

Time Series

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- 1 Stationary Process
 - Strict Stationary Process
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 - Revisiting ARMA Process

Stationary Process I

- Going beyond *i.i.d* stochastic process (time series)
- Stationary Process
 - Strict Stationary Process
 - Weak Stationary Process

Strict Stationary Process I

- $\{X_t\}$ is a strictly stationary process if

$$(X_{i_1}, X_{i_2}, \dots, X_{i_n})' \stackrel{D}{=} (X_{i_1+h}, X_{i_2+h}, \dots, X_{i_n+h})'$$

for all integers h and $n \geq 1$.

- i.e.

$$f_{X_{i_1}, X_{i_2}, \dots, X_{i_n}}(x_1, \dots, x_n) = f_{X_{i_1+h}, X_{i_2+h}, \dots, X_{i_n+h}}(x_1, x_2, \dots, x_n)$$

for all integers h and $n \geq 1$.

Strict Stationary Process II

- Properties of a Strictly Stationary Process $\{X_t\}$:
 - The random variables X_t are identically distributed
 - Not necessarily independent
 - An *i.i.d* sequence is also strictly stationary
 - $(X_t, X_{t+h})' \stackrel{D}{=} (X_1, X_{1+h})'$ for all integers t and h .

(Weak) Stationary Process I

- $\{X_t\}$ is a (weakly) stationary process if all of the following three conditions hold

① $E[X_t^2] < \infty$

- Finite second order moment

② $\mu_X(t)$ is independent of t ,

- where $\mu_X(t)$ is the **mean function** of $\{X_t\}$ and is defined as

$$\mu_X(t) = E(X_t)$$

③ $\gamma(t+h, t)$ is independent of t for each h ,

- where $\gamma(t+h, t)$ is the **covariance function** of $\{X_t\}$ and is defined as

$$\begin{aligned}\gamma(t+h, t) &= \text{Cov}(X_{t+h}, X_t) \\ &= E[(X_{t+h} - \mu_X(t+h))(X_t - \mu_X(t))]\end{aligned}$$

(Weak) Stationary Process II

- Remarks:

- If $\{X_t\}$ is strictly stationary and $E[X_t^2] < \infty$ for all t , then $\{X_t\}$ is also weakly stationary
- In this course, whenever we use the term stationary we shall mean weakly stationary, unless we specifically indicate otherwise.
- We use the term covariance function with reference to a stationary time series $\{X_t\}$ we shall mean the function γ_X of one variable defined by

$$\gamma_X(h) := \gamma_X(h, 0) = \gamma_X(t + h, t)$$

- The function $\gamma_X(\cdot)$ will be referred to as the **autocovariance function** (ACVF) of X_t and $\gamma_X(h)$ as its value at *lag* h .

(Weak) Stationary Process III

- Formally, the **autocovariance function** (ACVF) of $\{X_t\}$ at lag h is defined as

$$\gamma_X(h) = E[(X_{t+h} - \mu_X)(X_t - \mu_X)]$$

- Note that

$$\gamma_X(0) \geq 0$$

and

$$\gamma_X(h) = \gamma_X(-h)$$

- The **autocorrelation function** (ACF) of $\{X_t\}$ at lag h is defined as

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)}$$

- Basic Properties of $\rho(\cdot)$:
 - $|\rho(h)| \leq 1$
 - $\rho(\cdot)$ is even, i.e., $\rho(h) = \rho(-h)$ for all h .

(Weak) Stationary Process: Examples I

- Some Elementary Stationary processes

- ① iid noise: $\{X_t\} \sim IID(0, \sigma^2)$, with $E(X_t^2) = \sigma^2 < \infty$

- ACVF:

$$\gamma_X(t+h, t) = \gamma_X(h) = \begin{cases} \sigma^2, & \text{if } h = 0, \\ 0 & \text{if } h \neq 0. \end{cases}$$

- ACF

$$\rho_X(t+h, t) = \rho_X(h) = \begin{cases} 1, & \text{if } h = 0, \\ 0, & \text{if } h \neq 0. \end{cases}$$

(Weak) Stationary Process: Examples II

2 White Noise: $\{X_t\} \sim WN(0, \sigma^2)$

- It's a sequence of uncorrelated random variables, each with zero mean and variance σ^2 ,
- ACVF:

$$\gamma_X(h) = \begin{cases} \sigma^2, & \text{if } h = 0, \\ 0 & \text{if } h \neq 0. \end{cases}$$

- ACF

$$\rho_X(h) = \begin{cases} 1, & \text{if } h = 0, \\ 0, & \text{if } h \neq 0. \end{cases}$$

(Weak) Stationary Process: Examples III

Note that every $iid(0, \sigma^2)$ sequence is $WN(0, \sigma^2)$ but the converse is not true, e.g.

Let $\{Z_t\}$ be $iid \sim N(0, 1)$ noise and define

$$X_t = \begin{cases} Z_t, & \text{if } t \text{ is even,} \\ (Z_{t-1}^2 - 1)/\sqrt{2} & \text{if } t \text{ is odd.} \end{cases}$$

Here, that $\{X_t\}$ is $WN(0, 1)$ but not $iid(0, 1)$.

(Weak) Stationary Process: Examples IV

- 3 First-order moving average or MA(1) process:

$$X_t = Z_t + \theta Z_{t-1}, t = 0, \pm 1, \dots,$$

where $\{Z_t\} \sim WN(0, \sigma^2)$ and θ is real valued constant

- $EX_t = 0$
- ACVF

$$\gamma_X(h) = \begin{cases} \sigma^2(1 + \theta^2), & \text{if } h = 0, \\ \sigma^2\theta, & \text{if } h = \pm 1 \\ 0, & \text{if } |h| \geq 2. \end{cases}$$

- ACF

$$\rho_X(h) = \begin{cases} 1, & \text{if } h = 0, \\ \theta/(1 + \theta^2), & \text{if } h = \pm 1 \\ 0, & \text{if } |h| \geq 2. \end{cases}$$

(Weak) Stationary Process: Examples V

- 4 First-order autoregressive or AR(1) process:

$$X_t = \phi X_{t-1} + Z_t, t = 0, \pm 1, \dots,$$

where $\{Z_t\} \sim WN(0, \sigma^2)$, $|\phi| < 1$ and Z_t is uncorrelated with X_s for each $s < t$

- $EX_t = 0$
- ACVF

$$\gamma(h) = \phi^{|h|} \frac{\sigma^2}{1 - \phi^2}$$

- ACF

$$\rho(h) = \phi^{|h|}$$

(Weak) Stationary Process: Examples VI

5 First-order ARMA or ARMA(1,1) process:

$$X_t = \phi X_{t-1} + Z_t + \theta Z_{t-1}, t = 0, \pm 1, \dots,$$

where $\{Z_t\} \sim WN(0, \sigma^2)$, $|\phi| < 1$, Z_t is uncorrelated with X_s for each $s < t$ and $\phi + \theta \neq 0$

- $EX_t = 0$
- ACVF

$$\gamma_X(h) = \begin{cases} \sigma^2 \left[1 + \frac{(\theta + \phi)^2}{1 - \phi^2} \right], & \text{if } h = 0, \\ \sigma^2 \left[\theta + \phi + \frac{(\theta + \phi)^2 \phi}{1 - \phi^2} \right], & \text{if } h = \pm 1 \\ \phi^{|h|-1} \gamma_X(1), & \text{if } |h| \geq 2. \end{cases}$$

(Weak) Stationary Process: Examples VII

- Is Random Walk a stationary process?

$$X_t = X_{t-1} + Z_t,$$

where $Z_t \sim IID(\mu, \sigma^2)$

(Weak) Stationary Process: Examples VIII

- NO

- $\mu_X(t) = \mu t$
- $\text{Cov}(X_m, X_n) = \min(m, n) \times \sigma^2$

(Weak) Stationary Process: Examples IX

- Three higher order stationary processes

- 1 q -order moving average or $MA(q)$ process:

$$X_t = Z_t + \theta_1 Z_{t-1} + \dots + \theta_q Z_{t-q}, t = 0, \pm 1, \dots,$$

where $\{Z_t\} \sim WN(0, \sigma^2)$ and $\theta_1, \dots, \theta_q$ are real valued constants

- 2 p -order autoregressive or $AR(p)$ process:

$$X_t = \phi_1 X_{t-1} + \dots + \phi_p X_{t-p} + Z_t, t = 0, \pm 1, \dots,$$

where $\{Z_t\} \sim WN(0, \sigma^2)$, Z_t is uncorrelated with X_s for each $s < t$ and all roots of the polynomial $(1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p)$ lie outside the unit circle.

(Weak) Stationary Process: Examples X

3 ARMA(p, q) process:

$$X_t = \phi_1 X_{t-1} + \dots + \phi_p X_{t-p} + Z_t + \theta_1 Z_{t-1} + \dots + \theta_q Z_{t-q}, t = 0, \pm 1, \dots,$$

where $\{Z_t\} \sim WN(0, \sigma^2)$, with all roots of the polynomial $(1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p)$ lie outside the unit circle and the polynomials $(1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p)$ and $(1 + \theta_1 z + \dots + \theta_q z^q)$ have no common factors.

Linear Process I

- Linear processes: It includes the class of autoregressive moving-average (ARMA) process,
- Definition: The process $\{X_t\}$ is a linear process if it has the representation

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j},$$

for all t , where $\{Z_t\} \sim WN(0, \sigma^2)$ and $\{\psi_j\}$ is a sequence of constants with $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$

- Alternate representation by backward shift operator:

$$X_t = \Psi(B)Z_t,$$

where $\Psi(B) = \sum_{j=-\infty}^{\infty} \psi_j B^j$

- The operator $\Psi(B)$ can be thought of as a linear filter, which when applied to the white noise “input” series $\{Z_t\}$ produces the “output” $\{X_t\}$
- Note: every second-order stationary process is either a linear process or can be transformed to a linear process by subtracting a deterministic component.

Remarks

- ① The condition $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$ (absolute summability) ensures that the infinite sum

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j},$$

converges (with probability one)

- Sketch of proof:-

Let $X_t^n = \sum_{j=-n}^n \psi_j Z_{t-j}$, and for small ϵ , define

$$A_n(\epsilon) = \left\{ |X_t^n - X_t| > \epsilon \right\} = \left\{ \left| \sum_{|j|>n} \psi_j Z_{t-j} \right| > \epsilon \right\}.$$

By Chebyshev's inequality,

$$P(A_n) \leq E \left[\left| \sum_{|j|>n} \psi_j Z_{t-j} \right|^2 \right] / \epsilon^2.$$

Thus,

$$\begin{aligned}
 \sum_{n=1}^{\infty} P(A_n) &\leq \frac{1}{\epsilon^2} \sum_{n=1}^{\infty} E \left| \sum_{|j|>n} \psi_j Z_{t-j} \right|^2 \\
 &= \frac{1}{\epsilon^2} \sum_{n=1}^{\infty} E \left| \sum_{|i|>n} \sum_{|k|>n} \psi_i \psi_k (Z_{t-i} Z_{t-k}) \right| ; \text{ as } |ab| = |a||b| \\
 &\leq \frac{1}{\epsilon^2} \sum_{n=1}^{\infty} \left[\sum_{|i|>n} \sum_{|k|>n} |\psi_i| |\psi_k| E |(Z_{t-i} Z_{t-k})| \right] ; \text{ by triangular inequality} \\
 &\leq \frac{1}{\epsilon^2} \sum_{n=1}^{\infty} \left[\sum_{|i|>n} \sum_{|k|>n} |\psi_i| |\psi_k| E^{1/2} |(Z_{t-i}^2)| E^{1/2} |(Z_{t-k}^2)| \right] ; \text{ by Cauchy-Schwarz inequality} \\
 &\leq \frac{\sigma^2}{\epsilon^2} \sum_{n=1}^{\infty} \left[\sum_{|i|>n} \sum_{|k|>n} |\psi_i| |\psi_k| \right] ; \text{ Stationarity} \\
 &= \frac{\sigma^2}{\epsilon^2} \sum_{n=1}^{\infty} \left[\sum_{|j|>n} |\psi_j| \right]^2 < \infty ; \text{ absolute summability}
 \end{aligned}$$

Linear Process VI

Therefore, by Borel Cantelli Lemma

$$\sum_{n=1}^{\infty} P(A_n) < \infty \Rightarrow P(A^{(S)}) = 0, \text{ where } A^{(S)} = \cap_{n=1}^{\infty} \cup_{m=n}^{\infty} A_m = \limsup_n A_n$$

- Event $A^{(S)}$ is called the lim sup event of the infinite sequence $\{A_n\}$.
- Event $A^{(S)}$ occurs if and only if for all $n \geq 1$, there exists an $m \geq n$ such that A_m occurs,
- equivalently, Event $A^{(S)}$ occurs if and only if infinitely many of the A_n occur.

By definition of limit, $\omega \in \left\{ \lim_{n \rightarrow \infty} X_n = X \right\}$ if and only if for all $u \geq 1$ there exists $n \geq 1$ such that for every $m \geq n$, $|X_m(\omega) - X(\omega)| \leq \frac{1}{u}$. Equivalently, it holds if and only if

$$\omega \in \cap_{u=1}^{\infty} \cup_{n=1}^{\infty} \cap_{m=n}^{\infty} \left[A_m \left(\frac{1}{u} \right) \right]^c = \left(\cup_{u=1}^{\infty} \limsup_n A_n \left(\frac{1}{u} \right) \right)^c.$$

Thus,

$$\begin{aligned} P \left(\omega : \lim_{n \rightarrow \infty} X_t^n(\omega) = X_t(\omega) \right) &= P \left(\left(\cup_{u=1}^{\infty} \limsup_n A_n(1/u) \right)^c \right) = 1 - P \left(\cup_{u=1}^{\infty} \limsup_n A_n(1/u) \right) \\ &\geq 1 - \sum_{u=1}^{\infty} P \left(\limsup_n A_n(1/u) \right) = 1 \end{aligned}$$

Hence, $X_t^n \xrightarrow{a.s.} X_t$.

- ② The condition $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$ ensures that $\sum_{j=-\infty}^{\infty} \psi_j^2 < \infty$ therefore, the series $\sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}$ converges in mean square to X_t , i.e.,

$$X_t^n = \sum_{j=-n}^n \psi_j Z_{t-j} \xrightarrow{m.s.} X_t,$$

equivalently, $\lim_{n \rightarrow \infty} E|X_t^n - X_t|^2 = 0$.

Linear Process VIII

- ③ In generally, let $\{Y_t\}$ be a stationary process with mean 0 and covariance function γ_Y . If $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$, then the process

$$X_t = \Psi(B)Y_t = \sum_{j=-\infty}^{\infty} \psi_j Y_{t-j},$$

is also stationary with mean 0 and autocovariance function as

$$\begin{aligned}\gamma_X(h) &= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k E[Y_{t-j} Y_{t+h-k}] \\ &= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_Y(h - k + j) \\ &= \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h} \sigma^2, \text{ (if } Y_t \text{ is white noise)}\end{aligned}$$

- 4 The filters of the form $\alpha(B) = \sum_{j=-\infty}^{\infty} \alpha_j B^j$ and $\beta(B) = \sum_{j=-\infty}^{\infty} \beta_j B^j$ with

absolutely summable coefficients can be applied successively to a stationary series $\{Y_t\}$ to generate a new stationary series

$$\begin{aligned} W_t &= \sum_{j=-\infty}^{\infty} \alpha_j \left(\sum_{k=-\infty}^{\infty} \beta_k Y_{(t-j)-k} \right) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \alpha_j \beta_k Y_{(t-j)-k} \\ &= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \alpha_{j-k} \beta_k Y_{t-j}, \text{ replacing } j \text{ by } j-k \\ &= \sum_{j=-\infty}^{\infty} \psi_j Y_{t-j} \end{aligned}$$

- Therefore, $\psi_j = \sum_{k=-\infty}^{\infty} \alpha_{j-k} \beta_k = \sum_{k=-\infty}^{\infty} \alpha_k \beta_{j-k}$
- Alternate form $W_t = \alpha(B)\beta(B)Y_t = \beta(B)\alpha(B)Y_t = \psi(B)Y_t$

- Forms of (stable) linear process:

- Causal: A linear process $\{X_t\}$ is causal if X_t can be expressed in terms of the current and past values $Z_s, s \leq t$,

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}.$$

- Invertible: A linear process $\{X_t\}$ is invertible if Z_t can be expressed in terms of the current and past values $X_s, s \leq t$,

$$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}.$$

Revisiting ARMA Proces I

- Let X_t be an ARMA(p,q) process

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$$

where $\{Z_t\} \sim WN(0, \sigma^2)$, and the polynomials $(1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p)$ and $(1 + \theta_1 z + \cdots + \theta_q z^q)$ have no common factors.

- Representing X_t as linear process

$$X_t = (1 - \phi_1 B - \cdots - \phi_p B^p)^{-1} (1 + \theta_1 B + \cdots + \theta_q B^q) Z_t$$

Revisiting ARMA Proces II

- Condition for stability of X_t :

The coefficients of linear process expression of $X_t (= \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j})$ are absolutely summable.

- Equivalent Condition:

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0 \text{ for all } |z| = 1$$

- No roots of $\phi(z)$ on the unit circle

- Example: AR(1)

$$X_t - \phi X_{t-1} = Z_t$$

- if $|\phi| < 1$, $X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}$ is stable.
- if $|\phi| > 1$, $X_t = \sum_{j=1}^{\infty} \phi^{-j} Z_{t+j}$ is stable.

Revisiting ARMA Proces IV

- Condition for causality of X_t :

Process X_t can be expressed in terms of the current and past

values $Z_s, s \leq t$, (i.e., $X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$)

- Equivalent Condition

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0 \text{ for all } |z| \leq 1.$$

- No roots of $\phi(z)$ inside the unit circle.

- Example: AR(1)

$$X_t - \phi X_{t-1} = Z_t$$

- if $|\phi| < 1$, $X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}$ is stable and causal.
- if $|\phi| > 1$, $X_t = \sum_{j=1}^{\infty} \phi^{-j} Z_{t+j}$ is stable but non-causal.

Revisiting ARMA Proces VI

- Condition for invertibility of X_t :
Process Z_t can be expressed in terms of the current and past values $X_s, s \leq t$, (i.e., $Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}$)

- Equivalent Condition

$$\theta(z) = 1 + \theta_1 z + \dots + \theta_q z^q \neq 0 \text{ for all } |z| \leq 1.$$

- No roots $\theta(z)$ inside the unit circle.