

# Chain Matrix Multiplication

Advanced Programming and Algorithmic Design

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The background of the slide features a large, faint watermark of the University of Trieste logo. The logo is circular and contains the text "UNIVERSITA' DEGLI STUDI DI TRIESTE" around the perimeter and "E SPLENDI" in the center. In the middle of the logo is a detailed illustration of a building with a dome and a tower, likely a representation of the University's main building or a significant landmark.

## Problem Definition

# Intuition for the Matrix-chain Multiplication Problem

Consider the matrices  $A_1, A_2, A_3$

- $A_1$  having dimension  $50 \times 5$
- $A_2$  having dimension  $5 \times 100$
- $A_3$  having dimension  $100 \times 10$

How many scalar multiplications does  $A_1 \times A_2 \times A_3$  require?

# Intuition for the Matrix-chain Multiplication Problem

Matrix product is associative i.e.,  $(A_1 \times A_2) \times A_3 = A_1 \times (A_2 \times A_3)$



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- if we compute  $(A_1 \times A_2) \times A_3$

$$50 * 100 * 5 = 25000 \quad (\text{to compute } A_1 \times A_2)$$

$$50 * 10 * 100 = 50000 \quad (\text{to compute } (A_1 \times A_2) \times A_3)$$

- if we compute  $A_1 \times (A_2 \times A_3)$

$$5 * 10 * 100 = 5000 \quad (\text{to compute } A_2 \times A_3)$$

$$50 * 10 * 5 = 2500 \quad (\text{to compute } A_1 \times (A_2 \times A_3))$$

**75000**  $((A_1 \times A_2) \times A_3)$  vs

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**75000**  $((A_1 \times A_2) \times A_3)$  **vs** **7500**  $(A_1 \times (A_2 \times A_3))$

# Problem Definition

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Consider the **chain** of matrices  $\langle A_1, \dots, A_n \rangle$  where  $A_i$  has dimensions  $p_{i-1} \times p_i$  for all  $i \in [1, n]$

Compute a **parenthesization** that minimizes the # of scalar products for the chain multiplication

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## A Naïve Approach

# Recursive Solution

We may try to search among all the possible parenthesizations

- if  $n = 1$ , the parenthesization is obvious
- if  $n > 1$ , the chain can be parenthesized as

$$(A_1 \times \dots A_k) \times (A_{k+1} \times \dots A_n)$$

for any  $k \in [1, n - 1]$ . Recursively produce the parenthesizations for  $\langle A_1, \dots, A_k \rangle$  and  $\langle A_{k+1}, \dots, A_n \rangle$

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How many parenthesizations has  $\langle A_1, \dots, A_n \rangle$ ?

many of all the calls are computing the same things: is trying to compute a suboptimal problem

# Counting Parenthesizations

$\langle A_1, \dots, A_n \rangle$  has

$$P(n) = \begin{cases} 1 & \text{if } n = 1 \\ \sum_{k=1}^{n-1} P(k) * P(n-k) & \text{if } n > 1 \end{cases}$$

different parenthesizations

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different parenthesizations

It can be proved that  $P(n) \in \Omega(2^n)$

Too many parenthesizations to be enumerated!!!  
(if you don't believe it, try for  $n = 8$ )



# Some Breakthrough Observations

- if  $(A_1 \times \dots \times A_k) \times (A_{k+1} \times \dots \times A_n)$  is optimal for the chain,
  - the 1st part is optimal for  $\langle A_1, \dots, A_k \rangle$
  - the 2nd part is optimal for  $\langle A_{k+1}, \dots, A_n \rangle$
- many branches of the 'naïve' recursive approach perform the very same computation
  - e.g. for every parenthesization of  $A_1 \times \dots \times A_k$ , the parenthesizations for  $A_{k+1} \times \dots \times A_n$

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**Idea:**

Solution optimal  $\leftrightarrow$  every slice is suboptimal dynamic programming avoid search

Recursively compute optimal parenthesizations and use dynamic programming

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## A Dynamic Programming Solution

# Dynamic Programming Solution

!!! $A_i$  has dimension  $p_{i-1} \times p_i$

Suppose you have  $A_1 \times \dots \times A_i \times \dots \times A_j \times \dots \times A_n$ . Define  $m[i,j]$  = number of scalar products required

Store the minimum # of products for all the sub-chains in  $m$

Recursively, compute  $m[i, j]$  as:

$$m[i, j] = \begin{cases} 0 & \text{if } i = j \\ \min_{k \in [i, j-1]} \{m[i, k] + m[k+1, j] + p_{i-1}p_kp_j\} & \text{if } i < j \end{cases}$$

split problem into subproblems:  $A_i \dots A_j \rightarrow A_i \dots A_k \dots A_j$

number of scalar

# Dynamic Programming Solution

Store the minimum # of products for all the sub-chains in  $m$

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For each  $i, j$  also store in  $s[i, j]$  the  $k$  that minimizes

$$m[i, k] + m[k+1, j] + p_{i-1}p_kp_j$$

i.e., the parenthesization for the current level



# Dynamic Programming Solution: Example

Consider  $A_1$  ( $3 \times 5$ ),  $A_2$  ( $5 \times 10$ ),  $A_3$  ( $10 \times 2$ ) and  $A_4$  ( $2 \times 3$ ).

$A_1 \times A_2 \times A_3 \times A_4$

1	2	3	4	
0	?	?	?	1
	0	?	?	2
		0	?	3
			0	4

$m$

2	3	4	
?	?	?	1
	?	?	2
		?	3

$s$

# Dynamic Programming Solution: Example

Consider  $A_1$  ( $3 \times 5$ ),  $A_2$  ( $5 \times 10$ ),  $A_3$  ( $10 \times 2$ ) and  $A_4$  ( $2 \times 3$ ).

$$(A_1 \times A_2 \times A_3 \times A_4)$$

I want to put the p

1	2	3	4	
0	?	?	?	1
	0	?	?	2
		0	?	3
			0	4

$m$

2	3	4	
?	?	?	1
	?	?	2
		?	3

$s$

# Dynamic Programming Solution: Example

Consider  $A_1$  ( $3 \times 5$ ),  $A_2$  ( $5 \times 10$ ),  $A_3$  ( $10 \times 2$ ) and  $A_4$  ( $2 \times 3$ ).

$$((A_1) \times (A_2 \times A_3 \times A_4))$$

Isolate  $A_1$  as a try for a  $k$ .  $A_1$  isolated  $\rightarrow$  no multiplication:  $i=j=1 \rightarrow m[i,j]=0$ . The whole diagonal

1	2	3	4	
0	?	?	?	1
	0	?	?	2
		0	?	3
			0	4

$m$

2	3	4	
?	?	?	1
	?	?	2
		?	3

$s$

# Dynamic Programming Solution: Example

Consider  $A_1$  ( $3 \times 5$ ),  $A_2$  ( $5 \times 10$ ),  $A_3$  ( $10 \times 2$ ) and  $A_4$  ( $2 \times 3$ ).

$$((A_1) \times ((A_2) \times (A_3 \times A_4)))$$

1	2	3	4	
0	?	?	?	1
	0	?	?	2
		0	?	3
			0	4

$m$

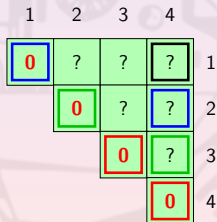
2	3	4	
?	?	?	1
	?	?	2
		?	3

$s$

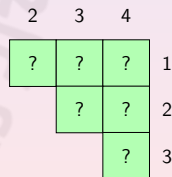
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$$((A_1) \times ((A_2) \times ((A_3) \times (A_4))))$$



$m$



$s$

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Consider  $A_1$  ( $3 \times 5$ ),  $A_2$  ( $5 \times 10$ ),  $A_3$  ( $10 \times 2$ ) and  $A_4$  ( $2 \times 3$ ).

$$((A_1) \times ((A_2) \times ((A_3) \times (A_4))))$$

1	2	3	4	
0	?	?	?	1
	0	?	?	2
		0	60	3
			0	4

$m$

$10 \times 2 \times 3$

2	3	4	
?	?	?	1
	?	?	2
		3	3

$s$

between  $A_3, A_4$  we can on

will store the  $k$  which minimizes  $m[i, j]$

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Consider  $A_1$  ( $3 \times 5$ ),  $A_2$  ( $5 \times 10$ ),  $A_3$  ( $10 \times 2$ ) and  $A_4$  ( $2 \times 3$ ).

$$((A_1) \times ((A_2) \times (A_3 \times A_4)))$$

1	2	3	4
0	?	?	?
	0	?	210
		0	60
			0

$m$

2	3	4
?	?	?
	?	2
		3

$s$

$5 \times 10 \times 3 + 60$

Seeing among  $A_2, A_4$

# Dynamic Programming Solution: Example

Consider  $A_1$  ( $3 \times 5$ ),  $A_2$  ( $5 \times 10$ ),  $A_3$  ( $10 \times 2$ ) and  $A_4$  ( $2 \times 3$ ).

$$((A_1) \times ((A_2 \times A_3) \times (A_4)))$$

1	2	3	4	
0	?	?	?	1
	0	?	210	2
		0	60	3
			0	4

$m$

2	3	4	
?	?	?	1
	?	2	2
		3	3

$s$

no we try  $s[4,2] = \text{parent}$



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$$((A_1) \times ((A_2) \times (A_3)) \times (A_4)))$$

1	2	3	4	
0	?	?	?	1
	0	?	210	2
		0	60	3
			0	4

$m$

2	3	4	
?	?	?	1
	?	2	2
		3	3

$s$

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$$((A_1) \times ((A_2) \times (A_3)) \times (A_4)))$$

1	2	3	4	
0	?	?	?	1
	0	100	210	2
		0	60	3
			0	4

$m$

2	3	4	
?	?	?	1
	2	2	2
		3	3

$s$

between  $A_2$  and  $A_3$  we c

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$$((A_1) \times ((A_2 \times A_3) \times (A_4)))$$

1	2	3	4	
0	?	?	?	1
	0	100	130	2
		0	60	3
			0	4

$m$

2	3	4	
?	?	?	1
	2	3	2
		3	3

$s$

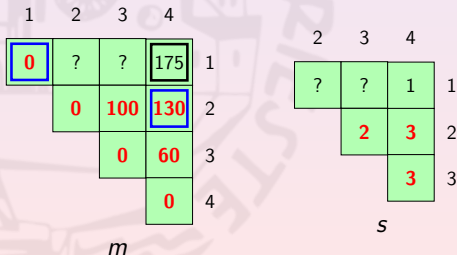
$(A_2 \times A_3) 100 \text{ op} + 5 \times 2 \times 3$

This is best way to parenthesize  $(A_2 A_3 A_4)$ . Will not be computed again

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$$((A_1) \times (A_2 \times A_3 \times A_4))$$



isolating  $A_1$ , the best parenthesisation gives 175 op at least

# Dynamic Programming Solution: Example

Consider  $A_1$  ( $3 \times 5$ ),  $A_2$  ( $5 \times 10$ ),  $A_3$  ( $10 \times 2$ ) and  $A_4$  ( $2 \times 3$ ).

$$((A_1 \times A_2) \times (A_3 \times A_4))$$

change case: parenthesis closed after 2 and 4. We already know how

1	2	3	4	
0	?	?	175	1
	0	100	130	2
		0	60	3
			0	4

$m$

2	3	4	
?	?	1	1
	2	3	2
		3	3

$s$

# Dynamic Programming Solution: Example

Consider  $A_1$  ( $3 \times 5$ ),  $A_2$  ( $5 \times 10$ ),  $A_3$  ( $10 \times 2$ ) and  $A_4$  ( $2 \times 3$ ).

$$(((A_1) \times (A_2)) \times (A_3 \times A_4))$$

210 > 175 -> Nope

	1	2	3	4	
1	0	150	?	175	1
2		0	100	130	2
3			0	60	3
4				0	4

$m$

	2	3	4	
1	1	?	1	1
2		2	3	2
3			3	3

$s$

between  $A_2$  and  $A_1$

# Dynamic Programming Solution: Example

Consider  $A_1$  ( $3 \times 5$ ),  $A_2$  ( $5 \times 10$ ),  $A_3$  ( $10 \times 2$ ) and  $A_4$  ( $2 \times 3$ ).

$$((A_1 \times A_2) \times (A_3 \times A_4))$$

1	2	3	4	
0	150	?	175	1
	0	100	130	2
		0	60	3
			0	4

$m$

2	3	4	
1	?	1	1
	2	3	2
		3	3

$s$

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$$((A_1 \times A_2 \times A_3) \times (A_4))$$

1	2	3	4	
0	150	?	175	1
	0	100	130	2
		0	60	3
			0	4

$m$

2	3	4	
1	?	1	1
	2	3	2
		3	3

$s$



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$$(((A_1) \times (A_2 \times A_3)) \times (A_4))$$

1	2	3	4	
0	150	130	175	1
	0	100	130	2
		0	60	3
			0	4

$m$

2	3	4	
1	1	1	1
	2	3	2
		3	3

$s$

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$$(((A_1 \times A_2) \times (A_3)) \times (A_4))$$

1	2	3	4	
0	150	130	175	1
	0	100	130	2
		0	60	3
			0	4

$m$

2	3	4	
1	1	1	1
	2	3	2
		3	3

$s$

# Dynamic Programming Solution: Example

Consider  $A_1$  ( $3 \times 5$ ),  $A_2$  ( $5 \times 10$ ),  $A_3$  ( $10 \times 2$ ) and  $A_4$  ( $2 \times 3$ ).

$$((A_1 \times A_2 \times A_3) \times (A_4))$$

Total: 130+ 3x2x3

1	2	3	4	
0	150	130	175	1
	0	100	130	2
		0	60	3
			0	4

$m$

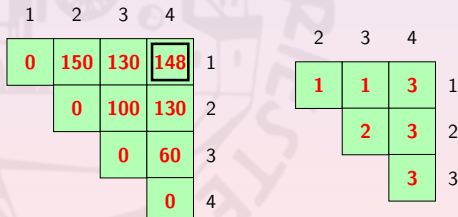
2	3	4	
1	1	1	1
	2	3	2
		3	3

$s$

# Dynamic Programming Solution: Example

Consider  $A_1$  ( $3 \times 5$ ),  $A_2$  ( $5 \times 10$ ),  $A_3$  ( $10 \times 2$ ) and  $A_4$  ( $2 \times 3$ ).

$$(A_1 \times A_2 \times A_3 \times A_4)$$



How to use s: I want to know  $A_1..A_4$ , look at 4,1 -> insert parenthesis now I have  $(A_1..A_3$

# Dynamic Programming Solution: Example

Consider  $A_1$  ( $3 \times 5$ ),  $A_2$  ( $5 \times 10$ ),  $A_3$  ( $10 \times 2$ ) and  $A_4$  ( $2 \times 3$ ).

$$(A_1 \times A_2 \times A_3) \times (A_4)$$

1	2	3	4	
0	150	130	148	1
	0	100	130	2
		0	60	3
			0	4

$m$

2	3	4	
1	1	3	1
	2	3	2
		3	3

$s$

# Dynamic Programming Solution: Example

Consider  $A_1$  ( $3 \times 5$ ),  $A_2$  ( $5 \times 10$ ),  $A_3$  ( $10 \times 2$ ) and  $A_4$  ( $2 \times 3$ ).

$$((A_1) \times (A_2 \times A_3)) \times (A_4)$$

	1	2	3	4	
1	0	150	130	148	1
2		0	100	130	2
3			0	60	3
4				0	4

$m$

	2	3	4	
1	1	1	3	1
2		2	3	2
3			3	3

$s$

# Dynamic Programming Solution: Example

Consider  $A_1$  ( $3 \times 5$ ),  $A_2$  ( $5 \times 10$ ),  $A_3$  ( $10 \times 2$ ) and  $A_4$  ( $2 \times 3$ ).

$$((A_1) \times ((A_2) \times (A_3))) \times (A_4)$$

1	2	3	4	
0	150	130	148	1
	0	100	130	2
		0	60	3
			0	4

$m$

2	3	4	
1	1	3	1
	2	3	2
		3	3

$s$

# Dynamic Programming Solution: An Iterative Version

Both  $m$  and  $s$  can be computed iteratively from the shortest sub-chains to the longest one.

U can work on the upperdiag of  $m$  so that we can avoid recursion.

1	2	3	4	
0	?	?	?	1
	0	?	?	2
		0	?	3
			0	4

$m$

2	3	4	
?	?	?	1
	?	?	2
		?	3

$s$



# Dynamic Programming Solution: An Iterative Version

Both  $m$  and  $s$  can be computed iteratively from the shortest sub-chains to the longest one.

	1	2	3	4	
	0	150	?	?	1
		0	100	?	2
			0	60	3
				0	4

$m$

	2	3	4	
	1	?	?	1
		2	?	2
			3	3

$s$

# Dynamic Programming Solution: An Iterative Version

Both  $m$  and  $s$  can be computed iteratively from the shortest sub-chains to the longest one.

	1	2	3	4	
1	0	150	130	?	1
2		0	100	130	2
3			0	60	3
4				0	4

$m$

	2	3	4	
1	1	1	?	1
2		2	3	2
3			3	3

$s$

# Dynamic Programming Solution: An Iterative Version

Both  $m$  and  $s$  can be computed iteratively from the shortest sub-chains to the longest one.

	1	2	3	4	
1	0	150	130	148	1
2		0	100	130	2
3			0	60	3
4				0	4

$m$

	2	3	4	
1	1	1	3	1
2		2	3	2
3			3	3

$s$

# Dynamic Programming Solution: Code

```
def MatrixChain(P):  
    m ← allocate(1..n, 1..n)  
    s ← allocate(1..n-1, 2..n)  
    for i ← 1..n:  
        m[i, i] ← 0  
    for l ← 2..n: l is the diagonal  
        for i ← 1..(n-l+1):  
            j ← i + l - 1  
            MatrixChainAux(P, m, s, i, j) subproblem  
        endfor  
    endfor  
  
    return (m, s)  
enddef
```

# Dynamic Programming Solution: Code

```
def MatrixChainAux(P,m,s,i,j):  
    m[i,j] ← INFINITY  
    for k ← i..(j-1):  
        q ← m[i,k] + m[k+1,j] +  
            P[i-1]*P[k]+P[j]  
        if q < m[i,j]:  
            m[i,j] ← q  
            s[i,j] ← k  
        endif  
    endfor  
enddef
```

# Dynamic Programming Solution: Complexity

The computation of  $m[i, j]$  takes time:

$$\sum_{k=i}^{(j-1)} \Theta(1) = \Theta(j - i)$$

Since  $i \in [1, n]$  and  $j \in [i, n]$ ,

$$\begin{aligned} T_C(n) &= \sum_{i=1}^n \sum_{j=i}^n \Theta(j - i) = \Theta \left( \sum_{i=1}^n \left( \sum_{j=i}^n j \right) - n * i \right) \\ &= \Theta \left( \sum_{i=1}^n \frac{n * (n + 1)}{2} - \frac{i * (i + 1)}{2} - n * i \right) = \Theta(n^3) \end{aligned}$$

much better than  $2^n$