TOPOLOGICAL DATA ANALYSIS

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CERTIFICATE

This is to certify that **Shreyas Samaga**, BS-MS (Mathematics), has worked on the project entitled **'Topological Data Analysis'** under my supervision and guidance. The content of this report is original and has not been submitted elsewhere for the award of any academic or professional degree.

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ABSTRACT

This thesis aims at giving a self-contained exposition of Topological Data Analysis. Topological Data Analysis (TDA) is deeply rooted in algebraic topology and has immense real-world applications and it offers an alternate approach to data analysis. We look at simplicial homology and some basic category theory, being the pre-requisites of TDA. Further, we look at the phenomenon of persistence, which forms one of the main branches of TDA. We also look at the limitations of single dimensional persistence theory to generalise to higher dimensions. We, then, study the sheaf theoretic approach to persistence which has the potential to address the multidimensional persistence problem. Persistence theory can be directly used for solving real-world problems. We look at an algorithm to compute Persistent Homology. Finally, we move into another branch of TDA which is useful for data visualisation. Algebraic topology concepts can be put to use to visualise a high dimensional dataset as a low dimensional simplicial complex. We study the method to do so, which is called the Mapper algorithm. This thesis concludes with an application of TDA to an Indian socio-economic dataset, which demonstrates the applicative nature of this technique.

LIST OF SYMBOLS OR ABBREVIATIONS

$\mathbb{N}$	The set of natural numbers
$\mathbb{Z}$	The set of integers
$\mathbb{R}$	The set of real numbers
$\mathbb{R}^n$	The n-dimensional Euclidean space
∂	The boundary operator in a chain complex
$C_k(X)$	The free group on the k simplices of X
$H_n(X)$	The nth homology group of X
$\check{\mathrm{C}}(X, \epsilon)$	The Čech complex of X corresponding to ϵ
$\mathrm{VR}(X, \epsilon)$	The Vietoris-Rips complex of X corresponding to ϵ
TDA	Topological Data Analysis

LIST OF FIGURES

CONTENTS

1. INTRODUCTION

Topological data analysis (TDA) is a method to analyse data using concepts from topology. Until now, we, primarily, had statistical tools and methods to analyse data. TDA offers an alternate approach to data analysis. TDA can also be complementary to the statistical analysis and can serve as an additional evidence to the claim being made by the statistical analysis of a given dataset.

In today's world of Watson, Data is Power and data is pouring in at an unprecedented rate. We require novel techniques and methods to analyse this huge amount of data. TDA is one such. Topological data analysis draws upon concepts from algebraic topology to determine the shape and pattern hidden in the data, which provides valuable insights into the dataset.

This thesis is aimed at providing an exposition of TDA, i.e, complete answers to questions such as 'What is topological data analysis?', 'What are the mathematical concepts that lie beneath the techniques of TDA?', 'What are the main highlights of TDA?' and 'What can TDA be used for?'

The thesis begins by building the necessary mathematical background required to understand the TDA techniques. Simplicial homology and some basic Category theory are covered in chapters 2 and 3 respectively.

Chapter 4 contains Persistent Homology, one of the biggest highlights of TDA, and is based on the foundational work by Gunnar Carlsson [1]. Persistent Homology provides information about the topological features that are present in the dataset and helps in understanding the topology of the data. Chapter 5 deals with the (co)sheaf theoretic approach to TDA, which is developed in [4]. This approach is useful in tackling one of the problems that arises, namely that of multidimensional persistence.

Chapter 6 has the details of an algorithm to compute Persistent Homology ([2]). Chapter 7 contains the Mapper algorithm, one of the other highlights of TDA. This algorithm can be used to visualise a high dimensional dataset to obtain some preliminary information. This is also based on concepts from algebraic topology. Finally, chapter 8 contains an application where TDA is applied to an Indian socio-economic dataset.

The packages used in this thesis are:

- R TDA - [19]
- Kepler Mapper - [20], [21]
- GeoPandas - [22]
- Scikit-learn - [25]

Let us now move on to the second chapter which covers the simplicial homology theory that is required to understand the basics of TDA.

2. HOMOLOGY

In this chapter, we shall look at simplicial homology theory. We shall begin with the definition of a simplicial complex and move on to the definition of a chain complex associated to a simplicial complex and subsequently define the nth simplicial homology group in section 2.1.

Further, we study induced homomorphisms in section 2.2 and look at the homotopy invariant nature of homology towards the end of this chapter, in section 2.3.

2.1 Simplicial Homology

Let us begin with the definition of an abstract simplicial complex.

Definition 2.1 (Abstract Simplicial Complex)**.** It is a double (V, Σ) where V is the set of vertices and Σ is a collection of subsets of V such that $\forall \sigma \in \Sigma$ and $\tau \subseteq \sigma, \tau \in \Sigma$.

We can naturally associate a topological space to a simplicial complex. This is called the **geometric realisation** of the simplicial complex and is denoted by $|(V, \Sigma)|$. It is defined as follows:
Consider a function $\phi\colon V \to \{1, 2, 3, \ldots, |V|\}$ where $|V|$ denotes the cardinality of the set V. Let $c(\sigma)$ denote the convex hull of $\{e_{\phi(v)}\}_{v\in\sigma}$ where e_i denotes the i-th basis vector in $\mathbb{R}^{|V|}$. Then $|(V, \Sigma)|$ is defined as $\bigcup\limits_{\sigma\in\Sigma} c(\sigma)$. This space can be topologised using the standard Euclidean topology of $\mathbb{R}^{|V|}$.

Remark 2.2. We note that the set V must be finite for the map ϕ to make sense. In this case, i.e, when V is a finite set, the simplicial complex is called a finite simplicial complex.

Remark 2.3. We say that $\sigma \in \Sigma$ is a k-simplex if $|\sigma| = k + 1$. The set of all k simplices are denoted by Σ_k.

Remark 2.4. We can use this ϕ to put an ordering on the vertices of the simplicial complex. The i-th vertex is the inverse image $\phi^{-1}(i)$ where $i \in \{1, 2, \ldots, |V|\}$.

Definition 2.5 (Chain Group)**.** Let X be a simplicial complex. The k-th Chain Group of X is the free abelian group on the set of k simplices of X. It is denoted by $C_k(X)$.

Consider the map $d_i \colon \Sigma_k \to \Sigma_{k-1}$ defined as

$$d_i(\sigma) = \sigma \setminus \{s_i\}$$

where s_i denotes the ith vertex of σ.

Definition 2.6 (Boundary Operator)**.** The boundary operator $\partial_k \colon C_k \to C_{k-1}$ is defined as

$$\partial_k = \sum_{i=0}^{k} (-1)^i d_i$$

where d_i is the map defined above.

Remark 2.7. ∂_k is defined on the basis elements and can be extended to elements of Σ_k linearly.

Proposition 2.8. $\partial_k \circ \partial_{k+1} = 0$.

Proof.

$$\begin{aligned}
\partial_{k+1} &= \sum_{i=0}^{k+1} (-1)^i d_i \\
\partial_k \circ \partial_{k+1} &= \sum_{j=0}^{k} (-1)^j d_j (\sum_{i=0}^{k+1} (-1)^i d_i) \\
&= \sum_{j<i} (-1)^i (-1)^j d_j d_i + \sum_{j>i} (-1)^i (-1)^{j-1} d_j d_i.
\end{aligned}$$

Now, the corresponding terms in the first summation and the second summation are negatives of each other. Hence, we get that $\partial_k \circ \partial_{k+1} = 0$. □

We now observe that the boundary operator connects the chain groups into a **Chain Complex**, $C_\star$ as follows:

$$\dots \xrightarrow{\partial_{k+2}} C_{k+1} \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} \dots$$

We now define the subgroups $Z_k = \text{Ker } \partial_k$ and $B_k = \text{Im } \partial_{k+1}$ of C_k. Owing to Proposition 2.8, we have $B_k \subset Z_k$. Since C_k is abelian, B_k is a normal subgroup of Z_k.

Definition 2.9 (Homology Group)**.** Let X be a simplicial complex. Let $C_\star$ be the associated chain complex. Then the kth homology group of the chain complex, H_k, is defined as:

$$H_k = Z_k / B_k$$

The elements of Z_k are called k-**cycles** and the elements of B_k are called k-**boundaries**. Two cycles which differ by a boundary represent the same homology class and are called **homologous**.

We saw the definition of a simplicial complex and that of simplicial homology defined on a simplicial complex. One can also look at a topological space and put a simplicial complex structure on it. This can be done as follows:

Definition 2.10 (Standard n-simplex)**.** The standard n-simplex, Δ^n, is defined as follows:

$$\Delta^n \colon = \left\{ (t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} | \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geq 0 \; \forall \; i \right\}$$

Let Δ^n_o denote the interior of Δ^n, i.e. $\Delta^n_o = \Delta^n \setminus \partial\Delta^n$.

Definition 2.11. A **simplicial complex structure** on a topological space X is a collection of maps $\sigma_\alpha \colon \Delta^n \to X$, with n depending on α, such that:

1. $\sigma_\alpha|_{\Delta^n_o}$ is injective.
2. Each restriction of σ_α to one of its faces is one of the maps $\sigma_\beta \colon \Delta^{n-1} \to X$.
3. $\sigma_\alpha = \sigma_\beta \implies \alpha = \beta$.
4. For any open set $A \subset X$, $\sigma_\alpha^{-1}(A)$ is open in Δ^n for all σ_α.

Now, there are other types of such complexes, for example, Δ-complexes and CW complexes.

We shall look at the definition of Δ-complexes. Though they may seem to be a slight generalisation of the simplicial complexes, they are useful from a computational point of view.

Definition 2.12. A **Δ-complex** structure on a topological space X is a collection of maps $\sigma_\alpha \colon \Delta^n \to X$, with n depending on α, such that:

1. $\sigma_\alpha|_{\Delta_o^n}$ is injective.
2. Each restriction of σ_α to one of its faces is one of the maps $\sigma_\beta \colon \Delta^{n-1} \to X$.
3. For any open set $A \subset X$, $\sigma_\alpha^{-1}(A)$ is open in Δ^n for all σ_α.

Remark 2.13. To put a simplicial complex on a torus, we would need 14 triangles, 21 edges and 7 vertices while the Δ-complex structure requires 2 triangles, 5 edges and 4 vertices. Thus, we see that fewer simplices are required in a Δ-complex.

Definition 2.14 (Singular n-simplex)**.** A **singular n-simplex** is any continuous map $\sigma \colon \Delta^n \to X$.

Boundary operators, chain complexes and, ultimately, homology groups can be defined in a similar way as it was defined for simplicial homology. This homology theory is called Singular Homology theory.

Remark 2.15. For a Δ-complex, simplicial homology groups are isomorphic to the singular homology groups. This can be seen on p128, [7].

2.2 Induced Homomorphisms

Let X and Y be topological spaces. Let $f \colon X \to Y$ be a continuous map. Then, f induces a group homomorphism, $f_\# \colon C_n(X) \to C_n(Y)$ for all n and is defined as follows:
$f_\#(\sigma) = f \circ \sigma$. Clearly, $f\sigma$ is a singular n-simplex in Y as $f\sigma \colon \Delta^n \to Y$. $f_\#$ can be linearly extended like $f_\#(\sum_i n_i \sigma_i) = \sum_i n_i f_\#(\sigma_i)$.

Proposition 2.16. *$f_{\#}\partial = \partial f_{\#}$ where ∂ denotes the boundary operator.*

Proof. $f_{\#}(\partial(\sigma)) = f_{\#}(\sum_i (-1)^i d_i(\sigma)) = \sum_i (-1)^i f d_i(\sigma) = \partial f_{\#}(\sigma)$. □

This can be represented as follows:

$$\begin{tikzcd}
\dots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "f_{\#}"'] & C_n(X) \arrow[r, "\partial"] \arrow[d, "f_{\#}"] & C_{n-1}(X) \arrow[r] \arrow[d, "f_{\#}"] & \dots \\
\dots \arrow[r] & C_{n+1}(Y) \arrow[r, "\partial"] & C_n(Y) \arrow[r, "\partial"] & C_{n-1}(Y) \arrow[r] & \dots
\end{tikzcd}$$

We now see that $f_{\#}$ takes cycles to cycles and boundaries to boundaries. For, if $\partial\gamma = 0$ then, $0 = f_{\#}\partial\gamma = \partial f_{\#}\gamma$, and boundaries to boundaries because $f_{\#}\partial\alpha = \partial f_{\#}\alpha$. Thus, this induces a homomorphism on the homology groups, $f_\star \colon H_n(X) \to H_n(Y)$.

Thus, a continuous map $f \colon X \to Y$ induces a homomorphism $f_\star \colon H_n(X) \to H_n(Y)$ for all n.

2.3 Homotopy Invariance

One of the most important properties of homology, for us, is its homotopy invariance. This property will be useful to us because this property implies that homotopy equivalent spaces have isomorphic homology groups.

Definition 2.17 (Homotopy)**.** Let X and Y be topological spaces. Let $f, g \colon X \to Y$ be two continuous maps. f and g are said to be homotopic if there exists a continuous function $H \colon X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$.

Definition 2.18 (Homotopy equivalent)**.** Two topological spaces X and Y are said to be homotopy equivalent if there exist functions $f\colon X \to Y$ and $g\colon Y \to X$ such that $f \circ g$ is homotopic to id_Y and $g \circ f$ is homotopic to id_X.

Example 2.19. Consider the closed disc $\mathbb{D}$ in $\mathbb{R}^2$, and the origin O. Consider the obvious maps $f\colon \mathbb{D} \to O$ and $g\colon O \to \mathbb{D}$. Now, $f \circ g = id_O$. Consider $H\colon \mathbb{D} \times [0,1] \to O$ defined as $H(x,t) = (1-t).x + t.(g \circ f)(x)$. This is a homotopy between $id_{\mathbb{D}}$ and $g \circ f$. Hence, the closed disc in $\mathbb{R}^2$ is homotopy equivalent to a point. This result is true even in higher dimensions and one can construct similar such homotopies. Such spaces, which are homotopy equivalent to a point, are called **contractible** spaces.

Now, we shall look at the homotopy invariance of the homology groups.

Theorem 2.20. *Let X and Y be topological spaces. If $f, g\colon X \to Y$ are homotopic, then $f_\star = g_\star\colon H_n(X) \to H_n(Y)$.*

Proof. Essentially, we need to prove that $f_\#$ and $g_\#$ differ by a boundary. Since, boundaries are undetectable by the homology groups, we would get that f and g induce the same homomorphism on the homology groups.

Let $F\colon X \times I \to Y$ be a homotopy between f and g. Let $\sigma\colon \Delta^n \to X$ be a singular n-simplex.

We divide $\Delta^n \times I$ into simplices. Let $\Delta^n \times \{0\} = \{v_0, v_1, \dots, v_n\}$ and $\Delta^n \times \{1\} = \{w_0, w_1, \dots, w_n\}$. We now pass from $\{v_0, v_1, \dots, v_n\}$ to $\{w_0, w_1, \dots, w_n\}$ in steps. In the first step, we change v_n to w_n, so that $\{v_0, v_1, \dots, v_n\}$ becomes $\{v_0, v_1, \dots, w_n\}$. In the next step, we change v_{n-1} to w_{n-1}.

Thus, in an intermediate step, $\{v_0, v_1, \dots, v_i, w_{i+1}, \dots w_n\}$ changes to $\{v_0, v_1, \dots, v_{i-1}, w_i, w_{i+1}, \dots, w_n\}$. The region between these two n-simplices is precisely the $(n+1)$-simplex $\{v_0, v_1, \ldots, v_i, w_i, w_{i+1}, \ldots, w_n\}$. By doing this, we see that $\Delta^n \times I$ is the union of these $(n+1)$-simplices.

Now, we consider the composition $F \circ (\sigma \times \mathbf{1})\colon \Delta^n \times I \to X \times I \to Y$. We define $P\colon C_n(X) \to C_{n+1}(Y)$ as follows:

$$P(\sigma) = \sum_i (-1)^i F \circ (\sigma \times \mathbf{1})|_{\{v_0, \ldots, v_i, w_i, \ldots, w_n\}}$$

Claim - $\partial P = g_\# - f_\# - P\partial$.

Proof of Claim -

$$\begin{aligned}\partial P(\sigma) =& \sum_{j \le i} (-1)^i (-1)^j F \circ (\sigma \times \mathbf{1})|\{v_0, \ldots, \hat{v}_j, \ldots, v_i, w_i, \ldots, w_n\} \\ &+ \sum_{j \ge i} (-1)^i (-1)^{j+1} F \circ (\sigma \times \mathbf{1})|\{v_0, \ldots, v_i, w_i, \ldots, \hat{w}_j, \ldots, w_n\}\end{aligned}$$

where, $\sigma|\{v_0, \ldots, \hat{v}_i, \ldots, v_n\} = d_i(\sigma)$ and $d_i(\sigma) = \sigma \setminus \{s_i\}$ as defined earlier. $\mathbf{1}$ is the identity map $\mathbf{1}\colon I \to I$.
We see that whenever $i = j$, the two summations are negatives of each other and hence the terms cancel out except $F \circ (\sigma \times \mathbf{1})|\{\hat{v}_0, w_0, \ldots, w_n\}$ and $F \circ (\sigma \times \mathbf{1})|\{v_0, \ldots, v_n, \hat{w}_n\}$ which are respectively $g \circ \sigma = g_\#(\sigma)$ and $-f \circ \sigma = -f_\#(\sigma)$.

We now observe that

$$\begin{aligned}P\partial(\sigma) =& \sum_{i > j} (-1)^{i-1} (-1)^j F \circ (\sigma \times \mathbf{1})|\{v_0, \ldots, \hat{v}_j, \ldots, v_i, w_i, \ldots, w_n\} \\ &+ \sum_{i < j} (-1)^i (-1)^j F \circ (\sigma \times \mathbf{1})|\{v_0, \ldots, v_i, w_i, \ldots, \hat{w}_j, \ldots, w_n\}\end{aligned}$$

Thus, we see that $\partial P = g_\# - f_\# - P\partial$.

Now, if $\gamma \in C_n(X)$ is a cycle, then $g_\#(\gamma) - f_\#(\gamma) = \partial P(\gamma) + P\partial(\gamma)$. But since, γ is a cycle, $\partial(\gamma) = 0$. Hence, $g_{\#}(\gamma) - f_{\#}(\gamma) = \partial P(\gamma)$ which is a boundary. Hence, $g_{\#}(\gamma)$ and $f_{\#}(\gamma)$ determine the same homology class, and thus, $f_\star = g_\star$ on the homology class of γ. □

Corollary 2.21. *If $h\colon X \to Y$ is a homotopy equivalence, then $h_\star\colon H_n(X) \to H_n(Y)$ is an isomorphism for all n.*

Proof. We use two properties of homology, namely:

1. If $f\colon X \to Y$ is a continuous map and $g\colon Y \to Z$ is a map, then $(gf)_\star = g_\star \circ f_\star\colon H_n(X) \to H_n(Z)$.

2. $(id)_\star = id$, where the id on the RHS is the group identity map.

These two properties of homology are called the functorial properties because they make homology a functor between the category of topological spaces and that of groups.

Let $k\colon Y \to X$ be a homotopy inverse to h. Then, $(hk)_\star = h_\star k_\star = id$ and $(kh)_\star = k_\star h_\star = id$. Hence, $H_n(X) \cong H_n(Y)$ for all n. □

This completes the knowledge of homology that we need to understand the basics of TDA.

In the next chapter, we shall look at some basic category theory, sheaf and (co)sheaf theory that will be required to understand the (co)sheaf theoretic approach to TDA that is discussed in Chapter 5.

3. A SHORT INTRODUCTION TO CATEGORY THEORY

We shall look at some basic category theory in this chapter. We begin with the definition of a category and move on to the notion of a functor between two categories. We shall also look at how the collection of functors between two categories form a category themselves. Further, we introduce the concept of a cone, cocone and subsequently define inverse limit and direct limit.

In the second section of this chapter, we look at the definition of pre-sheaves and pre-cosheaves. We shall also see conditions that a pre-sheaf and a pre-cosheaf must satisfy in order to become a sheaf and a cosheaf respectively. These conditions are called the sheaf condition and the cosheaf condition respectively.

3.1 Category Theory

Let us begin by looking at the definition of a category.

Definition 3.1 (Category)**.** A category C consists of a class of objects ($\mathrm{ob}(C)$) and a set of morphisms ($\mathrm{Hom}_C(a,b)$) between $a, b \in \mathrm{ob}(C)$ such that

- For all $f \in \mathrm{Hom}_C(a,b)$ and $g \in \mathrm{Hom}_C(b,c)$, $g \circ f \in \mathrm{Hom}_C(a,c)$.
- Composition is associative, i.e, $h \circ (g \circ f) = (h \circ g) \circ f$ for all $f \in \mathrm{Hom}_C(a,b)$, $g \in \mathrm{Hom}_C(b,c)$, $h \in \mathrm{Hom}_C(c,d)$ where $a, b, c, d \in \mathrm{ob}(C)$.

- For all $x \in \mathrm{ob}(C)$, there exists an identity morphism, id_x, such that $id_b \circ f = f$ and $f \circ id_a = f$ for all $f \in \mathrm{Hom}_C(a, b)$.

Remark 3.2. A category is called a **small** category if the class of objects is actually a set and it is called a **finite** category if the cardinality of the set of objects is finite.

Example 3.3. Consider the category of vector spaces where the objects are vector spaces themselves and the morphisms between them are the linear maps. This category is denoted by **Vect**. The category of finite dimensional vector spaces is denoted by **vect**.

Example 3.4. Let $(X, \leqslant)$ be a partially ordered set. Then it forms a category with the objects being the elements of the set and there exists a morphism from x to y if and only if $x \leq y$.

Example 3.5. Any set can be thought of as a **discrete category** where the objects are the members of the set and the morphisms are only the identity morphisms.

Example 3.6. Let X be a topological space. Let $\mathcal{U} = \{U_i\}_{i \in \Lambda}$ be a collection of open sets of X. Then the **Open Set Category** consists of the open sets as the objects and there exists a morphism from U_i to U_j if and only if $U_i \subset U_j$. This is denoted by **Open(X)**.

This particular category will be important to us in the later chapters.

Definition 3.7. For a given category C, we can define its **Opposite Category**, C^{op}, where the morphisms have been turned around, i.e, $\mathrm{ob}(C^{op}) = \mathrm{ob}(C)$ $\mathrm{Hom}_{C^{op}}(a, b) = \mathrm{Hom}_C(b, a)$ for all $a, b \in \mathrm{ob}(C)$.

Example 3.8. Open(X)^{op} is the category where there exists a morphism from U to V if and only if $U \supset V$.

We will now look at how to define the notion of limit in categories. In order to do that, we would need a few other preliminary definitions.

Definition 3.9. Let C be a category. The object x is called **initial** if there exists a unique morphism $f_y \colon x \to y$ for all $y \in \mathrm{ob}(C)$. The object y is called **terminal** if there exists a unique morphism $f_x \colon x \to y$ for all $x \in \mathrm{ob}(C)$.

Definition 3.10. Let C and D be categories. A **functor**, $F \colon C \to D$, is given by the following data:

- F assigns to each object of C an object of D, i.e, $F(c) \in \mathrm{ob}(D)$ for all $c \in \mathrm{ob}(C)$
- If f is a morphism from a to b, then $F(f)$ is a morphism from $F(a)$ to $F(b)$, i.e, $F(f) \in \mathrm{Hom}_D(F(a), F(b))$ for all $f \in \mathrm{Hom}_C(a, b)$.
- $F(f \circ g) = F(f) \circ F(g)$ for all $f \in \mathrm{Hom}_C(a, b)$ and $g \in \mathrm{Hom}_C(c, a)$.
- For all $a \in \mathrm{ob}(C)$, $F(\mathrm{id}_a) = F(\mathrm{id}_{F(a)})$. Here id_a denotes the identity morphism.

So functor is a homomorphism between categories. Just like group homomorphisms preserve the group structure and linear maps preserve the structure of the vector space, functors preserve the structure of the categories.

Definition 3.11. Let C and D be categories and let $F \colon C \to D$ be a functor. We use the notation $F(a, b) \colon \mathrm{Hom}_C(a, b) \to \mathrm{Hom}_D(F(a), F(b))$. We call the functor F **faithful** if $F(a, b)$ is injective for all $a, b \in \mathrm{ob}(C)$. The functor F is called **full** if $F(a, b)$ is surjective for all $a, b \in \mathrm{ob}(C)$.

The collection of functors between two categories has more structure to it than being a mere collection.

Definition 3.12. Let C and D be categories. Let $F, G\colon C \to D$ be two functors. A natural transformation $\eta\colon F \to G$ assigns to each object, a, of C a morphism, $\eta(a)\colon F(a) \to G(a)$ such that for all $f \in \mathrm{Hom}_C(a,b)$

$$\begin{tikzcd} F(a) \arrow[r, "\eta(a)"] \arrow[d, "F(f)"'] & G(a) \arrow[d, "G(f)"] \\ F(b) \arrow[r, "\eta(b)"] & G(b) \end{tikzcd}$$

the above diagram commutes, i.e, $G(f) \circ \eta(a) = \eta(b) \circ F(f)$.

Two functors, F and G, are said to be naturally isomorphic if there exists a natural transformation, η, such that $\eta(a)$ is an isomorphic for all objects a.

Remark 3.13. The functors as objects and natural transformations as morphisms also form a category, **Fun(C,D)**.

Now that we have looked at a way of navigating between two functors between the same two categories, we could define the notion of equivalence of categories.

Definition 3.14. A functor $F\colon C \to D$ is called an **equivalence** if there exists a functor $G\colon D \to C$ such that $\eta\colon GF \to id_C$ and $\epsilon\colon FG \to id_D$ are natural isomorphisms.

Remark 3.15. The above definition of equivalence essentially means that a functor $F\colon C \to D$ is an equivalence if

1. the map $C(X,Y) \to D(FX,FY)$ (the map between the set of morphisms between two objects) is a bijection for all objects X, Y of C. If this property is satisfied, the functor is said to be **fully faithful**.

2. for each object Y of D, there exists an isomorphism $FX \to Y$ for some object X of C. Any functor satisfying this property is called **essentially surjective**.

Definition 3.16. A **diagram** is a functor $F \colon I \to C$, where I is a small category and C is any arbitrary category.

Example 3.17. Let I be a small category and C be any category. Let $c \in \text{ob}(C)$. Then, $const_c \colon I \to C$ defined as $const_c(x) = const_c(y) = c$ for all $x, y \in I$ and every morphism goes to the identity morphism. Such a diagram is called a **constant diagram**.

Definition 3.18. Let $F \colon I \to C$ be a diagram. A **cone** on F is a natural transformation from a constant diagram to F, i.e, a choice of an object $L \in C$ and a collection of morphisms $\psi_x \colon L \to F(x)$ for each $x \in I$ such that if $g \colon x \to y$ is a morphism in I then the following diagram commutes

$$\begin{tikzcd} F(x) \arrow[rr, "F(g)"] & & F(y) \\ & L \arrow[ul, "\psi_x"] \arrow[ur, "\psi_y"'] & \end{tikzcd}$$

i.e, $F(g) \circ \psi_x = \psi_y$.

Example 3.19. Let $\Lambda = \{1, 2, \ldots, n\}$ be a discrete finite category. Let X be a topological space. Consider a functor $F \colon \Lambda \to \text{Open(X)}$. Recall that

Open(X) is the category of open sets of a topological space X where the objects are the open sets and the morphisms are characterised by inclusion of the open sets. This functor is, in essence, picking out n open sets from the collection of open sets. Hence, a cone to F would be an open set that would include into all the possible open sets picked out by F.

Remark 3.20. The collection of cones on a diagram F form a category. We will denote this by **Cone(F)**. The objects in this category are the cones, (L, ψ_x), and the morphisms between two cones, (L, ψ_x) and (L', ψ'_x), consists of a map $u \colon L \to L'$ such that $\psi_x = \psi'_x \circ u$.

We shall now define the notion of inverse limit.

Definition 3.21 (Inverse Limit)**.** The limit of a diagram $F \colon I \to C$ is the terminal object in Cone(F) which is denoted by $\varprojlim F$. In other words, $\varprojlim F$ is an object in C along with a collection of morphisms, $\psi_x \colon \varprojlim F \to F(x)$ which commute such that if there is another object L' and morphisms ψ'_x which also commute then there exists a unique morphism $u \colon L' \to \varprojlim F$ such that the following entire diagram commutes

$$\begin{tikzcd}
F(x) \arrow[rr, "F(g)"] & & F(y) \\
& \varprojlim F \arrow[ul, "\psi_x"'] \arrow[ur, "\psi_y"] & \\
& L' \arrow[uul, "\psi'_x"] \arrow[uur, "\psi'_y"'] \arrow[u, "\exists!"', "u"] &
\end{tikzcd}$$

i.e, $\psi_x \circ u = \psi'_x$ for all $x \in I$.

Example 3.22. Let $\Lambda = \{1, 2, \ldots, n\}$ be a discrete finite category. Let X be a topological space. Consider a functor $F \colon \Lambda \to \text{Open(X)}$. We saw how should a cone to this functor look like. Now, the limit of this diagram is the terminal object in Cone(F). Hence, it will be the largest open set that would include into all the possible open sets picked out by F. Hence, $\varprojlim F = \bigcap_{i=1}^{n} F(i)$.

Example 3.23. Consider the following category I and a representation $F\colon I \to Vect$ (a representation is a functor with the target category as the category of vectors spaces)

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \qquad \begin{tikzcd} & V \arrow[d, "S"] \\ W \arrow[r, "T"] & U \end{tikzcd}$$

Now, the limit of this diagram, L, would be an object which would make the following diagram commute

$$\begin{tikzcd} L \arrow[r] \arrow[d] & V \arrow[d, "S"] \\ W \arrow[r, "T"] & U \end{tikzcd}$$

One of the obvious starting steps towards finding the limit would be to consider $L = V \times W$ and the arrows would be the corresponding projection maps. Now, if the diagram has to commute, $T(w) = S(v)$ for $(w, v) \in W \times V$. Thus, we get that $L = \{(w, v) \in W \times V | T(w) = S(v)\}$ is a candidate for $\varprojlim F$. We will now show that if L' is any other such object, then there exists a unique morphism from L' to L. Let L' be any other object which makes the diagram commute

$$\begin{tikzcd} L' \arrow[r, "A"] \arrow[d, "B"] & V \arrow[d, "S"] \\ W \arrow[r, "T"] & U \end{tikzcd}$$

Clearly, there exists a map from L' to $W \times V$, namely, $l \mapsto (A(l), B(l))$. Thus, by definition and the universal property of the limit, we have that

$\varprojlim F = \{(w, v) \in W \times V | T(w) = S(v)\}$. This limit is called **pullback**.

Remark 3.24. Such a construction, in categories where products exist, is also known as **fibre product**.

We now look at the dual constructions of these above definitions, i.e, Cocone and Colimit.

Definition 3.25. Let $F\colon I \to C$ be a diagram. A **cocone** on F is a natural transformation from F to a constant diagram, i.e, a choice of an object $C \in C$ and a collection of morphisms $\phi_x\colon F(x) \to C$ one for each $x \in I$ such that if $g\colon x \to y$ is a morphism in I then the following diagram commutes

$$\begin{tikzcd} & C & \\ F(x) \arrow[ur, "\phi_x"] \arrow[rr, "F(g)"'] & & F(y) \arrow[ul, "\phi_y"'] \end{tikzcd}$$

i.e, $\phi_y \circ F(g) = \phi_x$.

Observe that the directions of the arrows have reversed.

Example 3.26. Let $\Lambda = \{1, 2, \ldots, n\}$ be a discrete finite category. Let X be a topological space. Consider a functor $F\colon \Lambda \to \text{Open(X)}$. A cocone to F would be an open set which would include each of the open sets picked out by F.

Remark 3.27. Similar to the case of the cones, the collection of cones to a functor F form a category which will be denoted by **Cocone(F)**.

Let us now look at the definition of Colimit.

Definition 3.28. The **colimit** of a diagram $F\colon I \to C$ is the initial object in Cocone(F) which is denoted by $\varinjlim F$. In other words, $\varinjlim F$ is an object in C along with a collection of morphisms $\phi_x\colon F(x) \to \varinjlim F$ which commute such that if there is another object C' and morphisms ϕ'_x which also commute then there exists a unique morphism $u\colon \varinjlim F \to C'$ such that the following entire diagram commutes

$$\begin{tikzcd}
 & C' & \\
 & \varinjlim F \arrow[u, "\exists!"', "u"] & \\
F(x) \arrow[uur, "\phi'_x"] \arrow[ur, "\phi_x"'] \arrow[rr, "F(g)"'] & & F(y) \arrow[uul, "\phi'_y"'] \arrow[ul, "\phi_y"]
\end{tikzcd}$$

i.e, $u \circ \phi_x = \phi'_x$ for all $x \in I$.

Example 3.29. Let $\Lambda = \{1, 2, \dots, n\}$ be a discrete finite category. Let X be a topological space. Consider a functor $F\colon \Lambda \to$ Open(X). We saw what a cocone to F would look like. Now, colimit of F would be such an open set which includes all the open sets picked out by F but is included in any other open set which includes the open sets picked out by F. So, it is the smallest open sets that contains all the open sets picked out by F. Hence, $\varinjlim F = \bigcup_{i=1}^{n} F(i)$.

Example 3.30. Consider the following small category I and a functor $F\colon I \to$ Vect.

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \\
\bullet &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
U \arrow[r, "S"] \arrow[d, "T"'] & V \\
W &
\end{tikzcd}$$

Consider the following diagram, which comes naturally looking at the given diagram:

$$\begin{tikzcd} U \arrow[r, "S"] \arrow[d, "T"'] \arrow[dr, "T\oplus S"] & V \arrow[d, "i_V"] \\ W \arrow[r, "i_W"'] & W \oplus V \end{tikzcd}$$

This is not a cocone because the diagram does not commute as $(Tu, 0) \neq (Tu, Su) \neq (0, Tu)$. However, the commutativity of the diagram can be forced by the following equivalence relation

$$(Tu, 0) \sim (0, Su) \text{ or equivalently } (Tu, -Su) \sim (0, 0).$$

Thus, $\varinjlim F = W \oplus V/im(T \oplus -S)$. This limit is called **pushout**.

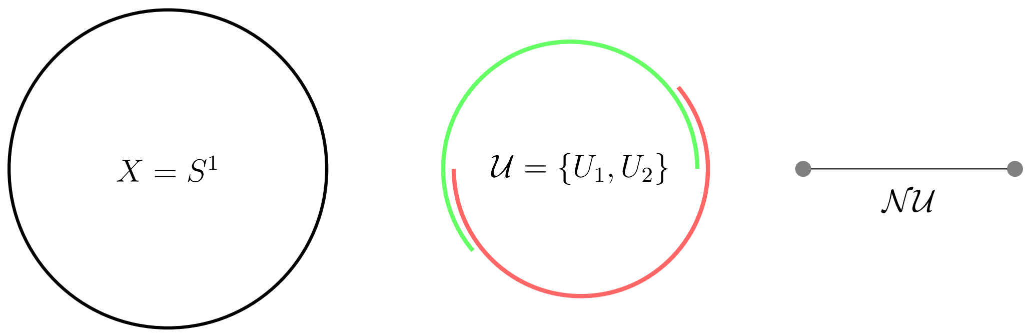
3.2 What are Sheaves and Cosheaves?

Definition 3.31 (Nerve of a cover)**.** Let X be a topological space. Let U be an open set. Let $\mathcal{U} = \{U_i\}_{i\in\Lambda}$ be an open cover for U. The **nerve** of this cover is defined to be the abstract simplicial complex $\mathcal{NU}$ with the vertex set as Λ (the indexing set of the open cover) and $\{i_0, i_1, \ldots, i_k\}$ span a k-simplex if and only if $U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_k} \neq \emptyset$.

We can think of $\mathcal{NU}$ as a category whose objects are $I = \{i_0, \ldots, i_k\}$ and a morphism from I to J if and only if $J \subseteq I$. We now observe that we get natural functors $N\colon \mathcal{NU} \to \text{Open}(X)$ and $N^{\mathbf{op}}\colon \mathcal{NU}^{\text{op}} \to \text{Open}(X)^{\text{op}}$ which would take $I = \{i_0, \ldots, i_k\}$ to $U_I = U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_k}$ and the corresponding morphisms to morphisms.

We get these functors because the intersections that we are considering are finite intersections and finite intersection of open sets is open.

Example 3.32.



Now, we shall define pre-sheaf and pre-cosheaf which will lead us to the definition of sheaf and cosheaf respectively.

Definition 3.33 (Pre-sheaf and pre-cosheaf)**.** A **Pre-Sheaf** is a functor $F\colon \mathrm{Open}(X)^{\mathrm{op}} \to D$ and a **Pre-Cosheaf** is a functor $\hat{F}\colon \mathrm{Open}(X) \to D$. Here D is any arbitrary category.

Remark 3.34. If $U \subset V$, then we write the restriction map $\rho_{U,V}\colon F(V) \to F(U)$ and the extension map $r_{V,U}\colon \hat{F}(U) \to \hat{F}(V)$.

Remark 3.35. For our purposes, we need D to be a category which has all the limits and colimits. Such a category is called **complete** and **co-complete**. E.g, the category of vector spaces, **Vect**, is both complete and co-complete (also called **bicomplete**). The poset of natural numbers, regarded as a category, is co-complete but not complete because there is no terminal object because natural numbers are a well-ordered set and hence there exists a minimum element. However, there is no maximum element.

We also observe that U $(= \bigcup U_i)$ is the direct limit of $N\colon \mathcal{NU} \to \mathrm{Open}(X)$. We thus want that the data associated to U be represented as the direct limit of the data associated to the nerve. This is the motivation to define a sheaf and a cosheaf.

Definition 3.36 (Sheaf and Cosheaf)**.** Let F and $\hat{F}$ be a pre-sheaf and a pre-cosheaf respectively taking values in a category D. With the same definitions of $U, \mathcal{U}$ as described earlier in this subsection, we say that F is a sheaf on $\mathcal{U}$ if the unique map from $F(U)$ to inverse limit of $F \circ N^{\mathrm{op}}$

$$F(U) \to \varprojlim_{I \in \mathcal{N}\mathcal{U}} F(U_I)$$

is an isomorphism.
We say that $\hat{F}$ is a cosheaf on $\mathcal{U}$ if the unique map from the direct limit of $\hat{F} \circ N$ to F(U)

$$\varinjlim_{I \in \mathcal{N}\mathcal{U}} \hat{F}(U_I) \to F(U)$$

is an isomorphism.

Example 3.37. Let $\mathcal{U} = \{U_1, U_2\}$. Consider the assignment $U \rightsquigarrow \{f \colon U \to \mathbb{R} | f \text{ is continuous}\}$. For this to become a sheaf, the two functions $f_1 \colon U_1 \to \mathbb{R}$ and $f_2 \colon U_2 \to \mathbb{R}$ corresponding to U_1 and U_2, respectively, have to agree on the intersection so that the function $f \colon U \to \mathbb{R}$ is well-defined. As we see, using local data in each of the open sets in the cover, we were able to move to a global setting to that of U. Hence, it is often said that sheaves and cosheaves mediate the passage from local to global.

This completes the pre-requisites of understanding the basics of TDA.

The subsequent chapters cover the techniques used in TDA. In the next chapter, we shall look at Persistent Homology, which is one of the highlights of TDA.

4. PERSISTENT HOMOLOGY

Now, let us look at one of the highlights of Topological Data Analysis, namely Persistent Homology. Persistent Homology is a tool which allows us to get a summary of the data at various different scales. By scales, we mean the way the dataset would appear to us, if we were to view it from different distances. This gives us insights of the various dimensional cycles (homological) present in the dataset and more importantly, it gives us a measure of how 'persistent' a cycle is. Hence, the name 'Persistent Homology'.

We shall now look at what is called the **Persistent Homology Pipeline**.

Point Cloud

$\downarrow$

Constructing Simplicial Complex

$\downarrow$

Computing **Persistent Homology** of this complex

$\downarrow$

Representing Persistent Homology in terms of **Barcodes**

To begin with, one has a dataset, i.e, a point cloud. From this dataset, one constructs a simplicial complex. Special types of such simplicial complexes can be constructed so that the simplicial complex is homotopy equivalent to the underlying space itself. We shall look at this in section 4.1.

Once we have such a simplicial complex, we can draw inference about the underlying space by computing the homology of the complex due to the homotopy invariance of homology. In section 4.2, we shall look at what Persistent Homology is. This would give us information about the features of the dataset which persist across different scales. Those features that persist across different scales are believed to be a feature of the dataset.

The final section of this chapter, section 4.3, will be devoted to explain a concise way to represent this Persistent Homology in the form of bars. This representation is called the barcode representation. This representation makes use of one of the classic theorems, namely the classification of finitely generated modules over a PID.

4.1 Constructing Simplicial Complex from a Point Cloud

Data is usually found in the form of a point cloud. A point could is a finite set of points equipped with a distance function. We need to make sense of this point cloud which might or might not even be embedded in some Euclidean space.

Hence, we build simplicial complexes out of this given data. We shall look at why such simplicial complex construction is useful in the later sections. We shall look at more than one way to build simplicial complexes out of a point cloud. Each type of simplicial complex has its own pros and cons. The type of complex that would give best results needs to be selected on a case by case basis.

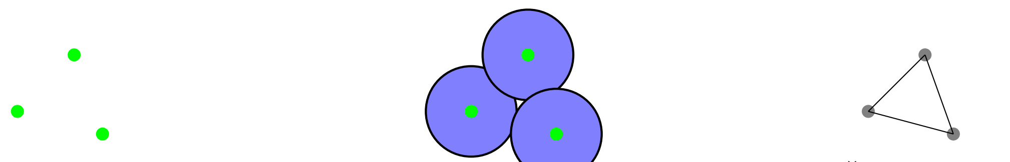
Definition 4.1 (Point Cloud)**.** A point cloud is a finite metric space embedded in a finite-dimensional Riemannian manifold.

However, in most applications point cloud is just a finite set of real vectors.

We have seen the definition of the nerve of a covering in Definition 3.31.

Definition 4.2 (Čech Complex)**.** Let (X, d) be a metric space. Let $\epsilon \in \mathbb{R}_+$. Let $\mathcal{U} = \{B_\epsilon(x)\}_{x \in X}$ be an open cover for X. The nerve of this cover is called the **Čech Complex** and is denoted by $\check{\mathrm{C}}(X, \epsilon)$.

Example 4.3.



X $\{B_{0.4}(x)\}_{x \in X}$ $\check{\mathrm{C}}(X, 0.4cm)$

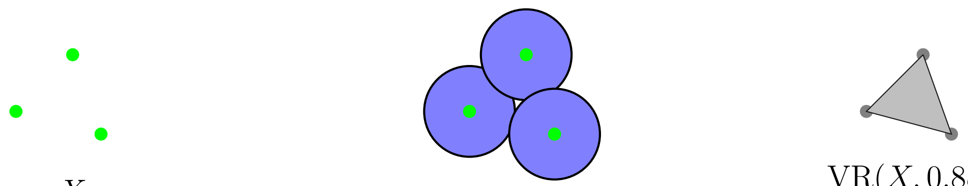
In the above example, we observe that there is no non-empty triple intersection. Hence, there is no 2-simplex in the Čech complex. The three 1-simplices correspond to the three double intersections.

Remark 4.4. $\check{\mathrm{C}}(X, \epsilon_i) \hookrightarrow \mathrm{C}(X, \epsilon_j)$ whenever $\epsilon_i < \epsilon_j$.

Remark 4.5. Čech Complexes are computationally expensive. This is because it becomes difficult to keep track of intersections. Moreover, information about simplices of all dimensions needs to be stored which is not very desirable.

Definition 4.6 (Vietoris-Rips Complex)**.** Let (X, d) be a finite metric space. Let $\epsilon \in \mathbb{R}_+$. Then the **Vietoris-Rips Complex** $VR(X, \epsilon)$ is the simplicial complex with X as the vertex set and $\{x_0, x_1, \ldots, x_k\}$ forms a k-simplex if and only if $d(x_i, x_j) \leq \epsilon \ \forall \ 0 \leq i, j \leq k$.

Example 4.7.



X $\qquad$ $\{B_{0.4}(x)\}_{x\in X}$ $\qquad$ $\mathrm{VR}(X, 0.8cm)$

Remark 4.8. $\mathrm{VR}(X, \epsilon_i) \hookrightarrow \mathrm{VR}(X, \epsilon_j)$ whenever $\epsilon_i < \epsilon_j$.

Remark 4.9. This is one of the most widely used simplicial complex constructions in the applications of TDA. However, even the Vietoris-Rips construction is computationally expensive because it has the whole metric space as the vertex set. Hence, for large datasets, this is not feasible.

Thus, we look at the Voronoi Decomposition of X and then the Delaunay Complex which depends on the Voronoi Decomposition of the space.

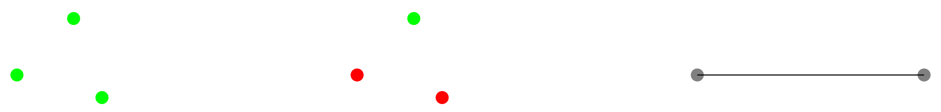
Definition 4.10. Let (X, d) be a metric space. Let L be a discrete subset of X. L is called the set of Landmark Points. For $l \in L$, we define the **Voronoi Cell**, V_l, as follows

$$V_l = \{x \in X | d(x, l) \leq d(x, l') \text{ for all } l' \in L\}$$

Remark 4.11. We can see that the voronoi cells form a cover for X because given any $x \in X$, one of the landmark points will be closest to it. Hence, x would belong to that cell. Since this is true for all $x \in X$, the Voronoi cells form a cover for X.

Definition 4.12 (Delaunay Complex)**.** Let (X, d) be a metric space and let a Voronoi Decomposition of X be given. Then, the nerve of the cover by Voronoi Cells is called the **Delaunay Complex**.

Example 4.13.



X Voronoi Decomposition of X Delaunay Complex

In the above figure, X denotes a 3-point space and the red points are the landmark points. The green point is equidistant from both the red points. In this example, the cover for X by Voronoi cells consists of two sets, each with one red point and the green point. Since, the green point is equidistant from both the red points, the nerve of this cover has a non-empty double intersection and hence a 1-simplex and two 0-simplices corresponding to the two sets in the cover by Voronoi cells.

Remark 4.14. One of the problems with Delaunay complex is that there are very few higher dimensional simplices. Mostly the simplicial complex is disconnected and there are just discrete points. This is because a point doesn't often belong to more than one Voronoi cell for the intersection of two Voronoi cells to be non-empty. If the intersection of two Voronoi cells is empty, there is no 1-simplex between the corresponding 0-simplices.

Hence, we consider a slightly modified versions of this construction, namely the Strong Witness Complex and the Weak Witness Complex.

Definition 4.15 (Strong Witness Complex)**.** Let (X, d) be a metric space and let L be a finite set of landmark points. Let $\epsilon \in \mathbb{R}_+$. For any $x \in X$, let m_x denote the distance of this point to the set L, i.e, $m_x = \min_{l_i \in L}\{d(x, l_i)\}$. Then the **strong witness complex** $\mathcal{W}^s(X, L, \epsilon)$ is defined as the simplicial complex with L as the vertex set and $\{l_0, \ldots l_k\}$ spans a k-simplex if and only if there is a point (witness) $x \in X$ such that $d(x, l_i) \leq m_x + \epsilon$ for all $0 \leq i \leq k$.

The weak witness complex is infact, a modified version of the strong witness complex construction.

Definition 4.16 (Weak Witness Complex)**.** Let (X, d) be a metric space. Let $L \subset X$ be a set of points. Let $\Lambda = \{l_0, \ldots, l_n\}$ be a finite subset of points of L. Then, we say that a point $x \in X$ is a **weak witness** for L if $d(x, l) \geq d(x, l_i)$ for all $l_i \in L$ and $l \in X \setminus \{x\}$. We say that x is an **ϵ-weak witness** if $d(x, l) + \epsilon \geq d(x, l_i)$ for all $l_i \in L$ and $l \in X \setminus \{x\}$. The **weak witness complex** $\mathcal{W}^w(X, L, \epsilon)$ is defined by declaring a family $\Lambda = \{l_0, l_1, \ldots, l_k\}$ span a k-simplex if and only if Λ and all its faces admit ϵ-weak witnesses.

Remark 4.17. We observe that for any of the above method of simplicial complex construction, for a given dataset, has the following property:

$$C(X, \epsilon_i) \hookrightarrow C(X, \epsilon_j) \text{ whenever } \epsilon_i < \epsilon_j$$

$C(X, \epsilon_i)$ has been used to denote any of the constructions above corresponding to the parameter ϵ_i.

All the above constructions are useful for computations in the light of the following two theorems.

Theorem 4.18 (Nerve Theorem)**.** *Let X be a topological space and $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be a countable open cover for X. If for all $\emptyset \neq S \subset A$ $\bigcap_{s \in S} U_s$ is either empty or contractible, then $\mathcal{NU}$ is homotopy equivalent to X.*

Theorem 4.19 (Hausmann, 1995). *If M is a compact Riemannian manifold, then there exists $e > 0$ such that $VR(M, \epsilon)$ is homotopy equivalent to M for all $\epsilon \leq e$.*

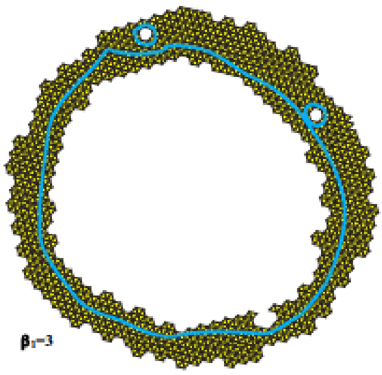
The proofs of these theorems can be found in Appendix A and Appendix B respectively.

As we have seen in the previous chapters that homology is homotopy invariant, theorems like these, in essence, give us information about the space by computing the homology of the simplicial complexes.

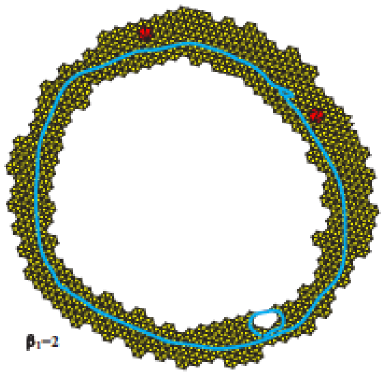
Computing simplicial homology is easy because a simplicial complex is a combinatorial object and the computation can be algorithmised.

4.2 Persistent Homology

Consider the following picture of a Čech Complex built from a dataset for some ϵ_1.



Now consider the following picture of the Čech Complex which corresponds to $\epsilon_2 > \epsilon_1$.



The above two pictures have been taken from [1].

We see that in the first picture, the rank of H_1 is 3 while in the second one, it is 2. But it is clearly evident from the picture that the actual rank of H_1 should be 1 because the data seems to be distributed in the shape of a circle. We need a way or a method to capture this information. We need a mathematical framework which would capture those features of the dataset that persist over a long range of ϵ's. Hence, we turn to Persistent Homology.

Definition 4.20. Let C be a category and let P be a partially ordered set, regarded as a category. Then, by **P-persistence object in C**, we mean a functor $\Phi\colon P \to C$.

Thus, by P-persistence object in C we mean a collection of objects $\{c_x\}_{x\in P}$ of C and a set of morphisms $\phi_{xy}\colon c_x \to c_y$ for all $x \leq y$ which satisfy the composition $\phi_{yz} \circ \phi_{xy} = \phi_{xz}$ for all $x \leq y \leq z$.

Remark 4.21. P-persistence objects in C form a category in their own right because the collection of functors between two given categories forms a category with the morphisms as the natural transformations.

Remark 4.22. A typical $\mathbb{N}$-persistence object, Q, would look like:

$$\dots \xrightarrow{f_{i-1}} Q_i \xrightarrow{f_i} Q_{i+1} \xrightarrow{f_{i+1}} Q_{i+2} \xrightarrow{f_{i+2}} \dots$$

Definition 4.23. Let $\mathbb{N}$ be the partially ordered set. Then **$\mathbb{N}$-persistence chain complex** is a family of chain complexes $\{C_\star^i\}_{i\geq 1}$ with the chain maps $f^i \colon C_\star^i \to C_\star^{i+1}$ as the morphisms.

Remark 4.24. A typical $\mathbb{N}$-persistence chain complex would look like:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\dots \arrow[r] & C_{n+1}^{i-1}(X) \arrow[r, "f_{n+1}^{i-1}"] \arrow[d, "\partial_{n+1}"] & C_{n+1}^{i}(X) \arrow[r, "f_{n+1}^{i}"] \arrow[d, "\partial_{n+1}"] & C_{n+1}^{i+1} \arrow[r] \arrow[d, "\partial_{n+1}"] & \dots \\
\dots \arrow[r] & C_{n}^{i-1}(X) \arrow[r, "f_{n}^{i-1}"] \arrow[d, "\partial_{n}"] & C_{n}^{i}(X) \arrow[r, "f_{n}^{i}"] \arrow[d, "\partial_{n}"] & C_{n}^{i+1}(X) \arrow[r] \arrow[d, "\partial_{n}"] & \dots \\
\dots \arrow[r] & C_{n-1}^{i-1}(X) \arrow[r, "f_{n-1}^{i-1}"] \arrow[d] & C_{n-1}^{i}(X) \arrow[r, "f_{n-1}^{i}"] \arrow[d] & C_{n-1}^{i+1}(X) \arrow[r] \arrow[d] & \dots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Similarly, one can define Persistence Modules.

Definition 4.25. Let $\mathbb{N}$ be the set of natural numbers which is a poset. Then an **$\mathbb{N}$-persistence module** is a family of modules $\{M_i\}_{i>0}$ with module homomorphisms $f_i \colon M_i \to M_{i+1}$ for each $i \in \mathbb{N}$.

Example 4.26. The homology of an $\mathbb{N}$-persistence chain complex would be an $\mathbb{N}$-persistence $\mathbb{Z}$-module.

By Remark 4.17, we can see that the construction of the simplicial complexes that we did in the previous section are $\mathbb{R}$-persistence simplicial complexes.

Now, we construct $\mathbb{R}$-persistence chain complex corresponding to the $\mathbb{R}$-persistence simplicial complex. The definition of $\mathbb{R}$-persistence chain complex is similar to Definition 4.23. The horizontal rows, as that in, Remark 4.24 would be"indexed" by $\mathbb{R}$.

One of the reasons we use homology as the tool to study topological spaces is that there is a classification theorem for finitely generated abelian groups. The object of interest to us, however, is $\mathbb{R}$-persistence abelian groups. If there was a classification theorem for $\mathbb{R}$-persistence abelian groups, we could have used that to classify the group and get the relevant information out of it but there isn't one.

However, there is a classification theorem for a subcategory of $\mathbb{N}$-persistence vector spaces. Hence, we need to transfer to the setting of that subcategory of $\mathbb{N}$-persistence vector spaces.

Let us now look at $\mathbb{N}$-persistence abelian groups and try to understand them. We can think of an $\mathbb{N}$-persistence abelian group as a graded module over a graded polynomial ring.

Definition 4.27 (Graded Ring). A ring $(R, +, \cdot)$ equipped with direct sum decomposition of abelian groups $R \cong \bigoplus\limits_{n \in \mathbb{Z}} R_n$ with the multiplication satisfying $R_n \cdot R_m \subseteq R_{n+m}$ for all $n, m \in \mathbb{Z}$.

Example 4.28. The ring of polynomials $R[x]$ can be graded by degree where each $R_n = R \cdot x^n$. Consider $ax^k \in R \cdot x^k (= R_k)$ and $bx^l \in R \cdot x^l (= R_l)$. Then $ax^k \cdot bx^l = abx^{k+l}$ and clearly $abx^{k+l} \in R \cdot x^{k+l} (= R_{k+l})$.

Definition 4.29 (Graded Module)**.** A graded module M over a graded ring R is a module equipped with the direct sum decomposition $M \cong \bigoplus_i M_i$ with the multiplication satisfying $R_n \cdot M_k \subseteq M_{n+k}$.

Example 4.30. A graded vector space over a field. Here, we consider the field with trivial grading, i.e, the graded parts are 0 and F for a field F.

Example 4.31. A graded ring is a graded module over itself. This is similar to how a ring is a module over itself.

Example 4.32. Let I be an ideal of a commutative ring R and let M be an R-module. Then $\bigoplus\limits_{k=0}^{\infty} I^k M/I^{k+1}M$ is a graded module over the graded ring $\bigoplus\limits_{k=0}^{\infty} I^k/I^{k+1}$.

We shall first see why $\bigoplus\limits_{k=0}^{\infty} I^k/I^{k+1}$ is a graded ring. Similar arguments would imply that $\bigoplus\limits_{k=0}^{\infty} I^k M/I^{k+1}M$ would be a graded module over $\bigoplus\limits_{k=0}^{\infty} I^k/I^{k+1}$.
Consider $a \in I^m/I^{m+1}$ and $b \in I^n/I^{n+1}$ for some $m, n \in \mathbb{N}$. Then,

$$
\begin{aligned}
a = c + I^{m+1} \text{ and } b = d + I^{n+1} &\text{ for some } a \in I^m \text{ and } b \in I^n \\
ab = cd + cI^{n+1} &+ dI^{m+1} + I^{m+n+2}
\end{aligned}
$$

We observe that $cd \in I^{m+n}$, $cI^{n+1}, dI^{m+1} \in I^{m+n+1}$ and we also know that $I^m \subseteq I^n \; \forall m > n$. Thus, $I^{m+n+2} \subseteq I^{m+n+1}$. Hence, we get $ab \in I^{m+n}/I^{m+n+1}$ $\bigoplus\limits_{k=0}^{\infty} I^k/I^{k+1}$ satisfies the condition for being a graded ring. A similar argument would show that $\bigoplus\limits_{k=0}^{\infty} I^k M/I^{k+1}M$ is a graded module over the graded ring $\bigoplus\limits_{k=0}^{\infty} I^k/I^{k+1}$.

Consider the functor $\Theta\colon \mathbb{N}_{\text{pers}}(\text{Ab}) \to \mathbb{Z}[\text{t}]\text{GrMod}$ that assigns to each $\mathbb{N}$-persistence abelian group $\{A_k\}$ the graded module

$$\Theta(\{A_s\}) = \bigoplus_{s\geq 0} A_s$$

where the nth graded part is the abelian group A_n. The action of the polynomial generator t is given by

$$t \cdot (a^0, a^1, \ldots) = (0, \phi_{0,1}(a^0), \phi_{1,2}(a^1), \ldots)$$

where $\phi_{n,n+1}\colon A_n \to A_{n+1}$ is the group homomorphism which is the image of the morphism $n < n+1$ and $a^i \in A_i$ $\forall i$. (Recall that $\mathbb{N}$-persistence abelian group is a functor). We note here that only finitely many a^i's are non-identity elements in $(a^0, a^1, \ldots)$ so that $(a^0, a^1, \ldots) \in \bigoplus_{s\geq 0} A_s$. In essence, t is just shifting the elements up in the gradation.

We can now define an inverse functor $\Omega\colon \mathbb{Z}[\text{t}]\text{GrMod} \to \mathbb{N}_{\text{pers}}(\text{Ab})$ where the morphisms $\phi_{m,n}$ are given by multiplying by t^{n-m}, i.e,

$$\Omega(\bigoplus_{s\geq 0} A_s) = \{A_s\}$$

Claim - Θ is an equivalence of categories.

Proof - We know that by Remark 3.15, it is enough to check that if the functor Θ is fully faithful and essentially surjective.
A map, f, between two $\mathbb{N}$-persistence abelian groups $\{A_n\}$ and $\{B_n\}$ would look like:

$$\begin{array}{ccccc} A_0 & \xrightarrow{\phi_{0,1}} & A_1 & \xrightarrow{\phi_{1,2}} & \dots \\ \Big\downarrow{\scriptstyle f_0} & & \Big\downarrow{\scriptstyle f_1} & & \\ B_0 & \xrightarrow{\psi_{0,1}} & B_1 & \xrightarrow{\psi_{1,2}} & \dots \end{array}$$

Each f_i is a group homomorphism and each square commutes. Hence, any map f between two $\mathbb{N}$-persistence abelian groups can be considered as a map between the corresponding $\mathbb{Z}[t]$-graded modules, $f'\colon \bigoplus_{s\geq 0} A_s \to \bigoplus_{s\geq 0} B_s$ defined as $f'(a_0, a_1, \dots) = (f_0(a_0), f_1(a_1), \dots)$. Similarly, any map between the $\mathbb{Z}[t]$-graded modules can be considered as a map between the corresponding $\mathbb{N}$-persistence abelian groups. Hence, full faithfullness is clear.

Now, we need to check essential surjectivity. Let $\bigoplus_{s\geq 0} A_s$ be an object in $\mathbb{Z}$[t]GrMod. Consider $\{A_n\} \in \mathbb{N}_{pers}(Ab)$. Then, clearly, there exists an isomorphism between $\Theta(\{A_n\})$ and $\bigoplus_{s\geq 0} A_s$, namely the identity. Hence, Θ is also essentially surjective.

Thus, Θ is an equivalence of categories.

Remark 4.33. Using Definition 3.14, we can directly see that Θ is an equivalence by defining Ω as an inverse functor. It is clear just by the definition of Ω that $\Theta\Omega$ is naturally isomorphic to $\mathrm{id}_{\mathbb{Z}[\mathrm{t}]\mathrm{GrMod}}$ and $\Omega\Theta$ is naturally isomorphic to $\mathrm{id}_{\mathbb{N}_{\mathrm{pers}(\mathrm{Ab})}}$.

We have identified the category of $\mathbb{N}$-persistence abelian groups with that of $\mathbb{Z}[t]$-graded modules and we also know that there exists a classification theorem for finitely generated modules over a PID.

Since, $\mathbb{Z}[t]$ is not a PID, we don't have the machinery to classify the graded modules over $\mathbb{Z}[t]$. However, we know that $F[t]$ is a PID where F denotes a field. So we now consider the equivalence between the category of $\mathbb{N}$-persistence F-vector spaces and that of graded $F[t]$ modules. (The category equivalence that we have established above is between the category of $\mathbb{N}$-persistence $\mathbb{Z}$-modules and that of graded $\mathbb{Z}[t]$ modules. We wish to transfer that setting to the equivalence of the category of $\mathbb{N}$-persistence F-vector spaces and that of graded $F[t]$ modules).

Let us look at the classification theorem for graded modules over a graded ring.

Theorem 4.34. *Let $M_\star$ be a finitely generated non-negatively graded $F[t]$ module where F is a field. Then there are integers $\{i_1, \ldots, i_m\}, \{j_1, \ldots, j_n\}$ and $\{l_1, \ldots, l_n\}$ and an isomorphism*

$$M_\star \cong \bigoplus_{s=1}^{m} \Sigma^{i_s} F[t] \oplus \bigoplus_{t=1}^{n} \Sigma^{j_t} F[t]/(t^{l_t})$$

where Σ^{i_s} denotes an upward shift in grading by i_s, which in this case is multiplying by t^{i_s}.

This theorem is only for finitely generated graded modules over a PID. Thus, we need to look at those $\mathbb{N}$-persistence F-vector spaces that correspond, under the equivalence of categories, to the finitely generated graded $F[t]$-modules.

Definition 4.35. An $\mathbb{N}$-persistence F-vector space $\{V_n\}$ is called **tame** if every vector space V_k is finite dimensional and $\exists N \in \mathbb{N}$ such that $\psi_{n,n+1} \colon V_n \to V_{n+1}$ is an isomorphism $\forall n \geq N$.

The following proposition would characterise the $\mathbb{N}$-persistence F-vector spaces that would correspond to the finitely generated graded modules under the equivalence of categories.

Proposition 4.36. *Let Θ be the functor defined above. Then $\Theta(\{V_n\})$ is a finitely generated graded module if and only if $\{V_n\}$ is tame.*

Proof. ($\Rightarrow$) Let $M = \Theta(\{V_n\})$ be a finitely generated graded module. Let $B = \{b_0, \ldots, b_n\}$ be a generating set for M. We divide B into disjoint subsets B_i where B_i only has the elements of degree i. (We are working with graded modules so there are elements of different degree). Thus, $B_k + \sum_{i=1}^{k-1} t^{k-i} B_i$ is the generating set for M_k where M_k denotes the kth graded part. By $B_k + \sum_{i=1}^{k-1} t^{k-i} B_i$, we mean the linear combination of elements of B_k and $t^{k-i}\cdot$ elements of B_i.
Since the generating set for M is finite, there exists an element of the highest degree. Let that degree be m. Thus, for all $j > m$, the generating set for M_j would be $\sum_{l=1}^{j} t^{j-l} B_l$. Hence, for all $j > m$, B is the generating set for M_j. Thus, if we let $\{V_n\} = \Omega(M)$, then dim $(V_n) < \infty$ for all n and moreover, for all $n > m$, $\psi_{n,n+1} \colon V_n \to V_{n+1}$ is an isomorphism.

($\Leftarrow$) Let $\{V_n\}$ be a tame vector space. Each V_n is finite dimensional and there exists an N such that for all $n > N$, $V_n \cong V_N$. Now, let $M = \Theta(\{V_n\})$. Let B_i be the basis for V_i for $i < N$. For all $n < N$, the generating set of M_n would be $\sum_{i=1}^{n} t^{n-i} B_i$. Since $V_n \cong V_N$ for all $n > N$, the generating set for M_n would be $\sum_{i=1}^{N} t^{N-i} B_i$ for all $n > N$. Therefore, the generating set for M consists of finitely many elements and thus, M is a finitely generated graded module. $\square$

Now that we have this proposition, we can obtain the summary of any tame $\mathbb{N}$-persistence F-vector space. However, we need to be able to represent this summary in an understandable manner so that even those without a lot of mathematical background can understand and use this technique for their analysis. We will now look at a way to represent this summary in an easy to understand and a visually pleasing way.

4.3 Barcodes

Definition 4.37 ($\mathcal{P}$-interval)**.** A **$\mathcal{P}$-interval** is an ordered pair (i, j) with $i, j \in \mathbb{Z} \cup \{\infty\}$ and $0 \leq i < j$.

We now associate a graded $F[t]$-module to a set of $\mathcal{P}$-intervals. We define a function Q as follows:

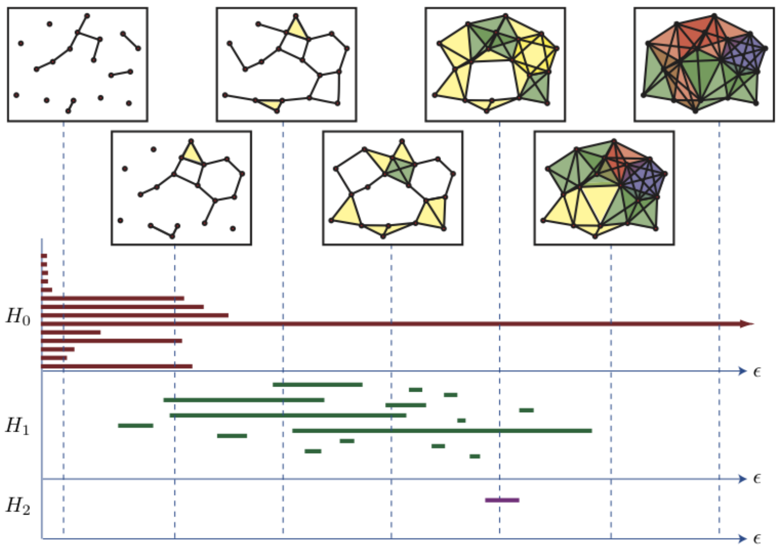
$$Q(i, j) = \Sigma^i F[t]/(t^{j-i}) \text{ and } Q(i, \infty) = \Sigma^i F[t]$$

Now for a set of $\mathcal{P}$-intervals, $S = \{(i_1, j_1), \ldots (i_k, j_k)\}$, define

$$Q(S) = \bigoplus_{n=1}^{k} Q(i_n, j_n)$$

This correspondence $S \to Q(S)$ defines a one-to-one correspondence between finite sets of $\mathcal{P}$-intervals and finitely generated graded $F[t]$-modules.

Hence, we have a correspondence between finite sets of $\mathcal{P}$-intervals and tame persistence F-vector spaces.



This image has been taken from the paper 'Barcodes: The persistent topology of data' by 'Robert Ghrist'.

The above image shows the evolution of the simplicial complex as the value of ϵ increases, which is plotted on the horizontal axis. It also shows the corresponding barcode which indicate the presence of different features. E.g, H_0 denotes the number of connected components. One can see that after a certain value of ϵ, everything becomes connected and there is just one long bar that remains. Similarly, one can also see that the bars in H_1 correspond to the birth and death of the 1-cycles in the simplicial complex. There are two 1-cycles in the second figure of the simplicial complex and there are two bars that begin in H_1 near that value of ϵ and stop as soon as those 1-cycles become the boundaries.

To summarise,

- From a given point cloud, we first construct the $\mathbb{R}$-persistence simplicial complex.
- We then choose an order preserving map $f\colon \mathbb{N} \to \mathbb{R}$.
- We construct the associated $\mathbb{N}$-persistence simplicial complex.
- We then construct the $\mathbb{N}$-persistence chain complex with coefficients in a field F and then compute the homology of this chain complex.
- We represent the homology as barcodes.

4.4 Multidimensional Persistence

In the previous sections, we have seen single-dimensional persistence, i.e, we were interested in just varying one parameter, namely ϵ, and obtain a summary over the range of values of ϵ. In many situations, one may need to vary more than one parameters.

Suppose one wants to study the behaviour of a real valued function f on $\mathbb{R}^n$ qualitatively, i.e, minima, maxima, saddle points, etc.
One of the efficient ways to do it to study how the topology of the sublevel sets, $S_L = \{x \in \mathbb{R}^n | f(x) \leq L\}$, changes with L.
If there is no explicit formula for f, but just some values of f on certain grids or a set S consisting of certain points, one can approximate the topology of sublevel sets by the Vietoris-Rips Complexes of $S \cap S_L$ and study how they evolve with L.

We observe that along with the changing parameter of L, one also needs to change ϵ, the scale parameter of the Vietoris-Rips Complexes. Hence, this is a two-parameter family of simplicial complexes, $\{VR(S \cap S_L, \epsilon)\}_{\epsilon,L}$.

There are many such examples where one would like to study multiparameter family of simplicial complexes. In order to deal with such situations, we need multidimensional persistence.

However, the machinery of one-dimensional persistence can not be generalised to higher dimensions because the classification of modules over a multivariate ring does not exist and thus, there is no parametrisation as such.

In [3], Curry develops the sheaf theoretic approach to Topological Data Analysis along with other things. Approaching TDA from a sheaf theoretic perspective has its own advantages, one of which being, it can help understand the multidimensional persistence problem better and tackle it in a more efficient way.

In the next chapter, we will look at the (co)sheaf theoretic approach to persistence and look at how it offers a different perspective of persistence.

5. (CO)SHEAF THEORETIC APPROACH TO PERSISTENCE

In the previous chapter, we saw the technique of one-dimensional persistence. However, towards the end of the chapter, we also saw the reason that machinery can not be generalised to higher dimensions. In order to tackle this, there is an alternate approach to persistence. Before we delve into that, we observe that the classification of finitely generated modules over a PID played a crucial role in representing $\mathbb{N}$-Persistent Homology as barcodes. However, the same theory does not work as is for other partially ordered sets such as $\mathbb{R}$. Thus, we shall look at how to deal with other partially ordered sets in section 5.1.

In the second section of this chapter, we shall look at barcodes obtained from sub-level sets of a function. We shall see how studying this is useful in obtaining a new perspective of persistence.

Further, in section 5.3, we shall study level set persistence and see how each sub-level set persistence problem can be cast as a level set persistence problem. Hence, it would be enough for us if we studied level set persistence.

Finally, in section 5.4, we shall look at the (co)sheaf theoretic approach to persistence which helps us study level set persistence problems. We will define simplicial cosheaves and look at a homology theory for simplicial cosheaves. Towards the end of this section, we shall see a theorem which would help us see the Persistent Homology of a filtered simplicial complex as the cosheaf homology of some cosheaf.

5.1 Barcode decomposition of $\mathbb{R}$-persistence modules

We now have a much stronger theorem which helps us classify and represent any persistence module, i.e, a functor from any partially ordered set to the category of modules. This theorem was proved by Crawley-Boevey in 2012 in [11].

Note that we will use persistence module to denote $\mathbb{R}$-persistence module unless otherwise specified.

Definition 5.1 (Interval Modules)**.** Let $I \subset \mathbb{R}$ be an interval. Then an **interval module** is a persistence module, k_I, which assigns to each $s \in I$ the vector space k and zero to the elements in $\mathbb{R} \setminus I$. All the maps $\phi_{s,t} = id$ for all $s, t \in I$ and $s \leq t$. For all other $s, t \in \mathbb{R}$, $\phi_{s,t} = 0$.

Remark 5.2. Interval modules are persistence modules which are zero everywhere except on an interval.

Theorem 5.3. *If $\{M\}$ is a persistence module such that each module M_t is finite dimensional for all $t \in \mathbb{R}$, then the module is isomorphic to direct sum of interval modules, i.e,*

$$M \cong \bigoplus_{I \in S} k_I$$

where S is a multiset of intervals, i.e, a set which can have more than one entries of the same interval.

Remark 5.4. Addition of two persistence module gives another persistence module. If $\{V\}$ and $\{W\}$ are two persistence modules, then $\{V \oplus W\}$ is defined as $V_t \oplus W_t$ for all $t \in \mathbb{R}$ and the maps in $\{V \oplus W\}$ will also be the direct sum of the corresponding maps, i.e, $\phi^{V \oplus W}_{s,t} = \phi^V_{s,t} \oplus \phi^W_{s,t}$.

5.2 Barcodes from sub-level sets

Up until now, we have considered a point cloud and computed its Persistent Homology. Now, let us look at a slightly different problem. Consider a space X equipped with a function $f\colon X \to \mathbb{R}$. We consider this problem because this generalises the previous setting that we were in. E.g, consider X to be a point cloud in $\mathbb{R}^n$. Now, for each point $p \in \mathbb{R}^n$, consider the function that returns its distance from the point cloud, i.e,

$$f(p) = \min_{x_i \in X} ||p - x_i||.$$

Now, consider the augmented point clouds X_{r_0}, i.e, it is the set of all points in $\mathbb{R}^n$ which are within a distance of r_0 from the point cloud. If we now consider an increasing sequence of r_i's, we get an inclusion of their corresponding point clouds

$$X_{r_0} \hookrightarrow X_{r_1} \hookrightarrow X_{r_2} \hookrightarrow \dots$$

We observe that this is exactly equal to

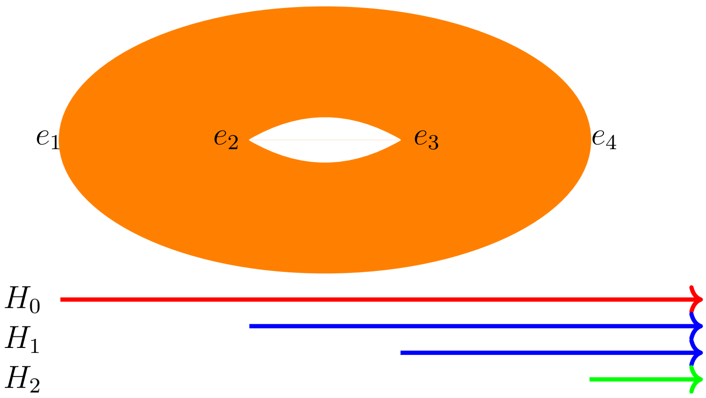
$$f^{-1}(-\infty, r_0) \hookrightarrow f^{-1}(-\infty, r_1) \hookrightarrow f^{-1}(-\infty, r_2) \hookrightarrow \dots$$

Computing the barcode for this sequence of point clouds would correspond to computing the Persistent Homology for the sublevel sets of f.

Note that the example given above was to explain what barcodes from sub-level sets means. It is not necessary that one has to consider functions which are similar to the one described in the example in the above discussion.

To make this more clear, we consider the following example.

Example 5.5. Consider the torus in $\mathbb{R}^3$. Consider the height function on the torus $h\colon \mathbb{T} \to \mathbb{R}$ given by $h(x, y, z) = x$.



The red bar represents H_0, the two blue bars H_1 and the green bar H_2.

If we consider the zeroth homology of $f^{-1}(-\infty, e_i)$, as e_i varies from left to right, we can see that there is one connnected component throughout and hence one long bar in H_0.

$H_1(f^{-1}(-\infty, e_1)) = 0$ because there is no 1-cycle, or a loop, that is present. The inverse image is like a curved disk which has trivial homology.

As we move rightwards, $H_1(f^{-1}(-\infty, e_2)) = k$ because the two 1-cycles present bound the same region and hence are homologous. Thus, homology can not differentiate between the two giving rise to a Betti number of 1 and one bar in H_1 starts at e_1.

Further, $H_1(f^{-1}(-\infty, e_3)) = k^2$. This is because, there are two homology classes of 1-cycles now. One of the class was previously present and another one (vertical loop) just got completed in the preimage. Hence, these give rise to two non-homologous 1-cycles.

As we move further right, H_1 doesn't change. However, at e_4, the second Betti number becomes 1 due to the void that arises in the preimage at e_4. The preimage at e_4 is essentially the full torus and we know that the torus has a non-trivial second homology.

Remark 5.6. One can also link Morse theory here and the critical points of the function are precisely at e_0, e_1, e_3 and e_4. Morse theory says that the homology of the space changes at its crictical points, if the function being considered is a Morse function which is exactly what we observe here.

5.3 Level set persistence

If the dataset that we want to analyse is very large, it becomes computationally quite expensive to use the data analytic techniques. Thus, one could sub-divide the data and run the techniques on each of the sub-division and somehow combine the results in order to obtain a comprehensive summary of the original dataset.

If one is using TDA techniques, it becomes imperative to not 'double-count' the features, i.e, it might so happen that there is a persistent non-trivial first homology in one of the sub-divisions and another sub-division also has a persistent non-trivial first homology. Then it might be very likely that the feature is present in the intersection of the two sub-divisions of the data (assuming that their intersection is non-empty). We need to be able to detect such a phenomenon. Hence, the concept of **Zig-zag** modules is useful. This was first done by Carlsson and Vin de Silva in 2009 in [12].

$$X_1 \to X_1 \cup X_2 \leftarrow X_2 \to X_2 \cup X_3 \leftarrow X_3 \ldots \leftarrow X_n$$

Similarly, if one has a function $f\colon X \to \mathbb{R}$ but not the computational resources to handle the data of $f^{-1}(-\infty, r)$, then one needs to be able to work with finite intervals and combine the data like in the above case. One can choose a partition $t_0 < t_1 \ldots < t_n$ and consider the zig-zag module of the pre-images.

$$f^{-1}(t_0) \to f^{-1}[t_0, t_1] \leftarrow f^{-1}(t_1) \to \ldots \leftarrow f^{-1}(t_n)$$

We see that in cases like these, one needs to be able to work with the level sets rather than the sub-level sets.
Moreover, every sub-level set persistence problem can be cast as a level set persistence problem as follows:
Given a map $f\colon X \to \mathbb{R}$, we construct a new space

$$Y = \{(x,t) \in X \times \mathbb{R} | f(x) \leq t\}$$

Now, if we consider the projection map $\pi_{\mathbb{R}}\colon Y \to \mathbb{R}$, then we can see that the fibres of this map are nothing but the sub-level sets of f.

Hence, we have transformed a sub-level set persistence problem into a level-set persistence problem. We now need to study level-set persistence. But in order to do so, we need to find a way to relate two fibres of a map $f\colon X \to Y$. This is because we need to be able to distinguish between persistent features and ephemeral ones using functoriality. This is the motivation to study the (co)sheaf theoretic approach to persistence.

5.4 (Co)sheaf theoretic approach

As we have seen in the earlier chapters, our primary objects of interest are the simplicial complexes. Thus, we need to be able to make sense of how to talk about (co)sheaves in the setting of simplicial complexes. For that reason, let us start by defining simplicial cosheaves.

However, before we do that, we would like to observe that we can put a partial order on the simplices of a simplicial complex in the following way.

$$\sigma \leq \tau \text{ if and only if } \sigma \subset \tau$$

Here, by $\sigma \subset \tau$, we mean the inclusion of subsets when considered as subsets of the vertex set of the simplicial complex.

Definition 5.7. Let X be a simplicial complex. A **simplicial cosheaf** is an assignment of a vector space $\hat{F}(\sigma)$ to each simplex σ of X and a linear map $r_{\tau,\sigma} \colon \hat{F}(\sigma) \to \hat{F}(\tau)$ to each relation $\sigma \leq \tau$ such that for all $\sigma \leq \tau \leq \rho$, the maps should satisfy $r_{\rho,\tau} \circ r_{\tau,\sigma} = r_{\rho,\sigma}$.

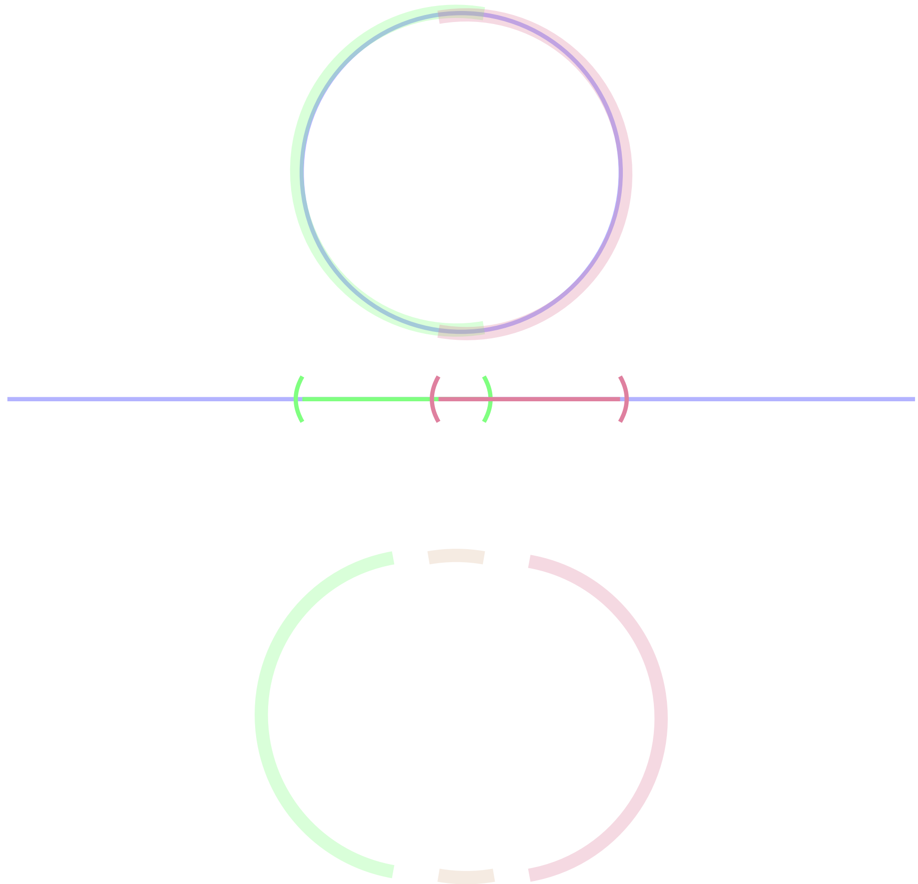
Now that we have seen a general definition of a simplicial cosheaf, we would like to define a particular type of simplicial cosheaf which would aid us in our study of Persistent Homology.

Definition 5.8 (Simplicial Leray cosheaf)**.** Let X and Y be two topological spaces. Let $f \colon X \to Y$ be a continuous map. Let $\mathcal{U}$ be a cover of $f(X) \subseteq Y$. Then, the **simplicial Leray cosheaf** in the n-th dimension over the nerve $\mathcal{N}\mathcal{U}$ is defined via the assignment $\sigma \mapsto H_i(f^{-1}(U_\sigma)$. By U_σ, we mean $\bigcap\limits_{s \in \sigma} U_s$.

We denote the n-th simplicial Leray cosheaf by $\hat{F}_n$.

Note that this is a definition and that the supposed 'simplicial Leray pre-cosheaf', if it exists, does not have to satisfy the cosheaf condition in order to become a simplicial Leray cosheaf.

Example 5.9. Consider the standard height function on the circle $f \colon S^1 \to \mathbb{R}$ given by $f(x, y) = x$.



$$H_0(f^{-1}(U_\sigma);k) \qquad k \longleftarrow k^2 \longrightarrow k$$

Let $\mathcal{U} = \{U_1, U_2\}$ be a cover for S^1. $f^{-1}(U_1)$ and $f^{-1}(U_2)$ both have one connected component each. Now, $U_1 \cap U_2 \hookrightarrow U_1$ and $U_1 \cap U_2 \hookrightarrow U_2$. Since the assignment $U \mapsto H_0(f^{-1}(U))$ is a pre-cosheaf, it is a covariant functor and thus, there are maps $H_0(f^{-1}(U_1 \cap U_2)) \to H_0(f^{-1}(U_1))$ and $H_0(f^{-1}(U_1 \cap U_2)) \to H_0(f^{-1}(U_2))$. We also observe that $f^{-1}(U_1 \cap U_2)$ has two connected components, the small arc at the top and the small arc at the bottom. Hence, its zeroth homology would have rank 2 (thus k^2) while that of $f^{-1}(U_1)$ and $f^{-1}(U_2)$ has rank 1.

Now that we have defined the notion of a simplicial cosheaf and also looked at an example, we can look at a homology theory for such cosheaves. However, in order to do so, we must need to have a notion of a chain complex and that of a boundary operator in this setting.

Before we dive into the definition of boundary operator, we will set up a notation that would be useful in understanding the definition better.

If $\sigma = \{v_{i_0}, \ldots, v_{i_n}\}$, then by $\partial\sigma_j$, we mean $\{v_{i_0}, \ldots, v_{i_{j-1}}, v_{i_{j+1}}, \ldots, v_{i_n}\}$, i.e, the jth face of the σ.

Definition 5.10. Let X be a simplicial complex. Let $\hat{F}$ be a simplicial cosheaf of vector spaces on X. Then the **boundary of a vector** $v \in \hat{F}(\sigma)$ is defined as

$$\partial(v) = (r_{\partial\sigma_0,\sigma}(v), -r_{\partial\sigma_1,\sigma}(v), \ldots, (-1)^n r_{\partial\sigma_n,\sigma}(v)) \in \bigoplus_{j=1}^{n} \hat{F}(\partial\sigma_j).$$

This idea of defining the boundary of a vector before defining the boundary map might seem bizarre at the first glance. But it is easier to understand the chain complex if one can see what the boundary map is doing to each vector. Moreover, once the chain complex is fully defined, one can see that this definition of the boundary of a vector is anything but natural.

Let us now look at the definition of the chain complex valued in $\hat{F}$.

Definition 5.11. Let X be a simplicial complex. Let $\hat{F}$ be a simplicial cosheaf of vector spaces on X. The n-th **group of chains** valued in $\hat{F}$ is defined to be the direct sum of the vector spaces that $\hat{F}$ assigns to each n-simplex, i.e,

$$C_n(X; \hat{F}) = \bigoplus_{\dim\,\sigma=n} \hat{F}(\sigma)$$

Now that this is defined, we define the boundary operator $\partial\colon C_n(X;\hat{F}) \to C_{n-1}(C;\hat{F})$ by extending the definition of the boundary of a vector, i.e,

$$\begin{aligned}\partial(v_1, v_2, \ldots, v_n) =&(r_{\partial\sigma_{1_0},\sigma_1}(v_1), -r_{\partial\sigma_{1_1},\sigma_1}(v_1), \ldots, (-1)^n r_{\partial\sigma_{1_n},\sigma_1}(v_1),\\ &r_{\partial\sigma_{2_0},\sigma_2}(v_2), -r_{\partial\sigma_{2_1},\sigma_2}(v_2), \ldots, (-1)^n r_{\partial\sigma_{2_n},\sigma_2}(v_2),\\ &\vdots\\ &r_{\partial\sigma_{n_0},\sigma_n}(v_n), -r_{\partial\sigma_{n_1},\sigma_n}(v_n), \ldots, (-1)^n r_{\partial\sigma_{n_n},\sigma_n}(v_n))\end{aligned}$$

where $(v_1, \ldots, v_n) \in \bigoplus_{i=1}^{n} \hat{F}(\sigma_i) = C_n(X;\hat{F})$ and each σ_i is of dimension n.

Proposition 5.12. *This boundary operator ∂ satisfies $\partial^2 = 0$.*

Proof. Let $v \in \hat{F}(\sigma)$ for some simplex σ.

$$\begin{aligned}\partial(v) =&(r_{\partial\sigma_0,\sigma}(v), -r_{\partial\sigma_1,\sigma}(v), \ldots, (-1)^n r_{\partial\sigma_n,\sigma}(v))\\ \partial^2(v) =&(r_{\partial\sigma_{0_0},\partial\sigma_0}(r_{\partial\sigma_0,\sigma}(v)), \ldots, (-1)^{n-1} r_{\partial\sigma_{0_{n-1}},\partial\sigma_0}(r_{\partial\sigma_0,\sigma}(v)),\\ &\vdots\\ &r_{\partial\sigma_{n_0},\partial\sigma_n}(r_{\partial\sigma_n,\sigma}(v)), \ldots, (-1)^{n-1} r_{\partial\sigma_{n_{n-1}},\sigma_n}(r_{\partial\sigma_n,\sigma}(v)))\end{aligned}$$

We know that $r_{\partial\sigma_{i_0},\partial\sigma_i} \circ r_{\partial\sigma_i,\sigma} = r_{\partial\sigma_{i_0},\sigma}$. We can now observe that the first row of $\partial^2(v)$ is nothing but $r_{\partial^2\sigma_0,\sigma}$. Similarly, the ith row is $r_{\partial^2\sigma_{i-1},\sigma}$.

From Proposition 2.8, we know that $\partial^2\sigma = 0$ for all simplices σ and hence the restriction map $r : \hat{F}(\sigma) \to 0$ is 0. Thus, we get that, $\partial^2(v) = 0$ for all $v \in \hat{F}(\sigma)$ for all simplices σ. $\square$

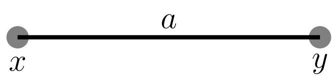
We are now in a position to define simplicial cosheaf homology.

Definition 5.13 (Simplicial cosheaf homology)**.** Let X be a simplicial complex. Let $\hat{F}$ be a cosheaf of vector spaces on X. Let $C_n(X;\hat{F})$ denote the n-th chain group valued in $\hat{F}$. Then, the nth **simplicial cosheaf homology**

is defined as

$$H_n(X; \hat{F}) = \ker\, \partial_n / \mathrm{im}\, \partial_{n+1}.$$

Let us now try to gain some hands-on understanding of how to compute simplicial cosheaf homology by considering three examples. In all these three examples, we shall consider the following simplicial complex, X, which is made up of two 0-simplices, x, y, and one 1-simplex, a. Let k be a field and consider it as a vector space over itself.



Example 5.14 (Closed Interval)**.** Consider the constant cosheaf $\hat{F}$ which assigns to each simplex the vector space k, i.e, $\hat{F}(a) = \hat{F}(x) = \hat{F}(y) = k$.

Consider the chain complex valued in $\hat{F}$

$$\ldots \to 0 \to \hat{F}(a) \to \hat{F}(x) \oplus \hat{F}(y) \to 0.$$

The chain complex looks like this because there are no simplices in dimension 2 or higher. Hence, the corresponding chain groups are trivial.
In this case, the chain complex looks like:

$$\ldots \longrightarrow 0 \xrightarrow{\partial_2} k \xrightarrow{\partial_1} k \oplus k \xrightarrow{\partial_0} 0$$

Now consider,

$$\partial_1 \colon \hat{F}(a) \to \hat{F}(x) \oplus \hat{F}(y)$$

Thus, by applying the definition, we get that

$$\partial_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Thus, we can see that the rank of this matrix is 1 and hence, im $\partial_1 \cong k$.

From the chain complex, we can see that ker $\partial_0 \cong k \oplus k \cong k^2$. Hence, we see that

$$H_0(X; \hat{F}) \cong k^2/k \cong k.$$

Now, using rank-nullity theorem, we observe that ker $\partial_1 = 0$. Hence,

$$H_1(X; \hat{F}) = \text{ker } \partial_1/\text{im } \partial_2 \cong 0/0 \cong 0.$$

Example 5.15 (Half-open interval)**.** Consider the simplicial cosheaf $\hat{F}$ which assigns to x and a the vector space k and 0 to y.

The chain complex in this case would look lke:

$$\ldots \longrightarrow 0 \xrightarrow{\partial_2} k \xrightarrow{\partial_1} k \oplus 0 \xrightarrow{\partial_0} 0$$

Thus, $\partial_1 = [1]$ and hence im $\partial_1 \cong k$. Now, from the chain complex, we can see that ker $\partial_0 = k$. Thus,

$$H_0(X; \hat{F}) \cong k/k = 0.$$

Now, using rank-nullity theorem again, we get that ker $\partial_1 = 0$. Hence,

$$H_1(X; \hat{F}) = 0/0 = 0.$$

Example 5.16 (Open Interval). Consider the simplicial cosheaf $\hat{F}$ which assigns to x and y the 0 vector space and k to a.

The chain complex would look like:

$$\dots \longrightarrow 0 \xrightarrow{\partial_2} k \xrightarrow{\partial_1} 0 \oplus 0 \xrightarrow{\partial_0} 0$$

Thus, in this case, $\partial_1 = 0$, i.e, im $\partial_1 = 0$. Hence,

$$H_0(X; \hat{F}) = 0/0 = 0.$$

Using the rank-nullity theorem, we observe that ker $\partial_1 \cong k$. Thus,

$$H_1(X; \hat{F}) \cong k/0 \cong k.$$

Remark 5.17. From the above three examples, one can make a rather simple observation that H_0 of a closed interval was non-trivial and so was H_1 of an open interval. Infact, this observation is, indeed, a very important one to make and we shall return to this later.

Simplicial cosheaf homology can, infact, be used to define the cosheaf axiom that a pre-cosheaf must satisfy in order for it to be called a cosheaf. Similarly, one could define simplicial sheaf cohomology in the exact similar way and use that to arrive at an alternate way of looking at the sheaf condition.

Let X be a topological space and let $\mathcal{U}$ be a cover for it. Let $\hat{F}$ be a pre-cosheaf of vector spaces over X. Then, we can restrict $\hat{F}$ to those open sets and their intersections appearing $\mathcal{N}\mathcal{U}$ (the nerve of the cover) and define a simplicial cosheaf. Now, we can compute the simplicial cosheaf homology. The zeroth simplicial cosheaf homology can be used to define the cosheaf condition as follows:

Definition 5.18. A pre-cosheaf $\hat{F}$ of vector spaces is a **cosheaf** if for every open set U and every open cover $\mathcal{U}$ of U,

$$\hat{F}(U) \cong H_0(\mathcal{N}\mathcal{U}; \hat{F}).$$

We need to check if the definitions of a cosheaf are consistent and are equivalent. In order to do so, we refer to Definition D.1 in Appendix D. This provides a better working definition of a cosheaf and we see that the equation being exact in Definition D.1 means precisely that

$$\begin{aligned} \hat{F}(U) \cong \ \text{coker } g = & \oplus_\alpha \hat{F}(U_\alpha)/ \text{ im } g \\ = & C_0(\mathcal{N}\mathcal{U}; \hat{F})/ \text{ im } \partial_1 \\ = & \text{ ker } \partial_0 / \text{ im } \partial_1 \\ = & H_0(\mathcal{N}\mathcal{U}; \hat{F}). \end{aligned}$$

Let us look at how we can use this definition to see if a pre-cosheaf is a cosheaf or not.

Let us begin by defining the Leray pre-cosheaf.

Definition 5.19. Let X and Y be topological spaces and let $f \colon X \to Y$ be a continuous function. For each $n \geq 0$, we can define the **Leray pre-cosheaf** as the assignment which, to each open set $U \subset Y$, assigns the ith homology group of the pre-image of U under f, i.e, $U \to H_i(f^{-1}(U))$. We denote the ith Leray pre-cosheaf by $\hat{P}_i$.

Claim - $\hat{P}_1$ is not a cosheaf.

Proof. Consider the standard height function on the circle as in Example 5.9.

Let $\mathcal{U} = U_i$ be a cover of $f(S^1)$ such that S^1 is not completely contained in the pre-image of any one of the U_i's, then,

$$\hat{P}_1(\bigcup U_i) = k \neq 0 = H_0(\mathcal{N}\mathcal{U}; \hat{P}_1)$$

Clearly, $\hat{P}_1(\bigcup U_i) = k$ because,

$$\bigcup U_i = f(S^1) \text{ and } H_1(f^{-1}(f(S^1))) = H_1(S^1) = k$$

However, $H_1(f^{-1}(U_i)) = 0$ for all i since $f(S^1)$ is not entirely contained in any one of the U_i's and thus, each of them is homeomorphic to an interval, which we know has trivial homology.

This leads to the fact that $C_0(\mathcal{N}\mathcal{U}; \hat{P}_1) = 0$ and thus, $H_0(\mathcal{N}\mathcal{U}; \hat{P}_1) = 0$. □

Claim- $\hat{P}_0$ is a cosheaf.

Proof. Let $f\colon X \to Y$ be a continuous function. Let $U = U_1 \cup U_2$. By the continuity of the map and the Mayer-Vietoris long exact sequence in homology, we have that

$$H_0(f^{-1}(U_1 \cap U_2)) \to H_0(f^{-1}(U_1)) \oplus H_0(f^{-1}(U_2)) \to H_0(f^{-1}(U)) \to 0$$

is exact.

$U_1 \cap U_2$ corresponds to the only 1-simplex in $\mathcal{N}\mathcal{U}$ and U_1 and U_2 correspond to the two 0-simplices.

Now, one can see that the first two terms of this sequence correspond to $\hat{P}_0(U_1 \cap U_2)$ and $\hat{P}_0(U_1) \oplus \hat{P}_0(U_2)$ respectively. Furthermore, these are exactly the terms used in computing $H_0(\mathcal{N}\mathcal{U}; \hat{P}_0)$ and the the sequence being exact says that

$$\hat{P}_0(U) = H_0(f^{-1}(U)) \cong (\hat{P}_0(U_1) \oplus \hat{P}_0(U_2))/\operatorname{im} \partial_1 \cong H_0(\mathcal{N}\mathcal{U}; \hat{P}_0)$$

where $\partial_1 \colon \hat{P}_0(U_1 \cap U_2) \to \hat{P}_0(U_1) \oplus \hat{P}_0(U_2)$.

By Theorem D.2, we have that $\hat{P}_0$ is a cosheaf. Hence, $\hat{P}_0$ is a cosheaf while $\hat{P}_1$ is not. □

Let us now return to the remark that we made earlier, Remark 5.17. We saw that when X was a simplicial complex and $\hat{F}$ was a cosheaf over it. Then, $H_0(X; \hat{F})$ counts the closed bars while $H_1(X; \hat{F})$ counts the open bars. By closed and open bars, we mean closed and open intervals like in the series of examples 5.14, 5.15 and 5.16.

However, the following theorem makes this idea precise. This can be found in [4].

Theorem 5.20. *Let $f \colon X \to Y$ be continuous. Let $\mathcal{U}$ be a cover for $f(X) \subset Y$ such that the nerve $\mathcal{N}\mathcal{U}$ is at most one-dimensional, i.e, the nerve has at most 1-simplices. Then, for each $i \geq 0$,*

$$H_i(X) \cong H_0(\mathcal{N}\mathcal{U}; \hat{F}_i) \oplus H_1(\mathcal{N}\mathcal{U}; \hat{F}_{i-1})$$

where $\hat{F}_i$ denotes the assginment $U \mapsto H_i(f^{-1}(U))$.

The proof of this theorem requires Leray Spectral sequences and Remak decomposition of the barcodes among other heavy mathematical machinery and hence is not possible to include in this thesis.

However, let us understand the consequences of this theorem. In essence, this theorem is saying that the homology of a space can be computed using appropriate cosheaf homology.

For our purposes, this theorem implies that the Persistent Homology of a filtered simplicial complex can be computed as some cosheaf homology for an appropriate cosheaf.

Thus, this approach offers a great deal of insight into the multidimensional persistence because this approach circumvents the problem of classification of PIDs over a multivariate polynomial ring.

We have now come to the end of the theoretical part of Persistent Homology. In the next chapter, we shall be looking at an algorithm to compute Persistent Homology. This was one of the first algorithms to appear for computing Persistent Homology.

6. COMPUTING PERSISTENT HOMOLOGY

In the previous chapters, we have seen what Persistent Homology is and what sort of information it captures, about a given dataset. One of the primary reasons for using simplicial homology is that the simplicial complex is a combinatorial object and computing simplicial homology is linear algebra, to be precise, matrix algebra. This information can be fed into a computer and the computer can be used to compute the simplicial homology.

In this chapter, we shall look at one of the first algorithms that appeared in [2] to compute Persistent Homology.

To begin with, we shall look at the standard algorithm to compute simplicial homology.

Since C_k (k-th chain group) is free abelian with the k-simplices forming a basis for it, we represent $\partial_k \colon C_k \to C_{k-1}$ as a matrix, M_k, with respect to the standard bases of the chain groups. Let $\{e_i\}$ denote the basis for C_k and $\{\hat{e_j}\}$ denote the basis for C_{k-1}.

The null space of the matrix corresponds to $Ker(\partial_k)$ (denoted by Z_k) and the image corresponds to $Im(\partial_k)$ (denoted by B_{k-1}).

The matrix is then reduced to its Smith-Normal form by using elementary row and column transformations.

A typical matrix in its Smith-Normal form looks like:

$$\tilde{M_k} = \left[\begin{array}{ccc|c} m_1 & \cdots & 0 & \\ \vdots & \ddots & \vdots & 0 \\ 0 & \cdots & m_{l_k} & \\ \hline & & & \\ & 0 & & 0 \\ & & & \end{array}\right].$$

We can now read off the homology from this matrix as follows:

- The torsion coefficients of H_{k-1} are precisely the diagonal entries which are greater than 1.

  This is true due to the following fact. When the classification of finitely generated modules over a PID is written in its elementary divisor form, precisely those d_i's in M/d_iM (M is a module over some ring R) contribute to the torsion part which are greater than 1.

- $\{e_i | l_k + 1 \leq i \leq m_k\}$ is a basis for Z_k, where m_k denotes the number of k-simplices. Thus, rank $Z_k = m_k - l_k$.

  This is just by the definition of the kernel of a matrix.

- $\{b_j \hat{e_j} | 1 \leq j \leq l_k\}$ is a basis for B_{k-1}. Hence, rankM_k = rank B_{k-1}. Equivalently, rank B_k = rank $M_{k+1} = l_{k+1}$.

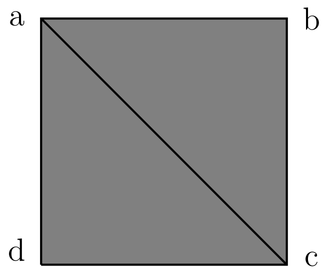
  This is also by definition.

- Hence, β_k = rank Z_k - rank $B_k = m_k - l_k - l_{k+1}$.

We see that we have determined the torsion coefficients and also the k-th Betti number for all k. This completely determines the k-th homology module over $\mathbb{Z}$, which is essentially the k-th homology group.

We will now look at an example.

Example 6.1. Consider the following simplicial complex with 4 vertices, 5 edges and 2 faces.



The standard matrix representation of ∂_1 is

$$M_1 = \begin{array}{cc} & \begin{matrix} ab & bc & cd & ad & ac \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} -1 & 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Upon reduction, the matrix looks like

$$\tilde{M}_1 = \begin{array}{cc} & \begin{matrix} cd & bc & ab & z_1 & z_2 \end{matrix} \\ \begin{matrix} d-c \\ c-b \\ b-a \\ a \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

where $z_1 = ad - bc - cd - ab$, $z_2 = ac - bc - ab$.

Thus, according to the bullet points stated above, z_1, z_2 form a basis for Z_1 and $\{d - c, c - b, b - a\}$ form a basis for B_0.

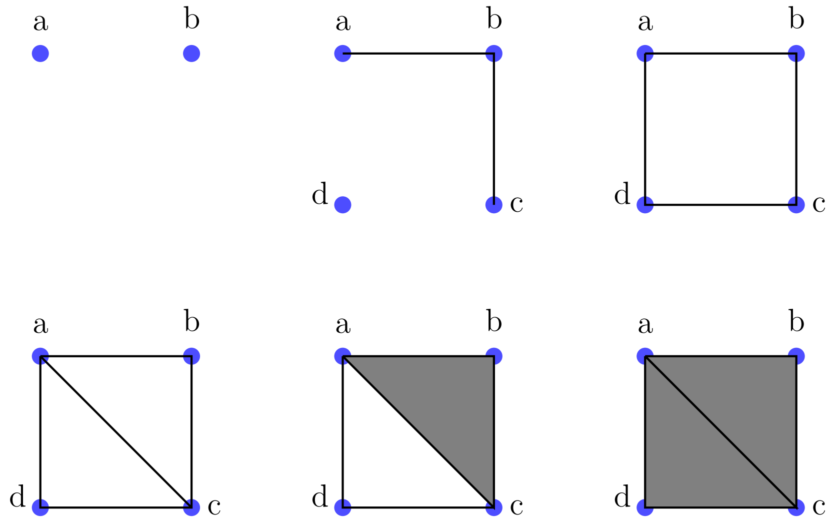
Clearly, $H_0 \cong \ker \partial_0 / \mathrm{im}\ \partial_1 \cong \mathbb{Z}^4/\mathbb{Z}^3 \cong \mathbb{Z}$ which is evident from the simplicial complex as it has one path-connected component.

We can not, however, use the above method to compute Persistent Homology. The issue with using it is that for each value of ϵ, the scale parameter, we would get a chain complex and we would need to apply the above algorithm to compute the homology of the chain complex. In order to compute the Persistent Homology, we would need to repeat the process for all values of ϵ and then find a way to combine the information that we have obtained for each ϵ in order to obtain a comprehensive summary of the dataset.

In [2], the authors have come up with an algorithm which avoids the above problem altogether. The algorithm computes the Persistent Homology, i.e, the homology information for each ϵ, simultaneously. This algorithm makes use of the fact that the category of persistence abelian groups are equivalent to that of graded modules over a graded polynomial ring. Hence, it uses the graded structure to compute the Persistent Homology directly.

Let us now look at the algorithm. In order to elucidate it better, we shall consider a running example.

Consider the following filtered simplicial complex.



To fill in the entries of the matrices of the standard representations of ∂_k's, we fill them up as follows:

$$(M_k)_{ij} = t^{\deg\, e_j - \deg\, \hat{e}_i} \tag{6.1}$$

where degree of a simplex is the step in which it came into existence. E.g, deg $a = 0$, deg $d = 1$, deg $cd = 2$ etc.

This method of filling up the matrix ensures that the simplices are shifted up in grading to the instant when they become the boundary of a higher dimensional simplex. E.g, a becomes the boundary of ab at $t = 1$. However, a has existed since $t = 0$. Thus, the correct equation would be

$$\partial_1(ab) = t \cdot a - t \cdot b$$

and **not**

$$\partial_1(ab) = a - b$$

The standard bases for chain groups are homogenous. The strategy, now, is to represent the matrix of ∂_k with respect to the standard basis for C_k and a homogenous basis for Z_{k-1}. Then, one can reduce the matrix and read off the homology description as done in the case of computing homology above.

This is done inductively in each dimension. The base case is trivial because, $Z_0 = C_0$ and the standard basis can be used for representing ∂_1. We now assume that the above statement is true for $n = k$. We have to prove it for $n = k + 1$, i.e, we have a matrix with respect to the standard basis of C_k and a homogenous basis of Z_{k-1} and we have to compute a homogenous basis for Z_k and represent ∂_{k+1} with respect to the standard basis of C_{k+1} and the computed homogenous basis of Z_k.

The row basis elements are sorted in the descending order according to their degrees. Elementary column operations are used to reduce the matrix to its column-echelon form. Column operations are used so that the homogeneity is maintained in the degrees of row basis elements.

We shall now make an observation in the form of a lemma.

Lemma 6.2. *The pivots in the column-echelon form are the same as the diagonal elements in the Smith-Normal form. Also, the degrees of the elements are also the same.*

Proof. The degrees of row elements decrease from top row to bottom. Within each column, the degree of the column basis element (e_j) is fixed. Hence, due to Equation (6.1), the degree of the elements in a column is increasing with row as deg $(M_k)_{ij}$ = deg e_j - deg $\hat{e_i}$. The non-zero elements below the pivots can be eliminated using row operations that does not change the degree of the pivot row elements. Then the matrix is placed in diagonal form with row and column swaps. This is, in essence, the Smith-Normal form of the matrix. □

We shall use $\mathbb{Z}_2[t]$ module for this running example.

The standard matrix representation of ∂_1 is:

$$M_1 = \left[\begin{array}{c|ccccc} & ab & bc & cd & ad & ac \\ \hline d & 0 & 0 & t & t & 0 \\ c & 0 & 1 & t & 0 & t^2 \\ b & t & t & 0 & 0 & 0 \\ a & t & 0 & 0 & t^2 & t^3 \end{array}\right]$$

The reduced matrix form would be:

$$\tilde{M_1} = \left[\begin{array}{c|ccccc} & cd & bc & ab & z_1 & z_2 \\ \hline d & t & 0 & 0 & 0 & 0 \\ c & 0 & 1 & 0 & 0 & 0 \\ b & 0 & 0 & t & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

where $z_1 = ad - cd - t \cdot bc - t \cdot ab$, $z_2 = ac - t^2 \cdot bc - t^2 \cdot ab$.

In order to read off the Persistent Homology from this matrix, we need the following observation which is a consequence of associating a set of $\mathcal{P}$-intervals to a graded module over a graded polynomial ring, which we have seen in section 3 of chapter 4.

Corollary 6.3. *Let $\tilde{M_k}$ be the reduced matrix of ∂_k relative to the bases of C_k and Z_{k-1}. If the ith row has as pivot $\tilde{M_k}(i,j) = t^n$, then it contributes $\Sigma^{deg\hat{e_i}} F[t]/(t^n)$ to the description of H_{k-1} else it contributes $\Sigma^{deg\hat{e_i}} F[t]$.*

The above statement put in other words, if the ith row has pivot as t^n, then the barcode corresponding to that would be $(deg\hat{e_i}, deg\hat{e_i} + n)$. Else it would be $(deg\hat{e_i}, \infty)$.

We have computed a homogenous basis of Z_k. We, now, have to represent ∂_{k+1} with respect to this basis and the standard basis of C_{k+1}.

We use the fact about the boundary operators that

$$\partial_k \circ \partial_{k+1} = 0.$$

Hence, we should have

$$M_k M_{k+1} = 0.$$

Lemma 6.4. *To compute the standard representation of ∂_{k+1} with respect to the basis of C_{k+1} and the computed basis of Z_k, delete the rows in M_{k+1} that correspond to pivot columns in $\tilde{M}_k$.*

Proof. Since we use elementary column operations to reduce the matrix, either column i is replaced by column j or column i is replaced by column $i +$ n column j. This is equivalent to replacing e_i by e_j or e_i by $e_i + ne_j$.

In order to have the same effect on the row basis for ∂_{k+1}, we would need to replace row j by row j - n row i.

However, row j eventually becomes 0 because of the corresponding column operations on column j and row i is unaffected.

Hence, we can just delete the rows in M_{k+1} that correspond to the pivot columns in M_k and replace the row basis of the remaining rows by the computed basis of Z_k. $\square$

In our example, the representation M_2 of ∂_2 with respect to the standard basis of C_2 and that of C_1 is

$$M_2 = \left[\begin{array}{c|cc} & abc & acd \\ \hline ac & t & t^2 \\ ad & 0 & t^3 \\ cd & 0 & t^3 \\ bc & t^3 & 0 \\ ab & t^3 & 0 \end{array}\right]$$

Using the lemma, the reduced form of M_2,

$$\tilde{M}_2 = \left[\begin{array}{c|cc} & abc & acd \\ \hline z_1 & t & t^2 \\ z_2 & 0 & t^3 \end{array}\right]$$

where z_1, z_2 is the computed basis for Z_1.

We observe from the above discussion that the $\mathcal{P}$-intervals can be calculated for an $F[t]$-module over field F. But, one can do better by just simulating the algorithm over the field without actually computing the $F[t]$-module.

Lemma 6.2 says that if the pivots are eliminated in the order of decreasing degree, then there is no need to compute the Smith-Normal form of the matrix.

Lemma 6.4 tells us that by noting the pivot columns in each dimension and by eliminating the corresponding rows in the next dimension, one can get the required basis change.

Due to these two observations, we don't explicitly need to compute the matrix in each dimension and rather just need column operations. The boundary operators are represented as a set of boundary chains corresponding to the columns of the matrix.

The data structure used in the algorithm is an array T with a slot for each simplex. An ordering is done on all the simplices based on their degree and the simplices with the same degree are arbitrarily assigned the ranks. The algorithm also requires the ability to mark simplices to indicate non-pivot columns.

COMPUTEINTERVALS(K) {
  for $k = 0$ to $\dim(K)$ {
    $L_k = \emptyset$;
  }
  for $j = 0$ to $m - 1$ {
    d=REMOVEPIVOTROWS(σ^j);
    if $d = \emptyset$
      Mark σ^j;
    else {
      $i =$ maxindex d; $k = \dim\sigma^i$;
      Store j and d in $T[i]$;
      $L_k = L_k \cup \{(\deg\, \sigma^i, \deg\, \sigma^j)\}$;
    }
  }
  for $j = 0$ to $m - 1$ {
    if σ^j is marked and $T[j]$ is empty
      $k =$dim σ^j; $L_k = L_k \cup \{(\deg\, \sigma^j, \infty)\}$;
    }
  }
}

```
REMOVEPIVOTROWS(σ) {
  k = dim σ; d = ∂_k σ;
  Remove unmarked terms in d;
  while d ≠ ∅ {
    i = maxindex d;
    if T[i] is empty
      break;

    Let q be the coefficient of σ^i in T[i];
    d = d − q^{-1} T[i];
  }
  return d;
}
```

The above two algorithms have been taken from [2].

COMPUTEINTERVALS computes the barcode for a given filtered simplicial complex. When a simplex σ^j gets added in the filtration, REMOVEPIVOTROWS checks if the boundary chain of σ^j corresponds to a zero column or a pivot column. If $d = \emptyset$, then it means that the boundary chain is empty and hence it corresponds to a zero column in the matrix representation and thus σ^j is marked. Else, the chain corresponds to a pivot column and $i = \text{maxindex } d$ stores the pivot. j and d are stored in $T[i]$ and by Corollary 6.3, we get the $\mathcal{P}$-interval $(\deg \sigma^i, \deg \sigma^j)$ which is stored in L_k for H_k.

Another loop is run to check for infinite $\mathcal{P}$-intervals where σ^j is marked and $T[j]$ is empty. This is essentially to check that the simplex σ^j is not contributing to the death of any homology class.

Let us now turn to the REMOVEPIVOTROWS algorithm. REMOVEPIVOTROWS algorithm computes the boundary chain d for the simplex σ. It then applies Lemma 6.4 to eliminate all terms involving unmarked simplices (which correspond to the pivot columns) and to get a representation in terms of the basis of Z_{k-1}. Then, it performs Gaussian elimination in the decreasing order of degree. $i = \text{maxindex}\ d$ is a potential pivot. If $T[i]$ is non-empty, then a pivot already exists in that row and inverse of its coefficient is used to remove the row from the chain. Else, we have found a pivot and the chain is a pivot column.

In the example that we considered, the marked 0-simplices are a, b, c, d which is not exactly surprising. However, the marked 1-simplices are ad and ac as was evident from the matrix representation of ∂_2 where the rows corresponding to the pivot columns in the previous dimensions were those corresponding to ab, bc and cd. REMOVEPIVOTROWS removes these rows and are not seen in the marked 1-simplices.

These marked simplices generate the set of $\mathcal{P}$-intervals $\{(0, \infty), (0, 1), (1, 1), (1, 2)\}$ for H_0 and $\{(2, 5), (3, 4)\}$ for H_1. This can be seen by applying Corollary 6.3 to the reduced forms of matrices $\tilde{M}_1$ and $\tilde{M}_2$.

This was one of the first algorithms to appear to compute Persistent Homology. Currently, there are libraries like GUDHI and Dionysus in C++ which provide a very efficient method to compute the barcodes. The python Ripser package and the R TDA package make use of such libraries while computing Persistent Homology for a given dataset.

We have seen the algorithm to compute Persistent Homology in this chapter. This completes the study of Persistent Homology, both the theoretical and the practical aspect of it. However, topology can also be used to visualise a dataset. It is often very difficult to get a grip on a high dimensional dataset. Visualising the dataset can prove useful and provide insights into the dataset. Concepts from algebraic topology can be used to visualise a high dimensional dataset as a low dimensional simplicial complex.

In the next chapter, we shall look at a method to do so. This method is called the Mapper algorithm.

7. THE MAPPER ALGORITHM

In the previous chapters, we have given an account of one of the major highlights of topological data analysis, namely Persistent Homology. We saw that Persistent Homology can be used to detect significant topological features that are present in the dataset. The length of the bars in the barcodes can be considered as a parameter to distinguish the true topological features from the transient ones. The common belief is that the longer the bar, more significant is the topological feature.

However, TDA has another important aspect, namely that of data visualisation. Often the data that is available is a very high dimensional one. Many a times, visualising the data gives some prima facie insights into the data which can then be improvised upon for further analysis. Topology can be used to develop a faithful method of visualising the data in 2 or 3 dimensions as is required by the user. This method is known as Mapper.

In this chapter, we shall look at this method Mapper which appeared in [6].

Let us first look at the properties that would be desirable for such a visualisation method:

- **Insensitivity to metric**: This is required because the metrics defined on many of the datasets are not particularly derived from any specific theory but are constructed as an indicator of the intuitive notion of situation the dataset is representative of.

- **Multiscale representations**: It is important to obtain a summary of the dataset at various different resolution levels and the features that are observed at multiple scales are more likely to be features of the dataset than the ones that occur only at certain scales.

Definition 7.1. Let X be a topological space. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be a finite covering of the space X. Let $\Delta[A]$ denote the standard simplex with vertex set A. Further, for any non-empty subset $S \subseteq A$, let $\Delta[S]$ be that face of $\Delta[A]$ that is spanned by the vertices of A present in S. Let $X[S] = \bigcap_{s \in S} U_s \subseteq X$. Then, the **Mayer-Vietoris blowup** of X associated to $\mathcal{U}$ is defined as the subspace

$$\bigcup_{\emptyset \neq S \subseteq A} \Delta[S] \times X[S] \subseteq \Delta[A] \times X.$$

It is denoted by $\mathcal{M}(X,\mathcal{U})$.

Now, there are two projection maps $f \colon \mathcal{M}(X,\mathcal{U}) \to X$ and $g \colon \mathcal{M}(X,\mathcal{U}) \to \Delta[A]$ with the following properties:

- The map f is a homotopy equivalence when X is a finite complex. One can construct an explicit homotopy inverse using partitions of unity subordinate to the cover $\mathcal{U}$ like it is done in the proof of the Theorem A.17.
- We observe that when all the sets $X[S]$ are either empty or contractible, then the image of g is the Čech complex of the cover $\mathcal{U}$, $\check{\mathrm{C}}(\mathcal{U})$, and the map g is a homotopy equivalence onto its image.

Both of these above steps are proved and used in the proof of Theorem A.17.

Hence, one obtains a map from X to $\check{\mathrm{C}}(X,\mathcal{U})$. Such a map can be viewed as a kind of coordinatisation of X. Usual coordinatisations provide maps into Euclidean spaces. However, such coordinatisations implicitly assume some sense of a metric function on X because the metric on the Euclidean space can be pulled back. Undoubtedly, such visualisations are quite important to get an idea of what the dataset is about. But, it would be good to visualise the data without implicitly assuming any metric on the space X.

Hence, the map obtained above from X to $\check{C}(X,\mathcal{U})$ can be used as one such candidate for coordinatisation that doesn't put any metric on the space X. Moreover, low dimensional simplicial complexes can be easily visualised and would provide useful insights into the high dimensional data.

There is a variant of the construction $\check{C}(X,\mathcal{U})$, namely, $\check{C}^{\pi_0}(X,\mathcal{U})$. We shall now understand this construction.

Let X be a topological space. Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ be a cover of X. Now, consider the path components of each open set in this cover and consider the cover $\mathcal{U}'$ which consists of path components of each open set of $\mathcal{U}$. This cover $\mathcal{U}'$ is indexed by $\{(\alpha,\zeta)\}$, where ζ is a path component of U_α.

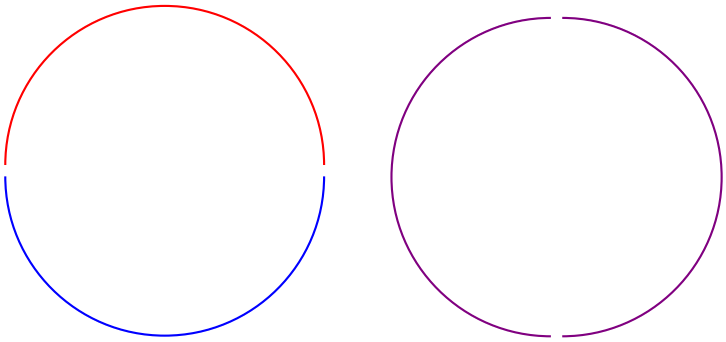
$\check{C}^{\pi_0}(X,\mathcal{U})$ is defined as the nerve of this cover $\mathcal{U}'$.

Clearly, there exits a set map from $\{(\alpha,\zeta)\} \mapsto \alpha$ which maps all the path components, ζ, in one U_α to α itself.

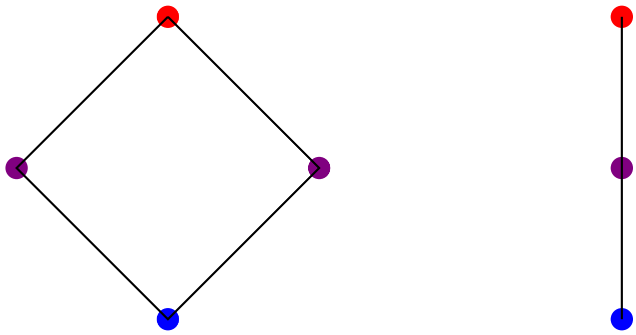
This set map gives rise to a map between the simplicial complexes $\check{C}^{\pi_0}(X,\mathcal{U}) \to \check{C}(X,\mathcal{U})$ because the set map is essentially a map between the vertex set of each simplicial complex and this map can be extended to a map between simplicial complexes.

Let us consider the following example:

Example 7.2. Let X be the unit circle. Let $\mathcal{U}$ be a covering of X by 3 open sets, namely $A = \{(x,y)|y < 0\}$, $B = \{(x,y)|y > 0\}$ and $C = \{(x,y)|y \neq \pm 1\}$.



The simplicial complexes $\check{\mathrm{C}}^{\pi_0}(X,\mathcal{U})$ and $\check{\mathrm{C}}(X,\mathcal{U})$ are given by



We see from the figures that $\check{\mathrm{C}}^{\pi_0}(X,\mathcal{U})$ is homotopic to the circle while $\check{\mathrm{C}}(X,\mathcal{U})$ is not. Hence, $\check{\mathrm{C}}^{\pi_0}(X,\mathcal{U})$ is a slightly more sensitive construction and captures the shape of the data with more fidelity.

The whole construction of $\check{\mathrm{C}}^{\pi_0}(X,\mathcal{U})$ is built on the covering that we considered. Hence, given a topological space, we need to be able to construct coverings for those topological spaces.

One of the well-known ways to do it is to consider a reference map to a metric space and consider a finite cover of the metric space and pull that cover back to cover the given topological space, i.e, a reference map $\rho\colon X \to M$ from the given topological space X to a metric space M. Let $\{U_n\}$ be a cover of M, then $\{\rho^{-1}(U_n)\}$ is a cover of X. Typical examples of metric spaces are $\mathbb{R}$, $\mathbb{R}^n$, S^1.

Example 7.3. Let $X = \mathbb{R}$. We could consider the covering $\mathcal{U}[R, e]$ which consists of intervals of the form $[kR - e, (k+1)R + e]$, $k \in \mathbb{Z}$. This is a two parameter family of covering, the parameters being R and e.

Similarly, one can look at the coverings for S^1.

Example 7.4. Let $X = S^1$, $N \in \mathbb{Z}$ such that $N \geq 2$ and $\epsilon \in \mathbb{R}$. Then the following forms a covering $\mathcal{U}[N, e]$ of S^1

$$U_j = \left\{ (\cos(x), \sin(x)) | x \in \left[\frac{2\pi j}{N} - \epsilon, \frac{2\pi j}{N} + \epsilon \right] \right\}$$

where $0 \leq j < N$ and $\epsilon > \frac{\pi}{N}$.

The above construction is in the setting of topological spaces. However, the starting point of Topological Data Analysis is a point cloud. Thus, we need to be able to transport the construction to the setting of point clouds.

The concepts involved in the above construction, like the construction of the covering, make sense in the setting of a point cloud except the notion of π_0 of a point cloud. Technically speaking, π_0 of a point cloud will have as many components as points in it. Neither is this interesting nor is it informative. Hence, we need an analogous construction in the setting of point clouds.

Thus, we look at **clustering**. There are many ways of clustering. One of the examples is **single-linkage clustering**. This is defined as follows:

Fix a parameter ϵ. The point cloud is then partitioned as the set of equivalence classes under the equivalence relation $\sim_\epsilon$ which is defined by $x \sim_\epsilon x'$ if and only if $d(x, x') \leq \epsilon$.

From the definition, it is clear that the clusters that are obtained after applying single-linkage clustering to a point cloud is the same as the components of the VR(X, ϵ), i.e, π_0 applied to VR(X, ϵ).

We can now define the construction $\check{\mathrm{C}}^{\pi_0}$ in the setting of a point cloud analogous to that in the case of a topological space. It can be done as follows:

1. Define a reference map $f\colon X \to Z$, where Z is a metric space and X is the given point cloud. This map is called the **filter function**.

2. Select a covering $\mathcal{U}$ of Z like in the case of Example 7.3 and Example 7.4.

3. Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$. Construct the subsets $X_\alpha = f^{-1}U_\alpha$ of X.

4. Select a value of ϵ for the single-linkage clustering and construct the clusters for each of the X_α's. Thus, we get a covering of X parametrised by two parameters, (α, c), where $\alpha \in A$ and c is used to denote a cluster in each X_α.

5. Construct the simplicial complex with vertex set as all possible pairs (α, c) where $\{(\alpha_0, c_0), \dots, (\alpha_k, c_k)\}$ spans a k-simplex if and only if those clusters have a point in common.

We can see that this construction depends on the reference map that is selected and also the parameter ϵ. Moreover, it also depends on the clustering scheme that is used for obtaining clusters of each X_α.

One of the important aspects of this construction is that it gives us a multiresolution or multiscale structure which allows us to distinguish the actual features from the transient ones. We shall now understand how does this construction give a multiscale structure.

Definition 7.5. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ and $\mathcal{V} = \{V_\beta\}_{\beta \in B}$ be two coverings of a metric space Z. A **map of coverings** is defined as a set map

$$\theta \colon A \to B \text{ such that } U_\alpha \subseteq V_{\theta(\alpha)} \text{ for all } \alpha \in A.$$

Remark 7.6. Associated to each map of coverings, there is a map between simplicial complexes

$$\mathcal{N}(\theta) \colon \mathcal{N}\mathcal{U} \to \mathcal{N}\mathcal{V}$$

which is defined by sending a vertex of $\mathcal{N}\mathcal{U}$ to a vertex of $\mathcal{N}\mathcal{V}$. This is because the vertex set of $\mathcal{N}\mathcal{U}$ is A and that of $\mathcal{N}\mathcal{V}$ is B and hence, that set map gives a map between the vertex sets of the two simplicial complexes. Also, the condition that $U_{\alpha_i} \subseteq V_{\theta(\alpha_i)}$ gives $\bigcap_{i=1}^{n} U_{\alpha_i} \subseteq \bigcap_{i=1}^{n} V_{\theta(\alpha_i)}$. Thus, this map between the simplicial complexes becomes a simplicial map because, for example, a 1-simplex in $\mathcal{N}\mathcal{U}$ corresponds to a double intersection, say $U_1 \cap U_2$ and that is a subset of $V_{\theta(1)} \cap V_{\theta(2)}$ which is a 1-simplex in $\mathcal{N}\mathcal{V}$.

Example 7.7. Consider the coverings $\mathcal{U}[R, e]$ of $\mathbb{R}$ defined in Example 7.3. The indexing set is the set of integers. Let $\mathcal{U}[R, e]$ and $\mathcal{U}[R, e']$ be two coverings of $\mathbb{R}$ with $e < e'$. Now, since the indexing set is the set of integers for both the coverings, consider the identity map from $\mathbb{Z}$ to itself. Hence, the map of coverings, in this case, consists of the inclusion of an interval into a bigger interval, though with the same center because we haven't changed R in both the coverings. Changing e would just change the interval size. This can be seen from the definition of $\mathcal{U}[R, e]$.

Example 7.8. The set map $k \mapsto \left[\frac{k}{2}\right]$, where $[\cdot]$ denotes the greatest integer function, defines a map of coverings between $\mathcal{U}[R, e]$ and $\mathcal{U}[2R, e]$. This map is a two-to-one map. This is because, under the set map $k \to \left[\frac{k}{2}\right]$, the interval $[kR - e, (k+1)R + e]$ would map to $[\frac{k}{2}R - e, \left(\frac{k+2}{2}\right) R + e]$. Hence, in $\mathcal{U}[2R, e]$, to see what this interval would look like, we need to replace R by $2R$ and by doing so, we get that the interval looks like $[kR - e, (k+2)R + e]$ which essentially contains $[kR - e, (k+1) + e]$ and $[(k+1)R - e, (k+2)R + e]$ in $\mathcal{U}[R, e]$. Hence, it is a two to one map.

Remark 7.9. Thus, if there is a family of coverings $\{\mathcal{U}_i\}_{i=1}^{n}$ and maps of coverings $f_i \colon \mathcal{U}_i \to \mathcal{U}_{i+1}$, then there is a diagram of simplicial complexes as follows:

$$\mathcal{N}\mathcal{U}_1 \xrightarrow{N(f_1)} \mathcal{N}\mathcal{U}_2 \xrightarrow{N(f_2)} \dots \xrightarrow{N(f_{n-1})} \mathcal{N}\mathcal{U}_n.$$

Let X be a point cloud and let a reference map or a filter function $\rho \colon X \to Z$ to a metric space Z be given. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ and $\mathcal{V} = \{V_\beta\}_{\beta \in B}$ be two coverings of Z. Let θ be a map of coverings $\theta \colon A \to B$. Since there is a map of coverings, there are inclusions $\rho^{-1}U_\alpha \subseteq \rho^{-1}V_{\theta(\alpha)}$. This is because there is an inclusion $U_\alpha \subseteq V_{\theta(\alpha)}$. Now, upon applying clustering scheme to these coverings, we get a map from the vertex set of $\check{\mathrm{C}}^{\pi_0}(X, \mathcal{U})$ to that of $\check{\mathrm{C}}^{\pi_0}(X, \mathcal{V})$ which also extends to a simplicial map as seen in Remark 7.6.

All these aspects of this construction give rise to the multiscale or multiresolution structure. Let us consider an example to understand this better.

We have seen in Example 7.7 that there exists a map of coverings between $\mathcal{U}[R, e]$ and $\mathcal{U}[2R, e]$. Thus, drawing on the ideas developed in the above paragraph, we get the following diagram of simplicial complexes:

$$\dots \longrightarrow \check{\mathrm{C}}^{\pi_0}(X, \mathcal{U}[R/4, e]) \longrightarrow \check{\mathrm{C}}^{\pi_0}(X, \mathcal{U}[R/2, e]) \longrightarrow \check{\mathrm{C}}^{\pi_0}(X, \mathcal{U}[R, e]).$$

As we move towards the left in the above diagram, the coverings of $\mathbb{R}$, and hence of X, become more and more refined and give a picture with finer resolution of the space. Studying such diagrams gives one a comprehensive summary at all the scales. This allows one to distinguish between the real features of the dataset and the spurious ones because the belief is that the features which appear at many different resolutions or scales is intrinsic to the data than the ones present at one particular scale.

Thus, topological methods can be used to study the shape of a dataset.

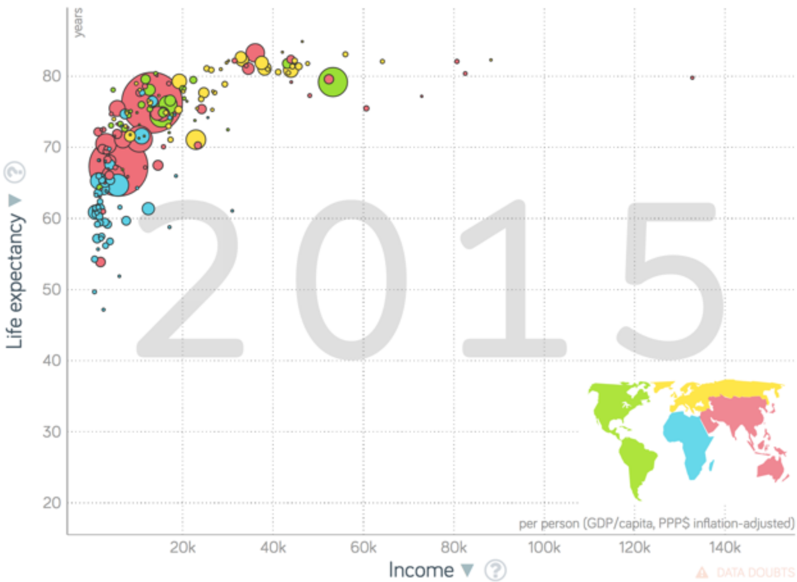
With this, we have completed the study of the two main highlights of TDA, Persistent Homology and Mapper algorithm.

In the next chapter, we shall look at an application of TDA. We shall look at how TDA can be applied to an Indian socio-economic dataset and what results does that give.

8. APPLICATION TO AN INDIAN SOCIO-ECONOMIC DATASET

In this chapter we shall look at an application of TDA. This work was carried out under Dr. Priyavrat Deshpande's supervision at the Chennai Mathematical Institute, India.

In [13], Andrew Banman and Lori Ziegelmeier used persistent homology on a socio-economic dataset of the world. They also studied the effect of incorporating the geography in this analysis. Drawing inspiration from [13], we decided to do a similar analysis in the Indian context. We wanted to use persistent homology to an Indian socio-economic dataset.



The above picture has been taken from [13]. The life expectancy of a person in a country is plotted against his/her per capita income. The countries are coloured broadly based on their geographical location. One can see that the European countries and some of the American countries are at the top of the graph and also rightwards as compared to the other countries. This shows that these countries are performing well with respect to these two parameters. However, the African countries are very close to the Y-axis and also do not have a very high life expectancy. Looking at the figure, one can only speak of broad geographic regions and it is not possible to draw inferences about sub-regions within this broad region. Persistent homology can be used to study the data at multiple scales and seek out the hidden structure present in the data.

In order to understand the method used to carry out the analysis, we would need a couple of definitions.

Definition 8.1 (Socio-economic space)**.** The **socio-economic space** of a socio-economic dataset D, $\mathcal{SE}(D)$, is a finite metric space embedded in a Euclidean space. This metric space inherits the Euclidean metric.

Remark 8.2. In the above definition, we need D to be a dataset consisting of numerical socio-economic variables.

In [13], for one of their analyses, the authors considered a 4-parameter dataset of 179 countries. These four parameters were:

- Life expectancy
- Gross Domestic Product per capita
- Infant mortality rate
- Gross National Income per capita

Hence, the socio-economic space of this dataset consisted of 179 points in $\mathbb{R}^4$ because there are 4 parameters.

Applying persistent homology on this dataset would give results which would be independent of the geographical location of the countries. Since the authors of [13] wanted to perform an analysis which would also be indicative of the geographical location of the countries, they defined a new distance function on the socio-economic space of the dataset.

Definition 8.3. Let D be a socio-economic dataset. Let $d(i, j)$ denote the Euclidean distance between country i and country j in $\mathcal{SE}(D)$. The new distance function is defined as follows:

$$D(i,j) = \begin{cases} d(i,j) & \text{if country } i \text{ and country } j \text{ share a border} \\ \infty & \text{else} \end{cases}$$

For practical purposes, the distance was set at a very high value in order to mimic infinity. Upon applying persistent homology with this new distance function on the socio-economic space, the authors observed that the 1-homology cycles pick up local developmental disparity, i.e, in any given cycle, there is a 'maximal' country which has the greatest value in one or more parameters, a 'minimal' country which has the least values in one or more parameters and the other countries in the cycle exist on a gradient between the two poles.

One of such cycles is the South American cycle as discussed on p12, [13]. This cycle passes through Chile, Peru, Bolivia, Brazil and Argentina. In this cycle, Chile has the greatest values in the Life Expectancy and GDP per capita while Bolivia has the lowest amongst the member countries of this cycle, as can be seen from Table 3, [13].

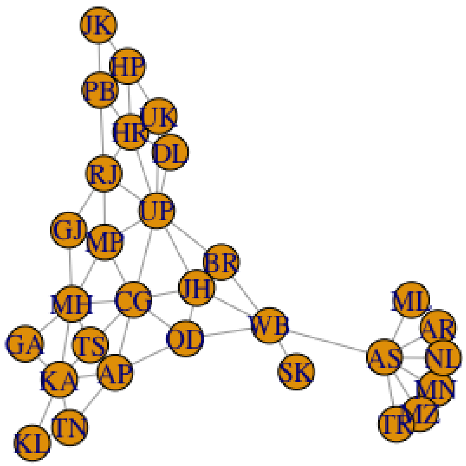
One can now observe, by looking at the definition of this distance function, that this particular distance only considers a country's immediate neighbours and not any of its next neighbours.

However, we believe that the geography has an important role to play and the socio-economic development of a state is affected by the socio-economic development of its neighbours and next neighbours.

In order to account for this, we defined a different distance function. Before we look at the definition of the new distance function, we will look at the definition of the graph of a region.

Definition 8.4. Let R denote a region, such as the world or a country. Let R be subdivided into s subregions, countries or states respectively. Then, by **graph of region**, $G(R)$, we mean a graph which has s vertices, which correspond to the s subregions, and there is an edge between two vertices if the corresponding subregions share a boundary in R.

Example 8.5.



In the above picture, we see the example of $G(R)$ where R is the country of India and the subregions are the states of India. The two letters used by the Regional Transport Offices to identify the states have been used to denote those states respectively, in the figure.

Now, let us look at the distance function that we defined.

Definition 8.6. Let R be a region. Let it be divided into s subregions $\{R_1, \ldots, R_s\}$. Let D be a socio-economic dataset of R consisting of socio-economic parameter values of these s subregions. Let $d(R_i, R_j)$ denote the Euclidean distance between R_i and R_j in $\mathcal{SE}(D)$. Let $G(R)$ be the graph of this region R with vertices $\{v_0, \ldots, v_s\}$ corresponding to the s subregions $\{R_1, \ldots, R_s\}$. Let $G(R_i, R_j)$ denote the graph distance between R_i and R_j, i.e, the value of $G(R_i, R_j)$ is the minimum number of steps needed to reach the vertex v_j when starting from the vertex v_i in $G(R)$. Then the distance is defined as follows:

$$D'(R_i, R_j) = G(R_i, R_j) \cdot d(R_i, R_j)$$

where $1 \leq i, j \leq s$.

This distance function gradually decreases the impact of the nth neighbours of a state as n increases because the graph distance also keeps on increasing. Thus, those states are being pushed further away in the socio-economic space according to this new distance function.

We created our own dataset which contains state-wise information about 9 socio-economic parameters. Since the numerical value of parameters had varying orders of magnitude, we scaled the data between 0 to 1 for each parameter with 0 being the minimum among the states for that parameter and 1 being the maximum. The 9 parameters we considered were as follows:

- Per capita income - Source - [18]

- Infant mortality rate - Source - [15]
- Road development - Source - [17]. The data that we have is the total kilometres of road present in each state. This is not a fair measure to compare states because the area of states is different and the bigger the state the more kilometres of road it would have. Dividing the kilometres of roads by the area of the state is not a very good solution because the shape of the states is not similar. For e.g, some state might be in the form of roughly a circle and some in the shape of an oval and have the same area. But the kilometres of roads would be different because of the shape of the state. Instead, we decided to divide the kilometres of roads by the number of Parliamentary Constituencies in that state as this would be a measure of road density.
- Sex ratio - Source - [15]
- Gross enrolment ratio - Source - [16]. We took the average of Rural Gross Enrolment Ratio and Urban Gross Enrolment Ratio.
- % of households with clean fuel for cooking - Source - [15]
- % of households with clean drinking water - Source - [15]
- % of households with sanitation facilities - Source - [15]
- % of households with electricity - Source - [15]

We chose these parameters so that our analysis is indicative of the overall socio-economic development of any given state and also the availability of data for the years $2015 - 2016$.

Thus, we have 9 parameters for a given state and there are 29 states in India. We also considered the NCR region in our analysis and therefore, we have 30 points in the socio-economic space which is a subset of $\mathbb{R}^9$ as there are 9 parameters.

Taking a closer look at these 9 parameters made us realise that about 5 of them are related to the health of individuals in a state and 2 of them related to the infrastructure present in the state. If all these 9 parameters are given equal weight, the health sector, as a whole, would get a lot of importance while the education and the wealth aspect of development would be undermined.

Thus, we decided to combine the health parameters into one index, called the health index. The following were combined to form the health index:

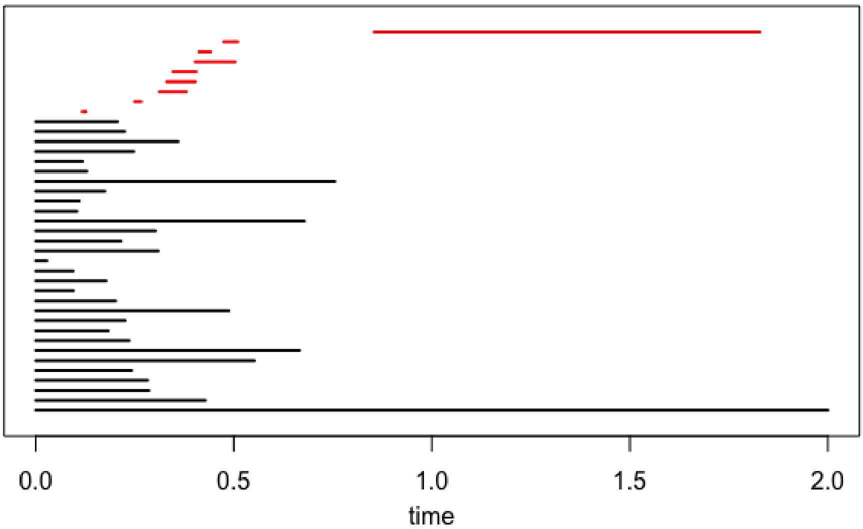
- Infant mortality rate
- % of households with clean fuel for cooking
- % of households with clean drinking water
- % of households with sanitation facilities
- Sex ratio

Similarly, for the infrastructure index we combined

- % of households with electricity
- Road development

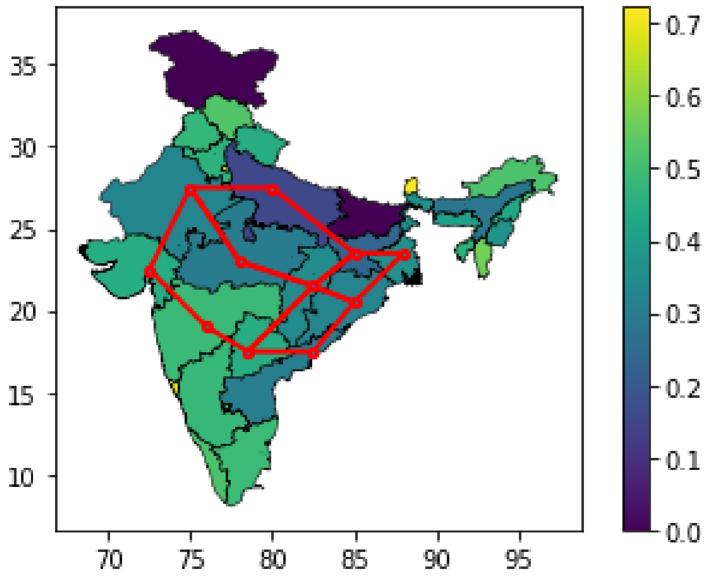
Per capita income was the sole contributor to the wealth index and gross enrolment ratio was the sole contributor to the education index. We used arithmetic mean as the aggregation function for combining the different parameters into one index.

Thus, for each state, we had 4 indices and hence our socio-economic space ended up being a subset of $\mathbb{R}^4$ having 30 points. However, the metric on the socio-economic space is not the Euclidean metric. It is the metric defined in Definition 8.6 because this metric allows for the geography also to play a role in the persistent homology analysis. We used the R TDA package to construct the Vietoris-Rips complex and compute the barcodes.



The black bars represent H_0 while the red bars represent H_1, i.e, the black bars have information about the connected components while the red bars have information about the 1-cycles.

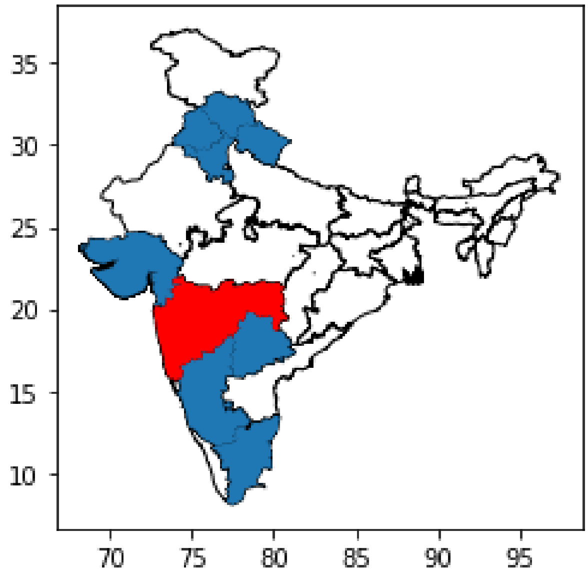
The long red bar visible in this barcode diagram is due to the topology of the graph and is not a feature of the dataset. The shorter bars are the local developmental disparity cycles. For example, the third red bar from the bottom passes through Chhattisgarh, Telangana, Maharashtra, Gujarat, Rajasthan and Madhya Pradesh. If we now look at the data points, we observe that Maharashtra has the maximum wealth index, infrastructure index and health index of all these states while Madhya Pradesh has the minimum wealth index and health index of all these states and the other states in this cycle have their values between these values.

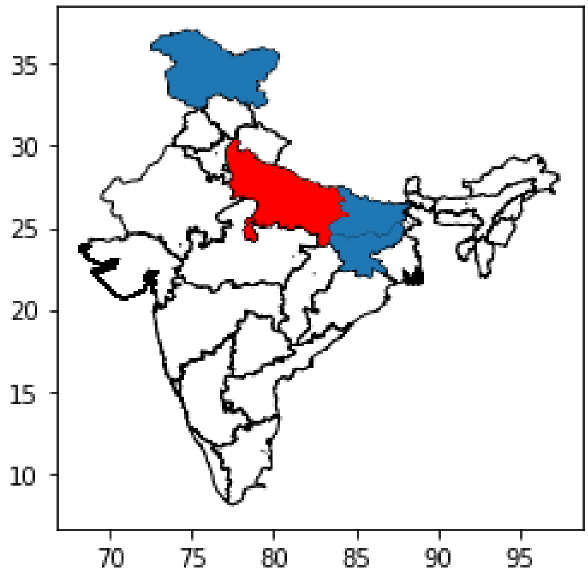


The four 1-cycles have been plotted on the map of India in the above picture. The colour shading is done on by aggregating the four indices by taking their geometric mean, as it is done while calculating the Human Development Index.

Recall the method of constructing the Vietoris-Rips complex from Definition 4.6. We can see that a 1-simplex is drawn between two 0-simplices as soon as the radius of the balls in the open cover reaches half the distance between the two points. As soon as this occurs, a bar dies in the barcode because, two different connected components have merged into one and only one bar continues in the diagram. Now, one also observes that the earlier in the diagram the bar dies the closer the points are, in the point cloud. Thus, in our case, this implies that the states, to which those points correspond, are similar, from a socio-economic perspective.

However, in order to look at the similarity between the states, one has to consider the Euclidean distance on the socio-economic space. We created the Vietoris-Rips filtration of the socio-economic space of the 4-indices dataset that we had and stored the order of appearance of simplices and the ϵ values at which they appear. We, then, calculated a similarity score between any two given states. This score is essentially the value of ϵ at which an edge between those two states is formed. Thus, the lesser the value the more similar the states are. Then, we created a function named 'Similar_States' which takes in a state name and a value as the two inputs and gives the map of India as the output with those states shaded which have a similarity score less than the input value with the input state name.

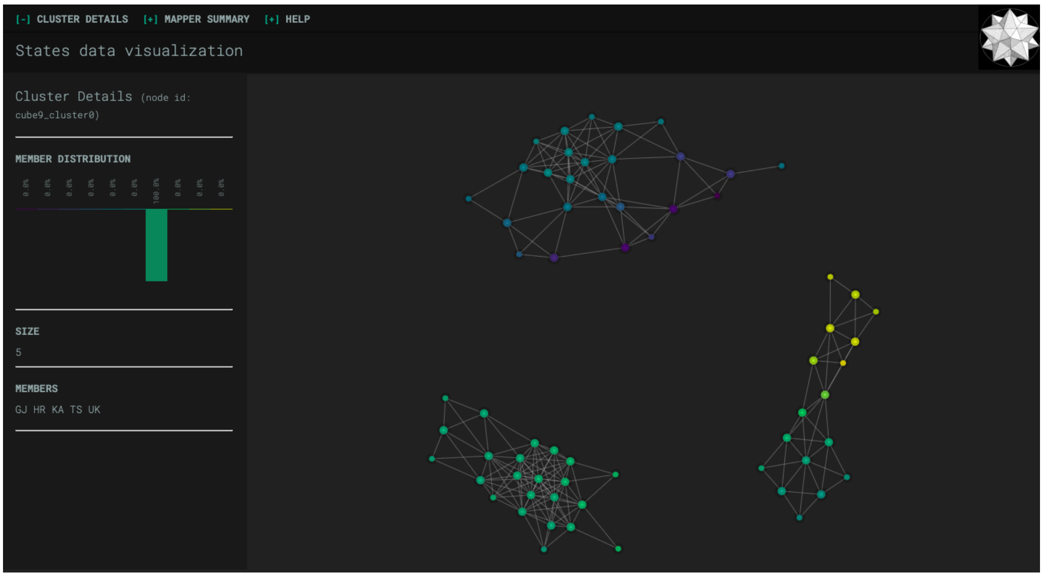




These two pictures show the output of the function' Similar_States'. The

'Similar_States' function took inputs as 'Maharashtra' and '0.2' and gave the first picture as output. Similarly, feeding in 'Uttar Pradesh' and '0.4' gave the second picture. From the pictures, it is clear that Karnataka, Telangana, Tamil Nadu, Gujarat, Punjab, Haryana, Himachal Pradesh and Uttarakhand have a similarity score less than 0.2 with Maharashtra which means that they are similar to Maharashtra. The second picture says that Bihar, Jammu and Kashmir and Jharkhand are more similar to Uttar Pradesh than the other states.

We, now, present the output of the Mapper algorithm applied to our dataset.



We used the Kepler Mapper module in Python to obtain this output. We used UMAP, [23], as the filter function. UMAP is a dimensionality reduction technique. The reader is referred to [24] for more details about this technique. We used 10 intervals to cover the range with a 60% overlap. We used DBSCAN, which is a function in the sklearn module in Python, as the clustering scheme. In the figure, we mainly see three distinct clusters. The colouring of the nodes is done based on the geometric mean of the 4 indices that we had with the colours ranging from purple for the lowest to yellow for the highest. As we can see, there are three distinct clusters in the image. The members of a cluster tend to have a similar colour shade as that of the other members of the same cluster.

Interested readers can refer to the author's GitHub page for the outputs of the Mapper algorithm for different filter functions.
https://github.com/samagashreyas/TDA_Indian_Socioeconomic_Dataset.
The GitHub page also contains the codes for the other algorithms used to obtain the various outputs discussed in this chapter.

With this, we come to the end of this chapter. In the next chapter, we shall briefly recapitulate the contents of this thesis and conclude by stating a few other applications of TDA.

9. CONCLUDING REMARKS

We started this thesis by covering the basics required to understand the basics of TDA which included simplicial homology and some basic category theory in Chapter 2 and Chapter 3 respectively.

In Chapter 4, we learnt about Persistent Homology and its mathematical foundations. We also saw the reason the machinery of single-dimensional persistence could not be generalised to higher dimensions. Subsequently, in Chapter 5, we saw the (co)sheaf theoretic approach to TDA and saw how it can potentially be used to circumvent the problem of multidimensional persistence.

In Chapter 6, we looked at an algorithm to compute Persistent Homology. Moving on, in Chapter 7, we looked at another aspect of TDA, namely the visualisation of a dataset. We learnt about an algorithm, the Mapper algorithm, which is widely used for this purpose. This algorithm also has its roots in algebraic topology, as we saw in that chapter. Finally, we looked at an application of the theory that we had learnt to an Indian socio-economic dataset in Chapter 8. This marked the end of our TDA journey.

Though to obtain a complete understanding of the underlying concepts involved in TDA would take quite some effort, using the packages such as TDA in R or Ripser in Python is not as difficult. Using the R TDA package, especially, is very straightforward and is accessible to non-mathematicians as well. One just needs to understand the construction of the Vietoris-Rips complex and the interpretation of barcodes. Upon gaining this knowledge, one can easily use such packages to do some interesting preliminary analysis.

However, to harness the full power and potential of such packages, one should be well-versed with other simplicial complex constructions and should have some domain knowledge about the problem in consideration so that one can use the construction that would be appropriate for that particular problem.

This was just one application of TDA. In the recent times, TDA is being used on various datasets, ranging from financial datasets to cancer datasets and even astrophysical datasets. TDA is useful in detecting the peculiarities attached to the shape of any dataset. More often than not, such peculiarities give a lot of information. For example, in the analysis of the stock market data, the topology of the dataset changes when the market is nearing a crash as demonstrated in [14].

We sincerely hope that the reader has developed an interest towards topological data analysis and also understood the basics, thus, fulfilling the aim of providing this exposition on TDA.

Appendices

Appendix A

We shall now look at a proof of Theorem 4.18, also known as the Nerve Theorem. Before we get to the theorem and its proof, we need to understand a few definitions and constructions.

Definition A.1. A **diagram of spaces** consists of an oriented graph Γ with a space X_v for each vertex v and a map $f_e \colon X_v \to X_w$ for each edge e between v and w.

Definition A.2. Let X denote a diagram of spaces over a graph Γ. Then, we define **amalgamation of diagram** X as

$$\amalg X = \bigsqcup_{v \in \Gamma} X_v / \sim$$

where the relation is given by $x \sim f_e(x)$.

Example A.3. If the diagram of spaces has the following form

$$X_0 \xleftarrow{f} A \hookrightarrow X_1$$

then $\amalg X$ is the space $X_0 \bigsqcup_f X_1$ obtained from X_0 by attaching X_1 via f.

Example A.4. If $\mathcal{U} = \{U_i\}$ is a cover of a space X by subspaces U_i, we can form a diagram of spaces $\amalg X_{\mathcal{U}}$ whose vertices are non-empty finite intersections $U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_k}$ and edges are inclusions of these sets into those obtained by omitting some of the subspaces in the intersection.

Remark A.5. One can see that if the subsets U_i are open subsets of X, then $\mathcal{U}$ will be an open cover and $\amalg X_{\mathcal{U}}$ would then correspond to the 1-skeleton of the nerve of the cover $\mathcal{N}\mathcal{U}$.

Definition A.6. Given a map $f\colon X \to Y$, the mapping cylinder of this map, M_f is defined as the quotient space of the disjoint union of $(X \times I) \sqcup Y$ obtained by identifying $(x, 1) \in X \times I$ to $f(x) \in Y$.

Definition A.7. Now, we can define the **realisation of** X by filling in the mapping cylinder M_f for each map f in the diagram instead of passing to the quotient space. By doing so, note that we are enlarging the space rather than shrinking it.

Definition A.8. One can generalise this definition and define **complex of spaces**. In this case, we start with a Δ complex. We associate to its $1-$ skeleton, a diagram of spaces such that the maps corresponding to each simplex form a commutative diagram.

Thus, if we now have a complex of spaces X, then for each n-simplex of Γ, we have a sequence of maps

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \ldots \xrightarrow{f_n} X_n.$$

We now define the iterated mapping cylinder $M(f_1, \ldots, f_n)$ inductively as follows:

$M(f_1, f_2)$ is the mapping cylinder of the composition

$$M(f_1) \xrightarrow{p} X_1 \xrightarrow{f_2} X_2$$

Similarly, for $n > 2$, $M(f_1, f_2, \ldots, f_n)$ is the mapping cylinder of the composition

$$M(f_1, \ldots, f_{n-1}) \xrightarrow{p} X_{n-1} \xrightarrow{f_n} X_n.$$

We see that there exists a natural projection from $M(f_1)$ onto Δ^1 because

$$M(f_1) = (X_0 \times I) \sqcup X_1 / \sim .$$

We see that I in $M(f_1)$ is nothing but Δ^1. Similarly, we have a natural projection map from $M(f_1, \ldots, f_n)$ onto Δ^n and on each face of Δ^n, one has the iterated mapping cylinder for the maps associated to the edges of the face.

All these mapping cylinders fit together to form a space ΔX with a canonical projection $\Delta X \to \Gamma$. ΔX is called the **realisation of the complex of spaces** and Γ is called the **base of X**.

Remark A.9. We have seen in Example A.4 that the diagram of spaces for an open cover $\mathcal{U}$ of a space X is a graph whose vertices are finite intersections of the members of $\mathcal{U}$ and the edges are inclusions. This diagram of spaces can be considered as a complex of spaces with n-simplices and $n-$ fold inclusions with the base Γ as the barycentric sub-division of the nerve of the cover.

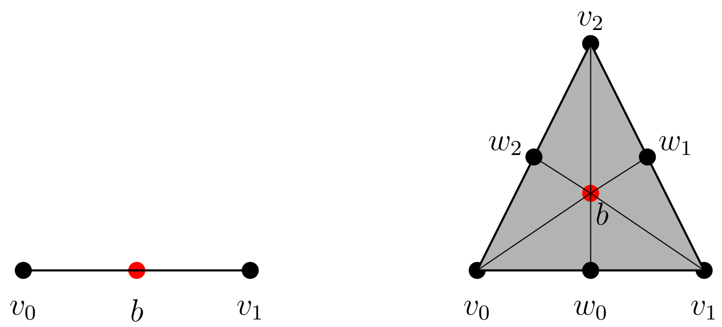
Let us look at the definition of barycentric subdivision.

Definition A.10. We recall the definition of a standard simplex, Definition 2.10. The **barycenter** of a simplex is defined as

$$b = \sum_{i=0}^{n} t_i v_i \text{ such that all } t_i\text{'s are equal, i.e, } t_i = \left(\frac{1}{n+1}\right) \text{ for all } 0 \leq i \leq n.$$

Definition A.11. **Barycentric subdivision** of a simplex $[v_0, v_1, \ldots, v_n]$ is the decomposition of $[v_0, v_1, \ldots, v_n]$ into n-simplices $[b, w_0, w_1, \ldots, w_{n-1}]$ where, inductively, $[w_0, w_1, \ldots, w_{n-1}]$ is an $n-1$ simplex in the barycentric subdivision of a face $[v_0, v_1, \ldots, \hat{v}_i, \ldots, v_n]$.

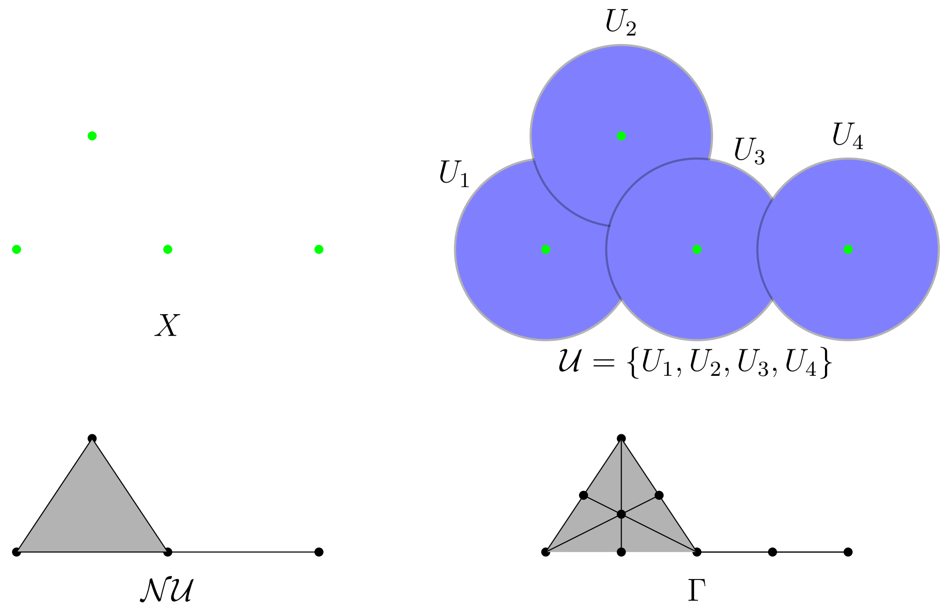
Example A.12.



In the above pictures, the red dot is the barycenter. We can get an idea, by looking at the second figure, of the induction step present in the definition. w_i's are the barycenters of the respective 1-simplices and b is the barycenter of the 2-simplex.

In order to understand Remark A.9 better, let us consider the following example:

Example A.13.



In the above pictures, X is a four-point space and we consider the cover $\mathcal{U}$ as shown.

According to the definition of the base Γ, it is constructed as follows. The point in the interior of the 2-simplex corresponds to the triple intersection $U_1 \cap U_2 \cap U_3$, the mid points of the 1-simplices correspond to the respective double intersections and the edges in the base Γ correspond to the inclusion of the open sets. E.g, the triple intersection includes into $U_1 \cap U_2$, $U_2 \cap U_3$, $U_3 \cap U_1$, U_1, U_2 and U_3. Hence, we see that there are 6 edges emerging (or going into) the point that corresponds to the triple intersection.

We clearly see that the base Γ is a barycentric subdivision of the nerve of the cover.

We shall now define what a locally finite cover is and then we shall state and look at a proof of the Nerve theorem.

Definition A.14. An open cover $\mathcal{U}$ of X is called **locally finite** if for all $x \in X$, there exists an open set U_x such that U_x intersects only finitely many members of $\mathcal{U}$.

Definition A.15. Let $\mathcal{U}$ and $\mathcal{V}$ be open covers of X. $\mathcal{V} = \{V_\beta\}$ is called a **refinement** of $\mathcal{U} = \{U_\alpha\}$ if for all $V_\beta \in \mathcal{V}$, there exists $U_\alpha \in \mathcal{U}$ such that $V_\beta \subseteq U_\alpha$.

Definition A.16. Let X be a topological space. X is called paracompact if every open cover has a locally finite open refinement.

Theorem A.17. *Let X be a paracompact topological space. Let $\mathcal{U} = \{X_i\}$ be an open cover of X. Let $X_\mathcal{U}$ denote the complex of spaces associated to $\mathcal{U}$. Then, the map $p\colon \Delta X_\mathcal{U} \to \amalg X_\mathcal{U}$ is a homotopy equivalence.*

Proof. We first observe that

$$\amalg\, X_\mathcal{U} = X \tag{A.1}$$

because $\mathcal{U}$ is an open cover and by the definition of amalgamation

$$\amalg X = \bigsqcup_{v \in \Gamma} X_v / \sim .$$

Here, the maps are the inclusion maps and taking the union of all X_v's amounts to taking the union of all the sets in the open cover which essentially gives us X.

Now, the realisation $\Delta X_\mathcal{U}$ can be thought of as the quotient space of the disjoint union of all the products

$$X_{i_0} \cap X_{i_1} \cap \ldots \cap X_{i_k} \times \Delta^k$$

with the identifications over the faces of Δ^k using inclusions

$$X_{i_0} \cap \ldots \cap X_{i_k} \hookrightarrow X_{i_0} \cap \ldots \cap \hat{X}_{i_j} \cap \ldots \cap X_{i_k}.$$

To get a grip on what the above statement means, we shall now consider a 2-simplex and the associated diagram of spaces.

Consider a 2 simplex $= [v_0, v_1, v_2]$. We associate the open set $X_0 \cap X_1 \cap X_2$ to v_0, $X_0 \cap X_1$ to v_1 and X_0 to v_2. To the three edges of the simplex, we associate the inclusion maps, i_1, i_2 and i_3 respectively. Now,

$$M(i_1) = (((X_0 \cap X_1 \cap X_2) \times I) \sqcup X_0 \cap X_1)/\sim$$

We observe that maps associated to the edges are inclusion maps and hence the relation would essentially mean that we identify $x \in X_0 \cap X_1 \cap X_2$ to $x \in X_0 \cap X_1$.

$$M(i_1, i_2) = ((((((X_0 \cap X_1 \cap X_2) \times I) \sqcup X_0 \cap X_1)/\sim) \times I) \sqcup X_0)/\sim$$

Using the above observation, we can simplify the above expression as

$$M(i_1, i_2) = (X_0 \cap X_1 \cap X_2) \times I^2 \sqcup (X_0 \cap X_1) \times I \sqcup X_0$$

We now observe that I^2 is the union of two Δ^2's because a square can be decomposed as two triangles.

Hence, we see that $M(i_1, i_2)$ can be seen as the union of $X_0 \cap X_1 \cap X_2 \times \Delta^2$, $X_0 \cap X_1 \times \Delta^1$ and $X_0 \times \Delta^0$. This is essentially what the statement above is saying. It works similarly in higher dimensions as well.

From this viewpoint, the points of $\Delta X_{\mathcal{U}}$ in a given 'fiber' $p^{-1}(x)$ can be written as finite linear combinations $\sum_i t_i x_i$ where $\sum_i t_i = 1$ and x_i is x regarded as a point of X_i for those X_i's that contain x.

Since X is paracompact, there exists a partition of unity subordinate to $\mathcal{U}$, i.e, a collection of maps $\phi_\alpha \colon X \to [0,1]$ such that

1. $\mathrm{supp}(\phi_\alpha) \subseteq X_{i_\alpha}$
2. only finitely any ϕ_α's are non-zero at each point of $x \in X$
3. $\sum_\alpha \phi_\alpha(x) = 1$ for all $x \in X$.

Now, define a section $s \colon X \to \Delta X_{\mathcal{U}}$ of p defined as

$$s(x) = \sum_\alpha \phi_\alpha(x) x_{i_\alpha}.$$

We observe that s embeds X as a retract of $\Delta X_{\mathcal{U}}$. This is because the canonical map $\Delta X_{\mathcal{U}} \to \amalg X_{\mathcal{U}} = X$ can be composed with the retractions of the mapping cylinders onto their target ends.

This is also a deformation retract because the points in the fiber $p^{-1}(x)$ can move linearly along line segments to $s(x)$. This is because of the first observation we made that $\Delta X_{\mathcal{U}}$ can be thought of as made up of product of n-simplices and finite intersections of X_i's along with appropriate identifications.

Since, it is a deformation retract, $\Delta X_{\mathcal{U}} \simeq \amalg X_{\mathcal{U}} = X$. $\square$

Corollary A.18 (Nerve Theorem)**.** *Let X be a paracompact topological space. Let $\mathcal{U} = \{X_i\}$ be an open cover of X. If the cover is such that the non-empty finite intersections are contractible, then $\mathcal{N}\mathcal{U} \simeq X$.*

Proof. By the previous theorem, we know that $\Delta X_{\mathcal{U}} \simeq X$.

Now, since each of the non-empty finite intersection is contractible, the map which contracts each of the space in the mapping question to a point is a homotopy equivalence and upon doing so, we will just get the 1-skeleton, i.e, the base Γ.

We have seen, in Remark A.9, that the base of $\Delta X_{\mathcal{U}}$ is the barycentric subdivision of the nerve of the cover $\mathcal{NU}$.

By the definition of barycentric subdivision, it is clear that the barycentric subdivision of a simplex is homotopic to the simplex.

Thus, we get that $\Gamma \simeq \mathcal{NU}$.

Combining all the observations, we get that

$$\mathcal{NU} \simeq \Gamma \simeq \Delta X_{\mathcal{U}} \simeq X.$$

$\square$

Appendix B

We shall look at a proof of Theorem 4.19 in this chapter. We shall need to understand a few concepts in order to do so. Let us begin by understanding the Edge Path Group of a Simplicial Complex.

X denotes a simplicial complex in this chapter.

Definition B.1. An **edge** is an ordered pair of vertices (v, v') which belong to some simplex σ of X. v is called the **origin** of the edge and v' is called the **end** of the edge.

Definition B.2. Edge Path z of X is a finite non-empty sequence of edges $e_1e_2\dots e_n$ of X such that end $e_i =$ orig e_{i+1}.

If $z_1 = e_1 \dots e_k$ and $z_2 = f_1 \dots f_l$ such that end $e_k =$ orig f_1, then

$$z_1z_2 = e_1 \dots e_k f_1 \dots f_l.$$

We can now see that

$$z_1(z_2z_3) = (z_1z_2)z_3.$$

Thus, we see that the operation of concatenation of edge paths forms an associative operation on the collection of edge paths of a simplicial complex. However, there are no left or right identity elements. Therefore, to obtain a category, we shall now look at an equivalence relation (analogous to the equivalence relation defined on the collection of paths between two points while defining the fundamental group of a space).

Definition B.3. Two edge paths z, z' are **simply equivalent** if there exist vertices v, v', v'' belonging to some simplex σ of X such that the unordered pair $\{z, z'\}$ equals one of the following:

1. Unordered pair $\{(v, v''), (v, v')(v', v'')\}$
2. Unordered pair $\{z_1(v, v''), z_1(v, v')(v', v'')\}$ for some edge path z_1 in X with end $z_1 = v$
3. Unordered pair $\{(v, v'')z_2, (v, v')(v', v'')z_2\}$ for some edge path z_2 in X with orig $z_2 = v''$
4. combination of (2) and (3)

Definition B.4. Two edge paths z and z' are said to be **equivalent** if there exists a finite sequence of edge paths $z_0 \dots z_n$ such that $z = z_0$, $z' = z_n$ and z_{i-1}, z_i are simply equivalent.

The relation defined above satisfies reflexivity trivially.

For symmetry, if $z \sim z'$, there exists a finite sequence of edge paths $z_0 \dots z_n$ such that $z_0 = z$, $z' = z_n$ and z_{i-1}, z_i are simply equivalent. Since the definition of simply equivalent considers only unordered pairs, if z_{i-1}, zi are simply equivalent, then so are z_i and z_{i-1}. Hence, by letting $w_i = z_{n-i}$, we have a sequence of edge paths $w_0 \dots w_n$ such that $w_0 = z'$, $w_n = z$ and w_{i-1}, w_i are simply equivalent.

For transitivity, if $z \sim z'$ and $z' \sim z''$, there exist finite sequences of edge paths $z_0 \dots z_n$ such that $z_0 = z$, $z' = z_n$ and z_{i-1}, z_i are simply equivalent and $z'_0 \dots z'_m$ such that $z'_0 = z'$, $z'' = z'_m$ and z'_{i-1}, z'_i are simply equivalent. Thus, if we consider the sequence $z_0 \dots z_n \dots z'_0 \dots z'_m$, we see that $z \sim z''$.

Thus the relation defined is an equivalence relation. Since it is so, we observe the following points:

1. $z \sim z' \Rightarrow \text{orig } z = \text{orig } z'$ and $\text{end } z = \text{end } z'$

2. $z_1 \sim z'_1$, $z_2 \sim z'_2$ and $\text{end } z_1 = \text{orig } z_2 \Rightarrow z_1 z_2 \sim z'_1 z'_2$

3. if $\text{orig } z = v_1$ and $\text{end } z = v_2$, then $(v_1, v_1)z \sim z \sim z(v_2, v_2)$.

If z is an edge path, let $[z]$ denote its equivalence class. Then, from (1) above, it follows that orig $[z]$ and end $[z]$ are well defined. From (2), it follows that $[z_1] \circ [z_2] = [z_1 z_2]$ is well defined.

We have the following theorem:

Theorem B.5. *There is a category $Z(X)$ whose objects are vertices of X and morphisms v_1 to v_0 are the equivalence classes $[z]$ with orig $[z] = v_0$, end $[z] = v_1$ and composite is $[z_1] \circ [z_2]$.*

We now see why $Z(X)$ is a groupoid, i.e, a category where all the morphisms are invertible.

If $e = (v, v')$ is an edge, define $e^{-1} = (v', v)$.

For an edge path $z = e_1 \dots e_r$ is an edge path in X, define $z^{-1} = e_r^{-1} \dots e_1^{-1}$.

We can now see that

1. $(z^{-1})^{-1} = z$

2. $\text{orig } z^{-1} = \text{end } z$

3. $z_1 \sim z_2 \Rightarrow z_1^{-1} \sim z_2^{-1}$

4. orig $z = v_1$ and end $z = v_2 \Rightarrow zz^{-1} \sim (v_1, v_1)$ and $z^{-1}z \sim (v_2, v_2)$

Hence, we see that $[z]^{-1} = [z^{-1}]$.

Thus, $Z(X)$ is a groupoid called the **edge path groupoid** of X.

If v_0 is a vertex of X, with the operation $[z] \circ [z']$ on the collection of elements with start and end at v_0, we get a group denoted by $E(X, v_0)$. This group is called the **edge path group** of X with base vertex v_0.

Now, we can compare $E(X, v_0)$ with $\pi_1(|X|, v_0)$ as follows:

Let $[z] \in E(X, v_0)$ and let $z = e_1 \ldots e_r$.

Let I be the unit interval and let I_r denote the subdivision of I into r subintervals.

$$I_r = \left\{ \frac{i}{r} | 0 \leq i \leq r \right\} \cup \left\{ \left(\frac{i-1}{r}, \frac{i}{r} \right) | 1 \leq i \leq r \right\}.$$

I_r is a simplicial complex made of 0-cells and 1-cells.

Let $\phi_z \colon I_r \to X$ be the simplicial map defined as:

$$\phi_z \left(\frac{i}{r} \right) = \begin{cases} \text{orig } e_{i+1} \ 0 \leq i \leq r-1 \\ \text{end } e_i \ 1 \leq i \leq r \end{cases}$$

We now observe that $|I_r| = I$. Hence, given the above simplicial map, we get a map

$$|\phi_z| \colon I \to |X|.$$

This is a path in $|X|$. This is how one can identify $E(X, v_0)$ with $\pi_1(|X|, v_0)$.

We also have the following proposition.

Proposition B.6. *On the category of pointed simplicial complexes X with base vertex v_0, there is an equivalence between the functor $E(X, v_0)$ and $\pi_1(|X|, v_0)$.*

All the above results can be found on p134,[8].

Let us now look at the definition of a Riemannian manifold.

The following three definitions can be found in [9].

Definition B.7 (Differentiable manifold)**.** A **differentiable manifold** of dimension n is a set M and a family of injective mappings $x_\alpha \colon U_\alpha \subset \mathbb{R}^n \to M$ such that

1. $\bigcup\limits_\alpha x_\alpha(U_\alpha) = M$
2. for any pair α, β with $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) = W \neq \emptyset$, $x_\alpha^{-1}(W)$ and $x_\beta^{-1}(W)$ are open in $\mathbb{R}^n$ and $x_\alpha \circ x_\beta^{-1}$ is differentiable.
3. The family $\{(U_\alpha, x_\alpha)\}_\alpha$ is maximal with respect to the conditions (1) and (2).

Definition B.8 (Riemannian metric)**.** **Riemannian metric** on a differentiable manifold M is a correspondence which associates to each point p of M, an inner product, $<,>_p$ (i.e, a positive, symmetric bilinear form) on the tangent space T_pM which varies differentiably in the following sense:
If

$$x \colon U \subset \mathbb{R}^n \to M$$

is a system of coordinates around p, with $x(x_1, \dots, x_n) = q \in x(U)$ and $\frac{\partial q}{\partial x_i} = dx_q(0, \dots, 1, \dots, 0)$, then

$$\left\langle \frac{\partial q}{\partial x_i}, \frac{\partial q}{\partial x_j} \right\rangle_q = g_{ij}(x_1, \dots, x_n)$$

is a differentiable function on U.

Definition B.9 (Riemmanian manifold)**.** A **Riemannian manifold** is a differentiable manifold with a Riemannian metric on it.

The below discussion can be found in [5].

Before stating the theorem and giving a proof, we shall look at the following.

We shall define $r(M)$ for a given compact Riemannian manifold M and this will serve the purpose of e in the statement of Theorem 4.19.

Define $r(M) \in \bar{\mathbb{R}_+}$ as the supremum of the set of real numbers which satisfy the following conditions:

1. for all $x, y \in M$, such that $d(x, y) < 2r$, then there exits a unique shortest geodesic joining x and y. Its length is $d(x, y)$.
2. let x, y, z, u with $d(x, y) < r$, $d(u, x) < r$, $d(u, y) < r$ and z be a point on the shortest geodesic joining x to y, then $d(u, z) \leq \max\{d(u, x), d(u, y)\}$.
3. if γ, γ' are arc length parametrised geodesics such that $\gamma(0) = \gamma'(0)$ and if $0 \leq s, s' < r$ and $0 \leq t \leq 1$, then $d(\gamma(ts), \gamma'(ts')) \leq d(\gamma(s), \gamma'(s'))$

Example B.10. $r(S^n) = \pi/2$ for the unit sphere.

Now, choose a total ordering on M. Consider a finite set $\{x_0, \ldots, x_q\}$ such that $x_0 < x_1 < \ldots < x_q$. Suppose $r(M) > 0$ and let $\epsilon < r(M)$. We now have M_ϵ as the Vietoris-Rips Complex of M (recall the definition of Vietoris-Rips Complex, Definition 4.6).

Now, to each q-simplex, $\sigma = \{x_0, \ldots, x_q\}$ of M_ϵ, we associate a singular q-simplex $T_\sigma \colon \Delta^q \to M$.

T_σ is defined inductively: $T(e_0) = x_0$. Suppose T_σ is defined for $y = \sum_{i=0}^{p-1} t_i e_i$. Let $z = \sum_{i=0}^{p} t_i e_i$. If $t_p = 1$, then define $T_\sigma(z) = x_p$. Else, let $x = T_\sigma\left(\frac{1}{1-t_p} \sum_{i=0}^{p-1} t_i e_i\right)$.

Then define $T_\sigma(z)$ as the point on the shortest geodesic joining x to x_p with

$$d(x, T_\sigma(z)) = t_p \cdot d(x, x_p)$$

We note that this shortest geodesic exists because $\epsilon < r(M)$ and by condition (2) above we get the existence of this shortest geodesic.

With all this background, we need to state two lemmas before stating the theorem and giving a proof.

Lemma B.11. *Let M be a compact Riemannian manifold with $r(M) > 0$. Let $0 < \epsilon < \epsilon' < r(M)$. Then, the inclusion $M_\epsilon \subset M'_\epsilon$ induces isomorphism on all the homology groups*

The proof of this lemma makes use of condition (3) in the definition of $r(M)$.

We see that the map we defined $\sigma \mapsto T_\sigma$ induces a chain map $T_\# \colon C_\star(M_\epsilon) \to SC_\star(M)$ where $SC_\star(M)$ denotes the singular chain complex of M.

Lemma B.12. *If* $0 \leq \epsilon \leq r(M)$*, then the chain map* $T_\star$ *induces isomorphism on homology groups.*

The proof of this lemma makes use of the previous lemma. Both the proofs can be found in [5].

We now state the theorem.

Theorem B.13. *Let* M *be a compact Riemannian manifold with* $r(M) > 0$*. Let* $0 < \epsilon \leq r(M)$*. Then the map*

$$T\colon |M_\epsilon| \to M$$

is a homotopy equivalence.

Proof. Step 1 - We want to show that T induces isomorphism on the fundamental group.

Let $\gamma\colon [0,1] \to M \in \pi_1(M)$. Let $p \in \mathbb{N}$ such that $1/p$ is a Lebesgue number for the covering $\{\gamma^{-1}(B(x, \epsilon/2))\}_{x \in M}$.

$\gamma|_{[k/p,(k+1)/p]}$ is canonically homotopic to a parametrisation of the shortest geodesic joining k/p to $(k+1)/p$. This is because we are considering balls of radius $\epsilon/2$ and $\epsilon \leq r(M)$. Thus, by the properties of $r(M)$, we get this.

Hence,

$$\begin{aligned} \gamma \simeq& \gamma_1 \star \gamma_2 \star \ldots \star \gamma_n \\ \simeq& \gamma' \end{aligned}$$

Such a γ' represents the image under T of an element of $\pi_1(|M_\epsilon|)$. One can see this by considering the identification of $\pi_1(|M_\epsilon|)$ with the edge path group of the simplicial complex (Proposition B.6). Thus, we get that $T_\star$ is surjective on the fundamental group.

A similar argument using the triangulation of $[0,1] \times [0,1]$ can be made to see that $T_\star$ is injective on the fundamental group.

Step 2 - Let $\tilde{M}$ be the universal cover of M endowed with Riemannian metric such that the covering map is a local isometry.

Claim - $r(\tilde{M}) \geq r(M)$.
This is because the covering map is a local isometry and hence, $r(\tilde{M})$ also satisfies all the properties that $r(M)$ does on the points of M. Since $r(M)$ is the least upper bound, $r(M) \leq r(\tilde{M})$.

Since $0 < \epsilon \leq r(M) \leq r(\tilde{M})$, by Lemma B.12, we get that the map

$$\hat{T} \colon |(\tilde{M}_\epsilon)| \to \tilde{M}$$

is a homology isomorphism. By the previous step, we also get that $\pi_1(|(\tilde{M}_\epsilon)|) \cong \pi_1(\tilde{M}) \cong 0$ as $\tilde{M}$ is the universal cover of M. Hence, by Hurewicz theorem we get that $\pi_n(|(\tilde{M}_\epsilon)|) \cong \pi_n(\tilde{M})$ for all n. Thus, by Whitehead theorem, we get that the map $\hat{T}$ is a homotopy equivalence.

Step 3- $p_\epsilon \colon |\tilde{M}_\epsilon| \to M_\epsilon$ be the universal covering projection over M_ϵ.

By Step 1, the following commutative diagram is a pull-back diagram.

$$\begin{tikzcd}
|\tilde{M}_\epsilon| \arrow[r, "\tilde{T}"] \arrow[d, "p_\epsilon"'] & \tilde{M} \arrow[d, "p"] \\
|M_\epsilon| \arrow[r, "T"'] & M
\end{tikzcd}$$

Since, this is a pull-back diagram, the map $\hat{T} \colon |(\tilde{M}_\epsilon)| \to \tilde{M}$ factors through

$|\tilde{M}_\epsilon|$ as

$$|(\tilde{M}_\epsilon)| \xrightarrow{\check{T}} |\tilde{M}_\epsilon| \xrightarrow{\tilde{T}} \tilde{M}.$$

If we show that $\check{T}$ is a homeomorphism and we know that $\hat{T}$ is a homotopy equivalence, then $\tilde{T}$ would be a homotopy equivalence. Hence, T would be a homotopy equivalence because covering maps preserve homotopies.

Step 4 - We want to show that $\check{T}$ is a homeomorphism.

Let L be the simplicial complex defined as follows:

- The vertex set of L, $L^{(0)} = \{p_\epsilon^{-1}(v) | v$ is a vertex of $M_\epsilon\}$
- $\bar{\sigma} = \{\bar{v}_0, \ldots, \bar{v}_q\}$ is a q-simplex of L if there exists a q-simplex $\sigma = \{v_0, \ldots, v_q\}$ of M_ϵ and a lift $f_{\bar{\sigma}}\colon |\sigma| \to |\tilde{M}_\epsilon|$ such that $f_{\bar{\sigma}}(v_i) = \bar{v}_i$ for all $0 \le i \le q$.

We claim that the collection $\{f_{\bar{\sigma}} | \bar{\sigma} \in L\}$ produces a homeomorphism $f\colon |L| \to |\tilde{M}_\epsilon|$.

The collection $\{f_{\bar{\sigma}} | \bar{\sigma} \in L\}$ defines a continuous map as follows

$$f\colon |L| \to |\tilde{M}_\epsilon|$$

defined as

$$f(\sum \alpha_i \tilde{v}_i) = f_{\bar{\sigma}}(\sum \alpha_i p_\epsilon(\tilde{v}_i))$$

for $\sum \alpha_i \tilde{v}_i \in |\bar{\sigma}|$.

Let $\phi\colon |L| \to |M_\epsilon|$ be the simplicial map $\phi(\tilde{v}) = p(\tilde{v})$.

$$\begin{tikzcd}
|L| \arrow[rr, "f"] \arrow[dr, "|\phi|"'] & & |\tilde{M}_\epsilon| \arrow[dl, "p"] \\
& |M_\epsilon| &
\end{tikzcd}$$

If v is a vertex of M_ϵ, then star(v), being contractible, is evenly covered by p. (Star of a vertex is defined as the union of all the simplices of which the vertex is a part). For $\tilde{v} = p^{-1}(v)$, let $U_{\tilde{v}}$ be the component of $p^{-1}(star(v))$ containing $\tilde{v}$. Then, $p|_{U_{\tilde{v}}}$ is a homeomorphism of $U_{\tilde{v}}$ onto star(v).

By definition of ϕ, $|\phi|(\mathrm{star}(\tilde{v})) \cong \mathrm{star}(v)$.

Thus, by the previous two statements, we get that $f|_{\mathrm{star}(v)}$ is a homeomorphism of star(v) onto $U_{\tilde{v}}$.

Hence, $f|_{\phi^{-1}(\mathrm{star}(v))} \cong U_{\tilde{v}}$.

Since this is true for all the vertices of L, f is a homeomorphism.

Now, we have obtained that $f\colon |L| \to |\tilde{M}_\epsilon|$ is a homeomorphism.

Using the pull-back diagram, we get a bijection $\tilde{M} \to L^{(0)}$. As $(\tilde{M})^{(0)}_\epsilon = \tilde{M}$, there is a bijection

$$\check{t}\colon (\tilde{M})^{(0)}_\epsilon \to L^{(0)}.$$

$\check{t}$ extends to a simplicial isomosphism between these two simplicial complexes and $|\check{t}| = \check{T}$.

Hence, $\check{T}$ is a homeomorphism.

Thus, by the argument stated just before the beginning of Step 4, we get that T is a homotopy equivalence. $\square$

Appendix C

We shall look at the notion of a directed system and direct limits. We shall then look at a proposition which gives the result that homology commutes with direct limits.

Definition C.1. A **directed set**, $\Lambda = \{\alpha\}$, is a partially ordered set such that for all $\alpha, \beta \in \Lambda$, there exists a $\gamma \in \Lambda$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$.

Definition C.2. A **direct system** of sets is a collection of sets $\{A^\alpha\}$ indexed by a directed set $\Lambda = \{\alpha\}$ and a collection of functions $f_\alpha^\beta \colon A^\alpha \to A^\beta$ for all $\alpha \leq \beta$ such that

1. $f_\alpha^\alpha = 1_{A^\alpha}$
2. $f_\alpha^\gamma = f_\beta^\gamma \circ f_\alpha^\beta \colon A^\alpha \to A^\gamma$ for $\alpha \leq \beta \leq \gamma \in \Lambda$.

Definition C.3. **Direct limit** of a direct system, $\varinjlim_\alpha \{A^\alpha\}$, is the set of equivalence classes with respect to the relation $a^\alpha \sim a^\beta$ if there exists a γ with $\alpha \leq \gamma$ and $\beta \leq \gamma$ such that $f_\alpha^\gamma a^\alpha = f_\beta^\gamma a^\beta$. For each α, there is a map $i_\alpha \colon A^\alpha \to \varinjlim_\alpha \{A^\alpha\}$ and if $\alpha \leq \beta$ then $i_\alpha = i_\beta \circ f_\alpha^\beta$.

Lemma C.4. *Given a direct system of sets $\{A^\alpha, f_\alpha^\beta\}$ and a set B and given that for every $\alpha \in \Lambda$, a function $g_\alpha \colon A^\alpha \to B$ such that $g_\alpha = g_\beta \circ f_\alpha^\beta$ if $\alpha \leq \beta$,* *then there is a unique map* $g\colon \varinjlim_{\alpha}\{A^\alpha\} \to B$ *such that* $g \circ i_\alpha = g_\alpha$.

Further, this map g *is a bijection if and only if the following two conditions hold*

1. $B = \bigcup g_\alpha(A^\alpha)$
2. $g_\alpha(a^\alpha) = g_\beta(a^\beta)$ *if and only if there is a* γ *with* $\alpha \leq \gamma$ *and* $\beta \leq \gamma$ *such that* $f_\alpha^\gamma(a^\alpha) = f_\beta^\gamma(a^\beta)$.

The following theorem can be found on p162,[8].

Theorem C.5. *The homology functor commutes with direct limits*

Proof. Let $\{C^\alpha, \tau_\alpha^\beta\}$ be a direct system of chain complexes and let $\{C, i_\alpha\}$ be its direct limit. Then, $\{H(C^\alpha), \tau_{\alpha\star}^\beta\}$ is a direct system of groups.

We now want to show that the direct limit of this direct system of groups is $(H(C), i_{\alpha\star})$.

We will make use of Lemma C.4 and show that this direct system satisfies both the conditions mentioned with the set B as $(H(C), i_{\alpha\star})$.

(1) Let $[z] \in H_q(C)$ for some q. Then $z = i_\alpha c^\alpha$ for some $c^\alpha \in (C^\alpha)_q$. Since,

$$0 = \partial_q z = \partial_q i_\alpha c^\alpha = i_\alpha \partial_q^\alpha c^\alpha,$$

there exists a β with $\alpha \leq \beta$ such that $\tau_\alpha^\beta \partial_q^\alpha c^\alpha = 0$. Thus, $\partial_q \tau_\alpha^\beta c^\alpha = 0$ and hence, $\tau_\alpha^\beta c^\alpha$ is a cycle in $(C^\beta)_q$ and $i_\beta \tau_\alpha^\beta c^\alpha = i_\alpha c^\alpha = z$. Hence, $i_{\beta\star}[\tau_\alpha^\beta c^\alpha] = [z]$. Hence, we see that $(H(C), i_{\alpha\star}) = \bigcup i_{\alpha\star}(H(C^\alpha), \tau_{\alpha\star}^\beta)$.

(2) Since we are dealing with a direct system of groups, in order to check the second condition it is enough to check that if $[z^\alpha] \in H_q(C^\alpha)$ is in the kernel of $i_{\alpha\star}$, then there is a γ with $\alpha \leq \gamma$ such that $[z^\alpha]$ is in the kernel of $\tau_{\alpha\star}^\gamma$.

$i_{\alpha\star}[z^\alpha] = 0 \Rightarrow i_\alpha z^\alpha = \partial_{q+1} c$ for some $c \in C_{q+1}$.

Since $c = i_\beta c^\beta$ for some β, we have $i_\alpha z^\alpha = i_\beta \partial_{q+1} c^\beta$.

We now choose γ' such that $\alpha, \beta \leq \gamma'$.Then, $i_{\gamma'}(\tau_\alpha^{\gamma'} z^\alpha - \tau_\beta^{\gamma'} \partial_{q+1}^\beta c^\beta) = 0$. Thus, there is γ with $\gamma' \leq \gamma$ such that $\tau_{\gamma'}^\gamma(\tau_\alpha^{\gamma'} z^\alpha - \tau_\beta^{\gamma'} \partial_{q+1}^\beta c^\beta) = 0$. Then, $\tau_\alpha^\gamma z^\alpha = \partial_{q+1}^\beta(\tau_\beta^\gamma c^\beta)$. Hence, $\tau_{\alpha\star}^\gamma[z^\alpha] = 0$. $\square$

Appendix D

We now look at an alternate definition of the cosheaf condition. This can be found on p281,[10].

Definition D.1. A pre-cosheaf $\hat{F}$ on X is a cosheaf if the sequence

$$\bigoplus_{\alpha,\beta} \hat{F}(U_\alpha \cap U_\beta) \to \bigoplus_{\alpha} \hat{F}(U_\alpha) \to \hat{F}(U) \to 0$$

is exact, for all collection of open sets $\{U_\alpha\}$ such that $U = \bigcup\limits_{\alpha} U_\alpha$ with the first morphism being $\sum\limits_{\alpha,\beta} r_{U\alpha,U_\alpha \cap U_\beta} - r_{U_\beta,U_\beta \cap U_\alpha}$ and the second one being $\sum\limits_{\alpha} r_{U,U_\alpha}$.

A similar proposition can be found on p418 in [10] and on p31 in [3].

Theorem D.2. *Let $\hat{F}$ be a pre-cosheaf of vector spaces. $\hat{F}$ is a cosheaf if the following two conditions are satisfied*

1. $\hat{F}(U \cap V) \xrightarrow{\ g\ } \hat{F}(U) \oplus \hat{F}(V) \xrightarrow{\ f\ } \hat{F}(U \cup V) \longrightarrow 0$ *is exact for all U and V, where $g = (-r_{U,U\cap V}, r_{V,U\cap V})$ and $f = r_{U\cup V,U} + r_{U\cup V,V}$.*
2. *If $\{U_\alpha\}$ is directed upwards by inclusion, then the canonical map $\varinjlim_\alpha \hat{F}(U_\alpha) \to \hat{F}(\bigcup_\alpha U_\alpha)$ is an isomorphism*

Proof. We shall first prove that if (1) is satisfied, then the sequence

$$\bigoplus_{i,j} \hat{F}(U_i \cap U_j) \to \bigoplus_{i} \hat{F}(U_i) \to \hat{F}(U) \to 0$$

is exact, where $\{U_0, \ldots, U_k\}$ is a finite collection of open sets such that $U = \bigcup_i U_i$.

The exactness on the right hand side can be proved by induction on k. Exactness is trivial when $k = 1$. Thus, let the sequence be exact on the right for $k = n$. We will show that the sequence is exact on the right for $k = n+1$ as well.

Let $U' = U_1 \cup U_2 \cup \ldots \cup U_n$ and let $U = U_0 \cup U'$.

Consider the following sequence,

$$\bigoplus_{i=1}^{n} \hat{F}(U_i) \to \hat{F}(U') \to 0.$$

This sequence is exact by our induction hypothesis.

Now, consider the following sequence,

$$\hat{F}(U_0) \oplus \hat{F}(U') \to \hat{F}(U) \to 0.$$

This equation is also exact by condition (1).

Thus, we have two surjective maps

$$f_1\colon \hat{F}(U_0) \oplus \hat{F}(U') \to \hat{F}(U)$$

and

$$f_2\colon \bigoplus_{i=1}^{n} \hat{F}(U_i) \to \hat{F}(U').$$

We note that $f_1 = r_{U,U_0} + r_{U,U'}$ and $f_2 = r_{U',U_1} + r_{U',U_2} + \ldots + r_{U',U_n}$ Now, consider the map

$$g\colon \hat{F}(U_0) \oplus \bigoplus_{i=1}^{n} \hat{F}(U_i) \to \hat{F}(U_0) \oplus \hat{F}(U')$$

which is the identity map on the first coordinate and f_2 on the rest. Clearly, g is a surjection.

Now, consider $f_1 \circ g$.

$$\begin{aligned} f_1 \circ g &= (r_{U,U_0} \circ r_{U_0,U_0}) + (r_{U,U'} \circ (r_{U',U_1} + \ldots r_{U',U_n})) \\ &= r_{U,U_0} + r_{U,U'} \circ r_{U',U_1} + \ldots + r_{U,U'} \circ r_{U',U_n} \\ &= r_{U,U_0} + r_{U,U_1} + \ldots + r_{U,U_n} \end{aligned}$$

This is a surjection since both f_1 and g are. Thus, we get that the map $\bigoplus_{i=0}^{n} \hat{F}(U_i) \to \hat{F}(U)$ is a surjection thus making the following sequence exact

$$\bigoplus_{i=0}^{n} \hat{F}(U_i) \to \hat{F}(U) \to 0$$

where $U = \bigcup_{i=0}^{n} U_i$.

The exactness in the middle will also be proved by induction on k. Let us assume that the above sequence is exact for $k = n$. We will prove it for $k = n + 1$.

Let $U' = U_1 \cup U_2 \cup \ldots \cup U_n$, $U = U_0 \cup U'$ and $V = U_0 \cap U'$.
Let $s_j \in \hat{F}(U_j)$ for $0 \leq j \leq n$ such that $\sum_{j=0}^{n} r_{U,U_j}(s_j) = 0$ in $\hat{F}(U)$.

Let $t' = \sum_{j=1}^{n} r_{U',U_j}(s_j)$. Then,

$$r_{U,U_0}(s_0) + r_{U,U'}(t') = 0. \tag{D.1}$$

By induction hypothesis, (for $k = n$ the equation is exact), we get that there is an element $v \in \hat{F}(V)$ such that

$$r_{U_0,V}(v) = s_0 \text{ and } r_{U',V}(v) = -t'. \tag{D.2}$$

Now, $V = (U_0 \cap U_1) \cup (U_0 \cap U_2) \cup \ldots \cup (U_0 \cap U_n)$. Thus, there exist $v_j \in \hat{F}(U_0 \cap U_j)$ such that

$$v = \sum_{j=1}^{n} r_{V,U_0 \cap U_j}(v_j). \tag{D.3}$$

Now,

$$g(\bigoplus_{0,j} v_j) = (\sum_{j=1}^{n} r_{U_0,U_0 \cap U_j}(v_j), -r_{U_1,U_0 \cap U_1}(v_1), \ldots, -r_{U_n,U_0 \cap U_n}(v_n)) \tag{D.4}$$

Now, consider $h = \bigoplus_{i=0}^{n} s_i - g(\bigoplus_{0,j} v_j) \in \bigoplus_{i=0}^{n} \hat{F}(U_i)$. This clearly has zero component in $\hat{F}(U_0)$ because

$$s_0 = r_{U_0,V}(\sum_{j=1}^{n} r_{V,U_0 \cap U_j}(v_j)) = \sum_{j=1}^{n} r_{U_0,U_0 \cap U_j}(v_j)$$

which we get by Equation (D.2) and Equation (D.3). Also this vector projects to the zero vector in $\hat{F}(U')$. This is because Equation (D.2), Equation (D.3) and the definition of t' give

$$r_{U',V}(\sum_{j=1}^{n} r_{V,U_0 \cap U_j}(v_j)) = -\sum_{j=1}^{n} r_{U',U_j}(s_j)$$

Hence,

$$\sum_{j=1}^{n} r_{U',U_j}(s_j) + \sum_{j=1}^{n} r_{U',U_0 \cap U_j}(v_j) = 0$$

Written in a different form, we get

$$r_{U',U_j}\left(\sum_{j=1}^{n} s_j + \sum_{j=1}^{n} r_{U_j,U_0 \cap U_j}(v_j)\right) = 0$$

which essentially says that h projects to the zero vector in $\hat{F}(U')$. Hence, we have showed that ker $f =$ im g and leading to the proof of exactness of the equation for $k = n+1$. Thus, by mathematical induction, the statement is true for all $n \in \mathbb{N}$.

Now, we shall use the second condition to complete the proof.

Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an arbitrary cover indexed by any ordered set Λ. By the first half of the proof we know that the first condition is true for any finite cover. Thus, for each finite subset $I \subset \Lambda$, we have that

$$\bigoplus_{\alpha<\beta\in I} \hat{F}(U_\alpha \cap U_\beta) \to \bigoplus_{\alpha\in I} \hat{F}(U_\alpha) \to \hat{F}(\bigcup_{\alpha\in I} U_\alpha) \to 0$$

is exact.

Now, we make use of the fact that all finite subsets form a directed system and that direct limits is an exact functor(i.e, it preserves exactness) in the category of vector spaces. Hence, we see that

$$\varinjlim_{I} \bigoplus_{\alpha<\beta\in I} \hat{F}(U_\alpha \cap U_\beta) \to \varinjlim_{I} \bigoplus_{\alpha\in I} \hat{F}(U_\alpha) \to \varinjlim_{I} \hat{F}(\bigcup_{\alpha\in I} U_\alpha) \to 0$$

is also exact.

Now we use the second condition and the fact that the direct limit of all the finite subset of Λ is Λ itself, to get that

$$\bigoplus_{\alpha<\beta\in \Lambda} \hat{F}(U_\alpha \cap U_\beta) \to \bigoplus_{\alpha\in \Lambda} \hat{F}(U_\alpha) \to \hat{F}(\bigcup_{\alpha\in \Lambda} U_\alpha) \to 0$$

is exact.

We now see that this is precisely the condition, as stated in Definition D.1, when a pre-cosheaf is a cosheaf. $\square$

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