Time Series

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Outline I

1. Estimation for Stationary Time Series
   - Sample Mean
   - Sample ACF
   - Sample PACF

Sample Mean I

- The mean μ of a stationary process, $\{X_n\}$, is estimated by its sample mean, defined as follows

$$\bar{X}_n = \frac{1}{n}(X_1 + \ldots + X_n)$$

Sample Mean II

- Remarks:
  - Expectation of $\bar{X}_n$ is

$$
\begin{aligned}
E[\bar{X}_n] &= \frac{1}{n}(E[X_1] + \ldots + E[X_n]) \\
&= \mu
\end{aligned}
$$

  - Thus, $\bar{X}_n$ is unbiased

Sample Mean III

- Mean squared error of $\bar{X}_n$ is

$$\begin{aligned}
E(\bar{X}_n - \mu)^2 = Var(\bar{X}_n) &= \frac{1}{n^2}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} Cov(X_i, X_j)\right] \\
&= n^{-2} \sum_{i-j=-n}^{n} (n - |i-j|)\,\gamma(i-j) \\
&= n^{-1} \sum_{h=-n}^{n} \left(1 - \frac{|h|}{n}\right)\gamma(h) \\
&= n^{-1} \sum_{h=-\infty}^{\infty} f_n(h),
\end{aligned}$$

where $f_n(h) = I_{(|h| \leq n)} \left(1 - \frac{|h|}{n}\right)\gamma(h)$.

Sample Mean IV

- Remark 1: $f_n(h) \leq |\gamma(h)|$, for all h, n.
  - Therefore, If $\gamma(n) \to 0$, as $n \to \infty$ then

$$\begin{aligned}
\lim_{n\to\infty} Var(\bar{X}_n) &= \lim_{n\to\infty} n^{-1} \sum_{h=-\infty}^{\infty} f_n(h) \\
&\leq \lim_{n\to\infty} n^{-1} \sum_{|h|\leq n} |\gamma(h)| \\
&= 2 \lim_{n\to\infty} |\gamma(n)| = 0.
\end{aligned}$$

Hence, $E(\bar{X}_n - \mu)^2 \to 0$, equivalently, $\bar{X}_n \xrightarrow{\mathcal{L}^2} \mu$.

- Remark 2: $f_n(h) = I_{(|h| \leq n)} \left(1 - \frac{|h|}{n}\right) \gamma(h) < |\gamma(h)|$, for all n.
  - Therefore, if $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$, then

$$\begin{aligned} \lim_{n\to\infty} nVar(\bar{X}_n) &= \lim_{n\to\infty} \sum_{h=-\infty}^{\infty} I_{(|h|\leq n)} \left(1 - \frac{|h|}{n}\right) \gamma(h) \\ &\overset{DCT}{=} \sum_{h=-\infty}^{\infty} \lim_{n\to\infty} I_{(|h|\leq n)} \left(1 - \frac{|h|}{n}\right) \gamma(h) \\ &= \sum_{h=-\infty}^{\infty} \gamma(h). \end{aligned}$$

  - DCT: If $f_n(x) < |g(x)|$ for all n, x and $\int |g(x)| dx < \infty$, then

$$\lim_{n\to\infty} \int f_n dx = \int \lim_{n\to\infty} f_n dx.$$

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Sample Mean VI

- For linear and ARMA models time series,

$$n^{1/2}(\bar{X}_n - \mu) \xrightarrow{D} N\left(0, \sum_{|h|<\infty} \gamma(h)\right)$$

  - $\bar{X}_n$, for large n, is approximately normal with mean μ and variance $n^{-1} \sum_{|h|<\infty} \gamma(h)$

Sample ACF I

- The autocorrelation function at lag h [i.e., $\rho(h)$] of a stationary process, $\{X_n\}$, is estimated by its sample autocorrelation function which is defined as follows

$$\hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)},$$

where, $\hat{\gamma}(h) = n^{-1} \sum_{t=1}^{n-|h|} (X_{t+|h|} - \bar{X}_n)(X_t - \bar{X}_n)$, the sample autocovariance of $\{X_n\}$ at lag h and $h = 0, \pm 1, \ldots$

Sample ACF II

- Remarks
  - The estimator $\hat{\rho}(h)$ is biased (even if the factor n^{-1} is replaced by $(n-h)^{-1}$)
    - Nevertheless, under general assumptions they are nearly unbiased for large sample sizes.

Sample ACF III

- The sample ACVF has the desirable property that for each $k \geq 1$ the $k-$dimensional sample covariance matrix

$$\hat{\Gamma}_k = \begin{bmatrix} \hat{\gamma}(0) & \hat{\gamma}(1) & \dots & \hat{\gamma}(k-1) \\ \hat{\gamma}(1) & \hat{\gamma}(0) & \dots & \hat{\gamma}(k-2) \\ \vdots & \vdots & \dots & \vdots \\ \hat{\gamma}(k-1) & \hat{\gamma}(k-2) & \dots & \hat{\gamma}(0) \end{bmatrix}$$

  is non-negative definite.
  - The sample autocorrelation matrix $\hat{R}_k = \hat{\Gamma}_k/\hat{\gamma}(0)$, is also non-negative definite.
  - If the factor n^{-1} is replaced by $(n-h)^{-1}$ in the definition of $\hat{\gamma}(h)$, the resulting covariance and correlation matrices $\hat{\Gamma}_k$ and $\hat{R}_k$ may not then be non-negative definite.
- The matrices $\hat{\Gamma}_k$ and $\hat{R}_k$ are in fact non-singular (hence, positive definite) if there is at least one nonzero $X_i - \bar{X}_n$

Sample ACF IV

- As h goes closer to n, the estimates $\hat{\gamma}(h)$ and $\hat{\rho}(h)$ becomes unreliable, since there are so few pairs (X_{t+h}, X_t) available
- A rule of thumb, provided by Box and Jenkins: n should be at least about 50 and $h \leq n/4$.

Sample ACF V

- For linear and ARMA models time series,

$$\begin{bmatrix} \hat{\rho}(1) \\ \hat{\rho}(2) \\ \vdots \\ \hat{\rho}(h) \end{bmatrix} = \hat{\rho} \xrightarrow{D} N\left(\rho = \begin{bmatrix} \rho(1) \\ \rho(2) \\ \vdots \\ \rho(h) \end{bmatrix}, n^{-1}W = n^{-1}[w_{ij}]_{h\times h} \right)$$

  - $\hat{\rho}$, for large n, is approximately normal with mean ρ and covariance matrix $n^{-1}W$, where

$$w_{ij} = \sum_{k=1}^{\infty}\{\rho(k+i)+\rho(k-i)-2\rho(i)\rho(k)\}\{\rho(k+j)+\rho(k-j)-2\rho(j)\rho(k)\}$$

    (Bartlett's formula)

- In particular, when sample size is large

$$\hat{\rho}(l) \sim N(\rho(l), n^{-1}w_{ll}),$$

  for $l = 1, \ldots, h$.

- Examples
  - $WN(0, \sigma^2)$

$$\rho_X(h) = \begin{cases} 1, & \text{if } h = 0, \\ 0, & \text{if } h \neq 0. \end{cases}$$

Thus,

$$w_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Hence,

$$\hat{\rho}(l) \sim N(0, n^{-1}),$$

for $l = 1, \ldots, h$.

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Sample ACF VII

- $MA(1)$ process,

$$w_{ii} = \begin{cases} 1 - 3\rho^2(1) + 4\rho^4(1), & \text{if } i = 1, \\ 1 + 2\rho^2(1), & \text{if } i > 1. \end{cases}$$

Hence,

$$\hat{\rho}(1) \sim N\left(\rho(1), n^{-1}\left[1 - 3\rho^2(1) + 4\rho^4(1)\right]\right)$$

and

$$\hat{\rho}(l) \sim N\left(0, n^{-1}\left[1 + 2\rho^2(1)\right]\right),$$

for $l = 2, \ldots, h$.

Sample ACF VIII

- $AR(1)$ process,

$$w_{ii} = (1 - \phi^{2i})(1 + \phi^2)(1 - \phi^2)^{-1} - 2i\phi^{2i}$$

Hence,

$$\hat{\rho}(l) \sim N\left(\phi^l, n^{-1}\left[(1 - \phi^{2l})(1 + \phi^2)(1 - \phi^2)^{-1} - 2l\phi^{2l}\right]\right),$$

for $l = 1, \ldots, h$.

Sample PACF I

- The partial autocorrelation function $\alpha(h)$ of a stationary process, $\{X_n\}$, is estimated by its sample partial autocorrelation function which is defined as follows

$$\hat{\alpha}(0) = 1$$

and

$$\hat{\alpha}(h) = \hat{\phi}_{hh}, \; h \geq 1$$

where $\hat{\phi}_{hh}$ is the last component of $\hat{\phi}_{\mathbf{h}} = \hat{\Gamma}_h^{-1}\hat{\gamma}_{\mathbf{h}}$.