Quantum Field Theory
Summary Revision Notes

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1 Classical Field Theory

Definition 1.1 *Lagrangian density and action*

The **Lagrangian density** of a field theory, which can be thought of *as* the field theory itself is a function

$$\mathcal{L} = \mathcal{L}(\varphi_a, \partial_\mu \varphi_a). \tag{1.1}$$

This is used mainly to define the **action** of the field theory

$$S[\varphi_a] = \int_{\mathbb{M}} \mathrm{d}^4 x \, \mathcal{L}(\varphi_a, \partial_\mu \varphi_a). \tag{1.2}$$

Proposition 1.1 (*Euler-Lagrange equations*)

The action of a field theory with Lagrangian density $\mathcal{L}$ is minimised when the **Euler-Lagrange equations**

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \right) - \frac{\partial \mathcal{L}}{\partial \varphi_a} = 0 \tag{1.3}$$

are satisfied.

Proof. See appendix A.1 for proof. □

Example 1.1 (*Klein-Gordon Field*)

The **Klein-Gordon** Lagrangian for a real scalar field φ is

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \tfrac{1}{2} m^2 \varphi^2. \tag{1.4}$$

The Euler-Lagrange equation for which, called the *Klein-Gordon equation* is

$$\partial^\mu \partial_\mu \varphi + m^2 \varphi = (\Box + m^2)\varphi = 0. \tag{1.5}$$

Example 1.2 (*Electromagnetic Field*)

In a vacuum (i.e. no charged particles), **Maxwell's electromagnetism** is given by the Lagrangian

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{1.6}$$

Variation of which produces Maxwell's equations

$$\partial_\mu F^{\mu\nu} = 0. \tag{1.7}$$

Proposition 1.2 (*Noether's Theorem*)

If a field theory has an action which is invariant under the action of some Lie group, then there is an associated *conserved current* j^μ such that

$$\partial_\mu j^\mu = 0, \qquad \Rightarrow \qquad \frac{d}{dt}\left(\int_V \mathrm{d}^3x\, j^0\right) = \int_{\partial V} j^i\, \mathrm{d}S_i. \tag{1.8}$$

Proof. See appendix A.2 for proof. □

Example 1.3 (*The Energy-Momentum Tensor*)

A common external symmetry in classical mechanics is that of spatial translation. Consider the action of an infinitesimal translation

$$x^\mu \to X^\mu + \epsilon^\mu \qquad \Rightarrow \qquad \varphi_a \to \varphi_a + \epsilon^\mu \partial_\mu \varphi_a \tag{1.9}$$

A Proofs

A.1 Proof of Proposition 1.1

Proof. For now, we just treat φ_a and $\partial_\mu \varphi_a$ as variables, not functions. As such the variation of the Lagrangian density is

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \varphi_a} \delta \varphi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \delta(\partial_\mu \varphi_a). \tag{A.1.1}$$

Next, for any reasonable variation of φ_a, we would expect that $\delta(\partial_\mu \varphi_a) = \partial_\mu(\delta \varphi_a)$. Assuming this is the case, then integrating by parts (noting that $\mathbb{M}$ has no boundary), the variation of the action is

$$\delta S = \int_{\mathbb{M}} \mathrm{d}^4 x \left[\frac{\partial \mathcal{L}}{\partial \varphi_a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \right] \delta \varphi_a. \tag{A.1.2}$$

Assuming this vanishes for any variation $\delta \varphi_a$ means, by the fundamental lemma of the calculus of variations, that the term in square brackets must also vanish. This results in the Euler-Lagrange equations. □

A.2 Proof of Proposition 1.2 (Nother's Theorem)

Proof. The action of (part of) a Lie group can be represented by action of the corresponding Lie algebra. Basically, we consider an 'infinitesimal' transformation

$$\varphi_a \to \varphi_a + \delta \varphi_a = \varphi_a + X_a(\varphi). \tag{A.2.1}$$

The condition for invariance, that $\int_{\mathbb{M}} \delta \mathcal{L} \, \mathrm{d}^4 x$ vanishes, allows us to say that $\delta L = \partial_\mu F^\mu$ (ignoring cohomology). Varying the Lagrangian using (A.2.1) gives us

$$\begin{aligned} \delta \mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \varphi_a} X_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \partial_\mu X_a \\ &= \left[\frac{\partial \mathcal{L}}{\partial \varphi_a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \right) \right] X_a + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} X_a \right). \end{aligned} \tag{A.2.2}$$

□