SCHOOL OF MATHEMATICS AND STATISTICS

STAT 393 **Assignment 1** **Due: 11:59pm Thursday 28 July 2022**

Linear Models

This assignment is worth 10%

1. In assessing the impact of rainfall on the growth pattern of tussock grass (*Chionochloa rubra*), several square metre plots were observed in 9 locations. 4 locations were in one block A, and 5 in a second block B. In each plot the annual rainfall (x) and the number of shoots per square metre (y) were recorded.

Block	A	A	A	A	B	B	B	B	B
Annual rainfall (mm), x	47	26	116	178	19	75	160	31	12
No. shoots/m^2, y	15.1	14.1	12.3	12.7	14.6	13.8	11.9	14.8	15.3

   (a) Draw a scatterplot of the data, using distinct symbols for each block. Attach the plot created in R.

   (b) Ignoring the grouping of the plots into blocks, fit a linear regression model $y = \alpha + \beta x$ in R. State the equation of the fitted regression line to 4 significant figures.

   (c) Draw the regression line on your scatterplot. Attach the plot created in R.

   (d) Write down the model/design matrix (X) and $\boldsymbol{\beta}$ for the regression model in (b) in the form $\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\varepsilon}$.

   (e) The ANCOVA model allowing for different regression relationships in each block is

$$y_{ij} = \mu + \alpha_i + \beta_0 x_{ij} + \beta_i x_{ij} + \varepsilon_{ij},$$

   where i refers to Block: $i = 1$ (A), and 2 (B). Treating Block B as the reference block. Write down the **constraints** on the parameters.
   Hint: In R, use

```
Block <- factor(Block)
lm(y~ relevel(Block, ref="B") + x + relevel(Block, ref="B"):x, x=T)
```

   to fit the model.

   (f) The model in (e) can be expressed in the form $\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\varepsilon}$. Write down $\mathbf{y}$, X, and $\boldsymbol{\beta}$ for the rainfall data.

   (g) Fit the model in (e) in R. State the parameter estimates, including $\widehat{\sigma}$ to 4 significant figures.

   (h) Are the slopes of two blocks the same? Justify your answer.

2. Let

$$B = \begin{bmatrix} 1 & 5 & 0 \\ 1 & 0 & 5 \\ 1 & 0 & 5 \end{bmatrix}$$

   (a) Are the column vectors of B linearly independent?

   (b) What is the rank of B?

   (c) Is B full row rank?

3. For a $n \times p$ matrix A of full column rank ($n \geq p$), we define the **generalised inverse** of A by

$$A^- = (A^T A)^{-1} A^T$$

   (a) State the dimensions of the following matrices:

$$\begin{array}{cccc} A & A^T & A^T A & (A^T A)^{-1} \\ A^- & AA^- & A^- A & \end{array}$$

   (b) Show that AA^- is symmetric

   (c) Show that AA^- is idempotent

   (d) What is the rank of AA^-?

   (e) Show that $A^- A = I$

4. If $\mathbf{y} \sim N(\boldsymbol{\mu}, \Sigma)$ and A has full row rank, it follows that $A\mathbf{y} \sim N(A\boldsymbol{\mu}, A\Sigma A^T)$ (Theorem 2 in the notes on p. 39).

   Use this result to find the joint distribution of w_1 and w_2, where $w_1 = y_1 + y_2 - 2y_3$ and $w_2 = y_1 + y_2$ given that $\mathbf{y} \sim N(\boldsymbol{\mu}, \Sigma)$, with

$$\boldsymbol{\mu} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \Sigma = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}.$$

5. Let $\mathbf{y}_{n\times 1} \sim N(\boldsymbol{\mu}, \sigma^2 I)$, where $\boldsymbol{\mu}^T = (\mu, \mu, \ldots, \mu)$. Find the distribution of the mean $\bar{y}$ using Theorem 2 in the notes on p. 39.

SCHOOL OF MATHEMATICS AND STATISTICS

STAT 393 **Assignment 1**

Linear Models
SOLUTIONS

1.

Assignment 1: Q1. ANCOVA solutions

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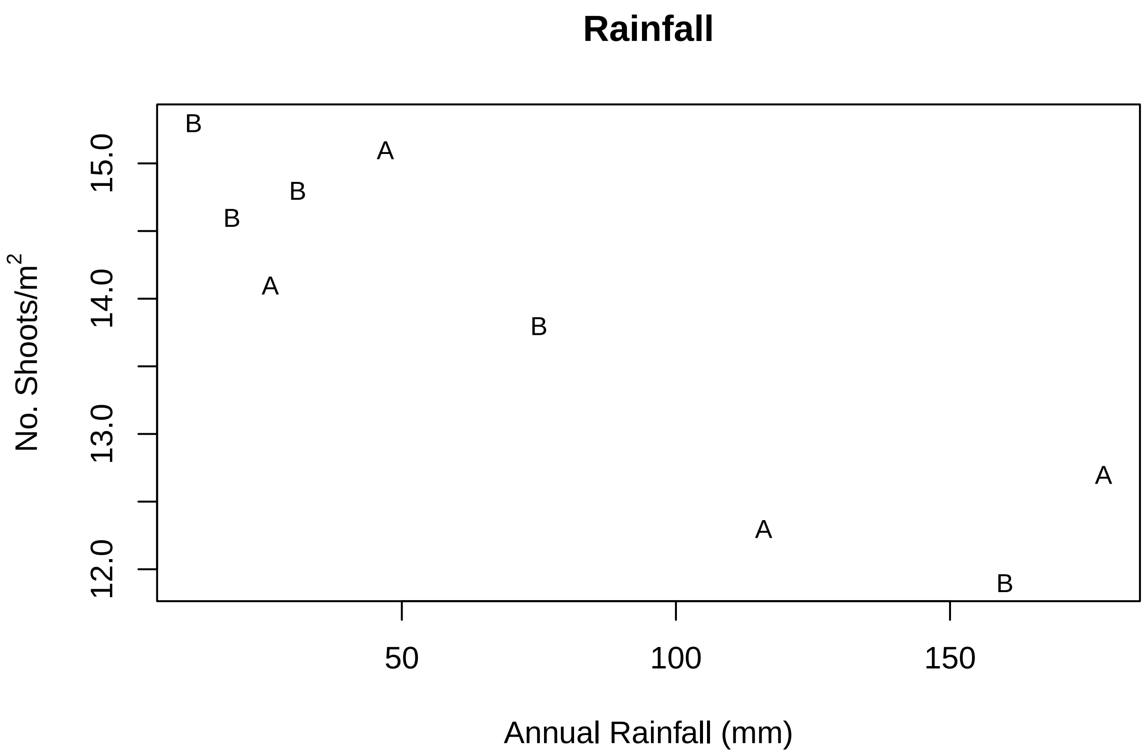
1(a). Scatterplot

```
y <- c(15.1, 14.1, 12.3, 12.7, 14.6, 13.8, 11.9, 14.8, 15.3)
x <- c(47,26,116,178,19,75,160,31,12)
Block <- c(rep("A",4),rep("B",5))
plot(x,y, pch=16, main="Rainfall",
     xlab="Annual Rainfall (mm)", ylab=expression("No. Shoots/"*m^2), type="n")
text(x,y, labels=Block, cex=0.8, font=1)
```



1 (b). Fit a linear regression model $y = \alpha + \beta x$

```
# Fit the regression line
lm1 <- lm(y~x, x=T)
summary(lm1)
```

```
##
## Call:
```

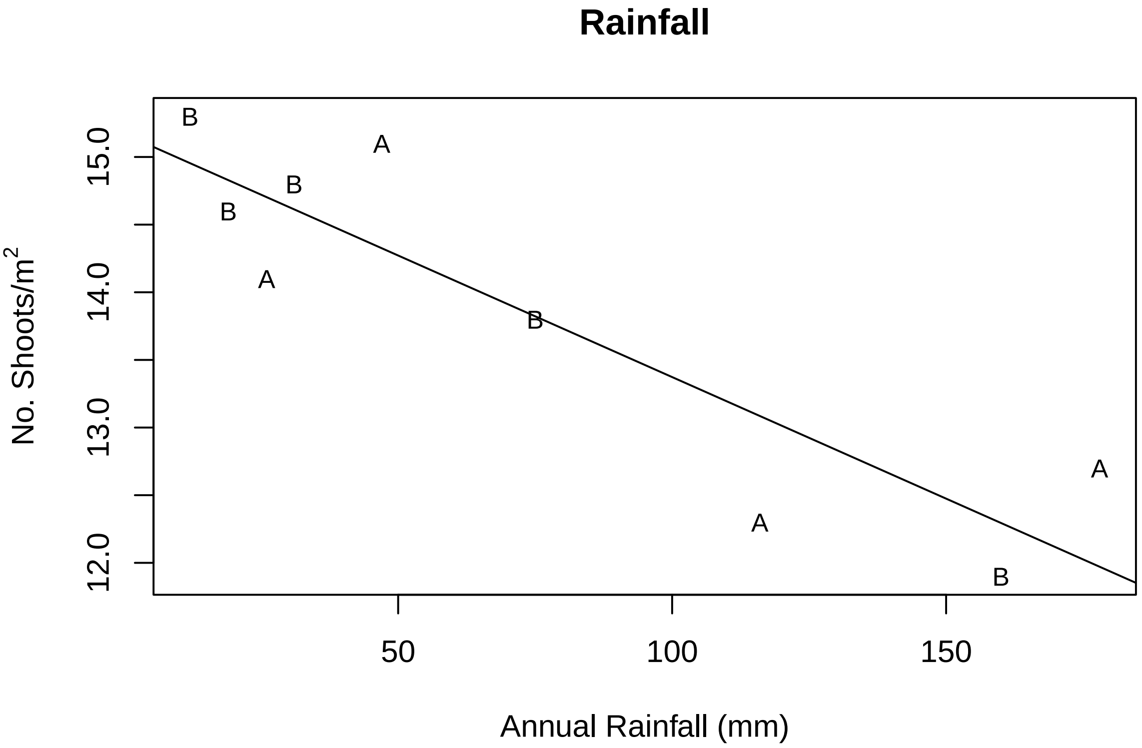
```
## lm(formula = y ~ x, x = T)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -0.78562 -0.39485 -0.02248  0.34528  0.77430
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) 15.170387   0.316096  47.993 4.46e-10 ***
## x           -0.017972   0.003338  -5.384  0.00103 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.5945 on 7 degrees of freedom
## Multiple R-squared:  0.8055, Adjusted R-squared:  0.7777
## F-statistic: 28.99 on 1 and 7 DF,  p-value: 0.001026
```

The fitted model is $\widehat{y} = 15.17 - 0.01797x$.

1 (c). Draw the regression line on the scatterplot.

```
plot(x,y, pch=16, main="Rainfall",
     xlab="Annual Rainfall (mm)", ylab=expression("No. Shoots/"*m^2), type="n")
text(x,y, labels=Block, cex=0.8, font=1)

# Draw on the regression line
abline(lm1)
```



1 (d). The model/design matrix (X) and the parameter vector $\boldsymbol{\beta}$.

```
lm1$x
```

```
##   (Intercept)   x
## 1           1  47
## 2           1  26
## 3           1 116
## 4           1 178
## 5           1  19
## 6           1  75
## 7           1 160
## 8           1  31
## 9           1  12
## attr(,"assign")
## [1] 0 1
```

$$X = \begin{pmatrix} 1 & 47 \\ 1 & 26 \\ 1 & 116 \\ 1 & 178 \\ 1 & 19 \\ 1 & 75 \\ 1 & 160 \\ 1 & 31 \\ 1 & 12 \end{pmatrix}, \quad \boldsymbol{\beta} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

1 (e). ANCOVA

The ANCOVA model allowing for different regression relationships in each block is

$$y_{ij} = \mu + \alpha_i + \beta_0 x_{ij} + \beta_i x_{ij} + \varepsilon_{ij},$$

where i refers to Block: $i = 1$ (A), and 2 (B).

Treating Block B as the reference block.

This model matrix has constraints $\alpha_2 = 0$ and $\beta_2 = 0$.

1 (f). Write down y, X, and $\boldsymbol{\beta}$ for the ANCOVA model.

$$\mathbf{y} = \begin{pmatrix} 15.1 \\ 14.1 \\ 12.3 \\ 12.7 \\ 14.6 \\ 13.8 \\ 11.9 \\ 14.8 \\ 15.3 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & 1 & 47 & 47 \\ 1 & 1 & 26 & 26 \\ 1 & 1 & 116 & 116 \\ 1 & 1 & 178 & 178 \\ 1 & 0 & 19 & 0 \\ 1 & 0 & 75 & 0 \\ 1 & 0 & 160 & 0 \\ 1 & 0 & 31 & 0 \\ 1 & 0 & 12 & 0 \end{pmatrix}, \quad \boldsymbol{\beta} = \begin{pmatrix} \mu \\ \alpha_1 \\ \beta_0 \\ \beta_1 \end{pmatrix}$$

1 (g). Fit the ANCOVA model.

```
Block <- factor(Block)
lm2 <- lm(y~ relevel(Block, ref="B") + x + relevel(Block, ref="B"):x, x=T)
summary(lm2)
```

```
##
## Call:
## lm(formula = y ~ relevel(Block, ref = "B") + x + relevel(Block,
##     ref = "B"):x, x = T)
##
## Residuals:
##        1        2        3        4        5        6        7        8
##  0.89790 -0.40812 -0.89663  0.40685 -0.34700  0.05478 -0.02110  0.11053
##        9
##  0.20278
##
## Coefficients:
##                              Estimate Std. Error t value Pr(>|t|)
## (Intercept)                 15.354740   0.429138  35.780 3.21e-07 ***
## relevel(Block, ref = "B")A   -0.467748   0.733993  -0.637  0.55198
## x                           -0.021460   0.005307  -4.044  0.00989 **
## relevel(Block, ref = "B")A:x  0.006888   0.007596   0.907  0.40609
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.6511 on 5 degrees of freedom
## Multiple R-squared:  0.8334, Adjusted R-squared:  0.7335
## F-statistic: 8.338 on 3 and 5 DF,  p-value: 0.02166
```

$\widehat{\mu} = 15.35$, $\widehat{\alpha}_1 = -0.4677$, $\widehat{\beta}_0 = -0.02146$, $\widehat{\beta}_1 = 0.006888$, and $\widehat{\sigma} = 0.6511$.

Note: The model matrix X can also be obtained from `lm()`.

```
lm2$x
```

```
##   (Intercept) relevel(Block, ref = "B")A   x relevel(Block, ref = "B")A:x
## 1           1                          1  47                           47
## 2           1                          1  26                           26
## 3           1                          1 116                          116
## 4           1                          1 178                          178
## 5           1                          0  19                            0
## 6           1                          0  75                            0
## 7           1                          0 160                            0
## 8           1                          0  31                            0
## 9           1                          0  12                            0
## attr(,"assign")
## [1] 0 1 2 3
## attr(,"contrasts")
## attr(,"contrasts")$`relevel(Block, ref = "B")`
## [1] "contr.treatment"
```

1 (h).

When the slopes of two blocks are the same, β_1 should be zero. It means that there is no interaction term (block:x) in the model. Thus, we need to test

$$H_0 : \beta_1 = 0 \text{ vs. } H_1 : \beta_1 \neq 0$$

From 1 (g), the p-value of the test is 0.406 (> 0.05).

We conclude that we do not reject H_0. The slopes of two blocks are not significantly different at a 5% level.

2. Let

$$B = \begin{bmatrix} 1 & 5 & 0 \\ 1 & 0 & 5 \\ 1 & 0 & 5 \end{bmatrix}$$

(a) Are the column vectors of B linearly independent?

No: the first column is a linear combination of the second two:

$$\mathbf{b}_1 = \frac{1}{5}\mathbf{b}_2 + \frac{1}{5}\mathbf{b}_3$$

(b) What is the rank of B?

The rank of a square matrix is the number of linearly independent columns: in this case two: $r(B) = 2$.

(c) Is B full row rank?

No: if a square matrix is not of full column rank it cannot be of full row rank. (Its column rank is 2 but its dimension is 3).

We can also see this by noting that the second and third rows of B are equal, so it does not have 3 linearly independent rows, and so it is not of full row rank.

3. For a $n \times p$ matrix A of full column rank ($n \geq p$), we define the **generalised inverse** of A by

$$A^- = (A^TA)^{-1}A^T$$

(a) State the dimensions of the following matrices:

$$\begin{array}{llllllll} A & (n \times p) & A^T & (p \times n) & A^TA & (p \times p) & (A^TA)^{-1} & (p \times p) \\ A^- & (p \times n) & AA^- & (n \times n) & A^-A & (p \times p) & & \end{array}$$

(b) Show that AA^- is symmetric

$$\begin{aligned} (AA^-)^T &= (A(A^TA)^{-1}A^T)^T \\ &= A(A^TA)^{-1}A^T \\ &= AA^- \end{aligned}$$

hence AA^- is symmetric.

(c) Show that AA^- is idempotent

$$\begin{aligned} (AA^-)(AA^-) &= (A(A^TA)^{-1}A^T)(A(A^TA)^{-1}A^T) \\ &= A(A^TA)^{-1}A^TA(A^TA)^{-1}A^T \\ &= A(A^TA)^{-1}A^T \\ &= AA^- \end{aligned}$$

hence AA^- is idempotent.

(d) What is the rank of AA^-?

Since AA^- is idempotent $r(AA^-) = \mathrm{tr}(AA^-)$:

$$\begin{aligned} r(AA^-) &= \mathrm{tr}(AA^-) \\ &= \mathrm{tr}(A(A^TA)^{-1}A^T) \\ &= \mathrm{tr}(A^TA(A^TA)^{-1}) \\ &= \mathrm{tr}(I_{p \times p}) \\ &= p \end{aligned}$$

(e) Show that $A^- A = I$

$$A^- A = (A^T A)^{-1} A^T A = I$$

4. If $\mathbf{y} \sim N(\boldsymbol{\mu}, \Sigma)$ and A has full row rank, it follows that $A\mathbf{y} \sim N(A\boldsymbol{\mu}, A\Sigma A^T)$.

   Use this result to find the joint distribution of w_1 and w_2, where $w_1 = y_1 + y_2 - 2y_3$ and $w_2 = y_1 + y_2$ given that $\mathbf{y} \sim N(\boldsymbol{\mu}, \Sigma)$, with

$$\boldsymbol{\mu} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \Sigma = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}.$$

   First define

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & 0 \end{bmatrix}$$

   so that

$$A\mathbf{y} = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} y_1 + y_2 - 2y_3 \\ y_1 + y_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \mathbf{w}$$

   A has linearly independent rows, so it is of full row rank, so by the Theorem $\mathbf{w} \sim N(\boldsymbol{\mu}_w, \Sigma_w)$, where

$$\boldsymbol{\mu}_w = A\boldsymbol{\mu} = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

   and

$$\Sigma_w = A\Sigma A^T = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 6 \\ 6 & 8 \end{bmatrix}$$

5. Let $\mathbf{y}_{n\times 1} \sim N(\boldsymbol{\mu}, \sigma^2 I)$, where $\boldsymbol{\mu}^T = (\mu, \mu, \ldots, \mu)$. Find the distribution of the mean $\bar{y}$.

   We express the mean as:

$$\bar{y} = \frac{1}{n}\sum_i y_i = \frac{1}{n}\mathbf{1}_n^T \mathbf{y}$$

   so $\bar{y} = A\mathbf{y}$ where $A = \frac{1}{n}\mathbf{1}_n^T$ in which case $\bar{y} \sim N(\mu_{\bar{y}}, \Sigma_{\bar{y}})$, where

$$\mu_{\bar{y}} = A\boldsymbol{\mu} = \frac{1}{n}\mathbf{1}_n^T \boldsymbol{\mu} = \frac{1}{n} n\mu = \mu$$

   and

$$\Sigma_{\bar{y}} = A\Sigma A^T = \frac{1}{n}\mathbf{1}_n^T \sigma^2 I \frac{1}{n}\mathbf{1}_n = \frac{\sigma^2}{n^2}\mathbf{1}_n^T \mathbf{1}_n = \frac{\sigma^2}{n^2} n = \frac{\sigma^2}{n}$$

   i.e. $\bar{y} \sim N(\mu, \sigma^2/n)$.