Identification and Estimation of a Dynamic Multi-Object Auction Model

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Abstract

Auctions rarely take place in isolation. Often, many heterogeneous lots are auctioned simultaneously, and auctions are repeated as new lots become available. In this paper I develop an empirical model of bidding in repeated rounds of simultaneous first-price auctions. Incorrectly modelling bidders as myopic or as having additive preferences over lots can lead to inaccurate counterfactuals and welfare conclusions. I prove non-parametric identification of primitives in this model, and introduce a computationally feasible procedure to estimate this type of game. I then apply my model to data on Michigan Department of Transportation highway procurement auctions. I investigate the extent of cost-synergies across lots and use counterfactual simulations to compare equilibrium efficiency when contracts are auctioned sequentially rather than simultaneously.

1 Introduction

First-price Auctions, which are regularly used to allocate government procurement contracts, rarely take place in isolation. Multiple lots (contracts) are often auctioned simultaneously, and auctions are repeated whenever new contracts become available. In real world environments bidders' values may be non-additive across different lots. For example, bidders may face capacity constraints, facing higher costs the larger their current backlog. Or, they may benefit from economies of scale, facing lower costs when working on many of the same type of contract at once. The structure of these non-additive values is highly relevant for auction design — should similar contracts be auctioned simultaneously, or spaced out over time? When capacity constraints are the dominant factor, auctioning a large number of contracts simultaneously may create inefficiencies by depressing competition. However, if firms are able to exploit economies of scale it may be worth auctioning similar contracts simultaneously, or even bundling the lots together.

In this paper I develop an empirical model of forward looking bidding in repeated rounds of simultaneous first-price auctions, and study identification and estimation in this framework. I then apply my model to data on Michigan Department of Transportation (MDOT)'s procurement auctions and investigate the empirical and policy relevance of these complementarities.

Previous research has either studied forward looking bidders and assumed auctions are single-object, or studied auctions of multiple objects and assumed bidders are myopic. For example, both Jofre-Bonet and Pesendorfer (2003) and Gentry et al. (2023) study synergies in bidding behaviour in repeated simultaneous first-price auctions for highway maintenance contracts.[1] Jofre-Bonet and Pesendorfer (2003) estimate a dynamic single object model, assuming that payoffs are additive in lots auctioned simultaneously, and find significant negative effects of capacity constraints on bids. Gentry et al. (2023) study simultaneous first-price auctions, assuming myopic bidding, and find similar capacity constraint effects. However, they also find evidence of positive synergies among similar contracts that allow firms to exploit economies of scale. The implication is that neither paper accurately models the non-additive

[1] Other examples from the literature on combinatorial and heterogenous multi-object auctions include Cantillon and Pesendorfer (2007) on London bus routes and Fox and Bajari (2013) on FCC spectrum licenses. Other examples from the dynamic single object literature include Kong (2021) on oil and gas leases and Backus and Lewis (2016) on online marketplaces.

values and the effect on bidding. To the best of the author's knowledge this paper is the first to unify the dynamic and multi-object approaches to empirical auctions.

I develop a structural empirical model of forward looking bidding in repeated simultaneous first-price auctions, where lots are heterogeneous and payoffs are non-additive across lots. The model is fundamentally the union of the models presented in Jofre-Bonet and Pesendorfer (2003) and Gentry et al. (2023), henceforth referred to as JP and GKS respectively. Bidder pay-offs are represented as the sum of privately known and potentially correlated *lot specific* values, a *combination specific* flow pay-off, and a *combination specific* continuation value. Following GKS, the combination specific flow payoff is treated as a deterministic function of state variables. This is a natural framework that reflects known capacity constraints or economies of scale. The model primitives consist of the distribution of lot specific values and the combination specific flow payoff function.[2] The central difficulty for both identification and estimation is that there is not a one-to-one relationship between bids and values. Therefore we cannot invert equilibrium bidding functions to point identify values, as in Guerre et al. (2000). We also cannot write the continuation value as a function of the equilibrium distribution of bids only, as in Jofre-Bonet and Pesendorfer (2003).

Building on this framework I make three key contributions to the empirical auction literature. First, I show that variation in state variables, such as backlogs or contract characteristics, allow us to non-parametrically identify bidders' combinatorial flow payoffs.[3] Intuitively, identification arises because variation in the state causes variation in bidders' combination values, which in turn causes variation in their bidding behaviour. If lots are substitutes we expect to observe more aggressive bidding when backlogs are low. Extending the approach presented in GKS to the dynamic setting I translate the inverse bidding system, conditional on a given state, into a system of linear equations in the unknown combinatorial flow payoffs. Key to the identification argument is that we combine these systems of equations across state variables,

[2]Like GKS this assumption allows me to separately identify complementarities and affiliations, the central problem studied by Kong (2021). Affiliation across lots comes through correlation in the lot specific pay-offs, while the synergies remain deterministic. However, like both papers I assume the lot-specific pay-offs are independent across players.

[3]My identification results do not collapse down to GKS even in a static setting. GKS's identification argument was based around exploiting exclusion restrictions — that one bidder's backlog does not directly enter another's combination value, and only effects their bidding behaviour indirectly through their equilibrium beliefs. However these exclusion restrictions are violated in a dynamic setting: Every state variable enters each bidder's continuation value, thereby directly impacting bidding behaviour. Instead, I show that variation in non-excluded variables is sufficient for identification.

essentially stitching together observations of bidding behaviour from different states. I prove that, under mild conditions, this system has a unique solution.

Second, I outline a computationally feasible three step procedure for estimating the model, generalising JP's estimation procedure. While the continuation value cannot be written as a function of the equilibrium distribution bids *only*, I show that it can be written as a function of the bid distribution *and* a term that corrects for the complementarities between lots. This correction term is a function of the sum of the combinatorial flow payoff and the discounted continuation value. The novelty of the estimation procedure then concerns how we estimate this correction term. I refer to this term as the 'pseudo-static' payoff, as it is essentially the object we would estimate if we incorrectly estimated a misspecified static model. This suggests an extremely simple estimation procedure. In the first step one estimates bidders' equilibrium beliefs, or the equilibrium distribution of bids. In the second step we estimate the pseudo-static payoff, that is the sum of the flow payoff and the continuation value, by essentially estimating the multi-object auction model *almost as if* it were a static model. In the final step we evaluate the continuation value following JP's procedure, incorporating the estimated correction term, before separating out the combinatorial flow payoff from the estimated pseudo-static payoff. This procedure is little more computationally costly than estimating a static multi-object model. In Appendix E I present the results of a simulation study examining the performance of this estimator.

Finally, I apply this framework to data from Michigan Department of Transport (MDOT)'s procurement auctions. In this setting around 45 contracts for highway maintenance and construction projects are auctioned simultaneously in each round, and rounds are repeated roughly every fortnight. I focus on contracts that require use of either hot-mix asphalt, concrete, or both. I use firms' backlogs of asphalt and concrete projects as their state variables, and consider how backlogs impact their cost functions, driving complementarities between lots. For asphalt specialist firms in particular I find evidence of increasing returns to specialising in asphalt contracts: Every one standard deviation increase their asphalt backlog increases the cost of completing a concrete contract by around 10%, and decreases the cost of an asphalt contract by roughly the same amount. I use counterfactual simulations to consider how the procurement cost to MDOT and the total cost to firms differs when contracts are auctioned sequentially instead of simultaneously.

The structure of this paper is as follows: In section 1.1 I discuss my contribution to the literature. Section 2 introduces the auction game that is the focus of this paper. Section 3 introduces the identification framework and proves that model primitives are point identified. Section 4 outlines the proposed three step estimation procedure, and Section 5 applies this procedure to data from MDOT procurement auctions. Several additional results are presented in the Appendices. Appendices A - C present technical proofs. Appendix D presents several extensions to the identification and estimation framework, including extensions for second-price auctions, reservation prices, endogenous entry, and stochastic combination values. Appendix E presents the results of a simulation experiment evaluating the proposed estimation procedure, and F presents additional analysis related to the empirical application.

1.1 Related Literature

My key contribution is to bring together the literatures on the identification and estimation of both dynamic auction models and multi-object auction models.[4] Jofre-Bonet and Pesendorfer (2003) was the first to estimate a dynamic auction game, analysing sequential highway procurement auctions and find backlog effects to be determinants of future bidding behaviour. Several papers have built on this framework, including Jeziorski and Krasnokutskaya (2016) on dynamic auctions with subcontracting, Groeger (2014) on participation in repeated auctions, Balat (2013) who introduce unobserved heterogeneity in lot quality, and Raisingh (2021) who study the effect of pre-announcements of auctions in the MDOT data. These papers generally study settings in which multiple auctions are held simultaneously, and assume payoffs are additively separable across auctions within a period. This assumption is unpalatable given they find evidence of non-additivities across auctions held in different periods.

Cantillon and Pesendorfer (2007) were the first to estimate a model of simultaneous auctions. They use combination bids to identify complementarities in simultaneous first-price auctions, studying procurement auctions for London bus routes. Kim et al. (2014) use this framework to study the allocation of contracts for Chilean school meals. Fox and Bajari (2013) study an auction environment *without* combination

[4] This work is also tangentially related to the literature on empirical Multi-unit auctions, which focuses on divisible homogenous units (see e.g. Hortaçsu and McAdams (2018)). The estimation procedure presented in section 4 easily extends to dynamic multi-unit auctions. Another related literature analyses forward looking behaviour in second-price auctions, including Backus and Lewis (2016), Bodoh-Creed et al. (2021), and Marra (2021). In Appendix D.1 I extend my identification and estimation results to the multi-object second-price setting.

bidding. However their equilibrium stability condition, which is used to identify the complementarities, cannot be applied in general. Gentry et al. (2023) also focus on simultaneous first-price auctions without combination bidding. They prove the model is identified using variation in 'excluded' variables: Variables that are excluded from the a bidder's combinatorial payoff, such as characteristics of their rivals, and only indirectly affect bidding behaviour through bidders' equilibrium beliefs. However, exclusion restrictions fail in a dynamic environment. Bidders' forward looking behaviour ensures every state variable directly effects their continuation value, and hence bidding behaviour. These exclusion restrictions are not necessary for identification.

2 The general model

2.1 Setup

Rules: Each period t, over an infinite horizon, n risk-neutral players i compete in a series of first-price Sealed Bid auctions. Lots are indexed by l, and player i wins lot l in period t if $b_{itl} \geq \max_{j \neq i} \{b_{jtl}\}$. Sealed bids are placed simultaneously, then winners are announced. Winners pay their bids, and every player observes the bids and identities of winners. Define the $L \times 1$ vector $\mathbf{w}_t$ as the outcome at time t, where $w_{tl} = i$ if i won lot l at time t. Ex-ante hypothetical outcomes are denoted by $\mathbf{w}_t^a$.

Reservation Prices and Ties: I assume reservation prices do not bind, that auction entry is exogenous, and that ties occur with probability zero.[5]

Lots and Lot Characteristics: $L < \infty$ lots are auctioned each period. Allowing L to vary across periods does not impact results. Each lot l is characterised by a row-vector of characteristics $\mathbf{x}_{tl}$, writing $\mathbf{X}_t$ for the stacked characteristics of all lots in period t. Characteristics may include the size and location of a particular contract, for example. The set of characteristics, $\mathbb{X}$, is assumed finite. Finally stack the lot characteristics and other common state variables into $\mathbf{s}_{0t} \in \mathbb{S}_0$.

2.1.1 Primitives

Individual States: Player i begins the period in state $\mathbf{s}_{it}$. This may represent a player's existing stock of the good, or backlog of contracts. The set of possible

[5]However these assumptions are relaxed in appendices D.2 and D.3.

individual states, $\mathbb{S}_i$, is assumed to be finite.[6] If the outcome at t is $\mathbf{w}_t^a$ then player i ends the period in state $\mathbf{s}_{it}^a$, referred to as the ex-post state. $\mathbf{s}_{it} = \mathbf{s}_{it}^a$ if and only if the player does not win a single lot. For notational convenience, define the set $\mathbb{S}_i^a(\mathbf{s}_i, \mathbf{s}_0)$ as the set of possible individual ex-post states $\mathbf{s}_i^a$ having started in state $\mathbf{s}_i$, given the available lots, lot characteristics, and other common state variables $\mathbf{s}_0$.

Total States: Stack the individual states $\{\mathbf{s}_{it}\}_{i\in\mathbb{I}}$, and $\mathbf{s}_{0t}$, into the total state variable $\mathbf{s}_t \in \mathbb{S}$, where $|\mathbb{S}| = S$ is finite. In section 3.6 I set out sufficient conditions on the set $\mathbb{S}$ which ensure identification. Similarly, Stack the ex-post states $\{\mathbf{s}_{it}^a\}_{i\in\mathbb{I}}$ and $\mathbf{s}_{0t}$ into the total ex-post state $\mathbf{s}_t^a \in \mathbb{S}$.

Transition Functions: At the beginning of each period, the state $\mathbf{s}_t$ is drawn stochastically from $T_{\mathbf{s}}(.|\mathbf{s}_{t-1}^a)$. Because $|\mathbb{S}|$ is finite, the transition probabilities can be described by transition matrix T, such that $P(\mathbf{s}_{it} = \mathbf{s}_m|\mathbf{s}_{t-1}^a = \mathbf{s}_n) = T_{mn}$.

Actions: Each player plays an L dimensional vector of bids each period, denoted $\mathbf{b}_{it}$. The set of possible bids is convex and compact, so that $b_{itl} \in [\underline{b}, \bar{b}]$.

Lot Specific Values: I focus on an independent private value framework. If i wins lot l at t they receive a lot specific payoff, υ_{itl}. Stacking these values $\boldsymbol{\upsilon}_{it}$, a $L \times 1$ vector, is drawn from cumulative density function $F_i(.|\mathbf{s}_t)$ with support $[\underline{\boldsymbol{\upsilon}}_i, \bar{\boldsymbol{\upsilon}}_i]$.

Combination Value: The combination value is given by $J_i(\mathbf{s}_t)$, a $2^L \times 1$ vector. Each row $J_{ia}(\mathbf{s}_t)$ gives the mean flow pay-off corresponding to a different outcome $\mathbf{w}_t^a$, ending the period in state $\mathbf{s}_{it}^a$. J_i is assumed a deterministic function of $\mathbf{s}$, and assumed to be finite. A player's type is characterised by the tuple $(\boldsymbol{\upsilon}_i, J_i)$. I assume that F_i and J_i are both common knowledge.

2.1.2 The Bidder's Problem

Strategies: A (pure) Markovian strategy σ_i consists of a mapping from a player's type $(\boldsymbol{\upsilon}_i, J_i)$ and the state of the world $\mathbf{s}$ onto a series of bids $\mathbf{b}_{it}$. Ex-ante a player's strategy admits a distribution of bids according to F_i, J_i, and $\mathbf{s}$.

Marginal Win Probabilities: Denote $G_{jl}(.;\sigma_j)$ and $g_{jl}(.;\sigma_j)$ respectively the marginal cdf and pdf of individual j's bid on lot l according to their strategy σ_j. Denote $\Gamma_i(\mathbf{b};\sigma_{-i})$ the $L \times 1$ vector where row l contains the probability that i wins lot l, given their bid and entry decision, taking as given the strategies of other players.

[6] This is predominantly for mathematical convenience, but is likely to hold in practice. Highway maintenance companies likely have a maximum number of contracts they can feasibly hold at any given time, and their backlog of contracts can be arbitrarily discretised into days of work remaining.

Because ties occur with zero probability we can write:

$$\Gamma_{il}(b_{ilt};\sigma_{-i}) = \prod_{j\neq i} G_{jl}(b_{ilt};\sigma_j)$$

Combination Win Probabilities: Denote $P_i(\mathbf{b};\sigma_{-i})$ the $2^L \times 1$ vector where row a contains the probability, conditional on their bid and the strategies of other players, that player i's ex-post state will be $\mathbf{s}_{it}^a$.

Overall Combination Probabilities Denote $Q_i(\mathbf{b};\sigma_{-i})$ the $n^L \times 1$ vector where row a contains the probability, conditional on their bid and the strategies of other players, that the outcome from period t is $\mathbf{w}_t^a$, and so the overall ex-post state is $\mathbf{s}_t^a$. This object is extremely similar to the combination win probabilities presented previously, except this object also takes into account exactly which player $j \neq i$ wins each lot. Importantly, $P_a = \sum_{a' \ s.t.\ \mathbf{s}^{a'}=\mathbf{s}^a} Q_{a'}$. That is, summing $Q_{a'}$ over all the ex-post outcomes for which player i's state is the same ($\mathbf{s}^{a'} = \mathbf{s}^a$) gives P_a.

Discounting: Players have temporally additively separable preferences, and discount future payoffs using known discount factor $\beta \in (0,1)$..

Expected Flow Pay-off: Bidders are risk neutral and payoffs are assumed quasi-linear in payments. Consider player i with a realisation of $\boldsymbol{\upsilon} = \boldsymbol{\upsilon}_{it}$ who places bid $\mathbf{b}$ against players bidding according to strategies σ_{-i}:

$$\Pi(\mathbf{b}|\boldsymbol{\upsilon}_i, \mathbf{s};\sigma_{-i}) = \Gamma_i(\mathbf{b};\sigma_{-i})^T(\boldsymbol{\upsilon}_i - \mathbf{b}) + P_i(\mathbf{b};\sigma_{-i})^T J_i(\mathbf{s})$$

Value Function: Denote $\{n^L\}$ the power set of possible combination wins. The bellman equation is given by:

$$W_i(\boldsymbol{\upsilon}_{it}, \mathbf{s}_t;\sigma_{-i}) = \max_{\mathbf{b}} \left\{ \Pi(\mathbf{b}|\boldsymbol{\upsilon}_{it}, \mathbf{s}_t;\sigma_{-i}) + \beta \sum_{a\in\{n^L\}} Q_{ia}(\mathbf{b};\sigma_{-i}) \int_{\bar{\mathbf{s}}} \int_{\boldsymbol{\upsilon}_{it}} W_i(\boldsymbol{\upsilon}_i, \bar{\mathbf{s}};\sigma_{-i}) dF(\boldsymbol{\upsilon}_i|\bar{\mathbf{s}}) dT(\bar{\mathbf{s}}|\mathbf{s}_t^a) \right\} \tag{1}$$

Continuation Value: It is useful to define the continuation value: $V_{ia}(\mathbf{s}_t;\sigma_{-i}) = \int_{\bar{\mathbf{s}}} \int_{\boldsymbol{\upsilon}_{it}} W_i(\boldsymbol{\upsilon}_i, \bar{\mathbf{s}};\sigma_{-i}) dF(\boldsymbol{\upsilon}_i|\bar{\mathbf{s}}) dT(\bar{\mathbf{s}}|\mathbf{s}_t^a)$. The combination continuation value is given by $V_i(\mathbf{s}_t;\sigma_{-i})$, a $n^L \times 1$ vector. Each element a of this vector contains the continuation value corresponding to a different allocation, ending the period in a different state $\mathbf{s}_t^a$.

2.2 Equilibrium

I now discuss equilibrium, and the assumptions required for existence of an equilibrium. A full and general proof of equilibrium existence is beyond the scope of this paper.[7] In place of a complete proof I present a proof of existence under the conjecture that a pure-strategy equilibrium exists in the static game. The lack of full proof should be considered only a theoretical issue, rather than a practical problem.

Markov Perfect Equilibrium I focus on symmetric markov perfect equilibria consisting of a set of strategies σ^* such that for any $(\boldsymbol{\upsilon}, J, \mathbf{s})$: 1) Each player's strategy σ_i^* is a best response to the strategies of rival bidders $\boldsymbol{\sigma}^*_{-i}$, 2) Players' beliefs are consistent with $\boldsymbol{\sigma}^*$, and 3) All players play the same strategies.

2.2.1 Equilibrium Existence

To prove equilibrium existence, I rely on the following conjecture:

Conjecture 1. Existence and Uniqueness of a continuous Static Equilibrium

There exists a unique symmetric (non co-operative) Pure Strategy Bayesian Nash Equilibrium of the (myopic) stage game, such that for all i and l the expected pay-off is continuous in $\boldsymbol{\upsilon}_i$ and J_i.

This conjecture takes essentially the same form as the assumption that a continuous and unique equilibrium exists in Gentry et al. (2023).

Proposition 1. *Under the assumptions of the game, and under Conjecture 1, a Symmetric Markov Perfect Equilibrium exists.*

Proof is relegated to Appendix C, as existence is not the main focus of this paper. The proof consists of showing that the equilibrium pay-off in the stage game is consistent with the continuation value, employing Kakutani's fixed point theorem.

[7] To my knowledge, no complete proof of equilibrium existence exists even for the static game without entry. This paper joins the papers studying sufficiently complex auction games in which neither existence, nor uniqueness of equilibrium can be guaranteed. For example, Gentry et al. (2023) on simultaneous first-price auctions, Fox and Bajari (2013) on simultaneous ascending auctions, or Jofre-Bonet and Pesendorfer (2003) on dynamic single-object first-price auctions. If the bid space were discrete, then static equilibrium existence follows from Milgrom and Weber (1985).

3 Identification

I now demonstrate that the distribution of lot specific values F, and the combination value J are non-parametrically point identified. The intuition behind this argument is that variation in $\mathbf{s}$ causes variation in payoffs which, in turn, cause variation in bidding behaviour. I then use the observed bidding behaviour, as well as information about bidders' equilibrium beliefs, to essentially 'back out' the distribution of values.

A model is point identified if, given the implications of equilibrium behaviour, the joint distribution of bidder's pay-offs, $\{F_i, J_i\}_{i \in \mathbb{I}}$, are uniquely determined by the joint distribution of observables (Athey and Haile, 2002). A model is non-parametrically identified if the identified objects are functions (Lewbel, 2019). This is in the sense that we do not assume a functional form, but identify the entire function $J_i(\mathbf{s})$ for every $\mathbf{s} \in \mathbb{S}$ and $F_i(\boldsymbol{v}|\mathbf{s})$ for every pair $(\boldsymbol{v}, \mathbf{s})$.

I begin by introducing the assumptions necessary for identification in subsection 3.1. In subsections 3.2 - 3.3 I use the bidder's optimisation problem to derive the Inverse Bid System. In 3.4 - 3.5 I show how we combine this system across states to form a system of simultaneous linear equation in J, so that identification of J collapses down to a rank condition. In subsection 3.6 I prove that only mild restrictions on the state space are sufficient for this rank condition to hold. In subsection 3.7 I then consider identification under several extensions of the model presented.

3.1 Assumptions necessary for identification

Define the objects $G_i(.|\mathbf{s}), \Gamma_i(.|\mathbf{s}), P_i(.|\mathbf{s}),$ and $Q_i(.|\mathbf{s})$ as the empirical counterparts to the objects presented previously.

Assumption 1. *For each t, the econometrician has a set of observations as follows:*

$$\mathbb{O}_t = \left\{ \mathbf{w}_t, \mathbf{s}_t, \{\mathbf{b}_{it}\}_{i \in \{1,2,\ldots,n\}} \right\}$$

I assume the econometrician observes all bids and entry decisions, not just the winning bid. Under this assumption $G, \Gamma, P, Q,$ and T are all non-parametrically identified. Therefore, in the remainder of this section I treat these objects as known.

Assumption 2. *The data $\{\mathbb{O}_t\}_{t=1\ldots T}$ are generated by strategy profile $\boldsymbol{\sigma}^*$ which is a symmetric Markov perfect equilibrium of the dynamic auction game.*

This assumption requires that the same equilibrium is played throughout the observed period, ensuring that strategies can be written as a function of the state. As a result, the continuation value can be written as a function of the state. We can then express the continuation value in vector form as $\mathbf{V}$, with elements corresponding to the expectation from ending a period in any particular ex-post state. It is then useful to define the relationship between the n^L vector $V(\mathbf{s})$ defined previously and $\mathbf{V}$:

$$\begin{pmatrix} V(\mathbf{s}_1) \\ \vdots \\ V(\mathbf{s}_S) \end{pmatrix} = A\mathbf{V}$$

Where A is an $Sn^L \times S$ dimensional matrix. I also use the notation $V(\mathbf{s}) = A_{\mathbf{s}}\mathbf{V}$ for the $n^L \times S$ submatrix $A_{\mathbf{s}}$. This contains a 1 in entry am if the potential outcome $\mathbf{w}^a$ yields ex-post state $\mathbf{s}^a = \mathbf{s}_m$, selecting the relevant continuation values corresponding to the possible ex-post states.

Assumption 3. *For all* $\mathbf{s}$*,* i*, and* l $G_i(\mathbf{b}_i|\mathbf{s};\sigma^*)$ *is absolutely continuous in* b_{il}*.*

This assumption ensures that the marginal, combination, and over-all combination win probabilities are continuous and differentiable in $\mathbf{b}$, enabling us to take first order conditions. As shown in GKS, when this assumption does not hold we lose point-identification, though the model primitives generally remain partially identified.

Assumption 4.

i) *Element* a *of* $J_i(\mathbf{s})$ *can be written as:* $J_{ia}(\mathbf{s}) = j_i(\mathbf{s}_i^a)$ *for some* $j_i : \mathbb{S}_i \to \mathbb{R}$.

ii) $E[\boldsymbol{\upsilon}|\mathbf{s}] = 0$.

Part i) of this assumption is relatively weak. I require the immediate combinatorial pay-off from ending the period in state $\mathbf{s}^a$ depends only on this final state.[8] Part ii) of the assumption just ensures that the mean of $\boldsymbol{\upsilon}$ is absorbed into J.

By stacking J over $\mathbf{s}$ and j over $\mathbf{s}_i$ I define a mapping between $J(\mathbf{s})$ and $j(\mathbf{s}_i)$:

$$\underbrace{\mathbf{J}}_{S2^L\times 1} = \begin{pmatrix} J(\mathbf{s}_1) \\ \vdots \\ J(\mathbf{s}_S) \end{pmatrix} \qquad \underbrace{\mathbf{j}}_{S_i\times 1} = \begin{pmatrix} j(\mathbf{s}_{i1}) \\ \vdots \\ j(\mathbf{s}_{iS}) \end{pmatrix} \qquad \mathbf{J} = B\mathbf{j}$$

[8]This differs from GKS' approach in which J_{ia} is able to depend on both $\mathbf{s}^a$, $\mathbf{s}$, and potentially even other $\mathbf{s}^{a'}$. However, in their empirical example they do impose this restriction.

Where B is an $S2^L \times S_i$ transformation matrix with rank S_i. I also write $J(\mathbf{s}) = B_{\mathbf{s}}\mathbf{j}$ using just the $2^L \times S_i$ sub-matrix $B_{\mathbf{s}}$. This matrix selects elements of $\mathbf{j}$ according to the possible ex-post states for player i, given they started the period in state $\mathbf{s}$. We can define the relationship $P(\mathbf{b}|\mathbf{s})^T B_{\mathbf{s}} = Q(\mathbf{b}|\mathbf{s})^T A_{\mathbf{s}} C$ for the $S \times S_i$ matrix C. entry mn of C is equal to 1 if $\mathbf{s}_i^m = \mathbf{s}_i^n$, and zero otherwise. Therefore, each row of C contains a single non-zero entry, while column n contains a 1 in rows for which $\mathbf{s}_i = \mathbf{s}_i^n$. This relationship holds because C collapses Q over states with the same $\mathbf{s}_i$.

Finally, we must normalise $j(\mathbf{s}_{i1})$ because only marginal payoffs are identified. Based on these assumptions, I will prove the following proposition:[9]

Proposition 2. *Under assumptions 1 - 4, the model primitives F and $\mathbf{j}$ are non-parametrically identified up to β and $j(\mathbf{s}_{i1})$.*

3.2 First Order Conditions

The agent's problem is to maximise their expected discounted pay-off, and so in each period the agent maximises the following object, with respect to $\mathbf{b}$:

$$\begin{aligned}\tilde{\Pi}(\mathbf{b}|\boldsymbol{v};\mathbf{s}) =& \Gamma(\mathbf{b}|\mathbf{s})^T(\boldsymbol{v}-\mathbf{b}) + P(\mathbf{b}|\mathbf{s})^T J(\mathbf{s}) + \beta Q(\mathbf{b}|\mathbf{s})^T V(\mathbf{s}) \\ =& \Gamma(\mathbf{b}|\mathbf{s})^T(\boldsymbol{v}-\mathbf{b}) + P(\mathbf{b}|\mathbf{s})^T B_{\mathbf{s}}\mathbf{j} + \beta Q(\mathbf{b}|\mathbf{s})^T A_{\mathbf{s}}\mathbf{V} \end{aligned} \tag{2}$$

Assumption 3 ensures that $P(\mathbf{b}|\mathbf{s})$, $Q(\mathbf{b}|\mathbf{s})$, and $\Gamma(\mathbf{b}|\mathbf{s})$ are continuously differentiable in $\mathbf{b}$. Necessary First Order Conditions of optimal bidding are then given as:

$$\underbrace{\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})}_{L\times L}\underbrace{(\boldsymbol{v}-\mathbf{b}^*)}_{L\times 1} = \underbrace{\Gamma(\mathbf{b}^*|\mathbf{s})}_{L\times 1} - \underbrace{\nabla_{\mathbf{b}}P(\mathbf{b}^*|\mathbf{s})}_{L\times 2^L}\underbrace{B_{\mathbf{s}}\mathbf{j}}_{2^L\times 1} - \beta\underbrace{\nabla_{\mathbf{b}}Q(\mathbf{b}^*|\mathbf{s})}_{L\times n^L}\underbrace{A_{\mathbf{s}}\mathbf{V}}_{n^L\times 1} \tag{3}$$

As above, under the assumption of zero probability ties (or exogenous tie-breaking), $\Gamma_{il}(\mathbf{b}|\mathbf{s}) = \prod_{j\neq i} G_{jl}(b_{il}|\mathbf{s})$. Therefore $\nabla\Gamma$ must be a diagonal matrix with entry ll equal to $\sum_{j\neq i} g_{jl}(b_{il}|\mathbf{s})\prod_{k\neq j,i} G_{kl}(b_{il}|\mathbf{s})$, and so $\nabla\Gamma$ must be invertible for most $\mathbf{b}$.

[9] This result differs from GKS' identification result even when bidders are myopic ($\beta = 0$), differing in the source of variation used to identify J. They prove identification using excluded variables which cause 'exogenous' variation in Γ and P. I use *included* variation in the state variable, which creates variation in Γ and P but also directly enter $J + \beta V$.

3.3 The Inverse Bidding System and Identification of F

F is identified, conditional on J and βV, by inverting the first order conditions to obtain $\boldsymbol{\upsilon}$ as a function of bids, J, and βV. This argument is almost precisely the same as that presented by GKS, which is a simple multi-object extension of Guerre et al. (2000) identification result from inverting the first order conditions. Invert the first order conditions for the inverse bid system:

$$\boldsymbol{\xi}(\mathbf{b}^*|J,\beta V;\mathbf{s}) = \underbrace{\mathbf{b}^*}_{\text{observed}} + \underbrace{\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}[\Gamma(\mathbf{b}^*|\mathbf{s})}_{Identified} - \underbrace{\nabla_{\mathbf{b}}P(\mathbf{b}^*|\mathbf{s})}_{Identified} B_{\mathbf{s}}\mathbf{j} - \underbrace{\nabla_{\mathbf{b}}Q(\mathbf{b}^*|\mathbf{s})}_{Identified} A_{\mathbf{s}}\beta\mathbf{V}] \tag{4}$$

This system of equations is a natural extension of the standard inverse bid function. At the optimum the lot specific value is equal to bids $\mathbf{b}^*$ plus a lot specific markup $\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\Gamma(\mathbf{b}^*|\mathbf{s})$, minus a combination markup $\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\nabla_{\mathbf{b}}P(\mathbf{b}^*|\mathbf{s})B_{\mathbf{s}}\mathbf{j}$, minus the dynamic markup which depends on precisely who won each combination of lots $\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\nabla_{\mathbf{b}}Q(\mathbf{b}^*|\mathbf{s})A_{\mathbf{s}}\beta\mathbf{V}$.

We can evaluate this inverse bid function at the observed bids, which holds for a particular candidate $(J,\beta V)$. If this candidate $(J,\beta V)$ is correct, then $\boldsymbol{\xi}(\mathbf{b}^*|J,\beta V;\mathbf{s}) = \boldsymbol{\upsilon}$. From here it is simple to non-parametrically identify $F(.)$.

3.4 Identification of V

We can write $\mathbf{V}$ as a function of the distribution of bids and $\mathbf{j}$ only:

Proposition 3. *Under assumptions 1 - 4, the expected stage pay-off is given by:*

$$\begin{aligned}\tilde{\Pi}(\mathbf{b}^*|\boldsymbol{\upsilon};\mathbf{s}) =& \Gamma(\mathbf{b}^*|\mathbf{s})^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\Gamma(\mathbf{b}^*|\mathbf{s}) \\ &+ [P(\mathbf{b}^*|\mathbf{s})^T - \Gamma(\mathbf{b}^*|\mathbf{s})^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\nabla_{\mathbf{b}}P(\mathbf{b}^*|\mathbf{s})]B_{\mathbf{s}}\mathbf{j} \\ &+ [Q(\mathbf{b}^*|\mathbf{s})^T - \Gamma(\mathbf{b}^*|\mathbf{s})^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\nabla_{\mathbf{b}}Q(\mathbf{b}^*|\mathbf{s})]A_{\mathbf{s}}\beta\mathbf{V}\end{aligned} \tag{5}$$

Proof of this proposition is given in Appendix A. This relation generalises Proposition 1 in JP. The first term on the right hand side of the equation can be written as $\sum_l \frac{\Pi_{j\neq i}G_{jl}(b_{il})}{\sum_{j\neq i}g_{jl}(b_{il})}$ — the first term in JP's proposition. Unlike in the single unit case there is a correction for the non-additivity.

From Proposition 3, employing the identity $P(\mathbf{b}|\mathbf{s})^TB_{\mathbf{s}} = Q(\mathbf{b}|\mathbf{s})^TA_{\mathbf{s}}C$, and taking an expectation of the observed bids, we can write the ex-ante value function as:

$$\begin{aligned}V^e(\mathbf{s}) &= \Phi(\mathbf{s}) + \Omega(\mathbf{s})[C\mathbf{j}+\beta\mathbf{V}]\\ \text{Where}\quad \Phi(\mathbf{s}) &= E_{\mathbf{b}}[\Gamma(\mathbf{b}^*|\mathbf{s})^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\Gamma(\mathbf{b}^*|\mathbf{s})|\mathbf{s}]\\ \Omega(\mathbf{s}) &= E_{\mathbf{b}}[Q(\mathbf{b}^*|\mathbf{s})^T - \Gamma(\mathbf{b}^*|\mathbf{s})^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\nabla_{\mathbf{b}}Q(\mathbf{b}^*|\mathbf{s})|\mathbf{s}]A_{\mathbf{s}}\end{aligned}$$

Stacking over $\mathbf{s}$ write the continuation value as $\mathbf{V} = T\mathbf{V}^e = T\Phi + T\Omega[C\mathbf{j}+\beta\mathbf{V}]$ Which we invert for: $\mathbf{V} = (I_S - \beta T\Omega)^{-1}[T\Phi + T\Omega C\mathbf{j}]$.[10] This ensures that, conditional on $\mathbf{j}$ being known, the continuation value is point identified.

3.5 Identification of j

Impose the mean zero property of $\boldsymbol{\upsilon}$ for:

$$\begin{aligned}0 =& E_{\boldsymbol{\upsilon}}[\boldsymbol{\upsilon}|\mathbf{s}] = E_{\mathbf{b}^*}[\boldsymbol{\xi}(\mathbf{b}^*;\mathbf{s},(\mathbf{j},\mathbf{V}))|\mathbf{s}]\\ =& E_{\mathbf{b}^*}[\mathbf{b}^* + \nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\Gamma(\mathbf{b}^*|\mathbf{s})|\mathbf{s}] - E_{\mathbf{b}^*}[\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\nabla_{\mathbf{b}}Q(\mathbf{b}^*|\mathbf{s})|\mathbf{s}]A_{\mathbf{s}}[C\mathbf{j}+\beta\mathbf{V}]\\ =& \Upsilon(\mathbf{s}) - \Psi(\mathbf{s})[C\mathbf{j}+\beta\mathbf{V}] \qquad (6)\end{aligned}$$

Stacking over $\mathbf{s}$, then substituting in the expression for $\mathbf{V}$ and simplifying, we get:

$$\begin{aligned}0 =& \Upsilon - \Psi[C\mathbf{j}+\beta\mathbf{V}]\\ =& \Upsilon - \beta\Psi(I_S - \beta T\Omega)^{-1}T\Phi - \Psi(I_S-\beta T\Omega)^{-1}C\mathbf{j} \qquad (7)\end{aligned}$$

This system of LS equations in $S_i - 1$ unknowns overcomes the standard order condition discussed in GKS. There exists a unique solution to this system ($\mathbf{j}$ is point identified) if and only if the $LS \times S_i$ matrix $\Psi(I_S - \beta T\Omega)^{-1}C$ has rank $S_i - 1$.

3.6 Rank of $\Psi(I_S - \beta T\Omega)^{-1}C$

This rank condition requires that observations of bidding behaviour, across all S states, produces sufficient information about $\mathbf{j}$ to uniquely pin down all $S_i - 1$ elements. We gain information about $j(\mathbf{s}_i)$ from how bidding behaviour changes when $\mathbf{s}_i$

[10] Non-singularity of $(I_S - \beta T\Omega)$ follows from the matrix being strictly diagonally dominant (Levy-Desplanques Theorem). Strict diagonal dominance arises because every element of $T\Omega$ is weakly positive, and rows sum to 1. Therefore, off diagonals of $I - \beta T\Omega$ lie in the interval $(-\beta, 0]$, while diagonals are strictly positive, and rows sum to $1-\beta$.

is a possible outcome from the round of auctions. By stacking the moment conditions in equation 7 we stitch together the information about $\mathbf{j}$ across different state observations. In addition to information as $\mathbf{s}_i$ varies, we also use information as $\mathbf{s}_{-i}$ varies, even when this is excluded from the function j, resulting in additional identifying variation. One additional assumptions is sufficient for this rank condition to hold:

Assumption 5. The set $\mathbb{S}_i$ is partially ordered according to the strict partial ordering $\succeq$, such that if $\mathbf{s}_i' \in \mathbb{S}_i^a(\mathbf{s}_i, \mathbf{s}_0)$ then $\mathbf{s}_i' \succeq \mathbf{s}_i$. In addition, the maximal elements of $\mathbb{S}_i$ do not outnumber the non-maximal elements.

The partial ordering assumption is mild, really only imposing the transitivity of partially ordered sets. A requirement for these partial orderings is that winning an auction is monotonic: one cannot gain an object from winning one auction and give it away by winning a different auction. I limit the number of maximal elements because observations of bidding from maximal elements are not informative.[11]

Proposition 4. *Under assumption 1 - 5* $\Psi(I_S - \beta T\Omega)^{-1}C$ *has rank* $S_i - 1$

Proof of this proposition is given in Appendix B. It is omitted from the main text for brevity. This rank condition is not trivial, since Ψ is certainly rank deficient. Likewise, it is not ex-ante obvious whether stacking $\Psi(\mathbf{s})$ across states yields information about every element of $\mathbf{j}$. The bulk of the proof requires establishing the rank of Ψ and finding it's null space. As we stitch together observations of bidding from each state, stacking $\Psi(\mathbf{s})$ across $\mathbf{s}$, the rank increases by at least two each time. I then consider the image of $(I_S - \beta T\Omega)^{-1}C$, proving that the only element in the intersection of this image and the null space of Ψ is the constant vector.[12]

3.7 Extensions

Second-price auctions: In Appendix D.1 I extend these results to simultaneous second-price auctions. Holding constant all but one bid, optimal bidding requires

[11] An element $\mathbf{s}_i$ is defined as maximal if there does not exist an $\mathbf{s}_i' \in \mathbb{S}_i$ such that $\mathbf{s}_i' \succ \mathbf{s}_i$. One interpretation is that these maximal elements are the largest (under $\succeq$) states that are observed as possible ex-post outcomes, but never as ex-ante outcomes. In this way, we want to try to identify j for these states, but do not get to use observations beginning in these states.

[12] This proof only holds for the setting when the state space is finite. However the underlying argument extends to the case with infinite states: Even though the rank of an infinitely large matrix is undefined, it is clear how the logic of combining observations across states yields identification.

bidding the expected marginal value of winning that additional lot, given the other bids. This yields an inverse bid system that is identical to the first-price case, except that it is missing the markup term $\nabla_{\mathbf{b}}\Gamma(\mathbf{b})^{-1}\Gamma(\mathbf{b})$. Proposition 3 extends similarly. From here the identification and estimation arguments extend intuitively.

Binding reservation prices: In Appendix D.2 I consider how the presence of binding reservation prices impact identification. Essentially, they cause censoring in the data so that we immediately lose point identification of both F and j. However, F remains partially identified, using a similar argument as presented in subsection 3.3. We can no longer use moment conditions to identify j, as in subsection 3.5, and instead use quantile conditions. While reservation prices are a mathematical nuisance, they do not have a meaningful impact on identification.

Endogenous Entry: In Appendix D.3 I consider an additional stage in-which the bidder choose a subset of auctions to enter, where each entering each subset has an associated cost. This creates a minor change to the representation of V as a function of j. The identification of j and F follows from previous arguments. Identification of the entry cost distribution then follows from standard results.

Stochastic Combination Value: In Appendix D.4 I consider the case with stochastic combination values. Specifically, when the combination value is a function of low dimensional ($< L$) unobservables, such as unobserved states or unknown parameters. The only necessary restriction is that this function is strictly monotonic in the unobservables. Identification arises from proving that the first order conditions can be inverted to point identify the unobservables.

4 Estimation Method

Having established non-parametric identification, I now describe a computationally feasible procedure to estimate F and j. Because we cannot write maximised expected payoffs as a function of bids only (Proposition 3), JP's estimation method for dynamic auction models inapplicable. I begin with a general description, outlining the key intuition. I then detail the three estimation steps and discuss asymptotics.

4.1 The Premise

The central premise of the procedure exploits that, under the assumption that payoffs are additively separable over time, we can write the continuation value as a function of: (1) Primitives of the transition process, (2) the observed distribution of equilibrium actions, and (3) the sum of the flow pay-off function and the discounted continuation value. I refer to this sum as the 'pseudo-static' pay-off; it is essentially what we estimate if we incorrectly assume myopic bidding. This relationship is given by:

$$
\begin{aligned}
V(\mathbf{s}') = &\int_{\mathbf{s}}\int_{\mathbf{b}} \Pi(\mathbf{b}|\mathbf{s};K)dG(\mathbf{b}|\mathbf{s})dT(\mathbf{s}|\mathbf{s}') \\
\text{Where:}\quad \Pi(\mathbf{b}|\mathbf{s};K) =&\Gamma(\mathbf{b}|\mathbf{s})^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}|\mathbf{s})^{-1}\Gamma(\mathbf{b}|\mathbf{s}) \\
&+ [Q(\mathbf{b}|\mathbf{s})^T - \Gamma(\mathbf{b}|\mathbf{s})^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}|\mathbf{s})^{-1}\nabla_{\mathbf{b}}Q(\mathbf{b}|\mathbf{s})]K(\mathbf{s}) \\
\text{and}\quad [K(\mathbf{s})]_a =&k(\mathbf{s}^a) = j(\mathbf{s}_i^a) + \beta V(\mathbf{s}^a)
\end{aligned}
\tag{8}
$$

This equation restates Proposition 3 as a function of G and T, as well as the pseudo-static pay-off function k. Both G and T can be estimated using standard methods. Therefore, if we had a consistent estimate for the function $k : \mathbb{S} \to \mathbb{R}$, then we would have a consistent estimate for V, and then j $(= k - \beta V)$. Like the distribution of equilibrium bids, this function $k(.)$ is not a model primitive but an equilibrium object. The central estimation problem then concerns estimating k.

The procedure generalises JP. We write the Continuation Value as a function of the distribution bids and this additional combinatorial term, correcting for the non-additivity across lots. Unlike JP we require an extra estimation step to estimate this correction term. When payoffs are additively separable $\Pi(\mathbf{b}|\mathbf{s};K) = \Gamma(\mathbf{b}|\mathbf{s})^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}|\mathbf{s})^{-1}\Gamma(\mathbf{b}|\mathbf{s})$ and the procedure collapses down to JP.[13]

This procedure is closely related to the estimation of a static model. If players are myopic, so that $\beta = 0$, then $k = j$. While we may expect that j is independent of $\mathbf{s}_{-i}$, k in general is not. The procedure involves estimating the model *as if* it were a generalised static model, in which payoffs are allowed to depend on elements of the state space that enters the continuation value.[14]

[13] In a dynamic discrete choice setting this procedure is equivalent to Conditional Choice Probability (CCP) estimation. It only differs in a 'choice over lotteries' setting when multiple (non-ordered) types may choose the same action, violating the conditions for CCP inversion.

[14] This permits a test of forward looking behaviour. If the model is correctly specified and $(\mathbf{s}_{-i}, \mathbf{s}_0)$ is excluded from j, then observing that k varies with $(\mathbf{s}_{-i}, \mathbf{s}_0)$ is sufficient to reject myopia.

I now outline the key steps. The procedure can be written succinctly as:

1. Estimate equilibrium win probabilities Γ and Q, and transition process T.
2. Given $\hat{\Gamma}$ and $\hat{Q}$, estimate F and k - the primitives of the pseudo-static model.
3. Given $\hat{\Gamma}, \hat{Q}, \hat{T}, \hat{F}$, and $\hat{k}$, evaluate $\hat{V}$ then back out $\hat{j}$

4.2 The Procedure

4.2.1 Step 1.

The First Step constitutes the standard first step in the empirical auction literature. There are several possible approaches the researcher might take. As in GKS and JP One might estimate the conditional joint distribution of bids G_i, then form $\Gamma(\mathbf{b})$, $P(\mathbf{b})$, and $Q(\mathbf{b})$ respectively. Otherwise the researcher may directly estimate these objects, essentially estimating the joint distribution of maximum rival bids. This is the approach taken in Cantillon and Pesendorfer (2007) and Raisingh (2021). A non-parametric approach is possible, using kernel density estimators or sieve-estimators, though make inference in later steps non-trivial. Given we must condition on state variables, either parametric or semi-nonparametric (e.g. B-splines) approaches are preferred. Imposing monotonicity and differentiability are simple using either approach. The transition process $T_{\mathbf{s}}(.|\mathbf{s}^a_{t-1})$ must be estimated similarly.

4.2.2 Step 2.

In the second step we estimate the pseudo-static pay-off function $k(\mathbf{s}) = j(\mathbf{s}_i) + \beta V(\mathbf{s})$. This broadly follows the second stage in the estimation procedure presented in GKS, estimating the model as if it were static. Practically, we employ a Generalised Method of Moments (GMM) procedure to impose the identifying conditions from Section 3: $E[\boldsymbol{\upsilon}|\mathbf{s}] = 0$. Set $\hat{k}$ such that, for all l and all $\mathbf{s}$, $E[\xi_l(\mathbf{b}^*|k;\mathbf{s})|\mathbf{s}] = 0$.

We can interpret the estimation problem as an Instrumental Variable model. Rewrite the Inverse Bid System as a (potentially non-linear) regression equation:

$$\underbrace{b_{lt} + \frac{\Gamma_l(b_{lt}|\mathbf{s}_t)}{\nabla_{b_l}\Gamma_l(b_{lt}|\mathbf{s}_t)}}_{y_t} = \underbrace{-\,[\frac{\nabla_{\mathbf{b}} Q(\mathbf{b}_t|\mathbf{s}_t)}{\nabla_{b_l}\Gamma_l(b_{lt}|\mathbf{s}_t)}]_{l.} K(\mathbf{s}_t)}_{\mathbf{x}_t\beta} + \upsilon_{lt}$$

We could estimate $K(\mathbf{s}_t)$ using least squares; set k to minimise the sum of squared residuals $\sum_t \sum_l \upsilon_{lt}^2$. In general $E[\upsilon_{lt}[\frac{\nabla_{\mathbf{b}} Q(\mathbf{b}_t|\mathbf{s}_t)}{\nabla_{b_l}\Gamma_l(b_{lt}|\mathbf{s}_t)}]_{l.}] \neq 0$ because $E[\upsilon_{lt} b_{l't}] \neq 0$, an endogeneity problem. We have a set of instruments $\mathbf{h}(\mathbf{s})$, where $\mathbf{h}$ is a known vector valued function of $\mathbf{s}$, which are mean independent of υ_l. The first stage is then:

$$\underbrace{-[\frac{\nabla_{\mathbf{b}} Q(\mathbf{b}_t|\mathbf{s}_t)}{\nabla_{b_l}\Gamma_l(b_{lt}|\mathbf{s}_t)}]_{l.}}_{\mathbf{x}_t} = \boldsymbol{\pi}_l \underbrace{\mathbf{h}(\mathbf{s}_t)}_{\mathbf{z}_t} + \varepsilon_{lt}$$

This interpretation enables the researcher to employ standard instrumental variable procedures, such as analysing the relevance and validity of the instruments.

One can estimate k either parametrically or non-parametrically. In the application presented in section 5 I use a parametric specification, assuming K takes a linear-in-parameters form, with $k(\mathbf{s}) = \mathbf{h}(\mathbf{s})^T\theta^k$. The linear-in-parameters specification allows the researcher to use standard software packages. B-spline specification is a convenient non-parametric method, again due to the linearity in parameters. Having estimated k we back out the distribution F using the empirical cdf of inverse bids ξ.

4.2.3 Step 3.

Given the estimated distribution of bids, transition process, and the pseudo-static pay-off function, use these objects to evaluate the continuation value using equation 8. For a candidate vector of bids $\mathbf{b}$ the researcher forms $\hat{\Pi}(\mathbf{b}|\mathbf{s};\hat{K})$. Integrating over the distribution of bids and transitions can be done either using simulation (drawing drawing $\mathbf{s}$ from $T(\mathbf{s}|\bar{\mathbf{s}})$, $\mathbf{b}$s from $G_i(\mathbf{b}|\mathbf{s})$, and averaging), or by taking a conditional expectation over observed bids and transitions. This latter approach means the distribution of bids never has to be explicitly estimated.

Finally, $\hat{j}$ is backed out using the identity $\hat{j} = \hat{k} - \beta\hat{V}$ given β.[15] We must average $\hat{j}$s over $(\mathbf{s}_{-i}, \mathbf{s}_0)$. With a correctly specified model and infinite data there will be no variation. The choice of weighting is important, placing less weight on estimates for $(\mathbf{s}_{-i}, \mathbf{s}_0)$ with fewer observations, so are more imprecisely estimated. One convenient method involves fitting another linear-in-parameters specification to the conditional expectation for V. Essentially, regress $\mathbf{s}_t$ on $\hat{\Pi}(\mathbf{b}_t|\mathbf{s}_t;\hat{K})$, weighting each 'observation'

[15] In the spirit of Magnac and Thesmar (2002) β is actually identified from our exclusion restrictions on j. We *could* set β such that $\hat{j}$ is independent of $\mathbf{s}_{-i}$. This is left for future work.

proportional to the standard error.[16]

4.3 Large Sample Properties

I now briefly discuss the asymptotic properties of this estimator. I appeal to standard asymptotic results for GMM estimators and the delta method. I focus on parametric and semi-nonparametric approaches. Due to data limitations these are the most widely used in practice. The semi-nonparametric approach involves approximating the cdf G and function k using, for example, B-splines with prespecified knot vector.[17]

As in GKS, steps 1 and 2 can be considered a two-step GMM procedure. The parametric or semi-nonparametric approach ensures that the equilibrium belief objects can be written as a function of a high dimensional parameter vector $\boldsymbol{\theta}$. The first step then either sets $\hat{\boldsymbol{\theta}}$ to maximise a likelihood function, which can be interpreted as a GMM estimator setting the expected score vector to zero, or to explicitly solve the moment equation $E_{\mathbf{b}_t}[G(\mathbf{b}|\mathbf{s};\boldsymbol{\theta}) - \prod_l \mathbb{I}[b_{lt} \leq b_l]|\mathbf{s}] = 0$ for all $\mathbf{b} \in \mathbb{B}$ and $\mathbf{s} \in \mathbb{S}$. The same approach is taken in the second step which explicitly employs GMM to estimate $\mathbf{k} = \mathbf{j} + \beta\mathbf{V}$. Under standard regularity conditions the consistency, $\sqrt{T}$ convergence, and asymptotic normality of objects estimated in these steps follows from Hansen (1982). The regularity conditions impact possible parametrisations. For example, both G and $\boldsymbol{\xi}$ must be continuously differentiable in $\boldsymbol{\theta}$ and $\mathbf{k}$. This rules out estimating parameters for maximum or minimum bids. In practice, the state space may be very large, and so parametrising $k(\mathbf{s})$ may be preferable.

In step 3 if the researcher estimates the continuation value by averaging over observed bids this is another GMM step that depends on the previous estimates. If they use simulation then asymptotic properties can be derived using the delta method, which follows from Mises (1947) as Equation 8 is smooth and differentiable in Γ and K, and hence also in parameters from the first two estimation steps. Finally, appeal to the delta method again for the properties of $\hat{j} = \hat{k} - \beta\hat{V}$.

[16] Another possible estimator, closer to the classic quasi-maximum likelihood estimators, is to not back out j in this step, and instead estimate it directly using an additional GMM step. Essentially, redo the second estimation step but plugging the estimate for $V(\mathbf{s})$ into the inverse bid system.

[17] From the Stone Weierstrass Theorem B-splines can approximate any continuous function to an arbitrary degree of precision given sufficiently many knots. B-splines are increasingly used in empirical auction studies as a flexible alternative to fully non-parametric approaches, with more convenient asymptotic properties for multi-stage estimation procedures than kernel or sieve estimators. Recent examples include Hickman et al. (2017) and Bodoh-Creed et al. (2021).

In Appendix E I present the results of a simulation study examining the performance of both the semi-parametric and semi-nonparametric estimator.

5 Application

I now apply this model to data from Michigan Department of Transport's procurement auctions for highway construction and maintenance contracts. This setting and data has been considered in several previous studies, including Groeger (2014), Somaini (2020), Raisingh (2021), and GKS. Contracts are allocated using simultaneous low-price sealed bid auctions, averaging around 45 contracts auctioned in each round, with rounds taking place every 2-4 weeks. 56 percent of bidders submit bids on more than one auction in a given round.

A large body of previous work has found evidence of cost complementarities in highway procurement, for example JP who find evidence of capacity constraints. Several previous papers, including GKS, have found evidence of complementarities in MDOT procurement specifically. GKS find evidence that a firm's cost of taking on a new project are increasing in their backlog, but the more similar their current projects the less the dis-economies of scale. Both Raisingh (2021) and Groeger (2014) find evidence of forward looking behaviour in the MDOT auctions. This suggests the need to use a dynamic multi-object auction model to estimate firm's costs.

I focus on road construction and paving projects. These projects either involve hot-mix asphalt, concrete construction, or both. I consider how firm's backlogs of both asphalt and concrete projects impact their costs, and how the two backlogs interact. Given GKS's findings, the expectation is that costs for all projects will be increasing in both backlogs, but that the cost of an asphalt contract increases faster in concrete backlog, and vice versa. Understanding the degree of complementarities is important for auction design — if the complementarities are especially large, it may be useful for MDOT to make sure similar auctions are held together.[18]

[18] In the current application I ignore the firm's entry problem. Entry is computationally difficult as firms maximise over combinations of auctions to enter. Instead I assume that firms face negligible entry costs, as bid preparation costs have been found to vary from \$5,000 to \$10,000, around %1 of the estimated contract cost (Raisingh, 2021). Given a set of available lots, the optimal entry combination is then non-probabilistic. Therefore, by taking an expectation over maximised payoffs from the auctions bidders were observed entering I correctly recover the equilibrium continuation value. This is a major simplification, but one that allows me to focus on just the dynamic multi-object auction problem — an important first step.

5.1 Data

I use the same data as GKS, using their data on bids, contracts, and competing firms.[19] This includes information on almost every auction run between 2002 and 2014. The contract data includes project descriptions, locations, the engineer's estimate of project cost, and the list of participating firms and their bids.

The firm level data includes details on the sub-sample of firms who submit at least 50 bids. This details the number and location of plants, and a description of the type of company. Following GKS's classification system, a large bidder is defined as one with at least 6 plants in Michigan. A regular bidder is defined as one that submits more than 100 bids in the sample period, otherwise they are designated a fringe bidder. In the final sample there are 36 regular bidders, 8 large (regular bidders) and 686 fringe bidders. I further categorise regular bidders into one of three types of firm: General contractors, Paving companies, and Construction companies.

Figure 1: Auction level summary statistics.

	Asphalt	Concrete	Both
Number	3563	712	1974
Auctions per Round	20.13	4.02	11.15
(p25 - p75)	(5 - 30)	(1 - 6)	(3 - 17)
Project Duration (days)	134.11	216.52	200.08
	(46 - 151)	(79.75 - 261.25)	(70 - 235.25)
Engineer's Estimate ($100,000s)	12.61	22.4	19.88
	(2.92 - 11.16)	(3.65 - 12.16)	(4.29 - 17.29)
Bidders per Auction	4.39	5.46	5.94
	(2 - 5)	(4 - 7)	(3 - 8)
Average Bid ($100,000s)	12.75	19.93	18.28
	(3.02 - 11.46)	(3.78 - 11.85)	(4.56 - 16.96)
Winning Bid ($100,000s)	11.98	21.19	18.69
	(2.69 - 10.46)	(3.34 - 11.46)	(3.99 - 16.27)

Note: Aside from the number of auctions, the numbers presented are means. For mixed projects the mean winning bid is higher than the mean bid. This is caused by the sizes of projects being very skewed — A number of small projects attract many low bids, bringing down the average losing bid more than this brings down the average winning bid.

[19] I kindly received my data directly from GKS. The auction level data is freely posted on the MDOt webpage: http://www.michigan.gov/mdot. Meanwhile their firm data is taken from a variety of sources including OneSource North America Business Browser and firms' websites.

Contract level descriptives are summarised in Figure 1. Around 20 asphalt projects are auctioned simultaneously each period, predominantly highway maintenance projects. But, these tend to be smaller projects, in both duration and predicted costs, than the concrete and mixed projects. These contracts involve construction or bridge maintenance projects, and so the engineers estimates exhibits a major right skew.

Figure 2: Bidder level summary statistics.

	General	Paving	Construction	Fringe
Plants	1.73	6.71	1.5	1.43
Bids per Round	2.07	2.8	1.8	0.24
(p25 - p75)	(0 - 3)	(0 - 4)	(0 - 3)	(0 - 0)
Backlog: Asphalt	5.57	5.61	2.97	0.24
(millions)	(0.25 - 3.88)	(0.96 - 7.6)	(0.48 - 4.39)	(0 - 0.2)
Backlog: Concrete	3.41	2.18	2.79	0.2
(millions)	(0.18 - 3.41)	(0.11 - 3.83)	(0.23 - 1.35)	(0 - 0.09)
Distance to project	105.65	84.18	121.42	119.27
Distance given Bid	71.21	47.03	87.18	69.33
Distance given Won	65.53	45.01	82.51	58.63

Note: Project locations are coded to the centroid of the county they are based in. Distance is calculated as the minimum distance (across plant locations) between a firm and the project location. A firm's backlog at t is calculated as the sum, over current contracts, of the engineer's estimate for each project multiplied by the fraction of project duration remaining. Backlogs are calculated separately for each type of project, assuming that mixed projects increase asphalt and concrete backlogs equally. I exclude the first two years of the data to construct backlogs.

Bidder level descriptive statistics are summarised in Figure 2. Regular bidders' backlogs are much larger than fringe bidders'. Asphalt backlogs are also generally higher than concrete backlogs due to the larger number of asphalt projects. Backlogs generally exhibit rightward skews, indicative of the right skewed project sizes.[20] Paving firms are closer to projects than fringe bidders because they have more plants. As expected, bidders bid on projects that are closer to them, and are more likely to win closer projects due to more aggressive bidding.

[20] There is lag between contracts being won and their start date. I must assume that every project begins before the next round of auctions. Otherwise while firms are bidding they already know that in several periods time their backlog will increase. This breaks the Markovian property of the game — at any given time a firm must consider its current backlog *and* its expected backlog in every future period. When bidding on a project that doesn't begin for several months, the firm must consider how their backlog is likely to change in those months as they take on additional projects.

5.2 Empirical model, revisited

I now apply and estimate the empirical model presented above. While a fully non-parametric approach is possible, I follow the literature and take a semi-parametric approach.[21] I apply the full dynamic multi-object model to regular bidders only, given that I need to observe sufficient observations of bidding to be able to estimate my objects of interest. I estimate separate parameters for each type of regular bidder. I assume fringe bidders are myopic, and that their costs are additive.

In the low-bid auction the bidder with the lowest bid receives their bid and pays their private cost. This involves a minor relabelling of the model presented in Section 2. The individual state is the Firm's backlog of asphalt and concrete contracts. The common state consists of the set of lots on offer, including both their locations and other contract characteristics, such as size, duration, and type. I also allow the lot specific costs distribution F to depend on additional lot specific factors such as distance to the project, and project type $\times$ firm fixed effects.

5.2.1 The State Space Approximation

The state $\mathbf{s}$ should include every firms' backlogs and information on every auction held each period, which is computationally intractable. It is unlikely that firms would track such a large state space. I follow the approach taken by Raisingh (2021) and Aradillas-Lopez et al. (2022). They condense $(\mathbf{s}_{-i}, \mathbf{s}_0)$ into a one dimensional index λ_{it}, approximating the degree of competition a firm faces on a given day. For each firm I only need to track three states — two backlogs and this competition index.

I construct λ_{it} using a random forest to predict the minimum rival bid using $(\mathbf{s}_{-i}, \mathbf{s}_0)$. λ_{it} is then a function of: *i*) the mean backlog of rival bidders, *ii*) the number of rival bidders, *iii*) the number of auctions held that period.[22] Full details of how the index is constructed, and additional results, are given in Appendix F.1.

[21] It is important to allow the pseudo-static payoff k to depend on states, auction level observables, and common/rival states. Non-parametric approaches suffers from a curse-of-dimensionality. The chosen parameterisation enables simple tests of additively separable payoffs and myopic bidding.

[22] The index assumption also implies that a firm's continuation value does not depend on which combination of lots each rival bidder wins. Therefore the firm only has to consider 2^L outcomes from the round of auctions (which combination they win themselves), rather than all n^L possible outcomes. This is reasonable — it is unlikely bidders consider how their bids impact the likelihood of their rivals winning different combinations of contracts. I do not take into account sampling uncertainty in estimating the competition index.

5.2.2 Empirical Specification

First Stage

To simplify estimation I assume firms believe that, conditional on auction characteristics and firms' states, the probability they win one auction is independent of whether they win another auction. This ensures the joint probabilities P can be written as products of the marginal probabilities.[23] I then specify the distribution of minimum rival bids as a three parameter Weibull distribution, with a support parameter as $\frac{1}{3}$ of the engineer's estimate for that contract.[24] This assumption is sensible as the Weibull distribution is the limiting distribution of the minimum of multiple independent random variables. The scale is written as a function of auction-level characteristics and the competition index, which I denote using the vector $\mathbf{x}_{tl}$:

$$F(\underline{b}_{lt}; \beta_1, \alpha) = 1 - e^{-(\frac{\underline{b}_{lt} - \frac{1}{3}}{\exp(\mathbf{x}_{lt}\beta_1)})^{\alpha}}$$

I assume that states transition according to an autoregressive order (1) process:

$$\begin{pmatrix} \lambda_{it} \\ \mathbf{s}_{it} \end{pmatrix} = \boldsymbol{\alpha}_i + \boldsymbol{\alpha} \begin{pmatrix} \lambda_{it-1} \\ \mathbf{s}_{it-1} \end{pmatrix} + \boldsymbol{\varepsilon}_{it}$$

Where $\boldsymbol{\alpha}_i$ are firm specific intercepts, $\boldsymbol{\alpha}$ is a 3×3 dimension matrix, that is allowed to vary by firm type, and $\boldsymbol{\varepsilon}_{it}$ is a white noise innovation.[25]

[23] While I can reject the null hypothesis of independence, the extent of this dependence is extremely small. I introduce dependence in the below procedure using a Gaussian Copula to allow correlation in these minimum rival bids. This correlation is allowed to depend on whether the contracts are the same type or in the same county. The maximum estimated correlation between any two winning bids is 0.0272, which I take as negligible.

[24] As discussed in Raisingh (2021) this is because several projects appear to have miscalculated estimates. These are treated as outliers and removed. This occurred in around 0.1% of cases. The minimum observed bid above $\frac{1}{3}$ is 0.46. I also impose that the shape parameter is greater than 1, however this restriction does not bind.

[25] By construction backlogs actually transition deterministically. However, not all projects are completed at the same rate. Therefore I must take into account future deterministic backlogs in the state variable. I assume this transition function for simplicity, as $AR(1)$ processes are often used to model the transitions of inclusive value indices.

Second Stage

I assume the pseudo-static pay-off is quadratic in backlogs. Testing for complementarities reduces to testing the significance of the quadratic terms. I normalise backlogs by the standard deviation of each firm's observed backlogs, so that backlog effects are estimated using within firm variation. Parameters can vary across the three firm types, so for a firm of type n the specification for the pseudo-static pay-off is:

$$k_n(\mathbf{s}_t) = \lambda_{it}\theta_n^{\lambda} + \mathbf{h}(\mathbf{s}_{it})^T\theta_n^h + \lambda_{it}\mathbf{h}(\mathbf{s}_{it})^T\theta_n^{h\lambda}$$

$$\text{Where} \qquad \mathbf{h}(\mathbf{s}_{it})^T = \begin{pmatrix} s_{it}^a & s_{it}^c & (s_{it}^a)^2 & (s_{it}^c)^2 & s_{it}^a \times s_{it}^c \end{pmatrix}$$

This step is estimated using GMM.[26] I also make use of additional moments to facilitate estimation. If $\mathbf{s}_t$ does not substantially shift bidding behaviour there may be a weak instrument problem. This occurs if a firm's observed backlog does not vary relatively much, but they bid on many contracts simultaneously so that the possible ex-post states $\mathbf{s}_t^a$ vary much more than $\mathbf{s}_t$. In this case we are trying to estimate k in regions where there is little variation in our instrument. This would be a problem if firms are successfully smoothing their backlogs.[27]

I include several additional instruments, or moment conditions, to ameliorate this problem. Write $\overrightarrow{\mathbf{s}}_l$ as the amount a firm's backlog will increase if they win lot l. This is the engineer's estimate of the project completion cost, split according to the type of contract. I make the additional assumption $E[\upsilon_{ilt}|\mathbf{s} + \overrightarrow{\mathbf{s}}_l] = 0$, using the ex-post state from only winning lot l as an additional instrument.[28] Many more potential instruments are available, using additional ex-post states as instruments. For illustrative purposes I also consider a specification that makes use of ex-post

[26]The distribution of contract sizes is very skewed, with a small number of extremely large contracts. These contracts impact backlogs much more than small contracts, and attract higher bids. These observations have a lot of leverage. To reduce the weight on these observations I weight observations by the inverse of the engineer's estimate of lot l (EE_l). This is equivalent to using of moment conditions of the form $E[\frac{\upsilon_{ilt}}{EE_l}|\mathbf{s}_t] = 0$. Furthermore, it is standard to normalise bids and associated costs by the size of the lot, which makes a similar assumption.

[27]This problem is alleviated if we do not use normalised backlogs, using variation across bidders to aid identification. However for this application this is undesirable.

[28]However, it is also possible that the larger the contract, so the larger $\overrightarrow{\mathbf{s}}_l$, the larger the lot-specific cost — meaning the instrument could be invalid. This is unlikely. First, I already control for the size of the contract through the linear term in k. Second, the weighting procedure I use, weighting observations by the inverse contract size, means this assumption is more reasonable. Finally, as the system is over-identified I perform additional Hansen tests of over-identifying restrictions.

states from winning pairs of contracts, increasing the number of instruments ten-fold. However this risks overfitting the first stage.

Third Stage

After forming the expected maximised period pay-off $\hat{\Pi}(\mathbf{b}_t|\hat{k};\mathbf{s}_t)$ I evaluate the ex-ante value function by approximating the conditional expectation over $\mathbf{b}_t$ using a linear in parameters prediction of Π_t given $\mathbf{h}(\mathbf{s}_t)$.[29] This is convenient as it ensures the ex-ante Value Function, for a firm of type n, can be written as: $E[\hat{\Pi}_i(\mathbf{b}_t|\hat{k};\mathbf{s}_t)|\mathbf{s}_t] = \mu_i + \mathbf{h}(\mathbf{s}_t)^T\theta_n^V$. Observations are weighted according to their inverse variance, using $var(\hat{\theta}_n^k)$. The combination of the quadratic form of $\mathbf{h}$ and the AR(1) transition process ensures I can write $E[\mathbf{h}(\mathbf{s}_t)|\mathbf{s}_{t-1}] = \mathbf{h}(\mathbf{s}_{t-1})^T\theta_n^\tau$, where θ_n^τ is a $|\mathbf{h}| \times |\mathbf{h}|$ dimensional matrix function of $\boldsymbol{\alpha}_n$ estimated in the first stage. This yields:

$$j(\mathbf{s}_{it}) = \mathbf{h}(\mathbf{s}_{it})^T(\theta_n^k + \beta\theta_n^\tau\theta_n^V) = \mathbf{h}(\mathbf{s}_{it})^T\theta_n^j$$

5.3 Structural Estimates

5.3.1 First Estimation Step

Results from the first stage of estimation are given in Figure 3. I present three specification, including varying sets of Fixed Effects. In later steps I make use of only the County Fixed Effects, dropping the time fixed effects. The shape parameter is estimated to be well above one, ensuring that the Markup is monotonically increasing in bids. Note that mean of the distribution is increasing in the scale. For each of the scale parameters I include separate slope coefficients for each type of auction. For all three types of auction the winning bid is increasing in the competition index: When λ is large, so there is little competition, bids are less aggressive. Meanwhile the magnitude for Asphalt projects is in line with the results presented in Raisingh (2021). Magnitudes for concrete and mixed projects are similar.

Because the dependent variable (lowest rival bid) is normalised by the engineer's estimate, the coefficients on engineer's estimate can be interpreted as returns to scale.

[29] This assumption is technically incompatible with the parametric assumption made above. However we can test the extent of the misspecification error using a standard RESET test. I am unable to reject the null of no specification error (at the 10% significance level) using a RESET test of order 10. Meanwhile no explicit parametric assumptions were made on the distributions of b or υ.

Figure 3: First Stage Results (1)

		Coefficient	SE	Coefficient	SE	Coefficient	SE
Shape							
	$\log(\alpha - 1)$	2.029	0.001	2.083	0.001	2.093	0.001
Scale	$(= e^{\mathbf{x}_{lt}\beta_1})$						
	Concrete	-0.48	0.001	-0.484	0.002	-0.495	0.003
	Asphalt	-0.458	0.001	-0.449	0.002	-0.461	0.003
	Both	-0.44	0.001	-0.45	0.002	-0.462	0.003
	Major Road	-0.013	0.001	-0.007	0.001	-0.007	0.001
	Bridge	-0.001	0.001	0.005	0.001	0.003	0.001
	MR $\times\lambda$	0.048	0.001	0.042	0.001	0.041	0.001
	Bridge $\times\lambda$	0.027	0.001	0.024	0.001	0.021	0.001
	Concrete $\times\lambda$	0.183	0.001	0.186	0.001	0.187	0.001
	Asphalt $\times\lambda$	0.196	0.001	0.198	0.001	0.196	0.001
	Both $\times\lambda$	0.172	0.001	0.18	0.001	0.181	0.001
	Concrete $\times$ log(EE)	0.008	0.001	0.001	0.001	0	0.001
	Asphalt $\times$ log(EE)	-0.008	0.001	-0.011	0.001	-0.012	0.001
	Both $\times$ log(EE)	-0.006	0.001	-0.009	0.001	-0.01	0.001
	Concrete $\times\lambda\times$ log(EE)	-0.006	0.001	-0.004	0.001	-0.004	0.001
	Asphalt $\times\lambda\times$ log(EE)	0.001	0.001	0.001	0.001	0.001	0.001
	Both $\times\lambda\times$ log(EE)	-0.002	0.001	-0.001	0.001	-0.001	0.001
	Fixed Effects						
County				$\checkmark$		$\checkmark$	
Year						$\checkmark$	
Month						$\checkmark$	
	Observations	193545		193545		193545	

The persistent negative coefficient on asphalt suggests increasing returns, in line with GKS and Raisingh's results.

5.3.2 Second Estimation Step

Figure 4 presents the results from the second estimation step and includes estimates from a least squares specification as well as three sets of instruments. Parameter interactions with the competition index are included in Appendix F.2. Estimates from the third column are used for the remainder of this application. Results are presented in thousands of dollars. So, for example, every kilometre increase in distance between a general contractor's plant (t1) and the project increases costs by around \$170.

The coefficients on combinatorial objects can be interpreted in terms of how they impact the pseudo-static cost function: Every one standard deviation increase in

Figure 4: Second Stage Results

Instruments		none (OLS)		$\mathbf{s}_{it}$		$\mathbf{s}_{it} + \overrightarrow{\mathbf{s}}_{ilt}$		$\mathbf{s}_{it} + \overrightarrow{\mathbf{s}}_{ilt} + \overrightarrow{\mathbf{s}}_{imt}$	
		$\hat{\theta}$	SE	$\hat{\theta}$	SE	$\hat{\theta}$	SE	$\hat{\theta}$	SE
Combinatorial									
s_t^a	t1	488	19.3	495	346	451	26.6	463	20.4
	t2	905	35	1940	2270	852	37	869	25.4
	t3	113	5.61	114	1190	113	5.96	113	4.75
s_t^c	t1	392	17.2	254	826	406	20.1	398	17.6
	t2	216	40.5	-4,380	9370	233	43.5	221	37.5
	t3	55.8	6.2	-10.1	5010	57	6.6	56.7	5.74
$(s_t^a)^2$	t1	-4.36	1.85	-21.6	67.3	-0.967	2.72	-1.76	2.27
	t2	-26	4.01	-221	481	-18	4.56	-21.5	3.35
	t3	-0.236	0.0645	-0.189	24.6	-0.26	0.0698	-0.246	0.0629
$(s_t^c)^2$	t1	-12	1.96	0.179	189	-15.9	2.4	-15.5	2.3
	t2	-18.6	6.04	741	1530	-26.1	6.91	-23.9	6.34
	t3	-0.245	0.0638	7.28	74.3	-0.344	0.119	-0.311	0.093
$s_t^a \times s_t^c$	t1	0.464	3.04	28.2	125	5.3	3.23	7.03	3.15
	t2	54.9	12.5	1.27	431	73.1	13.8	72.5	10.2
	t3	0.277	0.176	-21.6	103	0.553	0.366	0.482	0.279
Lot specific									
Distance	t1	0.159	0.0814	0.238	0.187	0.188	0.0779	0.164	0.0823
	t2	0.0597	0.104	-0.0511	0.599	0.0999	0.108	0.0667	0.102
	t3	0.159	0.0946	-0.018	2.87	0.166	0.0943	0.159	0.094
Fixed Effects									
County		$\surd$		$\surd$		$\surd$		$\surd$	
Firm × Type		$\surd$		$\surd$		$\surd$		$\surd$	
Tests				(stat)	(p-val)				
Hansen		36.5	(0.192)	-	(-)	19	(0.393)	637	(0)
Cragg-Donald		-		0.00257		178		119	
R^2		0.6		-10.8		0.597		0.599	
Observations									
T		3919		3919		3919		3919	
$\sum_t L_t$		14691		14691		14691		14691	

Note: Column 1 Hansen test is a Hausman test of endogeneity, using instruments from column 3. Figures are given in 000s of dollars, so that every one standard deviation increase in a general contractor's (t1) backlog of asphalt projects increases their pseudo-cost (cost + expected future opportunity cost) by around \$470,000. Standard errors are adjusted for the two-step estimation procedure, and clustered within bidder days. I winsorise the bottom percentile of estimated $\frac{\Gamma_l(b_{ilt})}{\nabla_b \Gamma_l(b_{ilt})}$, since beliefs in the tails of the distribution are likely to be poorly estimated.

a general contractor's (t1) backlog of asphalt projects increases their pseudo-cost (cost + expected future opportunity cost) by around $\$470,000$. Coefficients can also be interpreted as how they impact the aggressiveness of the firm's bidding. The coefficients on linear backlogs are all positive, suggesting firms bid less aggressively on larger projects. We cannot interpret the quadratic coefficients from the second stage as evidence of returns to scale. However they give evidence of non-additivities across lots: The null hypothesis of additive values is rejected with p-value < 0.001.

The post-estimation tests demonstrate that the choice of instruments is important. The Hansen test of over-identifying restrictions presented in column 4 rejects the null at the 1% significance level, suggesting these additional instruments are invalid. However we cannot reject the validity of the additional instruments used in column 3. Likewise, the Hausman test for endogeneity in column 1 also fails to reject. Meanwhile, The (adjusted) Cragg-Donald statistic in column 2 suggests that the initial state alone is a weak instrument. Therefore, even though we do not have reason to believe the estimates from column 1 are consistent, they are almost certainly better estimates than those presented in column 2. This suggests that problems caused by irrelevant instruments may be more damaging than failing to instrument at all.

5.3.3 Third Estimation Step

Figure 5: Third Stage Results

Object		$j(\mathbf{s}_i)$		$V(\mathbf{s})$		$k(\mathbf{s})$	
		$\hat{\theta}$	SE	$\hat{\theta}$	SE	$\hat{\theta}$	SE
λ	t1	0	(-)	5.39	0.63	6.79	0.763
	t2	0	(-)	15.6	1.58	6.47	1.69
	t3	0	(-)	6.81	0.713	1.81	0.228
s_t^a	t1	123	7.01	-451	26.6	451	26.6
	t2	285	11.3	-839	37.1	852	37
	t3	40	1.92	-103	6.07	113	5.96
s_t^c	t1	107	5.35	-405	20.1	406	20.1
	t2	89.1	11.9	-207	43.5	233	43.5
	t3	15.6	1.91	-57.8	6.63	57	6.6
$(s_t^a)^2$	t1	-0.337	1.29	1.74	2.73	-0.967	2.72
	t2	-9.26	2.46	18.6	4.65	-18	4.56
	t3	-1.34	0.147	-2.13	0.292	-0.26	0.0698
$(s_t^c)^2$	t1	-7.6	1.13	15.5	2.41	-15.9	2.4
	t2	-14	3.38	23.4	6.95	-26.1	6.91
	t3	-0.479	0.102	-0.262	0.202	-0.344	0.119
$s_t^a \times s_t^c$	t1	1.38	1.52	-6.7	3.25	5.3	3.23
	t2	33.4	7.12	-80.4	13.9	73.1	13.8
	t3	0.432	0.199	-0.237	0.405	0.553	0.366
Fixed Effects							
Firm				$\surd$		$\surd$	

Figure 5 presents results from the third stage of estimation. Costs are increasing in both types of linear backlog for all three types of firm. However, the magnitudes are much smaller than the linear coefficients estimated in the second stage. This suggests large anticipated opportunity costs from high backlogs. This result is sensible since projects have very long durations.

By considering the quadratic terms we see that general contractors only exhibit increasing returns to scale, or increasing returns to specialisation, in concrete contracts. Meanwhile, both paving and construction companies exhibit increasing returns for both types of contracts, but with a negative cost interaction. Taking on concrete (asphalt) projects come with additional costs for these firms already specialised in asphalt (concrete) projects. In Appendix F.3 I consider how my results compare to results from misspecified dynamic single-object, and static multi-object models. I find that the dynamic single-object model under-estimates the degree of non-additivity across lots. The static multi-object model over-estimates the effect of backlogs on costs, mistaking expected future costs for present costs.

5.4 Counterfactual

I now consider how procurement costs and efficiency change when contracts are allocated using sequential first-price auctions. This is an interesting counterfactual as it speaks to the importance of the 'exposure problem' as well as the value of 'batching'. Furthermore, many empirical dynamic auction papers assume contracts are auctioned sequentially anyway, making this a useful comparison for researchers.

Theoretical results suggest sequential allocation will be less efficient than simultaneous allocation (batching).[30] Bidders do not know what types of contracts will be auctioned in the near future, making it more difficult to exploit cost synergies. However, batching contracts but not allowing firms to place combinatorial bids also limits their ability to exploit synergies. Sequential allocation may improve efficiency by giving bidders greater control over their cost synergies, reducing the likelihood that bidders accidentally win too many or too few contracts (the exposure problem). These effects will be more pronounced the larger the degree of complementarities across lots. The effects of this alternate procurement mechanism are ex-ante unclear.

[30] See Akbarpour et al. (2020) as an example. I ignore that collusion is easier to sustain in sequential auctions (Hendricks and Porter, 1989), further increasing procurement costs.

5.4.1 The Counterfactual Mechanism

I now briefly discuss how I simulate equilibrium bidding under the counterfactual mechanism. See Appendix F.4 for full details. Contracts are auctioned sequentially, in random order, within each 14 day period. Auctions are low-price sealed bid, as in the main application. Consistent with the estimated model I assume projects begin before the next auction. I use the same competition index λ_{it} to capture changes in competition within these periods. Firms have beliefs about the probability they win any given lot, conditional on lot characteristics and λ_{it}. Firms place bids conditional on their beliefs, backlogs, and their continuation value, defined as in the main model.[31]

I find equilibrium beliefs and value functions using fixed point iteration. For a given beliefs and value functions I simulate the auction process, numerically maximising expected payoffs conditional on beliefs and continuation values. I then take a conditional expectation of the maximised expected pay-off using a linear-in-parameters prediction as I did in the main model. I then form the new continuation value. I repeat this process until the value function converges for each type of firm. I then find the equilibrium distribution of winning bids as I did in the main specification, repeating this outer-loop until the distribution of winning bids converges.

5.4.2 Results

Figure 6: Counterfactual Results

Mechanism	Outcome	Estimate ($000s)	S.E.
Simultaneous Auctions	Procurement Cost	1470	-
	Completion Cost	1170	4.28
Sequential Auctions	Procurement Cost	1489	3
	Completion Cost	1280	22.6

Note: The results are based on 60 draws of parameters from their estimated asymptotic distribution. Equilibrium Beliefs and Value Functions are computed for each draw.

Figure 6 presents the results from the counterfactual simulations. The table

[31] I assume firms only place bids on the set of auctions they actually bid on. Given my assumption of negligible entry costs, firms were only observed bidding on the contracts they have the largest cost advantages in. If their cost advantages were mostly additive, such as due to low v_{ilt} draws, they will have the same advantage under the sequential mechanism, and so bidding on this set of lots will remain optimal. Therefore my estimates can, to an extent, be considered lower bounds on costs.

presents estimates of the average cost per contract for firms and MDOT, in thousands of dollars, under the simultaneous auction regime and the counterfactual sequential auction regime. The key takeaway is that the sequential mechanism decreases efficiency and raises procurement cost. The procurement cost is estimated to increase by an average of $\$19,000$ per contract (1.3%), while for firms completion costs increase by an average of $\$110,000$ per contract (9.4%). This suggests the batching effect dominates the exposure effect. This arises because the cost complementarities are relatively small. The non-additivity in payoffs across lots only explains 11.5% of the variation in payoffs, while the remainder is lot specific variation. Furthermore, this figure includes both the positive complementarities between same type contracts, and the negative complementarities between different type contracts. Consequently, the exposure risk is only small.

Finally, the increase in procurement cost is much smaller than the increase in completion costs because firms face more competition for each contract. At any one time, instead of n firms compete for L contracts there are n firms competing for 1 contract, unsure of when any future contracts will be auctioned. However, this finding strongly relies on the assumption of a non-collusive equilibrium.

6 Conclusion

In this paper I did three things: First, I set-up a dynamic multi-object auction model and proved that the model primitives are identified from standard bidding data. Second, in order to overcome the technical challenges of estimating model primitives in this setting, I proposed a computationally feasible estimation procedure. Finally, I applied the model to data from Michigan Department of Transport's procurement data and evaluated the efficiency and revenue of holding repeated rounds of simultaneous auction relative to auctioning all contracts sequentially.

This paper was motivated by the prevalence of such repeated, multi-object auctions. Significant complementarities between auctioned objects have been found in both the dynamic single-object literature, and the static multi-object literature, most notably in JP and GKS. However, these two types of model had not, until this point, been unified in a single framework. Future work should attempt to take into account the firms' entry decisions, as this was a major simplification in this paper.

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Appendices

A Proof of Proposition 3

In this Appendix I essentially extend Proposition 1 from JP to the multi-object case. In Appendix A.1 I prove Proposition 3 from the main text. In Appendix A.2 I present (and prove) that the ex-ante value function also has an analytic expression.

For the remainder of this section I use the definition of $\mathbf{k} = C\mathbf{j} + \beta\mathbf{V}$, and equivalently $K(\mathbf{s}) = J(\mathbf{s}_i) + \beta V(\mathbf{s})$. I also use the result that the Inverse Bid System is strictly monotonic in bids. This result, as is standard, follows from the maximising behaviour that ensures a negative definite Hessian matrix at the optimum. This ensures a globally positive definite Jacobian matrix of the inverse bid system: $Jac(\mathbf{b}^*) = -\nabla_{\mathbf{b}}\Gamma_l(\mathbf{b}^*)Hess(\mathbf{b}^*)$. Full proof is omitted due to lack of novelty.

A.1 Proof of Proposition 3

Proposition 3 : Under assumptions 1 - 4, the expected stage pay-off is given by:

$$\begin{aligned}\tilde{\Pi}(\mathbf{b}^*|\boldsymbol{v};\mathbf{s}) =&\Gamma(\mathbf{b}^*|\mathbf{s})^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\Gamma(\mathbf{b}^*|\mathbf{s})\\ &+[P(\mathbf{b}^*|\mathbf{s})^T \;-\; \Gamma(\mathbf{b}^*|\mathbf{s})^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\nabla_{\mathbf{b}}P(\mathbf{b}^*|\mathbf{s})]B_{\mathbf{s}}\mathbf{j}\\ &+[Q(\mathbf{b}^*|\mathbf{s})^T \;-\; \Gamma(\mathbf{b}^*|\mathbf{s})^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\nabla_{\mathbf{b}}Q(\mathbf{b}^*|\mathbf{s})]A_{\mathbf{s}}\beta\mathbf{V}\end{aligned}$$

Proof: 1. Necessary First Order Conditions are given by:

$$\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})(\boldsymbol{v}-\mathbf{b}^*) = \Gamma(\mathbf{b}^*|\mathbf{s}) - \nabla_{\mathbf{b}}P(\mathbf{b}^*|\mathbf{s})B_{\mathbf{s}}\mathbf{j} - \beta\nabla_{\mathbf{b}}Q(\mathbf{b}^*|\mathbf{s})A_{\mathbf{s}}\mathbf{V}$$

2. Left multiplying by $\Gamma(\mathbf{b}^*|\mathbf{s})^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}$

$$\Gamma(\mathbf{b}^*|\mathbf{s})^T(\boldsymbol{v}-\mathbf{b}^*) = \Gamma(\mathbf{b}^*|\mathbf{s})^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}[\Gamma(\mathbf{b}^*|\mathbf{s})-\nabla_{\mathbf{b}}P(\mathbf{b}^*|\mathbf{s})B_{\mathbf{s}}\mathbf{j}-\beta\nabla_{\mathbf{b}}Q(\mathbf{b}^*|\mathbf{s})A_{\mathbf{s}}\mathbf{V}]$$

3. Substituting $\Gamma(\mathbf{b}^*|\mathbf{s})^T(\boldsymbol{\upsilon} - \mathbf{b}^*)$ into equation 2 gives the result.

□

A.2 Ex-ante Value Function

Building on Proposition 3 the ex-ante Value function can be written as:

$$\begin{aligned} V_i^E(\mathbf{s}_t) =& E_{\mathbf{b}}[\,\Gamma(\mathbf{b}|\mathbf{s}_t)^T \nabla_{\mathbf{b}}\Gamma(\mathbf{b}|\mathbf{s}_t)^{-1}\Gamma(\mathbf{b}|\mathbf{s}_t)] \\ &+ E_{\mathbf{b}}[Q(\mathbf{b}|\mathbf{s}_t)^T - \Gamma(\mathbf{b}|\mathbf{s}_t)^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}|\mathbf{s}_t)^{-1}\nabla_{\mathbf{b}}Q(\mathbf{b}|\mathbf{s}_t)\;|\mathbf{s}_t]K(\mathbf{s}_t) \end{aligned} \tag{9}$$

Proof that this is the case should be trivial, however it requires that we apply a change of variable, changing from integrating over $\boldsymbol{\upsilon}$ to integrating over $\mathbf{b}$.

Proof: 1. To obtain the ex-ante value function from equation 5 we take an expectation over both sides with respect to $\boldsymbol{\upsilon}$ for:

$$\begin{aligned} &E_{\boldsymbol{\upsilon}}[\Gamma(\mathbf{b}^*|\mathbf{s}_t)^T(\boldsymbol{\upsilon} - \mathbf{b}^*) + P(\mathbf{b}|\mathbf{s}_t)^T K(\mathbf{s}_t)] \\ &\qquad =E_{\boldsymbol{\upsilon}}[\Gamma(\mathbf{b}^*|\mathbf{s}_t)^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s}_t)^{-1}\Gamma(\mathbf{b}^*|\mathbf{s}_t)] \\ &\qquad\quad + E_{\boldsymbol{\upsilon}}[Q(\mathbf{b}|\mathbf{s}_t)^T - \Gamma(\mathbf{b}^*|\mathbf{s}_t)^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s}_t)^{-1}\nabla_{\mathbf{b}}Q(\mathbf{b}^*|\mathbf{s}_t)]K(\mathbf{s}_t) \end{aligned}$$

2. By applying the Law of the Unconscious Statistician (change of variables for expectations) the right hand side of this equation is equal to

$$E_{\mathbf{b}}[\,\Gamma(\mathbf{b}|\mathbf{s}_t)^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}|\mathbf{s}_t)^{-1}\Gamma(\mathbf{b}|\mathbf{s}_t) + [Q(\mathbf{b}|\mathbf{s}_t)^T - \nabla_{\mathbf{b}}\Gamma(\mathbf{b}|\mathbf{s}_t)^{-1}\nabla_{\mathbf{b}}Q(\mathbf{b}|\mathbf{s}_t)]K(\mathbf{s}_t)\;|\mathbf{s}_t]$$

□

Importantly, in order to apply the Law of the Unconscious Statistician we require that the mapping $\boldsymbol{\xi}(\mathbf{b}^*|K;\mathbf{s})$ is monotonic in $\mathbf{b}$.

B Proof of Proposition 4

In this Appendix I prove Proposition 4. The proposition is given by:

Proposition 4. *Under assumption 1 - 5* $\Psi(I_S - \beta T\Omega)^{-1}C$ has rank $S_i - 1$

The proof is split into three parts. First, I establish the rank of Ψ, and then find its null space. I then demonstrate that the intersection of this null space and the image of $(I_S - \beta T\Omega)^{-1}C$ only contains a single element.

B.1 Rank of Ψ

B.1.1 Additional Definitions

Define the partial ordering $\succeq^*$ such that if $\mathbf{s}_i \succeq \mathbf{s}'_i$ then $\mathbf{s} \succeq^* \mathbf{s}'$. This simply extends the partial ordering of the individual state to the overall state. Next, similar to the graph theoretic definition, define a 'component' $\mathbb{S}^c$ of the set $\mathbb{S}$ as follows:

Definition B.1 (Component)**.** $\mathbb{S}^c \subset \mathbb{S}$ is such that two states $\mathbf{s}, \mathbf{s}'$ are in component c if and only if there exists a state $\bar{\mathbf{s}}$ such that one of the following holds:

$$\mathbf{s} \succeq^* \bar{\mathbf{s}} \;\;\&\;\; \mathbf{s}' \succeq^* \bar{\mathbf{s}} \quad \text{or} \quad \mathbf{s} \succeq^* \bar{\mathbf{s}} \succeq^* \mathbf{s}' \quad \text{or} \quad \mathbf{s}' \succeq^* \bar{\mathbf{s}} \succeq^* \mathbf{s} \quad \text{or} \quad \bar{\mathbf{s}} \succeq^* \mathbf{s} \;\;\&\;\; \bar{\mathbf{s}} \succeq^* \mathbf{s}'$$

A component is a subset of $\mathbb{S}$ that is 'connected' by this partial ordering. By definition $\mathbf{s}_0$ does not vary within a component, and in general there is one component corresponding to each element of $\mathbb{S}_0$. The S^c components form a partition of $\mathbb{S}$.

Finally, denote $\tilde{\min}(\mathbb{S})$ as the subset of $\mathbb{S}$, such that $\forall \mathbf{s} \in \tilde{\min}(\mathbb{S}) \;:\; \nexists\, \mathbf{s}' \in \mathbb{S} \;:\; \mathbf{s} \in \mathbb{S}^a(\mathbf{s}')$. This definition is primarily for notational convenience, and does not necessarily coincide with the set of minimal elements of $\mathbb{S}$. Instead, this is the (potentially empty) set of states that never occur as possible ex-post states. Intuitively, pay-offs from ending in these states will not be identified.

B.1.2 Additional Lemmas

Lemma B.1. *From any two distinct, non-maximal, states,* $\mathbf{s}$ *and* $\mathbf{s}'$*, if* $\mathbf{s}' \not\succeq^* \mathbf{s}$ *then there exists a state* $\mathbf{s}^a$ *such that* $\mathbf{s}^a \in \mathbb{S}^a(\mathbf{s}) \;\&\; \mathbf{s}^a \notin \mathbb{S}^a(\mathbf{s}')$

This Lemma states that if one non-maximal state is not 'higher' in the partial ordering than another, then their set of ex-post states cannot perfectly overlap. The proof examines whether the unique element of $\mathbb{S}^a(\mathbf{s})$ that consists of bidder i winning every lot, denoted $\mathbf{s}^{all_i}$, can be an element of $\mathbb{S}^a(\mathbf{s}')$. I makes use of the following property of the partial ordering $\succeq$ from assumption 5: For any two incomparable states $\mathbf{s}_i, \mathbf{s}'_i$ and any $\mathbf{s}_0$ there must exist some $\mathbf{s}^a \in \mathbb{S}^a_i(\mathbf{s}_i, \mathbf{s}_0)$ such that $\mathbf{s}^a \notin \mathbb{S}^a_i(\mathbf{s}'_i, \mathbf{s}_0)$.

This result arises from the same set of lots being available in each state, ensuring that the sets $\mathbb{S}_i^a(\mathbf{s}_i, \mathbf{s}_0)$ and $\mathbb{S}_i^a(\mathbf{s}_i', \mathbf{s}_0)$ cannot totally overlap.

Proof:

1. Suppose $\mathbf{s}' \not\succeq^* \mathbf{s}$. Therefore either $\mathbf{s} \succeq^* \mathbf{s}'$, or the states are incomparable.
2. If $\mathbf{s} \succeq^* \mathbf{s}'$ then they must lie in the same component, hence $\mathbf{s}_0 = \mathbf{s}_0'$. Hence the same lots are available in both states. The result follows trivially — it cannot be the case that $\mathbf{s}^{all_i} \in \mathbb{S}^a(\mathbf{s}')$ for non-maximal states.
3. If they are incomparable then we have another two options: Either $\mathbf{s}$ and $\mathbf{s}'$ belong to different components, or they belong to the same component.
4. If they belong to different components then by definition $\mathbb{S}^a(\mathbf{s})$ and $\mathbb{S}^a(\mathbf{s}')$ must be mutually exclusive.
5. If they belong to the same component then, by definition of the partial ordering $\succeq^*$, $\mathbf{s}_i$ and $\mathbf{s}_i'$ are incomparable under the ordering $\succeq$. Therefore exists some $\mathbf{s}^a \in \mathbb{S}_i^a(\mathbf{s}_i, \mathbf{s}_0)$ such that $\mathbf{s}^a \notin \mathbb{S}_i^a(\mathbf{s}_i', \mathbf{s}_0)$. Therefore exists some $\mathbf{s}^a \in \mathbb{S}^a(\mathbf{s})$ such that $\mathbf{s}^a \notin \mathbb{S}^a(\mathbf{s}')$.

$\square$

Lemma B.2. *$\Psi(\mathbf{s})A_{\mathbf{s}}$ has rank at least 2 if, for all $\mathbf{s}, \boldsymbol{v}, l$, $\Gamma_{il}(\mathbf{b}(\boldsymbol{v}, \mathbf{s})|\mathbf{s}) \in (0,1)$*

The proof proceeds by first showing that $rank(\Psi(\mathbf{s}))$ is weakly greater than two, then using the full rank property of the transformation matrix $A_{\mathbf{s}}$.[32]

Proof:

1. Consider the $L \times L(n-1)^{L-1}$ sub-matrix of $\Psi(\mathbf{s})$ that consists of only the columns of $\Psi(\mathbf{s})$ corresponding to outcomes in which player i wins exactly one lot. Call this matrix $\tilde{\Psi}$.
2. Row l, column a of $\tilde{\Psi}$ is strictly positive for columns corresponding to outcomes $\mathbf{w}^a$ in which bidder i wins lot l This is because the probability that i wins lot l, and no other lot, is strictly increasing in b_l.
3. Row l, column a of $\tilde{\Psi}$ is strictly negative for columns corresponding to outcomes $\mathbf{w}^a$ in which i does not win lot l. This is because the probability lot l is won, and no other, is strictly decreasing in b_m for $m \neq l$.

[32]In general $\Psi(\mathbf{s})A_{\mathbf{s}}$ has rank L. Essentially, each state gives us L pieces of information, rather than just two pieces of information. However, proof that the rank is always L has proven elusive.

4. Any two rows of $\tilde{\Psi}$ are linearly independent: Each row contains one positive entry, each in a distinct column.[33] Therefore, $\tilde{\Psi}$, and hence $\Psi(\mathbf{s})$ have rank ≥ 2.

5. Matrix $A_{\mathbf{s}}$ is a rank n^L transformation matrix for any non-maximal $\mathbf{s}$. Therefore, from step 4, $\Psi(\mathbf{s})A_{\mathbf{s}}$ for non-maximal $\mathbf{s}$ has rank at least 2.

$\square$

B.1.3 Rank(Ψ)

Proposition 5. *Rank*$(\Psi) = S - S^c - |\tilde{\min}(\mathbb{S})|$

The proof involves demonstrating that as we stack these $\Psi(\mathbf{s})A_{\mathbf{s}}$ matrices for non-maximal $\mathbf{s}$, the rank increases by *at least* two each time. However, by definition columns corresponding to elements in $\tilde{\min}(\mathbb{S})$ are all zero, ensuring the rank must be deficient by at least $|\tilde{\min}(\mathbb{S})|$. Likewise, for each submatrix of Ψ made up of rows corresponding to states that are all within the same component (denoted by Ψ^c, a $|\mathbb{S}^c| \times S$ matrix), the rows all sum to zero. This ensures each Ψ^c is rank deficient by at least one, and so Ψ is rank deficient by at least S^c.

Proof:

1. Order elements of $\mathbb{S}$ (likewise, columns of Ψ) according to the partial ordering $\succeq^*$. Incomparable states are ordered at random. So, for each $\mathbf{s}$, the furthest left non-zero column of $\Psi(\mathbf{s})A_{\mathbf{s}}$ is in the column corresponding to the ex-post state in which player i wins every lot $\mathbf{s}^{all_i}$.

2. Focus on one component, $\mathbb{S}^c$. Find the 'smallest' state within $\mathbb{S}^c$, $\mathbf{s}_1^c$ (i.e. right most column index of Ψ). This must be a minimal element of $\mathbb{S}^c$.

3. Find the second smallest state $\mathbf{s}_2^c$, which may also be a minimal element. Vertically stack the matrices $\Psi(\mathbf{s}_1^c)A_{\mathbf{s}_1^c}$ and $\Psi(\mathbf{s}_2^c)A_{\mathbf{s}_2^c}$, for $\Psi^c_{\{1,2\}}$.

4. $\Psi^c_{\{1,2\}}$ has rank ≥ 4. Lemma B.2 ensures that both matrices have rank 2, while lemma B.1 ensures that each row of $\Psi(\mathbf{s}_1^c)A_{\mathbf{s}_1^c}$ is linearly independent of each row of $\Psi(\mathbf{s}_2^c)A_{\mathbf{s}_2^c}$. This last point arises because lemma B.1 ensures that since $\mathbf{s}_1^c \not\succeq^* \mathbf{s}_2^c$ there must be at least one column of non-zero entries in $\Psi(\mathbf{s}_2^c)A_{\mathbf{s}_2^c}$ that matches up to an all-zero column of $\Psi(\mathbf{s}_1^c)A_{\mathbf{s}_1^c}$.

[33] This only holds for $L \geq 3$. For $L = 2$ we must also assume $E[\Gamma_1 + \Gamma_2] \neq 1$.

5. Continue this process for each non-maximal state in component $\mathbb{S}^c$. At each stage, based on the ordering of elements in $\mathbb{S}$ at step 1, and from lemmas B.2 and B.1, $\Psi(\mathbf{s}^c_n)A_{\mathbf{s}^c_n}$ must always contain at least one non-zero column that matches up to an all-zero column of $\Psi^c_{\{1,2\ldots n-1\}}$. Typically this is the furthest left column, corresponding to $\mathbf{s}^{c\,all_i}_n$. Therefore, the rank increases by at least 2 each step.

6. The final matrix $\Psi^c_{\{1,2\ldots\}}$ has non-zero entries *somewhere* in each of the $|\mathbb{S}^c|$ columns corresponding to states in this set, except for columns correspond to elements of $\tilde{\min}(\mathbb{S}^c)$. These columns are all zeros — there is always zero probability of ending in these states. As the rank of this matrix increased by $\geq$ two at each additional non-maximal state, and because we have at least as many non-maximal states as maximal states, this matrix must have rank $\geq |\mathbb{S}^c| - |\tilde{min}(\mathbb{S}^c)| - 1$. The rank cannot exceed this, and must be strictly less than $|\mathbb{S}^c| - |\tilde{min}(\mathbb{S}^c)|$ because the row sum for each row of this final matrix equals zero, a property inherited from the fact that $Q^T\boldsymbol{\iota} = 1$.

7. Any two components $\mathbb{S}^c$ and $\mathbb{S}^{c'}$ are mutually exclusive. Therefore, the two matrices for any two components $\Psi^c_{\{1,2\ldots\}}$ do not share any non-zero columns. Therefore, when we stack these matrices across different components, the ranks must sum together at each step.

8. Therefore $rank(\Psi) = \sum_{\mathbb{S}^c \subset \mathbb{S}} |\mathbb{S}^c| - |\tilde{min}(\mathbb{S}^c)| - 1 = S - |\tilde{\min}(\mathbb{S})| - S^c$

□

B.2 nullspace of Ψ

B.2.1 The $|\tilde{\min}(\mathbb{S})|$ elements

Ψ contains only zeros in columns corresponding to states in $\tilde{\min}(\mathbb{S})$. Any vector $\mathbf{y}$ containing non-zero entries only in rows corresponding to elements of this set is in this null space. Denote this set of vectors $\mathbb{Y}^1$, with $|\tilde{\min}(\mathbb{S})|$ distinct elements.

B.2.2 The S^c elements

Consider the vector $\mathbf{y}$ such that $y_{\mathbf{s}} = y_{\mathbf{s}'}$ if $\mathbf{s}$ and $\mathbf{s}'$ belong to the same component. Denote this set of vectors $\mathbb{Y}^2$, containing S^c distinct elements.

As established above, columns of the submatrix $\Psi^c_{\{1\ldots|\mathbb{S}^c|\}}$ that correspond to states in different components contain all zeros, from the definition of a component.

Therefore, for any $\mathbf{y} \in \mathbb{Y}^2$ we have $\Psi^c\mathbf{y} = 0$. Entries of $\mathbf{y}$ are constant across rows that correspond to the non-zero entries of $\Psi^c_{\{1\ldots|\mathbb{S}^c|\}}$. This holds for any c. Therefore, as we stack the $\Psi^c_{\{1\ldots|\mathbb{S}^c|\}}$s into Ψ we will have $\Psi\mathbf{y} = 0$ for any $\mathbf{y} \in \mathbb{Y}^2$.

B.3 Image of $(I_S - \beta T\Omega)^{-1}C$

I have established that the null space of Ψ is given by $\mathbb{Y}^1 \cup \mathbb{Y}^2$. I now show that the intersection of this space and the image of $(I_S - \beta T\Omega)^{-1}C$ only contains the constant vector, denoted ι_{S^i}. This result requires three additional lemmas:

B.3.1 Three Additional Lemmas

Lemma B.3. *For any $\mathbf{y} \in \mathbb{Y}^1$ we have $\Omega\mathbf{y} = 0$.*

Proof:

1. Recall that $\Omega(\mathbf{s}) = E_{\mathbf{b}}[Q(\mathbf{b}^*|\mathbf{s})^T - \Gamma(\mathbf{b}^*|\mathbf{s})^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\nabla_{\mathbf{b}}Q(\mathbf{b}^*|\mathbf{s})|\mathbf{s}]A_{\mathbf{s}}$
2. $A_{\mathbf{s}}\mathbf{y} = 0$ for $\mathbf{y} \in \mathbb{Y}^1$, since $A_{\mathbf{s}}$ selects elements of $\mathbf{y}$ corresponding to possible ex-post states, given beginning in $\mathbf{s}$. But $\mathbf{y}$ only contains non-zero entries for states that are never observed as ex-post states.

□

Lemma B.4. *For any $\mathbf{y} \in \mathbb{Y}^2$ we have $\Omega\mathbf{y} = \mathbf{y}$.*

Proof:

1. For $\mathbf{y} \in \mathbb{Y}^2$ we have $A_{\mathbf{s}}\mathbf{y} = y_{\mathbf{s}}\iota_{2^L}$, where ι_{2^L} is a $2^L \times 1$ vector of ones. This holds because $A_{\mathbf{s}}$ selects the elements of the vector $\mathbf{y}$ that correspond to states that are possible outcomes from an auction round beginning in state $\mathbf{s}$. By definition these ex-post states are all in the same component, while $\mathbf{y}$ is constant within components.
2. As the rows of $Q(\mathbf{b}^*|\mathbf{s})^T$ sum to one, we have $E_{\mathbf{b}}[Q(\mathbf{b}^*|\mathbf{s})^T|\mathbf{s}]\iota_{2^L} = \iota_{2^L}$.
3. As rows of $\nabla_{\mathbf{b}}Q(\mathbf{b}^*|\mathbf{s})$ sum to zero (derivative of a vector with rows summing to one) we have: $E[\Gamma(\mathbf{b}^*|\mathbf{s})^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\nabla_{\mathbf{b}}Q(\mathbf{b}^*|\mathbf{s})|\mathbf{s}]\iota_{2^L} = \mathbf{0}$
4. Therefore $\Omega(\mathbf{s})\mathbf{y} = y_{\mathbf{s}}\iota_{2^L}$ for $\mathbf{y} \in \mathbb{Y}^2$. Stacking over $\mathbf{s}$ yields the result.

□

Finally, for $\mathbf{y} \in \mathbb{Y}^2$ we can write $\mathbf{y} = M\bar{\mathbf{y}}$ Where $\bar{\mathbf{y}}$ is an $S^c \times 1$ vector containing the constant elements of $\mathbf{y}$ from component. Meanwhile M is an $S \times S^c$ dimensional matrix that contains a 1 in a row corresponding to state $\mathbf{s}$ and column corresponding to component c if $\mathbf{s} \in \mathbb{S}^c$, and zero otherwise. Each row of M contains a single 1.

Lemma B.5. *Let the matrix N be any $S^c \times S^c$ submatrix of $(I - \beta T)M$ that is formed by selecting one row from each of the S^c components. N is non-singular.*

Proof:

1. Select S^c states, one from each component, and denote the corresponding set of rows of by $\mathbb{M}$. The sub-matrix of interest is denoted $M_{\mathbb{M},.} - \beta T_{\mathbb{M},.} M$
2. Recognise that $M_{\mathbb{M},.} = I$. This is because we chose one row of M associated with each component. Each row of M contains a single 1, therefore so must $M_{\mathbb{M},.}$. Because every row is associated with a different component, each row contains a 1 in a different column.
3. Elements of the $S^c \times S^c$ sub matrix $T_{\mathbb{M},.} M$ are just transition probabilities, so $T_{\mathbb{M},.} M \iota_{S^c} = 1$. This is because right multiplying by M causes us to sum over states within a component. For a particular row t we have element c of the row vector $T_{t,.} M$ is equal to $\sum_{\mathbf{s}\,:\,\mathbf{s}^c = \mathbf{s}^{\tilde{c}}} P(\mathbf{s}|\mathbf{s}^t)$. That is, the probability, given ending a period in state $\mathbf{s}^t$, that they begin the next period in component c.
4. Diagonal entries of the matrix $I - \beta T_{\mathbb{M},.} M$ are strictly positive, as $\beta \times$ a probability is strictly less than 1 (for $\beta < 1$). Likewise, off diagonal entries are weakly negative, as we have $-\beta \times$ a probability. Last, rows must sum to $1 - \beta$ because rows of both I and $T_{\mathbb{M},.} M$ sum to 1.
5. This ensures this matrix is strictly diagonally dominant. Therefore, from the Levy–Desplanques theorem, the matrix must be non-singular.

□

B.3.2 Proof of Image Result

Proposition 6. $Image((I_S - \beta T\Omega)^{-1} C) \cap null(\Psi) = \iota_{S^i}$

The proof employs the result $T\iota_S = \iota_S$ (rows of a transition matrix sum to one). The proof proceeds by first demonstrating that the image of $(I_S - \beta T\Omega)^{-1} C$ does not intersect $\mathbb{Y}^1$. Next, that the intersection with $\mathbb{Y}^2$ only contains the constant vector.

Proof:

1. Suppose there exists an $\mathbf{x}$ such that for some $\mathbf{y} \in \mathbb{Y}^1$ we could write $\mathbf{y} = (I_S - \beta T\Omega)^{-1}C\mathbf{x}$. Equivalently, $(I_S - \beta T\Omega)\mathbf{y} = C\mathbf{x}$.
2. From Lemma B.3 this implies $\mathbf{y} = C\mathbf{x}$. In turn, from the definition of C this requires $\mathbf{x}$ contains zeros in every entry except the first.
3. However this cannot be the case, since we always normalise this first entry to zero. Therefore $image((I_S - \beta T\Omega)^{-1}C) \cap \mathbb{Y}^1 = \emptyset$
4. Next, Suppose there exists an $\mathbf{x}$ such that for some $\mathbf{y} \in \mathbb{Y}^2$ we could write $\mathbf{y} = (I - \beta T\Omega)^{-1}C\mathbf{x}$. Equivalently $(I - \beta T\Omega)\mathbf{y} = C\mathbf{x}$
5. From Lemma B.4 $C\mathbf{x} = (I - \beta T)\mathbf{y} = (I - \beta T)M\bar{\mathbf{y}}$. In matrix form:

$$\begin{pmatrix} M - \beta\bar{T} & -C \end{pmatrix} \begin{pmatrix} \bar{\mathbf{y}} \\ \mathbf{x} \end{pmatrix} = 0$$

   Where $\bar{T} = TM$, the probability of transitioning to any component from any state. If $(M - \beta\bar{T}, -C)$, the $S \times (S_C + S_i)$ matrix has rank $S_C + S_i - 1$ then there is a unique $\mathbf{y}$ and $\mathbf{x}$ where this relationship holds.

6. I now show the first column of $-C$ is linearly independent of $(M - \beta\bar{T})$. $-C_{.,1}$ contains -1 in every element associated with states such that $\mathbf{s}_i = \mathbf{s}_i^1$ and zeros otherwise. No linear combination of the columns for the corresponding rows of $(M - \beta\bar{T})$ can match these zeros. Choose S_c rows of $(M - \beta\bar{T})$ such that each row is associated with a state from a different component. E.g. rows such that in each component $\mathbf{s}_i = \mathbf{s}_i^{S_i}$ — the 'final' individual state. Call the corresponding $S_c \times S_c$ submatrix of $M - \beta\bar{T}$ N. From Lemma B.5 N is non-singular. No $S_c \times 1$ vector $\mathbf{z}$ exists such that $N\mathbf{z} = 0$. Therefore columns of $(M - \beta\bar{T})$ are linearly independent of $-C_{.,1}$. By concatenating this column, the rank increases by one.
7. Repeat this process for columns $n = 2 ... S_i - 1$ of $-C$. That is, every column *except* the final column which is the only column to contain non-zeros in entries associated with $\mathbf{s}_i^{S_i}$.[34] Each of these columns must be linearly independent of $M - \beta\bar{T}$ - no linear combination of its columns

[34]This assumes one individual state exists within each component (I used $\mathbf{s}_i^{S_i}$). This holds if $\mathbb{S} = \mathbb{S}^0 \times \prod_i \mathbb{S}_i$. This is not necessary — the only requirement is that at each step n I can select one state from each component such that the corresponding rows of $-C_{.,n}$ are all zero.

can match the zero entries of $-C_{.,n}$, since any $S^c \times S^c$ submatrix that consists of one row from each component must be non-singular.

8. Columns of $-C$ are linearly independent. So, at each step n the rank increases by 1. Therefore $rank(M - \beta\bar{T}, -C) \geq S_C + S_i - 1$.

9. The vector $(\bar{\mathbf{y}} = \iota_{S^c}, \mathbf{x} = (1-\beta)\iota_{S^i})$ lies in the null space of $(M-\beta\bar{T}, -C)$. This is evident since $(M - \beta\bar{T})\iota_{S^c} = (1-\beta)\iota_S$ while we also have $C(1-\beta)\iota_{S_i} = (1-\beta)\iota_S$. Therefore, applying the rank-nullity theorem ensures that $Image((I_S - \beta T\Omega)^{-1}C) \cap null(\Psi) = \iota_{S^i}$

□

C Proof of Proposition 1

In this Appendix I prove Proposition 1, which states that under the assumptions of the game, and under Conjecture 1, a Symmetric Markov Perfect Equilibrium exists.

First I prove that, conditional on Conjecture 1, a Pure Strategy Bayesian Nash Equilibrium exists in the stage game. I then show that the equilibrium pay-off in the stage game is consistent with the continuation value, employing Kakutani's fixed point theorem. This requires showing the existence, convex-valuedness, and upper hemicontinuity of the continuation value. While I assume entry is costless in my identification framework, bidders still make a strategic decision over which auctions to enter. Therefore I consider the entry game when discussing equilibrium existence.

Proof: **Equilibrium of the entry game**: player i chooses entry decision $\mathbf{d}$ to maximise their expected payoff, taking an expectation over rivals' entry decisions given their strategies. This is a standard game of incomplete information.

A symmetric equilibrium in distributional strategies exists (Milgrom and Weber, 1985). Because types are atomless, existence of a Pure Strategy equilibrium follows from their purification result. This equilibrium may not be unique, so the value function may not be continuous. Continuity arises by augmenting entry strategies to be a function of the realisation of a public random variable (Fudenberg and Maskin, 1991). Public randomisation enables players to coordinate equilibria. Conditional on this public random variable the set of equilibrium pay-offs is convex (Aumann, 1974).

Equilibrium existence of the dynamic game requires that the equilibrium pay-off in the stage game is consistent with the continuation value.[35] That is, can we write the ex-ante value function $\mathbf{V}_t^E$, stacked over $\mathbf{s}$, as a function of $\mathbf{V}_{t+1}^E$, so that $\mathbf{V}_t^E = \Omega(\mathbf{V}_{t+1}^E)$ (existence). Stationarity requires the correspondence Ω has a fixed point such that $\mathbf{V}^E = \Omega(\mathbf{V}^E)$.

Existence of $\mathbf{V}_t^E = \Omega(\mathbf{V}_{t+1}^E)$: Equation 1 writes the value function recursively. Taking an expectation over $\boldsymbol{\upsilon}_{it}$ ensures we can also write the ex-ante value function recursively. Existence then follows from the assumption that pay-offs are bounded. This ensures the set $\Omega(\mathbf{V}_{t+1}^E)$ is non-empty.

(non-)Uniqueness of $\Omega(\mathbf{V}_{t+1}^E)$: The possibility of multiple equilibria in the entry game imply the value function is non-unique. So the ex-ante value function is also non-unique. Fortunately Ω must be convex valued, as the set of equilibrium pay-offs, conditional on the public random variable, is convex.

Upper-hemi continuity of $\Omega(.)$: The continuation value is continuous in $\mathbf{V}_{t+1}^E$, taking an expectation over the transition process. Next, consider the conditional value function, conditional on entry decision $\bar{\mathbf{d}}$:

$$\tilde{W}_i(\bar{\mathbf{d}}, \boldsymbol{\upsilon}_{it}, \mathbf{s}_t; \sigma_{-i}) = \\ \max_{\mathbf{b}} \left\{ \Gamma_i(\mathbf{b}, \bar{\mathbf{d}}; \sigma_{-i})^T (\boldsymbol{\upsilon}_{it} - \mathbf{b}) + Q_i(\mathbf{b}, \bar{\mathbf{d}}; \sigma_{-i})^T [J_i(\mathbf{s}_t) + \beta V_i(\mathbf{s}_t; \sigma_{-i})] \right\}$$

Continuity of $\tilde{\mathbf{W}}_t$ in $\mathbf{V}_{t+1}^E$ is guaranteed by conjecture 1, which requires equilibrium expected pay-offs are continuous in $J_i + \beta V_i$. The value function can then be written as $W_i(\boldsymbol{\upsilon}_{it}, \mathbf{s}_t; \sigma_{-i}) = \max_{\mathbf{d}} \left\{ \tilde{W}_i(\mathbf{d}, \boldsymbol{\upsilon}_{it}, \mathbf{s}_t; \sigma_{-i}) \right\}$. Upper-hemi continuity of $\mathbf{W}_t$ in $\tilde{\mathbf{W}}_t$, and hence in $\mathbf{V}_{t+1}^E$, arises from our public random variable (Fudenberg and Maskin, 2009).[36] Upper-hemi continuity of $\mathbf{V}_t^E$ arises from the ex-ante value function taking an expectation over states.

Existence of a stationary dynamic equilibrium: In order to show existence of a stationary equilibrium we must show that there exists a fixed point

[35] Symmetry of the dynamic equilibrium arises because equilibrium in the stage game is symmetric, with strategies depending on states not identities or time periods.

[36] Public randomisation ensures that the set of equilibrium pay-offs is convex. Public randomisation means $\mathbf{W}_t$ is the convex hull of possible equilibrium pay-offs from entry, $\tilde{\mathbf{W}}_t$. Therefore, so long as $\tilde{\mathbf{W}}_t$ is compact valued, $\mathbf{W}_t$ is upper hemicontinuous (Charalambos and Aliprantis, 2013). Compact valuedness comes from pay-offs being drawn from a compact set.

of the correspondence $\mathbf{V}^E = \Omega(\mathbf{V}^E)$. As $\Omega()$ is non-empty, convex valued, and upper-hemi continuous, we can apply Kakutani's fixed point theorem. Therefore, a Markov Perfect Equilibrium exists.

□

D Extensions

D.1 Second-Price Auctions

My identification results extend, almost trivially, to second price auctions. In Appendix D.1.1 I set up the bidder's optimisation problem in the second price framework. In Appendix D.1.2 I show how optimal bidding yields First Order Conditions, and an Inverse Bid System, similar to the first-price case. In D.1.3 I extend Proposition 3 from the main text to the second-price case. In Appendix D.1.4 I prove that $\mathbf{j}$ is point identified from the same moment condition assumed in the main text.

I do not discuss estimation of the dynamic multi-object second price model. However the estimation procedure presented in Section 4 can be applied to the second price setting, making use of the inverse bid system presented below.

D.1.1 Setup

In the second price setting, player i wins lot l at time t if $b_{ilt} > \max_{i'} \{b_{i'lt}\}$. As in the text, let $\Gamma(\mathbf{b}|\mathbf{s})$ denote the $L \times 1$ equilibrium marginal probabilities of winning each lot. Define the vectors P and Q similarly. The Value Function is given by:

$$W_i(\boldsymbol{v}_{it}, \mathbf{s}_t; \sigma_{-i}) = \\ \max_{\mathbf{b}} \left\{ \Gamma_i(\mathbf{b}; \sigma_{-i})^T(\boldsymbol{v}_i - \tilde{\mathbf{b}}(\mathbf{b}; \mathbf{s}_t)) + P_i(\mathbf{b}; \sigma_{-i})^T J_i(\mathbf{s}_t) + \beta Q_i(\mathbf{b}; \sigma_{-i})^T V_i(\mathbf{s}_t; \sigma_{-i}) \right\} \tag{10}$$

Element a of the $n^L \times 1$ continuation value V_i is $V_{ia}(\mathbf{s}_t; \sigma_{-i}) = \int_{\mathbf{s}} \int_{\boldsymbol{v}} W_i(\boldsymbol{v}, \mathbf{s}; \sigma_{-i}) dF(\boldsymbol{v}|\mathbf{s}) dT(\mathbf{s}|\mathbf{s}_t^a)$. $\tilde{\mathbf{b}}(\mathbf{b}; \mathbf{s}_t)$ gives the expected second highest bid, given that b_{ilt} is the highest. Since the cdf of the highest rival bids is $\Gamma_l(x|\mathbf{s})$, we can write $\Gamma_l(b_l|\mathbf{s})\tilde{b}_l(\mathbf{b}; \mathbf{s})) = \int_{\underline{b}_l}^{b_{ilt}} \bar{b}_l \nabla_{b_l} \Gamma_l(\bar{b}_l|\mathbf{s}) d\bar{b}_l$.

D.1.2 First Order Conditions and Inverse Bid System

Rearrange the maximand for: $\Gamma(\mathbf{b}|\mathbf{s})^T\boldsymbol{\upsilon}-\sum_l\int_{\underline{b}_l}^{b_l}\bar{b}_l\nabla_{b_l}\Gamma_l(\bar{b}_l|\mathbf{s})d\bar{b}_l+P(\mathbf{b}|\mathbf{s})J(\mathbf{s})+\beta Q(\mathbf{b}|\mathbf{s})V(\mathbf{s})$

Differentiate for FOCs: $0=\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})(\boldsymbol{\upsilon}-\mathbf{b}^*)+\nabla_{\mathbf{b}}P(\mathbf{b}^*|\mathbf{s})J(\mathbf{s})+\beta\nabla_{\mathbf{b}}Q(\mathbf{b}^*|\mathbf{s})V(\mathbf{s})$.[37] We then invert the FOCs for the inverse bid system:

$$\boldsymbol{\xi}(\mathbf{b}_{it}|J,\beta V;\mathbf{s})=\mathbf{b}_{it}-\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}[\nabla_{\mathbf{b}}P(\mathbf{b}^*|\mathbf{s})B_{\mathbf{s}}\mathbf{j}+\nabla_{\mathbf{b}}Q(\mathbf{b}^*|\mathbf{s})A_{\mathbf{s}}\beta\mathbf{V}]$$

This is similar to the inverse bid system presented in text, omitting the mark-up term $\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\Gamma(\mathbf{b}^*|\mathbf{s})$. Consequently, conditional on $\mathbf{j}$ and $\beta\mathbf{V}$, the distribution of lot specific values F is point identified from the empirical quantiles of $\boldsymbol{\xi}(\mathbf{b}_{it}|J,\beta V;\mathbf{s})$.

D.1.3 Extension of Proposition 3

I now extend Proposition 3 to the second price case. Note that there are many ways I could prove this general second price identification argument. I use this structure for the purposes of outlining the similarity to the first price case.

Proposition 7. *Under assumptions 1 - 4, the expected stage pay-off is given by:*

$$\begin{aligned}\tilde{\Pi}(\mathbf{b}^*|\boldsymbol{\upsilon};\mathbf{s})=&\Gamma(\mathbf{b}^*|\mathbf{s})^T(\mathbf{b}^*-\tilde{\mathbf{b}}(\mathbf{b}^*;\mathbf{s}))\\&+[P(\mathbf{b}^*|\mathbf{s})^T\ -\ \Gamma(\mathbf{b}^*|\mathbf{s})^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\nabla_{\mathbf{b}}P(\mathbf{b}^*|\mathbf{s})]B_{\mathbf{s}}\mathbf{j}\\&+[Q(\mathbf{b}^*|\mathbf{s})^T\ -\ \Gamma(\mathbf{b}^*|\mathbf{s})^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\nabla_{\mathbf{b}}Q(\mathbf{b}^*|\mathbf{s})]A_{\mathbf{s}}\beta\mathbf{V}\end{aligned}$$

This is similar to the expression given in Proposition 3, except that the optimal lot specific surplus term is given by $\mathbf{b}^*-\tilde{\mathbf{b}}(\mathbf{b}^*;\mathbf{s})$ instead of $\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\Gamma(\mathbf{b}^*|\mathbf{s})$. Proof is omitted due to its simplicity - simply substitute the inverse bid function $\boldsymbol{\xi}(\mathbf{b}_{it}|J,\beta V;\mathbf{s})$ for $\boldsymbol{\upsilon}$ into the maximand of the value function in equation 10.

From Proposition 7, employing the identity $P(\mathbf{b}|\mathbf{s})^TB_{\mathbf{s}}=Q(\mathbf{b}|\mathbf{s})^TA_{\mathbf{s}}C$, and taking an expectation over the observed bids, write the ex-ante value function as:

$$\begin{aligned}V^e(\mathbf{s})&=\Phi(\mathbf{s})+\Omega(\mathbf{s})[C\mathbf{j}+\beta\mathbf{V}]\\\text{Where}\quad\Phi(\mathbf{s})&=E_{\mathbf{b}}[\Gamma(\mathbf{b}^*|\mathbf{s})^T(\mathbf{b}^*-\tilde{\mathbf{b}}(\mathbf{b}^*;\mathbf{s}))|\mathbf{s}]\\\Omega(\mathbf{s})&=E_{\mathbf{b}}[Q(\mathbf{b}^*|\mathbf{s})^T-\Gamma(\mathbf{b}^*|\mathbf{s})^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\nabla_{\mathbf{b}}Q(\mathbf{b}^*|\mathbf{s})|\mathbf{s}]A_{\mathbf{s}}\end{aligned}$$

[37] This condition can equivalently be derived by requiring that, at the optimum, b^*_{lt} equals the marginal expected pay-off from winning lot l, conditional on bids for lots $m\neq l$.

Stacking this equation over $\mathbf{s}$ allows us to write the continuation value as: $\mathbf{V} = T\Phi + T\Omega[C\mathbf{j} + \beta\mathbf{V}]$ Which we can invert for: $\mathbf{V} = (I_S - \beta T\Omega)^{-1}[T\Phi + T\Omega C\mathbf{j}]$. This yields a stationary solution for the continuation value. This is precisely the equation derived in the text, except I have defined the matrix $\Phi(\mathbf{s})$ slightly differently.

D.1.4 Identification

As in the main text I impose the mean zero property of $\boldsymbol{\upsilon}$ for:

$$\begin{aligned}0 = E_{\mathbf{b}^*}[\boldsymbol{\xi}(\mathbf{b}^*;\mathbf{s},(\mathbf{j},\mathbf{V}))|\mathbf{s}] &= E_{\mathbf{b}^*}[\mathbf{b}^*|\mathbf{s}] - E_{\mathbf{b}^*}[\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbf{s})^{-1}\nabla_{\mathbf{b}}Q(\mathbf{b}^*|\mathbf{s})|\mathbf{s}]A_{\mathbf{s}}[C\mathbf{j} + \beta\mathbf{V}] \\ &= \Upsilon(\mathbf{s}) - \Psi(\mathbf{s})[C\mathbf{j} + \beta\mathbf{V}]\end{aligned}$$

Again, using a slightly differently defined $\Upsilon(\mathbf{s})$ from the text. Stack over $\mathbf{s}$, and substituting in the expression for the $\mathbf{V}$ found in subsection D.1.3, we get:

$$0 = \Upsilon - \beta\Psi(I_S - \beta T\Omega)^{-1}T\Phi - \Psi(I_S - \beta T\Omega)^{-1}C\mathbf{j}$$

There is a unique solution to this system ($\mathbf{j}$ is point identified) if and only if the $LS \times S_i$ matrix $\Psi(I_S - \beta T\Omega)^{-1}C$ has rank $S_i - 1$. This matrix is the same as in the main text. Proposition 4 holds in this case as well, ensuring the rank condition.

D.2 Binding Reservation Prices

I now introduce binding reservation prices. A reservation price is binding if, in equilibrium, there is non-zero probability of winning a lot at the reservation price. This also extends to endogenous entry with zero entry costs — where reservation prices are necessary to prevent arbitrarily low bids. Binding reservation prices do not pose a substantive problem, though do introduce additional mathematical complexity.

In the presence of reservation prices a bidder with a low value may choose not to bid strictly above the reservation price. This results in corner solutions as bids clump at the reservation price. We lose point identification as the FOCs no longer point identify $\boldsymbol{\upsilon}_i$. This is a problem, even in a single object context.

The identification argument presented below diverges from the argument presented in 3. Instead, it is closer to the estimation method presented in Section 4. Identification is demonstrated in an additional step. First I show that F is (partially) identified conditional on $(J,\ V,\ \beta)$, but in particular it is partially identified conditional on $J + \beta V$. I then show that the object $j(\mathbf{s}_i) + \beta V(\mathbf{s})$ is partially identified, (for some $\mathbf{s}_i$ it is only bounded). This is shown using quantile moment conditions: Instead finding the $j + \beta V$ such that $E[\boldsymbol{\xi}(\mathbf{b}; \mathbf{s}, j + \beta V)|\mathbf{s}] = 0$ I find it such that $P(\xi_l(\mathbf{b}; \mathbf{s}, j + \beta V) \leq 0|\mathbf{s}) = 0.5$, imposing a zero conditional median assumption. Finally, I show that conditional on the identification of F and $J + \beta V$, V is identified, and hence J can be backed out given an assumption about β.

D.2.1 Changes to the Model

Denote the reservation price as R. This could vary across lots, bidders, and time. Denote player i's entry decisions as the vector $\mathbf{d}_{it}$ with entry $d_{itl} = 1$ if they enter lot l, and zero otherwise. Adjust objects G, Γ, P and Q to be functions of bids and entry — if a player does not enter a lot, they trivially lose that lot with probability 1. Identification requires one additional assumption:

Assumption 6. $\frac{\partial \Gamma_{il}(\mathbf{b}_i, \mathbf{d}_i|\mathbf{s})}{\partial b_{im}} = 0$ for $m \neq l$

I assume that the probability an individual wins any given lot, conditional on the state and rivals' equilibrium strategies, only depends on their bid for that lot. This implies $\nabla \Gamma_i(\mathbf{b}_i, \mathbf{d}_i|\mathbf{s})$ is a diagonal matrix. This assumption was not previously necessary for identification. If ties happen with zero probability or if tie breaking is exogenous, then this assumption will hold.[38] Finally, I assume the lot specific values have zero conditional median, replacing the previous zero conditional mean assumption. I am then able to prove the following:

Proposition 8. *Given assumption 1, 2, 3, 4, and 6, both $F_i(.|\mathbf{s})$ and $K_i(\mathbf{s})$ are non-parametrically partially identified. $k(\mathbf{s}^a)$ is point identified if we observe the individual bidding $b > R$ on a lot that may yield pay-off $k(\mathbf{s}^a)$.*

That is, we will point identify the truncated distribution $F_i(.|\boldsymbol{v} >= A_1(\mathbf{b}^*, \mathbf{d}^*, \mathbf{s}); \mathbf{s})$,

[38] For mathematical convenience I assume ties occur in equilibrium with zero probability. The argument below can be easily extended to allow for ties at the reservation price. All that changes is that it introduces a discontinuity in the inverse bidding system at the reservation price, so that as the bidder goes from bidding the reserve to just above it, their payoff changes discontinuously. This slightly changes how we identify F, as we must essentially introduce an additional discrete choice of whether the bidder bids the reservation compared to bidding just above it. This additional discrete choice then restores the (upper-hemi) continuity of equilibrium, payoffs.

as well as the objects $F_i(A_1(\mathbf{b}^*, \mathbf{d}^*, \mathbf{s}); \mathbf{s}) - F_i(A_2(\mathbf{b}^*, \mathbf{d}^*, \mathbf{s}); \mathbf{s})$ and $F_i(A_2(\mathbf{b}^*, \mathbf{s}); \mathbf{s})$ for some (known) $A_1(\mathbf{b}^*, \mathbf{d}^*, \mathbf{s}), A_2(\mathbf{b}^*, \mathbf{d}^*, \mathbf{s})$.

While I assume players play pure strategies conditional on entry, I must allow for the possibility that players play mixed strategies in their entry decisions. However, I am able to use bidders' entry decisions to bound the pay-offs of un-entered auctions. I exploit the fact that, at the equilibrium mixing strategy, players can not *strictly* prefer to enter any other combination of auctions.

D.2.2 Identification of F, conditional on K.

I now show that under assumptions 1, 2, 3, 4, and 6, and conditional on K being point identified, the cdf F is non-parametrically partially identified. I do this by by proving that, similar to case 6.3.1.2 described in Athey and Haile (2007), we can invert observed bids, point identifying υ_l such that $b_l > R$. Meanwhile, for bids at the reservation price, so that $b_l = R$, we can use the first order conditions from the Lagrangian to find an upper bound on υ_l. Then, that they still prefer to enter this auction and bid low, than not enter, implies a lower bound on υ_l. Lastly, for unentered auctions we exploit that at the margin, they prefer not to enter than to enter and bid low, yielding an upper bound on these υ_l.

First, reformulate the problem to include entry decisions. The player's problem is to decide which auctions to enter ($\mathbf{d}$), then set their bids ($\mathbf{b}$) to maximise payoffs, subject to their bids being weakly above reservation prices. The Lagrangian and corresponding FOCs for this problem, conditional on entry $\mathbf{d}^*$, is given as:

$$L(\mathbf{b}, \mathbf{d}^*, \boldsymbol{\upsilon}, \boldsymbol{\lambda}|\mathbf{s}) = \Gamma(\mathbf{b}, \mathbf{d}^*|\mathbf{s})^T(\boldsymbol{\upsilon} - \mathbf{b}) + P(\mathbf{b}, \mathbf{d}^*|\mathbf{s})^T K + \boldsymbol{\lambda}^T(\mathbf{b} - R)$$

$$0 = \nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})(\boldsymbol{\upsilon} - \mathbf{b}^*) - \Gamma(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s}) + \nabla_{\mathbf{b}}P(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})^T K + \boldsymbol{\lambda}^*$$

Entry ll of $\nabla_{\mathbf{b}}\Gamma(\mathbf{b}, \mathbf{d}|\mathbf{s})$ and entry la of $\nabla_{\mathbf{b}}P(\mathbf{b}, \mathbf{d}|\mathbf{s})$ are as they were in section 3 if $d_l = 1$, and normalised to 0 otherwise. Rearrange this equation for:

$$\begin{aligned}\boldsymbol{\xi}(\mathbf{b}^*, \mathbf{d}^*, \boldsymbol{\lambda}|K; \mathbf{s}) = \mathbf{b}^* + \nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})^{-1}[\Gamma(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s}) - \nabla_{\mathbf{b}}P(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})K] \\ - \nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})^{-1}[\boldsymbol{\lambda}^*]\end{aligned}$$

At the true K we have $\xi_l(\mathbf{b}^*, \mathbf{d}^*, \boldsymbol{\lambda}^*|K; \mathbf{s}) = \upsilon_l$. But we do not observe $\boldsymbol{\lambda}^*$. Therefore, define $\boldsymbol{\xi}(\mathbf{b}^*, \mathbf{d}^*|K; \mathbf{s}) = \mathbf{b}^* + \nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})^{-1}[\Gamma(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s}) - \nabla_{\mathbf{b}}P(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})K]$. Next, I consider what can be inferred for each of the four possible entry/bidding possibilities: *i)* $b_l > R$, *ii)* $b_l = R$, *iii)* $d_l = 0$, and the null case $l \notin \mathbb{L}$.

i) l **such that** $b_l^* > R$**:**

Any entry l such that $b_l^* > R_l$, $\lambda_l^* = 0$. By Assumption 6, entry l of $\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})^{-1}[\boldsymbol{\lambda}^*]$ equals zero, and $\xi_l(\mathbf{b}^*, \mathbf{d}^*|K; \mathbf{s}) = \upsilon_l$ is point identified.

ii) l **such that** $b_l^* = R$**:**

For entry l with $b_l^* = R_l$, $\lambda_l^* > 0$. From Assumption 6 entry l of $\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})^{-1}[\boldsymbol{\lambda}^*]$ is greater than zero, and we attain the following bound:

$$\upsilon_l \leq \xi_l(\mathbf{b}^*, \mathbf{d}^*|K; \mathbf{s}) = R_l + \nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})^{-1}[\Gamma(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s}) - \nabla_{\mathbf{b}}P(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})K]_l$$

For vector M, $[M]_l$ denotes row l. As $(\mathbf{b}^*, \mathbf{d}^*)$ maximises expected payoffs, payoffs are (weakly) higher from playing $(\mathbf{b}^*, \mathbf{d}^*)$ than not entering auction l, playing $(\mathbf{b}^{l-}, \mathbf{d}^{l-})$ (the only difference between these actions is that $d_l^{l-} = 0$). Therefore:

$$\Gamma(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})^T(\boldsymbol{\upsilon} - \mathbf{b}^*) + P(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})^T K \geq \Gamma(\mathbf{b}^{l-}, \mathbf{d}^{l-}|\mathbf{s})^T(\boldsymbol{\upsilon} - \mathbf{b}^{l-}) + P(\mathbf{b}^{l-}, \mathbf{d}^{l-}|\mathbf{s})^T K$$

Which rearranges for: $\Gamma_l(b_l^*, d_l^*|\mathbf{s})(\upsilon_l - R_l) + [P(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s}) - P(\mathbf{b}^{l-}, \mathbf{d}^{l-}|\mathbf{s})]^T K \geq 0$, and hence $\upsilon_l \geq R_l - \frac{1}{\Gamma_l(b_l^*, d_l^*|\mathbf{s})}[P(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s}) - P(\mathbf{b}^{l-}, \mathbf{d}^{l-}|\mathbf{s})]^T K$.

iii) l **such that** $d_l^* = 0$**:**

Consider l such that $d_l = 0$. They must attain a greater payoff from not bidding than from bidding the reservation price. Consider alternate action $(\mathbf{b}^{l+}, \mathbf{d}^{l+})$ where the only difference between this and $(\mathbf{b}^*, \mathbf{d}^*)$ is that $b_l^{l+} = R_l$ and $d_l^{l+} = 1$. Therefore:

$$\Gamma(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})^T(\boldsymbol{\upsilon} - \mathbf{b}^*) + P(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s})^T K \geq \Gamma(\mathbf{b}^{l+}, \mathbf{d}^{l+}|\mathbf{s})^T(\boldsymbol{\upsilon} - \mathbf{b}^{l+}) + P(\mathbf{b}^{l+}, \mathbf{d}^{l+}|\mathbf{s})^T K$$

Rearranging for: $-\Gamma_l(b_l^{l+}, d_l^{l+}|\mathbf{s})(\upsilon_l - R_l) + [P(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s}) - P(\mathbf{b}^{l+}, \mathbf{d}^{l+}|\mathbf{s})]^T K \geq 0$ Yielding the bound: $\upsilon_l < R_l - \frac{1}{\Gamma_l(b_l^{l+}, d_l^{l+}|\mathbf{s})}[P(\mathbf{b}^*, \mathbf{d}^*|\mathbf{s}) - P(\mathbf{b}^{l+}, \mathbf{d}^{l+}|\mathbf{s})]^T K$

D.2.3 Identification of k under binding reservation prices

I now prove that, under assumptions 1, 2, 3, 4, and 6, the function k is non-parametrically partially identified up to standard normalisations. $k(\bar{\mathbf{s}})$ is point identified at $\mathbf{s} = \bar{\mathbf{s}}$ if we observe bidding strictly above R on a combination of goods that would have the outcome $\mathbf{s}^a = \bar{\mathbf{s}}$. I prove this by exploiting multiple observations for every state to establish a necessary rank condition, similar to the one presented in Section 3. This enables me to overcome the inherent order condition. Whereas the previous proof employed a condition on the mean of $\boldsymbol{\xi}(\mathbf{b}, \mathbf{d})$, this proof employs a condition on the marginal quantiles of $\boldsymbol{\xi}(\mathbf{b}, \mathbf{d})$. I set $k(\mathbf{s})$ such that the median (or some other quantile) is equal to zero. As above, binding reservation prices cause our FOCs to break down, so that even at the true $\mathbf{k}$ $(= C\mathbf{j} + \beta\mathbf{V})$ we can only write:

$$\boldsymbol{\upsilon} \leq \boldsymbol{\xi}(\mathbf{b}, \mathbf{d}|k; \mathbf{s}) = \mathbf{b} + \nabla_{\mathbf{b}}\Gamma(\mathbf{b}, \mathbf{d}|\mathbf{s})^{-1}[\Gamma(\mathbf{b}, \mathbf{d}|\mathbf{s}) - \nabla_{\mathbf{b}}P(\mathbf{b}, \mathbf{d}|\mathbf{s})A_{\mathbf{s}}\mathbf{k}]$$

Which only holds with equality for rows l with $b_l > R$. Stack these over $\mathbf{s}$ for:

$$\underbrace{\underline{\boldsymbol{\upsilon}}}_{LS\times 1} \leq \underline{\boldsymbol{\xi}}(\underline{\mathbf{b}}, \underline{\mathbf{d}}|k) = \underbrace{\underline{\mathbf{b}}}_{LS\times 1} + \underbrace{\nabla_{\underline{\mathbf{b}}}\underline{\Gamma}(\underline{\mathbf{b}}, \underline{\mathbf{d}})^{-1}}_{LS\times LS}[\underbrace{\underline{\Gamma}(\underline{\mathbf{b}}, \underline{\mathbf{d}})}_{LS\times 1} - \underbrace{\nabla_{\underline{\mathbf{b}}}\underline{P}(\underline{\mathbf{b}}, \underline{\mathbf{d}})}_{LS\times S}\mathbf{k}] \tag{11}$$

$$\underline{\boldsymbol{\xi}}(\underline{\mathbf{b}}, \underline{\mathbf{d}}|k) = \begin{pmatrix} \boldsymbol{\xi}(\mathbf{b}_1, \mathbf{d}_1|k; \mathbf{s}_1) \\ \vdots \\ \boldsymbol{\xi}(\mathbf{b}_S, \mathbf{d}_S|k; \mathbf{s}_S) \end{pmatrix} \qquad \underline{\mathbf{b}} = \begin{pmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_S \end{pmatrix}$$

$$\underline{\Gamma}(\underline{\mathbf{b}}, \underline{\mathbf{d}}) = \begin{pmatrix} \Gamma(\mathbf{b}_1, \mathbf{d}_1|\mathbf{s}_1) \\ \vdots \\ \Gamma(\mathbf{b}_S, \mathbf{d}_S|\mathbf{s}_S) \end{pmatrix} \qquad \nabla_{\underline{\mathbf{b}}}\underline{P}(\underline{\mathbf{b}}, \underline{\mathbf{d}}) = \begin{pmatrix} \nabla_{\mathbf{b}}P(\mathbf{b}_1, \mathbf{d}_1|\mathbf{s}_1)A_{\mathbf{s}_1} \\ \vdots \\ \nabla_{\mathbf{b}}P(\mathbf{b}_S, \mathbf{d}_S|\mathbf{s}_S)A_{\mathbf{s}_S} \end{pmatrix}$$

I require a rank condition on $\nabla_{\underline{\mathbf{b}}}\underline{\Gamma}(\underline{\mathbf{b}}, \underline{\mathbf{d}})^{-1}\nabla_{\underline{\mathbf{b}}}\underline{P}(\underline{\mathbf{b}}, \underline{\mathbf{d}})$. If this has full rank then each $\underline{\boldsymbol{\xi}}$ implies a unique $\mathbf{k}$, so that if I observed just one observation of $\underline{\boldsymbol{\upsilon}}$ I could solve for $\mathbf{k}$. Note that $E[\nabla_{\underline{\mathbf{b}}}\underline{\Gamma}(\underline{\mathbf{b}}, \underline{\mathbf{d}})^{-1}\nabla_{\underline{\mathbf{b}}}\underline{P}(\underline{\mathbf{b}}, \underline{\mathbf{d}})] = \Psi$, the matrix presented in text. Importantly, the proof presented in B.1, that $Rank(\Psi) = S - S^c - |\tilde{\min}(\mathbb{S})|$ extends trivially to $\nabla_{\underline{\mathbf{b}}}\underline{\Gamma}(\underline{\mathbf{b}}, \underline{\mathbf{d}})^{-1}\nabla_{\underline{\mathbf{b}}}\underline{P}(\underline{\mathbf{b}}, \underline{\mathbf{d}})$. The proof never exploited the fact we had taken an expectation, and entirely used the partial ordering structure of the state space.

With binding reservation prices and entry, certain states may never be outcomes that *could have* occurred with positive probability, so the corresponding elements of $\mathbf{k}$ are not point identified. These entries of $\mathbf{k}$ do not appear in the above equation, having a coefficient of zero. These states will only be partially identified.

Next, fix an $LS \times 1$ vector of probabilities $\mathbf{p}$.By definition of the marginal CDF, the following relationship holds:

$$\begin{pmatrix} p_1 \\ \vdots \\ p_{LS} \end{pmatrix} = \begin{pmatrix} F_1(\tilde{\upsilon}_1|\mathbf{s}_1) \\ \vdots \\ F_L(\tilde{\upsilon}_{LS}|\mathbf{s}_S) \end{pmatrix} = \begin{pmatrix} E_{\upsilon_1}[\ \mathbb{I}[\upsilon_1 \leq \tilde{\upsilon}_1]\ \ |\mathbf{s}_1] \\ \vdots \\ E_{\upsilon_L}[\ \mathbb{I}[\upsilon_L \leq \tilde{\upsilon}_{LS}]\ \ |\mathbf{s}_S] \end{pmatrix}$$

Employ a change of variables, taking expectations over the observed random variables $(\mathbf{b}, \mathbf{d})$ instead of υ_l. This change is only valid for state-lot combinations such that when $\upsilon_l = \tilde{\upsilon}_l$, $b_l > R$, when $\xi_l(\mathbf{b}, \mathbf{d}; k) = \upsilon_l$ holds with equality and the mapping from $\mathbf{b}$ to υ_l is continuous, smooth, and monotonic.[39] Drop rows where this condition fails, as we lose identifiability of corresponding elements of $\mathbf{k}$. If, even when υ_l is as large as $\tilde{\upsilon}_l$, the elements of $K(\mathbf{s})$ corresponding to winning lot l are so small that they never bid strictly above R on lot l, these elements of $K(\mathbf{s})$ are not identified.

This change of variables yields:

$$\mathbf{p} = \begin{pmatrix} E_{\upsilon_1}[\ \mathbb{I}[\upsilon_1 \leq \tilde{\upsilon}_1]\ \ |\mathbf{s}_1] \\ \vdots \\ E_{\upsilon_L}[\ \mathbb{I}[\upsilon_L \leq \tilde{\upsilon}_{LS}]\ \ |\mathbf{s}_S] \end{pmatrix} = \begin{pmatrix} E_{\mathbf{b},\mathbf{d}}[\ \mathbb{I}[\xi_1(\mathbf{b}_1, \mathbf{d}_1; k) \leq \tilde{\upsilon}_1]\ \ |\mathbf{s}_1] \\ \vdots \\ E_{\mathbf{b},\mathbf{d}}[\ \mathbb{I}[\xi_L(\mathbf{b}_S, \mathbf{d}_S; k) \leq \tilde{\upsilon}_{LS}]\ \ |\mathbf{s}_S] \end{pmatrix}$$

Proving point identification of $\mathbf{k}$ requires we show that the $\mathbf{p}$th quantiles of $\underline{\boldsymbol{\xi}}(\underline{\mathbf{B}}, \underline{\mathbf{D}}|k)$ equals $\tilde{\boldsymbol{\upsilon}}$ only at the true $\mathbf{k}$. But, from our rank condition, a unique $\underline{\boldsymbol{\xi}}(\underline{\mathbf{B}}, \underline{\mathbf{D}}|k)$ implies a unique $\mathbf{k}$. Therefore, only a unique $\mathbf{k}$ is such that the $\mathbf{p}$th quantiles of $\underline{\boldsymbol{\xi}}(\underline{\mathbf{B}}, \underline{\mathbf{D}}|k)$ equals $\tilde{\boldsymbol{\upsilon}}$. Therefore, there exists a unique $\mathbf{k}$ such that this equation holds.[40]

[39] This is essentially an application of the Law of the Unconscious Statistician. Monotonicity of the inverse bid function for bids strictly above the reservation price is discussed in A.

[40] It should be noted that $\mathbf{k}$ is unique up to $|\tilde{min}(\mathbb{S})| + S^c$ elements of $\mathbf{k}$ that must be normalised to to the rank deficiency of the matrix Ψ. These elements are the entries associate with states $\mathbf{s} \in \tilde{min}(\mathbb{S})$ that are never observed as possible ex-post states, and one additional state from each component - associated with $\mathbf{s}_i = \mathbf{s}_i^1$. We will see in Appendix D.2.4 that these normalisations do not impact the identification of $\mathbf{j}$.

D.2.4 Identification of j and βV

Up to this point I have proven the non-parametric (partial) identification of F_i and $K_i = J_i + \beta V_i$. We also previously established that the ex-ante value function is just a function of beliefs, F_i, and $K_i = J_i + \beta V_i$, all of which are identified. Therefore so too is the ex-ante value function. The continuation value V is then just a function of the ex-ante value function and the transition process, both of which we have established are identified. Finally, fixing β, J_i is a function of K_i, β, and V_i, ensuring that J_i is also non-parametrically partially identified.[41]

D.3 Endogenous Entry

In this Appendix I introduce endogenous entry in which entry is costly and v_{ilt} is not observed before entry, though I assume that the entry decisions of other players is observed before bidding.[42] I focus on the case with non-binding reservation prices, though it will be clear how the results from Appendix D.2 extend to this case.

The identification argument presents a minor generalisation on the one presented in the main text. The argument proceeds as follows: F is non-parametrically point identified conditional on $\mathbf{k} = C\mathbf{j} + \beta\mathbf{V}$. As in the previous Appendix, $\mathbf{k}$ remains non-parametrically identified conditional on the identification of Γ and P using observed variation in $\mathbf{s}$, relying on our rank condition on the matrix Ψ. Given identification of $\mathbf{k}, \Gamma$, and P, Proposition 3 ensures that the expected payoff from each entry structure is also non-parametrically identified. Given these expected payoffs, the entry problem is then a multinomial discrete choice problem, so I rely on standard results for the identification of entry costs. Identification of expected entry payoffs and costs ensures the ex-ante value function, and hence the continuation value $\mathbf{V}$, is identified, thereby identifying $\mathbf{j} = C^{-1}(\mathbf{k} - \beta\mathbf{V})$.

[41] If there are values of $J_i(\mathbf{s}_t) + \beta V_i(\mathbf{s}_t)$ that were only ever bid on at the reservation price, then the value function is only partially identified. However, this non-identified region will generally be very small. Likewise, elements of k corresponding to states which never appear as possible ex-post states will be trivially zeroed out in this equation, so it does not matter how they are normalised. Finally, the normalised elements corresponding to one (minimal, with $\mathbf{s}_i = \mathbf{s}_i^0$) element from each component $\mathbb{S}^c \subset \mathbb{S}$. These normalisations constitute location shifts of Π for all elements in that component, as we essentially made the normalisations because only marginal payoffs are identified. Finally, when we back out $\mathbf{j}$, we will normalise $j(\mathbf{s}_i^0) = 0$, in line with these location normalisations.

[42] Allowing the 'entry structure' to be unknown before bidding does not change anything substantive, since we simply have to alter the objects Γ_l P and Q to additionally take an expectation over the entry decisions of other players.

I proceeds as follows: In Appendix D.3.1 I introduce changes to the main model, and demonstrate that the previous identification results for F and $\mathbf{k}$ also apply. In Appendix D.3.2 I show that the distribution of entry costs is non-parametrically identified, and finally that $\mathbf{V}$, and hence $\mathbf{j}$ are also identified.

D.3.1 Changes to the Model

All objects below should be treated as functions of the state $\mathbf{s}$. Conditional on an entry structure $\mathbb{D}$ and having observed the lot specific values $\boldsymbol{v}$ the agent places bids to maximise the following:

$$\Pi(\mathbf{b}|\boldsymbol{v};\mathbb{D}) = \Gamma(\mathbf{b}|\mathbb{D})^T(\boldsymbol{v}-\mathbf{b}) + P(\mathbf{b}|\mathbb{D})^T J + Q(\mathbf{b}|\mathbb{D})^T \beta V$$

Given the agent's behaviour conditional on entry, the agent's problem is to choose an entry structure $\mathbb{D}_i$ to maximise their expected pay-off. I assume that agent's entry costs, a $2^L \times 1$ vector $\mathbf{c}$, are drawn independently and privately from $C(.|\mathbf{s}_i)$ (independent of $\mathbf{s}_{-i}$). I assume that C is common knowledge.

The agent observes $\mathbf{s}$ and, given knowledge of F and $\mathbf{k}$ and their equilibrium beliefs about other players, forms and maximises an expected pay-off associated with any given entry structure:

$$W(\mathbb{D}_i|\mathbf{c}) = E_{\mathbb{D}_{-i}}[E_{\boldsymbol{v}}[\underset{\mathbf{b}}{max}\ \{\Pi(\mathbf{b}|\boldsymbol{v};\mathbb{D})\}\,]|\mathbb{D}_i] - c_{\mathbb{D}_i}$$

The continuation value associated with ending the period in state $\mathbf{s}^a$ is then:

$$V(\mathbf{s}^a) = E_{\mathbf{s}}[E_{\mathbf{c}}[\max_{\mathbb{D}_i}\{W(\mathbb{D}_i|\mathbf{c})\}\,|\mathbf{s}]|\mathbf{s}^a]$$

Identification of F conditional on the identification of K

The Inverse Bid System, as given in equation 4, where the state variable has simply been augmented to include the entry structure. Hence F remains non-parametrically identified conditional on the identification of Γ, Q, and $\mathbf{k}$.

Identification of k

As in the main text, we can take a conditional expectation of the inverse bid system, setting this equal to zero: $E[\boldsymbol{\xi}|\mathbf{s},\mathbb{D}] = 0$. We can then again stack this system of equations across states and entry structures for $0 = \Upsilon - \Psi\mathbf{k}$. Non-parametric point identification of $\mathbf{k}$ then requires the same rank condition on Ψ proven previously.[43]

Identification of $E_{\boldsymbol{\upsilon}}[\tilde{\Pi}(\mathbf{b}^*|\boldsymbol{\upsilon};\mathbf{s},\mathbb{D})]$

Recognise that Proposition 3 continues to hold, and so we can write the expected maximised payoff, conditional on $\mathbb{D}$, as

$$\begin{aligned}\bar{\Pi}(\mathbf{s},\mathbb{D}) = E_{\boldsymbol{\upsilon}}[\tilde{\Pi}(\mathbf{b}^*|\boldsymbol{\upsilon};\mathbb{D})] =&\Gamma(\mathbf{b}^*|\mathbb{D})^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbb{D})^{-1}\Gamma(\mathbf{b}^*|\mathbb{D}) \\ &+ [Q(\mathbf{b}^*|\mathbb{D})^T \; - \; \Gamma(\mathbf{b}^*|\mathbb{D})^T\nabla_{\mathbf{b}}\Gamma(\mathbf{b}^*|\mathbb{D})^{-1}\nabla_{\mathbf{b}}Q(\mathbf{b}^*|\mathbb{D})]A_{\mathbf{s}}\beta\mathbf{k}\end{aligned}$$

D.3.2 Identification of C

At the entry stage, the agent sets their entry structure $\mathbb{D}_i$ such that:

$$E_{\mathbb{D}_{-i}}[E_{\boldsymbol{\upsilon}}[\underset{\mathbf{b}}{max} \;\; \{\Pi(\mathbf{b}|\boldsymbol{\upsilon};\mathbb{D})\}\,]|\mathbb{D}_i] - c_{\mathbb{D}_i} \geq \underset{\mathbb{D}_i'\neq\mathbb{D}_i}{max}\left\{E_{\mathbb{D}_{-i}}[E_{\boldsymbol{\upsilon}}[\underset{\mathbf{b}}{max} \;\; \{\Pi(\mathbf{b}|\boldsymbol{\upsilon};\mathbb{D})\}\,]|\mathbb{D}_i'] - c_{\mathbb{D}_i'})\right\}$$

Similar to how we identify G, because we observe entry decisions, we therefore observe the equilibrium distribution of $\mathbb{D}_i$ for all i. Therefore, following from the above, $E_{\mathbb{D}_{-i}}[E_{\boldsymbol{\upsilon}}[\underset{\mathbf{b}}{max} \;\; \{\Pi(\mathbf{b}|\boldsymbol{\upsilon};\mathbb{D})\}\,]|\mathbb{D}_i]$ is non-parametrically point identified. Normalising that the entry cost of entering zero auctions is always zero, I now exploit the exclusion restriction that the distribution of $\mathbf{c}$ is independent of $\mathbf{s}_{-i}$. Therefore, variation in $\mathbf{s}_{-i}$ leads to known variation in $E_{\mathbb{D}_{-i}}[E_{\boldsymbol{\upsilon}}[\underset{\mathbf{b}}{max} \;\; \{\Pi(\mathbf{b}|\boldsymbol{\upsilon};\mathbb{D})\}\,]|\mathbb{D}_i]$, thereby tracing out the distribution $C(.|\mathbf{s}_i)$, ensuring we have non-parametric identification.[44]

The ex-ante value function $V^e(\mathbf{s}) = E[\underset{\mathbb{D}_i}{max}\left\{E_{\mathbb{D}_{-i}}[E_{\boldsymbol{\upsilon}}[\underset{\mathbf{b}}{max} \;\; \{\Pi(\mathbf{b}|\boldsymbol{\upsilon};\mathbb{D})\}\,]|\mathbb{D}_i] - c_{\mathbb{D}_i})\right\}]$, and hence the continuation value $V(\mathbf{s})$ are then also non-parametrically identified, which in turn yields identification of the flow payoff function j.

[43] We must normalise elements of $\mathbf{k}$ that correspond to states which are either the minimal element of their component (connected subset of $\mathbb{S}$, connected by our partial ordering), or never appear as possible ex-post states. By definition, there will be $S^c + |min(\mathbb{S})|$ of these. In Appendix B.1 we found previously that Ψ has rank $S - S^c - |min(\mathbb{S})|$.

[44] Technically, identification is partial: The set of states is finite, so we will only actually be point identifying $C(.|\mathbf{s}_i)$ at a finite set of points across its support. We can achieve full point identification either by assuming discrete support, or introducing one continuously varying element of $\mathbf{s}_{-i}$.

D.4 Stochastic Combination Value

I now present two identification results for the case when the combination value is stochastic, when $j(\mathbf{s})$ is not a function but a probability distribution. I focus on the static setting for two reasons. First, these results are novel even in the static case. Second, as we have seen throughout this paper, identification of the primitives of a generalised static model (where primitives are allowed to depend on $\mathbf{s}_0$ and $\mathbf{s}_{-i}$), is sufficient for identification of the primitives of a dynamic model. This is because identification of the Pseudo-Static payoff function k implies identification of j.

I focus on two cases: First, when J is a function of low-dimensional un-observables M, such as stocks, where $M \leq L$.[45] Second, I consider a case when $M > L$, but elements of the unobservable vector are constant over time (e.g. constant parameters).

These extensions both centre on the theme of finding some way to reduce the dimensionality of the unknowns. The key idea is this: Each observation of bidding on an auction yields L pieces of information. Therefore, in order to have any hope at point identifying unobservables, there cannot be more than L unobservables. However, as in the main text, we can combine observations of bidding across period (or bidders) to identify unobservables that remain constant across the observations.

D.4.1 Case 1: Known function of low dimensional un-observables

Suppose the combinatorial value can be written as $\mathbf{J}(\mathbf{m}_t)$ where $\mathbf{m}_t \in \mathbb{M}$ is an unobserved (potentially) stochastic random variable of dimension $M \leq L$. I require that $\mathbf{J} : \mathbb{M} \to \mathbb{J}$ is a known function (with range $\mathbb{J} \subset \mathbb{R}^{2^L}$). Importantly, some elements of $\mathbf{m}$ may represent fixed parameters associated with the functional form J.

Normalise the first element of this vector valued function (corresponding to player i losing every lot) to zero, so that I focus on the marginal combinatorial pay-off $\mathbf{J}(\mathbf{m})_{2:2^L} - \mathbf{J}(\mathbf{m})_1$. The expected payoff is $\Pi(\mathbf{b}) = P(\mathbf{b})^T \mathbf{J}(\mathbf{m}_t) - \Gamma(\mathbf{b})^T \mathbf{b}$. Necessary first order conditions are then given by:

$$0 = \nabla_{\mathbf{b}} P(\mathbf{b}) \mathbf{J}(\mathbf{m}_t) - \nabla_{\mathbf{b}} \Gamma(\mathbf{b}) \mathbf{b} - \Gamma(\mathbf{b})$$

[45] Importantly, in this case, for the dynamic setting, we will require that the continuation value, and hence k, does not depend on the unobservables of the other players. So, if we are thinking about an unobserved stock model, we will require that the continuation value for player i does not depend on the unobserved stocks of player j. This will be the case if, for example, the distribution of equilibrium winning bids is independent of player's stocks.

Because $\mathbf{m}$ is the only non-identified object in this equation, the problem is to show this is point identified. I make two assumptions about this function that are sufficient for $\mathbf{m}_t$ to be point identified:

Assumption 7.

1. $\mathbf{J}(\mathbf{m})$ is continuous and continuously differentiable for all $\mathbf{m}_t$.
2. For any $\mathbf{m}$ and $\mathbf{m}'$ there exists a set $\mathbb{U} \subset \{1, 2, \ldots, 2^L\}$ with $|\mathbb{U}| = M$ that defines the vector value function $\mathbf{F}^{\mathbb{U}}$ where $F_n^{\mathbb{U}}(\mathbf{m}) = J_{U_n}(\mathbf{m})$ such that

$$(\mathbf{m} - \mathbf{m}')^T(\mathbf{F}^{\mathbb{U}}(\mathbf{m}) - \mathbf{F}^{\mathbb{U}}(\mathbf{m}')) > 0$$

The second part of this assumption is essentially an extension of strict monotonicity to the case of 2^L dimensional functions in M dimensional variables. The assumption states that for any two distinct $\mathbf{m}$ and $\mathbf{m}'$ we can find a set of rows of $\mathbf{J}(.)$ such that this inner product is strictly positive.[46] A key result of this property is that the function $\mathbf{J}(.)$ is a bijection: Each $\mathbf{m}$ maps onto a unique $\mathbf{J}$, and the condition ensures that for any two distinct $\mathbf{m}$ and $\mathbf{m}'$ it must be the case that $\mathbf{J}(\mathbf{m}) \neq \mathbf{J}(\mathbf{m}')$ (since otherwise we could not find a $\mathbb{U}$ such that $(\mathbf{m} - \mathbf{m}')^T(\mathbf{F}^{\mathbb{U}}(\mathbf{m}) - \mathbf{F}^{\mathbb{U}}(\mathbf{m}')) > 0$). This ensures that the inverse $\mathbf{J}^{-1}(.)$ exists, such that for all $\mathbf{m} \in \mathbb{M}$ $\mathbf{m} = \mathbf{J}^{-1}(\mathbf{J}(\mathbf{m}))$. Furthermore, because $\mathbf{J}(.)$ is continuous and continuously differentiable everywhere, so that $\mathbf{J}^{-1}(.)$ must be differentiable everywhere, $\mathbf{J}^{-1}(.)$ must also be continuous.

Proposition 9. *Under assumptions 1, 2, and 7, $\mathbf{m}_t$ is identified up to normalisation.*

For example, if the second a third elements of $\mathbf{m}_t$ are parameters describing the mean and standard deviation of m_{1t}, then $\mathbf{m}_t$ is identified up to location and scale.

The proof requires arguing that with L equations in only M unknowns there exists a unique solution to the system. The proof proceeds by recognising that the set of vectors $\mathbf{J}$ which satisfy the FOCs is convex. From the continuity of the inverse function $\mathbf{J}^{-1}(.)$ and the (generalised) intermediate value theorem, this implies that the set of $\mathbf{m}$ for which the FOCs hold is path connected. So, there must be a point arbitrarily close to $\mathbf{m}_t$ for which the FOCs hold. However, that $\nabla_{\mathbf{b}}P(\mathbf{b})$ has rank L and the function $\mathbf{J}(.)$ is invertible implies the system is locally unique.

[46]This property is satisfied when, for example, each element of J is weakly monotone in elements of $\mathbf{m}$, and strictly monotonic in at least one element.

Proof: 1. Consider the set of $2^L \times 1$ dimensional vectors which satisfy the system of equations $\nabla_{\mathbf{b}} P(\mathbf{b})\mathbf{J} = \nabla_{\mathbf{b}}\Gamma(\mathbf{b})\mathbf{b} + \Gamma(\mathbf{b})$. This set, denoted $\tilde{\mathbb{J}}$, is convex, and hence path-connected, as for two vectors $\mathbf{J}, \mathbf{J}' \in \tilde{\mathbb{J}}$:

$$\lambda\nabla_{\mathbf{b}} P(\mathbf{b})\mathbf{J} + (1-\lambda)\nabla_{\mathbf{b}} P(\mathbf{b})\mathbf{J}' = (\lambda + (1-\lambda))(\nabla_{\mathbf{b}}\Gamma(\mathbf{b})\mathbf{b} + \Gamma(\mathbf{b}))$$
$$\therefore \quad \nabla_{\mathbf{b}} P(\mathbf{b})(\lambda\mathbf{J} + (1-\lambda)\mathbf{J}') = \nabla_{\mathbf{b}}\Gamma(\mathbf{b})\mathbf{b} + \Gamma(\mathbf{b})$$

2. This implies that the image of the intersection of $\tilde{\mathbb{J}}$ and $\mathbb{J}$ defined by the continuous function $\mathbf{J}^{-1}(.)$ (the set of $\mathbf{m}$ for which the FOCs hold) must also be path connected. This follows from the generalised intermediate value theorem, which states that for a continuous function $f : \mathbb{X} \to \mathbb{Y}$, if the set $\mathbb{X}$ is path-connected, then so is the image $f(\mathbb{X})$.

3. Therefore, if the intersection of $\tilde{\mathbb{J}}$ and $\mathbb{J}$ contains more than a single element, then for any $\mathbf{m}$ which satisfies the FOCs, there must exist an arbitrarily nearby $\mathbf{m}'$ which also satisfies these conditions.

4. However, from the inverse function theorem, the FOCs are locally unique. The Jacobian of these FOCs, with respect to $\mathbf{m}$ are given by:

$$\nabla_{\mathbf{b}} P(\mathbf{b})\nabla_{\mathbf{m}}\mathbf{J}(\mathbf{m})$$

This has rank M because $\nabla_{\mathbf{b}} P(\mathbf{b})$ has rank L (it consists of L linearly independent rows), and $\mathbf{J}(\mathbf{m})$ is invertible (so $\nabla_{\mathbf{m}}\mathbf{J}(\mathbf{m})$ has rank M). Therefore it is locally invertible, and so the set of $\mathbf{m}$ which satisfy the FOCs contain a single element.

□

D.4.2 Case 2: When M > L

When $M > L$ we can combine information across observations, instead of identifying everything from a single observation, so long as *enough* elements of M are constant across observations. This is relevant when $\mathbf{m}_t$ can be decomposed into $(\mathbf{m}_t^1, \mathbf{m}^0)$, where $\mathbf{m}^0$ are fixed parameters. Suppose $M \leq 2L$, and in particular, $|\mathbf{m}_t^1| < L$. Consider a pair of FOCs from two separate periods t_1 and t_2. Importantly, I still impose assumption 7. Combine the two sets of first order conditions as follows:

$$\begin{pmatrix} \nabla_{\mathbf{b}} P(\mathbf{b}_{t_1}) & 0 \\ 0 & \nabla_{\mathbf{b}} P(\mathbf{b}_{t_2}) \end{pmatrix} \begin{pmatrix} \mathbf{J}(\mathbf{m}_{t_1}) \\ \mathbf{J}(\mathbf{m}_{t_2}) \end{pmatrix} = \begin{pmatrix} \nabla_{\mathbf{b}} \Gamma(\mathbf{b}_{t_1}) \mathbf{b}_{t_1} + \Gamma(\mathbf{b}_{t_1}) \\ \nabla_{\mathbf{b}} \Gamma(\mathbf{b}_{t_2}) \mathbf{b}_{t_2} + \Gamma(\mathbf{b}_{t_2}) \end{pmatrix}$$

Uniqueness of the solution to this system follows the same logic as the previous proof with the added note that $\nabla_{(\mathbf{m}_{t_1}, \mathbf{m}_{t_2})} \begin{pmatrix} \mathbf{J}(\mathbf{m}_{t_1}) \\ \mathbf{J}(\mathbf{m}_{t_2}) \end{pmatrix}$ has rank $2|\mathbf{m}_t^1| + |\mathbf{m}^0|$, so that I can appeal to the inverse function theorem for local uniqueness.

This result allows us to add a large number of additional parameters to the function $\mathbf{J}(.)$ which are identified by using variation across observations. This employs a similar philosophy used to prove the identification results in the main paper.

E Monte Carlo Simulation

I now present the results of a Monte-Carlo study evaluating the estimator proposed in 4. As discussed in GKS, the difficulty in simulating these games is that solving for equilibrium bidding strategies is intractable. Meanwhile, numerically finding equilibrium bidding strategies - by iterating over equilibrium beliefs and actions until a fixed point is found - is extremely computationally intensive. For each hypothesised set of beliefs, we must find the equilibrium continuation value through iteration, which requires numerically optimising bids many times.

For simplicity I focus on the case where bidders are bidding against a parametric set of beliefs. That is, I essentially take the equilibrium as given. Furthermore I focus on an equilibrium in which equilibrium beliefs do not depend on each bidder's individual states $\{\mathbf{s}_{it}\}_{i \in \mathbb{N}}$. This is similar to many applications seen in practice, including GKS, Backus and Lewis (2016), Groeger (2014), Balat (2013).

E.0.1 Set up

Every period there are two auctions ($L = 2$) and two types of object, denoted x and y. Each lot contains one type of object, and one lot of each type of good is auctioned each period. However, some lots contain ten units of the good, while other lots contain only 5. The set of available lots is denoted (z^x, z^y), so lot 1 contains z^x units of x, and lot 2 contains z^y units of y. The possible characteristics of lots

$\mathbb{X}_t = \{(5,5),(10,5),(5,10)\}$. In this simple setting the common state is just $\mathbf{x}_t$. For simplicity, this transitions stochastically where each states occurs with equal probability, independent of previous states.

States consist of their stocks of the two objects, which come in integer values: $s^x_{it} \in \{\ 0,\ 1,\ \dots\ ,\ 100\}$, likewise for good y. At the end of each period bidders consume 3 units of good x with probability 0.4 and three units of good y with probability 0.3, until their stocks fall to 0. A bidder's combinatorial flow pay-off is given by:

$$j(s^x, s^y) = \theta_1 \log(s^x + 1) + \theta_2 \log(s^x + 1)\log(s^y + 1)$$

where (θ_1, θ_2) are parameters set to 20 and 10 respectively. θ_1 ensures pay-offs are not additively separable over time, while $\theta_2 > 0$ ensures the lots are complements. I can check that bidding behaviour varies strongly with $\mathbf{s}_{it}$, ensuring this baseline instrument is strong. The lot-specific pay-offs are drawn from:

$$\boldsymbol{v}_{it} \sim N\begin{pmatrix} 0 & & 900z^x_t & 100z^x_t z^y_t \\ 0 & , & 100z^x_t z^y_t & 400z^y_t \end{pmatrix}$$

I take as given the equilibrium distribution of the highest rival bids, which follows a type 2 extreme value distribution. The mean of this distribution is given by the average (across states) marginal payoff from each lot ($\approx (17.1z^x, 12.5z^y)$. The variance is tuned to the variance (across states and lot-specific payoffs) of the marginal payoffs from winning each lot. The shape is set to 0.1.

I perform value function iteration to find the continuation value under this distribution of pay-offs and these equilibrium beliefs. Having found a continuation value, I can then simulate a dataset. Given the set-up the state space consists of $30,000$ unique elements. Focusing on a large number of elements is intended to simulate my real world application when the state space will be treated as continuous.

I simulate $1,000$ datasets of bids and states, with $N \in \{\ 1000,\ 10,000,\ 100,000\ \}$ observations uniformly sampled from the state space. I consider 3 estimators: 1) a parametric estimator using the same functional form as j, and 2) a quadratic polynomial estimator, and 3) a semi-nonparametric cubic spline estimator. For the spline, I use uniformly spaced knots, setting the knots to ensure at least 30 observations per knot. For each estimator I consider estimates from using no instruments, the baseline "initial state" instruments, and all the possible ex-post states as instruments.

E.0.2 Results

Results are presented in figure 7. Each estimator yields estimates of $\hat{j}(\mathbf{s}_i)$ for each $\mathbf{s}_i \in \mathbb{S}_i$. I then fit the correctly specified j across these states, extracting $\hat{\theta}_1$ and $\hat{\theta}_2$.

The non-parametric estimator (3) outperforms the two semi-parametric estimators, even in relatively small samples. However, it is very computationally intensive, with estimation taking almost 100 times longer than the semi-parametric estimators. Semi-parametric estimator (1), which fits the true functional form of j to both k and V, performs poorest. This is because we should not expect either k or V to inherit the functional form of j. Likewise, estimator (2), the flexible polynomial, performs reasonably well despite being misspecified. The choice of instruments is also shown to be important. For the most part, using no instruments ($\emptyset$) out performs the initial state instrument. This arises for the combination of two reasons. First, as discussed previously, the initial state instruments may suffer from weak instrument problems, as variation in the initial state may not induce enough variation in bidding behaviour. Second, the degree of bias in the least squares estimation is expected to be small, depending on the correlation between $\Gamma_l(b_l)$ and $\upsilon_{l'}$. This correlation is relatively small because b_l varies much more with other variables, such as υ_l and the state variables. Finally, using the ex-post states as instruments performs much better, but does not dominate (nor is dominated by) the no-instrument estimator.

F Estimation Details and Additional Results

F.1 Constructing the Index Function

The index is constructed as in Aradillas-Lopez et al. (2022) and Raisingh (2021), using most of the same covariates for the random forest as in Raisingh (2021).

The aim is to predict the minimum rival bid in each auction using various elements of the state. To capture rivals' states I classify the rivals of each bidder according to their distance from the bidder using distance bins (near, 0-25km, medium, 25-50km, and far, >50km), and take the average general backlog of rivals within each bin. The features I include as predictors to form λ_{it} are: The number and average backlog of rivals in each distance bin, the number of asphalt / concrete projects auctioned that period, as well as interactions between the type of contract (concrete/asphalt) and the number of concrete / asphalt projects auctioned each period.

Figure 7: Monte Carlo Study

Instrument =			$\emptyset$			$\mathbf{s}_t$			$\{\mathbf{s}_t^a\}$		
	θ	N	Mean	SD	rMSE	Mean	SD	rMSE	Mean	SD	rMSE
(1)	θ_1	$1,000$	5.79	5.11	15.1	4.46	6.11	16.7	4.74	5.64	16.3
		$10,000$	6.19	3	14.1	4.9	3.54	15.5	4.8	3.25	15.5
		$100,000$	6.57	2.63	13.7	5.56	2.95	14.7	5.5	2.75	14.8
	θ_2	$1,000$	5.76	0.596	4.28	6.71	0.775	3.38	6.33	0.733	3.75
		$10,000$	5.83	0.347	4.19	6.76	0.436	3.27	6.41	0.401	3.61
		$100,000$	5.98	0.274	4.03	6.9	0.374	3.12	6.53	0.34	3.49
(2)	θ_1	$1,000$	24.2	1.79	4.55	24.3	4.74	6.38	22.8	4.63	5.4
		$10,000$	24.2	0.602	4.27	24.7	1.52	4.89	22.9	1.49	3.23
		$100,000$	24.2	0.301	4.25	24.6	0.529	4.64	22.9	0.52	2.98
	θ_2	$1,000$	12.1	0.289	2.12	11.2	0.717	1.38	12.1	0.692	2.23
		$10,000$	12.1	0.0963	2.15	11.1	0.224	1.16	12.1	0.227	2.14
		$100,000$	12.2	0.0458	2.17	11.2	0.0803	1.16	12.1	0.0802	2.15
(3)	θ_1	$1,000$	20	2.04	2.04	20.7	14.9	14.9	19	3.73	3.85
		$10,000$	21.2	1.06	1.58	22.4	4.07	4.71	20.4	1.54	1.59
		$100,000$	21.3	0.413	1.32	22.5	1.19	2.74	20.4	0.48	0.653
	θ_2	$1,000$	10.5	0.306	0.558	10.5	2.17	2.22	10.7	0.587	0.928
		$10,000$	10	0.157	0.157	9.93	0.614	0.617	10.1	0.257	0.29
		$100,000$	10	0.0481	0.0486	9.92	0.172	0.189	10.1	0.0741	0.159

Note: The true values for θ_1 and θ_2 are 20 and 10 respectively. For each study I use $1,000$ simulated datasets. The three instruments are: $\emptyset$ = no instrument (OLS), $\mathbf{s}_t$ = initial states, $\{\mathbf{s}_t^a\}$ = all the possible ex-post states, given the period began in $\mathbf{s}_t$. Estimator (1) is a semi-parametric estimator, using the true functional form of j to fit k and V. Estimator (2) fits a cubic polynomial, while Estimator (3) fits a cubic spline.

I now detail the random forest I use to estimate the competition index λ, given the covariates outlined above. For a detailed description of the algorithm, see Appendix B.2 of the full random forest algorithm Raisingh (2021). The key distinction, relative to a standard random forest, is the need to avoid over-fitting when making predictions on the training data. Broadly, the algorithm proceeds as follows:

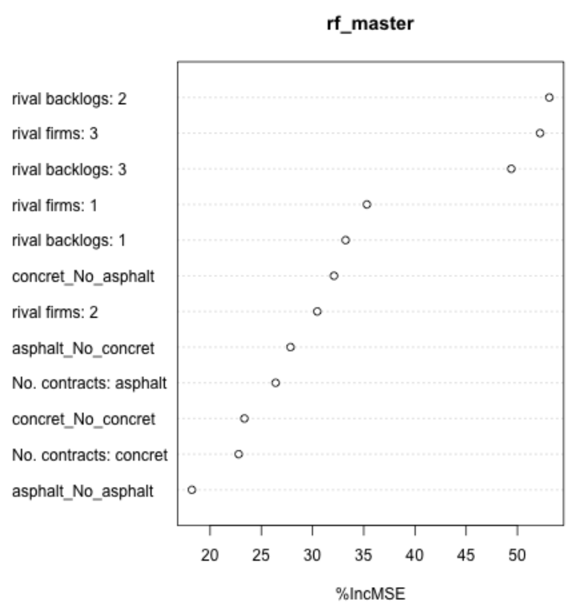
1. Split the data into K equal sized folds
2. Estimate K random forests, each with Q trees, on data from $K-1$ of the folds. This way each fold is excluded from one random forest.
3. Combine the K random forests.

4. Repeat steps $1-3$ L times, yielding L random forests, each with $Q \times K$ trees.

5. Combine the L random forests.

Following Raisingh (2021) I set $L = 24$, $K = 2$, and $Q = 50$. As a consequence, every data-point is used to train around $\frac{1}{3}$ of trees. Figure 9 gives a variable importance plot, highlighting which variables have the most predictive power for the minimum rival bid, and so what most strongly influence the estimated competition index. As in Raisingh (2021), rival backlogs have the most predictive power, followed by the number of rivals. Further away rivals appear to more strongly influence the index, perhaps because they are more likely to be larger firms.

Figure 8: Variable Importance Plot



Note: This plot shows the reduction in sum of squared residuals that occurs from splitting the data on each variable. Higher numbers demonstrate more predictive power.

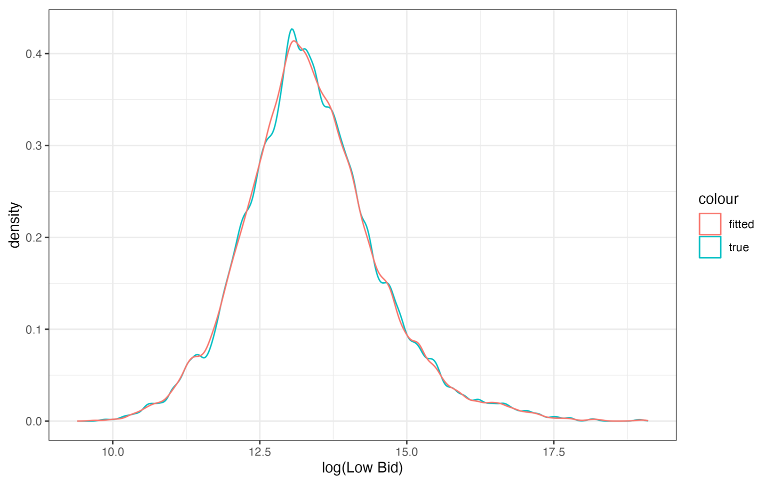
Because the index is auction specific I average across auctions to form the period $\times$ bidder specific competition index. Since the most important predictors are all period $\times$ bidder specific the index varies much more across periods than with periods.

F.2 Additional Results

F.2.1 First Stage

Figure 9 plots the observed distribution of minimum rival bids against the estimated distribution. The three parameter Weibull distribution fits the data well.

Figure 9: First Stage Fit



F.2.2 Second Stage

Figure 10 displays additional results from the second estimation step, demonstrating how the pseudo-static cost function varies with the competition index λ_{it}. The estimated parameters can be interpreted as follows: Holding fixed a general contractor's (t1) backlog of asphalt projects, every one standard deviation in λ, as competition *decreases*, increases the opportunity cost of winning by around $\$90,000$. Estimated parameters generally have the expected signs, with pseudo-costs increasing in the degree of competitiveness (coefficients are positive for positive coefficients in Figure 4, and negative for negative coefficients in Figure 4).

Furthermore, the estimated interaction parameters are jointly significant ($p < 0.01$) for all but the specification with weak instruments. Under the exclusion restriction that $j(\mathbf{s}_i)$ is independent of λ_i, we can therefore reject the null hypothesis that $\beta = 0$, rejecting the myopic model. The association between the degree of competition and bidding behaviour is strong, even when we account for equilibrium beliefs.

Figure 10: Second Stage Results: λ interactions

Instruments		none (OLS)		$\mathbf{s}_{it}$		$\mathbf{s}_{it} + \overrightarrow{\mathbf{s}}_{ilt}$		$\mathbf{s}_{it} + \overrightarrow{\mathbf{s}}_{ilt} + \overrightarrow{\mathbf{s}}_{imt}$	
		$\hat{\theta}$	SE	$\hat{\theta}$	SE	$\hat{\theta}$	SE	$\hat{\theta}$	SE
Combinatorial									
$s_t^a \times \lambda_t$	t1	84.6	23.8	361	510	87.2	27.4	91.9	25.7
	t2	157	35.8	1230	2670	136	37.4	146	32.5
	t3	16.4	5.45	1170	2580	16.5	6.48	15.8	5.11
$s_t^c \times \lambda_t$	t1	55	15.6	207	931	68	16.6	57.4	16.3
	t2	-56	39.6	6640	15,700	-80	45.7	-75	39
	t3	3.03	4.87	-1,260	3160	2.17	5.2	3.29	4.87
$(s_t^a)^2 \times \lambda_t$	t1	-2.5	3.03	-74.1	147	-2.19	3.5	-4.06	3.2
	t2	-8.38	4.87	-586	1370	-4.91	5.94	-7.38	4.93
	t3	0.0262	0.124	-65.4	123	0.126	0.144	0.0681	0.123
$(s_t^c)^2 \times \lambda_t$	t1	-2.19	3.38	-42.3	302	-2.65	3.48	-1.45	3.52
	t2	-2.13	2.35	-44.3	327	-3.74	2.74	-3.29	2.5
	t3	-0.24	0.139	-7.07	28.2	-0.00431	0.366	-0.19	0.2
$s_t^a \times s_t^c \times \lambda_t$	t1	-4.63	4.36	8.17	142	-8.43	4.87	-5.67	4.52
	t2	44.2	15.5	133	704	60.2	16.8	56.6	14.8
	t3	0.888	0.42	105	244	0.141	1.15	0.724	0.617
Fixed Effects									
County		✓		✓		✓		✓	
Firm × Type		✓		✓		✓		✓	
R^2		0.6		-10.8		0.597		0.599	
Observations									
T		3919		3919		3919		3919	
$\sum_t L_t$		14691		14691		14691		14691	

Note: Figures are given in 000s of dollars. Holding fixed a general contractor's (t1) backlog of asphalt projects, every one standard deviation in λ, as competition *decreases*, increases the opportunity cost of winning by around \$90,000.

F.3 Comparison to Misspecified Models

I now compare estimates of $j(\mathbf{s}_i)$ from the dynamic multi-object model presented above, to two misspecified models: A dynamic single object model, and a static multi-object model. Results are presented in figure 11.

F.3.1 Mispecified Models

Static Model

The static model is nested within the dynamic multi-object model, imposing $\beta = 0$. Estimation involves the same first and second steps presented in section 5.

Single Object Model

Even though bidders place multiple bids each period, the static single-object model ignores possible cost-synergies between lots, even when it allows costs to be non-linear in backlogs. One interpretation is that separate groups within the firm bid simultaneously, without communication among one another. Therefore bidding groups do not take into account how their payoff depends not only on their own bid, but also the bids of other groups within the firm.

I estimate the model using the general approach from JP, albeit evaluating the expected maximised payoff using an average over observed bids rather than using numerical integration. I complete the first estimation step as in the text, then skip to the third estimation step and evaluating the continuation value as in JP, taking an expectation over observed bids instead of using estimated bid distributions. Because, in practice, multiple auctions occur each period I evaluate the expected period profit by taking the sum of the expected (additive) profit from each auction occurring that period. Finally, I back out $j(\mathbf{s}_i)$ from the inverse bid function.

F.3.2 Results

Estimates for the static model are off by an order of magnitude, but are extremely similar to the results for the pseudo-static pay-off presented in figure 5. This is because we essentially mistake the sum of current costs and discounted future costs (and opportunity costs) for just current costs. The results for the dynamic single-object model are more more similar to the dynamic multi-object model. However this misspecified model generally under estimates the extent of the returns to scale, generally underestimating the degree of non-additivity across lots.

F.4 Counterfactual Simulations

I now detail how I simulate the sequential auction regime. Time is discrete, and each period in the simultaneous regime (14 days) is split into 100 sub-periods. Auctions are distributed randomly across sub-periods.

To map the estimated $AR(1)$ transition process from 14 day-long periods into 100 sub-periods I assume the sub-period transition process remains $AR(1)$, such that the mean and variance of the process is the same as the estimated process over the 100

Figure 11: Model comparison

Model		DMO		DSO		SMO	
	$j(\mathbf{s}_i)$	$\hat{\theta}$	SE	$\hat{\theta}$	SE	$\hat{\theta}$	SE
s_t^a	t1	123	7.01	39.5	7.77	423	23.6
	t2	285	11.3	18.7	12.7	835	36.3
	t3	40	1.92	145	7.58	108	5.77
s_t^c	t1	107	5.35	49.4	9.86	378	17.3
	t2	89.1	11.9	25.9	14.3	153	53.3
	t3	15.6	1.91	89.7	6.05	55.2	6.44
$(s_t^a)^2$	t1	-0.337	1.29	0.0669	1.59	-0.116	2.44
	t2	-9.26	2.46	-1.68	2.5	-16.2	4.4
	t3	-1.34	0.147	-2.08	1.01	-0.229	0.0872
$(s_t^c)^2$	t1	-7.6	1.13	-1.39	0.889	-14.5	2.12
	t2	-14	3.38	-2.48	1.99	-4.93	1.82
	t3	-0.479	0.102	-0.671	0.672	-0.328	0.11
$s_t^a \times s_t^c$	t1	1.38	1.52	-0.364	2	7.94	2.97
	t2	33.4	7.12	-1.6	3.44	58.8	14.4
	t3	0.432	0.199	0.801	0.876	0.534	0.34
R^2		0.597		0.595		0.581	

sub-periods, ensuring the long run process is the same. Likewise, estimated payoffs $j(\mathbf{s}_i)$ are only defined on 14 day long intervals. To evaluate payoffs in the sub-periods I find a function $\tilde{j}(\mathbf{s}_i)$ such that the expected sum of these sub-period payoffs across 100 sub-periods equals $j(\mathbf{s}_i)$. Finally, I use the same estimated competition index as in the text, capturing the amount of competition for each contract.

For each parameter draw, beginning at an initial set of equilibrium beliefs, I numerically find bidders' continuation values. I iteratively loop through auctions numerically maximising bidders' payoffs. I make the simplifying assumption that bidders only enter the auctions they were actually observed entering, assuming these are the auctions they have the largest cost advantage in, regardless of the choice of mechanism. In finding the continuation value, to facilitate convergence, I fix bidders' states at their observed levels. Just as in estimation I fit a quadratic form to bidders' maximum expected payoffs, and so evaluate the next the continuation value. I continue this process until the continuation value converges. I also use Newton-Kantorovich iterations to improve convergence, employing the envelope theorem to evaluate the derivative of the maximum expected payoffs.

I then simulate the system again, this time allowing bidders states to vary as they win, and gradually complete, contracts. I then fit the same Weibull form to minimum rival bids, as in the first estimation step. While the payoffs of Fringe bidders do not change in the counterfactual scenario, their beliefs do. I continue this process until achieving convergence. While there may be multiple equilibria, by beginning with the equilibrium beliefs from the simultaneous regime I ensure we search for equilibria close to to this regime. Therefore any equilibrium will be relatively nearby that from simultaneous auctions, ensuring estimates are conservative.