Identification and Estimation of a Dynamic Multi-Object Auction Model

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[PRELIMINARY AND INCOMPLETE]

Abstract

In this paper I develop an empirical model of bidding in repeated rounds of simultaneous first-price auctions. The model is motivated by the fact that auctions rarely take place in isolation; they are often repeated over time, and multiple heterogeneous lots are regularly auctioned simultaneously. Incorrect modelling of bidders as myopic or as having additive preferences over lots can lead to inaccurate counterfactuals and welfare conclusions. I prove non-parametric identification of primitives in this model, and introduce a computationally feasible procedure to estimate this type of game. I then apply my model to data on Michigan Department of Transportation highway procurement auctions. I investigate the extent of cost-synergies across lots and use counterfactual simulations to compare equilibrium efficiency when contracts are auctioned sequentially rather than simultaneously.

1 Introduction

First-price Auctions, which are regularly used to allocate government procurement contracts, rarely take place in isolation. Multiple lots (contracts) are often auctioned simultaneously, and auctions are repeated whenever new contracts become available. In real world environments bidders' values may be non-additive across different lots. For example, bidders may face capacity constraints, facing higher costs the larger their current backlog. Or, they may benefit from economies of scale, facing lower costs when working on many of the same type of contract at once. The structure of these non-additive values is highly relevant for auction design - should similar contracts be auctioned simultaneously, or spaced out over time? When capacity constraints are the dominant factor, auctioning a large number of contracts simultaneously may create inefficiencies by depressing competition. However, if firms are able to exploit economies of scale it may be worth auctioning similar contracts simultaneously, or even bundling the lots together.

In this paper I develop an empirical model of bidding and entry in repeated simultaneous first-price auctions, and study identification and estimation in this framework. I then apply my model to data on Michigan Department of Transportation (MDOT) highway procurement auctions and investigate the empirical and policy relevance of these complementarities.

Previous research has either studied forward looking bidders and assumed auctions are single-object, or studied auctions of multiple objects and assumed bidders are myopic.[1] For example, both Jofre-Bonet and Pesendorfer (2003) and Gentry et al. (2018) study synergies in bidding behaviour in repeated simultaneous first-price auctions for

[1]These types of complementarities are regularly studied in a static context in the literature on combinatorial and heterogenous multi-object auctions, for example Gentry et al. (2018) on highway maintenance contracts, Cantillon and Pesendorfer (2007) on London bus routes, and Fox and Bajari (2013) on FCC spectrum licenses. Likewise, the relationship between forward looking behaviour and complementarities have been studied on a number of prominent occasions, for example Jofre-Bonet and Pesendorfer (2003) on highway maintenance contracts, Kong (2016) on oil and gas leases, and Backus and Lewis (2016) on online marketplaces. However, the two have not yet (to the best of the author's knowledge) been studied together, despite being highly relevant.

highway maintenance contracts. Jofre-Bonet and Pesendorfer (2003), who estimate a dynamic single object model, find significant negative effects of capacity constraints on bids. Meanwhile Gentry et al. (2018), who study simultaneous first-price auctions in a static setting, find similar capacity constraint effects; however, they also find evidence of positive synergies among similar contracts that allow firms to exploit economies of scale. The implication is that neither paper accurately models the effect of non-additive values on bidding decisions. To the best of the author's knowledge this paper is the first to consider both a dynamic and multi-object approach to an empirical auction model.

In order to evaluate the properties of any auction mechanism, which is generally done by estimating a model and simulating counterfactuals, it is necessary to model relevant components of the data generating process. For example, if objects are strongly substitutable then bidders may only bid aggressively on a small subset of auctions to avoid accidentally winning multiple objects. Likewise, having just won one lot a bidder may not bid aggressively on a similar lot. This can generate patterns of data that are difficult to replicate in simulations, leading to inaccurate predictions and descriptions of welfare. However, the more accurately we attempt to capture the true data generating process, the more difficult estimation becomes. With L possible auctions and n possible bidders there are n^L possible outcomes. Combining this high dimensionality with already computationally burdensome value function iteration algorithms can make the models intractable.

I develop a structural empirical model of forward looking bidding in repeated simultaneous first-price auctions, where objects are heterogeneous and preferences are non-additive over objects. The model is fundamentally a blend of the models presented in Jofre-Bonet and Pesendorfer (2003) and Gentry et al. (2018), henceforth referred to as JP and GKS respectively. Bidder pay-offs are represented as the sum of privately known and potentially correlated *lot specific* values, a *combination specific* flow payoff, and a *combination specific* continuation value. Following GKS, the combination specific flow payoff is treated as a deterministic function of state variables.[2] This is a natural framework that reflects known capacity constraints or economies of scale. The model primitives consist of the distribution of lot specific values and the combination specific flow payoff function.[3]

Building on this framework I make four key contributions to the empirical auction literature. First, I extend GKS' identification framework to make use of variation in a bidder's individual state variables, such as their backlog of contracts, to non-parametrically identify their combination specific value function *without* the need for any exclusion restrictions. Intuitively, identification arises because variation in the state causes variation in bidders' combination values, which in turn causes variation in their bidding behaviour. If lots are substitutes we expect to observe more aggressive bidding when backlogs are low. Following the approach presented in GKS I translate the inverse bidding system, conditional on a given state, into a system of linear equations in the unknown combination values. I then show that the combination values are identified by combining these systems of linear equations across state variables, proving that the combined system has a unique solution. This result is important because it ensures the combination value is identified without the need for exclusion restrictions that prohibit identification of forward looking behaviour.[4]

Second, I extend this identification framework to a dynamic setting. Building on JP I show that the continuation value can be written as a recursive function of the observed distribution of bids and the sum of the immediate combination value and the discounted continuation value. This recursive formulation can be solved

[2]In Appendix D I consider several extensions when the combination value is stochastic. Most notably, I consider the case when the combination value is a function of low-dimensional unobservables. For example, if we do not observe firm backlogs. I show that model primitives remain identified in this case, so long as the function has a strictly monotonic first derivative.

[3]Like GKS this assumption allows me to separately identify complementarities and affiliations, the central problem studied by Kong (2016). Affiliation across lots comes through correlation in the lot specific pay-offs, while the synergies remain deterministic. However, like both papers I assume the lot-specific pay-offs are independent across players.

[4]Without additional restrictions, any excluded variable (such as common state variables or the states of other bidders) enter each bidder's continuation value, thereby directly affecting their bidding behaviour and violating the exclusion restriction, rendering the model non-identified.

for the continuation value as a function of the observed distribution of bids and the immediate combination value. Substituting this identity into the system of first order conditions, combined across state variables, yields a system of equations in the unknown combination values. I prove that, under mild conditions, this system has a unique solution.

Third, I outline a three step procedure for estimating the model. The estimation procedure generalises that of JP for estimating dynamic auctions, overcoming the problem that the continuation value cannot be written as a function of the observed distribution of bids only. In the first step one estimates the probability that each bidder wins any given lot - the first step in most empirical auction studies.[5] In the second step one estimates the distribution of underlying lot-specific values, and the sum of the immediate combination value function and the discounted continuation value function. I refer to this sum as the 'pseudo-static' pay-off function. It is essentially the object one would estimate if we incorrectly estimated the model as if it was a static model.[6] In the third step one disentangles this sum, separately estimating both the immediate combination value and the continuation value. This exploits the fact that the continuation value can be written as a recursive function of the distribution of observed bids and the pseudo-static pay-off function.[7] This estimation procedure is shown to be computationally feasible, particularly in an environment where the interior fixed point algorithm is extremely slow.[8] In Appendix E I present the results of a simulation study examining the performance of this estimator.

Finally, I apply this framework to data from Michigan Department of Transport

[5]In practice, this will generally involve estimating the (censored) distribution of each bidders bids, as per the first step in the estimation procedure of Cantillon and Pesendorfer (2007).

[6]This step is similar to the second step in the estimation procedure presented by GKS. As in GKS, if one is non-parametrically estimating the model then this estimation step will actually consist of two substeps: First, estimating the deterministic combinatorial valuations, then estimating the distribution of stochastic lot specific valuations.

[7]This step is similar to the second step in JP's procedure, wherein the continuation value is written as a function of the distribution of observed bids, allowing us to back out the cost function.

[8]In a dynamic discrete choice context this procedure is numerically equivalent to the Conditional Choice Probability methods of Hotz and Miller (1993) when the econometrician fits a parametric function to the CCPs.

(MDOT)'s procurement auctions. In this setting around 45 contracts for highway maintenance and construction projects are auctioned simultaneously in each round, and rounds are repeated roughly every fortnight. I focus on contracts that require use of either hot-mix asphalt, concrete, or both. I use firms' backlogs of asphalt and concrete projects as their state variables, and consider how backlogs impact their cost functions, driving complementarities between lots. For asphalt specialist firms in particular I find evidence of increasing returns to specialising in asphalt contracts: Every one standard deviation increase their asphalt backlog increases the cost of completing a concrete contract by around 10%, and decreases the cost of an asphalt contract by roughly the same amount. I use counterfactual simulations to consider how the procurement cost to MDOT and the total cost to firms differs when contracts are auctioned sequentially instead of simultaneously.

The structure of this paper is as follows: In section 1.1 I discuss this paper's contribution to the literature. Section 2 introduces the auction game that is the focus of this paper. Section 3 introduces the identification framework and proves that model primitives are point identified. Section 4 outlines the proposed three step estimation procedure, and Section 5 applies this procedure to data from MDOT procurement auctions. Several additional results are presented in the Appendices. Appendices A - C present technical proofs. Appendix D presents several extensions to the identification and estimation framework.[9] Appendix E presents the results of a simulation experiment evaluating the proposed estimation procedure, and **??** presents additional analysis related to the empirical application.

1.1 Related Literature

To the best of the authors knowledge, this paper is the first to consider the identification or estimation of dynamic multi-object auction games. However, I build

[9]These include extensions for second-price auctions, reservation prices, ties, endogenous entry, inter-temporal budget constraints, and stochastic combination values.

strongly on a number of literatures, most notably the literature on the estimation and identification of dynamic auction games, and multi-object auction games.[10]

Cantillon and Pesendorfer (2007) was the first paper to consider identification and estimation of simultaneous first-price auctions. They focus on first-price auctions in which bidders submit combination bids over London bus routes. Similarly, Kim et al. (2014) study combinatorial first-price auctions in the allocation of contracts for Chilean school meals. In both cases, observations of combination bids are necessary to identify complementarities between lots. Gentry et al. (2018)'s key contribution to the literature is to consider simultaneous first-price auctions *without* combination bidding. They showed that variation in the characteristics of rival bidders could be used similarly to an instrumental variable to identify the bidders' combination values. However, exclusion restrictions fail in a dynamic environment since bidders' forward looking behaviour ensures that their continuation values, and hence bidding behaviour, are generally affected by every state variable. This paper builds on their work by demonstrating that these exclusion restrictions are not necessary for identification. Fox and Bajari (2013) also study an auction environment without combination bidding. In their application the equilibrium allocation is shown to be 'stable', enabling the identification of preferences and partly enabling them to get around the dimensionality problem. However, this stability property cannot be applied in general.

Jofre-Bonet and Pesendorfer (2003) was the first paper to empirically analyse dynamic auction games, focusing on repeated first-price auctions. They analyse highway contracts, finding backlog effects to be determinants of future bidding behaviour. A

[10] This work is also strongly related to the literature on empirical Multi-unit auctions, which is reviewed in detail in Hortaçsu and McAdams (2018). Multi-unit auctions generally focus on divisible homogenous units. The focus in this paper is on auctions of multiple heterogenous, indivisible, objects. However, the identification framework presented in 3 can be extended to Multi-unit auctions. This is left for future work. The estimation procedure presented in section 4 easily extends to these dynamic multi-unit games, since the procedure essentially involves first estimating a generalised version of the static game (which can be done by making use of results presented in Hortaçsu and McAdams (2018)), and then making use of this third estimation step, in which the primitives are backed out of the primitives from the estimates of the 'pseudo-static' game.

number of papers have built on this framework, for example Jeziorski and Krasnokutskaya (2016) who study dynamic auctions with subcontracting. Groeger (2014) study participation in first-price auctions in a dynamic setting. They assume (justified by the data) that while entry decisions are forward looking, bidding behaviour is myopic. Balat (2013) generalise participation in the dynamic setting, additionally allowing for unobserved heterogeneity in lot quality. These papers generally study data in which multiple auctions are held simultaneously - assuming that pay-offs are additively separable across auctions. I build on this literature by relaxing the assumption that pay-offs are additively separable. Finally, Raisingh (2021) studies the effect of pre-announcements of auctions in the MDOT data, using a dynamic model of bidding and entry behaviour. My empirical application builds on their model, making use of similar computationally convenient functional form specifications.

Kong (2016) study an intermediate case in which gas and oil leases for a given tract are auctioned sequentially, finding evidence of both complementarities and affiliaiton in the values of sequentially auctioned leases. There is also an extensive literature on dynamic second price auctions, often studying on-line market places such as eBay. Sailer (2006) consider a model of identical objects, presenting various non-parametric identification results. Backus and Lewis (2016) extend this framework to allow objects to be heterogeneous and allow for a 'random coefficient demand' style model. Their framework has since been applied on a number of occasions, for example Bodoh-Creed et al. (2021) who study the efficiency of on-line marketplaces. This paper contributes to the general dynamic auction literature with the estimation procedure introduced in section 4. As well as being applicable more broadly than JP's procedure, the procedure is generally more feasible in the presence of a large state space - a problem the dynamic second price literature has struggled with. Finally, Balat et al. (2015) study a dynamic multi-object auction model to analyse how dynamic strategising affects bidder's behaviour in electricity markets. The model they present is used to derive several hypotheses, which they test using reduced form analyses.

2 The general model

I now present the dynamic auction model that is the focus of this paper. The model builds on the models of Gentry et al. (2018) and Jofre-Bonet and Pesendorfer (2003). Assumptions necessary for identification are discussed in section 3.

Setup: Each period t, over an infinite horizon, n risk-neutral players i compete in a series of first-price Sealed Bid auctions. Player i wins lot l in period t if $b_{itl} \geq \max_{j \neq i} \{b_{jtl}\}$. Sealed bids are placed simultaneously, then winners are announced simultaneously. Winners pay their bids, and every player observes the bids and identities of winners. Define the $L \times 1$ vector $\mathbf{w}_t$ as the outcome at time t, where $w_{tl} = i$ if i won lot l at time t. Ex-ante hypothetical outcomes are denoted by $\mathbf{w}_t^a$. The index l is a label and need not reflect anything substantive.

Reservation Prices and Ties: I assume reservation prices do not bind, that auction entry is exogenous, and that ties occur with probability zero.[11]

Lots and Lot Characteristics: Each period a set of lots $\mathbb{L}_t$ comes to auction. L possible lots can be auctioned in a given period, where L is finite. Each lot l is characterised by a row-vector of characteristics $\mathbf{x}_{tl}$, writing $\mathbf{X}_t$ for the stacked characteristics of all lots available in period t.[12] The set of characteristics, $\mathbb{X}$, is assumed finite. If $l \notin \mathbb{L}_t$ then $\mathbf{x}_{tl} = \emptyset$. Finally stack the lot characteristics, set of available lots, and other common state variables into $\mathbf{s}_{0t} \in \mathbb{S}_0$.

Individual States: Player i begins the period in state $\mathbf{s}_{it}$. This may represent a player's existing stock of the good, or backlog of contracts. The set of possible individual states, $\mathbb{S}_i$, is assumed to be finite.[13] If the outcome at t is $\mathbf{w}_t^a$ then player i ends the period in state $\mathbf{s}_{it}^a$, referred to as the ex-post state. $\mathbf{s}_{it} = \mathbf{s}_{it}^a$ if and only if the player does not win a single lot. For notational convenience, define the set $\mathbb{S}_i^a(\mathbf{s}_i, \mathbf{s}_0)$

[11] However these assumptions are relaxed in appendices D.2, D.3, and D.6 respectively.

[12] Characteristics may include details of contracts being auctioned. Such as the project's location.

[13] This is predominantly for mathematical convenience, but is likely to hold in practice. For example, Highway maintenance companies likely have a maximum number of contracts they can feasible hold at any given time, and the measurement of their backlog of contracts can be arbitrarily discretised into days of work remaining.

as the set of possible individual ex-post states $\mathbf{s}_i^a$ having started in state $\mathbf{s}_i$, given the available lots, lot characteristics, and other common state variables $\mathbf{s}_0$.

Total States: Stack the individual states $\{\mathbf{s}_{it}\}_{i\in\mathbb{I}}$, and $\mathbf{s}_{0t}$, into the total state variable $\mathbf{s}_t \in \mathbb{S}$, where $|\mathbb{S}| = S$ is finite. In section 3.6 I set out sufficient conditions on the set $\mathbb{S}$ which ensure identification. Similarly, Stack the ex-post states $\{\mathbf{s}_{it}^a\}_{i\in\mathbb{I}}$ and $\mathbf{s}_{0t}$ into the total ex-post state $\mathbf{s}_t^a \in \mathbb{S}$.

Transition Functions: I assume that at the beginning of each period, the state $\mathbf{s}_t$ is drawn stochastically from $T_{\mathbf{s}}(.|\mathbf{s}_{t-1}^a)$. Because $|\mathbb{S}|$ is finite, the transition probabilities can be described by a standard transition matrix T, such that $P(\mathbf{s}_{it} = \mathbf{s}_m|\mathbf{s}_{t-1}^a = \mathbf{s}_n) = T_{mn}$.

Actions: Each player plays an L dimensional vector of bids each period, denoted $\mathbf{b}_{it}$. The set of possible bids conditional on available lots $\mathbb{B}(\mathbb{L})$ is assumed to be convex and compact, so that $b_{itl} \in [\underline{b}, \bar{b}]$ for l such that $l \in \mathbb{L}_t$.

Lot Specific Values: I focus on an independent private value framework. Player i's lot specific valuation, υ_{itl}, conditional on $l \in \mathbb{L}_t$ is a mean zero random variable observed before they make their entry decisions.[14] If $l \notin \mathbb{L}_t$ then υ_{itl} is normalised to zero. Stacking these values $\boldsymbol{\upsilon}_{it}$, a $L \times 1$ vector, is drawn from cumulative density function $F_i(.|\mathbf{s}_t)$ with support $[\underline{\boldsymbol{\upsilon}}_i, \bar{\boldsymbol{\upsilon}}_i]$.[15]

Combination Value: The combination value is given by $J_i(\mathbf{s}_t)$, a $2^L \times 1$ vector. Each row $J_{ia}(\mathbf{s}_t)$ gives the mean flow pay-off corresponding to a different outcome $\mathbf{w}_t^a$, ending the period in state $\mathbf{s}_{it}^a$.[16] Entries corresponding to winning lots that are not available are normalised to zero. J_i is assumed a deterministic function of $\mathbf{s}$, and assumed to be finite. A player's type is characterised by the tuple $(\boldsymbol{\upsilon}_i, J_i)$. I assume

[14]The mean zero assumption is without loss of generality. The mean of the variable will be absorbed into the combination value.

[15]In the current paper I assume these values are, conditional on the state, independent over time. This is predominantly for comprehensibility. The identification results carry over to the case of dependence over time, however depending on the structure of this dependence estimation may become more involved.

[16]Even though $J_i(\mathbf{s}_t)$ varies with $\mathbf{s}$ because $\mathbf{s}_{0t}$ dictates the type of lots available and hence the possible ex-post states. However I will assume that the values of $J_{ia}(\mathbf{s}_t)$ only depends on $\mathbf{s}_{it}^a$.

that F_i and J_i are both common knowledge.

Strategies: A (pure) strategy consists of a mapping from a player's type $(\boldsymbol{v}_i, J_i)$ and the state of the world $\mathbf{s}$ onto a series of bids $\mathbf{b}_{it}$. Ex-ante a player's strategy admits a distribution of bids σ_i according to F_i as well as J_i and $\mathbf{s}$.

Marginal Win Probabilities: Denote $G_{jl}(.;\sigma_j)$ and $g_{jl}(.;\sigma_j)$ respectively the marginal cdf and pdf of individual j's bid on lot l according to their strategy σ_j. Denote $\Gamma_i(\mathbf{b};\sigma_{-i})$ the $L \times 1$ vector where row l contains the probability that i wins lot l, given their bid and entry decision, taking as given the strategies of other players. Because ties occur with zero probability we can write:

$$\Gamma_{il}(b_{ilt};\sigma_{-i}) = \prod_{j \neq i} G_{jl}(b_{ilt};\sigma_j)$$

Combination Win Probabilities: Denote $P_i(\mathbf{b};\sigma_{-i})$ the $2^L \times 1$ vector where row a contains the probability, conditional on their bid and the strategies of other players, that player i's ex-post state will be $\mathbf{s}^a_{it}$.

Overall Combination Probabilities Denote $Q_i(\mathbf{b};\sigma_{-i})$ the $n^L \times 1$ vector where row a contains the probability, conditional on their bid and the strategies of other players, that the outcome from period t is $\mathbf{w}^a_t$, and so the overall ex-post state is $\mathbf{s}^a_t$. This object is extremely similar to the combination win probabilities presented previously, except this object also takes into account exactly which player $j \neq i$ wins each lot. Importantly, $P_a = \sum_{a' \text{ s.t. } \mathbf{s}^{a'} = \mathbf{s}^a} Q_{a'}$. That is, summing $Q_{a'}$ over all the ex-post outcomes for which player i's state is the same ($\mathbf{s}^{a'} = \mathbf{s}^a$) gives P_a. For both the marginal and combination win probabilities entries of both vectors that correspond to winning unavailable or un-entered lots are normalised to zero.

Discounting: Players are assumed to have temporally additively separable preferences, and make forward looking decisions with discount parameter $\beta \in (0,1)$. β is assumed known by players and the econometrician.

Expected Flow Pay-off: Payoffs are assumed quasi-linear in payments.[17] Consider player i with a realisation of $\boldsymbol{v} = \boldsymbol{v}_{it}$ who places bid $\mathbf{b}$ against players bidding according to strategies σ_{-i}. For notational convenience I also suppress subscripts.

$$\Pi(\mathbf{b}|\boldsymbol{v}_i, \mathbf{s}; \sigma_{-i}) = \Gamma_i(\mathbf{b}; \sigma_{-i})^T(\boldsymbol{v}_i - \mathbf{b}) + P_i(\mathbf{b}; \sigma_{-i})^T J_i(\mathbf{s}) \tag{1}$$

Value Function: The Value Function for player i is then given as:

$$W_i(\boldsymbol{v}_{it}, \mathbf{s}_t; \sigma_{-i}) = \max_{\mathbf{b}} \left\{ \Gamma_i(\mathbf{b}; \sigma_{-i})^T(\boldsymbol{v}_i - \mathbf{b}) + P_i(\mathbf{b}; \sigma_{-i})^T J_i(\mathbf{s}_t) + \beta Q_i(\mathbf{b}; \sigma_{-i})^T V_i(\mathbf{s}_t; \sigma_{-i}) \right\} \tag{2}$$

Were $V_i(\mathbf{s}_t; \sigma_{-i})$ is the continuation value, to be discussed shortly.

Ex-Ante Value Function: Define the (scalar) ex-ante value function by taking an expectation over private information $\boldsymbol{v}_{it}$: $V_i^E(\mathbf{s}_t; \sigma_{-i}) = \int_{\boldsymbol{v}_{it}} W_i(\boldsymbol{v}_i, \mathbf{s}_t; \sigma_{-i}) dF(\boldsymbol{v}_i|\mathbf{s}_t)$

Continuation Value: The combination continuation value is given by $V_i(\mathbf{s}_t; \sigma_{-i})$, a $n^L \times 1$ vector. Each element a of this vector contains the continuation value $V_{ia}(\mathbf{s}_t; \sigma_{-i})$ corresponding to a different allocation of lots and ending the period in a different state $\mathbf{s}_t^a$. This relates to the ex-ante value function by taking an expectation over the next period state $\mathbf{s}_{t+1}$, given $\mathbf{s}_t^a$: $V_{ia}(\mathbf{s}_t; \sigma_{-i}) = \int_{\mathbf{s}_{t+1}} V_i^E(\mathbf{s}; \sigma_{-i}) dT(\mathbf{s}|\mathbf{s}_t^a)$

2.1 Equilibrium

I now discuss the dynamic equilibrium, and the assumptions required for existence of an equilibrium. A full and general proof of equilibrium existence is beyond the scope of this paper.[18] In place of a complete proof I present a proof of equilibrium existence

[17] However, in appendix D.4 I extend the model to allow for an inter-temporal budget constraint, proving that quasi-linearity is observationally equivalent to the inter-temporal budget constraint model when players have constant marginal utility of wealth.

[18] To my knowledge, no complete proof of equilibrium existence exists even for the static game without entry. Therefore this paper joins the long line papers studying sufficiently complex auction games in which neither existence, nor uniqueness of equilibrium can be guaranteed. For example, Gentry et al. (2018) on simultaneous first-price auctions, Fox and Bajari (2013) on simultaneous ascending auctions, or Jofre-Bonet and Pesendorfer (2003) on dynamic single-object first-price auctions. If the bid space were discrete, then the static equilibrium existence follows from Milgrom and

under the conjecture that a pure-strategy Bayesian Nash Equilibrium exists in the static game without entry. The lack of full existence proof should be considered only a theoretical issue, rather than a practical problem.

Markov Perfect Equilibrium I focus on symmetric markov perfect equilibria consisting of a set of strategies σ^* such that for any $(\boldsymbol{v}, J, \mathbf{s})$:

$$\mathbf{b}_i^{\sigma^*} = arg \quad \max_{\mathbf{b}} \left\{ \; \Gamma_i(\mathbf{b}; \sigma^*_{-i})^T(\boldsymbol{v}_i - \mathbf{b}) + P_i(\mathbf{b}; \sigma^*_{-i})^T J_i(\mathbf{s}_i) + \beta Q_i(\mathbf{b}; \sigma^*_{-i}) V_i(\mathbf{s}; \sigma^*_{-i}) \; \right\} \tag{3}$$

2.2 Equilibrium Existence

To prove equilibrium existence, I rely on the following conjecture:

Conjecture 1. Existence and Uniqueness of a continuous Static Equilibrium

There exists a unique symmetric (non co-operative) Pure Strategy Bayesian Nash Equilibrium of the (myopic) stage game, such that for all i and l the expected pay-off is continuous in $\boldsymbol{v}_i$ and J_i.

This conjecture takes essentially the same form as the assumption that a continuous and unique equilibrium exists in Gentry et al. (2018).[19]

Proposition 1. *Under the assumptions of the game, and under Conjecture 1, a Symmetric Markov Perfect Equilibrium exists.*

Proof of Proposition 1 is relegated to Appendix A, as existence is not the main focus of this paper. The proof consists of showing that the equilibrium pay-off in the stage game is consistent with the continuation value, employing Kakutani's fixed point theorem.

Weber (1985).

[19]The stage game refers to the game which is repeated each period.

3 Identification

I now demonstrate that the distribution of lot specific values F, and the combination value J are non-parametrically point identified. The intuition behind this argument is that variation in $\mathbf{s}$ causes variation in payoffs which, in turn, cause variation in bidding behaviour. I then use the observed bidding behaviour, as well as information about bidders' equilibrium beliefs, to essentially 'back out' the distribution of values.

A brief discussion of what I mean by non-parametric identification is helpful: A model is point identified if, given the implications of equilibrium behaviour, the joint distribution of bidder's pay-offs, $\{F_i, J_i\}_{i \in \mathbb{I}}$, are uniquely determined by the joint distribution of observables (Athey and Haile, 2002). Using the terminology of Lewbel (2019) we say that a model is non-parametrically identified if the identified objects are functions. This is in the sense that we do not assume a functional form, but identify the entire function $J_i(\mathbf{s})$ for every $\mathbf{s} \in \mathbb{S}$ and $F_i(\boldsymbol{v}|\mathbf{s})$ for every pair $(\boldsymbol{v}, \mathbf{s})$.

I begin by introducing the assumptions necessary for identification in subsection 3.1. In subsection 3.2 I consider the agent's optimisation problem, and establish necessary First Order Conditions. Next, I prove that the model primitives are identified in 4 steps. First, in subsection 3.3, I show that conditional on identification of J and V, F is non-parametrically identified from the inverse bidding system. This argument is only a minor extension to the argument presented in GKS. Second, in subsection 3.4 I show that conditional on identification of J, V is non-parametrically identified. Third, in subsection 3.5 I show that identification of J collapses down to a rank condition. Finally, in subsection 3.6 I demonstrate that only very mild restrictions on the state space are sufficient for this rank condition to hold. In subsection 3.7 I then consider identification under several extensions of the model presented.

3.1 Assumptions necessary for identification

Define the objects $G_i(.|\mathbf{s}), \Gamma_i(.|\mathbf{s}), P_i(.|\mathbf{s}),$ and $Q_i(.|\mathbf{s})$ as the empirical counterparts to the objects presented previously.

Assumption 1. *For each t, the econometrician has a set of observations as follows:*

$$\mathbb{O}_t = \left\{ \mathbf{w}_t, \mathbf{s}_t, \{\mathbf{b}_{it}\}_{i \in \{1,2,\ldots,n\}} \right\} \tag{4}$$

I assume the econometrician observes all bids and entry decisions, not just the winning bid. Under this assumption $G, \Gamma, P, Q,$ and T are all non-parametrically identified. Therefore, in the remainder of this section I treat these objects as known.

Assumption 2. *The data $\{\mathbb{O}_t\}_{t=1\ldots T}$ are generated by strategy profile $\boldsymbol{\sigma}^*$ which is a symmetric Markov perfect equilibrium of the dynamic auction game.*

This assumption requires that the same equilibrium is played throughout the observed period, ensuring that strategies can be written as a function of the state. As a result, the continuation value can be written as a function of the state. We can then express the continuation value in vector form as $\mathbf{V}$, with elements corresponding to the expectation from ending a period in any particular ex-post state. It is then useful to define the relationship between the n^L vector $V(\mathbf{s})$ defined previously and $\mathbf{V}$:

$$\begin{pmatrix} V(\mathbf{s}_1) \\ \vdots \\ V(\mathbf{s}_S) \end{pmatrix} = A\mathbf{V}$$

Where A is an $Sn^L \times S$ dimensional matrix. I often also use the relationship $V(\mathbf{s}) = A_{\mathbf{s}}\mathbf{V}$ for the $n^L \times S$ submatrix $A_{\mathbf{s}}$. This matrix contains a 1 in entry am if the potential outcome $\mathbf{w}^a$ yields ex-post state $\mathbf{s}^a = \mathbf{s}_m$. That is, this matrix just selects the relevant continuation values that correspond to possible ex-post states.

Assumption 3. *For all i and $l \in \mathbb{L}$ $\Gamma_{il}(b_{il}|\mathbf{s};\sigma^*_{-i})$ is strictly increasing and differentiable in b_{il}. Similarly, for all i, j, and $l \in \mathbb{L}$ the object $Prob(j \text{ wins } l|b_{ilt};\sigma^*_{-i})$ is continuous and differentiable in b_{ilt}.*

This assumption ensures that the marginal, combination, and over-all combination win probabilities are continuous and differentiable in $\mathbf{b}$, enabling us to take first order conditions. As shown in GKS, when this assumption does not hold we lose point-identification, though the model primitives generally remain partially identified.

Assumption 4.

i) *Element a of $J_i(\mathbf{s})$ can be written as: $J_{ia}(\mathbf{s}) = j_i(\mathbf{s}_i^a)$ for some $j_i : \mathbb{S}_i \to \mathbb{R}$.*

ii) $E[\boldsymbol{\upsilon}|\mathbf{s}] = 0$.

Part i) of this assumption is relatively weak. I require the immediate combinatorial pay-off from ending the period in state $\mathbf{s}^a$ depends only on this final state.[20] Part ii) of the assumption just ensures that the mean of $\boldsymbol{\upsilon}$ is absorbed into J.

By stacking J over $\mathbf{s}$ and j over $\mathbf{s}_i$ I define a mapping between $J(\mathbf{s})$ and $j(\mathbf{s}_i)$:

$$\underbrace{\mathbf{J}}_{S2^L\times 1} = \begin{pmatrix} J(\mathbf{s}_1) \\ \vdots \\ J(\mathbf{s}_S) \end{pmatrix} \qquad \underbrace{\mathbf{j}}_{S_i\times 1} = \begin{pmatrix} j(\mathbf{s}_{i1}) \\ \vdots \\ j(\mathbf{s}_{iS}) \end{pmatrix} \qquad \mathbf{J} = B\mathbf{j} \tag{5}$$

Where B is an $S2^L \times S_i$ transformation matrix with rank S_i. We will also write $J(\mathbf{s}) = B_{\mathbf{s}}\mathbf{j}$ using just the $2^L \times S_i$ sub-matrix $B_{\mathbf{s}}$. The idea is that this matrix selects elements of $\mathbf{j}$ according to the possible ex-post states for player i, given they started the period in state $\mathbf{s}$. We can define the relationship $P(\mathbf{b}|\mathbf{s})^T B_{\mathbf{s}} = Q(\mathbf{b}|\mathbf{s})^T A_{\mathbf{s}} C$ for the $S \times S_i$ matrix C. entry mn of C is equal to 1 if $\mathbf{s}_i^m = \mathbf{s}_i^n$, and zero otherwise. Therefore, each row of C contains a single non-zero entry, while column n contains

[20] This differs from GKS' approach in which J_{ia} is able to depend on both $\mathbf{s}^a$, $\mathbf{s}$, and potentially even other $\mathbf{s}^{a'}$. For example, their approach permits identification of reference dependent preferences. However, in their empirical example they do impose this restriction. Furthermore, while I impose that j is independent of $\mathbf{s}_{-i}$, this restriction is not necessary for identification or estimation.

a 1 in rows for which $\mathbf{s}_i = \mathbf{s}_i^n$. This relationship holds because C collapses Q over states with the same $\mathbf{s}_i$.

Based on these assumptions, I will prove the following proposition:[21]

Proposition 2. *Under assumptions 1 - 4, the model primitives F and $\mathbf{j}$ are non-parametrically identified up to β and $j(\mathbf{s}_{i1})$.*

As is standard, I take β as given. Meanwhile, we must normalise $j(\mathbf{s}_{i1})$ because the level of pay-offs is not-identified - only marginal pay-offs are identified. Just as, in the single-object case, we normalise the pay-off from losing to zero.

3.2 First Order Conditions

The agent's problem is to maximise their expected discounted pay-off, and so in each period the agent maximises the following object, with respect to $\mathbf{b}$:

$$\begin{aligned}\tilde{\Pi}(\mathbf{b}|\boldsymbol{v};\mathbf{s}) &= \Gamma(\mathbf{b}|\mathbf{s})^T(\boldsymbol{v}-\mathbf{b}) + P(\mathbf{b}|\mathbf{s})^T J(\mathbf{s}) + \beta Q(\mathbf{b}|\mathbf{s})^T V(\mathbf{s}) \\ &= \Gamma(\mathbf{b}|\mathbf{s})^T(\boldsymbol{v}-\mathbf{b}) + P(\mathbf{b}|\mathbf{s})^T B_{\mathbf{s}}\mathbf{j} + \beta Q(\mathbf{b}|\mathbf{s})^T A_{\mathbf{s}}\mathbf{V}\end{aligned}$$

Assumption 2 ensures that $P(\mathbf{b}|\mathbf{s})$, $Q(\mathbf{b}|\mathbf{s})$, and $\Gamma(\mathbf{b}|\mathbf{s})$ are continuously differentiable in $\mathbf{b}$. Necessary First Order Conditions of optimal bidding are then given as:

$$\underbrace{\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})}_{L\times L}\underbrace{(\boldsymbol{v}-\mathbf{b}^*)}_{L\times 1} = \underbrace{\Gamma(\mathbf{b}^*|\mathbf{s})}_{L\times 1} - \underbrace{\nabla_{\mathbf{b}}P(\mathbf{b}^*|\mathbf{s})}_{L\times 2^L}\underbrace{B_{\mathbf{s}}\mathbf{j}}_{2^L\times 1} - \beta\underbrace{\nabla_{\mathbf{b}}Q(\mathbf{b}^*|\mathbf{s})}_{L\times n^L}\underbrace{A_{\mathbf{s}}\mathbf{V}}_{n^L\times 1} \tag{6}$$

As above, under the assumption of zero probability ties (or exogenous tie-breaking), $\Gamma_{il}(\mathbf{b}|\mathbf{s}) = \prod_{j\neq i} G_{jl}(b_{il}|\mathbf{s})$. Therefore $\nabla\Gamma$ must be a diagonal matrix with entry ll

[21] Importantly, this proposition does not collapse down to GKS' identification result, even when the game is static or bidders are myopic (e.g. $\beta = 0$). This proposition differs from GKS's key proposition in the source of variation, and exclusion restriction, used to identify J. They prove identification from variation in characteristics of non-bid on lots and characteristics of bidder i's competitors, which cause 'exogenous' variation in Γ and P. I focus on variation in the state variable, which creates variation in Γ and P but also directly creates variation in $J + \beta V$, all of which then cause variation in bids.

equal to $\sum_{j\neq i} g_{jl}(b_{il}|\mathbf{s})\prod_{k\neq j,i} G_{kl}(b_{il}|\mathbf{s})$, and so $\nabla\Gamma$ must be invertible for most $\mathbf{b}$.

3.3 The Inverse Bidding System and Identification of F

I now consider how the first order conditions can be inverted to give the inverse bidding system. This enables me to show that F is non-parametrically identified, conditional on J and βV. This argument is almost precisely the same as that presented by GKS, which is a simple multi-object extension of Guerre et al. (2000) identification result from inverting the first order conditions.

Proposition 3. *Under assumptions 1 - 3, and conditional on J and βV being known, the cdf F is non-parametrically identified.*

Suppose both J, and βV are already identified The First Order Conditions can then be inverted to obtain the inverse bid function:[22]

$$\boldsymbol{\xi}(\mathbf{b}^*|J,\beta V;\mathbf{s}) = \underbrace{\mathbf{b}^*}_{\text{observed}} + \underbrace{\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}}_{Identified}[\Gamma(\mathbf{b}^*|\mathbf{s}) - \underbrace{\nabla_{\mathbf{b}}P(\mathbf{b}^*|\mathbf{s})}_{Identified}B_{\mathbf{s}}\mathbf{j} - \underbrace{\nabla_{\mathbf{b}}Q(\mathbf{b}^*|\mathbf{s})}_{Identified}A_{\mathbf{s}}\beta\mathbf{V}] \tag{7}$$

This system of equations is a natural extension of the standard inverse bid function. At the optimum the lot specific value is equal to bids $\mathbf{b}^*$ plus a lot specific markup $\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\Gamma(\mathbf{b}^*|\mathbf{s})$, minus a combination markup $\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\nabla_{\mathbf{b}}P(\mathbf{b}^*|\mathbf{s})B_{\mathbf{s}}\mathbf{j}$, minus the standard dynamic markup which depends on precisely who won each combination of lots $\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\nabla_{\mathbf{b}}Q(\mathbf{b}^*|\mathbf{s})A_{\mathbf{s}}\beta\mathbf{V}$.

We can evaluate this inverse bid function at the observed bids, which holds for a particular candidate $(J,\beta V)$. If this candidate $(J,\beta V)$ is correct, that is, if we have already identified $(J,\beta V)$, then $\boldsymbol{\xi}(\mathbf{b}^*|J,\beta V;\mathbf{s}) = \boldsymbol{\upsilon}$ trivially. From here it is simple to non-parametrically identify $F(.)$.

[22]For rows l such that $l \notin \mathbb{L}$ and hence $b_l = \emptyset$, we will also have $\xi_l = \emptyset$

3.4 Identification of V

I now demonstrate that conditional on $\mathbf{j}$ being known, we can write $\mathbf{V}$ as a function of the distribution of bids and $\mathbf{j}$ only. This essentially extends Proposition 1 from JBP to the multi-object case.

Proposition 4. *Under assumptions 1 - 4, the expected stage pay-off is given by:*

$$\begin{aligned}\tilde{\Pi}(\mathbf{b}^*|\boldsymbol{v};\mathbf{s}) = \Gamma(\mathbf{b}^*|\mathbf{s})^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\Gamma(\mathbf{b}^*|\mathbf{s}) \\ + [P(\mathbf{b}^*|\mathbf{s})^T \;-\; \Gamma(\mathbf{b}^*|\mathbf{s})^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\nabla_{\mathbf{b}}P(\mathbf{b}^*|\mathbf{s})]B_{\mathbf{s}}\mathbf{j} \\ + [Q(\mathbf{b}^*|\mathbf{s})^T \;-\; \Gamma(\mathbf{b}^*|\mathbf{s})^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\nabla_{\mathbf{b}}Q(\mathbf{b}^*|\mathbf{s})]A_{\mathbf{s}}\beta\mathbf{V}\end{aligned} \quad (8)$$

Proof of this proposition is given in Appendix B. This relation is an extension of Proposition 1 presented in JBP. The first term on the right hand side of the equation can be written as $\sum_l \frac{\Pi_{j\neq i}G_{jl}(b_{il})}{\sum_{j\neq i} g_{jl}(b_{il})}$ - the first term in JBP's proposition. Unlike in the single unit case there is an additional adjustment for the non-additivity.

From Proposition 4, employing the identity $P(\mathbf{b}|\mathbf{s})^T B_{\mathbf{s}} = Q(\mathbf{b}|\mathbf{s})^T A_{\mathbf{s}}C$, and taking an expectation of the observed bids, we can write the ex-ante value function as:

$$\begin{aligned}V^e(\mathbf{s}) &= \Phi(\mathbf{s}) + \Omega(\mathbf{s})[C\mathbf{j} + \beta\mathbf{V}] \\ \text{Where} \qquad \Phi(\mathbf{s}) &= E_{\mathbf{b}}[\Gamma(\mathbf{b}^*|\mathbf{s})^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\Gamma(\mathbf{b}^*|\mathbf{s})|\mathbf{s}] \\ \Omega(\mathbf{s}) &= E_{\mathbf{b}}[Q(\mathbf{b}^*|\mathbf{s})^T - \Gamma(\mathbf{b}^*|\mathbf{s})^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\nabla_{\mathbf{b}}Q(\mathbf{b}^*|\mathbf{s})|\mathbf{s}]A_{\mathbf{s}}\end{aligned}$$

Stacking over $\mathbf{s}$ write the continuation value as $\mathbf{V} = T\mathbf{V}^e = T\Phi + T\Omega[C\mathbf{j} + \beta\mathbf{V}]$ Which we can invert for: $\mathbf{V} = (I_S - \beta T\Omega)^{-1}[T\Phi + T\Omega C\mathbf{j}]$. This yields a stationary solution for the continuation value. Non-singularity of $(I_S - \beta T\Omega)$ is a condition for stationarity, though this matrix is guaranteed to be non-singular anyway.[23] This

[23]Non-singularity follows from the matrix being strictly diagonally dominant. This result is known as the Levy-Desplanques Theorem. Strict diagonal dominance arises because every element of $T\Omega$ is weakly positive, and rows sum to 1. Therefore, off diagonals of $I - \beta T\Omega$ lie in the interval $(-\beta, 0]$, while diagonals are strictly positive, and rows sum to $1 - \beta$.

ensures that, conditional on $\mathbf{j}$ being known, the continuation value is point identified.

3.5 Identification of j

I now prove that identification of $\mathbf{j}$ collapses to a rank condition. Impose the mean zero property of $\boldsymbol{\upsilon}$ for:

$$\begin{aligned}0 = E_{\boldsymbol{\upsilon}}[\boldsymbol{\upsilon}|\mathbf{s}] = E_{\mathbf{b}^*}[\boldsymbol{\xi}(\mathbf{b}^*;\mathbf{s},(\mathbf{j},\mathbf{V}))|\mathbf{s}]\\ = E_{\mathbf{b}^*}[\mathbf{b}^* + \nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\Gamma(\mathbf{b}^*|\mathbf{s})|\mathbf{s}] - E_{\mathbf{b}^*}[\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\nabla_{\mathbf{b}}Q(\mathbf{b}^*|\mathbf{s})|\mathbf{s}]A_{\mathbf{s}}[C\mathbf{j}+\beta\mathbf{V}]\\ = \Upsilon(\mathbf{s}) - \Psi(\mathbf{s})[C\mathbf{j}+\beta\mathbf{V}] \quad (9)\end{aligned}$$

Stacking over $\mathbf{s}$, and substituting in the expression for $\mathbf{V}$, we get:

$$\begin{aligned}0 = \Upsilon - \Psi[C\mathbf{j}+\beta\mathbf{V}]\\ = \Upsilon - \beta\Psi(I_S-\beta T\Omega)^{-1}T\Phi - [\Psi C + \beta\Psi(I_S-\beta T\Omega)^{-1}T\Omega C]\mathbf{j}\\ = \Upsilon - \beta\Psi(I_S-\beta T\Omega)^{-1}T\Phi - \Psi(I_S-\beta T\Omega)^{-1}C\mathbf{j} \quad (10)\end{aligned}$$

This system of LS equations in $S_i - 1$ unknowns overcomes the standard order condition discussed in GKS. There exists a unique solution to this system ($\mathbf{j}$ is point identified) if and only if the $LS \times S_i$ matrix $\Psi(I_S-\beta T\Omega)^{-1}C$ has rank $S_i - 1$.

3.6 Rank of $\Psi(I_S-\beta T\Omega)^{-1}C$

This rank condition requires that observations of bidding behaviour, across all S states, produces sufficient information about $\mathbf{j}$ to uniquely pin down all $S_i - 1$ elements. We gain information about $j(\mathbf{s}_i)$ from how bidding behaviour changes when $\mathbf{s}_i$ is a possible outcome from the round of auctions. By stacking the moment conditions in equation 10 we stitch together the information about $\mathbf{j}$ across different state observations. In addition to information as $\mathbf{s}_i$ varies, we also use information as $\mathbf{s}_{-i}$ varies, even when this is excluded from the function j, resulting in additional identifying variation. One additional assumptions is sufficient for this rank condition to hold:

Assumption 5. The set $\mathbb{S}_i$ is partially ordered according to the strict partial ordering $\succeq$, such that if $\mathbf{s}_i' \in \mathbb{S}_i^a(\mathbf{s}_i, \mathbf{s}_0)$ then $\mathbf{s}_i' \succeq \mathbf{s}_i$. In addition, the maximal elements of $\mathbb{S}_i$ do not outnumber the non-maximal elements.

The partial ordering assumption is very mild, really only imposing the transitivity of partially ordered sets. A requirement for these partial orderings is that winning an auction is monotonic: one cannot gain an object from winning one auction and give it away by winning a different auction. I limit the number of maximal elements because observations of bidding from maximal elements are not informative.[24]

Proposition 5. *Under assumption 1 - 5* $\Psi(I_S - \beta T\Omega)^{-1}C$ *has rank* $S_i - 1$

Proof of this proposition is given in Appendix C. It is omitted from the main text for brevity. This rank condition is not trivial, since Ψ is certainly rank deficient. The proof proceeds by first establishing the rank of Ψ and finding it's null space. As we stitch together observations of bidding from each state, stacking $\Psi(\mathbf{s})$ across $\mathbf{s}$, the rank increases by at least two each time. I then consider the image of $(I_S - \beta T\Omega)^{-1}C$, proving that the only element in the intersection of this image and the null space of Ψ is the constant vector.

3.7 Extensions

I now consider several extensions to this identification framework. Some of these extensions are extremely mild, often relaxing some of the previously stated assumptions.

[24]An element $\mathbf{s}_i$ is defined as maximal if there does not exist an $\mathbf{s}_i' \in \mathbb{S}_i$ such that $\mathbf{s}_i' \succ \mathbf{s}_i$. The interpretation of how an element can be considered maximal is left ambiguous. One interpretation is that these maximal elements are the largest (in partial ordering terms) states that are observed as possible ex-post outcomes, but are never observed as ex-ante outcomes. In this way, we want to try to identify j for these states, but do not get to use observations beginning in these states.

Other extensions significantly alter the empirical and model framework.[25]

Second-price auctions: In Appendix D.1 I demonstrate that identification extends, almost trivially, to simultaneous second-price auctions. Holding constant all but one bid, optimal bidding requires bidding the expected marginal value of winning that additional lot, given the other bids. This yields a set of necessary first order conditions similar to the first-price case. These can then be inverted for an inverse bid system that is identical to the first-price case, except that it is missing the markup term $\nabla_{\mathbf{b}}\Gamma(\mathbf{b})^{-1}\Gamma(\mathbf{b})$. Proposition 4 is then extended in a similar fashion. From here the identification and estimation arguments extend intuitively.

Binding reservation prices: In Appendix D.2 I consider how the presence of binding reservation prices impact identification. The binding reservation prices essentially cause censoring in the data. We immediately lose point identification of both F and j. However, F remains partially identified, using a similar argument as presented in subsection 3.3. We can no longer use moment conditions to identify j, as we did in subsection 3.5, and instead make used of quantile conditions. As a result, some elements of j can only be bounded. While reservation prices are a mathematical nuisance, they do not have a meaningful impact on identification, particularly when econometricians make parametric assumptions on F.

Endogenous Entry: In Appendix D.3 I consider an additional stage in-which the bidder choose a subset of auctions to enter, where each entering each subset has an associated cost. This creates a minor change to the representation of V as a function of j. The identification of j and F then follow from previous arguments. The distribution of the entry cost distribution then follows from standard results.

Inter-temporal Budget Constraint: In Appendix D.4 I relax the assumption that pay-offs are quasi-linear in wealth and instead allow for an inter-temporal budget constraint, equivalent to the standard no-ponzi condition. I prove that the quasi-

[25] Various trivial extensions are also possible, such as when we allow cost interdependencies. The framework presented above allows us to easily adapt the framework of Somaini (2020), for example allowing j and F to depend on $\mathbf{s}_{-i}$, or even allowing correlations between v_{ilt} and $v_{i'lt}$.

linear model is observationally equivalent to the budget constraint model when the marginal value of wealth is constant. In this Appendix I also establish that, if we observe bidders' budgets, the estimation procedure discussed in section 4 can be similarly extended with little additional computational cost.

Stochastic Combination Value: In Appendix D.5 I allow the combination value to be stochastic. I consider several cases. First, when the combination value depends only on the number of lots won, not which lots are won. Second, when the combination value is a function of low dimensional ($< L$) unobservables, such as unobserved states or unknown parameters. The only necessary restriction is that this function is strictly monotonic in the unobservables. Third, when when the combination value is a function of low dimensional unobservables, and the bidder faces an inter-temporal budget constraint. In all three cases identification arises from proving that the first order conditions can be inverted to point identify the unobservables.

4 Estimation Method

Having established the non-parametric identification of the dynamic game, I now describe a computationally feasible procedure to estimate F and j.[26] I begin with a general description of the novel procedure, outlining the key intuition. I then describe in detail the three estimation steps.

The Premise

The central premise of the procedure exploits the fact that, under the assumption that payoffs are additively separable over time, we can write the continuation value

[26]Standard procedures are either infeasible or inapplicable. The method of JP is inapplicable since we cannot write the maximised expected payoff as a function of bids only. Interior fixed point methods are infeasible due to the computation difficulty of repeated numerically maximising expected payoffs. Finally, CCP style methods are applicable, but are generally more computationally expensive than the approach presented below. In the most abstract case presented in the previous section, fully non-parametric with a finite set of observed states, a CCP approach will be equivalent to the one presented below.

as a function of: (1) Primitives of the (observed) transition process, (2) the observed distribution of equilibrium actions, and (3) the sum of the mean flow pay-off function, and the discounted continuation value. I refer to this sum as the 'pseudo-static' pay-off, as it is essentially the object we estimate if we incorrectly assume myopic bidding. Notably, this relationship is weaker than the key theorem of Hotz and Miller (1993) that underpins CCP methods. In this setting, this relationship is given by:

$$V(\mathbf{s}') = \int_{\mathbf{s}} \int_{\mathbf{b}} \Pi(\mathbf{b}|\mathbf{s}; K) dG(\mathbf{b}|\mathbf{s}) dT(\mathbf{s}|\mathbf{s}') \tag{11}$$

$$\begin{aligned} \text{Where:} \quad \Pi(\mathbf{b}|\mathbf{s}; K) &= \Gamma(\mathbf{b}|\mathbf{s})^T \nabla_{\mathbf{b}} \Gamma(\mathbf{b}|\mathbf{s})^{-1} \Gamma(\mathbf{b}|\mathbf{s}) \\ &\quad + [Q(\mathbf{b}|\mathbf{s})^T - \Gamma(\mathbf{b}|\mathbf{s})^T \nabla_{\mathbf{b}} \Gamma(\mathbf{b}|\mathbf{s})^{-1} \nabla_{\mathbf{b}} Q(\mathbf{b}|\mathbf{s})] K(\mathbf{s}) \\ &\qquad \text{and} \quad [K(\mathbf{s})]_a = k(\mathbf{s}^a) = j(\mathbf{s}_i^a) + \beta V(\mathbf{s}^a) \end{aligned}$$

This equation writes Proposition 4 as a function of the pseudo-static pay-off function k, then takes an expectation over observed bids and the transition process. Both G and T are easily estimated using standard methods. Therefore, if we had a consistent estimate for the function $k : \mathbb{S} \to \mathbb{R}$, then we would have a consistent estimate for V, and then j $(= k - \beta V)$. Like CCPs or the distribution of equilibrium bids, this function $k(.)$ is not a model primitive but a function of primitives. The central estimation problem then concerns estimating k.

This procedure is closely related to the estimation of a static model. If players are myopic, so that $\beta = 0$, then $k = j$. Therefore, the procedure is almost like estimating the model *as if* it were static. While we may expect that j is independent of $\mathbf{s}_{-i}$, k in general is not. Therefore, the procedure involves estimating the model as if it were a generalised static model, in which pay-offs are allowed to depend on every element of the state space that enters the continuation value.[27]

In part, the procedure is a generalisation of JP, since we write the Continuation

[27] This permits a simple test of forward looking behaviour. If the model is correctly specified and $\mathbf{s}_{-i}$ is excluded from j, then observing that k varies with $\mathbf{s}_{-i}$ is sufficient to reject myopia.

Value as a function of the distribution bids and this additional combinatorial term. When k is additively separable $\Pi(\mathbf{b}|\mathbf{s};K) = \Gamma(\mathbf{b}|\mathbf{s})^T \nabla_{\mathbf{b}} \Gamma(\mathbf{b}|\mathbf{s})^{-1}\Gamma(\mathbf{b}|\mathbf{s})$, just as in JP. The procedure is also related to CCP methods. Both CCP methods and JP's procedure involve writing the continuation value as a function of the distribution of observed actions. The pseudo-static pay-off function, if it is identified from standard choice data, will itself be a function of the distribution of observed actions.[28] Therefore the continuation value is still a function of the distribution of observed actions, however it requires the additional intermediate step of estimating k.

Finally, in a discrete choice setting, this estimation procedure is numerically equivalent to CCP estimation when the researcher fits a parametric functional form to the conditional choice probabilities. For example, in a binary choice setting one might fit a logit link function, so that $p(d_t = 1|\mathbf{s}_t) = \frac{\exp(\delta \mathbf{s}_t^1)}{\exp(\delta \mathbf{s}_t^0)+\exp(\delta \mathbf{s}_t^1)}$. This is equivalent to specifying the choice specific value function $U(d_t = a|\varepsilon_t^a, \mathbf{s}_t) = \varepsilon_t^a + j(\mathbf{s}_t^a) + \beta V(\mathbf{s}_t^a)$, where ε_t^a is Type 1 extreme value, and $j(\mathbf{s}_t^a) + \beta V(\mathbf{s}_t^a) = k(\mathbf{s}_t^a) = \delta \mathbf{s}_t^a$. Then we have $V(\mathbf{s}_t) = E[\ln(\exp(k(\mathbf{s}_{t+1}^1)) + \exp(k(\mathbf{s}_{t+1}^0)))|\mathbf{s}_t] + \gamma$, where γ is Euler's constant. The difference is that we give the reduced form coefficients δ a structural interpretation.

I now outline the three key estimation steps. The discussion is kept general within the dynamic multi-object auction framework. In Section 5 I present an example with several convenient functional form assumptions. The procedure can be written succinctly as the following three step procedure:

1. Estimate equilibrium win probabilities Γ and Q, and transition functions T.

2. Given $\hat{\Gamma}$ and $\hat{Q}$, estimate F and k - the primitives of the pseudo-static model.

3. Given $\hat{\Gamma}, \hat{Q}, \hat{T}, \hat{F}$, and $\hat{k}$, evaluate $\hat{V}$ then back out $\hat{j}$

[28] In this application, identification of k just uses the assumption that $E[\boldsymbol{v}|\mathbf{s}] = 0$. Point Identification is then ensured by proposition 5, ensuring the first line of equation 10 is invertible, as the matrix Ψ is invertible up to normalisations.

4.1 Step 1.

The First Stage represents the standard first stage in the empirical auction literature. There are several possible approaches the research might take. The researcher could estimate the conditional cumulative distribution of bids G_{il}, then form $\Gamma_{il}(b)$, $P_{il}(b)$, and $Q_{il}(b)$ respectively. This is the approach taken in both GKS and JP. Otherwise the researcher may directly estimate Γ by estimating the probability that i wins any given lot given their bid, estimating $P(\max_{j \neq i} \{b_j\} \leq b|b)$, essentially estimating the distribution of highest bids other than bids belonging to i. This is the approach taken in Cantillon and Pesendorfer (2007) and Raisingh (2021).

The researcher can easily take either a parametric or non-parametric approach, so long as the ensuing estimated object is continuously differentiable. That is, the researcher must continue to impose the assumptions required for identification. The same goes for estimating the transition distribution $T_{\mathbf{s}}(.|\mathbf{s}^a_{t-1})$.

4.2 Step 2.

In the second step we estimate the sum of the immediate combination value and the discounted continuation value; the pseudo-static pay-off function $k(\mathbf{s})$. This step broadly follows the second stage in the estimation procedure presented in GKS. Practically, this step will be done by exploiting the moment conditions used in the identification argument: $E[\boldsymbol{\upsilon}|\mathbf{s}] = 0$. Set $\hat{k}$ such that, for all l and all $\mathbf{s}$, $E[\xi_l(\mathbf{b}^*|k; \mathbf{s})|\mathbf{s}] = 0$.

However, the identifying conditions gives rise to an additional set of moments:

$$E[\upsilon_l \mathbf{h}(\mathbf{s})] = E[E[\upsilon_l \mathbf{h}(\mathbf{s})|\mathbf{s}]] = E[E[\upsilon_l|\mathbf{s}]\mathbf{h}(\mathbf{s})] = 0$$

Where $\mathbf{h}(\mathbf{s})$ is a known vector valued function of $\mathbf{s}$, such as $\mathbf{h}(\mathbf{s}) = \mathbf{s}$ or $\mathbf{h}(\mathbf{s}) = \mathbb{I}[\mathbf{s} = \bar{\mathbf{s}}]$ (dummy variables for each possible state). That is, we can also just set $\hat{k}$ to ensure that $\xi_l(\mathbf{b}^*|k; \mathbf{s})$ is mean independent of $\mathbf{s}$. These moment conditions help us interpret the estimation problem as an instrumental variables procedure with $\mathbf{h}(\mathbf{s})$ as the instruments. Rewrite the FOCs as a regression equation:

$$\underbrace{b_{lt} + \frac{\Gamma_l(b_{lt}|\mathbf{s}_t)}{\nabla_{b_l}\Gamma_l(b_{lt}|\mathbf{s}_t)}}_{y_t} = \underbrace{-[\frac{\nabla_{\mathbf{b}}Q(\mathbf{b}_t|\mathbf{s}_t)}{\nabla_{b_l}\Gamma_l(b_{lt}|\mathbf{s}_t)}]_{l.}K(\mathbf{s}_t)}_{\mathbf{x}_t\beta} + \upsilon_{lt}$$

We could estimate $K(\mathbf{s}_t)$ using a least squares procedure; set k to minimise the sum squared residuals $\sum_t \sum_l \upsilon_{lt}^2$. But, in general $E[\upsilon_{lt}[\frac{\nabla_{\mathbf{b}}Q(\mathbf{b}_t|\mathbf{s}_t)}{\nabla_{b_l}\Gamma_l(b_{lt}|\mathbf{s}_t)}]_{l.}] \neq 0$ because $E[\upsilon_{lt}b_{l't}] \neq 0$. Therefore we have a standard endogeneity problem. Fortunately, we have a set of candidate instruments, $\mathbf{h}(\mathbf{s}_t)$, so we can write the first stage as:

$$\underbrace{-[\frac{\nabla_{\mathbf{b}}Q(\mathbf{b}_t|\mathbf{s}_t)}{\nabla_{b_l}\Gamma_l(b_{lt}|\mathbf{s}_t)}]_{l.}}_{\mathbf{x}_t} = \boldsymbol{\pi}_l \underbrace{\mathbf{h}(\mathbf{s}_t)}_{\mathbf{z}_t} + \varepsilon_{lt}$$

This interpretation is helpful as it enables the researcher to make use of a number of standard instrumental variable procedures, such as analysing the relevance and validity of our instruments.[29]

The econometrician is able to estimate k either parametrically or non-parametrically. In the application presented in section 5 I use a semi-parametric specification, assuming K takes a convenient linear-in-parameters form, with $k(\mathbf{s}) = \mathbf{h}(\mathbf{s})^T\theta^k$, and estimate θ^k using Generalised Method of Moments, imposing $E[\upsilon_l \mathbf{h}(\mathbf{s})] = 0$. Having estimated k we back out the distribution F using the empirical cdf of inverse bids ξ.

4.3 Step 3.

The Third Stage broadly corresponds to the second stage in JP's estimation procedure. Given the distribution of bids and the pseudo-static pay-off function K; use these objects to evaluate the continuation value.

Building on Proposition 4 we use equation 11 to evaluate the continuation value,

[29] Depending on the form of $\mathbf{h}$ the first stage may not need to be estimated and $\boldsymbol{\pi}_l$ may be known. This holds in the application presented in Section 5.

evaluated using estimates of T, G, and K. First, the researcher forms $\hat{\Pi}(\mathbf{b}_t|\mathbf{s};\hat{K})$ for each t. Next, they evaluate the ex-ante value function by either numerically integrating over the estimated distribution of bids or by taking a conditional expectation over observed bids. This latter approach is convenient as it ensures the distribution of bids never has to be explicitly estimated. The continuation value is then formed by taking a conditional expectation over the transition process, which can again be performed using numerical integration or by averaging over the observed distribution.

Finally, $\hat{j}$ can be backed out using the identity $\hat{j} = \hat{k} - \beta\hat{V}$ for some fixed value of β.[30] At this stage we must average our $\hat{j}$s over $\mathbf{s}_{-i}$. With a correctly specified model and an infinite amount of data there should be no variation. The choice of weighting procedure is important, since it will be necessary to place less weight on estimates for $\mathbf{s}_{-i}$ with fewer observations, so are more imprecisely estimated.[31]

4.4 Inference

Inference in this estimation procedure is complicated by the multiple stage nature. Inference on the first two stages is standard, making use of two step variance estimates for the second stage. Inference in the second stage enables hypothesis testing on model specification. For example, testing for the presence of non-additivity across lots, justifying the need for the dynamic multi-object approach.

In the third stage one must use the delta-method, taking as given the first two stages. Bootstrap and Monte-Carlo methods are also possible, though may be time intensive. In Section 5 I present an application employing the same linear-in-parameters specification for both the second and third stages. This makes inference in the third

[30]Note that at this stage one can vary β to see how $\hat{j}$ varies correspondingly. Also note, is β identified here? i.e. the β such that $\hat{j}$ is independent of $\mathbf{s}_{-i}$.

[31]This weighting problem is related to the standard small sample problem for CCP methods, which is generally alleviated by reducing the weight on observations with poorly estimated continuation values. Similarly, like how these problems can be alleviated by taking a parametric approach to estimating conditional choice probabilities, a parametric approach to estimating k can alleviate these problems.

stage extremely simple, as $j(\mathbf{s})$ inherits the same functional form as k and V. The variance of $\hat{j}$ is a function of the variances from the second and third stages, and their covariance. Appendix E discusses how the model can be estimated fully non-parametrically, and presents the results of a simulation study examining the performance of both the parametric and non-parametric estimator.

5 Application

I now consider an application of this model. I apply the model to data from Michigan Department of Transport's procurement auctions for highway construction and maintenance contracts. Contracts are allocated using simultaneous low-price sealed bid auctions, averaging around 45 contracts auctioned in each round, with rounds taking place every 2-4 weeks. 56 percent of bidders submit bids on more than one auction in a given round. Bidders must also pass a pre-qualification check to ensure they are eligible to work on that type of contract.

A large body of previous work has found evidence of cost complementarities in highway procurement, for example JP who find evidence of capacity constraints. Several previous papers, including GKS, have found evidence of complementarities in MDOT procurement specifically. GKS find evidence that a firm's cost of taking on a new project are increasing in their backlog, but the more similar their current projects the less the dis-economies of scale. Meanwhile, Raisingh (2021) finds evidence of forward looking bidding behaviour in the MDOT auctions. This suggests the need to use a dynamic multi-object auction model to estimate firm's costs.

For the purposes of this application I focus on road construction and paving projects. These projects either involve hot-mix asphalt, concrete construction, or both. I consider how firm's backlogs of both asphalt and concrete projects impact their costs, and likewise how the two backlogs interact. Given GKS's findings, the expectation is that costs for all projects will be increasing in both backlogs, but that the cost of an asphalt contract increases faster in concrete backlog, and vice versa. Understanding the degree of complementarities is important for auction design - if the complementarities are especially large, it may be useful for MDOT to make sure similar auctions are held together.[32]

5.1 Data

I use the same data used by GKS, making use of data on bids, the contracts being auctioned, and the competing firms.[33] This includes information on almost every auction run between 2002 and 2014. I exclude the first two years in order to construct backlogs. The contract data includes project descriptions, locations, pre-qualification requirements, the engineer's estimate of project cost, and the list of participating firms and their bids. I also make use of the project 'type' constructed by GKS. The bidding data includes the bids and identities of both winning and losing bids.

The firm level data includes details on the sub-sample of firms who submit at least 50 bids. This details the number and location of plants, and a description of the type of company. Following GKS's classification system, a large bidder is defined as one with at least 6 plants in Michigan. A regular bidder is defined as one that submits more than 100 bids in the sample period, otherwise they are designated a fringe bidder. In the final sample there are 36 regular bidders, 8 large (regular bidders) and 686 fringe bidders. I further categorise regular bidders into one of three types of firm: General contractors, Paving companies, and Construction companies.

[32]In the current application I ignore the firm's entry problem. The entry problem is computationally difficult as firms maximise over a set of auctions to enter. Instead I assume that firms face negligible entry costs. This is in line with the suggestion that bid preparation costs vary from $5,000 to $10,000, around %1 of the estimated contract cost (Raisingh, 2021). This ensures that, given a set of available lots, the optimal entry combination is non-probabilistic, and so in evaluating the ex-ante value function we can ignore the entry problem and use just the expected maximum expected pay-off from the auctions they were observed entering.

[33]I kindly received my data directly from GKS. The auction level data is freely posted on the MDOt webpage: http://www.michigan.gov/mdot. Meanwhile their firm data is taken from a variety of sources including OneSource North America Business Browser, Dun and Bradstreet, Hoover's, Yellowpages.com and firms' websites.

Figure 1: Auction level summary statistics.

	Asphalt	Concrete	Both
Number	3563	712	1974
Auctions per Round	20.13	4.02	11.15
(p25 - p75)	(5 - 30)	(1 - 6)	(3 - 17)
Project Duration (days)	134.11	216.52	200.08
	(46 - 151)	(79.75 - 261.25)	(70 - 235.25)
Start Date - Let Date (days)	73.13	55.4	55.06
	(10 - 128)	(10 - 80)	(10 - 73)
Engineer's Estimate ($100,000s)	12.61	22.4	19.88
	(2.92 - 11.16)	(3.65 - 12.16)	(4.29 - 17.29)
Bidders per Auction	4.39	5.46	5.94
	(2 - 5)	(4 - 7)	(3 - 8)
Average Bid ($100,000s)	12.75	19.93	18.28
	(3.02 - 11.46)	(3.78 - 11.85)	(4.56 - 16.96)
Winning Bid ($100,000s)	11.98	21.19	18.69
	(2.69 - 10.46)	(3.34 - 11.46)	(3.99 - 16.27)

Note: Aside from the number of auctions, the numbers presented are means.

Contract level descriptives are summarised in Figure 1. There are many more asphalt and mixed projects than their are concrete only projects, which include bridge construction and maintenance. The size of projects, as measured by the Engineer's Estimate and project duration, exhibit major right skew, while logs of both measures is fairly symmetric. Importantly, the bulk of mass for all types of projects are fairly balanced, as seen in the 25th and 75th percentiles of project durations and engineer's estimates. However, the project duration for concrete and mixed projects are generally longer.[34]

Project locations are coded to the centroid of the county they are based in. Distance is calculated as the minimum distance (across plant locations) between a firm and the project location. On average Paving firms are closer to projects than fringe

[34]One curiosity from this table is that the average winning bid for mixed projects is higher than the average losing bid for mixed projects. This occurs due to the skewed distribution of sizes. A number of small mixed projects attract a lot of very low bids, bringing down the average losing bid, whereas this winning bid only brings down the average winning bid by a small amount.

Figure 2: Bidder level summary statistics.

	General	Paving	Construction	Fringe
Plants	1.73	6.71	1.5	1.43
Bids per Round	2.07	2.8	1.8	0.24
(p25 - p75)	(0 - 3)	(0 - 4)	(0 - 3)	(0 - 0)
Backlog: Asphalt	5.57	5.61	2.97	0.24
(millions)	(0.25 - 3.88)	(0.96 - 7.6)	(0.48 - 4.39)	(0 - 0.2)
Backlog: Concrete	3.41	2.18	2.79	0.2
(millions)	(0.18 - 3.41)	(0.11 - 3.83)	(0.23 - 1.35)	(0 - 0.09)
Distance to project	105.65	84.18	121.42	119.27
Distance given Bid	71.21	47.03	87.18	69.33
Distance given Won	65.53	45.01	82.51	58.63

bidders - they have more plants on average. As expected, bidders tend to bid on projects that are closer to them, and are more likely to win closer projects - due to bidding more aggressively.

A firm's remaining backlog on particular project at time t is calculated as the fraction of the project remaining until the completion date, multiplied by the engineer's estimate of the project cost. Their total backlog is then the sum of backlogs across projects that are currently under-way. I calculate backlogs for projects involving asphalt and concrete separately, assuming that projects involving both asphalt and concrete is evenly divided between the two materials. For the purposes of estimation I work with normalised backlogs, dividing by the standard deviation of the firms' observed backlogs. This ensures that any estimated backlog effects are estimated using within firm variation.

Bidder level descriptive statistics are summarised in Figure 2. Regular bidders tend to have much larger backlogs than fringe bidders, Paving firms particularly. Likewise, asphalt backlogs are generally higher than concrete backlogs - due to the larger number of asphalt projects. Finally, backlogs generally exhibit rightward skews, indicative of the right skewed project sizes. However this skewed distribution also suggests that firms do not 'smooth' their backlogs by picking up new contracts before existing projects are finished, and so keeping backlogs at a near constant level.[35]

5.2 Empirical model, revisited

I now apply and estimate the empirical model presented above. I apply the full dynamic multi-object model to regular bidders only, given that I need to observe sufficient observations of bidding to be able to estimate my objects of interest. I estimate separate parameters for each type of regular bidder. I assume fringe bidders are myopic, and that their costs are additive.

Unlike the standard first-price context, in the low bid context bidders pay their costs and receive their bid, if their bid is the lowest. This essentially involves just a minor relabelling of the previous model. For the sake of clarity, the Bellman equation for this problem is given by:

$$W(\boldsymbol{\upsilon}, \mathbf{s}) = \max_{\mathbf{b}} \left\{ \Gamma(\mathbf{b}|\mathbf{s})^T(\mathbf{b} - \boldsymbol{\upsilon}) - P(\mathbf{b}|\mathbf{s})^T J(\mathbf{s}) + \beta \sum_a Q_a(\mathbf{b}|\mathbf{s}) \int_{\tilde{\mathbf{s}}} \int_{\tilde{\boldsymbol{\upsilon}}} W(\tilde{\boldsymbol{\upsilon}}, \tilde{\mathbf{s}}) dF(\tilde{\boldsymbol{\upsilon}}|\tilde{\mathbf{s}}) dT(\tilde{\mathbf{s}}|\mathbf{s}^a) \right\}$$

The pseudo-static pay-off, written in terms of costs, is given by $-k(\mathbf{s}) = -j(\mathbf{s}_i) + \beta V(\mathbf{s})$. The individual state is the Firm's backlog of asphalt and concrete contracts. The common state consists of the set of lots on offer, including both their locations and other contract characteristics, such as size, duration, and type. I also allow the lot specific costs $\boldsymbol{\upsilon}$ to depend on additional lot specific factors such as distance to the project, and other lot specific covariates including project type $\times$ firm fixed effects.[36]

[35] There is generally lag between most contracts being won, and projects actually beginning. For simplicity I assume that every project begins before the next round of auctions. Otherwise firms are bidding on projects, when they already know that in several periods time their backlog will increase. This is difficult as it breaks the Markovian property of the game, since at any given time a firm must consider its current backlog, as well as it's expected backlog in every future period. Then, when bidding on a project that doesn't begin for several months, the firm must consider how their backlog is likely to change in those months as they take on additional projects.

[36] Parameters on these covariates are estimated like in a standard regression model, treating them as deterministic, where υ is then an additive error component. This allows us to continue to impose the assumption that $E[\upsilon_l|\mathbf{s}] = 0$. In the identification framework, these covariates would appear in

5.2.1 The State Space approximation

The state $\mathbf{s}$ should to include every firms' backlogs, as well as information on every auction being held each period. Such a large state space is computationally intractable. It is also unlikely that firms would keep track such a large state space. I instead follow the index approach taken by Raisingh (2021) and Aradillas-Lopez et al (2020). They condense $(\mathbf{s}_{-i}, \mathbf{s}_0)$ into a one dimensional index λ_{it}, approximating the degree of competition a firm faces on a given day. This ensures that for each firm I only need to track three states - their two backlogs and this competition index.

Aradillas-Lopez et al construct λ_{it} using a random forest to predict the minimum rival bid using $(\mathbf{s}_{-i}, \mathbf{s}_0)$. They use a variant of the random forest that is less likely to over-fit when the training sample is also the prediction sample. λ_{it} is assumed to be a function of: *i*) the mean backlog of rival bidders, *ii*) the number of rival bidders, *iii*) the number of auctions held that period.[37]

The index assumption also ensures that a firm's continuation value only depends on which combination of lots they win. It does not depend on the combinations won by their rivals. Therefore the firm only has to consider 2^L outcomes from the round of auctions, rather than n^L possible outcomes. This ensures that the matrix Q collapses down to the matrix P in estimation, reducing the dimensionality of the problem. This is reasonable - it is unlikely bidders consider how their bids impact the likelihood of their rivals winning different combinations of contracts.

Full details of how the index is constructed, as well as results from estimating the index, is given in Appendix **??**.

J, albeit imposing the assumption that they appear in the model additively separable.

[37]Following Raisingh (2021) I use distance bins (near, 0-25km, medium, 25-50km, and far, >50km), and size bins (small, an engineer's estimate of $< \$0.75$ million, and large, an engineer's estimate of $\geq \$0.75$ million). In this way, the index depends on the number of close rivals, the mean backlog of far rivals, the number of small near auctions, etcetera. I do not take into account sampling uncertainty in estimating the competition index.

5.2.2 Specification

While a fully non-parametric approach is possible, I follow the literature and take semi-parametric approach. It is important to allow the pseudo-static pay-off function k to depend on states, auction level observables, and the competition index. A non-parametric approach would suffer from a curse-of-dimensionality. Parametrisation also gives me additional statistical power to reject the null-hypothesis that pay-offs are additively separable.

First Stage

I make the simplifying assumption that firms believe the probability they win one auction is conditionally independent of whether they win another auction, ensuring the joint probabilities P can be written as products of the marginal probabilities.[38] I then specify the distribution of minimum rival bids as a three parameter Weibull distribution, with a support parameter (i.e. minimum rival bid) as 0.7 of the engineer's estimate for that contract.[39] This assumption is sensible as the Weibull distribution is the limiting distribution of the minimum of multiple independent random variables. The scale parameter is written as a function of auction-level characteristics and the competition index, which I denote using the vector $\mathbf{x}_{tl}$:

$$F(\underline{b}_{lt}; \beta_1, \alpha) = 1 - e^{-(\frac{b_{lt} - 0.7}{\exp(\mathbf{x}_{lt}\beta_1})))^{\alpha}}$$

I assume that states transition according to an autoregressive order (1) process:

$$\begin{pmatrix} \lambda_{it} \\ \mathbf{s}_{it} \end{pmatrix} = \boldsymbol{\alpha}_i + \boldsymbol{\alpha} \begin{pmatrix} \lambda_{it-1} \\ \mathbf{s}_{it-1} \end{pmatrix} + \boldsymbol{\varepsilon}_{it}$$

[38]This assumption is tested in Appendix **??** by examining the extent of correlation between winning bids, controlling for the state variables. While I am able to reject the null hypothesis of independence, I show the extent of this dependence, when conditioning on state variables, is small.

[39]As discussed in Raisingh (2021) this is because several projects appear to have miscalculated estimates. These are treated as outliers and removed. This occurred in around 1.5% of cases. I also impose that the shape parameter is greater than 1, however this restriction does not bind.

Where $\boldsymbol{\alpha}_i$ are firm specific intercepts, $\boldsymbol{\alpha}$ is a 3×3 dimension matrix, that is allowed to vary by firm type, and $\boldsymbol{\varepsilon}_{it}$ is a white noise innovation.[40]

Second Stage

I now discuss the chosen parameterisation for k and the moments used in estimation. I assume the pseudo-static pay-off is quadratic in backlogs. Testing for complementarities reduces to testing the significance of the quadratic terms. This gives statistical power to reject additively separable values. It also simplifies the dimensionality of the problem, ensuring that complementarities between lots are pairwise, so that I never have to construct the full 2^L dimensional vectors K and P. k also varies with the competition index λ, including interactions with both the linear and quadratic backlog variables. I allow the parameters to vary across the three firm types.[41] For a firm of type n the specification for the pseudo-static pay-off is therefore:

$$k_n(\mathbf{s}_t) = \lambda_{it}\theta_n^{\lambda} + \mathbf{h}(\mathbf{s}_{it})^T\theta_n^h + \lambda_{it}\mathbf{h}(\mathbf{s}_{it})^T\theta_n^{h\lambda}$$
$$\text{Where} \qquad \mathbf{h}(\mathbf{s}_{it})^T = \begin{pmatrix} s_{it}^a & s_{it}^c & (s_{it}^a)^2 & (s_{it}^c)^2 & s_{it}^a \times s_{it}^c \end{pmatrix}$$

I also make use of additional moment equations to facilitate estimation. If $\mathbf{s}_t$ does not substantially shift bidding behaviour there may be a weak instrument problem. This will occur if, for example, a firm's observed backlog does not vary relatively much, but they bid on many contracts simultaneously, so that the possible ex-post states $\mathbf{s}_t^a$ vary much more than $\mathbf{s}_t$. This means we essentially are trying to estimate k in regions where there is little variation in our instrument. This would be a problem if firms are successfully smoothing their backlogs.

[40] By construction backlogs actually transition deterministically - how much of each contract will be left in 14 days time is determined by the length and size of the project. However, since not all projects are completed at the same rate this requires that I must take into account future deterministic backlogs in the state variable. I assume this simple transition function for simplicity.

[41] Because λ is constant within a period the coefficient on λ is not identified from the second estimation stage. Instead, it will be backed out from the third stage.

To alleviate the possible weak instrument problems I consider several additional instruments, or several additional moment conditions. Write $\overrightarrow{\mathbf{s}}_l$ as the amount a firm's backlog will increase if they win lot l. In this application, this is the engineer's estimate of the project completion cost, split according to the type of contract. I make the additional assumption that $E[\upsilon_{ilt}|\mathbf{s} + \overrightarrow{\mathbf{s}}_l] = 0$. This allows me to use the ex-post state from only winning lot l as an additional instrument. This is more likely to be a strong instrument, given bidding behaviour on lot l is almost certainly affected by how winning that lot will increase the backlog.[42] In principle I could use every possible ex-post state as an instrument, however these instruments are expected to be highly correlated and we risk over-fitting the first stage. For illustrative purposes I also consider a specification that makes use of ex-post states from winning pairs of contracts, increasing the number of instruments ten-fold.

This step is estimated using standard GMM methods, ensuring standard errors are appropriately adjusted to account for the two stage estimation procedure. I allow lot specific costs to be correlated within a period, and include fixed effects for the county where the contract is located, and also Firm $\times$ contract type fixed effects. I winsorise the bottom percentile of estimated $\frac{\Gamma_l(b_{ilt})}{\nabla_b \Gamma_l(b_{ilt})}$, which predominantly includes bids below 0.7 of the Engineer's Estimate.[43]

[42]However, it is also possible that the larger the contract, so the larger $\overrightarrow{\mathbf{s}}_l$, the larger the lot-specific cost - meaning the instrument could be invalid. This is not a major worry for several reasons. First, I already control for the size of the contract through the linear term in k. Second, because I am using multiple instruments, the system is over-identified, so I can perform a simple Hansen test of over-identifying restrictions. Finally, the contract size normalisation I make, weighting observations by the inverse contract size, means this assumption is more reasonable.

[43]The distribution of contract sizes is very skewed, with a small number of extremely large contracts. These contracts impact backlogs much more than small contracts, and attract higher bids. These observations have a lot of leverage. To reduce the weight on these observations I weight observations by the inverse of the engineer's estimate of lot l (EE_l). This is equivalent to using of moment conditions of the form $E[\frac{\upsilon_{ilt}}{EE_l}|\mathbf{s}_t] = 0$. Furthermore, it is standard to normalise bids and associated costs by the size of the lot, which makes a similar assumption.

Third Stage

After forming the expected maximised period pay-off $\hat{\Pi}(\mathbf{b}_t|\hat{k};\mathbf{s}_t)$ I evaluate the ex-ante value function by approximating the conditional expectation over $\mathbf{b}_t$ using a linear in parameters prediction of Π_t given $\mathbf{h}(\mathbf{s}_t)$, as well as firm fixed effects.[44] This is convenient as it ensures the ex-ante Value Function, for a firm of type n, can be written as: $E[\hat{\Pi}_i(\mathbf{b}_t|\hat{k};\mathbf{s}_t)|\mathbf{s}_t] = \mu_i + \mathbf{h}(\mathbf{s}_t)^T\theta_n^V$.

The AR(1) assumption on the transition process, coupled with the quadratic form of $\mathbf{h}$, ensures I can write $E[\mathbf{h}(\mathbf{s}_t)|\mathbf{s}_{t-1}] = \mathbf{h}(\mathbf{s}_{t-1})^T\theta_n^\tau$, where θ_n^τ is a $|\mathbf{h}|\times|\mathbf{h}|$ dimensional matrix that can be written as a function of $\boldsymbol{\alpha}_n$ estimated in the first stage. This yields:[45]

$$j(\mathbf{s}_{it}) = \mathbf{h}(\mathbf{s}_{it})^T(\theta_n^k + \beta\theta_n^\tau\theta_n^V) = \mathbf{h}(\mathbf{s}_{it})^T\theta_n^j$$

5.3 Structural Estimates

5.3.1 First Estimation Step

I now discuss the results from estimating the distribution of minimum rival bids. Results from the first stage of estimation are given in Figure 3. I present three specification, including varying sets of Fixed Effects. The shape parameter is estimated to be well above one, ensuring that the Markup is monotonically increasing in bids. Note that mean of the distribution is increasing in the scale. For each of the scale parameters I include separate slope coefficients for each type of auction. For all three types of auction the winning bid is increasing in the competition index: When λ is large, so there is little competition, bids are less aggressive. Meanwhile

[44] This assumption is technically incompatible with the parametric assumption made of k previously. However we can easily test whether the prediction is subject to misspecification error using a standard RESET test. I am unable to reject the null of no specification error (at the 10% significance level) using a RESET test of order 10. Meanwhile no explicit parametric assumptions were made on the distributions of b or v.

[45] The variance estimate for θ^V must take into account how it depends on θ^k as well as first stage estimates. For the standard errors of θ^j we use: $\hat{Var}(\hat{\theta}^j) = \hat{Var}(\hat{\theta}^k) + \beta\hat{Cov}(\hat{\theta}^k, \hat{\theta}^{VT})\hat{\theta}^{\tau T} + \beta\hat{\theta}^\tau\hat{Cov}(\hat{\theta}^V, \hat{\theta}^{kT}) + \beta^2 A$ Where $A = \hat{Var}(\hat{\theta}^\tau\hat{\theta}^V)$. Writing $\hat{\theta}_l^\tau$ for row l of $\hat{\theta}^\tau$, and exploiting that $\hat{\theta}^\tau$ and $\hat{\theta}^V$ are independent, we have $A_{lm} = \hat{\theta}_l^\tau\hat{Var}(\hat{\theta}^V)\hat{\theta}_m^{\tau T} + \hat{\theta}^{VT}\hat{Cov}(\hat{\theta}_l^\tau, \hat{\theta}_m^\tau)\hat{\theta}^V + tr[\hat{Var}(\hat{\theta}^V)\hat{Cov}(\hat{\theta}_l^\tau, \hat{\theta}_m^\tau)]$

the magnitude for Asphalt projects is in line with the results presented in Raisingh (2021), while magnitudes for concrete and mixed projects are significantly, though not meaningfully, smaller.

Because the dependent variable (lowest rival bid) is normalised by the engineer's estimate, the coefficients on engineer's estimate can be interpreted as returns to scale. The positive coefficient on concrete suggests decreasing returns, whereas the negative coefficients on mixed and asphalt loads are in line with GKS and Raisingh's results.

Figure 3: First Stage Results (1)

		Coefficient	SE	Coefficient	SE	Coefficient	SE
Shape							
	$\log(\alpha - 1)$	0.966	0.002	1.015	0.002	1.024	0.002
Scale	$(= e^{\mathbf{x}_{lt}\beta_1})$						
	Concrete	-1.419	0.002	-1.429	0.006	-1.442	0.006
	Asphalt	-1.366	0.001	-1.353	0.006	-1.367	0.006
	Both	-1.337	0.001	-1.362	0.006	-1.375	0.006
	Major Road	-0.041	0.002	-0.029	0.002	-0.027	0.002
	Bridge	-0.016	0.002	-0.006	0.002	-0.005	0.002
	MR $\times\lambda$	0.086	0.002	0.081	0.002	0.078	0.002
	Bridge $\times\lambda$	0.083	0.003	0.083	0.003	0.075	0.003
	Concrete $\times\lambda$	0.384	0.002	0.385	0.002	0.389	0.002
	Asphalt $\times\lambda$	0.436	0.001	0.442	0.001	0.438	0.001
	Both $\times\lambda$	0.376	0.001	0.391	0.001	0.395	0.001
	Concrete $\times$ log(EE)	0.376	0.001	0.391	0.001	0.395	0.001
	Asphalt $\times$ log(EE)	0.016	0.001	0.004	0.001	0.001	0.001
	Both $\times$ log(EE)	-0.016	0.001	-0.022	0.001	-0.025	0.001
	Concrete $\times\lambda\times$ log(EE)	-0.01	0.001	-0.016	0.001	-0.019	0.001
	Asphalt $\times\lambda\times$ log(EE)	0.004	0.002	0.008	0.002	0.01	0.002
	Both $\times\lambda\times$ log(EE)	0.031	0.001	0.031	0.001	0.031	0.001
	Fixed Effects						
County				$\checkmark$		$\checkmark$	
Year						$\checkmark$	
Month						$\checkmark$	
	Observations	193545		193545		193545	

5.3.2 Second Estimation Step

Estimates from the second stage are presented in Figure 4, presenting estimates from least squares estimation and three sets of instruments. Parameter interactions with the competition index are included in Appendix **??**. I focus on estimates from the third column, which are used for the remainder of this application. Results are presented in thousands of dollars. So, for example, every kilometre increase in distance between a plant and the project increases costs by around \$650.

The coefficients on combinatorial objects can be interpreted in terms of how they impact the pseudo-static cost function: Every one standard deviation increase in a general contractor's (t1) backlog of asphalt projects increases their pseudo-cost (cost + expected future opportunity cost) by around $\$680,000$. Coefficients can also be interpreted as how they impact the aggressiveness of the firm's bidding. The coefficients on linear backlogs are all positive, suggesting firms bid less aggressively on larger projects. We cannot interpret the quadratic coefficients from the second stage as evidence of returns to scale. However they do give evidence of non-additivities across lots. The null hypothesis of additive values is rejected with p-value < 0.001.

The post-estimation tests demonstrate that the choice of instruments is important. The Hausman test for endogeneity in column 1, and the Hansen test of over-identifying restrictions presented in column 4, are both rejected at 0.001 significance level. The (adjusted) Cragg-Donald statistics suggest that the initial state alone is a weak, if not irrelevant, instrument. Importantly, while estimated parameters in columns 1 and 4 are statistically significantly different from those in column 3, they are still fairly close. They are far closer than estimates presented in column 2. This suggests that problems caused by irrelevant instruments may be more damaging than the problems caused by invalid instruments, or failing to instrument at all.

5.3.3 Third Estimation Step

Figure 5 presents results from the third stage of estimation. I find evidence that costs are increasing in both linear backlogs for all three types of firm. However, the magnitudes are much smaller than the linear coefficients estimated in the second stage, presented in the third column of figure 5. This suggests large anticipated

Figure 4: Second Stage Results

Instruments		none (OLS)		$\mathbf{s}_{it}$		$\mathbf{s}_{it} + \overrightarrow{\mathbf{s}}_{ilt}$		$\mathbf{s}_{it} + \overrightarrow{\mathbf{s}}_{ilt} + \overrightarrow{\mathbf{s}}_{imt}$	
		$\hat{\theta}$	SE	$\hat{\theta}$	SE	$\hat{\theta}$	SE	$\hat{\theta}$	SE
Combinatorial									
s_t^a	t1	669	25.2	625	402	684	27.4	673	21.8
	t2	1250	57.6	1410	439	1210	58.6	1230	40.6
	t3	136	8.76	128	284	139	8.97	137	7.26
s_t^c	t1	484	24.6	-64	817	505	27.3	497	24.8
	t2	1040	78.8	-821	2190	352	82.4	341	76.8
	t3	55.1	8.11	101	417	53.9	8.04	55.5	7.82
$(s_t^a)^2$	t1	-12.7	2.24	8.48	87	-14.6	2.57	-13.4	2.23
	t2	-44.3	4.51	-51	89.5	-40.4	4.5	-43.3	3.57
	t3	-0.384	0.0453	1.07	6.63	-0.413	0.0456	-0.399	0.0367
$(s_t^c)^2$	t1	-17.1	2.31	102	239	-19.8	2.95	-19.2	2.67
	t2	-12.8	9.34	209	254	-11.5	10.3	-12.5	10.8
	t3	-0.204	0.0527	0.17	14.1	-0.198	0.0543	-0.197	0.0528
$s_t^a \times s_t^c$	t1	-3.58	3.1	-138	341	-3.73	3.99	-1.98	3.42
	t2	67.1	18.3	-76	212	75.6	19.5	85.2	18.3
	t3	0.193	0.11	-3.18	56	0.188	0.119	0.168	0.117
Lot specific									
Distance		0.661	0.0836	0.533	1.43	0.656	0.0839	0.66	0.0799
Fixed Effects									
County		$\surd$		$\surd$		$\surd$		$\surd$	
Firm $\times$ Type		$\surd$		$\surd$		$\surd$		$\surd$	
Tests				(stat)	(p-val)				
Hansen		-9.81	(1)	-	(-)	19.2	(0.381)	446	(0)
Cragg-Donald		-		0.00174		263		152	
R^2		0.589		-0.165		0.589		0.589	
Observations									
T		3919		3919		3919		3919	
$\sum_t L_t$		14691		14691		14691		14691	

Note: Column 1 Hansen test is a Hausman test of endogeneity, using instruments from column 3.

opportunity costs from having high backlogs. This can also be inferred from the linear estimated coefficient on the continuation value, presented in the second column. This result is sensible - projects have very long durations, meaning if backlogs are large at period t, they will also likely be large in period $t+1, t+2, t+3, ...$ Any backlog costs will propagate through time.[46]

[46]Because backlogs are expected to be high in future this may raise the opportunity costs of taking on additional projects now. This is because having large backlogs in future may prevent them from bidding on new, lucrative contracts. However, given the evidence of increasing returns to scale this interpretation is unlikely to be born out in this application.

By considering the quadratic terms we see that general contractors exhibit increasing returns to scale, or increasing returns to specialisation, in both types of contracts. Paving companies also exhibit increasing returns for asphalt contracts, but not for concrete contracts. Instead, paving companies exhibit a large negative cost complementarity between the two types of projects, suggesting that taking on concrete projects come with additional costs for these already specialised firms. In Appendix **??** I consider how cost estimates compare to those obtained from misspecified dynamic single-object, and static multi-object models. I find that the dynamic single-object model under-estimates the degree of non-additivity across lots. The static multi-object model over-estimates the effect of backlogs on costs, mistaking expected future costs for present costs.

Figure 5: Third Stage Results

Object		$j(\mathbf{s}_i)$		$V(\mathbf{s})$		$k(\mathbf{s})$	
		$\hat{\theta}$	SE	$\hat{\theta}$	SE	$\hat{\theta}$	SE
λ	t1	0	(-)	-2.83	2.88	5.22	1.17
	t2	0	(-)	-6.1	5.24	10.2	3.12
	t3	0	(-)	2.43	2.02	0.669	0.527
s_t^a	t1	186	11.2	-673	29.6	684	27.4
	t2	397	22.8	-1,200	61.9	1210	58.6
	t3	41	7.44	-138	13.1	139	8.97
s_t^c	t1	130	9.52	-503	28.6	505	27.3
	t2	131	27.2	-309	85.2	352	82.4
	t3	15.3	6.23	-53.6	11.2	53.9	8.04
$(s_t^a)^2$	t1	-8.91	1.95	10.6	3.82	-14.6	2.57
	t2	-21.7	3.61	40	7.45	-40.4	4.5
	t3	-0.713	0.943	-0.591	1.87	-0.413	0.0456
$(s_t^c)^2$	t1	-10.2	1.54	17.6	3.25	-19.8	2.95
	t2	-7.85	5.58	4.76	11.1	-11.5	10.3
	t3	-0.75	0.802	-1.06	1.55	-0.198	0.0543
$s_t^a \times s_t^c$	t1	-0.357	2.47	6.08	5.02	-3.73	3.99
	t2	34.1	10.5	-83.6	20.6	75.6	19.5
	t3	0.166	0.894	-0.0295	1.74	0.188	0.119
Fixed Effects							
Firm				$\surd$		$\surd$	

5.4 Counterfactual

I now consider how procurement costs and efficiency change as we vary the auction mechanism. I consider a counterfactual with contracts allocated using sequential first-price auctions. This is an interesting counterfactual as it speaks to the value from 'batching' contracts by allocating multiple contracts simultaneously. Furthermore, many empirical dynamic auction papers assume contracts are auctioned sequentially anyway, making this a useful comparison for researchers.

Standard theoretical results suggest sequential allocation will be less efficient than simultaneous allocation.[47] This is because bidders do not know what types of contracts will be auctioned in the near future, making it more difficult to exploit potential cost synergies. However, batching contracts but not allowing firms to place combinatorial bids also limits their ability to exploit cost synergies. Sequential allocation may actually improve efficiency by giving bidders greater control over their cost synergies, reducing the likelihood that bidders accidentally win too many or too few contracts. All these effects are likely to be more pronounced the larger the degree of complementarities across lots. The effects of this alternate procurement mechanism are ex-ante unclear.

5.4.1 The Counterfactual Mechanism

I now discuss how I set up the counterfactual mechanism. I split each 14 day period into 50 'slots', where at most one auction can occur in each slot. I randomly allocate auctions occurring within each letting period to one of the 50 slots. Auctions are low-price sealed bid, as in the main application. Consistent with the estimated model I assume that projects all begin before the next auction slot.[48]

[47]See Akbarpour et al. (2017) as an example. I also ignore that collusion is typically easier to sustain in sequential auctions, which is expected to further increase procurement costs.

[48]The estimated cost function $j(\mathbf{s}_i)$ was defined over 14 day intervals. For these counterfactuals this function must be redefined over 14/50 day intervals. I do this by dividing the cost function by 50. The discount factor is adjusted similarly, ensuring the same annual discount factor of 0.9.

5.4.2 Equilibrium

In each period, defined at 14/50 day intervals, I use the same competition index λ_{it} to capture changes in common and rival states. This is reasonable since the irregular bidders are treated as myopic single-object bidders. Firms have beliefs about the probability they win any given lot, conditional on lot characteristics and λ_{it}. Firms place bids conditional on their beliefs, backlogs, and their continuation value, defined as in the main model.[49]

I find equilibrium beliefs and value functions using fixed point iteration. For a given set of beliefs and continuation values I simulate the auction process, numerically maximising expected pay-offs conditional on beliefs and continuation values. I then take a conditional expectation of the maximised expected pay-off using a linear-in-parameters prediction as I did in the main model. This allows me to form the new continuation value. I repeat this process until the value function converges for each type of firm. I then find the equilibrium distribution of winning bids as I did in the main specification, repeating this outer-loop until the distribution of winning bids converges. See Appendix **??** for additional details on how I simulate counterfactuals.

5.4.3 Results

Figure 6 presents the results from the counterfactual simulations. The table presents the observed procurement cost, as well as the estimated cost to firms under the simultaneous mechanism, and simulated procurement and completion costs under the sequential mechanism. The key takeaway is that the sequential mechanism is less efficient for firms than the simultaneous mechanism, costing firms an average of $\$110,000$ more to complete each contract (10%). The sequential mechanism increases

[49] I assume firms continue to place bids only on the set of auctions they actually bid on, so I do not simulate the entry problem. Given my assumption of negligible entry costs, firms were only observed bidding on the contracts they have the largest cost advantages in. If their cost advantages were mostly additive, such as due to low v_{ilt} draws, they will have the same advantage under the sequential mechanism, and so bidding on this set of lots will remain optimal. Therefore my estimates can, to an extent, be considered lower bounds on costs.

procurement costs by an average of $\$30,000$ per auction (2%).[50]

The results are driven by firms bidding sub-optimally on contracts that occur earlier, because they do not know what contracts will be auctioned in later periods. When the contracts are auctioned simultaneously, they bid more aggressively on the contracts they have the cost advantages in. I observe that simulated bidding is relatively less aggressive on contracts that are allocated early in the day, relative to observed bids ($p = 0.043$). This demonstrates that firms bid less aggressively because they are unsure of whether more profitable contracts will be auctioned later in the period. However, when conditioning on their observed bid under the simultaneous mechanism, firms bid more aggressively on lots occurring earlier in the period. This shows that they are bidding inappropriately low, given that more profitable contracts will be auctioned later that period.

Figure 6: Counterfactual Results

Mechanism	Outcome	Estimate
Simultaneous	Procurement Cost	1,520,000
	Completion Cost	1,130,000
Sequential	Procurement Cost	1,550,000
	Completion Cost	1,240,000

6 Conclusion

In this paper I did three things: First, I set-up a dynamic multi-object auction model and proved that the model primitives are identified from standard bidding data. Second, in order to overcome the technical challenges of estimating model primitives in

[50] I find a larger difference in the completion cost for firms than the difference in procurement cost. This occurs because the fringe bidders' costs do not change (as I assume they only play a single object myopic game), putting them at a slight advantage over regular bidders. Regular bidders must then bid more aggressively, relative to their costs, than in the simultaneous auction environment. Consequently I only find a small change in the distribution of winning bids, however this result is highly reliant on the assumption of a non-collusive equilibrium.

this setting, I proposed a computationally feasible estimation procedure. Finally, I applied the model to data from Michigan Department of Transport's procurement data and evaluated the efficiency and revenue of holding repeated rounds of simultaneous auction relative to auctioning all contracts sequentially.

This paper was motivated by the prevalence of such repeated, multi-object auctions. Significant complementarities between auctioned objects have been found in both the dynamic single-object literature, and the static multi-object literature, most notably in JP and GKS. However, these two types of model had not, until this point, been unified in a single framework. Future work should attempt to take into account the firms' entry decisions, as this was a major simplification in this paper.

References

Akbarpour, M., Li, S., and Gharan, S. O. (2017). Thickness and information in dynamic matching markets.

Athey, S. and Haile, P. A. (2002). Identification of standard auction models. *Econometrica*, 70(6):2107–2140.

Athey, S. and Haile, P. A. (2007). Nonparametric approaches to auctions. *Handbook of econometrics*, 6:3847–3965.

Aumann, R. J. (1973). Subjecjwity and correlation in randomized strategies.

Backus, M. and Lewis, G. (2016). Dynamic demand estimation in auction markets. Technical report, National Bureau of Economic Research.

Balat, J. (2013). Highway procurement and the stimulus package: Identification and estimation of dynamic auctions with unobserved heterogeneity. *Johns Hopkins University Mimeo.*

Balat, J., Carranza, J. E., Martin, J. D., and Carranza, J. E. (2015). Dynamic and strategic behavior in hydropower-dominated electricity markets: empirical evidence for colombia. *Borradores de Economía; No. 886*.

Bodoh-Creed, A. L., Boehnke, J., and Hickman, B. (2021). How efficient are decentralized auction platforms? *The Review of Economic Studies*, 88(1):91–125.

Cantillon, E. and Pesendorfer, M. (2007). Combination bidding in multi-unit auctions.

Charalambos, D. and Aliprantis, B. (2013). *Infinite Dimensional Analysis: A Hitchhiker's Guide*. Springer-Verlag Berlin and Heidelberg GmbH & Company KG.

Fox, J. T. and Bajari, P. (2013). Measuring the efficiency of an fcc spectrum auction. *American Economic Journal: Microeconomics*, 5(1):100–146.

Fudenberg, D. and Maskin, E. (1991). On the dispensability of public randomization in discounted repeated games. *Journal of Economic Theory*, 53(2):428–438.

Fudenberg, D. and Maskin, E. (2009). The folk theorem in repeated games with discounting or with incomplete information. In *A Long-Run Collaboration On Long-Run Games*, pages 209–230. World Scientific.

Gentry, M. L., Komarova, T., and Schiraldi, P. (2018). Preferences and performance in simultaneous first-price auctions: A structural analysis. *Available at SSRN 2514995*.

Groeger, J. R. (2014). A study of participation in dynamic auctions. *International Economic Review*, 55(4):1129–1154.

Guerre, E., Perrigne, I., and Vuong, Q. (2000). Optimal nonparametric estimation of first-price auctions. *Econometrica*, 68(3):525–574.

Hortaçsu, A. and McAdams, D. (2018). Empirical work on auctions of multiple objects. *Journal of Economic Literature*, 56(1):157–84.

Hotz, V. J. and Miller, R. A. (1993). Conditional choice probabilities and the estimation of dynamic models. *The Review of Economic Studies*, 60(3):497–529.

Jeziorski, P. and Krasnokutskaya, E. (2016). Dynamic auction environment with subcontracting. *The RAND Journal of Economics*, 47(4):751–791.

Jofre-Bonet, M. and Pesendorfer, M. (2003). Estimation of a dynamic auction game. *Econometrica*, 71(5):1443–1489.

Kim, S. W., Olivares, M., and Weintraub, G. Y. (2014). Measuring the performance of large-scale combinatorial auctions: A structural estimation approach. *Management Science*, 60(5):1180–1201.

Kong, Y. (2016). Sequential auctions with synergy and a filiation.

Lewbel, A. (2019). The identification zoo: Meanings of identification in econometrics. *Journal of Economic Literature*, 57(4):835–903.

Milgrom, P. R. and Weber, R. J. (1985). Distributional strategies for games with incomplete information. *Mathematics of operations research*, 10(4):619–632.

Raisingh, D. (2021). The effect of pre-announcements on participation and bidding in dynamic auctions.

Sailer, K. (2006). Searching the ebay marketplace.

Somaini, P. (2020). Identification in auction models with interdependent costs. *Journal of Political Economy*, 128(10):3820–3871.

Appendices

A Proof of Proposition 1

In this Appendix I prove Proposition 1, which states that under the assumptions of the game, and under Conjecture 1, a Symmetric Markov Perfect Equilibrium exists.

The proof of Proposition 1 proceeds by first demonstrating that, conditional on Conjecture 1, a Pure Strategy Bayesian Nash Equilibrium exists in the stage game. I then show that the equilibrium pay-off in the stage game is consistent with the continuation value. I do so by employing Kakutani's fixed point theorem, which requires showing the existence, convex-valuedness, and upper hemicontinuity of the continuation value.

Proof: **Static Equilibrium of the entry game**: player i chooses their entry decision $\mathbf{d}$ to maximise their expected pay-off. If the player knew the entry decisions of other players in advance they would essentially choose an entry structure, then the expected pay-offs would just be the equilibrium expected payoffs associated with each entry structure. Instead, their expected pay-off is formed by taking expectation over the entry decisions of every other player, given the strategies of other players. The entry game is therefore a simple game of incomplete information.

A symmetric equilibrium in distributional strategies exists, thanks to Milgrom and Weber (1985).[51] However this equilibrium may not be unique, a problem for continuity of the value function. Continuity can be restored by following Fudenberg and Maskin (1991), entry strategies can be augmented to be a function of the realisation of a public random variable. The public random variable enables players to coordinate over which equilibrium will be played. Conditional on this public random variable the set of equilibrium pay-offs is convex (Aumann, 1973).

As in JP, equilibrium existence of the dynamic game then requires that the equilibrium pay-off in the stage game is consistent with the continuation value.[52] That is, can we write the ex-ante value function $\mathbf{V}_t^E$, stacked over states, as a function of $\mathbf{V}_{t+1}^E$, so that $\mathbf{V}_t^E = \Omega(\mathbf{V}_{t+1}^E)$ (existence). In addition, does the correspondence Ω have a fixed point such that $\mathbf{V}^E = \Omega(\mathbf{V}^E)$ (stationarity).

Existence of $\mathbf{V}_t^E = \Omega(\mathbf{V}_{t+1}^E)$: Writing the ex-ante value function in recursive form, substitute equation 2 for period $t+1$ into equation 2 into equation 2 back into equation 2 for period t. Existence then follows from the assumption that pay-offs are bounded. This ensures the set $\Omega(\mathbf{V}_{t+1}^E)$ is non-empty.

(non-)Uniqueness of $\Omega(\mathbf{V}_{t+1}^E)$: The possible existence of multiple equilibria in the entry game imply the value function is non-unique. therefore, the ex-ante value function is also non-

[51]Likewise, existence of a Pure Strategy equilibrium follows from their purification result, under the assumption that types are atomless. However, we will not require bidders to play pure strategies, and instead ensure our identification framework is consistent with distribution and/or pure entry strategies.

[52]This dynamic equilibrium will be symmetric since the equilibrium in the stage game is symmetric, so that strategies depend only on states, not player identities or time periods.

unique. Fortunately Ω must be convex valued, as the set of equilibrium pay-offs, conditional on the public random variable, is convex.

Upper-hemi continuity of $\Omega(.)$: The continuation value must be continuous in $\mathbf{V}^E_{t+1}$, shown by equation 2. Next, consider the conditional value function, conditional on entry decision $\bar{\mathbf{d}}$:

$$\tilde{W}_i(\bar{\mathbf{d}}, \boldsymbol{v}_{it}, \mathbf{s}_t; \sigma_{-i}) = \\ \underset{\mathbf{b}}{max} \left\{ \Gamma_i(\mathbf{b}, \bar{\mathbf{d}}; \sigma_{-i})^T(\boldsymbol{v}_{it} - \mathbf{b}) + P_i(\mathbf{b}, \bar{\mathbf{d}}; \sigma_{-i})^T[J_i(\mathbf{s}_t) + \beta V_i(\mathbf{s}_t; \sigma_{-i})] \right\}$$

Continuity of $\tilde{\mathbf{W}}_t$ in $\mathbf{V}^E_{t+1}$ is guaranteed by conjecture 1, which requires equilibrium expected pay-offs are continuous in $J_i + \beta V_i$. The value function can then be written as $W_i(\boldsymbol{v}_{it}, \mathbf{s}_t; \sigma_{-i}) = \underset{\mathbf{d}}{max}\left\{\tilde{W}_i(\mathbf{d}, \boldsymbol{v}_{it}, \mathbf{s}_t; \sigma_{-i})\right\}$. Upper-hemi continuity of $\mathbf{W}_t$ in $\tilde{\mathbf{W}}_t$, and hence in $\mathbf{V}^E_{t+1}$, arises from our public random variable (Fudenberg and Maskin, 2009).[53] Upper-hemi continuity of $\mathbf{V}^E_t$ arises from the ex-ante value function taking an expectation over states.

Existence of a stationary dynamic equilibrium: In order to show existence of a stationary equilibrium we must show that there exists a fixed point of the correspondence $\mathbf{V}^E = \Omega(\mathbf{V}^E)$. As $\Omega()$ is non-empty, convex valued, and upper-hemi continuous, we can apply Kakutani's fixed point theorem. Therefore, a Markov Perfect Equilibrium exists.

□

B Extension of Proposition 1 from JBP

In this Appendix I essentially extend Proposition 1 from JBP to the multi-object case. In Appendix B.1 I prove Proposition 4 from the main text. In Appendix B.2 I present (and prove) that the ex-ante value function also has an analytic expression. In Appendix B.3 I prove that even in the case of binding reservation prices the ex-ante value function still has an analytic expression. Finally, in Appendix B.3 I prove that the Inverse Bid System is strictly monotonic for bids strictly above the reservation price. This is an important proof as it allows me to employ the 'Law of the Unconscious Statistician'.

For the remainder of this section I regularly make use of the definition of $\mathbf{k} = C\mathbf{j} + \beta\mathbf{V}$, and equivalently $K(\mathbf{s}) = J(\mathbf{s}_i) + \beta V(\mathbf{s})$.

[53] Public randomisation ensures that the set of equilibrium pay-offs is convex. Public randomisation means $\mathbf{W}_t$ is the convex hull of possible equilibrium pay-offs from entry, $\tilde{\mathbf{W}}_t$. Therefore, so long as $\tilde{\mathbf{W}}_t$ is compact valued, $\mathbf{W}_t$ is upper hemicontinuous (Charalambos and Aliprantis, 2013). Compact valuedness comes from pay-offs being drawn from a compact set.

B.1 Proof of Proposition 4

Proposition 4 : Under assumptions 1 - 4, the expected stage pay-off is given by:

$$\begin{aligned}\tilde{\Pi}(\mathbf{b}^*|\boldsymbol{v};\mathbf{s}) = \Gamma(\mathbf{b}^*|\mathbf{s})^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\Gamma(\mathbf{b}^*|\mathbf{s}) \\ + [P(\mathbf{b}^*|\mathbf{s})^T \ - \ \Gamma(\mathbf{b}^*|\mathbf{s})^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\nabla_{\mathbf{b}}P(\mathbf{b}^*|\mathbf{s})]B_{\mathbf{s}}\mathbf{j} \\ + [Q(\mathbf{b}^*|\mathbf{s})^T \ - \ \Gamma(\mathbf{b}^*|\mathbf{s})^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\nabla_{\mathbf{b}}Q(\mathbf{b}^*|\mathbf{s})]A_{\mathbf{s}}\beta\mathbf{V}\end{aligned} \quad (12)$$

Proof:

1. Necessary First Order Conditions are given by:

$$\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})(\boldsymbol{v}-\mathbf{b}^*) = \Gamma(\mathbf{b}^*|\mathbf{s}) - \nabla_{\mathbf{b}}P(\mathbf{b}^*|\mathbf{s})B_{\mathbf{s}}\mathbf{j} - \beta\nabla_{\mathbf{b}}Q(\mathbf{b}^*|\mathbf{s})A_{\mathbf{s}}\mathbf{V}$$

2. Left multiplying by $\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}$

$$(\boldsymbol{v}-\mathbf{b}^*) = \nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}[\Gamma(\mathbf{b}^*|\mathbf{s}) - \nabla_{\mathbf{b}}P(\mathbf{b}^*|\mathbf{s})B_{\mathbf{s}}\mathbf{j} - \beta\nabla_{\mathbf{b}}Q(\mathbf{b}^*|\mathbf{s})A_{\mathbf{s}}\mathbf{V}]$$

3. Left multiplying by $\Gamma(\mathbf{b}^*|\mathbf{s})^T$

$$\begin{aligned}&\Gamma(\mathbf{b}^*|\mathbf{s})^T(\boldsymbol{v}-\mathbf{b}^*) \\ &\quad = \Gamma(\mathbf{b}^*|\mathbf{s})^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}[\Gamma(\mathbf{b}^*|\mathbf{s}) - \nabla_{\mathbf{b}}P(\mathbf{b}^*|\mathbf{s})B_{\mathbf{s}}\mathbf{j} - \beta\nabla_{\mathbf{b}}Q(\mathbf{b}^*|\mathbf{s})A_{\mathbf{s}}\mathbf{V}]\end{aligned}$$

4. The Maximised Expected Pay-off is given by:

$$\tilde{\Pi}(\mathbf{b}^*|\boldsymbol{v};\mathbf{s}) = \Gamma(\mathbf{b}^*|\mathbf{s})^T(\boldsymbol{v}-\mathbf{b}^*) + P(\mathbf{b}^*|\mathbf{s})B_{\mathbf{s}}\mathbf{j} + \beta Q(\mathbf{b}^*|\mathbf{s})A_{\mathbf{s}}\mathbf{V}$$

   Substituting in $\Gamma(\mathbf{b}^*|\mathbf{s})^T(\boldsymbol{v}-\mathbf{b}^*)$ gives the result.

□

B.2 Ex-ante Value Function

Building on Proposition 4 the ex-ante Value function can be written as:

$$\begin{aligned}V_i^E(\mathbf{s}_t) = E_{\mathbf{b}}[\ \Gamma(\mathbf{b}|\mathbf{s}_t)^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}|\mathbf{s}_t)^{-1}\Gamma(\mathbf{b}|\mathbf{s}_t)] \\ + E_{\mathbf{b}}[Q(\mathbf{b}|\mathbf{s}_t)^T - \Gamma(\mathbf{b}|\mathbf{s}_t)^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}|\mathbf{s}_t)^{-1}\nabla_{\mathbf{b}}Q(\mathbf{b}|\mathbf{s}_t)\ |\mathbf{s}_t]K(\mathbf{s}_t)\end{aligned} \quad (13)$$

Proof that this is the case should be trivial, however it requires that we apply a change of variable, changing from integrating over $\boldsymbol{v}$ to integrating over $\mathbf{b}$.

Proof:

1. To obtain the ex-ante value function from the equation presented in Proposition 4 we

then take an expectation over both sides with respect to $\boldsymbol{\upsilon}$ for:

$$
\begin{aligned}
&E_{\boldsymbol{\upsilon}}[\Gamma(\mathbf{b}^*|\mathbf{s}_t)^T(\boldsymbol{\upsilon}-\mathbf{b}^*)+P(\mathbf{b}|\mathbf{s}_t)^TK(\mathbf{s}_t)]\\
&\qquad\qquad\qquad\qquad = E_{\boldsymbol{\upsilon}}[\Gamma(\mathbf{b}^*|\mathbf{s}_t)^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s}_t)^{-1}\Gamma(\mathbf{b}^*|\mathbf{s}_t)]\\
&\qquad\qquad\qquad\qquad + E_{\boldsymbol{\upsilon}}[Q(\mathbf{b}|\mathbf{s}_t)^T-\Gamma(\mathbf{b}^*|\mathbf{s}_t)^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s}_t)^{-1}\nabla_{\mathbf{b}}Q(\mathbf{b}^*|\mathbf{s}_t)]K(\mathbf{s}_t)
\end{aligned}
$$

2. By applying the Law of the Unconscious Statistician (change of variables for expectations) the right hand side of this equation is equal to

$$
E_{\mathbf{b}}[\ \Gamma(\mathbf{b}|\mathbf{s}_t)^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}|\mathbf{s}_t)^{-1}\Gamma(\mathbf{b}|\mathbf{s}_t)+[Q(\mathbf{b}|\mathbf{s}_t)^T-\nabla_{\mathbf{b}}\Gamma(\mathbf{b}|\mathbf{s}_t)^{-1}\nabla_{\mathbf{b}}Q(\mathbf{b}|\mathbf{s}_t)]K(\mathbf{s}_t)\ |\mathbf{s}_t]
$$

□

Importantly, in order to apply the Law of the Unconscious Statistician we require that the mapping $\boldsymbol{\xi}(\mathbf{b}^*|K;\mathbf{s})$ is monotonic (the jacobian has non-zero determinant) in $\mathbf{b}$. I prove this in Appendix B.4. For those unfamiliar with the Law of the Unconscious Statistician, this intuitive law states that Given random variables U_l and $\mathbf{B}$ such that $U_l=h(\mathbf{B})$, where $||\nabla_{\mathbf{b}}h||>0$, the Law of the Unconscious Statistician states that $E_{U_l}[U_l]=\int \upsilon_l f_{U_l}(\upsilon_l)d\upsilon_l=\int_{b_1}...\int_{b_L}h(\mathbf{b})f_{\mathbf{B}}(\mathbf{b})db_L...db_1=E_{\mathbf{B}}[h(\mathbf{b})]$

B.3 Ex-ante Value Function (Reservation Prices)

I now consider the case with binding reservation prices. In this case, I must define a partition of the bidding space according to which bids are at the reservation price, and which bids are strictly above the reservation price. This partition essentially partitions which first order conditions hold exactly (Lagrangian multiplier is zero), and which do not. Each component m, denoted $\mathbb{A}_m$, will consist of a set of bids above the reservation price $\mathbb{A}_m^+=\{l\ :\ b_l>R\}$, a set at the reservation price $\mathbb{A}_m^-=\{l\ :\ b_l=R\}$, and a non-entered set $\mathbb{A}_m^c=\{l\ :\ b_l=\emptyset\}$. Importantly, for a given K, each component defines a corresponding set in lot-specific pay-off space, which I write as $\upsilon(\mathbb{A}_m)$. This set is known from the inequalities derived in section D.2.2. We can then write the ex-ante value function as follows:

$$
\begin{aligned}
V_i^E(\mathbf{s}_t) &= E_m[\sum_{l\in\mathbb{A}_m^-}\Gamma_l(R,d_l|\mathbf{s}_t)(E_{\upsilon_l}[\upsilon_l|\boldsymbol{\upsilon}\in\upsilon(\mathbb{A}_m)]-R)\\
&+\sum_{l\in\mathbb{A}_m^+}E_{\mathbf{b},\mathbf{d}}[\frac{\Gamma_l(b_l,d_l|\mathbf{s}_t)^2}{\nabla_{b_l}\Gamma_l(b_l,d_l|\mathbf{s}_t)}-\frac{\Gamma_l(b_l,d_l|\mathbf{s}_t)}{\nabla_{b_l}\Gamma_l(b_l,d_l|\mathbf{s}_t)}\nabla_{b_l}Q(\mathbf{b},\mathbf{d}|\mathbf{s}_t)K(\mathbf{s}_t)|\mathbf{b}\in\mathbb{A}_m]]\\
&+E_{\mathbf{b},\mathbf{d}}[Q(\mathbf{b},\mathbf{d}|\mathbf{s}_t)|\mathbf{b}\in\mathbb{A}_m]^TK(\mathbf{s}_t)\\
&=\tilde{\Phi}(\mathbf{s}_t)+\tilde{\Omega}(\mathbf{s}_t)K(\mathbf{s}_t) \qquad (14)
\end{aligned}
$$

This expression is extremely convenient for the econometrician, who is then able to evaluate the ex-ante value function by taking integrals with respect to observed bids. The only difficulty is in calculating the truncated means $E_{\upsilon_l}[\upsilon_l|\boldsymbol{\upsilon} \in \upsilon(\mathbb{A}_m)]$.

I now derive equation 14. For notational simplicity I drop the dependence on $\mathbf{s}$.

Proof:

1. We can split an expectation into partitions of conditional expectations, which follows from the law of iterated expectations. This allows us to write:

$$E_{\boldsymbol{\upsilon}}[\Gamma(\mathbf{b}^*,\mathbf{d}^*)^T(\boldsymbol{\upsilon}-\mathbf{b}^*)+P(\mathbf{b}^*,\mathbf{d}^*)^T K] \\ = \sum_m P(\boldsymbol{\upsilon} \in \upsilon(\mathbb{A}_m))E_{\boldsymbol{\upsilon}}[\Gamma(\mathbf{b}^*,\mathbf{d}^*)^T(\boldsymbol{\upsilon}-\mathbf{b}^*)+P(\mathbf{b}^*,\mathbf{d}^*)^T K|\boldsymbol{\upsilon} \in \upsilon(\mathbb{A}_m)]$$

2. As expectations are linear operators we can write the right hand side of this equation as:

$$= \sum_m P(\boldsymbol{\upsilon} \in \upsilon(\mathbb{A}_m))(\sum_l E_{\boldsymbol{\upsilon}}[\Gamma_l(b_l^*,d_l^*)(\upsilon_l - b_l^*)|\boldsymbol{\upsilon} \in \upsilon(\mathbb{A}_m)])+E_{\boldsymbol{\upsilon}}[Q(\mathbf{b}^*,\mathbf{d}^*)^T K|\boldsymbol{\upsilon} \in \upsilon(\mathbb{A}_m)]$$

3. Consider $E_{\boldsymbol{\upsilon}}[P(\mathbf{b}^*,\mathbf{d}^*)^T K|\boldsymbol{\upsilon} \in \upsilon(\mathbb{A}_m)]$. We will essentially apply the Law of the Unconscious Statistician to write this as an expectation over $\mathbf{b}$ and $\mathbf{d}$. Apply the Law of Iterated Expectations, so that we take expectations first over υ_l for $l \in \mathbb{A}_m^+$ first, and then over υ_l for $l \notin \mathbb{A}_m^+$. We have:

$$E_{\boldsymbol{\upsilon}}[P(\mathbf{b}^*,\mathbf{d}^*)^T K|\boldsymbol{\upsilon} \in \upsilon(\mathbb{A}_m)] \\ = E_{\upsilon_l\ l\in\mathbb{A}_m^+}[E_{\upsilon_l\ l\notin\mathbb{A}_m^+}[Q(\mathbf{b}^*,\mathbf{d}^*)^T K|\boldsymbol{\upsilon} \in \upsilon(\mathbb{A}_m)\ \&\ \upsilon_l\ l \in \mathbb{A}_m^+]|\boldsymbol{\upsilon} \in \upsilon(\mathbb{A}_m)]$$

4. By definition of $\mathbb{A}_m^+$ we have $E_{\upsilon_l\ l\notin\mathbb{A}_m^+}[Q(\mathbf{b}^*,\mathbf{d}^*)^T K|\boldsymbol{\upsilon} \in \upsilon(\mathbb{A}_m)\ \&\ \upsilon_l l \in \mathbb{A}_m^+] = Q(\mathbf{b}^*,\mathbf{d}^*)^T K$. This is because bids on lot $l \notin \mathbb{A}_m^+$ are constant for $\boldsymbol{\upsilon} \in \upsilon(\mathbb{A}_m)$.

5. Next, we are able to apply the law of the unconscious statistician, since the functions $b_l^*(\boldsymbol{\upsilon},K)$ for $l \in \mathbb{A}_m^+$ are monotonic in $\boldsymbol{\upsilon}$, as shown in appendix B.4.

$$E_{\upsilon_l\ l\in\mathbb{A}_m^+}[Q(\mathbf{b}^*,\mathbf{d}^*)^T K|\boldsymbol{\upsilon} \in \upsilon(\mathbb{A}_m)] = E_{b_l\ l\in\mathbb{A}_m^+}[Q(\mathbf{b}^*,\mathbf{d}^*)^T K|\boldsymbol{b} \in \mathbb{A}_m]$$

   The idea here is that we couldn't apply this change of variables in general, because some of the random variables (bids) we tried to take expectations over are in regions of $\boldsymbol{\upsilon}$ space where the bids are non-monotonic in $\boldsymbol{\upsilon}$, and actually do not vary with $\boldsymbol{\upsilon}$ at all. But, precisely because they do not vary in these regions, we can just integrate out these dimensions, before applying the change of variables. Last of all, recognise that because b_l for $l \notin \mathbb{A}_m^+$ is constant, we have:

$$E_{b_l\ l\in\mathbb{A}_m^+}[Q(\mathbf{b}^*,\mathbf{d}^*)^T K|\boldsymbol{b} \in \mathbb{A}_m] = E_{\mathbf{b}}[Q(\mathbf{b}^*,\mathbf{d}^*)^T K|\boldsymbol{b} \in \mathbb{A}_m]$$

6. Next, focus on $E_{\boldsymbol{\upsilon}}[\Gamma_l(b_l^*,d_l^*)(\upsilon_l - b_l^*)|\boldsymbol{\upsilon} \in \upsilon(\mathbb{A}_m)]$. For $l \in \mathbb{A}_m^c$, so that $d_l^* = 0$, we have $E_{\boldsymbol{\upsilon}}[\Gamma_l(b_l^*,d_l^*)(\upsilon_l - b_l^*)|\boldsymbol{\upsilon} \in \upsilon(\mathbb{A}_m)] = 0$, since for this l we have $\Gamma_l(b_l^*,d_l^*) = 0$ by definition.

7. Meanwhile, for $l \in \mathbb{A}_m^-$ we have

$$E_{\boldsymbol{\upsilon}}[\Gamma_l(b_l^*,d_l^*)(\upsilon_l - b_l^*)|\boldsymbol{\upsilon} \in \upsilon(\mathbb{A}_m)] = \Gamma_l(R,1)(E_{\boldsymbol{\upsilon}}[\upsilon_l|\boldsymbol{\upsilon} \in \upsilon(\mathbb{A}_m)] - R)$$

by definition of partition m.

8. Finally for $l \in \mathbb{A}_m^+$ we have

$$E_{\boldsymbol{\upsilon}}[\Gamma_l(b_l^*, d_l^*)(\upsilon_l - b_l^*)|\boldsymbol{\upsilon} \in \upsilon(\mathbb{A}_m)] \\ = E_{\boldsymbol{\upsilon}}[\frac{\Gamma_l(b_l^*, 1)^2}{\nabla_{b_l}\Gamma_l(b_l^*, 1)} - \frac{\Gamma_l(b_l^*, 1)}{\nabla_{b_l}\Gamma_l(b_l^*, 1)}\nabla_{b_l}Q(\mathbf{b}^*, \mathbf{d}^*)|\boldsymbol{\upsilon} \in \upsilon(\mathbb{A}_m)]$$

This arises because the lth first order condition of the lagrangian given in equation 19 is given by:

$$\nabla_{b_l}\Gamma_l(b_l^*, d_l^*)(\upsilon_l - b_l) - \Gamma_l(b_l^*, d_l^*) + \nabla_{b_l}Q(\mathbf{b}^*, \mathbf{d}^*)K + \lambda_l^* = 0$$

l such that $b_l^* > R$ we have $\lambda_l^* = 0$ (this solution is unconstrained). This equation then rearranges for

$$\Gamma_l(b_l^*, d_l^*)(\upsilon_l - b_l^*) = \frac{\Gamma_l(b_l^*, 1)^2}{\nabla_{b_l}\Gamma_l(b_l^*, 1)} - \frac{\Gamma_l(b_l^*, 1)}{\nabla_{b_l}\Gamma_l(b_l^*, 1)}\nabla_{b_l}Q(\mathbf{b}^*, \mathbf{d}^*)$$

9. Still considering $l \in \mathbb{A}_m^+$, the idea is that we now want to apply the Law of the Unconscious Statistician for

$$E_{\boldsymbol{\upsilon}}[\frac{\Gamma_l(b_l^*, 1)^2}{\nabla_{b_l}\Gamma_l(b_l^*, 1)} - \frac{\Gamma_l(b_l^*, 1)}{\nabla_{b_l}\Gamma_l(b_l^*, 1)}\nabla_{b_l}Q(\mathbf{b}^*, \mathbf{d}^*)|\boldsymbol{\upsilon} \in \upsilon(\mathbb{A}_m)] \\ = E_{\mathbf{b}}[\frac{\Gamma_l(b_l^*, 1)^2}{\nabla_{b_l}\Gamma_l(b_l^*, 1)} - \frac{\Gamma_l(b_l^*, 1)}{\nabla_{b_l}\Gamma_l(b_l^*, 1)}\nabla_{b_l}Q(\mathbf{b}^*, \mathbf{d}^*)|\mathbf{b} \in \mathbb{A}_m]$$

The problem is that $\nabla_{b_l}Q(\mathbf{b}^*, \mathbf{d}^*)$ in general depends on $b_{l'}$ for $l' \notin \mathbb{A}_m^+$. However, just as we did in steps 3 to 5, we can first apply the law of iterated expectations and integrate out these other l' dimensions, before employing the change of variables.

10. Putting together steps 1 through 5, 6, 7, and 8 through 9, we can write:

$$E_{\boldsymbol{\upsilon}}[\Gamma(\mathbf{b}^*, \mathbf{d}^*)^T(\boldsymbol{\upsilon} - \mathbf{b}^*) + Q(\mathbf{b}^*, \mathbf{d}^*)^T K] \\ = \sum_m P(\boldsymbol{\upsilon} \in \upsilon(\mathbb{A}_m)) \sum_{l \in \mathbb{A}_m^-} \Gamma_l(R, d_l|\mathbf{s}_t)(E_{\upsilon_l}[\upsilon_l|\boldsymbol{\upsilon} \in \upsilon(\mathbb{A}_m)] - R) \\ + \sum_{l \in \mathbb{A}_m^+} E_{\mathbf{b},\mathbf{d}}[\frac{\Gamma_l(b_l, d_l|\mathbf{s}_t)^2}{\nabla_{b_l}\Gamma_l(b_l, d_l|\mathbf{s}_t)} - \frac{\Gamma_l(b_l, d_l|\mathbf{s}_t)}{\nabla_{b_l}\Gamma_l(b_l, d_l|\mathbf{s}_t)}\nabla_{b_l}Q(\mathbf{b}, \mathbf{d}|\mathbf{s}_t)K(\mathbf{s}_t)|\mathbf{b} \in \mathbb{A}_m] \\ + E_{\mathbf{b},\mathbf{d}}[Q(\mathbf{b}, \mathbf{d}|\mathbf{s}_t)|\mathbf{b} \in \mathbb{A}_m]^T K(\mathbf{s})$$

□

B.4 Monotonicity of the Inverse Bid System

This argument proceeds in two parts, but is fundamentally just an application of the Envelope Theorem. First, I consider the second order conditions of the bidding problem, substituting in the first order conditions to find an particularly useful expression. I then consider the Jacobian of the inverse bidding system, substituting in the aforementioned expression to show that this jacobian matrix must have non-zero determinant.

Proof:

1. Differentiating equation 19 we obtain a hessian matrix of second derivatives with entry ij given by:

$$H_{ln} = \begin{cases} \nabla^2_{b_l}\Gamma_l(b_l^*, d_l^*)(\upsilon_l - b_l^*) & \text{if} \quad l = n \\ \qquad\qquad -2\nabla_{b_l}\Gamma_l(b_l^*, d_l^*) + \nabla^2_{b_l}Q(\mathbf{b}^*, \mathbf{d}^*)K & \\ \nabla_{b_l}\nabla b_n Q(\mathbf{b}^*, \mathbf{d}^*)K & \text{if} \quad l \neq n \end{cases}$$

2. From the first order conditions we then substitute in $\nabla^2_{b_l}\Gamma_l(b_l^*, d_l^*)(\upsilon_l - b_l^*) = \frac{\nabla^2_{b_l}\Gamma_l(b_l^*, d_l^*)}{\nabla_{b_l}\Gamma_l(b_l^*, d_l^*)}(\Gamma_l(b_l^*, d_l^*) - \nabla_{b_l}Q(\mathbf{b}^*, \mathbf{d}^*)K)$:

$$H_{ln} = \begin{cases} \frac{\nabla^2_{b_l}\Gamma_l(b_l^*, d_l^*)}{\nabla_{b_l}\Gamma_l(b_l^*, d_l^*)}(\Gamma_l(b_l^*, d_l^*) - \nabla_{b_l}Q(\mathbf{b}^*, \mathbf{d}^*)K) & \text{if} \quad l = n \\ \qquad\qquad -2\nabla_{b_l}\Gamma_l(b_l^*, d_l^*) + \nabla^2_{b_l}Q(\mathbf{b}^*, \mathbf{d}^*)K & \\ \nabla_{b_l}\nabla b_n Q(\mathbf{b}^*, \mathbf{d}^*)K & \text{if} \quad l \neq n \end{cases}$$

3. Recognise that if reservation prices were not binding, this Hessian matrix would have to be negative definite as a condition for utility maximising. Even as it is, for the subset of bids that are strictly above the reservation price, and so whose Lagrangian multipliers are zero, the corresponding sub-Hessian must also be negative definite.

4. Next, consider the inverse bidding system for $b_l > R$:

$$\xi_l(\mathbf{b}, \mathbf{d}; K) = b_l + \frac{\Gamma_l(b_l, d_l)}{\nabla_{b_l}\Gamma_l(b_l, d_l)} - \frac{1}{\nabla_{b_l}\Gamma_l(b_l, d_l)}\nabla_{b_l}Q(\mathbf{b}^*, \mathbf{d}^*)K$$

   Taking the Jacobian of this object yields:

$$J_{ln} = \begin{cases} 2 - \frac{\Gamma_l(b_l, d_l)\nabla^2_{b_l}\Gamma_l(b_l, d_l)}{\nabla_{b_l}\Gamma_l(b_l, d_l)^2} + \frac{\nabla^2_{b_l}\Gamma_l(b_l, d_l)}{\nabla_{b_l}\Gamma_l(b_l, d_l)^2}\nabla_{b_l}Q(\mathbf{b}^*, \mathbf{d}^*)K & \text{if} \quad l = n \\ \qquad\qquad -\frac{1}{\nabla_{b_l}\Gamma_l(b_l, d_l)}\nabla^2_{b_l}Q(\mathbf{b}^*, \mathbf{d}^*)K & \\ -\frac{1}{\nabla_{b_l}\Gamma_l(b_l, d_l)}\nabla_{b_l}\nabla_{b_n}Q(\mathbf{b}^*, \mathbf{d}^*)K & \text{if} \quad l \neq n \end{cases}$$

5. Substituting in our updated Hessian matrix considered previously, this yields the simple identity $J_{ln} = -\frac{1}{\nabla_{b_l}\Gamma_l(b_l, d_l)}H_{ln}$, or equivalently that $J = -\nabla_{\mathbf{b}}\Gamma_l(\mathbf{b}^*, \mathbf{d}^*)^{-1}H$. This is as expected given the envelope theorem, and definitely didn't come as an enormous surprise.

6. If every bid is strictly above the reservation price, or if reservation prices do not bind, then $det(J) = det(\nabla_{\mathbf{b}}\Gamma_l(\mathbf{b}^*, \mathbf{d}^*)^{-1})det(-H)$ which must be positive as H must be negative definite. Meanwhile, if only a subset of bids are strictly above the reservation price, we instead focus on just the sub-Hessian and Jacobians.

□

C Proof of Proposition 5

In this Appendix I prove Proposition 5. The proposition is given by:

Proposition 5. *Under assumption 1 - 5* $\Psi(I_S - \beta T\Omega)^{-1}C$ has rank $S_i - 1$

The proof is split into three parts. First, I establish the rank of Ψ, and then find its null space. I then demonstrate that the intersection of this null space and the image of $(I_S - \beta T\Omega)^{-1}C$ only contains a single element.

C.1 Rank of Ψ

This proposition employs assumption 5, as well as two additional lemmas stated below. But first, we need some additional tools to help us establish this result.

C.1.1 Additional Definitions

Define the partial ordering $\succeq^*$ such that if $\mathbf{s}_i \succeq \mathbf{s}_i'$ then $\mathbf{s} \succeq^* \mathbf{s}'$. This simply extends the partial ordering of the individual state to the overall state.

Next, define a 'component', written $\mathbb{S}^c$, of the set $\mathbb{S}$ as follows:[54]

Definition C.1 (Component)**.** The component $\mathbb{S}^c \subset \mathbb{S}$ is such that two states $\mathbf{s}, \mathbf{s}'$ are in component c if and only if there exists a state $\bar{\mathbf{s}}$ such that one of the following holds:

$$\mathbf{s} \succeq^* \bar{\mathbf{s}} \;\;\&\;\; \mathbf{s}' \succeq^* \bar{\mathbf{s}} \quad \text{or} \quad \mathbf{s} \succeq^* \bar{\mathbf{s}} \succeq^* \mathbf{s}' \quad \text{or} \quad \mathbf{s}' \succeq^* \bar{\mathbf{s}} \succeq^* \mathbf{s} \quad \text{or} \quad \bar{\mathbf{s}} \succeq^* \mathbf{s} \;\;\&\;\; \bar{\mathbf{s}} \succeq^* \mathbf{s}' \tag{15}$$

Therefore, a component is essentially a subset of $\mathbb{S}$ that are 'connected' by this partial ordering. By definition $\mathbf{s}_0$ does not vary within a component, and in general there will be one component corresponding to each element of the set $\mathbf{s}_0 \in \mathbb{S}_0$. The components are mutually exclusive and exhaustive subsets of $\mathbb{S}$. Suppose there are S^c components.

Finally, denote $\tilde{\min}(\mathbb{S})$ as the subset of $\mathbb{S}$, such that $\forall \mathbf{s} \in \tilde{\min}(\mathbb{S}) \;:\; \nexists\, \mathbf{s}' \in \mathbb{S} \;:\; \mathbf{s} \in \mathbb{S}^a(\mathbf{s}')$. This definition is primarily for notational convenience, and does not necessarily coincide with the set of minimal elements of $\mathbb{S}$. Instead, this is the (potentially empty) set of states that never occur as possible ex-post states. Intuitively, pay-offs from ending in these states will not be identified.

[54] Recognise that this definition corresponds to the graph theoretic definition of a component if we instead state that there exists a directed path between 'nodes' $\mathbf{s}, \mathbf{s}'$ if and only if $\mathbf{s}' \succeq^* \mathbf{s}$

C.1.2 Useful Lemmas

Lemma C.1. *From any two distinct, non-maximal, states,* $\mathbf{s}$ *and* $\mathbf{s}'$*, if* $\mathbf{s}' \not\succeq^* \mathbf{s}$ *then there exists a state* $\mathbf{s}^a$ *such that* $\mathbf{s}^a \in \mathbb{S}^a(\mathbf{s})$ & $\mathbf{s}^a \notin \mathbb{S}^a(\mathbf{s}')$

This Lemma states that if one non-maximal state is not 'higher' in the partial ordering than another, then their set of ex-post states cannot perfectly overlap. Proof of the lemma is very simple, and focuses on whether the unique element of $\mathbb{S}^a(\mathbf{s})$ that consists of bidder i winning every lot, denoted $\mathbf{s}^{all_i}$, can be an element of $\mathbb{S}^a(\mathbf{s}')$. The lemma makes use of the following property of the partial ordering $\succeq$ from assumption 5: For any two incomparable states $\mathbf{s}_i, \mathbf{s}'_i$ and any $\mathbf{s}_0$ there must exist some $\mathbf{s}^a \in \mathbb{S}^a_i(\mathbf{s}_i, \mathbf{s}_0)$ such that $\mathbf{s}^a \notin \mathbb{S}^a_i(\mathbf{s}'_i, \mathbf{s}_0)$. This result arises from the fact that the same set of lots are available in each state, ensuring that the sets $\mathbb{S}^a_i(\mathbf{s}_i, \mathbf{s}_0)$ and $\mathbb{S}^a_i(\mathbf{s}'_i, \mathbf{s}_0)$ cannot totally overlap. This enables proof of lemma C.1:

Proof:

1. Suppose that $\mathbf{s}' \not\succeq^* \mathbf{s}$. This gives us 2 initial options: Either $\mathbf{s} \succeq \mathbf{s}'$, or the two states are incomparable.
2. If $\mathbf{s} \succeq \mathbf{s}'$ then they must lie in the same component, implying that $\mathbf{s}_0 = \mathbf{s}'_0$. In turn, this implies that exactly the same lots must be available in each state. The result follows trivially - it cannot be the case that $\mathbf{s}^{all_i} \in \mathbb{S}^a(\mathbf{s}')$ for non-maximal states.
3. If they are incomparable then we have another two options: Either $\mathbf{s}$ and $\mathbf{s}'$ belong to different components, or they belong to the same component.
4. If they belong to different components then by definition $\mathbb{S}^a(\mathbf{s})$ and $\mathbb{S}^a(\mathbf{s}')$ must be mutually exclusive.
5. If they belong to the same component then by definition of the partial ordering $\succeq^*$ it must be that $\mathbf{s}_i$ and $\mathbf{s}'_i$ are incomparable under the ordering $\succeq$. Therefore there must exist some $\mathbf{s}^a \in \mathbb{S}^a_i(\mathbf{s}_i, \mathbf{s}_0)$ such that $\mathbf{s}^a \notin \mathbb{S}^a_i(\mathbf{s}'_i, \mathbf{s}_0)$. Therefore there must exist some $\mathbf{s}^a \in \mathbb{S}^a(\mathbf{s})$ such that $\mathbf{s}^a \notin \mathbb{S}^a(\mathbf{s}')$.

$\square$

Lemma C.2. $\Psi(\mathbf{s})A_{\mathbf{s}}$ *has rank at least 2 if, for all non-maximal* $\mathbf{s}, \boldsymbol{\upsilon}$*,* $\Gamma_{il}(\mathbf{b}(\boldsymbol{\upsilon}, \mathbf{s})|\mathbf{s}) \in (0, 1)$ *for each* l

The proof proceeds by first showing that $rank(\Psi(\mathbf{s}))$ is weakly greater than two, then using the full rank property of the transformation matrix $A_{\mathbf{s}}$.

Proof:

1. Consider the $L \times L(n-1)^{L-1}$ sub-matrix of $\Psi(\mathbf{s})$ that consists of only the columns of $\Psi(\mathbf{s})$ corresponding to outcomes in which player i wins exactly one lot. Call this matrix $\tilde{\Psi}$.
2. Row l, column a of $\tilde{\Psi}$ is strictly positive for columns corresponding to outcomes $\mathbf{w}^a$ in which bidder i wins lot l This arises because the probability that i wins lot l, and no other lot, is strictly increasing in b_l.

3. Row l, column a of $\tilde{\Psi}$ is strictly negative for columns corresponding to outcomes $\mathbf{w}^a$ in which bidder i does not win lot l. This arises because the probability that lot l is won, and no other lot is won, is strictly decreasing in b_m for $m \neq l$.

4. Any two rows of this matrix are linearly independent: Each row contains just one positive entry, each in a distinct column.[55] Therefore, $\tilde{\Psi}$, and hence $\Psi(\mathbf{s})$ has rank at least 2.

5. The matrix $A_{\mathbf{s}}$ is a rank n^L transformation matrix for any non-maximal $\mathbf{s}$.

6. From steps 5 and 4, $\Psi(\mathbf{s})A_{\mathbf{s}}$ for non-maximal $\mathbf{s}$ has rank at least 2.

$\square$

C.1.3 Rank(Ψ)

Proposition 6. *Rank*$(\Psi) = S - S^c - |\tilde{\min}(\mathbb{S})|$

Proving this proposition involves demonstrating that as we stack these $\Psi(\mathbf{s})A_{\mathbf{s}}$ matrices for non-maximal $\mathbf{s}$, the rank increases by *at least* two each time. However, by definition columns corresponding to elements in $\tilde{\min}(\mathbb{S})$ do not contain any non-zero entries, ensuring the rank must be deficient by at least $|\tilde{\min}(\mathbb{S})|$. Likewise, for each submatrix of Ψ made up of rows corresponding to states that are all within the same component (denoted by Ψ^c, a $|\mathbb{S}^c| \times S$ matrix), the rows all sum to zero. This ensures each Ψ^c is rank deficient by at least one, and so Ψ is rank deficient by at least S^c.

Proof:

1. Arbitrarily order, and label, the elements of $\mathbb{S}$ (and likewise the columns of Ψ) according to the partial ordering $\succeq^*$. Incomparable states can be ordered at random. This ensures that, for each $\mathbf{s}$, the furthest left non-zero column of $\Psi(\mathbf{s})A_{\mathbf{s}}$ is in the column corresponding to the ex-post state that corresponds to player i winning every lot having begun in state $\mathbf{s}$, $\mathbf{s}^{all_i}$.

2. Focus on one component, $\mathbb{S}^c$. Find the 'smallest' state within $\mathbb{S}^c$, smallest in terms of the ordering and labelling of step 1. Call this state $\mathbf{s}_1^c$. This element will be one of the minimal elements of $\mathbb{S}^c$, and, if the set is non-empty, will be an element of $\tilde{\min}(\mathbb{S}^c)$.

3. Next, find the second smallest state, which may also be a minimal element, and call this state $\mathbf{s}_2^c$. Vertically stack the two matrices $\Psi(\mathbf{s}_1^c)A_{\mathbf{s}_1^c}$ and $\Psi(\mathbf{s}_2^c)A_{\mathbf{s}_2^c}$, naming this matrix $\Psi^c_{\{1,2\}}$.

4. The matrix $\Psi^c_{\{1,2\}}$ has rank at least 4. Lemma C.2 ensures that both matrices have rank 2, while lemma C.1 ensures that each row of $\Psi(\mathbf{s}_1^c)A_{\mathbf{s}_1^c}$ is linearly independent of each row of $\Psi(\mathbf{s}_2^c)A_{\mathbf{s}_2^c}$. This last point arises because lemma C.1 ensures that since $\mathbf{s}_1^c \not\succeq^* \mathbf{s}_2^c$ there must be at least one column of non-zero entries in $\Psi(\mathbf{s}_2^c)A_{\mathbf{s}_2^c}$ that matches up to an all-zero column of $\Psi(\mathbf{s}_1^c)A_{\mathbf{s}_1^c}$.

[55] Note that this only holds for $L \geq 3$. For the case where $L = 2$ we must also assume $E[\Gamma_1 + \Gamma_2] \neq 1$, which holds generically.

5. Continue this process for each of the non-maximal states in component $\mathbb{S}^c$. At each stage, based on how we ordered the elements of $\mathbb{S}$ at step 1, and from lemmas C.2 and C.1, $\Psi(\mathbf{s}_n^c)A_{\mathbf{s}_n^c}$ must always contain at least one non-zero column that matches up to an all-zero column of $\Psi^c_{\{1,2\ldots n-1\}}$. In general, this will be the furthest left column, corresponding to the state $\mathbf{s}_n^{c\,all_i}$. Therefore, the rank will increase by at least 2 at each step.
6. The final matrix $\Psi^c_{\{1,2\ldots\}}$ has non-zero entries *somewhere* in each of the $|\mathbb{S}^c|$ columns corresponding to states in this set, except for columns correspond to elements of $\tilde{\min}(\mathbb{S}^c)$. These columns will contain all zeros, since there is always zero probability of ending a period in one of these states. As the rank of this matrix increased by at least two at each additional non-maximal state, and because we have at least as many non-maximal states as maximal states, this matrix must have rank at least $|\mathbb{S}^c| - |\tilde{min}(\mathbb{S}^c)| - 1$. The rank cannot be strictly greater than this. The rank must be strictly less than $|\mathbb{S}^c| - |\tilde{min}(\mathbb{S}^c)|$ because the row sum for each row of this final matrix must equal zero, a property inherited from the fact that $Q^T\boldsymbol{\iota} = 1$.
7. Any two components $\mathbb{S}^c$ and $\mathbb{S}^{c'}$ are mutually exclusive. Therefore, the two matrices for any two components $\Psi^c_{\{1,2\ldots\}}$ do not share any non-zero columns. Therefore, when we stack these matrices across different components, the ranks must sum together at each step.
8. Therefore $rank(\Psi) = \sum_{\mathbb{S}^c \subset \mathbb{S}} |\mathbb{S}^c| - |\tilde{min}(\mathbb{S}^c)| - 1 = S - |\tilde{\min}(\mathbb{S})| - S^c$

$\square$

One final thing of note: In general, the matrix $\Psi(\mathbf{s})A_{\mathbf{s}}$ has rank L. Essentially, this means that each state gives us L pieces of information, rather than just two pieces of information. However, proof of this proposition has proven elusive, even though it intuitively makes a lot of sense and has been observed empirically (that is, I have been able to show the rank is not weakly less than two). This feature ensures that we only need $1/L$ of the states to be non-maximal states, rather than requiring at least half be non-maximal.

C.2 nullspace of Ψ

The aim is to find the $|\tilde{\min}(\mathbb{S})| + S^c$ elements of this null space. Intuitively, there are two types of elements in this null space: Those corresponding to the $|\tilde{\min}(\mathbb{S})|$ elements, and those corresponding to the remaining S^c elements.

C.2.1 The $|\tilde{\min}(\mathbb{S})|$ elements

Immediately obvious is that any vector $\mathbf{y}$ that only contains non-zero entries in rows corresponding to elements of the set $\tilde{\min}(\mathbb{S})$ is in this null space. This is because Ψ contains all zeros in columns corresponding to these states. Call this set of vectors $\mathbb{Y}^1$, which evidently contains $|\tilde{\min}(\mathbb{S})|$ distinct elements.

C.2.2 The S^c elements

Consider the vector $\mathbf{y}$ such that $y_{\mathbf{s}} = y_{\mathbf{s}'}$ if $\mathbf{s}$ and $\mathbf{s}'$ belong to the same component. Call this set of vectors $\mathbb{Y}^2$, which evidently contains S^c distinct elements.

As above, denote Ψ^c the $|\mathbb{S}^c| \times S$ submatrix of Ψ that takes rows corresponding to states within component c. As established previously, columns of this matrix that correspond to states in different components only contain zeros, which follows from the definition of a component.

Therefore, for any $\mathbf{y} \in \mathbb{Y}^2$ we have $\Psi^c \mathbf{y} = 0$. This is because entries of $\mathbf{y}$ are constant across rows that correspond to the non-zero entries of Ψ^c. Clearly this holds for any c. Therefore, as we stack the Ψ^cs into Ψ we will have $\Psi \mathbf{y} = 0$ for any $\mathbf{y} \in \mathbb{Y}^2$.

C.3 Image of $(I_S - \beta T \Omega)^{-1} C$

We have previously established that the null space of Ψ is given by $\mathbb{Y}^1 \cup \mathbb{Y}^2$, which contains $|\tilde{\min}(\mathbb{S})| + S^c$ elements. I will now show that the intersection of this space and the image of $(I_S - \beta T \Omega)^{-1} C$ only contains a single element - the constant vector, denoted ι_{S^i}.

Establishing the necessary result requires three additional lemmas:

C.3.1 Three Additional Lemmas

Lemma C.3. *For any $\mathbf{y} \in \mathbb{Y}^1$ we have $\Omega \mathbf{y} = 0$.*

Proof:

1. Recall that $\Omega(\mathbf{s}) = E_{\mathbf{b}}[Q(\mathbf{b}^*|\mathbf{s})^T - \Gamma(\mathbf{b}^*|\mathbf{s})^T \nabla_{\mathbf{b}} \Gamma(\mathbf{b}^*|\mathbf{s})^{-1} \nabla_{\mathbf{b}} Q(\mathbf{b}^*|\mathbf{s})|\mathbf{s}] A_{\mathbf{s}}$
2. $A_{\mathbf{s}} \mathbf{y} = 0$ for $\mathbf{y} \in \mathbb{Y}^1$. This is because $A_{\mathbf{s}}$ selects the elements of $\mathbf{y}$ that correspond to possible ex-post states given beginning the period in state $\mathbf{s}$. But we know that $\mathbf{y}$ only contains non-zero entries in elements that correspond to states that are never observed as possible ex-post states.

□

Lemma C.4. *For any $\mathbf{y} \in \mathbb{Y}^2$ we have $\mathbf{y} = \Omega \mathbf{y}$. That is, every element of $\mathbb{Y}^2$ is an eignevector of the matrix Ω with eigenvalue 1.*

Proof:

1. Recall that $\Omega(\mathbf{s}) = E_{\mathbf{b}}[Q(\mathbf{b}^*|\mathbf{s})^T - \Gamma(\mathbf{b}^*|\mathbf{s})^T \nabla_{\mathbf{b}} \Gamma(\mathbf{b}^*|\mathbf{s})^{-1} \nabla_{\mathbf{b}} Q(\mathbf{b}^*|\mathbf{s})|\mathbf{s}] A_{\mathbf{s}}$
2. Note that for $\mathbf{y} \in \mathbb{Y}^2$ $A_{\mathbf{s}} \mathbf{y} = y_{\mathbf{s}} \iota_{2^L}$ Where ι_{2^L} is a 2^L dimensional vector of ones. This holds because $A_{\mathbf{s}}$ selects the elements of the vector $\mathbf{y}$ that correspond to states that are possible outcomes from an auction round beginning in state $\mathbf{s}$. By definition of a component every one of these elements of $\mathbf{y}$ lie in the same component.
3. As the rows of $Q(\mathbf{b}^*|\mathbf{s})^T$ sum to one, we have $E_{\mathbf{b}}[Q(\mathbf{b}^*|\mathbf{s})^T|\mathbf{s}] \iota_{2^L} = \iota_{2^L}$.

4. As the rows of $\nabla_{\mathbf{b}} Q(\mathbf{b}^*|\mathbf{s})$ sum to zero (derivative of a vector function with rows summing to one) we have:

$$E[\Gamma(\mathbf{b}^*|\mathbf{s})^T \nabla_{\mathbf{b}} \Gamma(\mathbf{b}^*|\mathbf{s})^{-1} \nabla_{\mathbf{b}} Q(\mathbf{b}^*|\mathbf{s})|\mathbf{s}]\iota_{2^L} = \mathbf{0}$$

5. Therefore $\Omega(\mathbf{s})\mathbf{y} = y_{\mathbf{s}}\iota_{2^L}$ for $\mathbf{y} \in \mathbb{Y}^2$
6. Therefore, as we stack our $\Omega(\mathbf{s})\mathbf{y}$s over $\mathbf{s}$ we are left with $\mathbf{y}$.

□

Finally, for $\mathbf{y} \in \mathbb{Y}^2$ we can write $\mathbf{y} = M\bar{\mathbf{y}}$ Where $\bar{\mathbf{y}}$ is an $S^c \times 1$ vector that contains each of the constant elements of $\mathbf{y}$, one for each component. Meanwhile M is an $S \times S^c$ dimensional matrix that contains a 1 in a row corresponding to state $\mathbf{s}$ and column corresponding to component c if $\mathbf{s} \in \mathbb{S}^c$, and zero otherwise. Therefore each row of M contains a single 1.

Lemma C.5. *Let the matrix N be any $S^c \times S^c$ submatrix of $(I - \beta T)M$ that is formed by selecting one row from each of the S^c components. N is non-singular.*

Proof:

1. Suppose we select S^c rows corresponding to states from different components, denoting this set of rows by $\mathbb{M}$. Therefore, the sub-matrix of interest is denoted $M_{\mathbb{M},.} - \beta T_{\mathbb{M},.}M$
2. We immediately have that $M_{\mathbb{M},.} = I$ from the definition of the matrix M. This is because we choose one row associated with each component. Each row of M contains a single 1, therefore so must $M_{\mathbb{M},.}$. Likewise, because every row is associated with a different component, each row contains a 1 in a different column.
3. Elements of the $S^c \times S^c$ sub matrix $T_{\mathbb{M},.}M$ are still just transition probabilities, so that $T_{\mathbb{M},.}M\iota_{S^c} = 1$. This is because right multiplying by M causes us to sum over states within a component. For row a particular row t we have element c of the row vector $T_{t,.}M$ is equal to $\sum_{\mathbf{s}\,:\,\mathbf{s}^c = \mathbf{s}^{\tilde{c}}} P(\mathbf{s}|\mathbf{s}^t)$. That is, it is the probability, given ending a period in state $\mathbf{s}^t$, that they begin the next period in any state from component c.
4. Diagonal entries of the matrix $I - \beta T_{\mathbb{M},.}M$ must be strictly positive, as $\beta\times$ a probability must be strictly less than 1 (for $\beta < 1$). Likewise, off diagonal entries must be weakly negative, as we have $-\beta\times$ a probability. Last, rows must sum to $1-\beta$. This is because rows of I evidently sum to 1, while rows of $T_{\mathbb{M},.}M$ also sum to 1.
5. This ensures this matrix is strictly diagonally dominant. Therefore, from the Levy–Desplanques theorem, the matrix must be non-singular.

□

C.3.2 Proof of Image Result

Proposition 7. $Image((I_S - \beta T\Omega)^{-1}C) \cap null(\Psi) = \iota_{S^i}$

Proof of this proposition also makes use of the standard result that $T\iota_S = \iota_S$, or that rows of a transition matrix must sum to one. The proof proceeds by first demonstrating that the image of $(I_S - \beta T\Omega)^{-1}C$ doesn't intersect $\mathbb{Y}^1$ at all. I then prove that the intersection with $\mathbb{Y}^2$ only contains a single element - the constant vector.

Proof:

1. Suppose there exists an $\mathbf{x}$ such that for some $\mathbf{y} \in \mathbb{Y}^1$ we could write $\mathbf{y} = (I_S - \beta T\Omega)^{-1}C\mathbf{x}$.
2. This implies $(I_S - \beta T\Omega)\mathbf{y} = C\mathbf{x}$.
3. From Lemma C.3 this implies $\mathbf{y} = C\mathbf{x}$. In turn, from the definition of C this requires $\mathbf{x}$ contains zeros in every entry except the first.
4. However this cannot be the case, since we always normalise this first entry to zero. Therefore $image((I_S - \beta T\Omega)^{-1}C) \cap \mathbb{Y}^1 = \emptyset$
5. I will now show that this image does not intersect with $\mathbb{Y}^2$ apart from a single element. Suppose there exists an $\mathbf{x}$ such that for some $\mathbf{y} \in \mathbb{Y}^2$ we could write $\mathbf{y} = (I - \beta T\Omega)^{-1}C\mathbf{x}$. Or, equivalently, such that $(I - \beta T\Omega)\mathbf{y} = C\mathbf{x}$
6. From Lemma C.4 This requires $(I - \beta T)\mathbf{y} = C\mathbf{x}$, or $(I - \beta T)M\bar{\mathbf{y}} = C\mathbf{x}$. We can then write this in matrix form:

   $$\begin{pmatrix} M - \beta\bar{T} & -C \end{pmatrix} \begin{pmatrix} \bar{\mathbf{y}} \\ \mathbf{x} \end{pmatrix} = 0$$

   Where $\bar{T} = TM$, essentially summing over the probability of transitioning to any given component from any given state. Therefore, if $(M - \beta\bar{T}, -C)$, the $S \times (S_C + S_i)$ matrix has rank $S_C + S_i - 1$ then there must be a unique $\mathbf{y}$ and $\mathbf{x}$ such that this relationship holds.
7. Consider whether the first column of $-C$ is linearly independent of the columns of $(M - \beta\bar{T})$. $-C_{.,1}$ contains -1 in every element associated with states such that $\mathbf{s}_i = \mathbf{s}_i^1$ and zeros otherwise. I now show that no linear combination of the columns for the corresponding rows of $(M - \beta\bar{T})$ can match these zeros. Choose S_c rows of $(M - \beta\bar{T})$ such that each row is associated with a state from a different component. For example, we might choose rows such that in each component $\mathbf{s}_i = \mathbf{s}_i^{S_i}$ - the 'final' individual state. Call the $S_c \times S_c$ submatrix of $M - \beta\bar{T}$ made of these rows (and all columns) N.
8. From Lemma C.5 N must be non-singular. Therefore there does not exist an $S_c \times 1$ vector $\mathbf{z}$ such that $N\mathbf{z} = 0$. Therefore columns of $(M - \beta\bar{T})$ must be linearly independent of $-C_{.,1}$. So, by catenating on this new column, the rank must increase by one.
9. Repeat this process for columns $n = 1...S_i - 1$ of $-C$. That is, every column *except* the final column which is the only column to contain non-zeros in entries associated with $\mathbf{s}_i^{S_i}$.[56] Each of these columns must be linearly independent of $M - \beta\bar{T}$ - no linear combination of its columns can match the zero entries of $-C_{.,n}$, since any $S^c \times S^c$ submatrix that consists of one row from each component must be non-singular.
10. Likewise, columns of $-C$ are all linearly independent of each other.
11. Therefore, at each step $n = 1...S_i - 1$, the rank of our catenated matrix increases by 1. Therefore $rank(M - \beta\bar{T}, -C) \geq S_C + S_i - 1$.
12. The vector $(\bar{\mathbf{y}} = \iota_{S^c}, \mathbf{x} = (1-\beta)\iota_{S^i})$ lies in the null space of $(M - \beta\bar{T}, -C)$. This is evident since $(M - \beta\bar{T})\iota_{S^c} = (1-\beta)\iota_S$ while we also have $C(1-\beta)\iota_{S_i} = (1-\beta)\iota_S$.

[56]This assumes that one individual state exists within each component (here, I used $\mathbf{s}_i^{S_i}$). This holds if for example, that $\mathbb{S} = \mathbb{S}^0 \times \prod_i \mathbb{S}_i$. However this is not strictly necessary. The only requirement is that at each step n I can select one row corresponding to a state from each component such that the corresponding rows of $-C_{.,n}$ are all zero.

Therefore, applying the rank-nullity theorem ensures that $Image((I_S - \beta T\Omega)^{-1}C) \cap null(\Psi) = \iota_{S^i}$

$\square$

D Extensions

D.1 Second-Price Auctions

In this Appendix I show that the previous identification results extend, almost trivially, to the second price case. In Appendix D.1.1 I set up the bidder's optimisation problem in the second price framework. In Appendix D.1.2 I show how optimal bidding yields a set of First Order Conditions, and hence an Inverse Bid System, similar to the first-price case. As before, this allows me to show that F is identified, conditional on $\mathbf{j}$ and V, from the inverse bid system. In Appendix D.1.3 I extend Proposition 4 from the main text to the second-price case, showing that maximised expected pay-off can be written as a function of the observed distribution of bids. This allows me to show that V is identified conditional on $\mathbf{j}$. In Appendix D.1.4 I prove that $\mathbf{j}$ is point identified from the same moment condition assumed in the main text.

I do not discuss estimation of the dynamic multi-object second price model. However it is clear that the estimation procedure presented in Section 4 can be applied to the second price setting, making use of the inverse bid system presented below.

D.1.1 Setup

In the second price setting, player i wins lot l at time t if $b_{ilt} > \max_{i'} \{b_{i'lt}\}$. As in the main text, let $\Gamma(\mathbf{b}|\mathbf{s})$ denote the $L \times 1$ equilibrium marginal probabilities of winning each lot. Define the vectors P and Q accordingly. The Value Function in the second price case can then be written as:

$$\begin{aligned} &W_i(\boldsymbol{v}_{it}, \mathbf{s}_t; \sigma_{-i}) \\ &\qquad = \max_{\mathbf{b}} \Big\{ \Gamma_i(\mathbf{b}; \sigma_{-i})^T (\boldsymbol{v}_i - \tilde{\mathbf{b}}(\mathbf{b}; \mathbf{s}_t)) + P_i(\mathbf{b}; \sigma_{-i})^T J_i(\mathbf{s}_t) + \beta Q_i(\mathbf{b}; \sigma_{-i})^T V_i(\mathbf{s}_t; \sigma_{-i}) \Big\} \end{aligned} \tag{16}$$

Where element a of the continuation value V_i is given by:

$$V_{ia}(\mathbf{s}_t; \sigma_{-i}) = \int_{\mathbf{s}} \int_{\boldsymbol{v}} W_i(\boldsymbol{v}, \mathbf{s}; \sigma_{-i}) dF(\boldsymbol{v}|\mathbf{s}) dT(\mathbf{s}|\mathbf{s}_t^a)$$

Finally, $\tilde{\mathbf{b}}(\mathbf{b};\mathbf{s}_t)$ gives the equilibrium expected second highest bid, given that b_{ilt} is the highest. Since the cdf of the highest rival bids is given by $\Gamma_l(x|\mathbf{s})$, we can write $\Gamma_l(b_l|\mathbf{s})\tilde{b}_l(\mathbf{b};\mathbf{s})) = \int_{\underline{b}_l}^{b_{ilt}} \bar{b}_l \nabla_{b_l}\Gamma_l(\bar{b}_l|\mathbf{s})d\bar{b}_l$.

D.1.2 First Order Conditions and Inverse Bid System

Fix the bid for every lot except lot l, and consider the marginal expected pay-off from winning lot l. This will be the expected pay-off conditional on winning lot l and on $\mathbf{b}^{-l}$ minus the expected pay-off conditional on losing lot l and $\mathbf{b}^{-l}$.

Following similar logic to the single-object second-price case, at the optimum, b_{lt}^* must equal this marginal expected pay-off, conditional on $\mathbf{b}_t^{-l}$. If they were to bid strictly below this figure, there is a non-zero chance they lose the lot, when they would have been willing to bid a little more, win the lot, and gain positive surplus. If they were to bid strictly above this figure, there is a non-zero chance they would win the lot and receive negative surplus, preferring instead to lower their bid.[57]

This marginal expected pay-off is given by:

$$\begin{aligned} &\upsilon_l + P(b_l = \bar{b}, \mathbf{b}^{-l}|\mathbf{s})J(\mathbf{s}) + Q(b_l = \bar{b}, \mathbf{b}^{-l}|\mathbf{s})\beta V(\mathbf{s}) \\ &\qquad\qquad - P(b_l = 0, \mathbf{b}^{-l}|\mathbf{s})J(\mathbf{s}) + Q(b_l = 0, \mathbf{b}^{-l}|\mathbf{s})\beta V(\mathbf{s}) \end{aligned}$$

From the definition of the vector P we get that:

$$\begin{aligned} &P_a(b_l = \bar{b}, \mathbf{b}^{-l}|\mathbf{s}) - P_a(b_l = 0, \mathbf{b}^{-l}|\mathbf{s}) \\ &\qquad = (-1)^{\mathbb{I}[w_l^a \neq i]} \times \prod_{m\neq l}^{L} [\prod_{j\neq i} G_{jm}(b_{imt})]^{\mathbb{I}[w_m^a = i]} \ \ [1 - \prod_{j\neq i} G_{jm}(b_{imt})]^{\mathbb{I}[w_m^a \neq i]} \\ &\qquad = \frac{1}{\nabla_{b_l}\Gamma_l(b_l|\mathbf{s})} \nabla_{b_l} P_a(b_l, \mathbf{b}^{-l}|\mathbf{s}) \end{aligned}$$

[57] This can easily be verified mathematically. Rearranging the maximand for:

$$\Gamma(\mathbf{b}|\mathbf{s})^T \boldsymbol{\upsilon} - \sum_l \int_{\underline{b}_l}^{b_l} \bar{b}_l \nabla_{b_l}\Gamma_l(\bar{b}_l|\mathbf{s})d\bar{b}_l + P(\mathbf{b}|\mathbf{s})J(\mathbf{s}) + \beta Q(\mathbf{b}|\mathbf{s})V(\mathbf{s})$$

and differentiating for FOCs:

$$0 = \nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})(\boldsymbol{\upsilon} - \mathbf{b}^*) + \nabla_{\mathbf{b}}P(\mathbf{b}^*|\mathbf{s})J(\mathbf{s}) + \beta\nabla_{\mathbf{b}}Q(\mathbf{b}^*|\mathbf{s})V(\mathbf{s})$$

Likewise, from the definition of Q we get that:

$$
\begin{aligned}
&Q_a(b_l = \bar{b}, \mathbf{b}^{-l}|\mathbf{s}) - Q_a(b_l = 0, \mathbf{b}^{-l}|\mathbf{s}) \\
&\qquad = (-1)^{\mathbb{I}[w_l^a \neq i]} \times \prod_{m \neq l}^{L} [\prod_{j \neq i} G_{jm}(b_{imt})]^{\mathbb{I}[w_m^a = i]} \;\; [Prob(w_m^a \text{ wins } m|b_{imt})]^{\mathbb{I}[w_m^a \neq i]} \\
&\qquad = \frac{1}{\nabla_{b_l}\Gamma_l(b_l|\mathbf{s})} \nabla_{b_l} Q_a(b_l, \mathbf{b}^{-l}|\mathbf{s})
\end{aligned}
$$

Therefore, at the optimum, conditional on $\mathbf{b}^{-l}$, we have:

$$
b_l^* = \upsilon_l + \frac{1}{\nabla_{b_l}\Gamma_l(b_l^*|\mathbf{s})} [\nabla_{b_l} P(b_l^*, \mathbf{b}^{-l}|\mathbf{s}) B_{\mathbf{s}}\mathbf{j} + \nabla_{b_l} Q(b_l^*, \mathbf{b}^{-l}|\mathbf{s}) A_{\mathbf{s}}\beta\mathbf{V}]
$$

Where the right hand side of this equation does not actually depend on b_l at all. Evidently, at the optimum, this condition must hold for each l. This gives us the following set of first order conditions:

$$
\mathbf{b}^* = \boldsymbol{\upsilon} + \nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}[\nabla_{\mathbf{b}} P(\mathbf{b}^*|\mathbf{s}) J(\mathbf{s}) + \nabla_{\mathbf{b}} Q(\mathbf{b}^*|\mathbf{s}) \beta V(\mathbf{s})]
$$

Where I have also made the substitution $V(\mathbf{s}) = A_{\mathbf{s}}\mathbf{V}$ and $J(\mathbf{s}) = B_{\mathbf{s}}\mathbf{j}$. These can be trivially inverted for the inverse bid function:

$$
\boldsymbol{\xi}(\mathbf{b}_{it}|J, \beta V; \mathbf{s}) = \mathbf{b}_{it} - \nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}[\nabla_{\mathbf{b}} P(\mathbf{b}^*|\mathbf{s}) B_{\mathbf{s}}\mathbf{j} + \nabla_{\mathbf{b}} Q(\mathbf{b}^*|\mathbf{s}) A_{\mathbf{s}}\beta\mathbf{V}]
$$

This is extremely similar to the inverse bid system presented in the main text, simply omitting the mark-up term $\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\Gamma(\mathbf{b}^*|\mathbf{s})$. It is then clear that, conditional on $\mathbf{j}$ and $\beta\mathbf{V}$, the distribution of lot specific values F is point identified from the empirical quantiles of $\boldsymbol{\xi}(\mathbf{b}_{it}|J, \beta V; \mathbf{s})$.

D.1.3 Extension of Proposition 4

I now extend Proposition 4 to the second price case. Note that there are many ways I could prove this general second price identification argument. I use this structure for the purposes of outlining the similarity to the first price case.

Proposition 8. *Under assumptions 1 - 4, the expected stage pay-off is given by:*

$$
\begin{aligned}
\tilde{\Pi}(\mathbf{b}^*|\boldsymbol{\upsilon}; \mathbf{s}) = {} & \Gamma(\mathbf{b}^*|\mathbf{s})^T(\mathbf{b}^* - \tilde{\mathbf{b}}(\mathbf{b}^*; \mathbf{s})) \\
& + [P(\mathbf{b}^*|\mathbf{s})^T \; - \; \Gamma(\mathbf{b}^*|\mathbf{s})^T \nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1} \nabla_{\mathbf{b}} P(\mathbf{b}^*|\mathbf{s})] B_{\mathbf{s}}\mathbf{j} \\
& + [Q(\mathbf{b}^*|\mathbf{s})^T \; - \; \Gamma(\mathbf{b}^*|\mathbf{s})^T \nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1} \nabla_{\mathbf{b}} Q(\mathbf{b}^*|\mathbf{s})] A_{\mathbf{s}}\beta\mathbf{V} \qquad (17)
\end{aligned}
$$

This is similar to the expression given in Proposition 4, except that the optimal lot specific surplus term is given by $\mathbf{b}^* - \tilde{\mathbf{b}}(\mathbf{b}^*; \mathbf{s})$ instead of $\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\Gamma(\mathbf{b}^*|\mathbf{s})$. Proof of this proposition is omitted due to its simplicity - one just substitutes the inverse bid function $\boldsymbol{\xi}(\mathbf{b}_{it}|J, \beta V; \mathbf{s})$ for $\boldsymbol{\upsilon}$ into the expression for the maximand of the value function in equation 16.

From Proposition 8, employing the identity $P(\mathbf{b}|\mathbf{s})^T B_{\mathbf{s}} = Q(\mathbf{b}|\mathbf{s})^T A_{\mathbf{s}} C$, and taking an expectation over the observed bids, write the ex-ante value function as:

$$
\begin{aligned}
V^e(\mathbf{s}) &= \Phi(\mathbf{s}) + \Omega(\mathbf{s})[C\mathbf{j} + \beta\mathbf{V}] \\
\text{Where} \quad \Phi(\mathbf{s}) &= E_{\mathbf{b}}[\Gamma(\mathbf{b}^*|\mathbf{s})^T(\mathbf{b}^* - \tilde{\mathbf{b}}(\mathbf{b}^*; \mathbf{s}))|\mathbf{s}] \\
\Omega(\mathbf{s}) &= E_{\mathbf{b}}[Q(\mathbf{b}^*|\mathbf{s})^T - \Gamma(\mathbf{b}^*|\mathbf{s})^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\nabla_{\mathbf{b}}Q(\mathbf{b}^*|\mathbf{s})|\mathbf{s}]A_{\mathbf{s}}
\end{aligned}
$$

Stacking this equation over $\mathbf{s}$ allows us to write the continuation value as

$$\mathbf{V} = T\Phi + T\Omega[C\mathbf{j} + \beta\mathbf{V}]$$

Which we can invert for: $\mathbf{V} = (I_S - \beta T\Omega)^{-1}[T\Phi + T\Omega C\mathbf{j}]$. This yields a stationary solution for the continuation value. Notice that this is exactly the same equation as in the main text, except I have defined the matrix $\Phi(\mathbf{s})$ slightly differently. It is therefore clear that $\mathbf{V}$ is point identified condition on $\mathbf{j}$.

D.1.4 Identification

As in the main text I impose the mean zero property of $\boldsymbol{\upsilon}$ for:

$$
\begin{aligned}
0 = E_{\mathbf{b}^*}[\boldsymbol{\xi}(\mathbf{b}^*; \mathbf{s}, (\mathbf{j}, \mathbf{V}))|\mathbf{s}] &= E_{\mathbf{b}^*}[\mathbf{b}^*|\mathbf{s}] - E_{\mathbf{b}^*}[\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\nabla_{\mathbf{b}}Q(\mathbf{b}^*|\mathbf{s})|\mathbf{s}]A_{\mathbf{s}}[C\mathbf{j} + \beta\mathbf{V}] \\
&= \Upsilon(\mathbf{s}) - \Psi(\mathbf{s})[C\mathbf{j} + \beta\mathbf{V}]
\end{aligned}
$$

Again, I have just used a slightly different definition of $\Upsilon(\mathbf{s})$. Stack over $\mathbf{s}$, and substituting in the expression for the continuation value found in subsection D.1.3, we get:

$$0 = \Upsilon - \beta\Psi(I_S - \beta T\Omega)^{-1}T\Phi - \Psi(I_S - \beta T\Omega)^{-1}C\mathbf{j}$$

There exists a unique solution to this system of equations ($\mathbf{j}$ is point identified) if and only if the $LS \times S_i$ matrix $\Psi(I_S - \beta T\Omega)^{-1}C$ has rank $S_i - 1$. This matrix is exactly the same as in the main text. Proposition 5 holds in this case as well, ensuring the sufficient rank condition.

D.2 Binding Reservation Prices

In this Appendix I introduce binding reservation prices. A reservation price is binding if, in equilibrium, there is a non-zero probability of winning a lot at the reservation price. Binding reservation prices do not pose a substantive problem, though do introduce additional mathematical complexity. I also introduce endogenous entry, but assume that entry is costless and the stochastic specific values are observed before an entry decision is made. When entry is costless reservation prices are necessary to prevent bidders submitting arbitrarily low bids. In Appendix D.3 I allow for costly entry where lot specific values are observed after entry.

In the presence of reservation prices a bidder with a low value may choose not to bid strictly above the reservation price. This results in corner solutions as bidders clump at the reservation price, or choose not to bid at all. This means we lose point identification. The First Order Conditions no longer point identify $\boldsymbol{v}_i$. This is a standard problem, even in a single object context.

The identification argument presented below diverges from the argument presented in the main text. Instead, it shares DNA with the estimation method presented in section 4. Identification is demonstrated in an additional step. First I show that F is (partially) identified conditional on (J, V, β), but in particular it is partially identified conditional on $J + \beta V$. I then show that the object $j(\mathbf{s}_i) + \beta V(\mathbf{s})$ is partially identified for all $\mathbf{s}$, making use of quantile moment conditions: Instead finding the $j+\beta V$ such that $E[\boldsymbol{\xi}(\mathbf{b}; \mathbf{s}, j+\beta V)|\mathbf{s}] = 0$ I find it such that $P(\xi_l(\mathbf{b}; \mathbf{s}, j+\beta V) \leq 0|\mathbf{s}) = 0.5$, imposing a zero conditional median assumption *instead* of the previous conditional mean assumption. Finally, I show that conditional on the identification of F and $J + \beta V$, V is point identified, and hence J can be backed out given an assumption about β.

D.2.1 Changes to the Model

Denote the reservation price as R. In principle this can be allowed to vary across lots, bidders, and time. Denote player i's entry decisions as the vector $\mathbf{d}_{it}$ with entry $d_{itl} = 1$ if they enter lot l, and zero otherwise. Adjust objects G, Γ, P and Q to be functions of entry decisions - if a player does not enter a lot, they trivially lose that lot with probability 1.[58]

Identification requires one additional assumption:

Assumption 6. $\frac{\partial \Gamma_{il}(\mathbf{b}_i|\mathbf{s})}{\partial b_{im}} = 0$ for $m \neq l$

I require that the probability an individual wins any given lot, conditional on the distribution

[58]In principle this change means that $\nabla_{\mathbf{b}}\Gamma$ will no longer be invertible, since it may contain zero rows/columns for non-entered auctions. However, this is not a problem at all, since a very simple generalised inverse can be used instead. This generalised inverse is discussed in appendix **??**.

of everyone else's bids, must depend only on their bid for that lot. This assumption implies that $\nabla\Gamma_i(\mathbf{b}_i|\mathbf{s})$ is a diagonal matrix. Up until this point this assumption was not strictly necessary for identification. Importantly, recognise that if ties happen with zero probability or if tie breaking is exogenous (based on the flip of a coin), then this assumption will hold.

I assume ties occur in equilibrium with zero probability. This is predominantly for mathematical convenience. In appendix D.6 I relax this assumption, allowing ties to be broken exogenously. The only thing this changes is how we identify F. Finally, I also assume the lot specific values have zero conditional median, replacing the previous zero conditional mean assumption. I am then able to prove the following proposition:

Proposition 9. *Given assumption 1 - 6, both $F_i(.|\mathbf{s})$ and $K_i(\mathbf{s})$ are non-parametrically partially identified. $k(\mathbf{s}^a)$ is point identified if we observe the individual bidding $b > R$ on a lot that may yield pay-off $k(\mathbf{s}^a)$.*

That is, we will point identify the truncated distribution $F_i(.|\boldsymbol{\upsilon} >= A_1(\mathbf{b}^*, \mathbf{s}); \mathbf{s})$, as well as the objects $F_i(A_1(\mathbf{b}^*, \mathbf{s}); \mathbf{s}) - F_i(A_2(\mathbf{b}^*, \mathbf{s}); \mathbf{s})$ and $F_i(A_2(\mathbf{b}^*, \mathbf{s}); \mathbf{s})$ for some (known) $A_1(\mathbf{b}^*, \mathbf{s}), A_2(\mathbf{b}^*, \mathbf{s})$.

I follow the same identification framework as above, As discussed in section 2, while I assume players play pure strategies conditional on entry, I must allow for the possibility that players play mixed strategies in their entry decisions. However, I am still able to use bidders' entry decisions to bound the pay-offs of un-entered auctions. I exploit the fact that, at the equilibrium mixing strategy, players can not *strictly* prefer to enter any other combination of auctions.

D.2.2 Identification of F, conditional on K.

Proposition 10. *Under assumptions 1 - 6, and conditional on K being point identified, the cdf F is non-parametrically partially identified.*

I prove Proposition 10 by proving that, similar to case 6.3.1.2 described in Athey and Haile (2007), we can invert observed bids, point identifying υ_l such that $b_l > R$. Meanwhile, for bids at the reservation price, so that $b_l = R$, we can use the first order conditions from the Lagrangian to find an upper bound on υ_l. I then use the fact that they still prefer to enter this auction and bid low, than not enter, to find a lower bound on υ_l. Lastly, for un-entered auctions we use the fact, at the margin, they prefer not to enter than to enter and bid low, to find an upper bound on these υ_l. Therefore, I am able to partially identify F.

First, reformulate the problem to include the entry decisions. The player's problem is to decide which auctions to enter ($\mathbf{d}$), and then set their bids ($\mathbf{b}$) to maximise utility, subject to the constraint that bids are weakly above reservation prices. The Lagrangian for this problem is given as:

$$L(\mathbf{b}, \mathbf{d}^*, \boldsymbol{\upsilon}, \boldsymbol{\lambda}|\mathbf{s}) = \Gamma(\mathbf{b}, \mathbf{d}^*|\mathbf{s})^T(\boldsymbol{\upsilon} - \mathbf{b}) + P(\mathbf{b}, \mathbf{d}^*|\mathbf{s})^T K + \boldsymbol{\lambda}^T(\mathbf{b} - R) \tag{18}$$

At the optimum their first order necessary conditions (with respect to bids) yields:

$$\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})(\boldsymbol{\upsilon} - \mathbf{b}^*) - \Gamma(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s}) + \nabla_{\mathbf{b}}P(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})^T K + \boldsymbol{\lambda}^* = 0 \tag{19}$$

Entry ll of $\nabla_{\mathbf{b}}\Gamma(\mathbf{b}, \mathbf{d}|\mathbf{s})$ is as it was in section 3 if $d_l = 1$, and normalised to 0 otherwise. Meanwhile entry la of $\nabla_{\mathbf{b}}P(\mathbf{b}, \mathbf{d}|\mathbf{s})$ is as it was in section 3 if $d_l = 1$, and normalised to 0 otherwise. Rearrange this equation for:[59]

$$\begin{aligned}\boldsymbol{\xi}(\mathbf{b}^*, \mathbf{d}^*, \boldsymbol{\lambda}|K; \mathbf{s}) = \mathbf{b}^* + \nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})^{-1}[\Gamma(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s}) - \nabla_{\mathbf{b}}P(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})K] \\ - \nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})^{-1}[\boldsymbol{\lambda}^*]\end{aligned} \tag{20}$$

As before at the true K we have $\xi_l(\mathbf{b}^*, \mathbf{d}^*, \boldsymbol{\lambda}^*|K; \mathbf{s}) = \upsilon_l$. However, we do not observe $\boldsymbol{\lambda}^*$. Therefore, define $\boldsymbol{\xi}(\mathbf{b}^*, \mathbf{d}^*|K; \mathbf{s}) = \mathbf{b}^* + \nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})^{-1}[\Gamma(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s}) - \nabla_{\mathbf{b}}P(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})K]$. Next, we consider what can be inferred for each entry/bidding decision. For each l there are four possibilities: *i)* $b_l > R$, *ii)* $b_l = R$, *iii)* $d_l = 0$, and the null case $l \notin \mathbb{L}$.

i) l such that $b_l^* > R$:

Importantly, any entry l such that $b_l^* > R_l$, $\lambda_l^* = 0$. Therefore, by Assumption 6, entry l of $\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})^{-1}[\boldsymbol{\lambda}^*]$ equals zero, and so $\xi_l(\mathbf{b}^*, \mathbf{d}^*|K; \mathbf{s}) = \upsilon_l$ is point identified.

ii) l such that $b_l^* = R$:

Next, for any entry l such that $b_l^* = R_l$, $\lambda_l^* > 0$. From Assumption 6 entry l of $\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})^{-1}[\boldsymbol{\lambda}^*]$ is greater than zero, and we attain the following bound:

$$\upsilon_l \leq \xi_l(\mathbf{b}^*, \mathbf{d}^*|K; \mathbf{s}) = R_l + \nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})^{-1}[\Gamma(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s}) - \nabla_{\mathbf{b}}P(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})K]_l$$

Where, for vector M, $[M]_l$ denotes row l. However, given that $(\mathbf{b}^*, \mathbf{d}^*)$ globally maximises expected utility, utility is (weakly) higher from playing $(\mathbf{b}^*, \mathbf{d}^*)$ than sitting out on auction l, by playing

[59] Using the pseudo-inverse for $\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*, \mathbf{d}^*|\mathbb{L})^{-1}$, given non-entry causes a particular row to be all zeros.

$(\mathbf{b}^{l-}, \mathbf{d}^{l-})$ (the only difference between these two actions is that $d_l^{l-} = 0$). Therefore:

$$\Gamma(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})^T(\boldsymbol{\upsilon} - \mathbf{b}^*) + P(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})^T K \geq \Gamma(\mathbf{b}^{l-}, \mathbf{d}^{l-}|\mathbf{s})^T(\boldsymbol{\upsilon} - \mathbf{b}^{l-}) + P(\mathbf{b}^{l-}, \mathbf{d}^{l-}|\mathbf{s})^T K \tag{21}$$

Which rearranges for:

$$\Gamma_l(b_l^*, d_l^*|\mathbf{s})(\upsilon_l - R_l) + [P(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s}) - P(\mathbf{b}^{l-}, \mathbf{d}^{l-}|\mathbf{s})]^T K \geq 0$$

Then:

$$\upsilon_l \geq R_l - \frac{1}{\Gamma_l(b_l^*, d_l^*|\mathbf{s})}[P(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s}) - P(\mathbf{b}^{l-}, \mathbf{d}^{l-}|\mathbf{s})]^T K \tag{22}$$

Therefore, we are able to bound υ_l between these two cut-offs: $\upsilon_l \in [A_1(\mathbf{b}^{l-}, \mathbf{s}), A_2(\mathbf{b}^*, \mathbf{b}^{l-}, \mathbf{s})]$.

$iii)$ l such that $d_l^* = 0$:

Finally, consider some l such that $d_l = 0$. They must attain greater utility from not bidding than from bidding the reservation price, all else equal. Consider an alternate action $(\mathbf{b}^{l+}, \mathbf{d}^{l+})$ where the only difference between this and their chosen action is that $b_l^{l+} = R_l$ and $d_l^{l+} = 1$. Therefore:

$$\Gamma(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})^T(\boldsymbol{\upsilon} - \mathbf{b}^*) + P(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})^T K \geq \Gamma(\mathbf{b}^{l+}, \mathbf{d}^{l+}|\mathbf{s})^T(\boldsymbol{\upsilon} - \mathbf{b}^{l+}) + P(\mathbf{b}^{l+}, \mathbf{d}^{l+}|\mathbf{s})^T K \tag{23}$$

Which rearranges for:

$$-\Gamma_l(b_l^{l+}, d_l^{l+}|\mathbf{s})(\upsilon_l - R_l) + [P(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s}) - P(\mathbf{b}^{l+}, \mathbf{d}^{l+}|\mathbf{s})]^T K \geq 0 \tag{24}$$

We then rearrange this for the bound:

$$\upsilon_l < R_l - \frac{1}{\Gamma_l(b_l^{l+}, d_l^{l+}|\mathbf{s})}[P(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s}) - P(\mathbf{b}^{l+}, \mathbf{d}^{l+}|\mathbf{s})]^T K \tag{25}$$

Therefore, these observations of bids and entry decisions enable partial non-parametric identification of F.

D.2.3 Identification of k under binding reservation prices

Proposition 11. *Under assumptions 1 - 6 the function k is non-parametrically partially identified up to standard normalisations. $k(\bar{\mathbf{s}})$ is point identified at $\mathbf{s} = \bar{\mathbf{s}}$ if we observe the bidder bid strictly above the reservation price on a combination of goods that would have the outcome $\mathbf{s}^a = \bar{\mathbf{s}}$.*

I prove this proposition by exploiting multiple observations for every state to establish a necessary rank condition, similar to the one presented in section 3. This enables me to overcome the inherent order condition. Whereas the previous proof then employed a condition on the mean of $\boldsymbol{\xi}(\mathbf{b},\mathbf{d})$, this proof essentially employs a condition on the marginal quantiles of $\boldsymbol{\xi}(\mathbf{b},\mathbf{d})$. I set $k(\mathbf{s})$ such that the median (or potentially some other quantile) is equal to zero.[60] As above, endogenous entry causes our first order conditions to break down, so that even at the true $\mathbf{k}$ $(= C\mathbf{j}+\beta\mathbf{V})$ we can only write:

$$\boldsymbol{\upsilon} \leq \boldsymbol{\xi}(\mathbf{b},\mathbf{d}|k;\mathbf{s}) = \mathbf{b} + \nabla_{\mathbf{b}}\Gamma(\mathbf{b},\mathbf{d}|\mathbf{s})^{-1}[\Gamma(\mathbf{b},\mathbf{d}|\mathbf{s}) - \nabla_{\mathbf{b}}P(\mathbf{b},\mathbf{d}|\mathbf{s})A_{\mathbf{s}}\mathbf{k}] \tag{26}$$

Where the inequality only holds with equality for rows l such that $b_l > R$. Stack these equations over $\mathbf{s}$ for:

$$\underbrace{\underline{\boldsymbol{\upsilon}}}_{LS\times 1} \leq \underline{\boldsymbol{\xi}}(\underline{\mathbf{b}},\underline{\mathbf{d}}|k) = \underbrace{\underline{\mathbf{b}}}_{LS\times 1} + \underbrace{\nabla_{\underline{\mathbf{b}}}\underline{\Gamma}(\underline{\mathbf{b}},\underline{\mathbf{d}})^{-1}}_{LS\times LS}[\underbrace{\underline{\Gamma}(\underline{\mathbf{b}},\underline{\mathbf{d}})}_{LS\times 1} - \underbrace{\nabla_{\underline{\mathbf{b}}}\underline{P}(\underline{\mathbf{b}},\underline{\mathbf{d}})}_{LS\times S}\,\mathbf{k}] \tag{27}$$

Where:

$$\underline{\boldsymbol{\xi}}(\underline{\mathbf{b}},\underline{\mathbf{d}}|k) = \begin{pmatrix} \boldsymbol{\xi}(\mathbf{b}_1,\mathbf{d}_1|k;\mathbf{s}_1) \\ \vdots \\ \boldsymbol{\xi}(\mathbf{b}_S,\mathbf{d}_S|k;\mathbf{s}_S) \end{pmatrix} \qquad \underline{\mathbf{b}} = \begin{pmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_S \end{pmatrix}$$

$$\underline{\Gamma}(\underline{\mathbf{b}},\underline{\mathbf{d}}) = \begin{pmatrix} \Gamma(\mathbf{b}_1,\mathbf{d}_1|\mathbf{s}_1) \\ \vdots \\ \Gamma(\mathbf{b}_S,\mathbf{d}_S|\mathbf{s}_S) \end{pmatrix} \qquad \nabla_{\underline{\mathbf{b}}}\underline{P}(\underline{\mathbf{b}},\underline{\mathbf{d}}) = \begin{pmatrix} \nabla_{\mathbf{b}}P(\mathbf{b}_1,\mathbf{d}_1|\mathbf{s}_1)A_{\mathbf{s}_1} \\ \vdots \\ \nabla_{\mathbf{b}}P(\mathbf{b}_S,\mathbf{d}_S|\mathbf{s}_S)A_{\mathbf{s}_S} \end{pmatrix} \tag{28}$$

We will require a rank condition on $\nabla_{\underline{\mathbf{b}}}\underline{\Gamma}(\underline{\mathbf{b}},\underline{\mathbf{d}})^{-1}\nabla_{\underline{\mathbf{b}}}\underline{P}(\underline{\mathbf{b}},\underline{\mathbf{d}})$. If this has full rank then each possible $\underline{\boldsymbol{\xi}}$ implies a unique $\mathbf{k}$, so that if we observed just one observation of $\underline{\boldsymbol{\xi}}$ we could solve for $\mathbf{k}$. Note that $E[\nabla_{\underline{\mathbf{b}}}\underline{\Gamma}(\underline{\mathbf{b}},\underline{\mathbf{d}})^{-1}\nabla_{\underline{\mathbf{b}}}\underline{P}(\underline{\mathbf{b}},\underline{\mathbf{d}})] = \Psi$, the matrix presented in the main text. Without binding reservation prices the rank proof and results for Ψ trivially carry over to $\nabla_{\underline{\mathbf{b}}}\underline{\Gamma}(\underline{\mathbf{b}},\underline{\mathbf{d}})^{-1}\nabla_{\underline{\mathbf{b}}}\underline{P}(\underline{\mathbf{b}},\underline{\mathbf{d}})$. This means we must normalise elements of $\mathbf{k}$ associated with the null space of Ψ.

However, with binding reservation prices, additional states will never be outcomes that were bid on, so the corresponding elements of $\mathbf{k}$ will not be point identified. These entries of $\mathbf{k}$ do not appear in the above equation, having a coefficient of zero. Consequently, partial identification requires the

[60]Note that additional exclusions restrictions on F also allow us to gain additional identification, particular if we observe a large degree of constrained bidding. For example, we might impose that F is independent of the set of lots on offer (i.e. marginal distributions do not vary with $\mathbb{L}_t$). We then gain additional power as we must also set k to ensure that the marginal distributions of inverse bids are invariant to $\mathbb{L}$. This argument appeared in a previous draft of this paper.

rank equals the number of states that are ever bid upon minus the number of normalised elements.

Next, fix an $LS \times 1$ vector of probabilities $\mathbf{p}$. Given the median zero assumption, every element of this vector should equal 0.5.[61] By definition of the marginal CDF, the following relationship holds:

$$\begin{pmatrix} p_1 \\ \vdots \\ p_{LS} \end{pmatrix} = \begin{pmatrix} F_1(\tilde{\upsilon}_1|\mathbf{s}_1) \\ \vdots \\ F_L(\tilde{\upsilon}_{LS}|\mathbf{s}_S) \end{pmatrix} = \begin{pmatrix} E_{\upsilon_1}[\ \mathbb{I}[\upsilon_1 \leq \tilde{\upsilon}_1]\ |\mathbf{s}_1] \\ \vdots \\ E_{\upsilon_L}[\ \mathbb{I}[\upsilon_L \leq \tilde{\upsilon}_{LS}]\ |\mathbf{s}_S] \end{pmatrix} \tag{29}$$

Next, we employ a change of variables, taking expectations over the observed random variables $(\mathbf{B}, \mathbf{D})$ rather than υ_l. This change is only valid for state-lot combinations such that when $\upsilon_l = \tilde{\upsilon}_l$, $b_l > R$, because only then $\xi_l(\mathbf{B}, \mathbf{D}; k) = \upsilon_l$ holds with equality, and so the mapping from $\mathbf{B}$ to υ_l is continuous, smooth, and monotonic.[62] We must to drop rows where this condition fails, as we will lose identifiability of corresponding elements of $\mathbf{k}$. The idea is that if, even when υ_l is as large as $\tilde{\upsilon}_l$, the elements of $K(\mathbf{s})$ corresponding to winning lot l are so small that they never bid strictly above R on lot l, then it is intuitive that these elements of $K(\mathbf{s})$ will not be identified.

This change of variables yields:

$$\mathbf{p} = \begin{pmatrix} E_{\upsilon_1}[\ \mathbb{I}[\upsilon_1 \leq \tilde{\upsilon}_1]\ |\mathbf{s}_1] \\ \vdots \\ E_{\upsilon_L}[\ \mathbb{I}[\upsilon_L \leq \tilde{\upsilon}_{LS}]\ |\mathbf{s}_S] \end{pmatrix} = \begin{pmatrix} E_{\mathbf{B},\mathbf{D}}[\ \mathbb{I}[\xi_1(\mathbf{B}_1, \mathbf{D}_1; k) \leq \tilde{\upsilon}_1]\ |\mathbf{s}_1] \\ \vdots \\ E_{\mathbf{B},\mathbf{D}}[\ \mathbb{I}[\xi_L(\mathbf{B}_S, \mathbf{D}_S; k) \leq \tilde{\upsilon}_{LS}]\ |\mathbf{s}_S] \end{pmatrix} \tag{30}$$

Proving point identification of $\mathbf{k}$ requires we show that this equation only holds at the true $\mathbf{k}$. We must show that the $\mathbf{p}$th quantiles of $\underline{\boldsymbol{\xi}}(\underline{\mathbf{B}}, \underline{\mathbf{D}}|k)$ equals $\tilde{\boldsymbol{\upsilon}}$ only at the true $\mathbf{k}$. But, from our rank condition, a unique $\underline{\boldsymbol{\xi}}(\underline{\mathbf{B}}, \underline{\mathbf{D}}|k)$ implies a unique $\mathbf{k}$. Therefore, only a unique $\mathbf{k}$ is such that the $\mathbf{p}$th quantiles of $\underline{\boldsymbol{\xi}}(\underline{\mathbf{B}}, \underline{\mathbf{D}}|k)$ equals $\tilde{\boldsymbol{\upsilon}}$. Therefore, there exists a unique $\mathbf{k}$ such that this equation holds.[63]

D.2.4 Non-Identified Elements

I now briefly discuss the elements of k that are not identified due to the binding reservation prices. There are three mutually exclusive types of non-identified elements. First, there are the elements

[61] In principal we might set this at a higher number, to ensure we identify as many elements of $\mathbf{k}$ as possible.

[62] This is essentially an application of the Law of the Unconscious Statistician. Monotonicity of the inverse bid function for bids strictly above the reservation price is proven in Appendix **??**.

[63] It should be noted that $\mathbf{k}$ is unique up to $|\tilde{min}(\mathbb{S})| + S^c$ elements of $\mathbf{k}$ that must be normalised to to the rank deficiency of the matrix Ψ. These elements are the entries associate with states $\mathbf{s} \in \tilde{min}(\mathbb{S})$ that are never observed as possible ex-post states, and one additional state from each component - associated with $\mathbf{s}_i = \mathbf{s}_i^1$. We will see in Appendix D.2.5 that these normalisations do not impact the identification of $\mathbf{j}$.

corresponding to outcome states $\mathbf{s}^a$ of various auctions that are never bid upon strictly above the reservation price, only at the reservation price. These elements are bounded above and below. If these elements were arbitrarily high then we could not rationalise the bidder only bidding the reservation price. If they were arbitrarily low then we could never rationalise them being bid on at all.

Second, there are elements corresponding to outcomes that are never bid on at all, but which there are occasions when only one additional bid would make this outcome a possibility. These elements are bounded above. If these elements were arbitrarily high then we could not rationalise the bidder never placing that one additional bid.

Third, there are the elements corresponding to $\mathbf{s}^a$ that are never bid on, and there are never occasions when, just by placing one additional bid, these outcomes become a possibility. These outcomes are neither bounded above, nor below. They are not bounded below because any arbitrarily low value would rationalise this outcome never being bid on. They are not bounded above because any arbitrarily high value can be rationalised by an arbitrarily low value corresponding to an 'adjacent' outcome from the second category, such that the risk of this arbitrarily low value is enough to put the bidder off from bidding on this arbitrarily high value. These bounds, where available, are detailed in Appendix D.2.6.

Therefore, $K_i(\mathbf{s})$ is partially identified.

D.2.5 Identification of j and βV

Up to this point I have proven the non-parametric (partial) identification of both F_i and $K_i = J_i + \beta V_i$. I now demonstrate that the primitives from the dynamic game, namely J and V are non-parametrically point-identified up to β. That is, I am essentially extending the implications of Proposition 1 from JP to the multi-object, binding reservation price, case.

Proposition 12. *Under assumptions 1 - 6, the objects $J_i(\mathbf{s})$, $V_i(\mathbf{s})$, and $F_i(|\mathbf{s})$ are non-parametrically (partially) identified up to $\beta = \bar{\beta}$.*

I have already proven that, under assumptions 1 - 6, both F and $J + \beta V$ are partially identified. Therefore, to prove Proposition 12 I only need to prove that, under these assumptions, J and V are separately identified conditional on both F and $J + \beta V$ being partially identified. I do so using a proof by construction. I show that V can be written as a function of only identified objects K and F. This is similar to JP's observation that in a single object context one can express the continuation value as a function of the distribution of bids alone.

First, consider maximised expected pay-off. Unlike the non-binding reservation prices case we cannot write down maximised expected utility as a function of bids and K only. In the presence of binding reservation prices the first order conditions break down when at least one bid is constrained. This means we then cannot substitute the inverse bid system into expected utility to find maximised expected utility. However, recognise that for any candidate $K_i = J_i + \beta V_i$ and any $\boldsymbol{\upsilon}_{it}$ we can still find maximised expected utility (Π^*) using numerical methods:

$$\Pi^*(\boldsymbol{\upsilon}_{it}, J_i + \beta V_i, \mathbf{s}_t) = \max_{(\mathbf{b}_t, \mathbf{d}_t)} \left\{ \Gamma_i(\mathbf{b}_t, \mathbf{d}_t|\mathbf{s}_t)^T(\boldsymbol{\upsilon}_{it} - \mathbf{b}_t) + P_i(\mathbf{b}_t, \mathbf{d}_t|\mathbf{s}_t)^T[J_i + \beta V_i] \quad s.t. \;\; b_l \geq R_{lt} \;\; \forall l \right\} \tag{31}$$

Therefore, because the function $K_i(\mathbf{s}_t) = J_i(\mathbf{s}_t) + \beta V_i(\mathbf{s}_t)$ is partially identified then so is maximised expected utility, and so the empirical counterpart to the Value Function, given by $W_i(\boldsymbol{\upsilon}, \mathbf{s})$, is also at least partially identified for all $\upsilon_l \in [\underline{\upsilon}, \bar{\upsilon}]$. In fact, this function is point identified, rather than only partially identified, because elements of $J_i(\mathbf{s}_t) + \beta V_i(\mathbf{s}_t)$ that were never bid on are not identified, only bounded. We know that the bidder would never bid on these combinations, so they will never enter the maximised expected utility.[64]

The ex-ante value function can be written as a function of the identified F and K only:

$$V_i^E(\mathbf{s}_t) = \int_{\boldsymbol{\upsilon}_i} W_i(\boldsymbol{\upsilon}_{it}, \mathbf{s}_t) dF_i(\boldsymbol{\upsilon}_{it}|\mathbf{s}_t) \tag{32}$$

Importantly, as shown in Appendix B the ex-ante value function can be written analytically as a function of the distribution of bids and entry decisions $G_i(\mathbf{b}, \mathbf{d}|\mathbf{s}_t)$, and the identified objects K and F, extending proposition 4 to the binding reservation prices case. As above, even though F is only partially identified the ex-ante value function remains point identified. This is because the non-identified (truncated) region of F is the region in which the bidder is never observed bidding. Therefore, since they would never bid for υ_l in this region their maximised expected utility will certainly just be the pay-off from not winning anything in this region.

We can then write the continuation value as follows:

$$V_{ia}(\mathbf{s}_t) = \int_{\mathbf{s}_{t+1}} V_i^E(\mathbf{s}) T_{\mathbf{s}}(\mathbf{s}|\mathbf{s}_t^a) \tag{33}$$

Therefore, given identification of $K_i(\mathbf{s})$ and $F(.|\mathbf{s})$ from the previous sections, we can evaluate this

[64]Note that if there are values of $J_i(\mathbf{s}_t) + \beta V_i(\mathbf{s}_t)$ that were only ever bid on at the reservation price, then the value function actually will be only partially identified. However, this non-identified region will generally be very small. ALSO NOTE THE NORMALISED ELEMENTS. SOME TRIVIALLY ZEROED OUT, OTHERS REPRESENT A LEVEL SHIFT, AND ONLY MARGINAL PAYOFFS ARE IDENTIFIED.

continuation value. Interestingly, however, V will be point identified even though F and K were not. The function J is then also non-parametrically identified as $J_i(\mathbf{s}) = K_i(\mathbf{s}) - \beta V_i(\mathbf{s})$. However, this will only be partially identified, given that K is only partially identified.

D.2.6 Bounds on non-identified elements of K

As discussed in the main text, there are three types of non-identified, non-normalised, elements of $\mathbf{k}$. First, elements corresponding to ex-post states which are never bid on strictly above the reservation price. That is, whenever the player bid on lots that made one of these states a possible outcome, they always bid exactly the reservation price. Second, elements corresponding to ex-post states that are never bid on, but which there are occasions where one additional bid placed would have made this state a possible outcome. Third, there are the elements corresponding to states that are never possible outcomes and there are no occasions where one additional bid would have made these states a possible outcome.

We consider each of these cases in reverse order.

Third type, unbounded

These elements of K are not-identified, and could take on any real value. These elements can take on any value without entering the bidders' observed first order conditions.

The coefficient on these entries that appears in the first order conditions - the corresponding columns of $\nabla_{\mathbf{b}} P(\mathbf{b}, \mathbf{d}|\mathbf{s})$ - are zero by definition, for any $\mathbf{s}$. Therefore, any arbitrarily high or arbitrarily low value of these elements are observationally equivalent.

Second type, bounded above

Suppose the element in question is $k(\mathbf{s}^a)$, which corresponds to element a of K. By definition of this type of element, we do observe the bidder bidding such that one additional bid, say on lot l, would make $\mathbf{s}^a$ a possible outcome. As discussed in section D.2.2 we are able to infer:

$$v_l < \bar{\xi}_l(\mathbf{b}|K;\mathbf{s}) = R_l - \frac{1}{\Gamma_l(b_l^{l+}, d_l^{l+}|\mathbf{s})} [P(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s}) - P(\mathbf{b}^{l+}, \mathbf{d}^{l+}|\mathbf{s})]^T K \tag{34}$$

Without loss of generality, while focusing on row l we can ignore the impact of every other non-bid on row. This is because columns of $[\nabla_{\mathbf{b}} P(\mathbf{b}, \mathbf{d}|\mathbf{s})]_l$ corresponding to winning other non-bid on lots are zero.

By definition, we know that $\upsilon_l \geq \underline{\upsilon}$. Therefore, $\bar{\xi}_l(\mathbf{b}|K;\mathbf{s}) \geq \underline{\upsilon}$. This, in turn, yields:

$$\underline{\upsilon} \leq R_l - \frac{1}{\Gamma_l(b_l^{l+}, d_l^{l+}|\mathbf{s})}[P(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s}) - P(\mathbf{b}^{l+}, \mathbf{d}^{l+}|\mathbf{s})]^T K \tag{35}$$

This equation rearranges for:

$$K_a \leq \frac{1}{[P(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s}) - P(\mathbf{b}^{l+}, \mathbf{d}^{l+}|\mathbf{s})]_a}(\Gamma_l(b_l^{l+}, d_l^{l+}|\mathbf{s})(R_l - \underline{\upsilon}) - [P(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s}) - P(\mathbf{b}^{l+}, \mathbf{d}^{l+}|\mathbf{s})]_{-a}^T K_{-a}) \tag{36}$$

Where $[P(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s}) - P(\mathbf{b}^{l+}, \mathbf{d}^{l+}|\mathbf{s})]_{-a}$ is the vector $[P(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s}) - P(\mathbf{b}', \mathbf{d}'|\mathbf{s})]$, minus row a. Importantly, recognise that $[P(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s}) - P(\mathbf{b}^{l+}, \mathbf{d}^{l+}|\mathbf{s})]_a \neq 0$. Meanwhile, K_{-a} is the vector K without element a, every element of which is either point identified, or gets multiplied by a zero coefficient in $[P(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s}) - P(\mathbf{b}^{l+}, \mathbf{d}^{l+}|\mathbf{s})]_{-a}$.

Hence, we have an upper bound on K_a. Note that a lowest upper bound can be found by searching over the space of $\mathbf{s}$ and non-bid on rows l that would also lead to outcome $\mathbf{s}^a$.

First type, bounded above and below

Suppose the element in question is $k(\mathbf{s}^a)$, which corresponds to element a of K. By definition of this type of element, we never observe them bidding strictly above the reservation price for each of the lots that make $\mathbf{s}^a$ a possible outcome. Therefore, focus on one of these occasions.

As we saw previously, we can write down the following inequality:

$$\begin{aligned} R_l - \frac{1}{\Gamma_l(b_l^*, d_l^*|\mathbf{s})}[P(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s}) - P(\mathbf{b}^{l-}, \mathbf{d}^{l-}|\mathbf{s})]^T K = \underline{\xi}_l(\mathbf{b}|K;\mathbf{s}) \leq \\ \upsilon_l \\ \leq \xi_l(\mathbf{b}|K;\mathbf{s}) = R_l + \nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})^{-1}[\Gamma(\mathbf{b}|\mathbf{s}) - \nabla_{\mathbf{b}}P(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})K]_l \end{aligned} \tag{37}$$

By recognising that $\underline{\upsilon} \leq \underline{\xi}_l(\mathbf{b}|K;\mathbf{s}) \leq \upsilon_l \leq \xi_l(\mathbf{b}|K;\mathbf{s}) \leq \bar{\upsilon}$ we can rearrange this inequality for:

$$\begin{aligned} K_a &\leq \frac{1}{[\nabla_{\mathbf{b}}\Gamma_i(\mathbf{b}|\mathbf{s})^{-1}\nabla_{\mathbf{b}}P(\mathbf{b}|\mathbf{s})]_{l,a}}(R - \underline{\upsilon} + \nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})^{-1}[\Gamma(\mathbf{b}|\mathbf{s})]_l - [\nabla_{\mathbf{b}}\Gamma_i(\mathbf{b}|\mathbf{s})^{-1}\nabla_{\mathbf{b}}P(\mathbf{b}|\mathbf{s})]_{l,-a}K_{-a}) \\ K_a &\geq \frac{1}{[P(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s}) - P(\mathbf{b}^{l-}, \mathbf{d}^{l-}|\mathbf{s})]_a}(\Gamma_l(b_l^*, d_l^*|\mathbf{s})(R_l - \bar{\upsilon}) - [P(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s}) - P(\mathbf{b}^{l-}, \mathbf{d}^{l-}|\mathbf{s})]_{-a}^T K_{-a}) \end{aligned} \tag{38}$$

Therefore bounding K_a both above and below. As considered previously, we can then search over the space of $\mathbf{s}$ and l to find the tightest bounds possible.

D.3 Endogenous Entry

RESULTS AND DISCUSSION TO BE ADDED

D.4 Inter-temporal Budget Constraint

In this Appendix I consider the case when bidders face an inter-temporal budget constraint, rather than assuming utility is quasi-linear in wealth. I focus on the case without binding reservation prices, and zero probability ties, for simplicity. It is clear how the model carries over to these other cases. Importantly, I show that a model with an inter-temporal budget constrain and constant marginal utility of wealth is observationally equivalent to the quasi-linear utility model discussed in the main text.

I do not conclusive prove non-parametric identification in this case, as such a proof goes well beyond the scope of this paper. I do, however, demonstrate that identification comes down to proving that there exists a unique solution to a system of first order differential moment equations. It is reasonably intuitive that observed variation in the state space should still obtain identification in this case, just as in the main text.

I do, however, show how such a model can be estimated with little more computational complexity than the estimation procedure presented in section 4 of the main text.

D.4.1 Budget Constraint

I will assume that bidders face a no-ponzi style budget constraint, such that their initial savings at $t+1$, A_{t+1} equal their savings at t, A_t plus their income y_t, less their realised expenditure in that period e_t. I assume bidders can save and borrow at interest rate r_t. This constraint is then given by:

$$\frac{A_{t+1}}{1+r_t} = A_t + y_t - e_t \tag{39}$$

For ease of notation I will treat r as constant. The only difference this makes is we remove the need for additional integrals.

D.4.2 The Bellman Equation

Writing down the Bellman equation in standard terms prohibits the use of matrix notation. For the sake of notation, define the set of possible allocations as $\mathbb{W}(\mathbf{s})$. The Bellman equation for this problem is given as:

$$W(\boldsymbol{v}, \mathbf{s}, A) = \max_{\mathbf{b}} \{\Pi(\mathbf{b}; \boldsymbol{v}, \mathbf{s}, A)\}$$

Where

$$
\begin{aligned}
\Pi(\mathbf{b};\boldsymbol{v},\mathbf{s},A) = &\sum_{l\in\mathbb{L}(\mathbf{s})}\Gamma_l(\mathbf{b}|\mathbf{s})v_l + \sum_{\mathbf{w}^a\in\mathbb{W}(\mathbf{s})} Q_a(\mathbf{b}|\mathbf{s})j(\mathbf{s}^a) \\
&+ \beta\sum_{\mathbf{w}^a\in\mathbb{W}(\mathbf{s})} Q_a(\mathbf{b}|\mathbf{s})\int_{\tilde{\mathbf{s}},\tilde{y}}\int_{\tilde{\boldsymbol{v}}} W(\tilde{\boldsymbol{v}},\tilde{\mathbf{s}},(1+r)[A-\sum_{l\in\mathbb{L}(\mathbf{s})}\mathbb{I}[w_l^a=i]b_l]+\tilde{y}))dF(\tilde{\boldsymbol{v}}|\tilde{\mathbf{s}})dF(\tilde{\mathbf{s}},\tilde{y}|\mathbf{s}^a)]
\end{aligned}
\tag{40}
$$

The optimal policy function, or optimal bidding function, $\mathbf{b}(\boldsymbol{v},\mathbf{s},A)$, must satisfy the following necessary First Order Conditions for each l:

$$
\begin{aligned}
0 = &\nabla_{b_l}\Gamma_l(\mathbf{b}^*|\mathbf{s})v_l + \sum_{\mathbf{w}^a\in\mathbb{W}(\mathbf{s})}\nabla_{b_l}Q_a(\mathbf{b}^*|\mathbf{s})j(\mathbf{s}^a) \\
&+ \sum_{\mathbf{w}^a\in\mathbb{W}(\mathbf{s})}\nabla_{b_l}Q_a(\mathbf{b}^*|\mathbf{s})\beta\int_{\tilde{\mathbf{s}},\tilde{y}}\int_{\tilde{\boldsymbol{v}}} W(\tilde{\boldsymbol{v}},\tilde{\mathbf{s}},(1+r)[A-\sum_{l\in\mathbb{L}(\mathbf{s})}\mathbb{I}[w_l^a=i]b_l^*]+\tilde{y}))dF(\tilde{\boldsymbol{v}}|\tilde{\mathbf{s}})dF(\tilde{\mathbf{s}},\tilde{y}|\mathbf{s}^a) \\
&- \beta\sum_{\mathbf{w}^a\in\mathbb{W}(\mathbf{s})}\mathbb{I}[w_l^a=i](1+r)Q_a(\mathbf{b}^*|\mathbf{s})\times \\
&\int_{\tilde{\mathbf{s}},\tilde{y}}\int_{\tilde{\boldsymbol{v}}}\nabla_A W(\tilde{\boldsymbol{v}},\tilde{\mathbf{s}},(1+r)[A-\sum_{l\in\mathbb{L}(\mathbf{s})}\mathbb{I}[w_l^a=i]b_l^*]+\tilde{y}))dF(\tilde{\boldsymbol{v}}|\tilde{\mathbf{s}})dF(\tilde{\mathbf{s}},\tilde{y}|\mathbf{s}^a)
\end{aligned}
\tag{41}
$$

Where, as standard with bellman equations, by applying the envelope theorem and the chain rule we find:

$$
\begin{aligned}
\nabla_A W(\boldsymbol{v},\mathbf{s},A) = &\beta(1+r)\sum_{\mathbf{w}^a\in\mathbb{W}(\mathbf{s})} Q_a(\mathbf{b}^*|\mathbf{s})\times \\
&\int_{\tilde{\mathbf{s}},\tilde{y}}\int_{\tilde{\boldsymbol{v}}}\nabla_A W(\tilde{\boldsymbol{v}},\tilde{\mathbf{s}},(1+r)[A-\sum_{l\in\mathbb{L}(\mathbf{s})}\mathbb{I}[w_l^a=i]b_l^*]+\tilde{y}))dF(\tilde{\boldsymbol{v}}|\tilde{\mathbf{s}})dF(\tilde{\mathbf{s}},\tilde{y}|\mathbf{s}^a)
\end{aligned}
\tag{42}
$$

D.4.3 Identification

Rather than working with the Bellman equations to consider identification, we will work with something slightly different. Define the continuation value as:

$$
V(\mathbf{s},A) = \int_{\tilde{\mathbf{s}},\tilde{y}}\int_{\tilde{\boldsymbol{v}}} W(\tilde{\boldsymbol{v}},\tilde{\mathbf{s}},(1+r)A+\tilde{y}))dF(\tilde{\boldsymbol{v}}|\tilde{\mathbf{s}})dF(\tilde{\mathbf{s}},\tilde{y}|\mathbf{s}^a)
$$

This gives the expected utility for the bidder at the point when $\mathbf{w}$ is announced. This allows us to rewrite our optimality conditions as:

$$0 = \nabla_{b_l}\Gamma_l(\mathbf{b}^*|\mathbf{s})v_l + \sum_{\mathbf{w}^a\in\mathbb{W}(\mathbf{s})} \nabla_{b_l}Q_a(\mathbf{b}^*|\mathbf{s})[j(\mathbf{s}^a) + \beta V(\mathbf{s}^a, A - \sum_{l\in\mathbb{L}(\mathbf{s})} \mathbb{I}[w_l^a = i]b_l^*))] \\ - \beta \sum_{\mathbf{w}^a\in\mathbb{W}(\mathbf{s})} \mathbb{I}[w_l^a = i]Q_a(\mathbf{b}^*|\mathbf{s})\nabla_A V(\mathbf{s}^a, A - \sum_{l\in\mathbb{L}(\mathbf{s})} \mathbb{I}[w_l^a = i]b_l^*])) \quad (43)$$

Next, as in the previous appendices, define the following object:

$$k(\mathbf{s}, A) = j(\mathbf{s}) + \beta V(\mathbf{s}, A) \quad (44)$$

Importantly, because we assume A does not directly enter $j()$, we can write $\nabla_A k(\mathbf{s}, A) = \beta\nabla_A V(\mathbf{s}, A)$.

D.4.4 Identification of F, conditional on k

Suppose the function k were known. The optimality conditions can be rewritten as:

$$0 = \nabla_{b_l}\Gamma_l(\mathbf{b}^*|\mathbf{s})v_l + \sum_{\mathbf{w}^a\in\mathbb{W}(\mathbf{s})} \nabla_{b_l}Q_a(\mathbf{b}^*|\mathbf{s})[k(\mathbf{s}^a, A - \sum_{l\in\mathbb{L}(\mathbf{s})} \mathbb{I}[w_l^a = i]b_l^*) \\ - \sum_{\mathbf{w}^a\in\mathbb{W}(\mathbf{s})} \mathbb{I}[w_l^a = i]Q_a(\mathbf{b}^*|\mathbf{s})\nabla_A k(\mathbf{s}^a, A - \sum_{l\in\mathbb{L}(\mathbf{s})} \mathbb{I}[w_l^a = i]b_l^*]) \quad (45)$$

Which we can invert, just as in the main text, for:

$$\xi_l(\mathbf{b}; k, \mathbf{s}) = -\sum_{\mathbf{w}^a\in\mathbb{W}(\mathbf{s})} \frac{\nabla_{b_l}Q_a(\mathbf{b}^*|\mathbf{s})}{\nabla_{b_l}\Gamma_l(\mathbf{b}^*|\mathbf{s})}[k(\mathbf{s}^a, A - \sum_{l\in\mathbb{L}(\mathbf{s})} \mathbb{I}[w_l^a = i]b_l^*) \\ + \sum_{\mathbf{w}^a\in\mathbb{W}(\mathbf{s})} \mathbb{I}[w_l^a = i]\frac{Q_a(\mathbf{b}^*|\mathbf{s})}{\nabla_{b_l}\Gamma_l(\mathbf{b}^*|\mathbf{s})}\nabla_A k(\mathbf{s}^a, A - \sum_{l\in\mathbb{L}(\mathbf{s})} \mathbb{I}[w_l^a = i]b_l^*]) \quad (46)$$

Therefore, conditional on $k(.)$ being identified, and evaluating this function at the true k, we have $\xi_l(\mathbf{b}; k, \mathbf{s}) = v_l$ and so the cdf $F(.|\mathbf{s})$ is non-parametrically identified.

D.4.5 Identification of j and V, conditional on k, F, and β

In this case, the function $W(\boldsymbol{v}, \mathbf{s}, A)$ is non-parametrically identified, since:

$$W(\boldsymbol{v}, \mathbf{s}, A) = \max_{\mathbf{b}} \left\{ \sum_{l\in\mathbb{L}(\mathbf{s})} \Gamma_l(\mathbf{b}|\mathbf{s})v_l + \sum_{\mathbf{w}^a\in\mathbb{W}(\mathbf{s})} Q_a(\mathbf{b}|\mathbf{s})[k(\mathbf{s}^a, A - \sum_{l\in\mathbf{L}(\mathbf{s})} \mathbb{I}[w_l^a = i]b_l^*)] \right\}$$

Which can be found by numerical maximisation. Likewise, if W is identified, the continuation value must also be identified since:

$$V(\mathbf{s}, A) = \int_{\tilde{\mathbf{s}},\tilde{y}} \int_{\tilde{\boldsymbol{v}}} W(\tilde{\boldsymbol{v}}, \tilde{\mathbf{s}}, (1+r)A + \tilde{y}))dF(\tilde{\boldsymbol{v}}|\tilde{\mathbf{s}})dF(\tilde{\mathbf{s}}, \tilde{y}|\mathbf{s}^a)$$

Where everything on the right hand side of this equation is identified. Therefore, $j(.)$ must also be identified, since $j(\mathbf{s}) = k(\mathbf{s}, A) = \beta V(\mathbf{s}, A)$.

D.4.6 Identification of k

Taking an expectation of both sides of equation 46, and setting this equation equal to zero yields the following:

$$\sum_{\mathbf{w}^a \in \mathbb{W}(\mathbf{s})} E[\frac{\nabla_{b_l} Q_a(\mathbf{b}^*|\mathbf{s})}{\nabla_{b_l} \Gamma_l(\mathbf{b}^*|\mathbf{s})} k(\mathbf{s}^a, A - \sum_{l \in \mathbb{L}(\mathbf{s})} \mathbb{I}[w_l^a = i] b_l^*)|\mathbf{s}]$$
$$= \sum_{\mathbf{w}^a \in \mathbb{W}(\mathbf{s})} E[\frac{\mathbb{I}[w_l^a = i] Q_a(\mathbf{b}^*|\mathbf{s})}{\nabla_{b_l} \Gamma_l(\mathbf{b}^*|\mathbf{s})} \nabla_A k(\mathbf{s}^a, A - \sum_{l \in \mathbb{L}(\mathbf{s})} \mathbb{I}[w_l^a = i] b_l^*])|\mathbf{s}] \quad (47)$$

This is simply a homogeneous system of first order differential equations. Therefore, non-parametric identification of k is equivalent to this system of equations having a unique solution. Proof that this system has a unique solution goes beyond the scope of this paper, and is unfortunately left for future work.

The idea behind any identification argument, however, would employ observed variation in both $\mathbf{s}$ and A in order to tease out a unique solution. Importantly, while the object k in the main text was only identified up to location (we had to normalise the level of utility), the object k will now only ever be identified up to location, while the marginal utility of wealth can only be identified up to location and scale. This is because, holding $\mathbf{s}$ constant, if $k(\mathbf{s}, A)$ is a solution to the above condition, so must be $\lambda k(\mathbf{s}, A)$ for $\lambda > 0$. This is a standard result in homogeneous first order differential equations.

D.4.7 Constant Marginal Utility of Wealth

Suppose we assume that the marginal utility of wealth is constant. That is, we assume that $\nabla_A W(v, \mathbf{s}, A) = \lambda$. Therefore, under this assumption we know that $W(\boldsymbol{v}, \mathbf{s}, A) = Y(\boldsymbol{v}, \mathbf{s}) + \lambda A$. The continuation value under this assumption can likewise be written as:

$$V(\mathbf{s}, A) = \int_{\tilde{\mathbf{s}},\tilde{y}} \int_{\tilde{\boldsymbol{v}}} Y(\tilde{\boldsymbol{v}}, \tilde{\mathbf{s}}) + \lambda[(1+r)A + \tilde{y}]dF(\boldsymbol{v}|\tilde{\mathbf{s}})dF(\tilde{\mathbf{s}}, \tilde{y}|\mathbf{s}) = \tilde{Y}(\mathbf{s}) + \lambda((1+r)A + E[\tilde{y}|\mathbf{s}])$$

We will also assume that $\beta(1+r)=1$, the standard equilibrium interest rate result from first year macroeconomics. This allows us to rewrite our first order optimality conditions as:

$$0 = \nabla_{b_l}\Gamma_l(\mathbf{b}^*|\mathbf{s})\upsilon_l + \sum_{\mathbf{w}^a \in \mathbb{W}(\mathbf{s})} \begin{cases} \nabla_{b_l} Q_a(\mathbf{b}^*|\mathbf{s})[j(\mathbf{s}^a) + \beta\tilde{Y}(\mathbf{s}^a) + \lambda\beta(1+r)(A - \mathbb{I}[w_l^a = i]b_l^* + E[\tilde{y}|\mathbf{s}^a])] \\ -\lambda\beta(1+r)\mathbb{I}[w_l^a = i]Q_a(\mathbf{b}^*|\mathbf{s}) \end{cases}$$

Imposing $\beta(1+r)=1$ and assuming that $E[\tilde{y}|\mathbf{s}^a]$ is invariant over $\mathbf{s}^a \in \mathbb{S}^a(\mathbf{s})$ (i.e. next period income is independent of who wins what in any given period), yields:

$$0 = \nabla_{b_l}\Gamma_l(\mathbf{b}^*|\mathbf{s})\upsilon_l + \sum_{\mathbf{w}^a \in \mathbb{W}(\mathbf{s})} \begin{cases} \nabla_{b_l} Q_a(\mathbf{b}^*|\mathbf{s})[j(\mathbf{s}^a) + \beta\tilde{Y}(\mathbf{s}^a) - \lambda\mathbb{I}[w_l^a = i]b_l^*] \\ -\lambda\mathbb{I}[w_l^a = i]Q_a(\mathbf{b}^*|\mathbf{s}) \end{cases}$$

Where we also recognise that $\sum_{\mathbf{w}^a \in \mathbb{W}(\mathbf{s})} \nabla_{b_l} Q_a(\mathbf{b}^*|\mathbf{s}) = 0$ so that the A and $E[\tilde{y}|\mathbf{s}^a]$ disappear. Finally, recognising that $\sum_{\mathbf{w}^a \in \mathbb{W}(\mathbf{s})} Q_a(\mathbf{b}^*|\mathbf{s})\mathbb{I}[w_l^a = i] = \Gamma_l(\mathbf{b}^*|\mathbf{s})$, we can write:

$$0 = \nabla_{b_l}\Gamma_l(\mathbf{b}^*|\mathbf{s})[\upsilon_l - \lambda b_l^*] - \lambda\Gamma_l(\mathbf{b}^*|\mathbf{s}) + \sum_{\mathbf{w}^a \in \mathbb{W}(\mathbf{s})} \nabla_{b_l} Q_a(\mathbf{b}^*|\mathbf{s})[j(\mathbf{s}^a) + \beta\tilde{Y}(\mathbf{s}^a)]$$

Which, given that the scale of the marginal utility of wealth is not identified, so normalised to 1, is exactly the same optimality conditions as those found in the main text, under the assumption that utility is quasi-linear in wealth. Therefore, a bidding model with an inter-temporal budget constraint and constant marginal utility of income is observationally equivalent to the bidding model with quasi-linear pay-offs.

D.4.8 Estimation

The estimation procedure outlined in section 4 maps neatly over to this case. The first estimation step is the same as it had been previously. Meanwhile, in the second estimation step the research jointly estimates both $F(.|\theta(\mathbf{s}))$, for some flexible distributional functional form F, and $k(\mathbf{s}, A)$ which can be done either parametrically or non-parametrically. As in the main text, at step k in the estimation we form the inverse bid function and find the conditional likelihood of these observations, given F and θ^k, k^k:

$$\xi_l(\mathbf{b};\mathbf{s};\hat{\Gamma},k^k) = \frac{1}{\nabla_{b_l}\hat{\Gamma}_l(\mathbf{b}^*|\mathbf{s})} \sum_{\mathbf{w}^a \in \mathbb{W}(\mathbf{s})} \begin{matrix} -\nabla_{b_l}\hat{Q}_a(\mathbf{b}^*|\mathbf{s})k^k(\mathbf{s}^a, A - \sum_{l \in \mathbb{L}(\mathbf{s})}\mathbb{I}[w_l^a = i]b_l^*) \\ +\mathbb{I}[w_l^a = i]\hat{Q}_a(\mathbf{b}^*|\mathbf{s})\nabla_A k(\mathbf{s}^a, A - \sum_{l \in \mathbb{L}(\mathbf{s})}\mathbb{I}[w_l^a = i]b_l^*]) \end{matrix}$$

In the third step, we then estimate the continuation value, and hence value function, by simulation, before backing out the function j.

D.5 Stochastic Combination Value

In this Appendix I present several identification results for the case when the combination value is stochastic. I focus on the case when the object $j(\mathbf{s})$ is not a function but a probability distribution. I initially focus on the static case, in which bidders are myopic ($\beta = 0$), before extending the framework to cover forward looking bidders.[65] I focus on the case with non-binding reservation prices and exogenous entry, since I established previously that these factors are only mathematically, not substantively, problematic.

I focus on two cases: First, when objects are homogenous, so that J only depends on the number of objects won. Second, when J is a function of low-dimensional un-observables. In both cases I consider several simple extensions that may be of interest to applied researchers in future.

These extensions both centre on the theme of finding some way to reduce the dimensionality of the unknowns. The key idea is this: Each observation of bidding on an auction yields L pieces of information. Therefore, in order to have any hope at point identifying unobservables, there cannot be more than L unobservables.

These extensions all require the following assumption:

Assumption 7. *Dimensionality reduction.* There exists an M dimensional 'aggregation' $\mathbf{h} : \mathbb{S}_{-i} \to \mathbb{R}^M$ such that for all l, $\mathbf{b}$, $\mathbf{s}_i$, and $\mathbf{s}_0$, $\Gamma_l(\mathbf{b}|\mathbf{s})$ can be written as $\Gamma_l(\mathbf{b}|\mathbf{s}_i, \mathbf{s}_0, \mathbf{h}(\mathbf{s}_{-i}))$. Likewise, the transition density for aggregation $\mathbf{h}(\mathbf{s}_{t+1})$ depends only on $\mathbf{h}(\mathbf{s}_t)$.

This assumption requires that for any two $\mathbf{s}_{-i}$ and $\mathbf{s}'_{-i}$ for which $h(\mathbf{s}_{-i}) = h(\mathbf{s}'_{-i})$ we have $\Gamma_l(\mathbf{b}|\mathbf{s}_i, \mathbf{s}_0, h(\mathbf{s}_{-i})) = \Gamma_l(\mathbf{b}|\mathbf{s}_i, \mathbf{s}_0, h(\mathbf{s}'_{-i}))$. Likewise, that the probability $\mathbf{h}(\mathbf{s}_{t+1}) = \mathbf{h}$ depends only on $\mathbf{h}(\mathbf{s}_t)$, not on individual elements of $\mathbf{s}_t$. This assumption simply reduces the dimensionality of the state space of rival bidders. It assumes that this can be done in such a way that bidder i only needs to consider aggregate statistics of the rival state, not each rival's state individually. The assumption is useful as it ensures that the bidder's continuation value does not depend on which rival won each combination of lots. Instead, they only need to consider the combinations of lots they win, reducing the dimensionality from n^L to 2^L. This assumption is employed in the application presented in section 5.

[65] This is because these cases with stochastic combination values are novel even in the static case.

D.5.1 Case 1. Homogenous lots (Static)

First, I consider the case when J only depends on the number of lots consumed. To begin with, assume bidders are myopic. Lots are allowed to differ (observably), but these differences cannot interact with the combinatorial pay-off. In this way, the pay-off associated with winning a combination of goods has two components: A component that depends on which lots are consumed, in an additively separable manner, and a component that depends just on how many lots are consumed. Expected utility is then given by:

$$EU(\mathbf{b}) = \Gamma(\mathbf{b})^T(\underbrace{\boldsymbol{\nu}}_{L\times 1} - \mathbf{b}) + \underbrace{P(\mathbf{b})^T}_{L+1\times 1}\mathbf{J}_t \tag{48}$$

Where ν give the deterministic lot-specific pay-off, and J_t gives the stochastic number of lots pay-off. Entry n gives the pay-off from winning $n-1$ lots.

We can rewrite expected utility as:

$$\Gamma(\mathbf{b})^T(\boldsymbol{\nu} - \mathbf{b}) + P(\mathbf{b})^T(\mathbf{J}_t - J_{1t}) + J_{1t}$$

Which follows from $\sum_a P_a(\mathbf{b}) = 1$. Recognise that row 1 of $(\mathbf{J}_t - J_{1t})$ is equal to zero, and so we can focus on just rows 2 through to $L+1$. For ease of notation, define $\tilde{P} = P_{2:L+1}$ and $\tilde{\mathbf{K}} = \mathbf{K}_{2:L+1}$

Necessary first order conditions are given by:

$$0 = \nabla_{\mathbf{b}}\Gamma(\mathbf{b})(\boldsymbol{\nu} - \mathbf{b}) - \Gamma(\mathbf{b}) + \nabla_{\mathbf{b}}\tilde{P}(\mathbf{b})(\tilde{\mathbf{J}}_t - J_{1t})$$

Importantly, $\nabla_{\mathbf{b}}\tilde{P}(\mathbf{b})$ is an $L \times L$ matrix with rank L.[66] Therefore, just as in the additively separable case we can invert these conditions for:

$$\tilde{\mathbf{J}}_t = J_{1t} + \nabla_{\mathbf{b}}\tilde{P}(\mathbf{b})^{-1}[\nabla_{\mathbf{b}}\Gamma(\mathbf{b})(\mathbf{b} - \boldsymbol{\nu}) + \Gamma(\mathbf{b})]$$

Therefore, conditional on J_{1t} and $\boldsymbol{\kappa}$, the joint distribution of $\tilde{\mathbf{J}}_t$ is non-parametrically point identified. Two important points are worth highlighting here: the fact that J_{1t} is not identified is not a problem. This is the same way that the pay-off from losing in the single-unit case is not identified. All we can identify is marginal pay-offs. Therefore, we will generally normalise the pay-off from losing to zero. Finally, $\boldsymbol{\nu}$ will be identified from variation in lot specific observables.

[66]Rank of this matrix follows from the fact that the matrix $\nabla_{\mathbf{b}}P(\mathbf{b})$ presented in section 3 has rank L, proven in Appendix C.1. The matrix $\tilde{P}$ is just a collapsed version of this, summed over the different lots.

D.5.2 Case 1b. Homogenous lots (dynamic)

Next, assume bidders are forward looking, and impose assumption 7. This ensures that we can write the value function as:

$$W(\mathbf{J}_t) = \max_{\mathbf{b}} \Gamma^T(\mathbf{b})(\boldsymbol{\nu} = \mathbf{b}) + P(\mathbf{b})^T[\mathbf{J}_t + \beta V]$$

That is, as a function of $P(\mathbf{b})$ rather than $Q(\mathbf{b})$.[67] Even though J is stochastic, the continuation value V remains deterministic (conditional on the state). We can then make the substitution $\mathbf{K}_t = \mathbf{J}_t + \beta V$ as we did in Appendix D.2, and see that the distribution of $\mathbf{K}_t$ is identified following the same arguments as in the previous section. Finally, recognise that just as in appendix D.2 we can write the ex-ante continuation value as:

$$V^E = \int_{\mathbf{b}} [(\Gamma(\mathbf{b})^T - P(\mathbf{b})^T \nabla \tilde{P}(\mathbf{b}) \nabla_{\mathbf{b}} \Gamma(\mathbf{b}))(\boldsymbol{\nu} - \mathbf{b}) + P(\mathbf{b})^T \nabla \tilde{P}(\mathbf{b}) \Gamma(\mathbf{b}) + K_1] dG(\mathbf{b})$$

Ensuring that we are again able to write the ex-ante continuation value as a function of K, rather than the objects J and βV individually. This ensures that we can separately identify J and V in a final step, just as we did previously.

D.5.3 Case 2: Known function of low dimensional un-observables

Suppose the combinatorial value can be written as $\mathbf{J}(\mathbf{m}_t)$ where $\mathbf{m}_t \in \mathbb{M}$ is an unobserved (potentially) stochastic random variable of dimension $M \leq L$. I require that $\mathbf{J} : \mathbb{M} \to \mathbb{J}$ is a known function (with range $\mathbb{J} \subset \mathbb{R}^{2^L}$). Importantly, some elements of $\mathbf{m}$ may represent fixed parameters associated with the functional form J, in this respect, this is essentially a semi-parametric identification argument.

Normalise the first element of this vector valued function (corresponding to player i losing every lot) to zero, so that I focus on the marginal combinatorial pay-off $\mathbf{J}(\mathbf{m})_{2:2^L} - \mathbf{J}(\mathbf{m})_1$. Expected utility is then given by:

$$EU(\mathbf{b}) = P(\mathbf{b})^T \mathbf{J}(\mathbf{m}_t) - \Gamma(\mathbf{b})^T \mathbf{b}$$

Necessary first order conditions are then given by:

$$0 = \nabla_{\mathbf{b}} P(\mathbf{b}) \mathbf{J}(\mathbf{m}_t) - \nabla_{\mathbf{b}} \Gamma(\mathbf{b}) \mathbf{b} - \Gamma(\mathbf{b})$$

Given that $\mathbf{m}$ is the only non-identified object in this equation, the question is then whether this is

[67] Technically we must also assume the equilibrium played in every period is such that the continuation value only depends on the number of lots won in previous periods.

point identified. I make two assumptions about this function that are sufficient for $\mathbf{m}_t$ to be point identified:

Assumption 8.

1. $\mathbf{J}(\mathbf{m})$ is continuous and continuously differentiable for all $\mathbf{m}_t$.
2. For any $\mathbf{m}$ and $\mathbf{m}'$ there exists a set $\mathbb{U} \subset \{1, 2, \ldots, 2^L\}$ with $|\mathbb{U}| = m$ that defines the vector value function $\mathbf{F}^{\mathbb{U}}$ where $F_n^{\mathbb{U}}(\mathbf{m}) = J_{U_n}(\mathbf{m})$ such that

$$(\mathbf{m} - \mathbf{m}')^T (\mathbf{F}^{\mathbb{U}}(\mathbf{m}) - \mathbf{F}^{\mathbb{U}}(\mathbf{m}')) > 0$$

The second part of this assumption is essentially an extension of strict monotonicity to the case of 2^L dimensional functions in M dimensional variables. The assumption states that for any two distinct $\mathbf{m}$ and $\mathbf{m}'$ we can find a set of rows of $\mathbf{J}(.)$ such that this inner product is strictly positive.[68] A key result of this property is that the function $\mathbf{J}(.)$ is a bijection: Each $\mathbf{m}$ maps onto a unique $\mathbf{J}$, and the condition ensures that for any two distinct $\mathbf{m}$ and $\mathbf{m}'$ it must be the case that $\mathbf{J}(\mathbf{m}) \neq \mathbf{J}(\mathbf{m}')$ (since otherwise we could not find a $\mathbb{U}$ such that $(\mathbf{m} - \mathbf{m}')^T (\mathbf{F}^{\mathbb{U}}(\mathbf{m}) - \mathbf{F}^{\mathbb{U}}(\mathbf{m}')) > 0$). This ensures that the inverse $\mathbf{J}^{-1}(.)$ exists, such that for all $\mathbf{m} \in \mathbb{M}$ $\mathbf{m} = \mathbf{J}^{-1}(\mathbf{J}(\mathbf{m}))$. Furthermore, because $\mathbf{J}(.)$ is continuous and continuously differentiable everywhere, so that $\mathbf{J}^{-1}(.)$ must be differentiable everywhere, $\mathbf{J}^{-1}(.)$ must also be continuous.

Proposition 13. *Under assumptions 1, 2, and 8, $\mathbf{m}_t$ is point identified up to normalisation.*

We may only identify $\mathbf{m}_t$ up to location and scale if, for example, the second and third elements of $\mathbf{m}_t$ are constant parameters describing the mean and standard deviation of m_{1t}.

The proof of Proposition 13 consists of arguing that we have L equations in only M unknowns, and that there exists a unique solution to the system of equations. The proof proceeds by recognising that the set of vectors $\mathbf{J}$ which satisfy the first order conditions must be convex. Which, from the continuity of the inverse function $\mathbf{J}^{-1}(.)$ and the (generalised) intermediate value theorem, implies that the set of $\mathbf{m}$ for which the first order conditions hold must be path connected. This implies there must be a point arbitrarily close to $\mathbf{m}_t$ for which the first order conditions hold. However, the fact that $\nabla_{\mathbf{b}} P(\mathbf{b})$ has rank L and the function $\mathbf{J}(.)$ is invertible implies that the system of equations must be locally unique.

[68]This property is satisfied when, for example, each element of J is weakly monotone in elements of $\mathbf{m}$, and strictly monotonic in at least one element.

Proof:

1. Consider the set of $2^L \times 1$ dimensional vectors which satisfy the system of equations $\nabla_{\mathbf{b}} P(\mathbf{b})\mathbf{K} - \nabla_{\mathbf{b}}\Gamma(\mathbf{b})\mathbf{b} - \Gamma(\mathbf{b}) = 0$. This set, which I will refer to as $\tilde{\mathbb{J}}$, must be convex, and hence path-connected, as for any two vectors in this set $\mathbf{J}, \mathbf{J}'$ we have:

$$\begin{aligned} \lambda \nabla_{\mathbf{b}} P(\mathbf{b})\mathbf{J} = \lambda(\nabla_{\mathbf{b}}\Gamma(\mathbf{b})\mathbf{b} + \Gamma(\mathbf{b})) \\ \& \qquad (1-\lambda)\nabla_{\mathbf{b}} P(\mathbf{b})\mathbf{J}' = (1-\lambda)(\nabla_{\mathbf{b}}\Gamma(\mathbf{b})\mathbf{b} + \Gamma(\mathbf{b})) \\ \therefore\ \lambda \nabla_{\mathbf{b}} P(\mathbf{b})\mathbf{J} + (1-\lambda)\nabla_{\mathbf{b}} P(\mathbf{b})\mathbf{J}' = \nabla_{\mathbf{b}}\Gamma(\mathbf{b})\mathbf{b} + \Gamma(\mathbf{b}) \\ \nabla_{\mathbf{b}} P(\mathbf{b})(\lambda\mathbf{J} + (1-\lambda)\mathbf{J}') = \nabla_{\mathbf{b}}\Gamma(\mathbf{b})\mathbf{b} + \Gamma(\mathbf{b}) \end{aligned} \tag{49}$$

2. This implies that the image of the intersection of $\tilde{\mathbb{J}}$ and $\mathbb{J}$ defined by the continuous function $\mathbf{J}^{-1}(.)$, that is the set of $\mathbf{m}$ for which the first order conditions hold, must also be path connected. This result follows from the generalised intermediate value theorem, which states that for a continuous function $f : \mathbb{X} \to \mathbb{Y}$, if the set $\mathbb{X}$ is path-connected, then so is the image $f(\mathbb{X})$.
3. Therefore, if the intersection of $\tilde{\mathbb{J}}$ and $\mathbb{J}$ contains more than a single element, then for any $\mathbf{m}$ which satisfies the first order conditions, there must exist an arbitrarily nearby $\mathbf{m}'$ which also satisfies the first order conditions.
4. However, from the inverse function theorem, the first order conditions are locally unique. The Jacobian of the first order conditions, with respect to $\mathbf{m}$ are given by:

$$\nabla_{\mathbf{b}} P(\mathbf{b})\nabla_{\mathbf{m}}\mathbf{J}(\mathbf{m})$$

which has rank M. This can be seen because $\nabla_{\mathbf{b}} P(\mathbf{b})$ has rank L, consisting of L linearly independent rows. Meanwhile, $\nabla_{\mathbf{m}}\mathbf{J}(\mathbf{m})$ has rank M, which arises because $\mathbf{J}(\mathbf{m})$ is invertible.
5. Therefore the set of $\mathbf{m}$ which satisfy the first order conditions must contain only a single element. Likewise, the intersection of $\tilde{\mathbb{J}}$ and $\mathbb{J}$ must also contain only a single element

□

Importantly, once $\mathbf{m}_t$ is point identified, we can identify its joint distribution over time. Information about previous winnings then allow us to pin down the location and scale of $\mathbf{m}_t$. For example, if $\mathbf{m}_t$ represents the bidder's stock of objects, which is assumed to follow an VAR(1) process along the lines of $\mathbf{m}_t = A\mathbf{m}_{t-1} + \mathbf{z}_{t-1} + \varepsilon_{t-1}$ then observation of winnings $\mathbf{z}_t$ immediately pin down the location and scale, while the distribution of ε_t and the auto-regressive matrix A can then be identified by running a simple auxiliary regression (assuming, for example, that ε_t is weakly exogenous with respect to $\mathbf{z}_t$).

D.5.4 Case 2: Extension 1. When M > L

This case is important when $\mathbf{m}_t$ can be decomposed into $(\mathbf{m}_t^1, \mathbf{m}^0)$, where $\mathbf{m}^0$ are fixed parameters that do not vary over time. Suppose $M \leq 2L$, and in particular, $|\mathbf{m}_t^1| < L$. In this case, rather than considering a single set of first order conditions, we can look at a pair of first order conditions from two separate periods t_1 and t_2. Importantly, we still impose assumption 8. Combine the two sets of first order conditions as follows:

$$\begin{pmatrix} \nabla_{\mathbf{b}} P(\mathbf{b}_{t_1}) & 0 \\ 0 & \nabla_{\mathbf{b}} P(\mathbf{b}_{t_2}) \end{pmatrix} \begin{pmatrix} \mathbf{J}(\mathbf{m}_{t_1}) \\ \mathbf{J}(\mathbf{m}_{t_2}) \end{pmatrix} = \begin{pmatrix} \nabla_{\mathbf{b}} \Gamma(\mathbf{b}_{t_1})\mathbf{b}_{t_1} + \Gamma(\mathbf{b}_{t_1}) \\ \nabla_{\mathbf{b}} \Gamma(\mathbf{b}_{t_2})\mathbf{b}_{t_2} + \Gamma(\mathbf{b}_{t_2}) \end{pmatrix}$$

The fact there exists a unique pair $(\mathbf{m}_{t_1}, \mathbf{m}_{t_2})$ that solves this system of equations follows the same logic as the previous proof with the added note that $\nabla_{(\mathbf{m}_{t_1}, \mathbf{m}_{t_2})} \begin{pmatrix} \mathbf{J}(\mathbf{m}_{t_1}) \\ \mathbf{J}(\mathbf{m}_{t_2}) \end{pmatrix}$ has rank $2|\mathbf{m}_t^1| + |\mathbf{m}^0|$, ensuring that we are again able to appeal to the inverse function theorem for local uniqueness.

This result is important as it allows us to add a large number of additional parameters to the function $\mathbf{J}(.)$ which are identified by essentially using variation across observations. This makes use of a similar philosophy used to prove the identification results in the main sections of this paper.

D.5.5 Case 2: Extension 2. Forward Looking Bidders

As before, I will impose assumption 7. In addition, I allow that the continuation value is a function of $\mathbf{m}$. In which case I write $\mathbf{K}(\mathbf{m}) = \mathbf{J}(\mathbf{m}) + \beta V(\mathbf{m})$. Extend assumption 8 as follows:

Assumption 9.

1. $\mathbf{K}(\mathbf{m}) = \mathbf{J}(\mathbf{m}) + \beta \mathbf{V}(\mathbf{m})$ is continuous and continuously differentiable for all $\mathbf{m}_t$.
2. For any $\mathbf{m}$ and $\mathbf{m}'$ there exists a set $\mathbb{U} \subset \{1, 2, \ldots, 2^L\}$ with $|\mathbb{U}| = m$ that defines the vector value function $\mathbf{F}^{\mathbb{U}}$ where $F_n^{\mathbb{U}}(\mathbf{m}) = K_{U_n}(\mathbf{m})$ such that

$$(\mathbf{m} - \mathbf{m}')^T (\mathbf{F}^{\mathbb{U}}(\mathbf{m}) - \mathbf{F}^{\mathbb{U}}(\mathbf{m}')) > 0$$

This assumption essentially rewrites the previous assumption in terms of K. It does not necessarily require assumption 8, but it is difficult to imagine a case of assumption 9 in which 8 does not hold. While the function K is assumed known, as in the previous section elements of $\mathbf{m}$ can be considered unknown constant parameters that allow K to be unknown up to parameters. First order conditions for optimal bidding are then given by:

$$\nabla_{\mathbf{b}} P(\mathbf{b})^T \mathbf{K}(\mathbf{m}_t) - \nabla_{\mathbf{b}} \Gamma(\mathbf{b})\mathbf{b} - \Gamma(\mathbf{b}) = 0$$

By 'inverting' the argument presented in previous sections ensures that $\mathbf{m}_t$ is point identified.

Conditional on $\mathbf{m}_t$ being point identified we can then write the ex-ante continuation value as:

$$V^E = P(\mathbf{b})^T A(\mathbf{b})\Gamma(\mathbf{b}) + (P(\mathbf{b})^T A(\mathbf{b})\nabla_{\mathbf{b}}\Gamma(\mathbf{b}) - \Gamma(\mathbf{b})^T)\mathbf{b}$$

Where $A(\mathbf{b})$ gives a $2^L \times L$ generalised left inverse of $\nabla_{\mathbf{b}} P(\mathbf{b})$. The left inverse is of course not unique. However, as we previously established that there exists a unique vector $\mathbf{K}$ within the image of the function $\mathbf{K}(\mathbf{m})$ such that the first order conditions hold, it is the generalised inverse associated with this vector to which I refer. This ensures that the ex-ante value function is identified, and consequently so is the continuation value.[69].

D.5.6 Case 2: Extension 3. When pay-offs are not quasi-linear

First, define the bid vector $\mathbf{y}$ as the 2^L dimensional vector of combinatorial bids, so that row a contains the payment from the associated outcome. This can be written as $\mathbf{y} = \mathbf{wb}$ where $\mathbf{w}$ is the $2^L \times L$ matrix with entry $(a, l) = 1$ if permutation a involves winning lot l, and zero otherwise. As in appendix D.4, define A as the bidders budget. As in the previous appendices I focus on the identification of this function $K = J + \beta V$ under assumption 7. The expected pay-off is then given by

$$\Pi(\mathbf{y}, \mathbf{b}, \mathbf{m}_t) = P(\mathbf{b})^T \mathbf{K}(\mathbf{m}_t, A - \mathbf{y})$$

Necessary first order conditions are then given by:

$$0 = \nabla_{\mathbf{b}} P(\mathbf{b})\mathbf{K}(\mathbf{m}_t, A - \mathbf{y}) - \mathbf{w}^T \nabla_{\mathbf{y}} \mathbf{K}(\mathbf{m}_t, A - \mathbf{y}) P(\mathbf{b})$$

Assumption 10.

1. $\mathbf{K}(\mathbf{m}, A - \mathbf{y})$ and $\nabla_A \mathbf{K}(\mathbf{m}, A - \mathbf{y})$ are both continuous and continuously differentiable functions in $\mathbf{m}$ for all $\mathbf{m}$ and A.

2. For any $A - \mathbf{y}$ and any pair $\mathbf{m}$ and $\mathbf{m}'$ there exists a set $\mathbb{U} \subset \{1, 2, \dots, 2^L\}$ with $|\mathbb{U}| = m$ that defines the vector value function $\mathbf{F}^{\mathbb{U}}$ where $F_n^{\mathbb{U}}(\mathbf{m}, A - \mathbf{y}) = K_{U_n}(\mathbf{m}, A - y_{U_n})$ such that

$$(\mathbf{m} - \mathbf{m}')^T (\mathbf{F}^{\mathbb{U}}(\mathbf{m}, A - \mathbf{y}) - \mathbf{F}^{\mathbb{U}}(\mathbf{m}', A - \mathbf{y})) > 0$$

Just like assumption 8, assumption 10 is essentially a multidimensional monotonicity condi-

[69] Could do with either clearing this up, or reducing and just focusing on the identification of K. In this case, just mention at the start of the section that I focus on this assumption, plus the identification of K

tion ensuring that both $\mathbf{K}(.)$ and $\nabla_A \mathbf{K}(.)$ are strictly monotonic, ensuring that both functions are bijections and so invertible. This ensures that Jacobians for both inverse have full rank.

Furthermore, just as in the proof of Proposition 13, the set of $\mathbf{K}$ and $\nabla_A \mathbf{K}$ satisfying the first order conditions is convex, and so path connected. Meanwhile, as assumption 10 ensures that the inverse functions are both continuous, the set of $\mathbf{m}$s satisfying the first order conditions must also be path connected. Therefore the set of $\mathbf{m}$s satisfying the first order conditions must contain only a single element.

Importantly, one can immediately see that additional normalisations will be required in this case. That is, the scale (as well as location) of $\mathbf{K}(.)$ must be normalised. In the quasi-linear case, this was done by assuming that the marginal utility of wealth was not only constant, but equal to one. Therefore, such a normalisation will generally be imposed in the function $\mathbf{K}(.)$.

D.6 Exogenously Broken Ties

In this appendix I extend the previous results to the case in which ties are broken exogenously. Because it is still assumed that, in equilibrium, $G_{il}(b|\mathbf{s})$ is strictly increasing in b, ties happen with probability zero for all $b > R$. However, ties at the reservation price occur with non-zero probability, as there is non-negligible probability that any given bidder bids the reservation price. Therefore this extension only has any bite in the presence of binding reservation prices. In keeping with the literature I assume that ties are broken with a coin flip, or some other randomisation device. I focus on identification of this object $K = J + \beta V$, recognising how the results of appendix D.2 demonstrate separate identification of J and V conditional on the identification of K (and β).[70]

D.6.1 Beliefs

Elements of both Γ and P change when we allow ties. The change concerns the probability that i wins any given lot at the reservation price. If player i bids the reservation price for lot l, and the highest opposing bid is also the reservation price, then the probability player i wins the lot is given by $\frac{1}{\sum_j \mathbb{I}[b_{jtl}=R]}$. As before, we write $g_{il}(R;\sigma_i)$ as the equilibrium probability that $d_{itl} = 1$ and $b_{itl} = R$. Unlike before, denote $G_{il}(R^-;\sigma_i)$ as the probability that $b_{itl} < R$, or equivalently, the

[70]Note that the results of appendix D.2.5 continue to apply, albeit with a greater region of partial identification caused by the relatively 'larger' breakdown of point identification (ϵ above the reservation price).

equilibrium probability that $d_{itl} = 0$. Γ can now be written as:

$$\Gamma_{il}(b_{ilt}, d_{ilt}; \sigma_{-i}) = \begin{cases} d_{ilt} \prod_{j \neq i} G_{jl}(b_{ilt}; \sigma_j) & \text{if} \quad b_{ilt} > R \\ d_{ilt} \sum_{\mathbb{C} \subset \mathbb{N}/\{i\}} \frac{\prod_{j \in \mathbb{C}} g_{jl}(R;\sigma_j) \prod_{j \notin \mathbb{C}} G_{jl}(R^-;\sigma_j)}{|\mathbb{C}|+1} & \text{if} \quad b_{ilt} = R \\ 0 & \text{if} \quad d_{ilt} = 0 \end{cases}$$

Likewise, Q can now be written as:

$$Q_{ia}(\mathbf{b}, \mathbf{d}; \sigma_{-\mathbf{i}}) = \prod_{l=1}^{L} [\Gamma_{il}(b_{ilt}, d_{ilt}; \sigma_{-i})]^{\mathbb{I}[w_l^a = i]} [Prob(w_l^a \text{ wins } l | b_{itl, d_{itl}; \sigma_{-i}})]^{\mathbb{I}[w_l^a \neq i]}$$

Where

$$Prob(w_l^a \text{ wins } l | b_{itl, d_{itl}; \sigma_{-i}}) = \begin{cases} \int_{b_{itl}}^{\bar{b}} \prod_{k \neq i, w_l^a} G_{kl}(b; \sigma_k) dG_{w_l^a l}(b; \sigma_{w_l^a}) \\ \qquad \text{if} \quad b_{ilt} > R \\ \int_{R}^{\bar{b}} \prod_{k \neq i, w_l^a} G_{kl}(b; \sigma_k) dG_{w_l^a l}(b; \sigma_{w_l^a}) \\ \qquad - \prod_{k \neq i, w_l^a} G_{kl}(R; \sigma_k) g_{w_l^a l}(R; \sigma_{w_l^a}) \\ \qquad + g_{w_l^a l}(R; \sigma_{w_l^a}) \sum_{\mathbb{C} \subset \mathbb{N}/\{-i, w_l^a\}} \frac{\prod_{k \in \mathbb{C}} g_{kl}(R;\sigma_j) \prod_{k \notin \mathbb{C}} G_{kl}(R^-;\sigma_j)}{|\mathbb{C}|+2} \\ \qquad \text{if} \quad b_{ilt} = R \\ \int_{R}^{\bar{b}} \prod_{k \neq i, w_l^a} G_{kl}(b; \sigma_k) dG_{w_l^a l}(b; \sigma_{w_l^a}) \\ \qquad - \prod_{k \neq i, w_l^a} G_{kl}(R; \sigma_k) g_{w_l^a l}(R; \sigma_{w_l^a}) \\ \qquad + g_{w_l^a l}(R; \sigma_{w_l^a}) \sum_{\mathbb{C} \subset \mathbb{N}/\{-i, w_l^a\}} \frac{\prod_{k \in \mathbb{C}} g_{kl}(R;\sigma_j) \prod_{k \notin \mathbb{C}} G_{kl}(R^-;\sigma_j)}{|\mathbb{C}|+1} \\ \qquad \text{if} \quad d_{ilt} = 0 \end{cases}$$

In $Prob(w_l^a \text{ wins } l | b_{itl, d_{itl}; \sigma_{-i}})$ for $b_{itl} = R$ and $d_{itl} = 0$, we must take an integral between $\bar{b}$ and R. However, the integrand at $b = R$ does not give the probability that individual w_l^a wins at the reservation price, since it does not take into account the ties. Therefore, we subtract the incorrect component of the integral ($\prod_{k \neq i, w_l^a} G_{kl}(R; \sigma_k) g_{w_l^a l}(R; \sigma_{w_l^a})$) and instead add on the true probability they win at the reservation price (times the probability they enter and bid the reservation price). Importantly, both Γ and Q are now no longer differentiable at the reservation price.

D.6.2 Identification of F given K

When ties are broken exogenously a bidder who otherwise bid the reservation price may prefer to instead bid just above the reservation price, ensuring that there is zero probability of a tie. Consequently, whereas before we observed bunching at the reservation price, we now expect bunching both at the reservation price, and bunching 'just above' the reservation price. Similarly, we must also take into account the non-differentiability at the reservation price.

Interior Solutions

As before, the bidder's optimisation problem can still be considered using the lagrangian:

$$L(\mathbf{b}, \mathbf{d}^*, \boldsymbol{\upsilon}, \boldsymbol{\lambda}|\mathbf{s}) = \Gamma(\mathbf{b}, \mathbf{d}^*|\mathbf{s})^T(\boldsymbol{\upsilon} - \mathbf{b}) + Q(\mathbf{b}, \mathbf{d}^*|\mathbf{s})^T K + \boldsymbol{\lambda}^T(\mathbf{b} - R) \tag{50}$$

At an interior optimum, with $b_l^* > R$, their first order necessary conditions (with respect to bids) yields:

$$\nabla_{b_l}\Gamma_l(b_l^*, d_l^*|\mathbf{s})(\upsilon_l - b_l^*) - \Gamma_l(b_l^*, d_l^*|\mathbf{s}) + \nabla_{b_l}Q(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})^T K + \lambda_l = 0 \tag{51}$$

This equation holds for all l such that $b_l^* > R$. Importantly, for l in this region $\lambda_l = 0$. As before we rearrange this equation for the inverse bidding system:

$$\xi_l(\mathbf{b}^*, \mathbf{d}^*|K; \mathbf{s}) = b_l^* + \frac{1}{\nabla_{b_l}\Gamma_l(b_l^*, d_l^*|\mathbf{s})}[\Gamma_l(b_l^*, d_l^*|\mathbf{s}) - \nabla_{b_l}Q(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})K] \tag{52}$$

As before at the true K we have $\xi_l(\mathbf{b}^*, \mathbf{d}^*|K; \mathbf{s}) = \upsilon_l$. Therefore, even with the possibility of ties we are able to invert the bidding function as we did before.

Bidding at the reservation price versus just above it

Next, consider what can be inferred from a decision to bid the reservation price, rather than just ε above the reservation price. Suppose $b_l^* = R$. As before, define the action $(\mathbf{b}^{l+}, \mathbf{d}^{l+})$ such that $(b_k^{l+}, d_k^{l+}) = (b_k^*, d_k^*)$ for $k \neq l$, and $(b_k^{l+}, d_k^{l+}) = (R + \varepsilon, 1)$. That is, we consider this small deviation. The optimality condition is given as:

$$\Gamma(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})^T(\boldsymbol{\upsilon} - \mathbf{b}^*) + Q(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})^T K \geq \Gamma(\mathbf{b}^{l+}, \mathbf{d}^{l+}|\mathbf{s})^T(\boldsymbol{\upsilon} - \mathbf{b}^{l+}) + Q(\mathbf{b}^{l+}, \mathbf{d}^{l+}|\mathbf{s})^T K \tag{53}$$

Which rearranges for:

$$\Gamma_l(R, 1|\mathbf{s})(\upsilon_l - R) + Q(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})^T K \geq \Gamma_l(R + \varepsilon, 1|\mathbf{s})(\upsilon_l - R - \varepsilon) + Q(\mathbf{b}^{l+}, \mathbf{d}^{l+}|\mathbf{s})^T K$$

or, when we recognise how Γ_l changes from the reservation price (positive probability of ties) to just above the reservation price:

$$\sum_{\mathbb{C}\subset\mathbb{N}_{-i}} \frac{\prod_{j\in\mathbb{C}} g_{jl}(R;\sigma_j)\prod_{j\notin\mathbb{C}} G_{jl}(R^-;\sigma_j)}{|\mathbb{C}|+1}(v_l - R) + Q(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})^T K$$

$$\geq$$

$$\prod_{j\neq i} G_j(R+\varepsilon|\mathbf{s})(v_l - R - \varepsilon) + Q(\mathbf{b}^{l+}, \mathbf{d}^{l+}|\mathbf{s})^T K \quad (54)$$

And so we can rearranges this to find an upper bound on v_l:

$$v_l \leq R + \frac{\varepsilon\prod_{j\neq i} G_j(R+\varepsilon|\mathbf{s}) + [Q(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s}) - Q(\mathbf{b}^{l+}, \mathbf{d}^{l+}|\mathbf{s})]^T K}{\prod_{j\neq i} G_j(R+\varepsilon|\mathbf{s}) - \sum_{\mathbb{C}\subset\mathbb{N}_{-i}} \frac{\prod_{j\in\mathbb{C}} g_{jl}(R;\sigma_j)\prod_{j\notin\mathbb{C}} G_{jl}(R^-;\sigma_j)}{|\mathbb{C}|+1}}$$

Not bidding versus bidding the reservation price

They optimality condition for entering auction l and bidding the reservation price, rather than not entering, is given as:

$$\Gamma(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})^T(\boldsymbol{v} - \mathbf{b}^*) + Q(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})^T K \geq \Gamma(\mathbf{b}^{l-}, \mathbf{d}^{l-}|\mathbf{s})^T(\boldsymbol{v} - \mathbf{b}^{l-}) + Q(\mathbf{b}^{l-}, \mathbf{d}^{l-}|\mathbf{s})^T K \quad (55)$$

Where the action $(\mathbf{b}^{l+}, \mathbf{d}^{l+})$ is such that $(b_k^{l-}, d_k^{l-}) = (b_k^*, d_k^*)$ for $k \neq l$, and $(b_l^{l-}, d_l^{l-}) = (\emptyset, 0)$. This equation rearranges, just as in the main text, to give the lower bound on v_l:

$$v_l \geq R - \frac{1}{\Gamma_l(R, 1|\mathbf{s})}[Q(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s}) - Q(\mathbf{b}^{l-}, \mathbf{d}^{l-}|\mathbf{s})]^T K \quad (56)$$

Therefore, we are able to bound v_l between these two cut-offs: $v_l \in [A_1(\mathbf{b}^*, \mathbf{b}^{l-}, \mathbf{s}), A_2(\mathbf{b}^*, \mathbf{b}^{l+}, \mathbf{s})]$. This means that even when ties happen with positive probability, and so expected utility becomes non-continuously differentiable at the reservation price, the econometrician is still able to partially identify the distribution function F.

D.6.3 Identification of K

The identification argument presented in appendix D.2.3 holds even when ties occur with positive probability. This is because K could only be point identified for states in which the bidder bids strictly above the reservation price with positive probability. Identification arises by matching the marginal quantiles of the implied inverse bid distribution to the marginal quantiles of the partially identified distribution from when only individual lots are available.

E Monte Carlo simulation

In this Appendix I present various Monte-Carlo simulations evaluating the estimator described in section 4. As discussed in GKS, the difficulty in simulating these types of games is that solving for equilibrium bidding strategies is intractable. Meanwhile, numerically finding equilibrium bidding strategies - by iterating over equilibrium beliefs and actions until a fixed point is found - is extremely computationally intensive. This is because, for each hypothesised set of beliefs, we must find the equilibrium continuation value through value function iteration.

For the sake of being able to consider the efficacy of the proposed estimator I focus on the case where bidders are bidding against a parametric set of beliefs. That is, I essentially take the equilibrium as given. For computational simplicity, in the current simulations, I also focus on an equilibrium in which equilibrium beliefs only depend on $\mathbf{s}_{0t}$, and do not depend on $\{\mathbf{s}_{it}\}_{i\in\mathbb{N}}$. This is similar to many applications seen in practice, including GKS, Backus and Lewis (2016), Groeger (2014), Balat (2013).

E.1 Set up

Every period there are two auctions ($L = 2$) and two types of object, denoted x and y. Each lot contains one type of object, and one lot of each type of good comes to auction each period. However, some lots contain two units of the good, rather than just one unit. The set of available lots is denoted (z^x, z^y), so lot 1 consists of z^x units of lot x, and lot 2 consists of z^y units of lot y. Each period the possible set of lots $\mathbb{L}_t \in \{(1,1),(2,1),(1,2)\}$. In this simplified setting the common state is just given by $\mathbb{L}_t$. This transitions stochastically such that $\mathbb{L}_t = \mathbb{L}_{t-1}$ with probability 0.5, and transitions to other menus with uniform probability 0.25.

Bidders' states consist of stocks of the two objects, both of which only come in integer values: $s^x_{it} \in \{\,0,\,1,\,\dots\,,\,9\}$, likewise for good y. At the end of each period, bidders consume one unit of good x with probability 0.8 and one unit of good y with probability 0.6, until their stocks fall to 0 for either good. A bidder's combinatorial flow pay-off is given by:

$$j(s^x, s^y) = a^x_1 s^x + a^y_1 s^y + a^x_2 \log(s^x+1) + a^y_2 \log(s^y+1) + a^{xy}_2 \log(s^x+1)\log(s^y+1)$$

where a^x_1 etc are parameters, which I set to $(a^x_1, a^y_1, a^x_2, a^y_2, a^{xy}_2) = (7.5, 5, 30, 20, 10)$. $(a^x_2, a^y_2, a^{xy}_2) \neq 0$ ensures pay-offs are not additively separable, while $a^{xy}_2 > 0$ ensures the two goods, and so the two lots, are complements. I can check that bidding behaviour varies strongly with $\mathbf{s}_{it}$, ensuring this baseline instrument is strong. Meanwhile, the lot-specific pay-offs are drawn from:

$$\boldsymbol{v}_{it} \sim N \begin{pmatrix} 0 & & 900z_t^x & 100z_t^x z_t^y \\ 0 & , & 100z_t^x z_t^y & 400z_t^y \end{pmatrix}$$

I take as given the equilibrium distribution of the highest rival bids, which is assumed to follow a type 2 extreme value distribution. The mean of this distribution is given by the average (across states) marginal pay-off from winning each lot ($\approx (17.1z^x, 12.5z^y)$). The standard deviation is tuned to the standard deviation (across states and lot-specific pay-offs) of the marginal pay-offs from winning each lot. The shape parameter is set to 0.1.

I perform value function iteration to find the continuation value under this distribution of pay-offs and these equilibrium beliefs. Having found a continuation value, I can then simulate a dataset. Given the set-up the state space consists of 300 unique elements, of which 295 states are non-minimal and 225 are non-maximal.

I simulate two sets of bidding datasets. The first simulates $N \in \{\, 3,\ 30,\ 300,\ 1000\}$ observations of bidding, beginning the period in every non-maximal state. This ensures that the combinatorial pay-offs from ending the period in each state are all identified. The second dataset instead simulates $N \in \{\, 300,\ 3000,\ 30000\}$ periods of the process, creating a dataset closer to the type of dataset used by researchers. However, as a result, fewer state-specific pay-offs will be non-parametrically identified since some states will never be observed. Each dataset is simulated 300 times.

E.2 Results

Results are presented in figure 7. For each dataset I present estimation results from three specifications: 1) a fully non-parametric estimator using baseline instruments, 2) a non-parametric estimator with no instruments (minimising squared residuals), and 3) a semi-parametric estimator which fits a log-quadratic functional form to $k(\mathbf{s}_i, \mathbf{s}_0)$, using baseline instruments. These specifications are used to consider how the use of instruments and functional form assumptions may impact estimation.

To evaluate the estimators I consider the proportional difference $\frac{j(\mathbf{s}_i) - \hat{j}(\mathbf{s}_i)}{j(\mathbf{s}_i)}$. I take both an average across identified states and a weighted average across identified states, weighting states according to their likelihood of being observed; these are presented as the left and right hand sets of results respectively.

One note on interpreting these figures - in the N-simulated periods case (lower set of results), as N increases, more states are sampled, increasing the likelihood that any given pay-off from ending in a particular state is identified. Therefore, for larger N, more pay-offs are non-parametrically identified. This explains why the standard deviation and root Mean Squared Error do not necessarily decrease as N increases. This also explains why the error for the semi-parametric estimator is so high for low N - because of the parametric assumptions, every state pay-off is identified. This estimator essentially tries to extrapolate pay-offs that are not non-parametrically identified.

Somewhat surprisingly, the no-instrument estimator dominates the baseline instrument estimator. This arises because even in this example with large complementarities the degree of bias is small, and the variance is smaller than when using instruments. This suggests that, in the bias-variance trade-off, it may be worth coming down on the side of more bias. Furthermore, in small samples, even the instrumental variable estimator exhibits substantial bias. The semi-parametric estimator performs well in both small and large samples, but does exhibit a non-negligible bias. This suggests that the choice of parametric functional form is very important.[71] For reasonably sized samples the estimators generally perform quite well, exhibiting rMSEs between 5 and 15 percent of the true parameters.

[71]The large errors in the weighted results, for top set of parametric results, is due to naturally large weight on maximal states (a quarter of states are maximal), which skews the results away for non-maximal states. This further emphasises the importance of the functional form assumptions.

Figure 7: MCMC evidence, case 1.

			Proportional			Proportional, weighted		
Estimator	Instruments	N	Mean bias	Std Deviation	rMSE	Mean bias	Std Deviation	rMSE
			N observations per state					
NP	$\mathbf{s}_t$	3	0.00359	0.211	0.211	0.0162	0.502	0.502
		30	-0.00323	0.145	0.145	0.0055	0.226	0.227
		300	-0.00444	0.123	0.123	0.00306	0.169	0.169
		1000	-0.00347	0.109	0.109	0.00428	0.164	0.164
NP	none	3	0.00543	0.164	0.164	0.0175	0.457	0.457
		30	-0.00273	0.0858	0.0859	0.00435	0.208	0.208
		300	-0.00379	0.0735	0.0736	0.00252	0.163	0.163
		1000	-0.00341	0.0727	0.0728	0.00328	0.161	0.161
SP	$\mathbf{s}_t$	3	-0.0402	0.105	0.112	-0.0886	0.369	0.38
		30	-0.044	0.0921	0.102	-0.0949	0.338	0.351
		300	-0.0433	0.0899	0.0998	-0.0936	0.331	0.344
		1000	-0.0432	0.0898	0.0996	-0.0934	0.331	0.344
			N simulated periods					
NP	$\mathbf{s}_t$	300	0.00312	0.17	0.17	0.0138	0.563	0.563
		3000	0.00302	0.128	0.128	0.00976	0.268	0.269
		30000	0.00588	0.156	0.156	0.00947	0.206	0.206
NP	none	300	8.93e-05	0.141	0.141	0.00674	0.402	0.402
		3000	-0.000794	0.0956	0.0956	0.00324	0.179	0.179
		30000	-0.00405	0.123	0.124	-0.000675	0.163	0.163
SP	$\mathbf{s}_t$	300	-0.195	0.462	0.501	-0.0367	0.231	0.234
		3000	-0.054	0.174	0.182	-0.0492	0.14	0.149
		30000	-0.0623	0.0547	0.0829	-0.0568	0.133	0.145