Last Time:

- Minimization with Equality Constraints

Today:

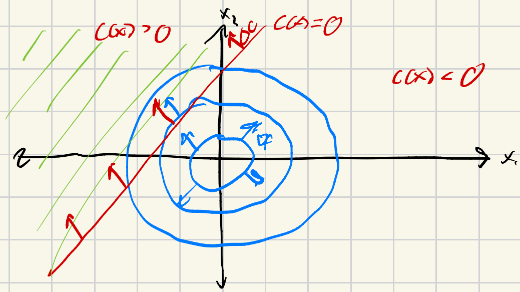
- Inequality Constraints

Inequality-Constrained Minimization:

$$\min_x f(x)$$
$$\text{s.t.} \quad c(x) \geq 0$$

- KKT Conditions:

$$\nabla f - \left(\frac{\partial c}{\partial x}\right)^T \lambda = 0 \qquad \text{(stationarity)}$$
$$c(x) \geq 0 \qquad \text{(primal feasibility)}$$
$$\lambda \geq 0 \qquad \text{(dual feasibility)}$$
$$\lambda \odot c(x) = 0 \qquad \text{(complementarity)}$$



- Unlike equality case, we can't directly solve KKT conditions with Newton

- Lots of solution methods:

  * Active Set:

    - Switch inequality constraints on/off and solve equality-constrained problem

    - Works well if you can guess active set well

  * Penalty Method:

    - Replace constraints with cost terms that penalize violation:

$$\left.\begin{array}{l}\min_x f(x) \\ \text{s.t. } c(x) \geq 0\end{array}\right\} \longrightarrow \min_x f(x) + \frac{\rho}{2}\left[\min(0, c(x))\right]^2$$

    - Easy to implement

    - has issues with ill-conditioning (have to crank $\rho \to \infty$)

    - Can't solve to high accuracy

    - Popular fix: estimate λ from penalty at each iteration → converge with finite ρ. Called "Augmented Lagrangian" (also closely related ADMM)

☆ Interior-Point / Barrier Methods

- Replace inequalities with barrier function in objective

$$\left.\begin{array}{l}\min_x \; f(x) \\ \text{s.t. } x \geq 0\end{array}\right\} \rightarrow \min_x \; f(x) - \rho \log(x)$$

- Gold standard for convex problems
- Fast convergence with Newton
- Strong theoretical properties
- Used in IPOPT

Primal-Dual Interior Point Method

$$\left.\begin{array}{l}\min \; f(x) \\ \text{s.t. } x \geq 0\end{array}\right\} \rightarrow \min_x \; f(x) - \rho \log(x)$$

$$\Rightarrow \quad \frac{\partial f}{\partial x} - \frac{\rho}{x} = 0$$

- This "primal" FON condition blows up as $x \to 0$
- We can fix this with the "primal-dual trick"

- Introduce new variable $\lambda = \frac{\rho}{x} \Rightarrow x\lambda = \rho$

$$\Rightarrow \begin{cases} \nabla f - \lambda = 0 \\ \underbrace{x\lambda = \rho}_{\text{relaxed complementarity from KKT}} \end{cases}$$

- Converges to exact KKT solution as $\rho \to 0$. We lower gradually as solver converges from $\rho \sim 1$ to $\rho \sim 10^{-8}$

- Note we still need to enforce $x \geq 0$ and $\lambda \geq 0$ (with line search)

- We will use another approach

Log-Domain Interior-Point Method:

- More general case:

$$\min_x f(x)$$
$$\text{s.t. } c(x) \geq 0$$

- Simplify by introducing a "slack variable"

$$\left.\begin{aligned} &\min_{x,s} f(x) \\ &\text{s.t. } c(x) - s = 0 \\ &\qquad\quad s \geq 0 \end{aligned}\right\} \rightarrow \begin{aligned} &\min_{x,s} f(x) - \rho \log(s) \\ &\text{s.t. } c(x) - s = 0 \end{aligned}$$

- Lagrangian:

$$L(x, s, \lambda) = f(x) - \rho \log(s) - \lambda^T(c(x) - s)$$

- FON Conditions:

$$\nabla_x L = \nabla f - \left(\frac{\partial c}{\partial x}\right)^T \lambda = 0$$

$$\nabla_s L = \frac{-\rho}{s} + \lambda = 0 \Rightarrow \underbrace{s \odot \lambda = \rho}_{\text{relaxed complementarity}}$$

$$\nabla_\lambda L = s - c(x) = 0$$

- To ensure $s \geq 0$ and $\lambda \geq 0$, introduce change of variables:

$$s = \sqrt{\rho}\, e^{\sigma} \Rightarrow \lambda = \sqrt{\rho}\, e^{-\sigma}$$

- Now (relaxed) complementarity is always satisfied

- Plug back into FON conditions:

$$\begin{cases} \nabla f - \left(\frac{\partial c}{\partial x}\right)^T \sqrt{\rho}\, e^{-\sigma} = 0 \\ c(x) - \sqrt{\rho}\, e^{\sigma} = 0 \end{cases}$$

- We can solve these with (Gauss) Newton:

$$\begin{bmatrix} H & \sqrt{\rho}\, C^T e^{-\sigma} \\ C & -\sqrt{\rho}\, e^{\sigma} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta \sigma \end{bmatrix} = \begin{bmatrix} -\nabla f + C^T \sqrt{\rho}\, e^{-\sigma} \\ -c(x) + \sqrt{\rho}\, e^{\sigma} \end{bmatrix}$$

* Example: Quadratic Program

$$\min_x \frac{1}{2} x^T Q x + q^T x, \quad Q \succ 0$$

$$\text{s.t.} \quad Ax \geq b$$
$$Cx = d$$

- Super useful in control

- Can be solved very fast (~kHz)