Symmetries, Fields and Particles
Summary Notes

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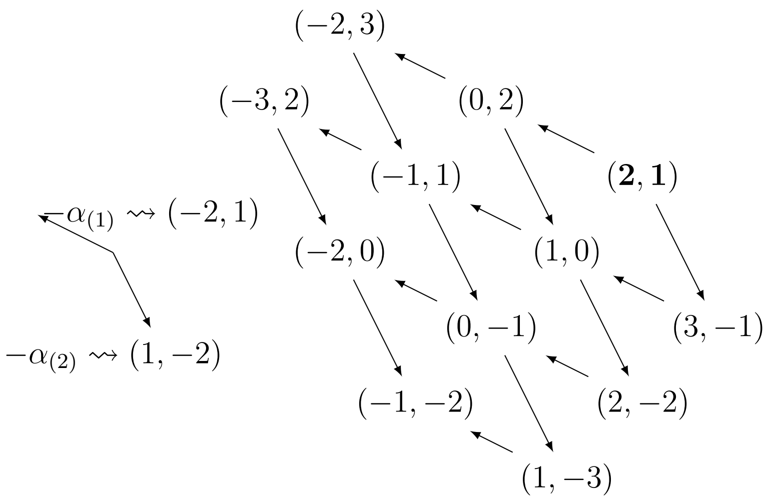


Contents

1 Lie Groups and Lie Algebras

1.1 Basic Definitions

Definition 1.1 *Lie group*

A **Lie group** is a group G that admits a smooth manifold structure, i.e. is locally diffeomorphic to $\mathbb{R}^n$, where n is the *dimension* of the Lie group. Furthermore, the maps L_g, R_g and Inv : $G \to G$ defined by

$$L_g(h) = gh, \qquad R_g(h) = hg, \qquad \text{Inv}(h) = h^{-1} \tag{1.1}$$

should be diffeomorphisms of G.

Definition 1.2 *Left invariant*

A vector field $X \in \mathfrak{X}(G)$ is **left invariant** if

$$(L_g)_* X|_h = X|_{gh}. \tag{1.2}$$

In words, the pushforward of the vector field by the L_g diffeomorphism is simply the value of the vector field at the target point.

Proposition 1.1

The space of left-invariant vector fields is an $n = \dim(G)$ dimensional vector space which is homeomorphic to T_eG, the tangent space to the identity of G.

Proof Omitted.

Definition 1.3 *Lie algebra*

A **Lie algebra** is a vector space $\mathfrak{g}$ equipped with a bilinear operator $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying

(i) (Anti-Commutativity) $[x, y] = -[y, x]$, $\forall\ x, y \in \mathfrak{g}$,

(ii) (Jacobi Identity) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$, $\forall\ x, y, z \in \mathfrak{g}$.

Proposition 1.2

The space of left-invariant vector fields on a Lie group G is closed under the Lie bracket $[X, Y] \circ f := X \circ (Y \circ f) - Y \circ (X \circ f)$, and thus forms a Lie algebra $\mathfrak{g}$.

Proof Omitted.

1.2 Matrix Groups

Most groups we will be dealing with will be subgroups of $GL(V)$, the space of invertible linear operators on a vector space V (which itself will often be $\mathbb{R}^n$ or $\mathbb{C}^n$). Here we will define some such Lie groups along with their associated algebras.

Remark. As a general rule, as we tend to consider subgroups of the form $H = f^{-1}(c)$, where $f : GL(V) \to \mathbb{F}^n$ is a smooth map with regular value c. In this case, we can define the associated Lie algebra $\mathfrak{h}$ as the set of matrices M such that

$$\lim_{\epsilon \to 0} \frac{f(\mathbb{1} + \epsilon M) - c}{\epsilon} = 0. \tag{1.3}$$

Example 1.1

The **special linear group** is the level set $SL(n,\mathbb{F}) = \det^{-1}(1) \subset GL(n,\mathbb{F})$. The Lie algebra is $\mathfrak{su}(n,\mathbb{F}) := \{M \in GL(n,\mathbb{F}) : \text{Tr}(M) = 0\}$

Example 1.2

The **orthogonal groups** are defined by $O(p,q) = \{M : M^T\eta M = \eta\}$ using a metric η of signature (p,q). The Lie algebra $\mathfrak{o}(p,q)$ consists of operators which are skew-adjoint with respect to the metric. There are also the **special orthogonal groups** $SO(p,q) = O(p,q) \cap SL(p+q,\mathbb{R})$, the Lie algebra being similarly defined. Finally, if the field is $\mathbb{C}$, with a Hermitian inner product, we instead define the **(special) unitary group** $(S)U(n)$.

1.3 Some Properties of Lie Algebras

Here we will define some objects associated with Lie algebras that will be useful for finding results later on.

Definition 1.4 *Structure constants*

For a finite dimensional Lie algebra $\mathfrak{g}$ with a basis $\{T^a\}_{a=1}^{\dim \mathfrak{g}}$, then we can describe the structure of the Lie algebra using the **structure constants**. A set of numbers f^{ab}_c such that

$$[T^a, T^b] = f^{ab}_c T^c. \tag{1.4}$$

Remark. Using a set of structure constants, we can express the properties of a Lie algebra from Definition 1.3 as

(i) $f^{ab}_c = -f^{ba}_c$,

(ii) $f^{[ab}_c f^{d]c}_c = 0$.

Definition 1.5 *Isomorphism (Lie algebra)*

Two Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ are **isomorphic** if there exists a homomorphism $\varphi : \mathfrak{g} \to \mathfrak{g}'$ which commutes with the Lie brackets, i.e.

$$[\varphi(x), \varphi(y)]' = \varphi([x,y]) \quad \forall\ x, y \in \mathfrak{g}. \tag{1.5}$$

Remark. As one might expect, isomophism is the natural notion of 'equivalence' between Lie algebras.

Definition 1.6 *Lie subalgebra*

A **Lie subalgebra** $\mathfrak{h} \subset \mathfrak{g}$ is a subspace of $\mathfrak{g}$ that is closed under the Lie bracket of $\mathfrak{g}$, i.e. is itself a Lie algebra.

Definition 1.7 *Ideal*

An **ideal** $\mathfrak{h}$ of $\mathfrak{g}$ is a Lie subalgebra which, for $h \in \mathfrak{h}, x \in \mathfrak{g}$ satisfies

$$[h, x] \in \mathfrak{h}. \tag{1.6}$$

Remark. Stealing some notation to be introduced later, an alternative definition of an ideal is a subspace $\mathfrak{h}$ such that $\text{Ad}_h : \mathfrak{g} \to \mathfrak{h}$, $\forall\ h \in \mathfrak{h}$.

Example 1.3

The two **trivial** ideals of *any* Lie algebra $\mathfrak{g}$ are $\{0\}$ and $\{\mathfrak{g}\}$ itself

Example 1.4

The **derived algebra** is the ideal

$$\mathfrak{i}(\mathfrak{g}) := \{[x,y] : x, y \in \mathfrak{g}\}. \tag{1.7}$$

A related ideal is the **centre**, defined by

$$J(\mathfrak{g}) := \{x \in \mathfrak{g} : [x,y] = 0 \forall\ y \in \mathfrak{g}\} \tag{1.8}$$

Remark. Again, we can steal the notation of [LINK] to define the ideal as

$$\mathfrak{i}(\mathfrak{g}) := \bigcup_{x \in \mathfrak{g}} \mathrm{Img}(\mathrm{Ad}_x). \tag{1.9}$$

And the centre as

$$J(\mathfrak{g}) := \bigcap_{x \in \mathfrak{g}} \mathrm{Ker}(\mathrm{Ad}_x). \tag{1.10}$$

Definition 1.8 *Abelian (Lie algebra)*

A Lie algebra $\mathfrak{g}$ is **Abelian** if its Lie bracket is the trivial map

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \{0\}. \tag{1.11}$$

Remark. For an Abelian Lie algebra, $\mathfrak{i}(\mathfrak{g}) = \{0\}$, and $J(\mathfrak{g}) = \mathfrak{g}$.

Definition 1.9 *Simple*

A Lie algebra $\mathfrak{g}$ is **simple** if it is not Abelian and has *only* trivial ideals

Remark. Similarly to the above, for a simple Lie algebra, $\mathfrak{i}(\mathfrak{g}) = \mathfrak{g}$, and $J(\mathfrak{g}) = \{0\}$.

1.4 The Exponential Map

2 Representations

2.1 Basic Definitions

Definition 2.1 *Representation (of a Lie algebra)*

A **representation** of a *Lie algebra* $\mathfrak{g}$ is an isomorphism

$$R : \mathfrak{g} \to \mathrm{End}(V), \tag{2.1}$$

where the Lie bracket for $\mathrm{End}(V)$ is a commutator. The vector space V is known as the **representation space** and the dimension of V is the **dimension** of the representation.

Definition 2.2 *Representation (of a group)*

A **representation** of a *group* G, which need *not* be a Lie group, is a group isomorphism

$$D : G \to GL(V). \tag{2.2}$$

Where, in this case, the isomorphism condition is that D commutes with the group multiplication $D(gh) = D(g)D(h)$. The representation space and dimension of the representation are defined exactly as above.

Proposition 2.1

If $\text{Exp}(\mathfrak{g}) = H \subset G$ is bijective, then a representation R of $\mathfrak{g}$ 'exponentiates' to the representation $D(\text{Exp}(x)) = \text{Exp}(R(x))$ of H.

Proof Omitted.

Example 2.1

(i) The **trivial representation** of a Lie algebra is the trivial map

$$R_0 : \mathfrak{g} \to \text{End}\left(\{0\}\right). \tag{2.3}$$

(ii) If $\mathfrak{g} \subset GL(V)$, then the **fundamental representation** of $\mathfrak{g}$ is the inclusion map

$$R_f \equiv i : \mathfrak{g} \hookrightarrow GL(V). \tag{2.4}$$

(iii) The **adjoint representation** of a Lie algebra utilises the fact that it is itself a vector space. The fact that

$$\text{Ad} : x \mapsto (\text{Ad}_x : y \mapsto [x, y]) \in GL(\mathfrak{g}) \tag{2.5}$$

is a Lie algebra isomorphism can be proved using the Jacobi identity.

Definition 2.3 *Isomorphism (representation)*

Two representations, R_1 and R_2, of a Lie algebra $\mathfrak{g}$ are **equivalent** (or **isomorphic**) if there is a linear bijection between the representation spaces $S : V_1 \to V_2$ such that

$$R_2(x) = SR_1(x)S^{-1}, \quad \forall\, x \in \mathfrak{g}. \tag{2.6}$$

Definition 2.4 *Invariant subspace*

Given a representation $R : \mathfrak{g} \to GL(V)$, an **invariant subspace** U of V is a vector subspace such that

$$R(x) : V \to U, \quad \forall\, x \in \mathfrak{g}. \tag{2.7}$$

Remark. The two trivial invariant subspaces of a representation space V are $\{0\}$ and V.

Definition 2.5 *Irreducible representation (irrep)*

A representation of a Lie algebra is **irreducible** (it is an **irrep**) if it *only* has trivial invariant subspaces.

2.2 The Representation Theory of $\mathfrak{su}(2)$

The Lie algebra $\mathfrak{su}(2)$ is actually a *real* vector space, as is simply by the fact that $x^\dagger = -x \Rightarrow (ix)^\dagger = ix$, thus the skew-adjoint condition is not $\mathbb{C}$ linear. The standard basis of $\mathfrak{su}(2)$ consists of the *Pauli matrices* $\{\sigma_i\}_{i=1}^3$. However, if we *complexify* the vector space of $\mathfrak{su}(2)$, we end up with a far more interesting Lie algebra.

Definition 2.6 *Cartan-Weyl basis*

The **Cartan-Weyl basis** of $\mathfrak{su}(2)$ consists of the three generators

$$H = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.8a}$$

$$E^+ = \tfrac{1}{2}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{2.8b}$$

$$E^- = \tfrac{1}{2}(\sigma_1 - i\sigma_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.8c}$$

In this basis, the Lie bracket is determined by

$$[H, E^\pm] = \pm 2E^\pm, \quad [E^+, E^-] = H. \tag{2.9}$$

Remark. With respect to this basis, the operator Ad_H is diagonal, as

$$\mathrm{Ad}_H H = 0, \quad \mathrm{Ad}_H E^\pm = \pm 2E^\pm. \tag{2.10}$$

For [REASONS], this means that for any finite dimensional representation R, $R(H)$ is also diagonalisable.

Proposition 2.2

Let $R : \mathfrak{su}(2) \to GL(V)$ be a finite dimensional *irreducible* representation. Then for any eigenvalue v of $R(H)$, the set

$$\{R(E^\pm)^n v \neq 0 : n \in \mathbb{Z}^+\}$$

forms an eigenbasis of V with respect to $R(H)$.

Proof. Firstly, we will show that, if $R(H)v = \lambda v$, the non-zero vectors $R(E^\pm)^n v$ are indeed eigenvectors of $R(H)$. Consider

$$\begin{aligned} R(H)\left[R(E^\pm)v\right] &= R(E^\pm)\left[R(H)v\right] + \left[R(H), R(E^\pm)\right] v, \\ &= R(E^\pm)(\lambda v) + R([H, E^\pm])v, \\ &= \lambda R(E^\pm)v \pm 2R(E^\pm)v = (\lambda \pm 2)\left[R(E^\pm)v\right]. \end{aligned} \tag{2.11}$$

A simple inductive argument then allows us to conclude that the result holds for any n such that $R(E^{\pm})^n v \neq 0$.

Secondly, as the $R(E^{\pm})^n v$ have different eigenvalues with respect to $R(H)$, they must be linearly independent. Therefore, as the representation is finite dimensional, there must be values n_+, n_- such that

$$R(E^{\pm})^{n_\pm} v \neq 0, \quad R(E^{\pm})^{n_\pm+1} v = 0. \tag{2.12}$$

Thus the set forms a basis of some subspace

$$U = \left(\bigoplus_{n=0}^{n_+} \big[R(E^+)^n v\big] \right) \oplus \left(\bigoplus_{n=1}^{n_-} \big[R(E^-)^n v\big] \right) \tag{2.13}$$

of V. Our final argument is to show that this subspace is *invariant* under the representation and thus, by irreducibility, must be V itself. Clearly $R(E^+)\left[R(E^+)^n v\right] = R(E^+)^{n+1} \in U$, and similar for E^-. But what about $R(E^{\pm})\left[R(E^{\mp})^n v\right]$? We can show that this is also an eigenvector of $R(H)$, as

$$\begin{aligned} R(H)\left[R(E^+)R(E^-)v\right] &= \big(R(E^+)R(H) + 2R(E^+)\big) R(E^-)v, \\ &= (\lambda - 2)R(E^+)R(E^-)v + 2R(E^+)R(E^-)v, \\ &= \lambda \left[R(E^+)R(E^-)v\right]. \end{aligned} \tag{2.14}$$

For the space U to be invariant, we must then have $R(E^+)R(E^-)v \propto v$, note this is *not* guaranteed by the vectors sharing an eigenvalue, as we have not shown/assumed $R(H)$ to be non-degenerate. To do this, we shall take our initial eigenvector to be v_Λ, the **highest weight vector**, which is defined such that $n_+ = 0$, implying $n_- = \text{Dim}(U) - 1$. In this case we have

$$\begin{aligned} R(E^+)R(E^-)v_\Lambda &= [R(E^+), R(E^-)]v_\Lambda, \\ &= R(H)v_\Lambda = \Lambda v_\Lambda. \end{aligned} \tag{2.15}$$

By induction, one can then prove that

$$R(E^+)R(E^-)v_{\Lambda-2\ell} \propto v_{\Lambda-2\ell} \tag{2.16}$$

where $v_{\Lambda-2\ell} := R(E^-)^\ell v_\Lambda$, i.e. $\Lambda - 2\ell$ is the eigenvalue of $v_{\Lambda-2\ell}$, known as the **weight**, with respect to $R(H)$. Thus we can rewrite U, which we have now proved to be an invariant subspace, and hence V, as

$$U = V = \bigoplus_{\ell=0}^{N-1} \text{Span}\{v_{\Lambda-2\ell}\}. \tag{2.17}$$

□

Remark. Firstly, it is worth pointing out that the component eigenspaces are known as **weight spaces**.

Secondly, as a corollary, we can relate Λ to the dimension of V as

2.3 Derived Representations

Definition 2.7 *Conjugate representation*

If R is a rep of $\mathfrak{g}$, then its **conjugate representation** is $\bar{R} : x \mapsto R(x)^*$. The meaning of the conjugation $R(x)^*$ is in general rather abstract. However, if we have a matrix representation of $R(x)$, then $R(x)^*$ is simply the matrix whose entries are the typical complex scalar conjugate of the corresponding entries of $R(x)$.[1]

[1] Perhaps for any finite dimensional Hilbert space $\mathcal{H}$, as any operator can be written as $\mathcal{O} = \sum_{ij} c_{ij} |i\rangle\langle j|$, the conjugate operator is $\mathcal{O}^* = \sum_{ij} c_{ij}^* |i\rangle\langle j|$, where $\{|i\rangle\}_{i=1}^n$ is an orthonormal basis of $\mathcal{H}$

Definition 2.8 *Direct sum & tensor product*

Given two representations $R_{1/2} : \mathfrak{g} \to GL(V_{1/2})$, we can combine them in two different ways

1. The **direct sum** $R_1 \oplus R_2$ has a dimension $\text{Dim}(V_1) + \text{Dim}(V_2)$, and is defined by
$$\left[\big(R_1 \oplus R_2\big)(x)\right](v_1 \oplus v_2) = \big(R_1(x)v_1\big) \oplus \big(R_2(x)v_2\big). \tag{2.18}$$

2. The **tensor product** $R_1 \otimes R_2$ had dimension $\text{Dim}(V_1)\text{Dim}(V_2)$, and is defined by
$$\left[\big(R_1 \otimes R_2\big)(x)\right](v_1 \otimes v_2) = \big(R_1(x)v_1\big) \otimes v_2 + v_1 \otimes \big(R_2(x)v_2\big). \tag{2.19}$$

Definition 2.9 *Fully reducible*

A representation is **fully reducible** if it can be written as a direct sum of finitely many *non-trivial* irreps.

Proposition 2.3

All finite number of tensor products of finite dimensional irreps of a complex simple Lie algebra are fully reducible. I.e., if $|\mathcal{R}|, |\mathcal{R}'| \in \mathbb{N}$
$$\bigotimes_{R \in \mathcal{R}} R = \bigoplus_{R' \in \mathcal{R}'} \mathfrak{M}(R')R', \tag{2.20}$$
where $\mathfrak{M}(R') \in \mathbb{Z}$ denotes the *multiplicity* of the rep R' in the decomposition.

Proof Omitted.

2.4 The Clebsch-Gordan Decomposition

Generally speaking, the Clebsch-Gordan decomposition is the process by which we explicitly perform the decomposition of (2.20) for the tensor product of a pair of representations (i.e. when $|\mathcal{R}| = 2$). However, we shall limit our attention to the case $\mathfrak{g} = \mathfrak{su}(2)$.

As irreps of $\mathfrak{su}(2)$ are determined up to similarity by their highest weights, we can express a decomposition generally as

$$R_\Lambda \otimes R_{\Lambda'} = \bigoplus_{\Lambda'' \in \mathbb{Z}^+} \mathfrak{M}^{\Lambda''}_{\Lambda\Lambda'} R_{\Lambda''}. \tag{2.21}$$

By considering eigenvalues of $R_\Lambda \otimes R_{\Lambda'}$, we see that the values of Λ'' for which $\mathfrak{M} \neq 0$ are those for which $\Lambda'' = \lambda + \lambda'$ for some $|\lambda| \leq \Lambda$, $|\lambda'| \leq \Lambda'$. Thus, the highest such is $\Lambda'' = \Lambda + \Lambda'$. The highest weight of $\left(R_\Lambda \otimes R_{\Lambda'}\right)/R_{\Lambda+\Lambda'}$[1] must then be $\Lambda + \Lambda' - 2$. Thus, we can assume that it contains $R_{\Lambda+\Lambda'-2}$. Recall that if V_Λ is the rep space of R_Λ, then $\mathrm{Dim}(V_\Lambda) = \Lambda + 1$, thus with the first two elements of the decomposition found thus far, we have identified a $2(\Lambda + \Lambda')$ dimensional subspace. If we continue the argument to $R_{\Lambda+\Lambda'-2\ell}$ then the dimension of the partial direct sum's rep space is

$$\sum_{i=0}^{\ell} (\Lambda + \Lambda' - 2i + 1) = (\ell + 1)(\Lambda + \Lambda' - \ell + 1). \tag{2.22}$$

Using the RHS of (2.22), we see that the partial sum has the same dimension as the tensor product if $\ell = \Lambda, \Lambda'$. Note that this argument only works when each of the $R_{\Lambda-\Lambda'-2\ell}$ are distinct, thus the correct solution is the lowest of Λ and Λ'. This gives us $\Lambda - \Lambda' - 2\ell = |\Lambda - \Lambda'|$ and hence

$$R_\Lambda \otimes R_{\Lambda'} = R_{\Lambda+\Lambda'} \oplus R_{\Lambda+\Lambda'-2} \oplus \cdots \oplus R_{|\Lambda-\Lambda'|+2} \oplus R_{|\Lambda-\Lambda'|}. \tag{2.23}$$

Example 2.2

The z component of an electron's spin has eigenvalues $\pm\frac{1}{2}\hbar$, thus it can be considered to form a 2 dimensional rep R_1 of $\mathfrak{su}(2)$. If we want to consider the spin eigenstates for a system containing a pair of electrons, from the Clebsch-Gordan decomposition we see that [WE GET A SINGLET (R_0) REP AND A TRIPLET (R_2)].

[1] This is a slight abuse of notation, this 'remainder' representation is basically defined such that it satisfies $R_{\Lambda+\Lambda'} \oplus \left(R_\Lambda \otimes R_{\Lambda'}\right)/R_{\Lambda+\Lambda'} = \left(R_\Lambda \otimes R_{\Lambda'}\right)$

3 The Cartan Classification

In which *all possible* finite dimensional semi-simple Lie algebras are determined and classified.

3.1 The Killing Form

Definition 3.1 *Killing form*

The **Killing form** of a Lie algebra $\mathfrak{g}$ over the field $\mathbb{F}$ is the symmetric bilinear map $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ defined by

$$\kappa(x, y) = \mathrm{Tr}\left(\mathrm{Ad}_x \circ \mathrm{Ad}_y\right). \tag{3.1}$$

The symmetry and linearity of the inner product are inherited from the cyclicity and linearity of the trace operation respectively.

Proposition 3.1

The Killing form of a Lie algebra is **invariant**, defined as the property

$$\kappa(\mathrm{Ad}_z x, y) = -\kappa(x, \mathrm{Ad}_z y), \tag{3.2}$$

i.e. Ad_z is a skew-adjoint operator $\forall\ z \in \mathfrak{g}$.

Proof. Basically use cyclicity and the fact that Ad is a rep of $\mathfrak{g}$. □

Definition 3.2 *Semisimple*

A Lie algebra is **semisimple** if it has no Abelian ideals. Equivalently, a semisimple Lie algebra is a direct sum of finitely many *simple* Lie algebras.

Theorem 3.1 *(Cartan's criterion)*

A finite dimensional Lie algebra is semisimple *if and only if* its Killing from is non-degenerate.

Partial Proof. We shall prove that a Lie algebra $\mathfrak{g}$ with an Abelian ideal $\mathfrak{j}$ (i.e. one that is not semisimple) has a degenerate Killing form. This is fairly easy, as we can show that $\kappa(j, x) = 0\ \forall\ j \in \mathfrak{j}, x \in \mathfrak{g}$. To do this, we prove that all eigenvalues of $\mathrm{Ad}_j \circ \mathrm{Ad}_x$ are 0. Suppose that $y \in \mathfrak{g}$ is an eigenvector. The fact that $\mathfrak{j}$ is an ideal means that

$\mathrm{Ad}_j\left(\mathrm{Ad}_x y\right) \in \mathfrak{j}$, thus either $y \in \mathfrak{j}$ or its eigenvalue is 0. Assuming the former to be true, then $\mathrm{Ad}_x y \in \mathfrak{j}$. Then, as $\mathfrak{j}$ is Abelian, we have that $\mathrm{Ad}_j\left(\mathrm{Ad}_x y\right) = 0$. Thus, again, the eigenvalue is 0. As the Killing form is the sum over the eigenvalues, it too is zero. □

3.2 Complexification

Definition 3.3 *Complexification*

Given a basis $\{T^a\}$ of a Lie algebra $\mathfrak{g}$ over $\mathbb{R}$, the **complexification** $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ is simply $\mathrm{Span}_{\mathbb{C}}\{T^a\}$.

Definition 3.4 *Real form*

Given a complex Lie algebra $\mathfrak{g}$, a real Lie algebra $\mathfrak{h}$ is said to be a **real form** of $\mathfrak{g}$ if its complexification is $\mathfrak{g}$, i.e. $\mathfrak{h}_{\mathbb{C}} = \mathfrak{g}$.

Remark. Note that, in general, a complex Lie algebra admits multiple *inequivalent* real forms. In fact, one can find real Lie algebras $\mathfrak{g}$ such that their complexifications $\mathfrak{g}_{\mathbb{C}}$ admit real forms inequivalent to the original $\mathfrak{g}$.

Example 3.1

To demonstrate the above remark, consider $\mathfrak{su}(2)_{\mathbb{C}}$. As the complexification invalidates the skew-Hermitian condition, this is simply the set of 2×2 traceless complex matrices, i.e. $\mathfrak{sl}(2, \mathbb{C})$. Now this also has the real form $\mathfrak{sl}(2, \mathbb{R})$, the set of traceless real matrices, which is inequivalent to $\mathfrak{su}(2)$.

Definition 3.5 *Compact type*

A real Lie algebra is of **compact type** if its Killing form is negative definite, i.e. $\kappa(x, x) < 0$, $\forall\ x \in \mathfrak{g}$.

Remark. Whilst we shall not prove it here, the reason for this name is that if a Lie group is topologically compact, then its associated Lie algebra will always be of compact type.

Theorem 3.2

Every complex, semisimple, finite-dimensional Lie algebra has a real form of compact type.

3.3 The Cartan-Weyl Basis

Definition 3.6 *Ad-diagonalisable*

An element $x \in \mathfrak{g}$ is **ad-diagonalisable** if Ad_x is diagonalisable.

Definition 3.7 *Cartan subalgebra*

A **Cartan subalgebra** (abbreviated CSA) $\mathfrak{h}$ is a *maximal Abelian subalgebra* of *ad-diagonalisable* elements of $\mathfrak{g}$. I.e. if, $\forall\ H, H' \in \mathfrak{h}$,

(i) Ad_H is diagonalisable

(ii) $[H, H'] = 0$

(iii) If $x \notin \mathfrak{h}$ is ad-diagonalisable, then $\exists\, \tilde{H} \in \mathfrak{h}$ such that $[x, \tilde{H}] \neq 0$.

Proposition 3.2

All CSAs of a Lie algebra have the same dimension.

Proof Omitted.

Definition 3.8 *Rank (Lie algebra)*

The **rank** of a Lie algebra is the dimension of its CSAs.

Example 3.2

The Lie algebra $\mathfrak{su}(2)_{\mathbb{C}}$ has rank 1, as H is ad-diagonalisable, but $E^{\pm}$ are not, thus $\mathfrak{h} = \text{Span}_{\mathbb{C}}\{H\}$.
In general, we can define a CSA of $\mathfrak{su}(n)_{\mathbb{C}}$ as $\mathfrak{h} = \text{Span}\{H^i\}_{i=1}^r$, where

$$(H^i)_{ab} = \delta_{ai}\delta_{bi} - \delta_{a(i+1)}\delta_{b(i+1)}. \tag{3.3}$$

From this it follows that $r = n - 1$ is the rank of $\mathfrak{su}(n)_{\mathbb{C}}$.

Remark. As $\mathfrak{h}$ is Abelian, $[\text{Ad}_H, \text{Ad}_{H'}] = 0,\ \forall\ H, H' \in \mathfrak{h}$. Thus, we can find a basis of $\mathfrak{g}$ which is an eigenbasis of *all* Ad_H simultaneously. Naturally the intersubsection of the kernels of these maps is $\mathfrak{h}$ itself, as this is just a restatement of the maximality condition.

Definition 3.9 *Step operators, roots & the root set*

Eigenvectors outside of $\mathfrak{h}$ are known as **step operators**. The are denoted E^α, where α is a linear functional on $\mathfrak{h}$, called a **root**, such that

$$\mathrm{Ad}_H E^\alpha = \alpha(H)E^\alpha. \tag{3.4}$$

In words, the root encodes the eigenvalues of E^α for all Ad_H, $H \in \mathfrak{h}$. The collection of all such roots is Φ, the **root set** of $\mathfrak{h}$.

Example 3.3

[DO EXAMPLE OF $\mathfrak{su}(2)_\mathbb{C}$ USING ABOVE BASIS OF CSA]

Definition 3.10 *Cartan-Weyl basis*

The **Cartan-Weyl basis** for a Lie algebra $\mathfrak{g}$ is the basis given by

$$\mathcal{B}(\mathfrak{g}) = \{H_i\}_{i=1}^r \cup \{E^\alpha\}_{\alpha\in\Phi}, \tag{3.5}$$

such that $\{H_i\}$ spans a Cartan subalgebra $\mathfrak{h} \in \mathfrak{g}$ with root set Φ.

Remark. The Lie algebra can then be expressed in this basis as

$$[H, H'] = 0, \ \forall\, H, H' \in \mathfrak{h}, \tag{3.6a}$$
$$[H, E^\alpha] = \alpha(H)E^\alpha, \ \forall\, H \in \mathfrak{h}, \alpha \in \Phi, \tag{3.6b}$$
$$[E^\alpha, E^\beta] = \begin{cases} N_{\alpha\beta}E^{\alpha+\beta} & \text{If } \alpha+\beta \in \Phi, \\ 0 & \text{If } \alpha+\beta \neq 0, \notin \Phi. \end{cases} \tag{3.6c}$$

The first two of these are easy enough to see. For (3.6c), consider the following application of the Jacobi identity

$$\begin{aligned} \big[H, [E^\alpha, E^\beta]\big] &= -\big[E^\alpha, [E^\beta, H]\big] - \big[E^\beta, [H, E^\alpha]\big], \\ &= (\alpha(H) + \beta(H))[E^\alpha, E^\beta]. \end{aligned} \tag{3.7}$$

Thus, if the RHS does not vanish, $[E^\alpha, E^\beta]$ is another step operator of $\mathfrak{g}$ with root $\alpha+\beta$. We shall return to the case $\beta = -\alpha$ later.

Proposition 3.3 (*Some facts step operators and the Killing form*)

(i) $\kappa(H, E^\alpha) = 0,\ \forall\, H \in \mathfrak{h}, \alpha \in \Phi$

(ii) $\kappa(E^\alpha, E^\beta) = 0,\ \forall\, \alpha \neq -\beta$

(iii) $\forall\, H \in \mathfrak{h},\ \exists\, H' \in \mathfrak{h}$ s.t. $\kappa(H, H') \neq 0$

(iv) $\forall\, \alpha \in \Phi, -\alpha \in \Phi$, and $\kappa(E^\alpha, E^{-\alpha}) \neq 0$.

(i) *Proof.* Recall that $\forall\, \alpha \in \Phi, \exists H' \in \mathfrak{h}$ such that $\alpha(H') \neq 0$, thus

$$\kappa\left(H, \alpha(H')E^\alpha\right) = \kappa\left(H, \mathrm{Ad}_{H'}E^\alpha\right) = -\kappa\left(\mathrm{Ad}_{H'}H, E^\alpha\right) = 0 \tag{3.8}$$

$\Rightarrow \kappa(H, E^\alpha) = 0$ □

(ii) *Proof.* As before, we multiply the expression by $\alpha(H')$ for some $H' \in \mathfrak{h}$ to get

$$\begin{aligned}\kappa\left(\alpha(H')E^\alpha, E^\beta\right) &= \kappa\left(\mathrm{Ad}_{H'}E^\alpha, E^\beta\right) \\ &= \kappa\left(E^\alpha, -\mathrm{Ad}_{H'}E^\beta\right) = -\beta(H')\kappa(E^\alpha, E^\beta).\end{aligned} \tag{3.9}$$

Thus, $\left(\alpha(H') + \beta(H')\right)\kappa(E^\alpha, E^\beta) = 0$. As our choice of H' was arbitrary, this can only hold $\forall\, H' \in \mathfrak{h}$ if either $\alpha = -\beta$, or the desired equation is satisfied. □

(iii) *Proof.* This is merely a consequence of (i) and the fact that κ is non-degenerate. □

(iv) *Proof.* Again, this is just a consequence of (i), (ii) and the non-degeneracy of κ. □

Remark. As, by definition, the adjoint operators of the CSA share eigenvalues, the restriction $\kappa|_{\mathfrak{h}\times\mathfrak{h}}$ has a particularly nice form. Using the fact that a trace is a sum over eigenvalues, we have that

$$\kappa(H, H') = \sum_{\delta\in\Phi} \delta(H)\delta(H'). \tag{3.10}$$

Remark. Note that (i) implies that $\kappa|_{\mathfrak{h}\times\mathfrak{h}}$ is non-degenerate, thus it induces an isomorphism

$$\begin{aligned}\hat{\kappa} : \mathfrak{h} &\to \mathfrak{h}^*, \\ H &\mapsto \left(H' \mapsto \kappa(H, H')\right).\end{aligned} \tag{3.11}$$

Moreover, we can use the *inverse* of this isomorphism to map $\alpha \in \Phi \subset \mathfrak{h}^*$ to unique elements of $\mathfrak{h}$. Thus we can define H^α such that

$$\kappa\left(H^\alpha, H\right) = \alpha(H). \tag{3.12}$$

We then use this map to define an inner product on Φ as

$$(\alpha, \beta) := \kappa(H^\alpha, H^\beta) = \beta(H^\alpha) = \alpha(H^\beta). \tag{3.13}$$

We can also rewrite this inner product using (3.10) as

$$(\alpha, \beta) = \sum_{\delta \in \Phi} (\delta, \alpha)(\delta, \beta). \tag{3.14}$$

Recall that earlier we wished to compute $[E^\alpha, E^{-\alpha}]$. From (3.7), we know that the result, should it be non-zero, commutes with all $H \in \mathfrak{h}$. Furthermore, observe that

$$\kappa([E^\alpha, E^{-\alpha}], H) = -\kappa(E^{-\alpha}, [E^\alpha, H]) = \alpha(H)\kappa(E^\alpha, E^{-\alpha}). \tag{3.15}$$

From property (iv), we know that $\kappa(E^\alpha, E^{-\alpha}) \neq 0$, thus we can divide throught to see that

$$\kappa\left(\frac{[E^\alpha, E^{-\alpha}]}{\kappa(E^\alpha, E^{-\alpha})}, H\right) = \alpha(H), \qquad \forall\, H \in \mathfrak{h}. \tag{3.16}$$

thus $[E^\alpha, E^{-\alpha}] = H^\alpha \kappa(E^\alpha, E^{-\alpha})$.

Definition 3.11

We can rescale E^α and H^α to

$$e^\alpha := \sqrt{\frac{2}{(\alpha,\alpha)\kappa(E^\alpha, E^{-\alpha})}} E^\alpha, \tag{3.17a}$$

$$h^\alpha := \frac{2}{(\alpha,\alpha)} H^\alpha. \tag{3.17b}$$

Remark. These new vectors satisfy the brackets

$$[h^\alpha, h^\beta] = 0, \tag{3.18a}$$

$$[h^\alpha, e^\beta] = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}, \tag{3.18b}$$

$$[e^\alpha, e^\beta] = \begin{cases} n_{\alpha\beta} e^{\alpha+\beta} & \text{If } \alpha + \beta \in \Phi, \\ h^\alpha & \text{If } \alpha + \beta = 0, \\ 0 & \text{Else.} \end{cases} \tag{3.18c}$$

Definition 3.12 *$\mathfrak{sl}(2)_\alpha$ Subalgebra*

Using the definitions of $h^\alpha, e^{\pm\alpha}$ above, the $\mathfrak{sl}(2)_\alpha$ **subalgebra** of $\mathfrak{g}$ is defined as $\text{Span}_{\mathbb{C}}\{h^\alpha, e^{+\alpha}, e^{-\alpha}\}$, which is naturally isomorphic to $\mathfrak{sl}(2;\mathbb{C})$.

3.4 Root Strings

Definition 3.13 *Root strings*

Let $\alpha, \beta \in \Phi$. The **α-string passing through β** is the set of roots

$$S_{\alpha,\beta} := \{\beta + n\alpha : n \in \mathbb{Z}, \beta + n\alpha \in \Phi\}. \tag{3.19}$$

Associated to the root string is the vector subspace of $\mathfrak{g}$

$$V_{\alpha,\beta} := \text{Span}_{\mathbb{C}}\{e^\gamma : \gamma \in S_{\alpha,\beta}\}. \tag{3.20}$$

Remark. One can fairly easily show that $V_{\alpha,\beta}$ is an invariant space of the adjoint representation of $\mathfrak{sl}(2)_\alpha$, and thus is itself a representation. From (3.18), we see that the weights of this rep are

$$\Lambda_{\alpha,\beta} = \left\{ \frac{2(\alpha,\gamma)}{\alpha,\alpha} : \gamma \in S_{\alpha,\beta} \right\}. \tag{3.21}$$

Using the general form of γ, we see these weights are actually

$$\frac{2(\alpha,\beta)}{(\alpha,\alpha)} + 2n, \tag{3.22}$$

where the valid values of n must fall between two finite values n_- and n_+. From here, we can employ our knowledge of $\mathfrak{su}(2)$ rep theory to declare that

$$\Lambda_{\alpha,\beta} = \{\Lambda, \Lambda - 2, \cdots, -\Lambda + 2, -\Lambda\}, \tag{3.23}$$

for some $\Lambda \in \mathbb{Z}$. Comparing these two definitions we see

$$\Lambda = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} + 2n_+, \quad -\Lambda = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} + 2n_-. \tag{3.24}$$

Adding these together, we arrive at the important result

$$R_{\alpha,\beta} := \frac{2(\alpha,\beta)}{(\alpha,\alpha)} = -(n_+ + n_-) \in \mathbb{Z}. \tag{3.25}$$

Substituting $(\alpha, \beta) = \frac{1}{2}(\alpha, \alpha)R_{\alpha,\beta}$ into (3.14), we see that

$$\begin{aligned}(\alpha, \beta) &= \tfrac{1}{4}(\alpha, \alpha)(\beta, \beta) \sum_{\delta \in \Phi} R_{\alpha,\delta} R_{\beta,\delta}, \\ &= \tfrac{1}{2}(\alpha, \alpha) \frac{(\alpha, \beta)}{R_{\beta,\alpha}} \sum_{\delta \in \Phi} R_{\alpha,\delta} R_{\beta,\delta}.\end{aligned} \tag{3.26}$$

Thus

$$(\alpha, \alpha) = \frac{2R_{\beta,\alpha}}{\sum_{\delta \in \Phi} R_{\alpha,\delta} R_{\beta,\delta}} \in \mathbb{Q}. \tag{3.27}$$

Meaning that the inner product between any two roots is not only real, but rational!

3.5 The Real Geometry of Roots

Proposition 3.4

The root set Φ spans $\mathfrak{h}^*$

Proof. Recall that the non-degeneracy of κ means that $\forall\, \lambda \in \mathfrak{h}, \exists H_\lambda \in \mathfrak{h}$ such that

$$\kappa(H_\lambda, H) = \lambda(H), \quad \forall\, H \in \mathfrak{h}. \tag{3.28}$$

Well, using (3.10), we can write this inner product as

$$\lambda(H) = \sum_{\delta \in \Phi} \delta(H_\lambda) \delta(H), \quad \forall\, H \in \mathfrak{h}. \tag{3.29}$$

Thus $\lambda = \sum_{\delta \in \Phi} \delta(H_\lambda)\, \delta \Rightarrow \lambda \in \text{Span}_{\mathbb{C}} \Phi$. □

Proposition 3.5

Let $\{\alpha_{(i)}\}_{i=1}^r \subset \Phi$ be any set of linearly independent roots, then $\Phi \subset \text{Span}_{\mathbb{R}}\{\alpha_{(i)}\} =: \mathfrak{h}_{\mathbb{R}}$.

Proof. As $\text{Span}_{\mathbb{C}}\{\alpha_{(i)}\} = \mathfrak{h}^* \supset \Phi$, we know that $\beta = \sum_i \beta^i \alpha_{(i)}$ for some potentially complex coefficients β. However, taking the inner product of each side with another $\alpha_{(j)}$, we see that

$$\begin{aligned}(\beta, \alpha_{(j)}) &= \sum_{i=1}^r \beta^i (\alpha_{(i)}, \alpha_{(j)}), \\ \Rightarrow \beta^i &= \sum_j \Delta^{-1}_{ij} (\beta, \alpha_{(j)}),\end{aligned} \tag{3.30}$$

where $\Delta_{ij} := (\alpha_{(i)}, \alpha_{(j)})$ is a matrix that is invertible by the non-degeneracy of κ. From the 'important result', we deduced that all inner products between roots are real. Hence the RHS, and β^i are real. $\square$

Proposition 3.6

Let $\mathfrak{h}_{\mathbb{R}}$ be as before, then the map $(\cdot,\cdot) : \mathfrak{h}_{\mathbb{R}} \times \mathfrak{h}_{\mathbb{R}} \to \mathbb{R}$ is a Euclidean inner product.

Proof. Using (3.14) for $\alpha = \beta$ we have

$$(\alpha, \alpha) = \sum_{\delta \in \Phi} (\delta, \alpha)(\delta, \alpha) = \sum_{\delta \in \Phi} [(\delta, \alpha)]^2 . \tag{3.31}$$

As (δ, α) is real, this sum is always greater than or equal to zero, with equality iff $(\delta, \alpha) = 0 \,\forall\, \delta \in \Phi$. However, the fact that Φ is in the real span of $\{\alpha_{(i)}\}$ means that this can only occur when $\alpha = 0$. $\square$

Remark. The significance of this result is that it allows us to define a norm on $\mathfrak{h}_{\mathbb{R}}$, $|\alpha| = \sqrt{(\alpha, \alpha)}$, as well as apply the Cauchy-Schwarz inequality to define an angle φ between roots such that $(\alpha, \beta) = |\alpha||\beta| \cos\varphi$. Moreover, returning to our integer $R_{\alpha,\beta}$ we now have

$$R_{\alpha,\beta} = 2\tfrac{|\beta|}{|\alpha|} \cos\varphi, \quad R_{\beta,\alpha} = 2\tfrac{|\alpha|}{|\beta|} \cos\varphi. \tag{3.32}$$

Multiplying these results in

$$4\cos^2\varphi = R_{\alpha,\beta} R_{\beta,\alpha} \Rightarrow \cos\varphi = \pm\frac{\sqrt{n}}{2}, \tag{3.33}$$

where $n \in \mathbb{N} \cap [0, 4] = \{0, 1, 2, 3, 4\}$. This restricts the possible angles to only

$$\varphi = \begin{cases} 0 & \text{If } \alpha = \beta, \\ \pi & \text{If } \alpha + \beta = 0, \\ \frac{\pi}{2} & \text{If } (\alpha, \beta) = 0, \\ \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3} & \text{If } (\alpha, \beta) > 0, \\ \frac{5\pi}{6}, \frac{3\pi}{4}, \frac{2\pi}{3} & \text{If } (\alpha, \beta) < 0. \end{cases} \tag{3.34}$$

3.6 Simple Roots

Remark. As $\alpha \in \Phi \Rightarrow -\alpha \in \Phi$, we can split our root system in half. By considering an arbitrary hyperplane of $\mathfrak{h}^*_{\mathbb{R}}$, we can label the roots as either 'positive' or 'negative', depending on which side of the hyperplane they fall on. We shall label each half of the root system under such a cut as $\Phi_{\pm}$ accordingly.

Definition 3.14 *Simple Root*

A **simple root** is a positive root which cannot be expressed as the sum of two other positive roots, i.e. it is an irreducible of the monoid Φ_+.

Proposition 3.7 (*Properties of Simple Roots*)

Let $\alpha, \beta \in \Phi$ be simple roots, then:

(i) $(\alpha - \beta) \notin \Phi$

(ii) The α-string through β has length

$$\ell_{\alpha,\beta} = 1 - 2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \tag{3.35}$$

(iii) $(\alpha, \beta) \leq 0$.

(iv) Any positive root can be written as a linear combination of simple roots with *integer* coefficients, thus the simple roots span $\mathfrak{h}^*_{\mathbb{R}}$

Proof.

(i) $\alpha = (\alpha - \beta) + \beta$, thus if $(\alpha - \beta) \in \Phi_+$, then α is not simple. If $(\alpha - \beta) \in \Phi_-$, then $(\beta - \alpha) \in \Phi_+ \Rightarrow \beta = (\beta - \alpha) + \alpha$ thus β is not simple. As both α and β are simple, $(\alpha - \beta) \notin \Phi_- \cup \Phi_+ = \Phi$

(ii) Consider the root string

$$S_{\alpha,\beta} \coloneqq \{\beta + n\alpha : n \in \mathbb{Z}, \beta + n\alpha \in \Phi\}. \tag{3.36}$$

The first result tells us that we cannot have $n = -1$, thus $n_- = 0$, from (3.25) that

$$n_+ = -2\frac{(\alpha,\beta)}{(\alpha,\alpha)}. \tag{3.37}$$

The desired result is then obtained as $\ell_{\alpha,\beta} = n_+ - n_- + 1$.

(iii) This result follows from the previous result, and the fact that $n_+ \geq 0$.

(iv) Select a root $\beta \in \Phi_+$, if it is simple, we are done. If not, then $\exists\, \beta_1, \beta_2 \in \Phi_+$ such that $\beta = \beta_1 + \beta_2$. For any of these roots which are not simple, repeat this step. When(/if) this process terminates, we will have expressed β as an integer sum of simple roots.

□

Proposition 3.8

Simple roots are linearly independent in $\mathfrak{h}^*_{\mathbb{R}}$.

Proof. Let $\lambda \in \mathfrak{h}^*_{\mathbb{R}}$ be some linear combination of simple roots $\{\alpha_{(i)}\}$. We can split this combination depending on whether the coefficients are positive or negative

$$\lambda = \sum_{c_i>0} c_i\alpha_{(i)} + \sum_{c_i<0} c_i\alpha_{(i)} =: \lambda_+ - \lambda_-. \tag{3.38}$$

Consider now the norm of λ

$$(\lambda, \lambda) = \underbrace{(\lambda_+, \lambda_+) + (\lambda_-.\lambda_-)}_{\geq 0} - 2(\lambda_+, \lambda_-). \tag{3.39}$$

We can show that this must be *strictly* greater than zero, thus proving that λ cannot vanish and the $\alpha_{(i)}$ are linearly independent. Looking at the final term, we have

$$-2(\lambda_+, \lambda_-) = 2\sum_{c_i>0}\sum_{c_j<0} \underbrace{c_i c_j}_{<0} \underbrace{\big(\alpha_{(i)}, \alpha_{(j)}\big)}_{i\neq j \Rightarrow \leq 0} > 0. \tag{3.40}$$

$\square$

Remark. As the simple roots also span Φ, this means that they must form a **basis** of $\mathfrak{h}^*_{\mathbb{R}}$, thus there are exactly r simple roots.

Definition 3.15

The **Cartan** matrix is more a list of integers, and is defined for a set of simple roots $\Phi_S = \{\alpha_{(i)}\}$ as

$$A^{ij} := 2\frac{\big(\alpha_{(i)}, \alpha_{(j)}\big)}{\big(\alpha_{(j)}, \alpha_{(j)}\big)}. \tag{3.41}$$

Remark. From the simple roots we can define the 'generators' - in a sense to be explained later - $\{h^i := h^{\alpha_{(i)}}, e^i_\pm := e^{\pm\alpha_{(i)}}\}_{i=1}^r$, which satisfy

$$[h^i, h^j] = 0, \qquad [h^i, e^j_\pm] = \pm A^{ji} e^j_\pm, \qquad [e^i_+, e^j_-] = \delta_{ij} h^i. \tag{3.42}$$

There is one more result we can obtain, using the rep. theory of $\mathfrak{sl}(2)_{\alpha_{(i)}}$. Specifically, we look at the representation obtained from the adjoint map acting on $e^j_\pm$. From result (ii) of Proposition 3.7, we know that the length of the $\alpha_{(i)}$ string through $\alpha_{(j)}$ terminates at $n_+ = -A^{ji}$. This gives us the maximum value of n for which $(\text{Ad}_{e^i_\pm})^n e^j_\pm \neq 0$,[2] as $n \leq n_+ \Rightarrow \alpha_{(j)} + n\alpha_{(i)} \in S_{ij} \Rightarrow (\text{Ad}_{e^i_\pm})^n e^j_\pm = e^{\pm(\alpha_{(j)}+n\alpha_{(i)})}$. Thus we know that

$$\left(\text{Ad}_{e^i_\pm}\right)^{(1-A^{ji})} e^j_\pm = 0. \tag{3.43}$$

The equations (3.42) and (3.43) together form the **Chevalley-Serre relations**. Note that the idiosyncrasies of these relations for a particular Lie algebra are encoded entirely within the Cartan matrix

Theorem 3.3 (*Cartan Classification - Partial Statement*)

A finite dimensional, simple, complex Lie algebra is determined up to isomorphism by its Cartan matrix.

Proposition 3.9 (*Constraints on the Cartan Matrix*)

(0) $A^{ji} \in \mathbb{Z}$,

(i) $A^{ii} = 2$,

(ii) $A^{ij} = 0 \Leftrightarrow A^{ji} = 0$,

(iii) $A^{ij} < 0 \, \forall \, i \neq j$,

(iv) $A = DS$ for some diagonal matrix D and some positive definite matrix S

Proof. Properties (0) through to (iii) follow immediately from (3.41). To prove (iv), note that A can be decomposed as $A^{ij} = \sum_{k=1}^r D_{ik} S_{kj}$, where

$$D_{ik} := \frac{2\delta_{ik}}{(r_i, r_j)}, \qquad S_{kj} := (r_i, r_j). \tag{3.44}$$

From Proposition 3.6 we know that $(\cdot, \cdot)$ is a Euclidean inner product, thus each entry on the diagonal of D is positive, and S is positive definite. $\square$

Remark. For the derived Lie algebra to be simple(?) we further impose that A cannot be expressed as $A^{(1)} \oplus A^{(2)}$ where $A^{(i)}$ are themselves Cartan matrices, if this *is* the case, we say that A is **reducible**.

Proposition 3.10

For $i \neq j$, the only valid pairs of values for (A^{ij}, A^{ji}) are (order irrelevant): $(0,0), (-1,-1), (-1,-2), (-1,-3)$.

[2] The sign of $e^i_\pm$ is assumed to be the same as that of $e^j_\pm$ as otherwise (i) of 3.7 tells us that the expression vanishes

Proof. Note that (3.33) tells us that $A^{ij} \cdot A^{ji}$ is some integer between 0 and 4. However, we can discount 4, as this implies that $\varphi_{ij} = 0$ which in turn implies that $\alpha_{(i)}$ and $\alpha_{(j)}$ are colinear. Property (iii) of Proposition 3.9 tells us that each of A^{ij}, A^{ji} is a negative integer. The above four pairs are the only way to express each integer between 0 and 3 as a product of two negative integers. $\square$

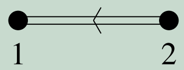
3.7 Dynkin Diagrams

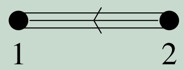
A Dynkin diagram is a pictorial representation of a Cartan matrix. Each *simple* root is represented by a node. The number of edges connecting the i^{th} node to the j^{th} is $\text{Max}\{|A^{ij}|, |A^{ji}|\}$, if this is greater than one then we include an arrow such that $|A^{ij}| > |A^{ji}|$ means the arrow points from the i^{th} node to the j^{th}. For example:

Example 3.4

The three possible Cartan matrices (up to permutation) have Dynkin diagrams:

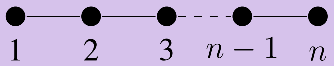
$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \qquad A = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$



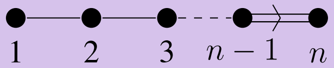


Theorem 3.4 *(Cartan Classification)*

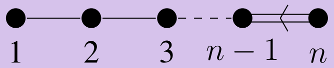
The only possible finite dimensional, simple, complex Lie algebras are the countable families

A_n: 

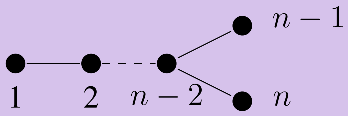
$\rightsquigarrow \mathfrak{su}(n+1)_{\mathbb{C}}$

B_n: 

$\rightsquigarrow \mathfrak{so}(2n+1)_{\mathbb{C}}$

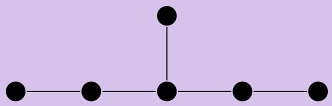
C_n: 

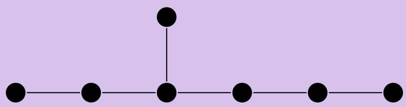
$\rightsquigarrow \mathfrak{sp}(2n)_{\mathbb{C}}$

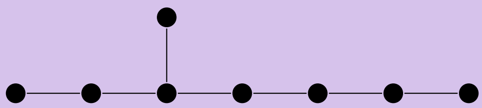
D_n: 

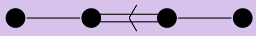
$\rightsquigarrow \mathfrak{so}(2n)_{\mathbb{C}}$

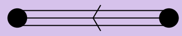
where $n = \text{Rank}(\mathfrak{g})$, as well as the special cases

E_6: 

E_7: 

E_8: 

F_4: 

G_2: 

Remark.

- $A_1 \simeq B_1 \simeq C_1 \simeq D_1$ ($\Rightarrow \mathfrak{su}(2)_{\mathbb{C}} \simeq \mathfrak{so}(3)_{\mathbb{C}} \simeq \mathfrak{sp}(2)_{\mathbb{C}}$),
- $B_2 \simeq C_2$ ($\Rightarrow \mathfrak{so}(5)_{\mathbb{C}} \simeq \mathfrak{sp}(4)_{\mathbb{C}}$),
- $D_2 \simeq A_1 \oplus A_1$ ($\Rightarrow \mathfrak{so}(4)_{\mathbb{C}} \simeq \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}}$),
- $D_3 \simeq A_3$ ($\Rightarrow \mathfrak{so}(6)_{\mathbb{C}} \simeq \mathfrak{su}(4)_{\mathbb{C}}$).

4 Reconstructing the Lie Algebra

Given a Dynkin diagram/Cartan matrix, we immediately know about the simple roots, and also have $3 \cdot \text{Rank}(\mathfrak{g})$ generators for the Lie algebra from the Chevalley-Serre relations (3.42) and (3.43). The challenge then is to construct the rest of the root system

THEOREM 4.1

To find the positive roots Φ_+ from the simple roots Φ_s of a simple Lie algebra, it is sufficient to consider root strings of the form

$$\delta + n\alpha_{(i)},$$

where $\delta \in \Phi_+$ and $\alpha_{(i)} \in \Phi_S$.

Example 4.1 *(Reconstructing $\mathfrak{su}(3)_\mathbb{C}$ from A_2)*

Reading off from the Dynkin diagram, we see that

$$A_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{4.1}$$

Label the simple roots α and β. For both the α string through β and vice versa, $n_+ = 1$, thus from either we obtain a single new root $\delta = (\alpha + \beta)$. Now, the result that $n_- = 0$ for simple root strings did *not* require the 'through' root to be simple, thus $n_- = 0$ for both the α string through δ and the β string, and

$$\begin{aligned} \ell_{\alpha\delta} &= 1 - 2\frac{(\alpha,\delta)}{(\alpha,\alpha)} \\ &= 1 - \left(2\frac{(\alpha,\alpha)}{(\alpha,\alpha)} + 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}\right) \\ &= 1 - \left(A^{11} + A^{21}\right) = 1 - (2 + (-1)) = 0. \end{aligned} \tag{4.2}$$

By symmetry, the same is true for $\ell_{\beta\delta}$. Thus, by the above theorem, we have exhausted all the possibilities for positive roots, and can write down our full root system

$$\Phi = \{\alpha, \beta, \delta = (\alpha + \beta), -\alpha, -\beta, -\delta\}. \tag{4.3}$$

From this root system, we can then write down the corresponding Cartan-Weyl basis for $\mathfrak{g}$

$$\mathfrak{g} = \text{Span}_\mathbb{C}\{h^\alpha, h^\beta, e^{\pm\alpha}, e^{\pm\beta}, e^{\pm\delta}\}, \tag{4.4}$$

where the Lie brackets are given by (3.18)

Remark. From property (iv) of Proposition 3.7, we know that any positive root can be expressed as $\delta = \sum_{i=1}^{r} m_i \alpha_{(i)}$, for some integers $m_i \geq 0$. Similarly to (4.2), we can deduce that

$$\ell_{\alpha_{(j)}\delta} = 1 - \sum_{i=1}^{r} A^{ij} m_i. \tag{4.5}$$

4.1 Representations of $\mathfrak{g}$

Definition 4.1 *Weight Space*

Suppose that we have some representation of a Lie algebra $R : \mathfrak{g} \to GL(V)$ such that $R(H)$ is diagonalisable $\forall\, H \in \mathfrak{h}$. Then a **weight space** is some subspace $V_\lambda \subset V$, indexed by a functional $\lambda \in \mathfrak{h}^*$, known as a **weight**, such that $\forall\, v \in V_\lambda$

$$R(H)v = \lambda(H)v. \tag{4.6}$$

Remark. If R is the adjoint rep of $\mathfrak{g}$, then the weights of the rep are just the roots of $\mathfrak{g}$.

Proposition 4.1 (*Some Facts About Weights of Representations*)

(i) Let S denote the set of weights of a representation, the representation space is then spanned by

$$V = \bigoplus_{\lambda \in S_R} V_\lambda, \tag{4.7}$$

(ii) For a weight λ and root α, if $\lambda + \alpha$ is also a weight, then

$$R(e^\alpha) : V_\lambda \to V_{\lambda+\alpha}. \tag{4.8}$$

(iii) For a weight λ and root α, $v \in V_\lambda$

$$R(h^\alpha)v = 2\frac{(\alpha, \lambda)}{(\alpha, \alpha)} \in \mathbb{Z} \tag{4.9}$$

Proof.

(i) This is simply a consequence of the assumption that all $R(H)$ are diagonalisable, as they share an eigenbasis which spans the vector space.

(ii) Let $v \in V_\lambda$, assuming $R(e^\alpha)v \neq 0$, we have

$$\begin{aligned} R(H)\left[R(e^\alpha)v\right] &= R(e^\alpha)R(H)v + R\left([H, e^\alpha]\right)v \\ &= \lambda(H)\left[R(e^\alpha)v\right] + \alpha(H)\left[R(e^\alpha)v\right] \\ &= (\lambda+\alpha)(H)\left[R(e^\alpha)v\right]. \end{aligned} \tag{4.10}$$

(iii) Recall (3.13), which tells us that $\lambda(H^\alpha) = (\alpha, \lambda)$, this works as κ induces an isomorphism between $\mathfrak{h}$ and *all of* $\mathfrak{h}^*$, not just roots. Then the rescaling of H^α (3.17b) gives the desired equality. To prove that this is integral, consider the $\mathfrak{sl}(2)_\alpha$ subalgebra of $\mathfrak{g}$. Let $S_{\alpha,\lambda} := \{\lambda + n\alpha \in S : n = \mathbb{Z}\}$, then

$$V_\alpha := \bigoplus_{\mu \in S_{\alpha,\lambda}} V_\mu \tag{4.11}$$

is a representation space of $\mathfrak{sl}(2;\mathbb{C})$ with weights $\frac{2(\alpha,\mu)}{(\alpha,\alpha)}$.[3] From our representation theory of $\mathfrak{sl}(2;\mathbb{C})$ we know that these weight must be integers.

□

4.2 Root and Weight Lattices

Remark. There was a glaring omission in (4.7), namely, *what weights does a given representation admit*?

Definition 4.2 *Root/Co-root Lattice*

Given a set of simple roots Φ_S for a root system Φ, the **root lattice** is the superset of Φ defined by

$$\mathcal{L}[\mathfrak{g}] := \mathrm{Span}_{\mathbb{Z}}\Phi_S. \tag{4.12}$$

If we define the **co-roots** associated to Φ_S as

$$\check{\alpha}_{(i)} := \frac{2\alpha_{(i)}}{(\alpha_{(i)}, \alpha_{(i)})}, \tag{4.13}$$

then the **co-root lattice** is

$$\check{\mathcal{L}}(\mathfrak{g}) := \mathrm{Span}_{\mathbb{Z}}\{\check{\alpha}_{(i)}\}_{i=1}^r. \tag{4.14}$$

[3] For completeness, the representation is $H \mapsto R(h^\alpha)$, $E^\pm \mapsto R(e^{\pm\alpha})$, note that the rescaled algebra was needed for the Lie algebra isomorphism to have the correct coefficients for the Lie brackets.

Definition 4.3 *Dual Lattice & Weight Lattice*

Given a vector space V with inner product $(\cdot,\cdot)$. The **dual** of a lattice $\mathcal{L}$ with respect to the inner product is defined by

$$\mathcal{L}^* := \{v \in V : (u,v) \in \mathbb{Z},\, \forall\, u \in \mathcal{L}\}. \tag{4.15}$$

The **weight lattice** $\mathcal{L}_W[\mathfrak{g}]$ is the dual of the co-root lattice, explicitly

$$\mathcal{L}_W[\mathfrak{g}] = \left\{ \lambda \in \mathfrak{h}^* : \frac{2(\alpha_{(i)},\lambda)}{(\alpha_{(i)},\alpha_{(i)})} \in \mathbb{Z}, i = 1, \cdots, r \right\} \tag{4.16}$$

Definition 4.4 *Fundamental weights*

A basis of the co-root lattice $\{\breve{\alpha}_{(i)}\}$ induces a basis for the weight lattice $\{\omega_{(i)}\}$ satisfying

$$(\breve{\alpha}_{(i)}, \omega_{(j)}) = \delta_{ij}. \tag{4.17}$$

These are then referred to as the **fundamental weights** of $\mathfrak{g}$

Definition 4.5 *Dynkin Labels*

Given a basis of fundamental weights for $\mathcal{L}_W[\mathfrak{g}]$, the coordinates of $\lambda \in \mathcal{L}_W[\mathfrak{g}]$ form a tuple (λ^i) known as the **Dynkin labels** of the weight λ.

Remark. As the simple roots form a basis of $\mathfrak{h}^*_{\mathbb{R}}$, we can express the fundamental weights as

$$\omega_{(i)} = \sum_{j=1}^{r} B_i^j \alpha_{(j)}. \tag{4.18}$$

Substituting this and (4.13) into (4.17) we see that

$$\sum_{k=1}^{r} \frac{2B_j^k}{(\alpha_{(i)},\alpha_{(i)})}(\alpha_{(i)},\alpha_{(k)}) = \sum_{k=1}^{r} B_j^k A_{ki} = \delta_{ij}. \tag{4.19}$$

Thus B is the inverse of A, i.e.

$$\alpha_{(i)} = \sum_{j=1}^{r} A_i^j \omega_{(j)}. \tag{4.20}$$

4.3 Highest Weight Representations

Proposition 4.2

Every finite dimensional irreducible representation $R : \mathfrak{g} \to GL(V)$ has a **highest weight** $\Lambda \in \mathcal{L}_W[\mathfrak{g}]$ with respect to some choice of Φ_+ such that, $\forall\, v \in V_\Lambda, \alpha \in \Phi_+$, $R(e^\alpha)v = 0$. Further more, all other weights of the representation are of the form

$$\lambda = \Lambda - \sum_{i=1}^{r} \mu^i \alpha_{(i)}, \tag{4.21}$$

for some $\mu^i \in \mathbb{Z}^+$. The highest weight characterises a representation uniquely up to isomorphism.

Proof (partial). The argument works similarly to that of the special case $\mathfrak{su}(2)_\mathbb{C}$. Let Λ be any weight of R. If $R(e^\alpha) : V_\Lambda \to \{0\}\,\forall\,\alpha \in \Phi_+$, then we are done. If not, then we can find some $\alpha \in \Phi_+$ such that $\Lambda + \alpha$ is a root. Relabel this to be our new Λ. We can repeat this process only finitely many times, as each weight space is linearly independent of the rest - by the same argument as for $\mathfrak{su}(2)_\mathbb{C}$ - and by hypothesis the rep space is finite dimensional. To prove (4.21), we rely on the fact that V is irreducible. Recall that if $\Lambda - \alpha$ is a weight, then $R(e^{-\alpha}) : V_\Lambda \to V_{\Lambda-\alpha}$. By acting on V_Λ by arbitrarily many step operators, we obtain a set of weights $\tilde{S}_R = \{\Lambda - \sum_i \mu^i \alpha_{(i)} \in S_R : \mu^i \in \mathbb{Z}\}$. We need only use simple roots as $\Phi \subset \mathcal{L}[\mathfrak{g}]$. As this is an invariant subspace, and R is irreducible, $U = V$, and $\tilde{S}_R = S_R$. Finally, that the integers μ^i must be positive follows from the definition of Λ, as if we have some $\mu^j < 0$, then $\Lambda + \mu^j \alpha_{(j)}$ is a weight, which contradicts the selection of Λ to be the highest weight. □

Remark. What is missing from this proof some sort of convexity argument, that $\lambda - n\alpha_{(i)} \in S_R \Rightarrow \lambda - m\alpha_{(i)}\,\forall\,0 < m < n$.

Proposition 4.3

If $\lambda = \sum_i \lambda^i \omega_{(i)} \in S_R$, then $\lambda - \sum_i m^i \alpha_{(i)} \in S_R \forall\, m^i \in \{0, 1, \cdots, \lambda^i\}$. In words, the Dynkin labels of a weight λ tell us how many times the corresponding *root* can be subtracted from that weight. Thus, if a weight has no positive roots, this result cannot be applied.

Example 4.2 *(Representation Theory of A_2)*

As this is a rank 2 Lie algebra, each weight has 2 Dynkin labels. Let the highest weight of a rep R be $\Lambda \rightsquigarrow (\Lambda_1, \Lambda_2)$. It helps to note that the Dynkin labels of the simple roots are $\alpha_{(1)} \rightsquigarrow (2,-1), \alpha_{(2)} \rightsquigarrow (-1,2)$. Thus, $\lambda = \Lambda - \sum_i \mu^i \alpha_{(i)}$ has Dynkin labels

$$(\lambda_1, \lambda_2) = (\Lambda_1 - 2\mu^1 + \mu^2, \Lambda_2 + \mu^1 - 2\mu^2). \tag{4.22}$$

The complete set of weights is then obtained through an application of the above proposition. Say we choose $\Lambda = (2,1)$, then we immediately have two 'strings' of weights, of length 2 and 1 respectively. By considering all the strings possible through these new weights, we can continue to obtain the full set of weights:

$$-\alpha_{(1)} \rightsquigarrow (-2,1)$$
$$-\alpha_{(2)} \rightsquigarrow (1,-2)$$

$$\begin{tikzcd}
 & (-2,3) \arrow[ddr] & & & & & \\
(-3,2) \arrow[ddr] & & & (0,2) \arrow[ull] \arrow[ddr] & & & \\
 & & (-1,1) \arrow[ull] \arrow[ddr] & & & (\mathbf{2},\mathbf{1}) \arrow[ull] \arrow[ddr] & \\
 & (-2,0) \arrow[ddr] & & & (1,0) \arrow[ull] \arrow[ddr] & & \\
 & & & (0,-1) \arrow[ull] \arrow[ddr] & & & (3,-1) \arrow[ull] \\
 & & (-1,-2) & & & (2,-2) \arrow[ull] & \\
 & & & & (1,-3) \arrow[ull] & &
\end{tikzcd}$$

Note: if you ever need to do this in an exam, it is sufficient to draw the lattice orthogonally, e.g. left for $\alpha_{(2)}$, down for $\alpha_{(1)}$. The result should be something like:

$$-\alpha_{(2)} \rightsquigarrow (1,-2)$$
$$-\alpha_{(1)} \rightsquigarrow (-2,1)$$

$$\begin{tikzcd}
 & & (3,-1) \arrow[d] & (2,1) \arrow[l] \arrow[d] \\
 & & (1,0) \arrow[d] & (0,2) \arrow[l] \arrow[d] \\
(1,-3) \arrow[d] & (0,-1) \arrow[l] & (-1,1) \arrow[l] \arrow[d] & (-2,3) \arrow[l] \\
(-1,-2) & (-2,0) \arrow[l] & (-3,2) \arrow[l] &
\end{tikzcd}$$

5 Symmetries in Quantum Mechanics

Remark. Traditionally, in quantum mechanics, the Hilbert space of quantum states is decomposed into a direct sum of eigenspaces of the Hamiltonian

$$\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n, \tag{5.1}$$

where $|n\rangle \in \mathcal{H}_n \Rightarrow \hat{H}|n\rangle = E_n|n\rangle$.

Definition 5.1 *Symmetry Transformation*

A **symmetry transformation** of a quantum system is an automorphism[1] $\hat{U}$ on its Hilbert space $\mathcal{H}$ such that $\hat{U}\hat{H}\hat{U}^\dagger = \hat{H}$.

[1] Here we tacitly define an isomorphism between Hilbert spaces to be a map which preserves addition (i.e. is linear) and the inner product. Thus an *auto*morphism is a unitary operator.

Definition 5.2 *Conserved Quantity*

A **conserved quantity** is an observable $\hat{I} = \hat{I}^\dagger$ such that $[\hat{I}, \hat{H}] = 0$.

Remark. If $\hat{I}$ commutes with $\hat{H}$, then so too does $e^{\alpha\hat{I}}\ \forall\, \alpha \in \mathbb{C}$. Further, if $\hat{I}$ is Hermitian, then $e^{is\hat{I}}$ is unitary $\forall\, s \in \mathbb{R}$. Thus, if $\hat{I}$ is a conserved quantity, then it generates a family of symmetry transformations $e^{is\hat{I}}$. This is essentially a quantum Noether's theorem.

Given a maximal set of linearly independent conserved quantities $\{\hat{I}^a\}_{a=1}^d$, then $\mathfrak{g} := \mathrm{Span}_{\mathbb{R}}\{\hat{I}^a\}_{a=1}^d$ forms a real Lie algebra, where the Lie bracket is given by commutation of the observables. The group G formed by exponentiating $\mathfrak{g}$ leaves each $\mathcal{H}_n$ invariant, due to all of the operators commuting with $\hat{H}$. *For reasons I cannot justify*, for each n, we can construct a representation of $R : \mathfrak{g} \to GL\left(\mathcal{H}_n\right)$ such that $R(x)$ is skew-Hermitian $\forall\, x \in \mathfrak{g}$. **It may have something to do with the fact that the exponential map gives unitary operators.**