}}{3} = z + x^2 - \frac{2x^2}{3} = z + \frac{x^2}{3} = Q.$$

Thus if $Q = 0$ at any point on a trajectory then it is zero for all time on that trajectory, so that $Q = 0$ is an invariant set.

The other interesting point is the fixed point $(0, 0, 0)$, which, by linearization, is a saddle because it has eigenvalues and eigenvectors given by:

$$\lambda_1 = 1, \quad \mathbf{w}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \lambda_2 = -1, \quad \mathbf{w}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \lambda_3 = -1, \quad \mathbf{w}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$



Section B

3. Let $A \in \mathcal{M}_n(\mathbb{R})$ be a semi-simple matrix (i.e. a $n \times n$ matrix with real coefficients that can be diagonalised) and let $\mathbf{x} = \mathbf{x}(t)$ be a solution of

$$\dot{\mathbf{x}} = A\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0 \neq \mathbf{0}.$$

Show that:

(i) If $\mathbf{x}_0 \in E^s$, then $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$ and $\lim_{t\to-\infty} |\mathbf{x}(t)| = \infty$

(ii) If $\mathbf{x}_0 \in E^u$, then $\lim_{t\to\infty} |\mathbf{x}(t)| = \infty$ and $\lim_{t\to-\infty} \mathbf{x}(t) = \mathbf{0}$

(iii) If $\mathbf{x}_0 \in E^c$, then $\exists M \in \mathbb{R}$ such that $\forall\ t\ \in \mathbb{R}$,

$$|\mathbf{x}(t)| \leq M.$$

(iv) [Harder] Which of these properties hold if A is not semi-simple? (prove or give a counter-example)

Solution:

For semi-simple A the solution is given by

$$\mathbf{x}(t) = \sum_{k=1}^{n} c_k e^{\lambda_k t} \mathbf{w}_k,$$

where λ_k and $\mathbf{w}_k$ are the eigenvalues and eigenvectors of A, and $\mathbf{x}_0 = c_k \mathbf{w}_k$. Arrange the eigenvalues so that

$$\begin{aligned}
\mathrm{Re}(\lambda_k) &< 0 \qquad \text{for } 1 \leq k \leq k_1,\\
\mathrm{Re}(\lambda_k) &= 0 \qquad \text{for } k_1 + 1 \leq k \leq k_2,\\
\mathrm{Re}(\lambda_k) &> 0 \qquad \text{for } k_2 + 1 \leq k \leq n.
\end{aligned}$$

(i) The stable subspace is spanned by $\{\mathbf{w}_1, \ldots \mathbf{w}_{k_1}\}$, so that if $\mathbf{x}_0 \in E^s$ then $c_k = 0$ for $k > k_1$ and

$$\mathbf{x}(t) = \sum_{k=1}^{k_1} c_k e^{\lambda_k t} \mathbf{w}_k.$$

Since $\mathrm{Re}(\lambda_k) < 0$ for $1 \leq k \leq k_1$, $|\mathbf{x}| \to 0$ as $t \to \infty$ and $|\mathbf{x}| \to \infty$ as $t \to -\infty$.

(ii) The unstable subspace is spanned by $\{\mathbf{w}_{k_2+1}, \ldots \mathbf{w}_n\}$, so that if $\mathbf{x}_0 \in E^u$ then $c_k = 0$ for $k \leq k_2$ and

$$\mathbf{x}(t) = \sum_{k=k_2+1}^{n} c_k e^{\lambda_k t} \mathbf{w}_k.$$

Since $\mathrm{Re}(\lambda_k) > 0$ for $k_2 < k \leq n$, $|\mathbf{x}| \to \infty$ as $t \to \infty$ and $|\mathbf{x}| \to 0$ as $t \to -\infty$.

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(iii) The centre subspace is spanned by $\{\mathbf{w}_{k_1+1}, \ldots \mathbf{w}_k\}$, so that if $\mathbf{x}_0 \in E^c$ then $c_k = 0$ for $k \leq k_1$ or $k > k_2$, and

$$\mathbf{x}(t) = \sum_{k=k_1+1}^{k_2} c_k e^{\lambda_k t} \mathbf{w}_k.$$

Since $\mathrm{Re}(\lambda_k) = 0$ for $k \leq k_1$ and $k > k_2$

$$\left| \sum_{k=k_1+1}^{k_2} c_k e^{\lambda_k t} \mathbf{w}_k \right| < \sum_{k=k_1+1}^{k_2} \left| c_k e^{\lambda_k t} \mathbf{w}_k \right| = \sum_{k=k_1+1}^{k_2} |c_k \mathbf{w}_k| = M,$$

say.

NB it is also true that $\mathbf{x}(t)$ is bounded away from the origin, but this is harder to show. Suppose for a contradiction that there does not exist $m > 0$ such that $|\mathbf{x}| \geq m$. Then there exists a sequence of values of t, $\{t_k\}$ say, such that $|\mathbf{x}(t_k)| \to 0$ as $k \to \infty$. The values $\mathrm{e}^{\lambda_k t_n}$ lie on the unit circle in the complex plane, which is closed, so that there must be a subsequence which converges $\mathrm{e}^{\lambda_k t_{n_i}} \to \mathrm{e}^{\mathrm{i}\theta_k}$, say. Applying this argument to each λ_k in turn gives a subsequence for which

$$\sum_{k=k_1+1}^{k_2} c_k e^{\lambda_k t} \mathbf{w}_k \to \sum_{k=k_1+1}^{k_2} c_k e^{\mathrm{i}\theta_k} \mathbf{w}_k$$

But since $|\mathbf{x}(t_k)| \to 0$ for this subsequence we must have

$$\sum_{k=k_1+1}^{k_2} c_k e^{\mathrm{i}\theta_k} \mathbf{w}_k = \mathbf{0},$$

which contradicts the linear independence of the eigenvectors.

(iv) If A is not semisimple (i.e. it is not diagonalizable), then that implies A has degenerate eigenvalues with possibly linearly dependent eigenvectors. The general solution will now include terms which are algebraic in t. Results (i) and (ii) will still hold because the exponential growth/decay dominates the algebraic growth. However, for case (iii) it is possible that orbits in the centre subspace grow in t or approach the origin, meaning that there are no longer lower or upper bounds on $\mathbf{x}(t)$.

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4. A **heteroclinic orbit** is an orbit that connects two fixed points. Find the value of α such that the system

$$\begin{aligned}\dot{x} &= x - y,\\ \dot{y} &= -\alpha x + \alpha x y,\end{aligned}$$

admits the first integral $I = (y - 2x + x^2)\mathrm{e}^{-2t}$. (A scalar function $I = I(\mathbf{x}, t)$ is a *first integral* if $\dot{I} = 0$ on all trajectories.) Compute the fixed points and show that a branch of the level set of this first integral is a heteroclinic orbit. Find a closed-form solution for this orbit.

Solution:

Differentiating $I(t) = (y - 2x + x^2)e^{-2t}$ gives

$$\begin{aligned}\dot{I}(t) &= (\dot{y} - 2\dot{x} + 2x\dot{x})e^{-2t} - 2(y - 2x + x^2)e^{-2t}\\ &= (-\alpha x + \alpha x y - 2(x - y) + 2x(x - y) - 2(y - 2x + x^2))e^{-2t}\\ &= \left[(2 - \alpha)\, x + (\alpha - 2)\, xy\right] e^{-2t},\end{aligned}$$

so that $\dot{I} = 0$ if $\alpha = 2$.

The fixed points are $(0, 0)$ and $(1, 1)$, regardless of the value for α. We see that $I = 0$ at both fixed points, so that both lie on this level set of I. When $I = 0$, $y = 2x - x^2$, so that

$$\dot{x} = x^2 - x,$$

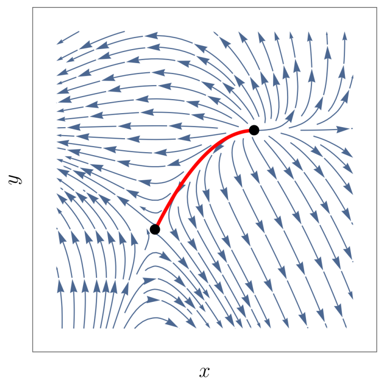
from which we can see that $\dot{x} < 0$ for $0 < x < 1$ so that there must be a heteroclinic orbit going from $(1, 1)$ when $t = -\infty$ to $(0, 0)$ at $t = +\infty$. To find the closed form of the heteroclinic orbit, separate the variables to give

$$\int \frac{\mathrm{d}x}{x(x-1)} = \int \left(-\frac{1}{x} + \frac{1}{x-1}\right) \mathrm{d}x = \log\left(\frac{x-1}{x}\right) = \int \mathrm{d}t = t + \log A$$

where we have written the constant of integration as $\log A$, so that

$$x = \frac{1}{1 - A\mathrm{e}^t}, \qquad y = \frac{1 - 2A\mathrm{e}^t}{(1 - A\mathrm{e}^t)^2}.$$

Note that (x, y) tends to $(1, 1)$ as $t \to -\infty$ and $(0, 0)$ as $t \to \infty$ for any A. The constant A just sets the origin of time along this trajectory.



5. The system

$$\begin{aligned}\dot{x} &= -y + x(1 - z^2 - x^2 - y^2)\\ \dot{y} &= x + y(1 - z^2 - x^2 - y^2)\\ \dot{z} &= 0\end{aligned}$$

defines a flow $\varphi_t : \mathbb{R}^3 \to \mathbb{R}^3$. Show that the union of the unit ball and the z-axis is an **attracting set** for this flow. Find its domain of attraction. (Hint: rewrite the system in cylindrical coordinates).

Solution:

Introduce the radius in cylindrical coordinates $r^2 = x^2 + y^2$ and differentiate with respect to time to give

$$r\dot{r} = x\dot{x} + y\dot{y} \qquad \Rightarrow \qquad \dot{r} = r\left(1 - r^2 - z^2\right)$$

which shows that for each $z \in [-1, 1]$ $r = 0$ is the α-limit set (asymptotically unstable) whilst $r = \sqrt{1 - z^2}$ is the ω-limit set (asymptotically stable). The union of these circles is the unit sphere. For each z with $|z| > 1$ we see $r = 0$ is attracting, so that an attracting set Ω is the union of the unit ball and z-axis.

Since $\dot{z} = 0$ it is easy to see that the domain of attraction is $D = \mathbb{R}^3$.

Note that Ω is not an attractor since there does not exist a dense orbit on Ω: orbits on the unit sphere are confined to circles at a particular z and will therefore never come arbitrarily close to a point with a different value of z.

Note also that the union of the unit sphere and the lines $\{(0, 0, z) \mid |z| > 1\}$ (call this Ω', say) is not an attracting set. Any neighbourhood U of Ω' must contain part of the z-axis between -1 and 1. These points are fixed points, so must also be contained in $\cap_{t>0}\varphi_t(U)$.

6. Consider the system

$$\begin{aligned}\dot{r} &= r(1 - r)\\ \dot{\theta} &= \sin^2\frac{\theta}{2},\end{aligned}$$

where (r, θ) are the usual polar coordinates of a point in the plane. Show that this system has two fixed points. Show that the fixed point $(x, y) = (1, 0)$ is the ω-limit set of almost all initial conditions, that is $\varphi_t(\mathbf{x}_0) \to (1, 0)$ for all initial conditions $\mathbf{x}_0 \neq (0, 0)$. Despite that, show that $(1, 0)$ is not stable. Is it an attracting set? Is the unit circle an attracting set? Find the domain of attraction (if any).

Solution:

Fixed points have $r = 0$ or $r = 1$ and $\theta = 2\pi n$ for $n \in \mathbb{Z}$, which correspond to 2 fixed points in Cartesian coordinates, $(x, y) = (0, 0)$ and $(1, 0)$. Since $\dot{r} > 0$ if $0 < r < 1$ and $\dot{r} < 0$ if $r > 1$ the

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value $r = 0$, corresponding to the fixed point $(0, 0)$, is asymptotically unstable whilst the value $r = 1$ is asymptotically stable and globally attracting as far as r is concerned. Since $\dot{\theta} > 0$ all trajectories proceed around the origin in an anti-clockwise direction and asymptotically approach $\theta = 2n\pi$ for some n. Therefore $(1, 0)$ is the ω-limit set for all initial conditions except the fixed point at the origin.

However, $\theta = 0$ is unstable since perturbing the steady state by $\theta = \epsilon > 0$ will result in the trajectory proceeding anti-clockwise around the unit circle until it reaches $(1, 0)$ again with $\theta \to 2\pi$.

The point $\mathbf{x}_0 = (1, 0)$ is not an attracting set. Because of the arguments above, an invariant open set $U \ni \mathbf{x}_0$ must include the whole circle (otherwise trajectories would not stay in U). But then

$$\bigcap_{t\in\mathbb{R}} \varphi_t(U)$$

is not the point $\mathbf{x}_0$ but the whole circle $r = 1$.

On the other hand, the unit circle *is* an attracting set because all trajectories that start with $r \in (1 - \epsilon, 1 + \epsilon)$, for $\epsilon > 0$, will stay in that annulus for all time.

Note that the unit disk is also an attracting set: trajectories that start with $r < 1 + \epsilon$, for $\epsilon > 0$, will stay in that disk for all time, and this time

$$\bigcap_{t\in\mathbb{R}} \varphi_t(U)$$

is the unit disk.

The domain of attraction for both the unit circle and $(1, 0)$ is $\mathbb{R}^2\backslash\{\mathbf{0}\}$.

Notes With the definitions we are using in the course, the unit circle is an attractor because it has a dense orbit: the orbit starting at $r = 1$ for any θ covers the unit circle when we take $t \in \mathbb{R}$ (note we take both positive and negative time). The point is not an attractor, since it is not an attracting set. The unit disk is not an attractor since it does not have a dense orbit.

With the definition of an attractor in Strogatz or Wikipedia, the circle is not an attractor (it is not minimal), but the point is an attractor (it is an invariant attracting set in the definition of Strogatz, who requires only that orbits starting in an open set $U \supset A$ converge to A).

With the definition of an attractor in Drazin, the circle is not an attractor (it is not an ω-limit set), and the point is not an attractor (it is not an attracting set).

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7. The equation for the simple pendulum is

$$\ddot{x} + \sin x = 0, \quad x \in \mathbb{R}$$

Find the potential for this system and use it to identify important orbits. In particular identify the fixed points and show that there exist heteroclinic orbits for this system. Sketch the phase portrait. Show that the orbits contained within a symmetric pair of heteroclinic orbits (called a *heteroclinic cycle*) form an invariant set. Is this an attracting set?

(This equation can be solved in terms of elliptic integrals of the first kind, but you should answer all these questions without solving the equation explicitly).

Solution:

The potential $V(x)$ is such that the force

$$-\sin x = -\frac{\mathrm{d}V}{\mathrm{d}x} \qquad \Rightarrow \qquad V = -\cos x.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\dot{x}^2 + V\right) = \dot{x}\ddot{x} + \dot{x}\frac{\mathrm{d}V}{\mathrm{d}x} = 0,$$

so that

$$\frac{1}{2}\dot{x}^2 + V = E = \text{constant}$$

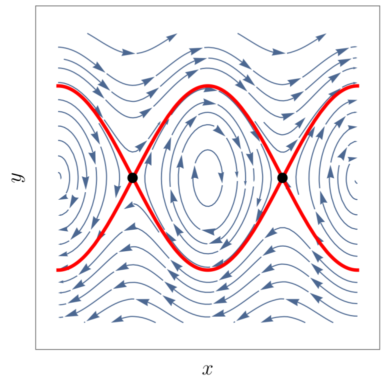
along trajectories.

The fixed points satisfy $\sin x = 0$ so that we have fixed points at $x = 2n\pi$ (pendulum is pointing down) and $x = (2n-1)\,\pi$ (pendulum is pointing up) for $n \in \mathbb{Z}$, with energy $E = -1$ and $E = 1$ respectively. The former are centers whilst the latter are saddle points.

Trajectories form level sets of E. Since the fixed points at $x = -\pi$ and $x = \pi$ have the same value of E, and since these points are saddle points, it is likely that there are heteroclinic connections. In fact there are two heteroclinic connections between the saddles at $x = -\pi$ and $x = \pi$: one forming part of the unstable manifold at $x = \pi$ and the stable manifold at $x = -\pi$, and one forming part of the stable manifold at $x = \pi$ and the unstable manifold at $x = -\pi$.

Since E is conserved along trajectories, any set of orbits $E_1 < E < E_2$ is an invariant set. In particular, orbits with $E \leq 1$ lie inside the pair of heteroclinic orbits just identified and form an invariant set.

Note that this set is not an attracting set.


y
x

Section C

One of the main tools of dynamical system is the linearisation of a nonlinear system close to its fixed points. From Part A, you should be familiar with the basic ideas. The following two **optional** exercises are meant as a refresher from last year.

8. Consider the systems below. Find the fixed points and determine their stability through linearisation whenever possible. For systems with parameters, discuss stability with respect to the parameters

(i)

$$\begin{aligned}\dot{x} &= 2x - 2xy \\ \dot{y} &= 2y - x^2 + y^2\end{aligned}$$

(ii)

$$\begin{aligned}\dot{x} &= -4y + 2xy - 8 \\ \dot{y} &= -x^2 + 4y^2\end{aligned}$$

(iii)

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0$$

Solution:

(i) There are 4 fixed points: $(x, y) = (0, -2)$, $(0, 0)$, $(-\sqrt{3}, 1)$, $(\sqrt{3}, 1)$. The Jacobian is

$$D\boldsymbol{f} = \begin{bmatrix} 2-2y & -2x \\ -2x & 2+2y \end{bmatrix}.$$

$$\begin{array}{l} x = 0 \\ y = -2 \end{array} \Rightarrow D\boldsymbol{f} = \begin{bmatrix} 6 & 0 \\ 0 & -2 \end{bmatrix} \Rightarrow \begin{array}{ll} \lambda_1 = 6, & \lambda_2 = -2, \\ \mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & \mathbf{w}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{array} \Rightarrow \text{saddle.} \quad .$$

$$\begin{array}{l} x = 0 \\ y = 0 \end{array} \Rightarrow D\boldsymbol{f} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \begin{array}{ll} \lambda_1 = 2, & \lambda_2 = 2, \\ \mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & \mathbf{w}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{array} \Rightarrow \begin{array}{l}\text{unstable} \\ \text{star node.}\end{array}$$

$$\begin{array}{l} x = -\sqrt{3} \\ y = 1 \end{array} \Rightarrow D\boldsymbol{f} = \begin{bmatrix} 0 & 2\sqrt{3} \\ 2\sqrt{3} & 4 \end{bmatrix} \Rightarrow \begin{array}{ll} \lambda_1 = -2, & \lambda_2 = 6, \\ \mathbf{w}_1 = \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix}, & \mathbf{w}_2 = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} \end{array} \Rightarrow \text{saddle.}$$

$$\begin{array}{l} x = \sqrt{3} \\ y = 1 \end{array} \Rightarrow D\boldsymbol{f} = \begin{bmatrix} 0 & -2\sqrt{3} \\ -2\sqrt{3} & 4 \end{bmatrix} \Rightarrow \begin{array}{ll} \lambda_1 = -2, & \lambda_2 = 6, \\ \mathbf{w}_1 = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}, & \mathbf{w}_2 = \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix} \end{array} \Rightarrow \text{saddle.}$$

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(ii) The fixed points are $(x, y) = (-2, -1)$, $(4, 2)$. The Jacobian is

$$D\boldsymbol{f} = \begin{bmatrix} 2y & 2x-4 \\ -2x & 8y \end{bmatrix}.$$

$$\begin{array}{l} x=-2 \\ y=-1 \end{array} \Rightarrow D\boldsymbol{f} = \begin{bmatrix} -2 & -8 \\ 4 & -8 \end{bmatrix} \quad \Rightarrow \quad \lambda^2 + 10\lambda + 48 = 0 \Rightarrow \lambda = -5 \pm \mathrm{i}\sqrt{23} \quad \Rightarrow \quad \begin{array}{l}\text{stable}\\ \text{spiral.}\end{array}$$

$$\begin{array}{l} x=4 \\ y=2 \end{array} \Rightarrow D\boldsymbol{f} = \begin{bmatrix} 4 & 4 \\ -8 & 16 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{ll} \lambda_1 = 8, & \lambda_2 = 12, \\ \mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, & \mathbf{w}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{array} \quad \Rightarrow \quad \begin{array}{l}\text{unstable}\\ \text{node.}\end{array}$$

(iii) Writing $y = \dot{x}$, gives can be written as:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\epsilon\left(x^2 - 1\right) y - x. \end{aligned}$$

There is a single fixed point at $(x, y) = (0, 0)$, with Jacobian

$$D\boldsymbol{f} = \begin{bmatrix} 0 & 1 \\ -1 & \epsilon \end{bmatrix},$$

and characteristic polynomial

$$0 = \lambda^2 - \epsilon\lambda + 1 \qquad \Rightarrow \qquad \lambda = \frac{1}{2}\left(\epsilon \pm \sqrt{\epsilon^2 - 4}\right).$$

There are 7 different cases:

(a) when $\epsilon > 2$ there are 2 real positive eigenvalues (unstable node);

(b) when $\epsilon = 2$ there is a repeated eigenvalue $\lambda = 1$, with a single eigenvector $\mathbf{w} = (1, 1)^T$ (unstable degenerate node);

(c) when $0 < \epsilon < 2$ there are complex conjugate eigenvalues with positive real part (unstable spiral);

(d) when $\epsilon = 0$ the eigenvalues are pure imaginary (center);

(e) when $-2 < \epsilon < 0$ there are complex conjugate eigenvalues with negative real part (stable spiral);

(f) when $\epsilon = -2$ there is a repeated eigenvalue $\lambda = -1$, with a single eigenvector $\mathbf{w} = (-1, 1)^T$ (stable degenerate node);

(g) when $\epsilon < -2$ there are 2 real negative eigenvalues (stable node).

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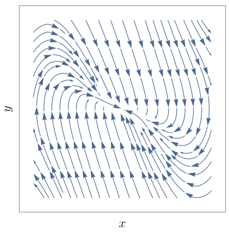
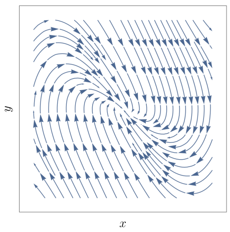
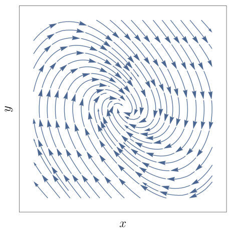


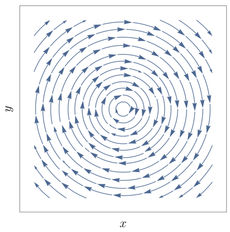
$\epsilon = -3$



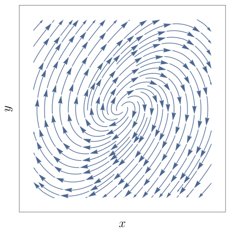
$\epsilon = -2$



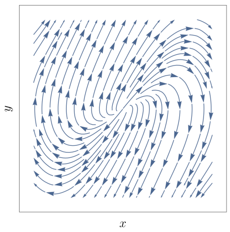
$\epsilon = -1$



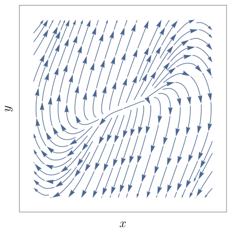
$\epsilon = 0$



$\epsilon = 1$



$\epsilon = 2$



$\epsilon = 3$

9. Consider the two systems

$$\dot{x} = f(x,y)$$
$$\dot{y} = g(x,y)$$

and

$$\dot{x} = x f(x,y)$$
$$\dot{y} = y\, g(x,y),$$

where f and g are both C^1 functions.

(i) Show that for the second system, the first positive quadrant (defined as the set $S = \{(x,y) \in \mathbb{R}^2 | x > 0, y > 0\}$) is an invariant set.

(ii) Show that if the first system has an exponentially stable or unstable fixed point in S of the form $(x,y) = (c,c)$ with $c > 0$, then the second one admits the same fixed point with the same stability properties (a fixed point is *exponentially stable* (resp. unstable) if the real parts of all its eigenvalues are strictly negative (resp. positive)). Is this property true for fixed points in the other quadrants (i.e. with fixed points $(\pm c, \pm c)$)? (Prove or give a counter-example)

(iii) [Harder] Are these properties true for the similar problem in n dimensions. Formulate the corresponding results.

Solution:

(i) Since $\dot{x} = 0$ on $x = 0$ and $\dot{y} = 0$ on $y = 0$ trajectories cannot cross either of the coordinate axes. Thus each quadrant is an invariant set.

(ii) Since (c, c) is a fixed point of the first system $f(c, c) = g(c, c) = 0$. The Jacobians are then

$$D\mathbf{f}_1 = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}, \qquad D\mathbf{f}_2 = \begin{bmatrix} c\frac{\partial f}{\partial x} & c\frac{\partial f}{\partial y} \\ c\frac{\partial g}{\partial x} & c\frac{\partial g}{\partial y} \end{bmatrix}$$

with all partial derivatives evaluated at (c, c). Since each entry in the second Jacobian is c multiplied by the corresponding entry in the first Jacobian, the eigenvalues of the second are c times the eigenvalues of the first (set $\lambda = c\lambda'$ to see that the characteristic polynomial of the second in λ' is c^2 times the characteristic polynomial of the first). Since $c > 0$ the stability properties of the second system are identical to those of the first.

The same result does not hold for other quadrants, which we show using a counter example. Suppose $f(x, y) = x + 1$ and $g(x, y) = y - 1$. The fixed point in this case is $(-1, 1)$, which has eigenvalues $\lambda = 1$ (twice). But

$$D\mathbf{f}_2 = \begin{bmatrix} 2x + 1 & 0 \\ 0 & 2y - 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

which has eigenvalues $\lambda = -1, 1$, so that the strictly positive or negative nature of the eigenvalues is not preserved.

(iii) The properties would hold in n dimensions. Consider the systems

$$\dot{x}_i = f_i(\mathbf{x}), \quad i = \{1, 2, ..., n\}, \tag{4}$$

and

$$\dot{x}_i = x_i f_i(\mathbf{x}), \quad i = \{1, 2, ..., n\}, \tag{5}$$

where $\mathbf{x} = (x_1, x_2, ..., x_n)^T$ and f_i is C^1 $\forall\, i \in \{1, 2, ..., n\}$.

The first quadrant $S = \{\mathbf{x} \in \mathbb{R}^n |\ x_i > 0\ \forall\, i \in \{1, 2, ..., n\}\}$ is invariant since $\dot{x}_i = 0$ for $x_i = 0$ means that no trajectory can cross a coordinate axis.

The Jacobians at a fixed point $\boldsymbol{x} = (c, c, \ldots, c)$ are given by

$$D\mathfrak{f}_1 = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_n} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}, \qquad D\mathbf{f}_2 = \begin{bmatrix} c\frac{\partial f_1}{\partial x_1} & c\frac{\partial f_1}{\partial x_2} & \cdots & c\frac{\partial f_1}{\partial x_n} \\ c\frac{\partial f_2}{\partial x_1} & c\frac{\partial f_2}{\partial x_2} & \cdots & c\frac{\partial f_2}{\partial x_n} \\ \vdots & \ddots & \ddots & \vdots \\ c\frac{\partial f_n}{\partial x_1} & c\frac{\partial f_n}{\partial x_n} & \cdots & c\frac{\partial f_n}{\partial x_n} \end{bmatrix},$$

so that the eigenvalues of the second are c times the eigenvalues of the first. For $c > 0$ the corresponding eigenvalues of the two systems therefore have the same sign.

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