Symmetries, Fields and Particles
Summary Notes

Sam Crawford

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1 Lie Groups and Lie Algebras

Proposition 1.1. The space of left-invariant vector fields is an $n = \dim(G)$ dimensional vector space which is homeomorphic to T_eG, the tangent space to the identity of G.

Proposition 1.2. The space of left-invariant vector fields on a Lie group G is closed under the Lie bracket $[X,Y] \circ f := X \circ (Y \circ f) - Y \circ (X \circ f)$, and thus forms a Lie algebra $\mathfrak{g}$.

2 Representations

Proposition 2.1. If $\text{Exp}(\mathfrak{g}) = H \subset G$ is bijective, then a representation R of $\mathfrak{g}$ 'exponentiates' to the representation $D(\text{Exp}(x)) = \text{Exp}(R(x))$ of H.

Proposition 2.2. Let $R : \mathfrak{su}(2) \to GL(V)$ be a finite dimensional *irreducible* representation. Then for any eigenvalue v of $R(H)$, the set

$$\{R(E^{\pm})^n v \neq 0 : n \in \mathbb{Z}^+\}$$

forms an eigenbasis of V with respect to $R(H)$.

Proposition 2.3. All finite number of tensor products of finite dimensional irreps of a complex simple Lie algebra are fully reducible. I.e., if $|\mathcal{R}|, |\mathcal{R}'| \in \mathbb{N}$

$$\bigotimes_{R \in \mathcal{R}} R = \bigoplus_{R' \in \mathcal{R}'} \mathfrak{M}(R')R', \tag{2.1}$$

where $\mathfrak{M}(R') \in \mathbb{Z}$ denotes the *multiplicity* of the rep R' in the decomposition.

3 The Cartan Classification

Proposition 3.1. The Killing form of a Lie algebra is **invariant**, defined as the property

$$\kappa(\mathrm{Ad}_z x, y) = -\kappa(x, \mathrm{Ad}_z y), \tag{3.1}$$

i.e. Ad_z is a skew-adjoint operator $\forall\ z \in \mathfrak{g}$.

Proposition 3.2. All CSAs of a Lie algebra have the same dimension.

Proposition 3.3 (Some facts step operators and the Killing form)**.** (i) $\kappa(H, E^\alpha) = 0,\ \forall\, H \in \mathfrak{h}, \alpha \in \Phi$

(ii) $\kappa(E^\alpha, E^\beta) = 0,\ \forall\, \alpha \neq -\beta$

(iii) $\forall\, H \in \mathfrak{h},\ \exists\, H' \in \mathfrak{h}$ s.t. $\kappa(H, H') \neq 0$

(iv) $\forall\, \alpha \in \Phi, -\alpha \in \Phi$, and $\kappa(E^\alpha, E^{-\alpha}) \neq 0$.

Proposition 3.4. The root set Φ spans $\mathfrak{h}^*$

Proposition 3.5. Let $\{\alpha_{(i)}\}_{i=1}^r \subset \Phi$ be any set of linearly independent roots, then $\Phi \subset \mathrm{Span}_{\mathbb{R}}\{\alpha_{(i)}\} =: \mathfrak{h}_{\mathbb{R}}$.

Proposition 3.6. Let $\mathfrak{h}_{\mathbb{R}}$ be as before, then the map $(\cdot,\cdot) : \mathfrak{h}_{\mathbb{R}} \times \mathfrak{h}_{\mathbb{R}} \to \mathbb{R}$ is a Euclidean inner product.

Proposition 3.7 (Properties of Simple Roots)**.** Let $\alpha, \beta \in \Phi$ be simple roots, then:

(i) $(\alpha - \beta) \notin \Phi$

(ii) The α-string through β has length

$$\ell_{\alpha,\beta} = 1 - 2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \tag{3.2}$$

(iii) $(\alpha, \beta) \leq 0$.

(iv) Any positive root can be written as a linear combination of simple roots with *integer* coefficients, thus the simple roots span $\mathfrak{h}^*_{\mathbb{R}}$

Proposition 3.8. Simple roots are linearly independent in $\mathfrak{h}^*_{\mathbb{R}}$.

Proposition 3.9 (Constraints on the Cartan Matrix)**.** (0) $A^{ji} \in \mathbb{Z}$,

(i) $A^{ii} = 2$,

(ii) $A^{ij} = 0 \Leftrightarrow A^{ji} = 0$,

(iii) $A^{ij} < 0 \,\forall\, i \neq j$,

(iv) $A = DS$ for some diagonal matrix D and some positive definite matrix S

Proposition 3.10. For $i \neq j$, the only valid pairs of values for (A^{ij}, A^{ji}) are (order irrelevant): $(0,0), (-1,-1), (-1,-2), (-1,-3)$.

4 Reconstructing the Lie Algebra

Proposition 4.1 (Some Facts About Weights of Representations)**.** (i) Let S denote the set of weights of a representation, the representation space is then spanned by

$$V = \bigoplus_{\lambda \in S_R} V_\lambda, \tag{4.1}$$

(ii) For a weight λ and root α, if $\lambda + \alpha$ is also a weight, then

$$R(e^\alpha) : V_\lambda \to V_{\lambda+\alpha}. \tag{4.2}$$

(iii) For a weight λ and root α, $v \in V_\lambda$

$$R(h^\alpha)v = 2\frac{(\alpha, \lambda)}{(\alpha, \alpha)} \in \mathbb{Z} \tag{4.3}$$

Proposition 4.2. Every finite dimensional irreducible representation $R : \mathfrak{g} \to GL(V)$ has a **highest weight** $\Lambda \in \mathcal{L}_W[\mathfrak{g}]$ with respect to some choice of Φ_+ such that, $\forall\, v \in V_\Lambda, \alpha \in \Phi_+$, $R(e^\alpha)v = 0$. Further more, all other weights of the representation are of the form

$$\lambda = \Lambda - \sum_{i=1}^{r} \mu^i \alpha_{(i)}, \tag{4.4}$$

for some $\mu^i \in \mathbb{Z}^+$. The highest weight characterises a representation uniquely up to isomorphism.

Proposition 4.3. If $\lambda = \sum_i \lambda^i \omega_{(i)} \in S_R$, then $\lambda - \sum_i m^i \alpha_{(i)} \in S_R \,\forall\, m^i \in \{0, 1, \cdots, \lambda^i\}$. In words, the Dynkin labels of a weight λ tell us how many times the corresponding *root* can be subtracted from that weight. Thus, if a weight has no positive roots, this result cannot be applied.

5 Symmetries in Quantum Mechanics