The Casimir Effect in pAQFT:

Another ‘Argument’ for $1 + 2 + 3 + \ldots = -1/12$

Sam Crawford

Supervisors: Kasia Rejzner and Benoît Vicedo

UNIVERSITY of York

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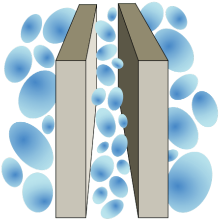
The Casimir Effect

- **Experimentally:**
  - Metal plates, placed close together in a vacuum, attract
  - Explained by the presence of 'fewer' quantum fluctuations of the vacuum within the cavity than without.
- **Theoretically:**
  - Discrepancy between the *normally ordered* energy-density of a spatially non-compact universe and a spatially compact one.



Classical Observables

- *Fields:* Smooth maps from spacetime $\mathcal{M}$ to $\mathbb{R}$
- $\mathcal{C}^\infty(\mathcal{M})$ space of field configurations
- *Classical observables:* Maps $\mathcal{F} : \mathcal{C}^\infty(\mathcal{M}) \to \mathbb{C}$
- $\mathfrak{F}_{\mathrm{loc}}(\mathcal{M})$ space of 'nice' classical observables.

Quantum Observables

- Actual quantum algebra, $\mathfrak{A}(\mathcal{M})$, defined abstractly
- Given a *Hadamard distribution* H, get an associated isomorphism

$$\alpha_H : \mathfrak{A}(\mathcal{M}) \to (\mathfrak{F}_{\mu c}(\mathcal{M}), \star_H) =: \mathfrak{A}^H(\mathcal{M}) \tag{1}$$

- For now, think of Hadamard distributions as **almost** smooth functions in $\mathcal{C}^\infty(\mathcal{M}^2)$
- α_H defined formally, but for two Hadamard distributions H, H', the map $\alpha_{H'} \circ \alpha_H^{-1} = \alpha_{H'-H}$ is a well defined automorphism of $\mathfrak{F}_{\mu c}(\mathcal{M})$.

Comparing Classical to Quantum

- $\star$ product of quantum algebra non-commutative unlike pointwise product $\cdot$ of classical
- $\rightsquigarrow$ ordering ambiguities when associating quantum counterpart to a classical observable
  - E.g. should $\mathcal{F} \cdot \mathcal{G}$ become $\mathcal{F} \star_H \mathcal{G}$, $\mathcal{G} \star_H \mathcal{F}$, etc...
- A choice of map $\mathfrak{F}_{\mathrm{loc}}(\mathcal{M}) \to \mathfrak{A}(\mathcal{M})$ called an *ordering prescription*
- Annoyingly denoted $\mathcal{F} \mapsto :\mathcal{F}:$

When Can We Compare Physics?

- Call a smooth map $\chi : \mathcal{M} \to \mathcal{N}$ an *admissible embedding* if it preserves geometry and causality
- Prototyipical example is the inclusion $\iota : U \hookrightarrow \mathcal{M}$ of a causally convex open subset $U \subset \mathcal{M}$
- Think of $\mathcal{M}$ as a "sub-spacetime" of $\mathcal{N}$

Comparing Classical Observables

- If $\phi \in \mathcal{C}^\infty(\mathcal{N})$, then $\chi^*\phi := \phi \circ \chi \in \mathcal{C}^\infty(\mathcal{M})$
- Hence, $\chi_*\mathcal{F}[\phi] := \mathcal{F}[\phi \circ \chi]$ is a classical observable in $\mathfrak{F}_{\mathrm{loc}}(\mathcal{N})$
- Hence for every admissible embedding of spacetimes, we have an associated embedding of classical observables

$$\mathfrak{F}_{\mathrm{loc}}(\mathcal{M}) \xrightarrow{\chi_*} \mathfrak{F}_{\mathrm{loc}}(\mathcal{N}) \tag{2}$$

Comparing Quantum Observables

- Difficult to define homomorphisms $\mathfrak{A}(\mathcal{M}) \to \mathfrak{A}(\mathcal{N})$ directly
- For $H \in \mathrm{Had}(\mathcal{M})$, $H' \in \mathrm{Had}(\mathcal{N})$, we **can** write down a homomorphism $\mathfrak{A}^H(\mathcal{M}) \to \mathfrak{A}^{H'}(\mathcal{N})$

$$\mathcal{F} \mapsto \chi_* \left(\alpha_{\chi^* H' - H} \mathcal{F}\right) \tag{3}$$

  Which works because $\chi^* H' \in \mathrm{Had}(\mathcal{M})$.

- Abstract homomorphism $\mathfrak{A}_\chi : \mathfrak{A}(\mathcal{M}) \to \mathfrak{A}(\mathcal{N})$ then defined to make the following diagram commute

$$\begin{tikzcd} \mathfrak{A}(\mathcal{M}) \arrow[r, "\mathfrak{A}_\chi"] \arrow[d, "\alpha_H"] & \mathfrak{A}(\mathcal{N}) \arrow[d, "\alpha_{H'}"] \\ \mathfrak{A}^H(\mathcal{M}) \arrow[r, "\chi_* \circ \alpha_{\chi^* H' - H}"'] & \mathfrak{A}^{H'}(\mathcal{N}) \end{tikzcd} \tag{4}$$

Normal Ordering

- Define an ordering prescription $:-:_\mathcal{M}\colon \mathfrak{F}_{\mathrm{loc}}(\mathcal{M}) \to \mathfrak{A}(\mathcal{M})$ for every spacetime $\mathcal{M}$

- *Locally covariant Wick ordering* 'compatible' with the maps defined earlier, i.e. the following diagram commutes

$$\begin{tikzcd} \mathfrak{F}_{\mathrm{loc}}(\mathcal{M}) \arrow[r, "\chi_*"] \arrow[d, ":-:_\mathcal{M}"'] & \mathfrak{F}_{\mathrm{loc}}(\mathcal{N}) \arrow[d, ":-:_\mathcal{N}"] \\ \mathfrak{A}(\mathcal{M}) \arrow[r, "\mathfrak{A}\chi"'] & \mathfrak{A}(\mathcal{N}) \end{tikzcd} \tag{5}$$

- Details of these maps actually irrelevant.

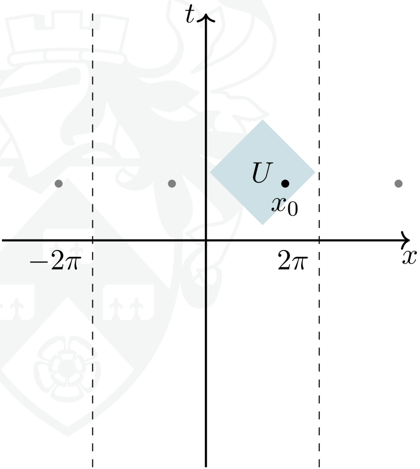
- We only need to know (i) they exist, and (ii) they make the above diagram commute

Stress-Energy on the Cylinder

- Spacetime: The Minkowski Cylinder
  - Minkowski space $\mathbb{M} \simeq \mathbb{R}^2$
  - Minkowski cylinder $\mathcal{M}_{\mathrm{cyl}} \simeq \mathbb{M}/\sim$, where $(t, x) \sim (t, x + 2\pi)$
  - Also introduce *null coordinates* $u = t + x$, $v = t - x$.
- For any point $\mathbf{x}_0 \in \mathbb{M}$, can define the **classical** observable

$$T_{\mathbf{x}_0}[\phi] := \frac{1}{2}\left(\left.\frac{\partial \phi}{\partial u}\right|_{\mathbf{x}=\mathbf{x}_0}\right)^2, \qquad (6)$$

physically interpreted as (part of) the energy density for a massless scalar field.



$$
\begin{tikzcd}
\mathfrak{F}_{\mathrm{loc}}(U) \arrow[r, "\chi_*"] \arrow[d, ":\!-\!:_U"'] & \mathfrak{F}_{\mathrm{loc}}(\mathcal{M}_{\mathrm{cyl}}) \arrow[d, ":\!-\!:_{\mathcal{M}_{\mathrm{cyl}}}"] \\
\mathfrak{A}(U) \arrow[r, "\mathfrak{A}\chi"'] \arrow[d, "\alpha_H"'] & \mathfrak{A}(\mathcal{M}_{\mathrm{cyl}}) \arrow[d, "\alpha_{H'}"] \\
\mathfrak{F}_{\mu c}(U) \arrow[r, "\chi_* \circ \alpha_{\chi^* H' - H}"'] & \mathfrak{F}_{\mu c}(\mathcal{M}_{\mathrm{cyl}})
\end{tikzcd}
$$

Minkowski's Special

- There is a special Hadamard distribution $H_{\mathbb{M}} \in \mathrm{Had}(\mathbb{M})$, called the *Minkowski vacuum*.
- In null coordinates, it can be written as

$$H_{\mathbb{M}}(u, v; u', v') = \frac{1}{4\pi} \int_{k=0}^{\infty} \frac{1}{k} \left(e^{-ik(u-u')} + e^{-ik(v-v')} \right) \mathrm{d}k$$

- Has the special property that $\alpha_{H_{\mathbb{M}}}(:\mathcal{F}:_{\mathbb{M}}) = \mathcal{F}$, $\forall \mathcal{F} \in \mathfrak{F}_{\mathrm{loc}}(\mathbb{M})$.
- So the "left-hand column" of the previous diagram becomes trivial.

Those α Maps...

- In general, the map α_h is defined for any $h \in \mathcal{C}^\infty(\mathcal{M}^2)$ by

$$\alpha_h(\mathcal{F}) = \sum_{n=0}^{\infty} \frac{\hbar^n}{2^n n!} \int_{\mathcal{M}^{2n}} \prod_{i=1}^{n} h(\mathbf{x}_i;\mathbf{y}_i) \frac{\delta^{2n}\mathcal{F}}{\delta\phi(\mathbf{x}_1)\cdots\delta\phi(\mathbf{y}_n)} \mathrm{dV}^{2n}$$

- For $T_{\mathbf{x}_0}$ we have

$$\frac{\delta^2 T_{\mathbf{x}_0}}{\delta\phi(\mathbf{x})\delta\phi(\mathbf{y})} = \partial_u \delta(\mathbf{x}_0 - \mathbf{x}) \partial_{u'} \delta(\mathbf{x} - \mathbf{y}) \tag{7}$$

- Which simplifies the above map to

$$\alpha_h(T_{\mathbf{x}_0}) = T_{\mathbf{x}_0} + \frac{\hbar}{2} \partial_u \partial_{u'} h(\mathbf{x}_0;\mathbf{x}_0) \tag{8}$$

Calculating the correction

- The two Hadamard distributions

$$\chi^* H_{\text{cyl}}(u, v; u', v') = \frac{1}{4\pi} \sum_{k=1}^{\infty} \frac{1}{k} \left(e^{-ik(u-u')} + e^{-ik(v-v')} \right)$$

$$H_{\mathbb{M}}(u, v; u', v') = \frac{1}{4\pi} \int_{k=0}^{\infty} \frac{1}{k} \left(e^{-ik(u-u')} + e^{-ik(v-v')} \right) \mathrm{d}k$$

- By defining $z_\epsilon = \epsilon + i(u - u')$, can approximate by smooth functions:

$$\partial_u \partial_{u'} H_{\text{cyl}}(u; u') = \frac{1}{4\pi} \lim_{\epsilon \to 0^+} \frac{e^{z_\epsilon}}{(1 - e^{z_\epsilon})^2} \tag{9a}$$

$$\partial_u \partial_{u'} H_{\mathbb{M}}(u; u') = \frac{1}{4\pi} \lim_{\epsilon \to 0^+} -\frac{1}{z_\epsilon^2}. \tag{9b}$$

Calculating the correction

- The RHS of (9a) can be written as a Laurent series

$$\frac{e^{z_\epsilon}}{(1-e^{z_\epsilon})^2} = -\frac{1}{z_\epsilon^2} + \sum_{n=0}^{\infty} \frac{\zeta(-n-1)}{n!} z_\epsilon^n \tag{10}$$

- Hence the quantum correction we want is

$$\begin{aligned} \frac{\hbar}{2} \lim_{u' \to u} \partial_u \partial_{u'} \left(H_{\mathrm{cyl}} - H_{\mathbb{M}}\right) &= \frac{\hbar}{8\pi} \lim_{z_\epsilon \to 0} \sum_{n=0}^{\infty} \frac{\zeta(-n-1)}{n!} z_\epsilon^n \\ &= \frac{\hbar}{8\pi} \frac{-1}{12} \end{aligned}$$

$1 + 2 + \cdots \overset{?}{=} -1/12$

- Taking the $u' \to u$ limit naïvely would instead give

$$\frac{\hbar}{2} \lim_{u' \to u} \partial_u \partial_{u'} \left(W_{\mathrm{cyl}} - W_{\mathbb{M}} \right) \text{“}=\text{”} \frac{\hbar}{8\pi} \left(\sum_{k=1}^{\infty} k - \int_{k=0}^{\infty} k \, \mathrm{d}k \right).$$

- So perhaps $\sum_{\mathbb{N}} k$ is not $-1/12$, but instead $\sum_{\mathbb{N}} k = -1/12 + \int_{\mathbb{R}_+} k \, \mathrm{d}k$...

Thanks for listening!