Last Time:

- Deterministic OC summary
- LQR vs. MPC
- DDP vs. DIRCOL
- Quaternion

Today:

- Optimization with Quaternions

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* Quaternion Recap:

- 4D Unit vectors

- Multiplication rule

$$q_1 * q_2 = \begin{bmatrix} s_1 \\ v_1 \end{bmatrix} * \begin{bmatrix} s_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} s_1 s_2 - v_1^T v_2 \\ s_1 v_2 + s_2 v_1 + v_1 \times v_2 \end{bmatrix}$$

$$L(q_1) = \begin{bmatrix} s_1 & v_1^T \\ v_1 & s_1 I + \hat{v}_1 \end{bmatrix} \Rightarrow \quad q_1 * q_2 = L(q_1) q_2$$
$$R(q_2) q_1$$

- Conjugate

$$q^+ = \begin{bmatrix} s \\ -v \end{bmatrix} = Tq \quad , \quad T = \begin{bmatrix} 1 & 0 \\ 0 & -I \end{bmatrix}$$

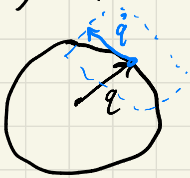
- Identity

$$q_I = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- "Hat map" for Quaternions:

$$\hat{\omega} = \begin{bmatrix} 0 \\ \omega \end{bmatrix} = H\omega \,, \qquad H = \begin{bmatrix} 0 \\ \hline I \end{bmatrix}$$

* Geometry of Quaternions

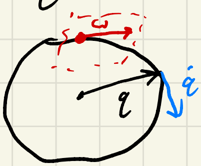


q lives on a sphere in $\mathbb{R}^4$

$\dot{q}$ lives in the tangent plane at q

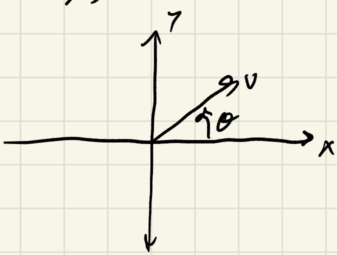
- Kinematics

$$\dot{q} \in \mathbb{R}^4 \,, \qquad \omega \in \mathbb{R}^3 \,, \qquad \dot{q} = \tfrac{1}{2} q * \begin{bmatrix} 0 \\ \omega \end{bmatrix} = \tfrac{1}{2} L(q) H \omega$$



ω is always written in the tangent plane at the identity, then kinematics rotates to tangent plane at q.

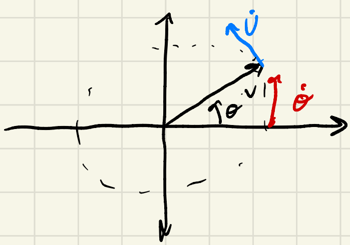
- Analogy with unit complex numbers in 2D



$$v = \cos(\theta) + i\sin(\theta)$$

$$= \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \Rightarrow v^T v = 1$$

$$\dot{v} = \frac{\partial v}{\partial \theta}\dot{\theta} = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}\dot{\theta} = \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}_{\text{rotation matrix}} \underbrace{\begin{bmatrix} 0 \\ \dot{\theta} \end{bmatrix}}_{\text{2D "hat map"}}$$



Kinematics rotates $\dot{\theta}$ from tangent plane at $\theta = 0$ to tangent at current v

* Differentiating Quaternions

- Two key facts

1) Derivatives are really 3D tangent vectors

2) Rotations compose by multiplication, not addition

- Infinitessimal Rotation

$$\delta q = \begin{bmatrix} \cos(\theta/2) \\ a\sin(\theta/2) \end{bmatrix} \approx \begin{bmatrix} 1 \\ \frac{1}{2}a\theta \end{bmatrix} \approx \begin{bmatrix} 1 \\ \frac{1}{2}\underbrace{\phi}_{\text{small axis-angle vector}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ \phi \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{2}H\phi$$

- Compose with q:

$$q' = q * \delta q = L(q)\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{2}H\phi\right)$$

$$= q + \frac{1}{2}\underbrace{L(q)H}_{G(q) \in \mathbb{R}^{4\times 3}}\phi$$

"Attitude Jacobian"

- Note: we can use any 3-parameter rotation representation we want for ϕ. They all linearize the same (up to a permutation/scaling).

$$q = \underbrace{\begin{bmatrix} \cos(\|\phi\|/2) \\ \frac{\phi}{\|\phi\|}\sin(\|\phi\|/2) \end{bmatrix}}_{\text{axis-angle}} = \underbrace{\begin{bmatrix} \sqrt{1-\phi^T\phi} \\ \phi \end{bmatrix}}_{\text{vector part of } q} = \underbrace{\frac{1}{\sqrt{1+\phi^T\phi}}\begin{bmatrix} 1 \\ \phi \end{bmatrix}}_{\text{Gibbs/ Rodrigues}}$$

- We'll use the vector part of q in class

- This lets us differentiate w.r.t. quaternions by inserting $G(q)$ in the right places:

$$f(q) : \underset{\substack{\uparrow \\ \text{quaternions} \\ (\text{"Hamilton"})}}{\mathbb{H}} \rightarrow \mathbb{R}$$ (gradient of a scalar-valued function)

$$\nabla f = \frac{\partial f}{\partial q} \frac{\partial q}{\partial \phi} = \frac{\partial f}{\partial q} G(q)$$

$$f(q) : \mathbb{H} \rightarrow \mathbb{H}$$ (Jacobian of a quaternion-valued function)

$$\phi' = \underbrace{\left[G(f(q))^T \frac{\partial f}{\partial q} G(q) \right]}_{\nabla f \in \mathbb{R}^{3\times 3}} \phi$$

transform output

transform input

- Hessian of $f(q) : \mathbb{H} \rightarrow \mathbb{R}$

$$\nabla^2 f = G(q)^T \frac{\partial^2 f}{\partial q^2} G(q) + \underbrace{\overbrace{I}^{3\times 3}\Big(\overbrace{\frac{\partial f}{\partial q} q}^{\text{scalar}}\Big)}_{\text{comes from } \frac{\partial G}{\partial q}}$$

- Now we can do Newton's method and DDP and SQP with quaternions

* Example: Pose Estimation

- Given a bunch of vectors to known landmarks in the environment, determine robot's attitude.

- Called "Wahba's Problem"

$$\min_{q} J(q) = \sum_{k=1}^{m} \| \underbrace{{}^N x_k}_{\substack{\text{known vectors} \\ \text{in world frame} \\ \text{(from map)}}} - Q(q)\, \underbrace{{}^B x_k}_{\substack{\text{Observed vectors} \\ \text{in body frame} \\ \text{(from camera)}}} \|_2^2 = \| r(q) \|_2^2 = r(q)^T \underbrace{r(q)}_{\substack{\text{"residual"} \\ \text{vector}}}$$

- ${}^N x_k$ and ${}^B x_k$ are unit vectors ("directions")

$$r(q) = \begin{bmatrix} {}^N x_1 - Q(q)\,{}^B x_1 \\ {}^N x_2 - Q(q)\,{}^B x_2 \\ \vdots \\ {}^N x_m - Q(q)\,{}^B x_m \end{bmatrix} \Rightarrow \underbrace{\nabla r(q)}_{3m \times 3} = \underbrace{\frac{\partial r}{\partial q}}_{3m \times 4} \underbrace{G(q)}_{4 \times 3}$$

- Background: Gauss-Newton for Least-Squares:

$$\min_{x} J(x) = \tfrac{1}{2} \| r(x) \|_2^2 = \tfrac{1}{2} r(x)^T r(x)$$

$$\frac{\partial J}{\partial x} = r(x)^T \frac{\partial r}{\partial x}$$

throw this out

$$\frac{\partial^2 J}{\partial x^2} = \left(\frac{\partial r}{\partial x}\right)^T \left(\frac{\partial r}{\partial x}\right) + \cancel{\left(I \otimes r(x)^T\right) \frac{\partial^2 \mathrm{vec}(r)}{\partial x^2}}$$

$$\Rightarrow \left(\frac{\partial^2 J}{\partial x^2}\right)^{-1} \nabla J \approx \left[\left(\frac{\partial r}{\partial x}\right)^T \left(\frac{\partial r}{\partial x}\right)\right]^{-1} \frac{\partial r}{\partial x}^T r(x)$$

* Gauss-Newton for Wahba's Problem:

$q \leftarrow q_0$ (initial guess)

do:

$$\nabla r(q) = \frac{\partial r}{\partial q} G(q)$$

$$\phi = -\left[(\nabla r^T \nabla r)^{-1} \nabla r^T\right] r(q)$$

$$q \leftarrow q * \begin{bmatrix} \sqrt{1-\phi^T\phi} \\ \phi \end{bmatrix} = L(q) \begin{bmatrix} \sqrt{1-\phi^T\phi} \\ \phi \end{bmatrix}$$

(multiplicative update)

(in general, do line search)

while $\|r(q)\| > tol$