General Relativity
Summary Notes

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May 31, 2018

Contents

1 Equivalence Principles

Equation 1.1 *(Newton's Law of Gravitation)*

The differential form of Newtonian gravity is

$$\Delta\Phi = 4\pi G\rho. \tag{1.1}$$

The integral solution to this is

$$\varphi(t, \mathbf{x}) = -G \int_{\mathbb{R}^3} \mathrm{d}^3 y \, \frac{\rho(t, \mathbf{y})}{|\mathbf{x} - \mathbf{y}|.} \tag{1.2}$$

2 Manifolds and Tensors

Proposition 2.1 *(Transformations of Vector and Covector Fields)*

Let X be a vector field, and ω a covector field on M. Further, let $(x^\mu), (x'^\mu)$ be two sets of coordinates on M with overlapping charts. If $X = X^\mu \partial_\mu = X'^\mu \partial'_\mu$ etc, then these coordinates are related to each other by

$$X'^\mu = \left(\frac{\partial x'^\mu}{\partial x^\nu}\right) X^\nu, \tag{2.1a}$$

$$\omega'_\mu = \left(\frac{\partial x^\nu}{\partial x'^\mu}\right) \omega_\nu. \tag{2.1b}$$

Definition 2.1 *Tensor*

A **tensor of rank** (r, s) on a vector field V is a multilinear map

$$T : \underbrace{V^* \times \cdots \times V^*}_{r \text{ times}} \times \underbrace{V \times \cdots \times V}_{s \text{ times}} \to \mathbb{R}. \tag{2.2}$$

Written in a basis $\{e_\mu\}$ with dual $\{f^\mu\}$, such a tensor has components written

$$T = T^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s} e_{\mu_1} \otimes \cdots \otimes e_{\mu_r} \otimes f^{\nu_1} \otimes \cdots \otimes f^{\nu_s}. \tag{2.3}$$

Remark. Remember a type (r, s) vector 'takes in' r covectors and s vectors in order to 'put out' a scalar. If we instead input $p < r$ covectors and $q < s$ vectors, the result can be considered a $(r - p, s - q)$ tensor.

Definition 2.2 *Tensor Product*

Given a rank (p, q) tensor S and a rank (r, s) tensor T on some vector space, their **tensor product**, a.k.a. **outer product**, is the rank $(p+r, q+s)$ tensor defined by

$$\begin{aligned}&(S \otimes T)(\omega_1, \dots, \omega_p, \eta_1, \dots, \eta_r, X_1, \dots, X_q, Y_1, \dots, Y_s)\\ &= S(\omega_1, \dots, \omega_p, X_1, \dots, X_q) T(\eta_1, \dots, \eta_r, Y_1, \dots, Y_s).\end{aligned} \tag{2.4}$$

3 The Metric Tensor

Equation 3.1 *(Geodesic Lagrangian)*

To find geodesics on a Lorentzian manifold, we use a functional formula for the proper time, treating this as an action, the 'Lagrangian' is

$$G(x(\lambda), \dot{x}(\lambda)) \coloneqq \sqrt{-g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu}. \tag{3.1}$$

The proper time for a curve $x : [0, 1] \hookrightarrow M$ is then

$$\tau[x] = \int_0^1 G(x(\lambda), \dot{x}(\lambda)) \mathrm{d}\lambda. \tag{3.2}$$

Remark. The relevant derivatives to extremise the proper time are

$$\frac{\partial G}{\partial \dot{x}^\mu} = -\frac{1}{G} g_{\mu\nu}\dot{x}^\nu, \tag{3.3a}$$

$$\frac{\partial G}{\partial x^\mu} = -\frac{1}{2G} g_{\nu\rho,\mu}\dot{x}^\nu \dot{x}^\rho. \tag{3.3b}$$

Equation 3.2 *(Geodesic Equation)*

The Euler-Lagrange equations for 3.1 reduce to the **geodesic equation**

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0, \tag{3.4}$$

where the **Christoffel symbols** are defined by

$$\Gamma^\mu_{\nu\rho} \coloneqq \tfrac{1}{2} g^{\mu\sigma}(g_{\sigma\nu,\rho} + g_{\sigma\rho,\nu} - g_{\nu\rho,\sigma}). \tag{3.5}$$

Remark. One can obtain (3.4) more directly by varying the Lagrangian

$$L = -g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}. \tag{3.6}$$

Note further that the fact L above has no explicit τ dependence, ${}^{\partial L}/_{\partial\tau} = 0$ which, along with the realisation that ${}^{dx^\mu}/_{d\tau}$ is a 4-velocity, leads to the conclusion that $L \equiv -1$ along geodesics.

Example 3.1 (*Schwarzschild Metric*)

One of the more famous solutions to Einstein's equations, the Schwarzschild metric can be written as

$$ds^2 = -f dt^2 + f^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2, \qquad f := 1 - \frac{2M}{r}. \tag{3.7}$$

4 Covariant Derivative

Definition 4.1

A **covariant derivative**, a.k.a. **connexion/connection** ∇ on a manifold M is a 'variable' tensor field, taking in a vector field X and a rank (r,s) tensor field T and producing a new rank (r,s) tensor field, written $\nabla_X T$, subject to the following properties

1. $\nabla_{fX+gY}T = f\nabla_X T + g\nabla_Y T$,
2. $\nabla_X(T+S) = \nabla_X T + \nabla_X S$,
3. $\nabla_X(T\otimes S) = (\nabla_X T)\otimes S + T\otimes(\nabla_X S)$.
4. $\nabla_X f = X\circ f$

I.e. the covariant derivative is $C^\infty(M)$ linear in the vector field argument, $\mathbb{R}$ linear in the tensor field argument, and satisfies the Leibniz rule for tensor products.

Remark. Sometimes we may wish to leave the vector field argument undefined, in which case we can consider ∇T as a type $(r, s+1)$ tensor field such that $\nabla T(X) = \nabla_X T$. In particular, for functions we have that $\nabla f = df$.

Equation 4.1 *(Tensor Coordinate Transformation)*

The generalisation of Proposition 2.1 for an arbitrary (r, s) tensor field is simply

$$T'^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s} = \left(\frac{\partial x'^{\mu_1}}{\partial x^{\rho_1}}\right)\cdots\left(\frac{\partial x'^{\mu_r}}{\partial x^{\rho_r}}\right)\left(\frac{\partial x^{\sigma_1}}{\partial x'^{\nu_1}}\right)\cdots\left(\frac{\partial x^{\sigma_s}}{\partial x'^{\nu_s}}\right)T^{\rho_1\cdots\rho_r}{}_{\sigma_1\cdots\sigma_s} \tag{4.1}$$

Definition 4.2

Given a local basis $\{e_\mu\}$ of $\mathfrak{X}(M)$, a covariant derivative can be defined using **connection components** defined by

$$\nabla_\mu(e_\nu) \coloneqq \nabla_{e_\mu}(e_\nu) = \Gamma^\rho_{\nu\mu}e_\rho. \tag{4.2}$$

The covariant derivative of a type $(1, 1)$ tensor field can then be written as

$$T^\mu_{\nu;\rho} \coloneqq (\nabla T)^\mu_{\nu\rho} = T^\mu_{\nu,\rho} + \Gamma^\mu_{\sigma\rho}T^\sigma_\nu - \Gamma^\sigma_{\nu\rho}T^\mu_\sigma. \tag{4.3}$$

Transformations for arbitrary tensor fields look similar, but a lot messier.

Remark. For a scalar function we have $f_{;\mu} = f_{,\mu}$, a further covariant derivative is then given by

$$f_{;\mu\nu} = f_{,\mu\nu} - \Gamma^\rho_{\mu\nu}f_{,\rho}. \tag{4.4}$$

Definition 4.3 *Torsion*

The **torsion tensor** associated to a connexion ∇ is a rank $(1, 2)$ tensor T defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]. \tag{4.5}$$

If the torsion tensor vanishes everywhere, we say that the connexion is **torsion free**, in which case in *any* basis $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$.

Theorem 4.1 *(Fundamental Theorem of (Pseudo-)Riemannian Geometry)*

For any Riemannian manifold (M, g), there exists a unique torsion-free connexion ∇ such that $\nabla g \equiv 0$, called the **Levi-Civita connexion**. Locally, its connexion components are the same Christoffel symbols as defined in (3.5).

Definition 4.4 *Affinely Parametrised Geodesic*

An **affinely parametrised geodesic** on a manifold M with connexion ∇ is a curve with an associated vector field X such that

$$\nabla_X X = 0. \tag{4.6}$$

Remark. This condition earns the '*affine*' tag as reparametrising the curve $t \to t(u)$ results in the associated vector field $X \to Y = t'X$. Whilst this describes the same curve, if $\nabla_X X = 0$, then $\nabla_Y Y = (X \circ t')Y \neq 0$.

THEOREM 4.2

Let M be a manifold with connexion ∇. Let $p \in M, X_p \in T_pM$. Then there exists a unique affinely parameterised geodesic through p with tangent vector $X|_p$ at p.

Proof. Existence and uniqueness of solutions to ODEs. Specifically we want to find a curve γ such that $\nabla_{\dot\gamma}\dot\gamma = 0$ subject to initial conditions $\gamma(0) = p$, $\dot\gamma(0) = X|_p$, which is a second order ODE. □

Definition 4.5 *Exponential Map*

Let M be a manifold with connexion ∇. For $p \in M$, the **exponential map** is a local diffeomorphism $\text{Exp} : T_pM \overset{\simeq}{\rightarrow} U \subset M$, defined such that if γ is the geodesic through p with $\dot\gamma_0 = X|_p$, then $\text{Exp}(X|_p) = \gamma(1)$.

Remark. It can be shown that, if $X|_p$ and γ are defined as above, for $t \in [0, 1]$

$$\text{Exp}(tX|p) = \gamma(t). \tag{4.7}$$

Definition 4.6 *Normal Coordinates*

Given a manifold M with connexion ∇, and a basis $\{e_\mu\}$ of T_pM, **normal coordinates** at p are defined as the inverse of the map

$$(x^\mu) \mapsto \text{Exp}(x^\mu e_\mu). \tag{4.8}$$

I.e. if $q = \text{Exp}(x^\mu e_\mu)$, then x^μ are the normal coordinates of q.

Lemma 4.1. For a manifold M with connexion ∇, in normal coordinates at $p \in M$, $\gamma^\mu_{(\nu\rho)} = 0$.

Proof. We can express any geodesic from p to $q = \text{Exp}(x^\mu e_\mu)$ in normal coordinates as

$$\gamma(t) = \text{Exp}(tx^\mu e_\mu), \tag{4.9}$$

i.e. the coordinates of the geodesic are (tx^μ). Thus the geodesic equation becomes

$$\ddot{\gamma}^\mu + \Gamma^\mu_{\nu\rho}\dot{\gamma}^\nu\dot{\gamma}^\rho = 0 + \Gamma^\mu_{\nu\rho}x^\nu x^\rho = 0. \tag{4.10}$$

As the Christoffel terms are being contracted with a symmetric term, we are free to symmetrise

$$\Gamma^\mu_{\nu\rho}x^\nu x^\rho = \Gamma^\mu_{(\nu\rho)}x^\nu x^\rho = 0. \tag{4.11}$$

□

Lemma 4.2. Let M be a manifold with Levi-Civita connexion ∇, then, in normal coordinates

$$g_{\mu\nu,\rho}(p) = 0. \tag{4.12}$$

Further more, one can select a basis of T_pM such that the metric at p is $\text{Diag}(-, \cdots, +, \cdots)$, depending on the signature of the metric.

Definition 4.7

For a Lorentzian manifold, a set of normal coordinates with respect to the Levi-Civita connexion such that $g_{\mu\nu}(p) = \eta_{\mu\nu}$ form a **local inertial frame** at p.

5 Physical Laws in Curved Spacetime

Definition 5.1 *Minimal Coupling*

Given a set of equations of motion on a flat space time, the process of **minimal coupling** is defined to be the following

(i) Replace the Minkowski metric with a (generally) curved spacetime metric.

(ii) Replace partial derivatives with covariant derivatives with respect to the Levi-Civita connexion

(iii) Replace coordinate basis indices with abstract indices.

Example 5.1

(i) [Klein-Gordon equation] $\nabla^a\nabla_a\Phi - m^2\Phi = 0$.

(ii) [Maxwell's Electromagnetism] The field strength tensor is defined by $F_{ab} = \nabla_a A_b - \nabla_b A_a$, in a specific basis, this is related to the physical fields by $F_{0i} = -E_i$, $F_{ij} = \epsilon_{ijk}B_k$. The vacuum equations are then

$$\nabla^a F_{ab} = 0, \qquad \nabla_{[a}F_{bc]} = 0. \tag{5.1}$$

The latter is a *Bianchi identity*, and holds even in the presence of sources. The coupling of matter to F is achieved through minimally coupling the Lorentz force law

$$u^b\nabla_b u^a = \tfrac{q}{m}F^a{}_b u^b. \tag{5.2}$$

Example 5.2 (*Stress-Energy Tensors*)

Each of the above theories has a corresponding stress-energy tensor which is symmetric and *conserved*, i.e. $\nabla^a T_{ab} = 0$.

(i) [Klein-Gordon Field] DO

(ii) [Maxwell's Electromagnetism] $T_ab = \frac{1}{4\pi}\left(F_{ac}F_b{}^c - \frac{1}{4}F^{cd}F_{cd}g_{ab}\right)$

(iii) [Perfect fluid] $T_{ab} = (\rho + p)u_a u_b + pg_{ab}$

6 Curvature

Definition 6.1 *Parallel Transport*

Given a manifold M with connexion ∇ and a vector field X, a tensor field T is said to have undergone **parallel transport** with respect to X if $\nabla_X T = 0$.

Remark. Given the value of T at $p \in M$, the parallel transport condition uniquely determines the value of T for all points along the integral curve of X through p.

Definition 6.2 *Riemann Curvature*

Given a manifold M with connexion ∇, the **Riemann curvature tensor** of the connexion is a type $(1,3)$ tensor defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{6.1}$$

Remark. The fact that this is indeed a tensor is not trivial. Basically we must prove that $R(fX + X', Y) = fR(X, Y) + R(X', Y)$ and similar for Z ($C^\infty(M)$ linearity in Y is given by the inherent skew-symmetry of R in X and Y).

Equation 6.1

By computing $R(e_\rho, e_\sigma)e_\nu$, we can obtain the basis-dependent form of the Riemann tensor

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\tau_{\nu\sigma}\Gamma^\mu_{\tau\rho} - \Gamma^\tau_{\nu\rho}\Gamma^\mu_{\tau\sigma}. \tag{6.2}$$

Definition 6.3 *Ricci Curvature Tensor*

The **Ricci curvature tensor** is a contraction of the Riemann tensor, defined by

$$R_{ab} := R^c{}_{acb}. \tag{6.3}$$

Equation 6.2 *(Ricci Identity)*

$$\nabla_c \nabla_d Z^a - \nabla_d \nabla_c Z^a = R^a{}_{bcd} Z^b. \tag{6.4}$$

(Proof: contract each side with X^c, Y^d)

Equation 6.3 *(Symmetries of the Riemann Tensor)*

$$R^a{}_{b(cd)} = 0. \tag{6.5}$$

For a torsion-free connexion:

$$R^a{}_{[bcd]} = 0. \tag{6.6}$$

Bianchi identity: (also for a torsion-free connexion)

$$R^a{}_{b[cd;e]} = 0. \tag{6.7}$$

If ∇ is the Levi-Civita connexion for some metric g_{ab}, then

$$R_{abcd} = R_{cdab}, \qquad R_{(ab)cd} = 0. \tag{6.8}$$

Equation 6.4 *(Geodesic Deviation)*

Let ∇ be a torsion-free connection, and let T, S be vector fields such that $\nabla_T T = 0$, and $[T, S] = 0$. Then

$$\nabla_T \nabla_T S = R(T, S)T. \tag{6.9}$$

Definition 6.4 *Einstein Tensor*

The **Einstein tensor** is a symmetric tensor of type $(0, 2)$ defined by

$$G_{ab} := R_{ab} - \frac{1}{2} R g_{ab} \tag{6.10}$$

Equation 6.5 *(Contracted Bianchi Identity)*

$$\nabla^a G_{ab} = 0, \quad \Leftrightarrow \quad \nabla^a R_{ab} - \frac{1}{2} \nabla_b R = 0. \tag{6.11}$$

Equation 6.6 *(Einstein Equation)*

$$G_{ab} = 8\pi G T_{ab}. \tag{6.12}$$

If a vacuum, this reduces to

$$R_{ab} = 0. \tag{6.13}$$

If we wish to include the cosmological constant, we then have

$$G_{ab} + \Lambda g_{ab} = 8\pi G T_{ab}. \tag{6.14}$$

7 Diffeomorphisms and the Lie Derivative

Definition 7.1 *Pullback (function)*

Let $\varphi : M \to N$ be a smooth map between manifolds (not necessarily a diffeomorphism), the **pullback** of a function $f \in C^\infty(N)$ is the $C^\infty(M)$ function defined by

$$\varphi^* f := f \circ \varphi. \tag{7.1}$$

Definition 7.2 *Pushforward*

Let $\varphi : M \to N$ be a smooth map between manifolds. Then the **pushforward** of φ is a linear map $\varphi_* : T_p M \to T_p N$ defined by

$$(\varphi_* v) f = v(\varphi^* f) \tag{7.2}$$

where $f \in C^\infty(N)$ is an arbitrary function.

Remark. An alternate definition is that the pushforward $\varphi_* X$ of X is the vector field on N such that if ρ_t is the flow of X on M, then $\varphi \circ \rho_t$ is the flow of $\varphi_* X$.

Definition 7.3 *Pullback (differential form)*

Using these, we can define the **pullback** of a p-form $\alpha \in \Omega^p(N)$ to be

$$(\varphi^* \alpha)(X_1, \cdots, X_p) = \alpha(\varphi_* X_1, \cdots, \varphi_* X_n), \tag{7.3}$$

for arbitrary $X_1, \cdots X_n \in \mathfrak{X}(M)$.

Remark. We can infact define pullbacks for *any* type $(0, s)$ tensors, not just those which are antisymmetric, and likewise we can define pushforwards for type $(r, 0)$ tensors using similar 'antidistributivity' relations. But these are less useful.

Lemma 7.1. The exterior derivative commutes with pullbacks, i.e.

$$\varphi^*(d\alpha) = d(\varphi^* \alpha). \tag{7.4}$$

(Only expected to prove this for 0-forms.)

Remark. The pullback also commutes with contractions.

Definition 7.4 *Diffeomorphism*

A map $\varphi : M \to N$ is a **diffeomorphism** if it is bijective, smooth, and has a smooth inverse

Definition 7.5 *Pullback/Pushforward (of a Diffeomorphism)*

Using a diffeomorphism, we can pullback vector fields, and pushforward differential forms using the inverse, i.e.

$$\varphi_* \alpha \coloneqq \left(\varphi^{-1}\right)^* \alpha, \qquad \varphi^* X \coloneqq \left(\varphi^{-1}\right)_* X. \tag{7.5}$$

Equation 7.1 *(Coordinate Based Pullback/Pushforward)*

Let $\varphi : M \to N$ be a diffeomorphism, and let (x^μ) be a set of coordinates on M, and (y^μ) a set of coordinates on N. Then

$$\left(\varphi_* \frac{\partial}{\partial x^\mu}\right) = \left(\frac{\partial y^\nu}{\partial x^\mu}\right) \frac{\partial}{\partial y^\nu}, \qquad (\varphi^* dy^\mu) = \left(\frac{\partial y^\mu}{\partial x^\nu}\right) dx^\nu. \tag{7.6}$$

The inverse operations, allowed by the fact that φ is a diffeomorphism, are then

$$\left(\varphi^* \frac{\partial}{\partial y^\mu}\right) = \left(\frac{\partial x^\nu}{\partial y^\mu}\right) \frac{\partial}{\partial x^\nu}, \qquad (\varphi_* dx^\mu) = \left(\frac{\partial x^\mu}{\partial y^\nu}\right) dy^\nu. \tag{7.7}$$

Thus, for a tensor S of type (r, s) on M, and T on N, we have

$$(\varphi_* S)^{\mu_1, \cdots, \mu_r}{}_{\nu_1, \cdots \nu_s} = \left(\frac{\partial y^{\mu_1}}{\partial x^{\alpha_1}}\right) \cdots \left(\frac{\partial y^{\mu_r}}{\partial x^{\alpha_r}}\right) \left(\frac{\partial x^{\beta_1}}{\partial y^{\nu_1}}\right) \cdots \left(\frac{\partial x^{\beta_s}}{\partial y^{\nu_s}}\right) S^{\alpha_1 \cdots \alpha_r}{}_{\beta_1 \cdots \beta_s}. \tag{7.8}$$

Remark. Essentially, this is just the chain rule. In fact, if we have charts for $U \subset M$ and $\varphi(U)$, then we can think of a diffeomorphism as essentially a change of chart.

Definition 7.6 *Pushforward (of a Connexion)*

Let $\varphi : M \to N$ be a diffeomorphism. Then if we have a connexion ∇ on M, we can define its **pushforward** $\tilde{\nabla}$ on N by

$$\tilde{\nabla}_X T = \varphi_* \left[\nabla_{\varphi^* X}(\varphi^* T)\right]. \tag{7.9}$$

Lemma 7.2. Let $\varphi : M \xrightarrow{\simeq} N$, and let $\tilde{\nabla}$ be the pushforward of ∇. Then

(i) The Riemann tensor of $\tilde{\nabla}$ is the pushforward of the Riemann tensor of ∇.

(ii) If ∇ is the Levi-Civita connexion of a metric g, then $\tilde{\nabla}$ is the Levi-Civita connexion of $\varphi_* g$.

Remark. From these results (and other similar results), we conclude that diffeomorphisms represent a *gauge freedom* in our description of GR.

Definition 7.7 *Symmetry Transformation/Isometry*

A diffeomorphism $\varphi : M \to M$ is a **symmetry transformation** of a tensor field T if $\varphi_* T \equiv T$. A symmetry transformation of the metric tensor is called an **isometry**.

Remark. A vector field $X \in \mathfrak{X}(M)$ generates a family of diffeomorphisms $\rho_t : M \xrightarrow{\simeq} M$ satisfying $\rho_t \circ \rho_s = \rho_{t+s}$, and $\rho_0 \equiv \text{Id}$. Thus $\rho_t^{-1} = \rho_{-t}$ and $\rho_t^* = (\rho_{-t})_*$.

Definition 7.8

The **Lie derivative** of a tensor field T with respect to a vector field X is defined using the flow ρ_t of X as

$$\mathcal{L}_X T = \frac{d}{dt}\left[\rho_t^* T\right]|_{t=0}. \tag{7.10}$$

Alternatively, we can define this more explicitly pointwise as

$$\mathcal{L}_X T|_p = \lim_{t \to 0} \frac{{\rho_t}^* T|_{\rho_t(p)} - T|_p}{t}. \tag{7.11}$$

Lemma 7.3. Given a vector field X on a manifold M, and a set of coordinates (t, x^i) adapted to X such that $X(t) = 1$, $X(x^i) = 0 \Leftrightarrow X = \partial_t$, we can define the Lie derivative in this basis as

$$(\mathcal{L}_X T)^{\mu_1, \cdots \mu_r}{}_{\nu_1, \cdots \nu_s} = \frac{\partial}{\partial t} T^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s}. \tag{7.12}$$

Remark. The Lie derivative also

- Obeys the Leibniz rule: $\mathcal{L}_X(S \otimes T) = (\mathcal{L}_X S) \otimes T + S \otimes (\mathcal{L}_X T)$.
- Interacts with the contraction operator as: $[\iota_X, \mathcal{L}_Y] = \iota_{[X,Y]}$.
- Commutes with 'internal' contraction: $\mathcal{L}_X(T^{\cdots a \cdots}{}_{\cdots a \cdots}) = (\mathcal{L}_X T)^{\cdots a \cdots}{}_{\cdots a \cdots}$.
- Coincides with the action of X on 0-forms: $\mathcal{L}_X f = X(f)$.
- Coincides with the Lie bracket on vector fields: $\mathcal{L}_X Y = [X, Y]$

Example 7.1 *(Lie Derivative vs Covariant Derivative (1-forms))*

The Lie derivative and covariant derivative agree on functions, we can use this to compare their action on generic tensors. For example, consider the action of the Lie derivative on a covariant vector field

$$\mathcal{L}_X(\iota_Y \omega) = X\left(\omega(Y)\right) = \iota_Y(\mathcal{L}_X \omega) = (\mathcal{L}_X \omega)(Y) \tag{7.13}$$

[TO FINISH: compute in normal coords of Levi-Civita]

Equation 7.2 *(Killing's Equation)*

A vector field X is a **Killing vector field** if its flows are isometries of g, equivalently $\mathcal{L}_X g = 0$. This can be show to be equivalent to *Killing's equation*

$$\nabla_a X_b + \nabla_b X_a = 0. \tag{7.14}$$

Or, equivalently

Lemma 7.4. If X is a Killing vector field, and γ is an affinely parametrised geodesic, then $g(X, \dot{\gamma})$ is constant along γ

Proof. $\dot{\gamma}\left(g(X, \dot{\gamma})\right) = g(\nabla_{\dot{\gamma}} X, \dot{\gamma}) =$ □

8 Linearised Theory

Equation 8.1 *(Linearly Perturbed Metric)*

Assuming that space is 'almost flat' we can find a chart such that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{8.1}$$

such that $h_{\mu\nu} \sim \mathcal{O}(\epsilon)$ and $\epsilon^2 \approx 0$. The inverse metric is then

$$g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}, \tag{8.2}$$

where $h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma} + \mathcal{O}(\epsilon^2)$.

Equation 8.2 *(Linearised Levi-Civita)*

Given the above linearisation, the Christoffel symbols of the Levi-Civita connexion can be written

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2}\eta^{\mu\sigma}(h_{\sigma\nu,\rho} + h_{\sigma\rho,\nu} - h_{\nu\rho,\sigma}). \tag{8.3}$$

This leads to the linearised Riemann tensor

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}(h_{\mu\sigma,\nu\rho} + h_{\nu\rho,\mu\sigma} - h_{\nu\sigma,\mu\rho} - h_{\mu\rho,\nu\sigma}). \tag{8.4}$$

And Ricci tensor

$$R_{\mu\nu} = \partial^{\rho}\partial_{(\mu}h_{\nu)\rho} - \frac{1}{2}\Box h_{\mu\nu} - \frac{1}{2}\partial_{\mu}\partial_{\nu}h, \tag{8.5}$$

where $h := h^{\mu}{}_{\mu}$. It is also useful to define the 'negative-traced' version of $h_{\mu\nu}$

$$\bar{h}_{\mu\nu} = h_{mu\nu} - \frac{1}{2}h\eta_{\mu\nu}. \tag{8.6}$$

Equation 8.3 *(Gauge Transformation of $h_{\mu\nu}$)*

We generate 'infinitesimal' diffeomorphisms using the flow of some vector field X for a sufficiently small parameter t, which is equivalent to taking the flow with unit parameter of the vector field $\xi = tX$. The result is

$$h_{\mu\nu} \to h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu}. \tag{8.7}$$

Equation 8.4 *(Linearised Einstein Equation (Harmonic Gauge))*

$$\Box\bar{h}_{\mu\nu} = -16\pi T_{\mu\nu} \tag{8.8}$$

Equation 8.5 *(Gravitational Wave)*

By seeking solutions of the linearised vacuum Einstein equation with the gauge conditions $\partial^\mu \bar{h}_{\mu\nu} = 0$ (Lorentz), $\bar{h}_{0\mu} = 0$ (longitudinal gauge) and $\bar{h} = 0$ (trace-free), which can all be imposed concurrently. For a plane wave solution with wave-vector $k_\rho = (\omega, 0, 0, \omega)$ the most general form has just two degrees of freedom

$$\bar{h}_{\mu\nu} = h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H_+ & H_\times & 0 \\ 0 & H_\times & -H_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{ik_\rho x^\rho}. \tag{8.9}$$

Equation 8.6 *(Linearised Geodesic Deviation)*

For a linear metric perturbation $h_{\mu\nu}$, and a parallelly transported orthonormal basis $\{(e_\alpha)^a\}$, we can approximate the geodesic deviation vector as

$$\frac{d^2 S_\alpha}{d\tau^2} \approx R_{\mu 00\nu} e^\mu_\alpha e^\nu_\beta S^\beta \approx \tfrac{1}{2} \frac{\partial^2 h_{\mu\nu}}{\partial t^2} e^\mu_\alpha e^\nu_\beta S^\beta. \tag{8.10}$$