Time Series

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Outline I

1. Components of Time Series
   - Additive Model
   - Examples

2. Estimation and Elimination of Trend and Seasonal Components
   - Estimation and Elimination of Trend in the Absence of Seasonality
   - Estimation and Elimination of Both Trend and Seasonality

Additive Model I

- Additive model of Time series $\{Y_t\}$,

$$Y_t = m_t + s_t + c_t + X_t$$

  - m_t (Trend)
  - s_t (Seasonality)
  - c_t (Cyclic)
  - X_t (Random)

Additive Model II

- Trend (m_t): Smooth, regular, long-term movement of the time series data.
  - Usually, most dominant component
  - Some series may exhibit an upward movement
  - Some series may exhibit a downward movement
  - Some series, after a period of growth (decline), may change its course and enter into a period of decline (growth)
  - Sudden or frequent changes are incompatible

Additive Model III

- Seasonality (s_t): A periodic movement, with period of movement less than one year
  - Periods and amplitudes are equal

Additive Model IV

- Cyclic (c_t): An oscillatory movement, with all periods of oscillation more than one year
  - Periods and amplitudes are not equal

Additive Model V

- Random (X_t): Irregular component of time series
  - Beyond human control

Multiplicative Model I

- Multiplicative model of Time series $\{Y_t\}$,

$$Y_t = m_t \times s_t \times c_t \times X_t$$

- Additive in logarithm

$$\log Y_t = \log m_t + \log s_t + \log c_t + \log X_t$$

Estimation and Elimination of Trend in the Absence of Seasonality I

- Nonseasonal Model with Trend:

$$Y_t = m_t + X_t, \text{ for } t = 1, \ldots, n,$$

where $EX_t = 0$.

  - Trend Estimation and then elimination
    - Smoothing with a finite moving average filter
    - Exponential smoothing
    - Smoothing by elimination of high-frequency components
    - Polynomial fitting
  - Direct Trend Elimination

Estimation and Elimination of Trend in the Absence of Seasonality II

- Smoothing with a finite moving average filter
  - Let q be a non-negative integer and consider the two-sided moving average

$$\hat{m}_t = (2q+1)^{-1} \sum_{j=-q}^{q} Y_{t-j}, \text{ for } q+1 \leq t \leq n-q$$

Estimation and Elimination of Trend in the Absence of Seasonality III

- Exponential smoothing
  - For any fixed $\alpha \in (0, 1)$, the one-sided moving averages $\hat{m}_t$, defined by the recursions

$$\hat{m}_t = \alpha Y_t + (1 - \alpha)\hat{m}_{t-1}, \text{ for } t = 2, \dots, n$$

    and

$$\hat{m}_1 = Y_1$$

  - Note: It is a weighted moving average of $Y_t, Y_{t-1}, \dots,$ with weights decreasing exponentially (except for the last one).

Estimation and Elimination of Trend in the Absence of Seasonality IV

- Smoothing by elimination of high-frequency components
  - Outside the scope of syllabus.

Estimation and Elimination of Trend in the Absence of Seasonality V

- Polynomial fitting
  - Regression

$$\hat{m}_t = \hat{a}_0 + \hat{a}_1 t + \hat{a}_2 t^2$$

  - Will not be discussed, here.

Estimation and Elimination of Trend in the Absence of Seasonality VI

- Once we estimate $\hat{m}_t$, we subtract it from Y_t to get the X_t (noise), i.e.

$$X_t = Y_t - \hat{m}_t$$

Estimation and Elimination of Trend in the Absence of Seasonality VII

- Trend Elimination by Differencing
  - Lag-1 difference operator

$$\nabla Y_t = Y_t - Y_{t-1} = (1 - B) Y_t$$

    - B is called backshift operator, i.e., $BY_t = Y_{t-1}$
  - In general,

$$B^j(Y_t) = Y_{t-j}$$

and

$$\nabla^j(Y_t) = \nabla(\nabla^{j-1} Y_t)$$

Estimation and Elimination of Trend in the Absence of Seasonality VIII

- The operator ∇, is sufficient to remove the linear trend function $m_t = a_0 + a_1 t$
- In the same way any polynomial trend of degree k can be removed by the application of the operator ∇^k

Estimation and Elimination of Both Trend and Seasonality I

- Model with Trend and Seasonality:

$$Y_t = m_t + s_t + X_t, \text{ for } t = 1, \ldots, n,$$

where $EX_t = 0, s_{t+d} = s_t$ and $\sum_{j=1}^{d} s_j = 0.$

  - Estimation and Elimination of Trend and Seasonal Components
  - Direct Elimination of Trend and Seasonal Components by Differencing

Estimation and Elimination of Both Trend and Seasonality II

- Estimation and Elimination of Trend and Seasonal Components
  1. Estimate the trend by applying a moving average filter

     $$\hat{m}_t = (0.5y_{t-q} + y_{t-q+1} + \ldots + y_{t+q-1} + 0.5y_{t+q})/d, \text{ for } q+1 \leq t \leq n-q,$$

     if d (i.e. length of season) is even ($d = 2q$)

     $$\hat{m}_t = (y_{t-q} + y_{t-q+1} + \ldots + y_{t+q-1} + y_{t+q})/d, \text{ for } q+1 \leq t \leq n-q,$$

     if d is odd ($d = 2q + 1$)

Estimation and Elimination of Both Trend and Seasonality III

2. Then estimate the seasonal component.
   - For each $k = 1, \ldots, d$, we compute the average w_k of the deviations $\{(y_{k+jd} - \hat{m}_{k+jd}),$ such that $j \geq 0$ and $q + 1 \leq k + jd \leq n - q\}$
   - Since these average deviations do not necessarily sum to zero, we estimate the seasonal component s_k as

$$\hat{s}_k = w_k - d^{-1} \sum_{i=1}^{d} w_i, \text{ for } k = 1, \ldots, d$$

   and

$$\hat{s}_k = \hat{s}_{k-d}, \text{ for } k > d$$

3. The deseasonalized data is then defined to be the original series with the estimated seasonal component removed, i.e.,

$$d_t = y_t - \hat{s}_t,$$

for $t = 1, \ldots, n$.

Estimation and Elimination of Both Trend and Seasonality IV

4. We reestimate the trend from the deseasonalized data $\{d_t\}$ using one of the methods of trend estimation and denote it by $\hat{m}_t$
5. Finally, subtract the estimated trend $\hat{m}_t$, from deseasonalized data $\{d_t\}$ and left with noise, i.e., $y_t - \hat{s}_t - \hat{m}_t$

Estimation and Elimination of Both Trend and Seasonality V

- Elimination of Trend and Seasonal Components by Differencing
  1. To reduce the seasonality of length d, apply the lag-d differencing operator ∇_d on Y_t , where
  $$\nabla_d Y_t = Y_t - Y_{t-d} = (1 - B^d) Y_t$$
     - Because, $Y_t = m_t + s_t + X_t \Rightarrow \nabla_d Y_t = m_t - m_{t-d} + Y_t - Y_{t-d}$
  2. The trend, $m_t - m_{t-d}$, can then be eliminated by applying suitable power of ∇
  3. As a result, we will be left with noise