Unified Necessary and Sufficient Conditions for the Robust Stability of Interconnected Sector-Bounded Systems

Saman Cyrus[1] Laurent Lessard[1]

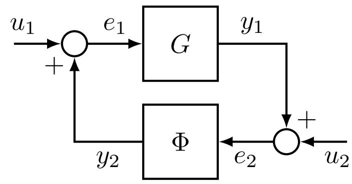
Abstract

Classical conditions for ensuring the robust stability of a linear system in feedback with a sector-bounded nonlinearity include the circle, small gain, passivity, and conicity theorems. In this work, we present a similar stability condition, but expressed in terms of operators on a semi-inner product space. This increased generality leads to a clean result that can be specialized in a variety of ways. First, we show how to recover both sufficient and necessary-and-sufficient versions of the aforementioned classical results. Second, we show that suitably choosing the semi-inner product space leads to a new necessary and sufficient condition for exponential stability with a given convergence rate. Finally, in the spirit of classical robust stability analysis, we provide linear matrix inequalities that allow for the efficient verification of the conditions of our theorem.

1 Introduction

In this paper, we consider the robust stability of the feedback interconnection shown in Figure 1, where a linear system G is in positive feedback with a sector-bounded nonlinearity Φ, and we are interested in finding conditions on the two systems such that the closed-loop system is stable. Different forms of this problem have been a point of study for over 75 years since the early work of Lur'e [1]. In general, one typically fixes conditions on one of the systems and describes required conditions on the other system that guarantees stability of the feedback interconnection. Examples of such results include the circle criterion [2,3], passivity [4], the small-gain theorem [4,5], the conic sector theorem [5], and the extended conic sector theorem [6]. These studies consider forced systems to study input-output stability [7, §6.6] and unforced systems to study asymptotic stability [7, §5.6].

The classical Lur'e problem is concerned with the case where G is a linear time-invariant (LTI) system and Φ is a sector-bounded memoryless nonlinearity. In this case, theorems such as the circle criterion are sufficient but not necessary for stability [8]. However, if Φ is allowed to



$$e_1 = u_1 + y_2 \tag{1a}$$
$$y_2 = \Phi e_2 \tag{1b}$$
$$e_2 = u_2 + y_1 \tag{1c}$$
$$y_1 = G e_1 \tag{1d}$$

Figure 1: Two interconnected systems. We assume in this work that G is linear and Φ is a sector-bounded nonlinearity.

have dynamics, the circle criterion becomes both necessary and sufficient [7, Thm. 6.6.126]. Likewise, passivity theory is sufficient but not necessary for stability when Φ is memoryless [7, Thm. 5.6.18] but becomes necessary and sufficient when Φ is allowed to have dynamics [9].

While robust stability is well-studied, there are relatively few results addressing robust *exponential* stability. This issue arose in recent works on the analysis of iterative algorithms [10,11], where it is desirable to have tight bounds on worst-case exponential convergence rates. It is known that under mild assumptions, input-output stability automatically implies exponential stability [12,13]. However, constructing an exponential rate via a gain bound is conservative in general [14]. Improved sufficient conditions appeared in [14,15] but to our knowledge, there are no results that obtain necessary and sufficient conditions for exponential stability.

Main contribution. Our main contribution is Theorem 2, a robust stability result in the spirit of the aforementioned classical results but expressed in a semi-inner product space (see Definition 1). This added generality leads to a clean result that avoids many of the technicalities associated with the typical extended spaces $\mathcal{L}_{2\mathrm{e}}$ or $\ell_{2\mathrm{e}}$. In Section 3, we show how Theorem 2 can be used to recover many classical results, including cases where we only have sufficiency. In Section 4, we apply Theorem 2 to obtain a new exponential stability result that is both necessary and sufficient. Finally, in Section 5, we present linear matrix inequalities (LMIs) that lead to efficient numerical verification of the conditions of Theorem 2.

Related work. Several prior works have also provided unified versions of classical robust stability results. We cite two examples: the extended conic sector theorem, which can handle the case where G is unstable [16] and a loop-shifting transformation that relates passivity, small-gain, and circle theorems [17]. Nevertheless, the present work is unique in its use of semi-inner product spaces and its ability to address exponential stability.

[1]S. Cyrus and L. Lessard are with the Department of Electrical and Computer Engineering and with the Wisconsin Institute for Discovery, both at the University of Wisconsin–Madison.
Email addresses: {cyrus2,laurent.lessard}@wisc.edu

This material is based upon work supported by the National Science Foundation under Grant No. 1710892.

There are also generalizations to cases where G is nonlinear or Φ is not sector-bounded. Examples include dissipativity theory [18], integral quadratic constraints [19, 20], and graph separation theorems [21, 22]. These efforts lie beyond the scope of the present work.

Before we state our main result, we recall the definition of a *semi-inner product space.*

Definition 1 (semi-inner product space)**.** *A semi-inner product space is a vector space $\mathcal{H}$ equipped with a semi-inner product $\langle\cdot,\cdot\rangle$. This is identical to an inner product except that it lacks definiteness. In other words, the associated semi-norm $\|x\| := \langle x, x\rangle$ satisfies $\|x\| \geq 0$ but $\|x\| = 0$ need not imply that $x = 0$. We say the semi-inner product space is nontrivial if there exists some $x \in \mathcal{H}$ such that $\|x\| > 0$. We refer the reader to [23].*

Theorem 2 (Main result)**.** *Let $\mathcal{H}$ be a nontrivial real semi-inner product space, let $M \in \mathbb{R}^{2\times 2}$ be given, and suppose $G : \mathcal{H} \to \mathcal{H}$ is a linear operator. The following statements are equivalent.*

(i) There exists $N \in \mathbb{R}^{2\times 2}$ satisfying $M + N \prec 0$, in the positive definite sense, such that G satisfies

$$\left\langle \begin{bmatrix} G\xi \\ \xi \end{bmatrix}, N \begin{bmatrix} G\xi \\ \xi \end{bmatrix} \right\rangle \geq 0 \qquad \text{for all } \xi \in \mathcal{H}. \tag{2}$$

(ii) There exists $\gamma > 0$ such that for all $\Phi : \mathcal{H} \to \mathcal{H}$ that satisfy

$$\left\langle \begin{bmatrix} \xi \\ \Phi\xi \end{bmatrix}, M \begin{bmatrix} \xi \\ \Phi\xi \end{bmatrix} \right\rangle \geq 0 \qquad \text{for all } \xi \in \mathcal{H}, \tag{3}$$

and for which the interconnection of G and Φ as in Figure 1 is well-posed, the following bound holds for all $u_1, u_2 \in \mathcal{H}$.

$$\|y_1\|^2 + \|y_2\|^2 \leq \gamma^2 \left(\|u_1\|^2 + \|u_2\|^2\right). \tag{4}$$

Here, well-posedness means that for any u_1, $u_2 \in \mathcal{H}$ there exists a solution $e_1, e_2, y_1, y_2 \in \mathcal{H}$ to (1). The solution need not be unique; (4) holds for all solutions.

Remark 3. *In* (2) *and* (3)*, inner products and matrix multiplications are overloaded to have intuitive interpretations in $\mathcal{H}^2$. Specifically, for any $\xi_1, \xi_2, \zeta_1, \zeta_2 \in \mathcal{H}$,*

$$N \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} := \begin{bmatrix} N_{11}\xi_1 + N_{12}\xi_2 \\ N_{21}\xi_1 + N_{22}\xi_2 \end{bmatrix} \in \mathcal{H}^2.$$

Likewise, inner products in $\mathcal{H}^2$ have the interpretation

$$\left\langle \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} \right\rangle := \langle \xi_1, \zeta_1\rangle + \langle \xi_2, \zeta_2\rangle.$$

Remark 4. *Theorem 2 applies to a general semi-inner product space, which need not include a notion of time. Therefore, the notions of causality, boundedness, or stability dot not come into play. Specializing $\mathcal{H}$ to a space that includes a notion of time such as the extended spaces $\ell_{2\mathrm{e}}$ or $\mathcal{L}_{2\mathrm{e}}$ is described in Section 3.*

2 Proof of Theorem 2

Sufficiency. We begin by proving *(i)* $\Longrightarrow$ *(ii)*. This part of the proof resembles [24]. Let $x = e_2$ in (3) and let $x = e_1$ in (2). Substituting the definitions for y_1 and y_2, Equations (3) and (2) become:

$$\left\langle \begin{bmatrix} e_2 \\ y_2 \end{bmatrix}, M \begin{bmatrix} e_2 \\ y_2 \end{bmatrix} \right\rangle \geq 0, \qquad \left\langle \begin{bmatrix} y_1 \\ e_1 \end{bmatrix}, N \begin{bmatrix} y_1 \\ e_1 \end{bmatrix} \right\rangle \geq 0.$$

Substitute $e_1 = u_1 + y_2$ and $e_2 = u_2 + y_1$, sum the inequalities, and collect terms to obtain

$$\left\langle \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, (M+N) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle + \left\langle \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \begin{bmatrix} -2N_{12} & 2M_{11} \\ -2N_{22} & 2M_{21} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right\rangle + \left\langle \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} N_{22} & 0 \\ 0 & M_{11} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right\rangle \geq 0.$$

Since $M + N \prec 0$ by assumption, There exists $\eta > 0$ such that $M + N \preceq -\eta I$. Applying this inequality together with Cauchy–Schwarz, we obtain:

$$-\eta\|y\|^2 + r\|y\|\|u\| + q\|u\|^2 \geq 0, \tag{5}$$

where $r := \left\| \begin{bmatrix} -2N_{12} & 2M_{11} \\ -2N_{22} & 2M_{21} \end{bmatrix} \right\|_2$, $q := \left\| \begin{bmatrix} N_{22} & 0 \\ 0 & M_{11} \end{bmatrix} \right\|_2$ are the standard spectral norms in $\mathbb{R}^{2\times 2}$, and we have defined concatenated signals, e.g. $y := \begin{bmatrix} y_1^\mathsf{T} & y_2^\mathsf{T} \end{bmatrix}^\mathsf{T} \in \mathcal{H}^2$. For any $\alpha \in (0, 1)$, Equation (5) is equivalent to

$$(1-\alpha)\eta\|y\|^2 \leq \left(q + \tfrac{r^2}{4\alpha\eta}\right)\|u\|^2 - \alpha\eta\left(\|y\| - \tfrac{r}{2\alpha\eta}\|u\|\right)^2$$

Manipulating the inequality above leads to:

$$\begin{aligned} \Longrightarrow \quad & (1-\alpha)\eta\|y\|^2 \leq \left(q + \tfrac{r^2}{4\alpha\eta}\right)\|u\|^2 \\ \Longleftrightarrow \quad & \|y\| \leq \sqrt{\tfrac{q}{(1-\alpha)\eta} + \tfrac{r^2}{4\alpha(1-\alpha)\eta^2}}\,\|u\| \\ \Longleftrightarrow \quad & \|y\| \leq \tfrac{1}{\sqrt{2}}\left(r + \sqrt{r^2 + 4q\eta}\right)\|u\|. \end{aligned}$$

In the last step, we chose $\alpha \in (0, 1)$ to minimize the right-hand side and obtain the tightest bound.

Necessity. We now prove *(ii)* $\Longrightarrow$ *(i)*. This part of the proof is based on the S-lemma, which relates sets of points defined by quadratic inequalities. See [25, 26] and references therein. We use a generalization of the S-lemma to semi-inner product spaces based on a Hilbert space version due to Hestenes [27] and similar to [9].

Definition 5. *Let $\mathcal{H}$ be a real vector space. A* quadratic form *Q is a function $Q : \mathcal{H} \to \mathbb{R}$ that has associated with it a function $\tilde{Q} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ such that the following properties hold for all $x, y, z \in \mathcal{H}$ and $a, b \in \mathbb{R}$.*

(P1) $Q(x) = \tilde{Q}(x, x)$

(P2) $\tilde{Q}(x, y) = \tilde{Q}(y, x)$

(P3) $\tilde{Q}(x, ay + bz) = a\tilde{Q}(x, y) + b\tilde{Q}(x, z)$

(P4) $Q(ax + by) = a^2 Q(x) + 2ab\tilde{Q}(x, y) + b^2 Q(y)$

Lemma 6. *Let $\mathcal{H}$ be a real vector space and let $S \subseteq \mathcal{H}$ be a subspace. Let σ_0 and σ_1 be quadratic forms. The following statements are equivalent.*

(S1) For all $x \in S$, we have $\sigma_1(x) \geq 0 \implies \sigma_0(x) \leq 0$.

(S2) There exists $\tau \geq 0$ such that for all $x \in S$, we have $\sigma_0(x) + \tau\sigma_1(x) \leq 0$.

Proof. Omitted. A similar result on Hilbert spaces is proved in [27] and extends immediately to semi-inner product spaces. In our version, no regularity conditions are required because all inequalities are non-strict. ■

We can now prove our result. We will use $\Theta \in \mathcal{H}^6$ as a symbol denoting a generic tuple $(u_1, u_2, y_1, y_2, e_1, e_2)$ of signals. Define the sets:

$$S := \left\{\Theta \in \mathcal{H}^6 \;\middle|\; e_1 = u_1 + y_2,\, e_2 = u_2 + y_1,\, y_1 = Ge_1\right\}$$
$$S'_\Phi := \left\{\Theta \in \mathcal{H}^6 \;\middle|\; \text{Equations (1) hold}\right\}.$$

Since G is linear, it follows that $S \subseteq \mathcal{H}^6$ is a subspace. Moreover, $\bigcup_{\Phi:\mathcal{H}\to\mathcal{H}} S'_\Phi = S$. To see why, first observe that $S'_\Phi \subseteq S$ for all Φ by definition. To prove the opposite inclusion, given any $\Theta \in S$, we can simply define Φ such that $y_2 = \Phi(e_2)$. Define the quadratic forms on $S \to \mathbb{R}$:

$$\sigma_0(\Theta) := \|y_1\|^2 + \|y_2\|^2 - \gamma^2\left(\|u_1\|^2 + \|u_2\|^2\right),$$
$$\sigma_1(\Theta) := \left\langle \begin{bmatrix} e_2 \\ y_2 \end{bmatrix}, M \begin{bmatrix} e_2 \\ y_2 \end{bmatrix} \right\rangle.$$

Item (ii) from Theorem 2 states that for all $\Phi \in \mathcal{H} \to \mathcal{H}$ and for all $\Theta \in S'_\Phi$, we have $\sigma_1(\Theta) \geq 0 \implies \sigma_0(\Theta) \leq 0$. But since $\bigcup_{\Phi:\mathcal{H}\to\mathcal{H}} S'_\Phi = S$, Item (ii) is equivalent to: for all $\Theta \in S$, we have $\sigma_1(\Theta) \geq 0 \implies \sigma_0(\Theta) \leq 0$. Since S is a subspace, we may apply Lemma 6 and conclude that there exists $\tau \geq 0$ such that

$$\text{for all } \Theta \in S, \qquad \sigma_0(\Theta) + \tau\sigma_1(\Theta) \leq 0. \tag{6}$$

Substituting the definitions of σ_0 and σ_1 into (6) yields

$$\|y_1\|^2 + \|y_2\|^2 - \gamma^2\left(\|u_1\|^2 + \|u_2\|^2\right) + \tau\left\langle \begin{bmatrix} e_2 \\ y_2 \end{bmatrix}, M \begin{bmatrix} e_2 \\ y_2 \end{bmatrix} \right\rangle \leq 0. \tag{7}$$

Let $\bar{S} \subseteq S$ be the subspace of S with $u_1 = u_2 = 0$. Restricting (7) to $\bar{S}$, we have $y_2 = e_1$ and $e_2 = y_1 = Ge_1$. Making these substitutions results in:

$$\|Ge_1\|^2 + \|e_1\|^2 + \tau\left\langle \begin{bmatrix} Ge_1 \\ e_1 \end{bmatrix}, M \begin{bmatrix} Ge_1 \\ e_1 \end{bmatrix} \right\rangle \leq 0,$$

for all $e_1 \in \mathcal{H}$. If $\tau = 0$, then from nontriviality of $\mathcal{H}$, the above inequality clearly cannot hold for all e_1, a contradiction. Therefore it must be the case that $\tau > 0$. Rearranging and dividing by τ, we obtain:

$$\left\langle \begin{bmatrix} Ge_1 \\ e_1 \end{bmatrix}, \left(-\tfrac{1}{\tau}I - M\right) \begin{bmatrix} Ge_1 \\ e_1 \end{bmatrix} \right\rangle \geq 0 \qquad \text{for all } e_1 \in \mathcal{H}.$$

Define $N := -\frac{1}{\tau}I - M$. Then we have $M + N = -\frac{1}{\tau}I \prec 0$, which is (2) and so we have proven (i) of Theorem 2. ■

Remark 7. *Theorem 2 also holds with (4) replaced by*

$$\|e_1\|^2 + \|e_2\|^2 < \gamma^2(\|u_1\|^2 + \|u_2\|^2)$$

and the proof is similar, possibly with a different $\gamma > 0$.

3 Specializing the main result

In this section, we show how Theorem 2 can be used to recover a variety of classical results via suitable choice of the semi-inner product space $\mathcal{H}$. We restrict our attention to discrete-time results in the interest of space, though continuous-time extensions are straightforward.

Recall the extended space $\ell_{2\mathrm{e}}$, which is the real vector space of semi-infinite sequences $\mathbb{Z}_+ \to \mathbb{R}^m$. Also recall the square-summable subset $\ell_2 \subset \ell_{2\mathrm{e}}$. Specifically,

$$\ell_{2\mathrm{e}} := \left\{(x[0], x[1], \dots) \;\middle|\; x[k] \in \mathbb{R}^n \text{ for } k = 0, 1, \dots\right\},$$
$$\ell_2 := \left\{x \in \ell_{2\mathrm{e}} \;\middle|\; \|x\| := \left(\sum_{k=0}^{\infty} \big\|x[k]\big\|^2\right)^{1/2} < \infty\right\}.$$

Here, the indices $[k]$ play the role of *time*. We now recall some standard definitions. For any $x \in \ell_{2\mathrm{e}}$, we define truncation $x_T \in \ell_2$ as follows.

$$x_T[k] := \begin{cases} x[k] & 0 \leq k \leq T \\ 0 & k \geq T+1 \end{cases}$$

An operator G is said to be *causal* if for any $T \geq 0$ and $f \in \ell_{2\mathrm{e}}$, it satisfies $(Af)_T = (Af_T)_T$. We will now apply Theorem 2 to the $\ell_{2\mathrm{e}}$ space equipped with a particular semi-inner product defined as follows.

Definition 8 (cumulative semi-inner product). *Define $\langle\cdot,\cdot\rangle_T$ to be the sum of the component-wise inner products up to time T. That is, $\langle x, y\rangle_T := \langle x_T, y_T\rangle$. Also define the associated semi-norm $\|x\|_T := \langle x, x\rangle_T$.*

Specializing Theorem 2 to $\ell_{2\mathrm{e}}$ equipped with $\langle\cdot,\cdot\rangle_T$ for every $T \geq 0$, we obtain the following result. Note that we still use the convention of Remark 3 here.

Corollary 9 (ℓ_2 stability). *Let $M \in \mathbb{R}^{2\times 2}$ be given and suppose $G : \ell_{2\mathrm{e}} \to \ell_{2\mathrm{e}}$ is a linear operator. The following statements are equivalent:*

(i) There exists $N \in \mathbb{R}^{2\times 2}$ satisfying $M + N \prec 0$ such that for all $\xi \in \ell_{2\mathrm{e}}$ and $T \geq 0$, G satisfies

$$\left\langle \begin{bmatrix} G\xi \\ \xi \end{bmatrix}, N \begin{bmatrix} G\xi \\ \xi \end{bmatrix} \right\rangle_T \geq 0. \tag{8}$$

(ii) There exists $\gamma > 0$ such that for all $\Phi : \ell_{2\mathrm{e}} \to \ell_{2\mathrm{e}}$ and $T \geq 0$ that satisfy

$$\left\langle \begin{bmatrix} \xi \\ \Phi\xi \end{bmatrix}, M \begin{bmatrix} \xi \\ \Phi\xi \end{bmatrix} \right\rangle_T \geq 0 \qquad \text{for all } \xi \in \ell_{2\mathrm{e}}, \tag{9}$$

and for which the interconnection of G and Φ as in Figure 1 is well-posed in the ℓ_2 sense, the following equation holds for all $u_1, u_2 \in \ell_2$

$$\|y_1\|^2 + \|y_2\|^2 \leq \gamma^2\left(\|u_1\|^2 + \|u_2\|^2\right). \tag{10}$$

Here, well-posedness in the ℓ_2 sense means that the map $(y_1, y_2) \to (u_1, u_2)$ has a unique causal inverse on $\ell_{2\mathrm{e}}$.

Proof. Applying Theorem 2 to ℓ_{2e} equipped with $\langle\cdot,\cdot\rangle_T$ for each T, we obtain the statement of Corollary 9 but instead of (10), we have for all $u_1, u_2 \in \ell_{2e}$ and $T \geq 0$

$$\|y_1\|_T^2 + \|y_2\|_T^2 \leq \gamma^2 \left(\|u_1\|_T^2 + \|u_2\|_T^2\right) \qquad (11)$$

From the proof of Theorem 2, γ only depends on the choice of M and N, and not on the choice of semi-inner product. Well-posedness also does not depend on the semi-inner product, so fixing Φ and $u_1, u_2 \in \ell_2$ always yields the same $y_1, y_2 \in \ell_{2e}$ regardless of the choice of T. Therefore, (10) holds for all $T \geq 0$ and γ, y_1, y_2 do not depend on T. It remains to show that (10) $\iff$ (11).

(11) $\implies$ (10). Since $u_1, u_2 \in \ell_2$, the right-hand side of (11) is a monotonically nondecreasing function of T and its limit is $\gamma^2\left(\|u_1\|^2 + \|u_2\|^2\right)$. Now the left-hand side of (11) is also bounded and monotonically nondecreasing, so $y_1, y_2 \in \ell_2$ holds and we have (10).

(10) $\implies$ (11). Define the map $H : (u_1, u_2) \mapsto (y_1, y_2)$. H is causal by well-posedness, so for any $u_1, u_2 \in \ell_{2e}$ and $T \geq 0$, we have $(y_1, y_2)_T = H(u_1, u_2)_T = H(u_{1T}, u_{2T})_T$. Applying (10) to the truncated signals, we obtain

$$\begin{aligned}\|(y_1, y_2)\|_T^2 &= \|(y_1, y_2)_T\|^2 = \|H(u_{1T}, u_{2T})_T\|^2 \\ &\leq \|H(u_{1T}, u_{2T})\|^2 \leq \gamma^2\|(u_{1T}, u_{2T})\|^2 \\ &= \gamma^2\|(u_1, u_2)\|_T^2\end{aligned}$$

which is the statement (11). ∎

3.1 Recovering necessary and sufficient results

Corollary 9 may now be applied to a variety of different scenarios by appropriately choosing $M, N \in \mathbb{R}^{2\times 2}$.

Remark 10 (sign convention)**.** *Although we used the positive feedback sign convention in Figure 1, using the negative feedback convention instead simply amounts to replacing N by $\tilde{N}$ in Theorem 2 and Corollary 9, where*

$$N := \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \quad \text{and} \quad \tilde{N} := \begin{bmatrix} N_{11} & -N_{12} \\ -N_{21} & N_{22} \end{bmatrix}.$$

Consider the classical passivity result by Vidyasagar, which may be found in [7, Thm. 6.7.3.43].

Theorem 11 (Vidyasagar)**.** *Consider the system*

$$\begin{cases} e_1 = u_1 - y_2, & y_1 = Ge_1 \\ e_2 = u_2 + y_1, & y_2 = \Phi e_2 \end{cases}$$

Suppose there exist constants ε_1, ε_2, δ_1, δ_2 such that for all $\xi \in \ell_{2e}$ and for all $T \geq 0$

$$\langle \xi, G\xi\rangle_T \geq \varepsilon_1\|\xi\|_T^2 + \delta_1\|G\xi\|_T^2 \qquad (12a)$$

$$\langle \xi, \Phi\xi\rangle_T \geq \varepsilon_2\|\xi\|_T^2 + \delta_2\|\Phi\xi\|_T^2 \qquad (12b)$$

Then the system is ℓ_2-stable if $\delta_1+\varepsilon_2 > 0$ and $\delta_2+\varepsilon_1 > 0$.

To obtain a corresponding necessary and sufficient result using Corollary 9, compare (8)–(9) to (12), which yields the following values of $\tilde{N}$, N, and M.

$$\tilde{N} = \begin{bmatrix} -\delta_1 & \frac{1}{2} \\ \frac{1}{2} & -\varepsilon_1 \end{bmatrix}, N = \begin{bmatrix} -\delta_1 & -\frac{1}{2} \\ -\frac{1}{2} & -\varepsilon_1 \end{bmatrix}, M = \begin{bmatrix} -\varepsilon_2 & \frac{1}{2} \\ \frac{1}{2} & -\delta_2 \end{bmatrix}.$$

For the definition of $\tilde{N}$, refer to Remark 10. Applying Corollary 9, we require $M + N \prec 0$; thus $\delta_1 + \varepsilon_2 > 0$ and $\delta_2 + \varepsilon_1 > 0$, which agrees with Theorem 11.

Similar specializations of Corollary 9 apply to the small-gain theorem [2, Thm. 5.6], extended conic sector theorem [6], circle criterion [25], and other versions of passivity such as Vidyasagar [7, Thm. 6.6.58] and Khong & Van der Shaft [9]. See Table 1 for a summary.

Remark 12. *Many results in the literature assume one of the systems is memoryless. As we will discuss in Section 3.2, this makes Corollary 9 sufficient-only. Nevertheless, the matrices M and N are similar in both cases.*

3.2 Recovering sufficient-only results

Here we discuss sufficient-only results from the literature that can also be obtained from Corollary 9 via a suitable relaxation. We discuss two such relaxations.

Memoryless systems. If we restrict $\Phi : \ell_{2e} \to \ell_{2e}$ to be time-invariant and memoryless, it is equivalent to an operator $\phi : \mathbb{R}^m \to \mathbb{R}^m$ that operates pointwise in time. Consequently, if Φ satisfies a sector bound of the form

$$\left\langle \begin{bmatrix} \xi \\ \phi(\xi) \end{bmatrix}, M \begin{bmatrix} \xi \\ \phi(\xi) \end{bmatrix} \right\rangle \geq 0 \quad \text{for all } \xi \in \mathbb{R}^m, \qquad (13)$$

then Φ also satisfies the cumulative relationship (9) for all T. Therefore, if we define condition *(iii)* to be the same as condition *(ii)* from Corollary 9, except (9) is replaced by (13), then we have *(i)* $\iff$ *(ii)* $\implies$ *(iii)*. So in general, if *(i)* fails to hold, there must exist some Φ satisfying (9) such that (10) fails, but such a Φ need not be time-invariant or memoryless. Examples of this case in the classical literature include [8, 28, 29].

Nested sector bounds. Another possible relaxation of Corollary 9 is to consider *nested sectors* for one of the systems. For example, define *(i')* to be the same as *(i)* except N is replaced by some $\hat{N} \preceq N$. Similarly, define *(ii')* to be the same as *(ii)* except M is replaced by some $\hat{M} \preceq M$. Then, we have the implications: *(i')* $\implies$ *(i)* $\iff$ *(ii)* $\implies$ *(ii')*. The implication *(i')* $\implies$ *(ii')* cannot be reversed in general, and is therefore a sufficient-only condition.

4 Exponential stability result

In this section, we present a specialization of Theorem 2 that leads to a new necessary and sufficient condition for exponential stability. A discrete-time system with state $x[k]$ is exponentially stable if there exists some $\rho \in (0, 1]$ and $c > 0$ such that

$$\big\|x[k]\big\| \leq c\rho^k\big\|x[0]\big\| \qquad \text{for } k = 0, 1, \ldots$$

For a fixed $\rho \in (0, 1]$, define the set $\ell_2^\rho \subset \ell_2$ of sequences $\{x[k]\}$ such that $\sum_{k=0}^{\infty} \rho^{-2k}\big\|x[k]\big\|^2 < \infty$. Define the

Name of Theorem	M	N	$M+N\prec 0$	Corresponding LMI
Extended conic sector theorem [6]	$\begin{bmatrix}-a & \frac{1}{2}(1+\frac{a}{b})\\ \frac{1}{2}(1+\frac{a}{b}) & -\frac{1}{b}\end{bmatrix}$	$\begin{bmatrix}-\frac{1}{d} & -\frac{1}{2}(1+\frac{c}{d})\\ -\frac{1}{2}(1+\frac{c}{d}) & -c\end{bmatrix}$	$d<-\frac{1}{a}, d\neq 0$ $c>-\frac{1}{b}, b\neq 0$	CS Lemma [30] ECS Lemma [6]
Conic sector theorem [5, Thm. 2a]	$\begin{bmatrix}-a & \frac{1}{2}(1+\frac{a}{b})\\ \frac{1}{2}(1+\frac{a}{b}) & -\frac{1}{b}\end{bmatrix}$, $b>0$	$\begin{bmatrix}-\frac{1}{d} & -\frac{1}{2}(1+\frac{c}{d})\\ -\frac{1}{2}(1+\frac{c}{d}) & -c\end{bmatrix}$	$d<-\frac{1}{a}, d\neq 0$ $c>-\frac{1}{b}, b\neq 0$	CS Lemma [30] ECS Lemma [6]
Circle criterion [3]	$\begin{bmatrix}-a & \frac{1}{2}(1+\frac{a}{b})\\ \frac{1}{2}(1+\frac{a}{b}) & -\frac{1}{b}\end{bmatrix}$ $b\geq a\geq 0$, G LTI	$\begin{bmatrix}-\frac{1}{d} & -\frac{1}{2}(1+\frac{c}{d})\\ -\frac{1}{2}(1+\frac{c}{d}) & -c\end{bmatrix}$	$d<-\frac{1}{a}, d\neq 0$ $c>-\frac{1}{b}, b\neq 0$	ECS Lemma [31]
Extended passivity [7, Thm. 6.6.58]	$\begin{bmatrix}-\varepsilon_2 & \frac{1}{2}\\ \frac{1}{2} & -\delta_2\end{bmatrix}$	$\begin{bmatrix}-\delta_1 & -\frac{1}{2}\\ -\frac{1}{2} & -\varepsilon_1\end{bmatrix}$	$\delta_1+\varepsilon_2>0$ $\delta_2+\varepsilon_1>0$	DPR lemma [32] ($\delta_1=\varepsilon_1=0$)
Extended passivity [4, Thm. 5.10]	$\begin{bmatrix}-\varepsilon_1+\lambda\rho_1 & \frac{1}{2}\\ \frac{1}{2} & -\lambda\end{bmatrix}$	$\begin{bmatrix}-\delta_2 & -\frac{1}{2}\\ -\frac{1}{2} & 0\end{bmatrix}$	$\varepsilon_1+\delta_2>0$	DPR lemma [32] ($\delta_2=0$)
Small gain theorem [2, Thm. 5.6]	$\begin{bmatrix}\gamma & 0\\ 0 & -\frac{1}{\gamma}\end{bmatrix}$	$\begin{bmatrix}-\frac{1}{\varepsilon} & 0\\ 0 & \varepsilon\end{bmatrix}$	$\gamma\varepsilon<1$	DBR lemma [33] ($\varepsilon=1$)

Table 1: Summary of some results in the literature that can be recovered with Corollary 9 and 14. We list values of M and N that correspond to the positive feedback convention. For negative feedback, replace N by $\tilde{N}$ as in Remark 10. We made use of the following abbreviations. CS: Conic sector lemma, ECS: Exterior conic sector lemma. DPR lemma: Discrete-time positive real lemma. DBR: Discrete-time bounded real lemma.

.

corresponding family of semi-inner product as

$$\langle x,\, y\rangle_T^\rho := \sum_{k=0}^{T} \rho^{-2k} \langle x[k],\, y[k]\rangle .$$

Rewriting Theorem 2 for this semi-inner product yields:

Corollary 13 (Exponential stability)**.** *Let $M \in \mathbb{R}^{2\times 2}$ and $\rho \in (0,1]$ be given and suppose $G : \ell_{2e} \to \ell_{2e}$ is LTI. The following statements are equivalent.*

(i) There exists $N \in \mathbb{R}^{2\times 2}$ satisfying $M + N \prec 0$ such that for all $\xi \in \ell_{2e}$ and $T \geq 0$, G satisfies

$$\left\langle \begin{bmatrix} G\xi \\ \xi \end{bmatrix},\, N \begin{bmatrix} G\xi \\ \xi \end{bmatrix} \right\rangle_T^\rho \geq 0. \tag{14}$$

(ii) Let $x[k]$ be the state of an observable realization of G. There exists $c > 0$ such that for all $\Phi : \ell_{2e} \to \ell_{2e}$ and $T \geq 0$ that satisfy

$$\left\langle \begin{bmatrix} \xi \\ \Phi\xi \end{bmatrix},\, M \begin{bmatrix} \xi \\ \Phi\xi \end{bmatrix} \right\rangle_T^\rho \geq 0 \qquad \text{for all } \xi \in \ell_{2e}, \tag{15}$$

and the interconnection of G and Φ as in Figure 1 is well-posed in the ℓ_2 sense, we have

$$\|x[k]\| \leq c\rho^k \|x[0]\| \quad \text{for } k = 0, 1, \ldots \tag{16}$$

Proof. Substitute $\mathcal{H} = \ell_2^\rho$ with the semi-inner product $\langle\cdot,\,\cdot\rangle_T^\rho$, we obtain Corollary 13 but instead of (16), we have for all $u_1, u_2 \in \ell_{2e}$ and $T \geq 0$

$$\|y_1\|_T^{\rho,2} + \|y_2\|_T^{\rho,2} \leq \gamma^2 \big(\|u_1\|_T^{\rho,2} + \|u_2\|_T^{\rho,2}\big) \tag{17}$$

As in the proof of Corollary 9, γ only depends on the choice of M and N. Therefore, (17) holds for a fixed γ, independent of T and ρ. Taking $T \to \infty$ and invoking well-posedness, we conclude that

$$\|y_1\|^{\rho,2} + \|y_2\|^{\rho,2} \leq \gamma^2 \left(\|u_1\|^{\rho,2} + \|u_2\|^{\rho,2}\right) \tag{18}$$

As in the proof of Corollary 9, we have (17) $\iff$ (18). Inequality (16) then follows from [14, Prop. 5]. ∎

5 Computational Verification

In this section, we provide a way to numerically check if a system satisfies a bound of the form (8) in Corollary 9, either via a linear matrix inequality (LMI) or frequency-domain inequality (FDI). The connection between LMIs, FDIs, and quadratic inequalities has been a point of interest for decades. Classical works include [34] and the KYP lemma [2]. More recent works in this area include the conic sector lemma [30] and exterior conic sector lemma [31]. Here is the main lemma.

Lemma 14 (verifying sector-boundedness)**.** *Suppose $G : \ell_{2e} \to \ell_{2e}$ is a discrete-time LTI system with controllable realization (A, B, C, D), initial state $x[0] = 0$, and transfer function $\tilde{G}(z)$. Let $N = N^\mathsf{T} \in \mathbb{R}^{2\times 2}$ be a given matrix. The following statements are equivalent:*

(i) For all $T \geq 0$ and $u \in \ell_{2e}$, G satisfies the inequality

$$\left\langle \begin{bmatrix} Gu \\ u \end{bmatrix},\, N \begin{bmatrix} Gu \\ u \end{bmatrix} \right\rangle_T \geq 0. \tag{19}$$

(ii) For all $z \in \mathbb{C}$ with $|z| \geq 1$, the following FDI holds

$$\begin{bmatrix} \tilde{G}(z) \\ I \end{bmatrix}^* N \begin{bmatrix} \tilde{G}(z) \\ I \end{bmatrix} \succeq 0. \tag{20}$$

(iii) There exists $P = P^\mathsf{T} \succeq 0$ satisfying the LMI

$$\begin{bmatrix} A^\mathsf{T} PA - P & A^\mathsf{T} PB \\ B^\mathsf{T} PA & B^\mathsf{T} PB \end{bmatrix} \preceq \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^\mathsf{T} N \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \quad (21)$$

Proof. This is a special case of [34, Thm. 3] where

$$w(x,u) = \begin{bmatrix} x[k] \\ u[k] \end{bmatrix}^\mathsf{T} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^\mathsf{T} N \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \begin{bmatrix} x[k] \\ u[k] \end{bmatrix}$$

Note that stability of G is not required. ∎

Remark 15. *In Item (i) of Lemma 14, if we add the constraint that $\lim_{k\to\infty} x[k] = 0$, then the domain of z in (20) changes to $|z| = 1$ and the semi-definiteness requirement on P in (21) changes to $P = P^\mathsf{T}$.*

Just as passivity, small-gain, and the circle criterion are special cases of Theorem 2, several existing results follow directly from Lemma 14 in a similar fashion. As an illustrative example, we show how Lemma 14 recovers the discrete bounded real lemma [33] below.

Theorem 16 (Discrete-time bounded real lemma)**.** *Let $G(z)$ be a real rational $p \times m$ transfer matrix, and let the real matrices (A, B, C, D) represent a minimal state-space realization of $G(z)$. Then $G(z)$ is bounded real if and only if there exist real matrices L, W and a real symmetric positive definite matrix P, such that*

$$\begin{aligned} A^\mathsf{T} PA + C^\mathsf{T} C + L^\mathsf{T} L &= P \\ B^\mathsf{T} PB + D^\mathsf{T} D + W^\mathsf{T} W &= I \\ A^\mathsf{T} PB + C^\mathsf{T} D + L^\mathsf{T} W &= 0 \end{aligned}$$

To recover this result using Lemma 14, choose

$$N = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

and let (C, A) be observable. The upper left block of (21) will force $P = P^\mathsf{T} \succ 0$. Inequality (19) states that the linear system should be nonexpansive. That is,

$$\sum_{k=0}^{T} y[k]^\mathsf{T} y[k] \le \sum_{k=0}^{T} u[k]^\mathsf{T} u[k]$$

The FDI (20) states that the transfer matrix $\tilde{G}(z)$ is discrete bounded real: $\tilde{G}(z)^* \tilde{G}(z) \le I$ for $|z| \ge 1$.

Similar specializations of Lemma 14 apply to the exterior conic sector lemma [31], discrete positive real lemma [32], and conic sector lemma [30]. Results are summarized in the last column of Table 1. We recover discrete-time counterparts in cases where the original results were in continuous time, though extensions to continuous time are straightforward.

The idea of Lemma 14 can be generalized to derive a similar result for verifying exponential stability.

Corollary 17 (Exponential stability)**.** *Suppose $G : \ell_{2e} \to \ell_{2e}$ is a discrete-time LTI system with controllable realization (A, B, C, D), initial state $x[0] = 0$, and transfer function $\tilde{G}(z)$. Let $N = N^\mathsf{T} \in \mathbb{R}^{2\times 2}$ and $\rho \in (0, 1]$ be given. The following statements are equivalent*

(i) For all $T \ge 0$ and $u \in \ell_{2e}$, G satisfies the inequality

$$\left\langle \begin{bmatrix} Gu \\ u \end{bmatrix}, N \begin{bmatrix} Gu \\ u \end{bmatrix} \right\rangle_T^\rho \ge 0. \quad (22)$$

(ii) For all $z \in \mathbb{C}$ with $|z| \ge 1$, the following FDI holds

$$\begin{bmatrix} \tilde{G}(\rho z) \\ I \end{bmatrix}^* N \begin{bmatrix} \tilde{G}(\rho z) \\ I \end{bmatrix} \succeq 0 \quad (23)$$

(iii) There exists $P = P^\mathsf{T} \succeq 0$ and $\rho \in (0, 1]$ which satisfies the LMI

$$\begin{bmatrix} A^\mathsf{T} PA - \rho^2 P & A^\mathsf{T} PB \\ B^\mathsf{T} PA & B^\mathsf{T} PB \end{bmatrix} \preceq \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^\mathsf{T} N \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \quad (24)$$

Proof. We can write (22) as

$$\sum_{k=0}^{T} \begin{bmatrix} \rho^{-k} y[k] \\ \rho^{-k} u[k] \end{bmatrix}^\mathsf{T} N \begin{bmatrix} \rho^{-k} y[k] \\ \rho^{-k} u[k] \end{bmatrix} \ge 0$$

Define $\tilde{u}[k] := \rho^{-k} u[k]$, $\tilde{y}[k] := \rho^{-k} y[k]$, $\tilde{A} := \rho^{-1} A$, $\tilde{B} := \rho^{-1} B$, and $\tilde{P} := \rho^{-2} P$. Noting that $\ell_{2e}^\rho = \ell_{2e}$, we can apply Lemma 14 to the system

$$\begin{aligned} \tilde{x}[k+1] &= \tilde{A}\tilde{x}[k] + \tilde{B}\tilde{u}[k] \\ \tilde{y}[k] &= C\tilde{x}[k] + D\tilde{u}[k] \end{aligned}$$

which yields (24). The associated transfer function is:

$$\tilde{C}(zI - \tilde{A})^{-1}\tilde{B} + \tilde{D} = C(\rho z I - A)^{-1} B + D = \tilde{G}(\rho z)$$

and applying Lemma 14 completes the proof. ∎

6 Conclusion

In this paper, we introduced a robust stability theorem (Theorem 2) framed in a general semi-inner product space. Our result unifies many existing results, including passivity, small-gain, and circle theorems. This includes both necessary-and-sufficient as well as sufficient-only versions. Moreover, our theorem leads to a new result on exponential stability (Corollary 13) and corresponding LMI and FDI conditions (Corollary 17) for efficient numerical verification.

7 Acknowledgments

The first author would like to thank L. Bridgeman, A. Packard, A. van der Schaft, B. Van Scoy, and M. Vidyasagar for helpful discussion.

References

[1] A. Lur'e and V. Postnikov, "On the theory of stability of control systems," *Applied mathematics and mechanics*, vol. 8, no. 3, pp. 246–248, 1944.

[2] H. Khalil, *Nonlinear systems.* Upper Saddle River, N.J: Prentice Hall, 2002.

[3] M. S. Fadali and A. Visioli, *Digital control engineering: analysis and design.* Academic Press, 2012.

[4] C. A. Desoer and M. Vidyasagar, *Feedback systems: input-output properties.* SIAM, 2009, vol. 55.

[5] G. Zames, "On the input-output stability of time-varying nonlinear feedback systems part one: Conditions derived using concepts of loop gain, conicity, and positivity," *IEEE Trans. Autom. Control*, vol. 11, no. 2, pp. 228–238, 1966.

[6] L. J. Bridgeman and J. R. Forbes, "The extended conic sector theorem," *IEEE Trans. Autom. Control*, vol. 61, no. 7, pp. 1931–1937, 2016.

[7] M. Vidyasagar, *Nonlinear Systems Analysis*, 2nd ed. Society for Industrial and Applied Mathematics, 2002.

[8] R. Brockett, "The status of stability theory for deterministic systems," *IEEE Trans. Autom. Control*, vol. 11, no. 3, pp. 596–606, 1966.

[9] S. Z. Khong and A. van der Schaft, "On the converse of the passivity and small-gain theorems for input–output maps," *Automatica*, vol. 97, pp. 58–63, 2018.

[10] L. Lessard, B. Recht, and A. Packard, "Analysis and design of optimization algorithms via integral quadratic constraints," *SIAM J. Optim.*, vol. 26, no. 1, pp. 57–95, 2016.

[11] S. Cyrus, B. Hu, B. V. Scoy, and L. Lessard, "A robust accelerated optimization algorithm for strongly convex functions," in *Amer. Control Conf.*, 2018, pp. 1376–1381.

[12] A. Rantzer and A. Megretski, "System analysis via integral quadratic constraints : Part II," Department of Automatic Control, Lund University, Sweden, Technical Report, Feb. 1997.

[13] U. Jönsson and A. Rantzer, "Systems with uncertain parameters—time-variations with bounded derivatives," *Int. J. Robust Nonlinear Contr.*, vol. 6, no. 9-10, pp. 969–982, 1996.

[14] R. Boczar, L. Lessard, A. Packard, and B. Recht, "Exponential stability analysis via integral quadratic constraints," *arXiv preprint arXiv:1706.01337*, 2017.

[15] B. Hu and P. Seiler, "Exponential decay rate conditions for uncertain linear systems using integral quadratic constraints," *IEEE Trans. Autom. Control*, vol. 61, no. 11, pp. 3631–3637, 2016.

[16] L. J. Bridgeman and J. R. Forbes, "A comparative study of input–output stability results," *IEEE Trans. Autom. Control*, vol. 63, no. 2, pp. 463–476, 2018.

[17] B. Anderson, "The small-gain theorem, the passivity theorem and their equivalence," *J. Franklin Inst.*, vol. 293, no. 2, pp. 105–115, 1972.

[18] J. C. Willems, "Dissipative dynamical systems part II: Linear systems with quadratic supply rates," *Archive for rational mechanics and analysis*, vol. 45, no. 5, pp. 352–393, 1972.

[19] A. Megretski and A. Rantzer, "System analysis via integral quadratic constraints," *IEEE Trans. Autom. Control*, vol. 42, no. 6, pp. 819–830, 1997.

[20] H. Pfifer and P. Seiler, "Integral quadratic constraints for delayed nonlinear and parameter-varying systems," *Automatica*, vol. 56, pp. 36–43, 2015.

[21] A. R. Teel, T. Georgiou, L. Praly, and E. Sontag, "Input-output stability," *The control handbook*, vol. 1, pp. 895–908, 1996.

[22] M. G. Safonov, *Stability and robustness of multivariable feedback systems.* MIT press, 1980.

[23] J. B. Conway, *A course in functional analysis.* Springer-Verlag, New York, 1990.

[24] M. J. McCourt and P. J. Antsaklis, "Control design for switched systems using passivity indices," in *American Control Conference.* IEEE, 2010, pp. 2499–2504.

[25] U. Jönsson, "A lecture on the s-procedure," *Lecture Note at the Royal Institute of technology, Sweden*, vol. 23, pp. 34–36, 2001.

[26] A. Megretski and S. Treil, "Power distribution inequalities in optimization and robustness of uncertain systems," *J. Math. Syst., Estimation Contr.*, vol. 3, no. 3, p. 301–319, 1993.

[27] M. R. Hestenes, *Optimization theory: The finite dimensional case.* Wiley, 1975.

[28] W. M. Haddad, E. G. Collins, and D. S. Bernstein, "Robust stability analysis using the small gain, circle, positivity, and popov theorems: A comparative study," *IEEE Trans. Control Syst. Technol.*, vol. 1, no. 4, pp. 290–293, 1993.

[29] K. Narendra and J. H. Taylor, *Frequency domain criteria for absolute stability.* New York: Academic Press, 1973.

[30] S. Gupta and S. M. Joshi, "Some properties and stability results for sector-bounded LTI systems," in *Proc. 33rd IEEE Conf. Decision Control*, vol. 3, 1994, pp. 2973–2978.

[31] L. J. Bridgeman and J. R. Forbes, "The exterior conic sector lemma," *Int. J. Control*, vol. 88, no. 11, pp. 2250–2263, 2015.

[32] L. Hitz and B. Anderson, "Discrete positive-real functions and their application to system stability," in *Proceedings of the Institution of Electrical Engineers*, vol. 116, no. 1. IET, 1969, pp. 153–155.

[33] P. Vaidyanathan, "The discrete-time bounded-real lemma in digital filtering," *IEEE Trans. Circuits Syst.*, vol. 32, no. 9, pp. 918–924, 1985.

[34] J. Willems, "Least squares stationary optimal control and the algebraic riccati equation," *IEEE Trans. Autom. Control*, vol. 16, no. 6, pp. 621–634, 1971.