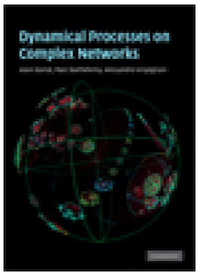

Dynamical Processes on Complex Networks

Alain Barrat, Marc Barthélemy, Alessandro Vespignani

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10

Social networks and collective behavior

The study of collective behavior in social systems has recently witnessed an increasing number of works relying on computational and agent-based models. These models use very simplistic schemes for the micro-processes of social influence and are more interested in the emerging macro-level social behavior. Agent-based models for social phenomena are very similar in spirit to the statistical physics approach. The agents update their internal state through an interaction with their neighbors and the emergent macroscopic behavior of the system is the result of a large number of these interactions.

The behavior of all of these models has been extensively studied for agents located on the nodes of regular lattices or possessing the ability to interact homogeneously with each other. But as described in Chapter 2, interactions between individuals and the structure of social systems can be generally represented by complex networks whose topologies exhibit many non-trivial properties such as small-world, high clustering, and strong heterogeneity of the connectivity pattern. Attention has therefore recently shifted to the study of the effect of more realistic network structures on the dynamical evolution and emergence of social phenomena and organization. In this chapter, we review the results obtained in four prototypical models for social interactions and show the effect of the network topology on the emergence of collective behavior. Finally, the last section of the chapter is devoted to recent research avenues taking into account that network topology and social interactions may be mutually related in their evolution.

10.1 Social influence

Social influence is at the core of social psychology and deals with the effect of other people on individuals' thoughts and behaviors. It underpins innovation adoption, decision-making, rumor spreading, and group problem solving which all unfold at a macro-level and affect disciplines as diverse as economics, political science, and anthropology. The overarching question in these phenomena is how the micro-processes between individuals are related to the macro-level behavior of groups or whole societies.

In particular, an important issue is the understanding of diversity or uniformity of attitudes and beliefs in a large number of interacting agents. If agents interact via linear assimilative influence, meaning that on average the recipient of influence moves some percentage toward the influencer's position, we immediately face the inevitable outcome of a complete uniformity of attitude in the system. This is not what we observe in reality, as minority opinions persist and we often see polarization of various kinds in politics and culture. The uniformity collapse, however, may be avoided by considering several other features of real-world social systems. First of all, social influence is not always a linear mechanism. An attitude that we could imagine as a continuous variable is not the same as a behavior, which is often a discrete variable. This is exemplified by the continuum of political attitudes with respect to the electoral behavior that often results, in the end, in a binary choice. Environmental influences can also be extremely relevant. Sometimes, social influence can generate a contrast that opposes assimilation. A typical example is provided by social identity and the motive to seek distinctiveness of subgroups. Finally, the patterns of connectivity among individuals may be very complex and foster or hinder the emergence of collective behavior and uniformity.

Several pioneering works use the agent-based models to explore how macro-level collective behavior emerges as a function of the micro-level processes of social influence acting among the agents of the system (Granovetter, 1978; Nowak, Szamrej and Latané, 1990; Axelrod, 1997b). These papers adopt an approach that is akin to the statistical physics approach, and the incursions by statistical physicists into the area of social sciences have become frequent. Nowadays, a vast array of agent-based models aimed at the study of social influence have been defined (see the recent review by Castellano, Fortunato and Loreto [2007]). A first class of models is represented by behavioral models where the attributes of agents are binary variables similar to Ising spins (see Chapter 5) as in the case of the Voter model (Krapivsky, 1992), the majority rule model (Galam, 2002; Krapivsky and Redner, 2003a), and the Sznajd model (Sznajd-Weron and Sznajd, 2000). In other instances additional realism has been introduced. Continuous opinion variables have been proposed by Deffuant *et al.* (2000) (see also Ben-Naim, Krapivsky and Redner [2003]) or by Hegselmann and Krause (2002). Along the path opened by Axelrod (1997b), models in which opinions or cultures are represented by vectors of cultural traits have introduced the notion of bounded confidence: an agent will not interact with any other agent independently of their opinions, but only if they are close enough. Finally, a large class of models bears similarities with the propagation of epidemics described in Chapter 9. These models aim at understanding the spread of rumors, information and the sharing of a common knowledge and can also be used to describe data dissemination or marketing campaigns. In all these models, the connectivity pattern among agents is extremely important in determining the macro-level behavior. A complete review of all the models is, however, beyond the scope of this book and in the next sections we focus on some classic models of social influence where the role of fluctuations and heterogeneities in the connectivity patterns has been studied in detail.

10.2 Rumor and information spreading

Rumors and information spreading phenomena are the prototypical examples of social contagion processes in which the infection mechanism can be considered of psychological origin. Social contagion phenomena refer to different processes that depend on the individual propensity to adopt and diffuse knowledge, ideas, or simply a habit. The similarity between social contagion processes and epidemiological models such as those described in Chapter 9 was recognized quite a long time ago (Rapoport, 1953; Goffman and Newill, 1964; Goffman, 1966). We also refer the reader interested in further details to the reviews by Dietz (1967) and Tabah (1999).[1] A simple translation from epidemiological to social vocabulary is in order here. A "susceptible" individual is an agent who has not yet learned the new information, and is therefore called "ignorant". An "infected" individual in epidemiology is a "spreader" in the social context, who can propagate rumors, habits, or knowledge. Finally "recovered" or "immunized" individuals correspond to "stiflers" who are aware (adopters) of the rumor (knowledge) but who no longer spread it. As in the case of epidemic modeling, it is possible to include other compartments at will such as latents or skeptics, and, when data is available, to compare the models with real propagation of ideas (Bettencourt *et al.*, 2006). It is also worth stressing the relevance of these modeling approaches in technological and commercial applications where rapid and efficient spread of information is often desired. To this end, epidemic-based protocols for information spreading may be used for data dissemination and resource discovery on the Internet (Vogels, van Renesse and Birman, 2003; Kermarrec, Massoulie and Ganesh, 2003) or in marketing campaigns using the so-called viral marketing techniques (Moreno, Nekovee and Pacheco, 2004a; Leskovec, Adamic and Huberman, 2006).

[1] An interesting modeling approach which instead considers the propagation of an idea as a cascade phenomenon similar to those of Chapter 11 can be found in Watts (2002).

Social and physiological contagion processes differ, however, in some important features. Some straightforward dissimilarities can be understood at a qualitative level: first of all, the spread of information is an intentional act, unlike a pathogen contamination. Moreover, it is usually advantageous to access a new idea or information and being infected is no longer just a passive process. Finally, acquiring a new idea or being convinced that a new rumor or information is grounded may need time and exposure to more than one source of information, which leads to the definition of models in which memory has an important role (Dodds and Watts, 2004; 2005). Such differences, which are important at the level of interpretation, do not necessarily change the evolution equations of the spreading model. On the other hand, a non-trivial modification to epidemiological models was proposed by Daley and Kendall (1964) in order to construct a stochastic model for the spread of rumors. This important modification takes into account the fact that the transition from a spreader state to a stifler state is not usually spontaneous: an individual will stop spreading the rumor if he encounters other individuals who are already informed. This implies that the process, equivalent to the recovery process in infectious diseases, is no longer a spontaneous state transition, but rather an interaction process among agents.

Following the parallel between disease and information spreading, the rumors model of Daley and Kendall (1964) (see also Daley and Kendall [1965]; Maki and Thompson [1973]; Daley and Gani [2000]) considers that individuals are compartmentalized in three different categories, ignorant, spreaders, and stiflers, described by their respective densities $i(t) = I(t)/N$, $s(t) = S(t)/N$, and $r(t) = R(t)/N$, where N is the number of individuals in the system and $i(t) + s(t) + r(t) = 1$. As in epidemic spreading, a transition from the ignorant (susceptible) to the spreader (infected) compartment is obtained at a rate λ when an ignorant is in contact with a spreader. On the other hand, and in contrast with basic epidemic models, the transition from spreader to stifler is not spontaneous. On the contrary, the recovery occurs only because of the contact of a spreader with either other spreaders or stiflers according to a transition rate α, introducing a key difference with respect to standard epidemic models.[2] The above spreading process can be summarized by the following set of pairwise interactions

$$\begin{cases} I + S \xrightarrow{\lambda} 2S \\ S + R \xrightarrow{\alpha} 2R \\ S + S \xrightarrow{\alpha} R + S. \end{cases} \tag{10.1}$$

[2] The inverse of the rate α can be also seen as a measure of the number of communication attempts with other individuals already knowing the rumor before the spreader turns into a stifler.

Note that these reaction processes correspond to the version of Maki and Thompson (1973) in which the interaction of two spreaders results (at rate α) in the conversion of the first spreader into a stifler; in the version of Daley and Kendall (1964), both are turned into stiflers (the last line is then $S + S \xrightarrow{\alpha} 2R$).

In order to provide some insight on the model's behavior, we first consider the homogeneous hypothesis in which all individuals are interacting at each time step with a fixed number of individuals $\langle k \rangle$ randomly selected in the population. The network of contacts is therefore assumed to have no degree fluctuations and the evolution equations of the densities of ignorants, spreaders and stiflers can be written as

$$\frac{\mathrm{d}i}{\mathrm{d}t} = -\lambda \langle k \rangle i(t) s(t) \tag{10.2}$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = +\lambda \langle k \rangle i(t) s(t) - \alpha \langle k \rangle s(t)[s(t) + r(t)] \tag{10.3}$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \alpha \langle k \rangle s(t)[s(t) + r(t)]. \tag{10.4}$$

These equations are derived by using the usual homogenous assumption as shown in Chapters 4 and 9. The terms $i(t)s(t)$ and $s(t)[s(t) + r(t)]$ are simply the mass action laws expressing the force of infection and recovery in the population. As anticipated in the model's description, the difference from a Susceptible–Infected–Removed model (see Chapter 9) lies in the non-linear decay rate $s(t)[s(t)+r(t)]$ of Equation (10.3), or equivalently in the right-hand side of Equation (10.4). Despite this non-linearity, the infinite time limit of these equations can be obtained. In the stationary regime, we can set the time derivatives equal to 0 which implies that $\lim_{t\to\infty} s(t) = 0$: individuals either have remained ignorants or are aware of the rumor but have stopped spreading it. The value of $r_\infty \equiv \lim_{t\to\infty} r(t)$ allows us to understand whether the information propagation process has reached a finite fraction of the population ($r_\infty > 0$) or not ($r_\infty = 0$), starting from e.g. one single initial spreader. This quantity defines the *reliability* r_∞ of the process. Given the initial conditions $s(0) = 1/N$, $i(0) = 1 - s(0)$ and $r(0) = 0$, the equation for the density of ignorants can be formally integrated, yielding

$$i(t) = i(0) \exp\left[-\lambda \langle k \rangle \int_0^t \mathrm{d}\tau\, s(\tau)\right]. \tag{10.5}$$

The insertion of $s(t) + r(t) = 1 - i(t)$ and of Equation (10.2) into Equation (10.4) yield the following relation valid for any time t

$$\int_0^t \mathrm{d}t \frac{\mathrm{d}r}{\mathrm{d}t} = \alpha \langle k \rangle \int_0^t \mathrm{d}\tau\, s(\tau) + \frac{\alpha}{\lambda} \int_0^t \frac{\mathrm{d}i}{\mathrm{d}t}, \tag{10.6}$$

or equivalently

$$\alpha\langle k\rangle \int_0^t \mathrm{d}\tau s(\tau) = r(t) - r(0) - \frac{\alpha}{\lambda}[i(t) - i(0)]. \tag{10.7}$$

Equation (10.5) leads then straightforwardly to

$$i_\infty \equiv \lim_{t\to\infty} i(t) = \exp\left(-\beta r_\infty\right), \tag{10.8}$$

with $\beta = 1 + \lambda/\alpha$ and where we have used $r(0) = 0$, $i(0) \approx 1$ and $i_\infty + r_\infty = 1$. The transcendental equation for the density of stiflers at the end of the spreading process finally reads (Sudbury, 1985)

$$r_\infty = 1 - \mathrm{e}^{-\beta r_\infty}. \tag{10.9}$$

The solutions of this equation can be obtained similarly to those of Equation (6.10) for the percolation phenomenon. It can indeed be written as $r_\infty = F(r_\infty)$, where $F(x) = 1 - \exp(-\beta x)$ is a monotonously increasing continuous function with $F(0) = 0$ and $F(1) < 1$. A non-zero solution can therefore exist if and only if the first derivative $F'(0) > 1$, leading to the inequality

$$\frac{\mathrm{d}}{\mathrm{d}r_\infty}(1 - \mathrm{e}^{-\beta r_\infty})\bigg|_{r_\infty=0} > 1, \tag{10.10}$$

which yields the condition on the spreading rate for the obtention of a finite density of stiflers at large times

$$\frac{\lambda}{\alpha} > 0. \tag{10.11}$$

Interestingly, and in strong contrast with the case of epidemic spreading, this condition is always fulfilled for any arbitrary positive spreading rate. The introduction of the non-linear term by Daley and Kendall (1964) in the description of the propagation has therefore the far-reaching consequence of removing the epidemic threshold: the rumor has always a non-zero probability of pervading a macroscopic fraction of the system whatever the values of the rates α and λ.

Information, like diseases, propagates along the contacts between individuals in the population. The transmission of information or rumors uses various types of social or technological networks (collaboration or friendship networks, email networks, telephone networks, WWW, the Internet...), which typically are far from being homogeneous random networks. Therefore the definition and structure of the contact network will be of primary importance in determining the properties of the spreading process. The first studies of rumors models in complex networks have focused on the effect of clustering and small-world properties. Zanette (2001) (see also Zanette [2002]) has investigated the spreading phenomenon (with $\lambda = \alpha = 1$) in Watts–Strogatz (WS) small-world networks through numerical simulations. In particular, the final fraction of stiflers, r_∞, has been measured as a function of the disorder in the network structure, quantified by the rewiring probability p of the WS network. As p increases, more and more shortcuts exist between distant regions of the initial one-dimensional chain, and one can expect that the information will propagate better. In fact, Zanette (2001) provided evidence of a striking transition phenomenon between a regime, for $p < p_c$, in which the rumor does not spread out of the neighborhood of its origin, and a regime where it is finally known to a finite fraction of the population, as shown in Figure 10.1. The value of p_c, which is close to 0.2 for a network with $\langle k \rangle = 4$, decreases when $\langle k \rangle$ increases. The rationale behind the appearance of this non-equilibrium phase transition is the following: at small p, the network is highly locally clustered (just as the one-dimensional chain at $p = 0$), so that many interactions take place on triangles. A newly informed spreader will therefore interact with large probability with other neighbors of the individual who has informed him, thereby easily becoming a stifler and stopping the propagation. The "redundant" links forming triangles therefore act as an inhibiting structure and keep the information localized. When enough links are instead devoted to long-range random connections, the information can spread out of a given neighborhood before the spreaders become stiflers. It is particularly interesting to notice that this dynamic transition occurs at a finite value of the disorder p, in contrast with the case of equilibrium dynamics described in Chapter 5, for which any strictly positive p leads to a transition. The dynamics at $p > p_c$ presents an additional feature which is also present in epidemics: for a given network and a given initial condition, different stochastic realizations of the spreading can lead either to a rapid extinction ($r_\infty = 0$), or to a propagation affecting a macroscopic

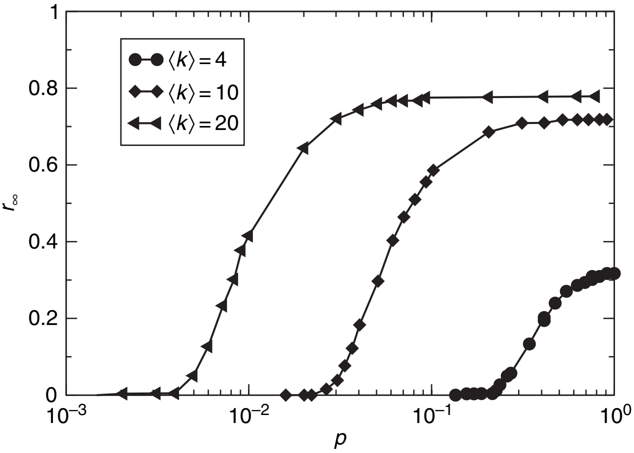

Fig. 10.1. Reliability of the rumor spreading process as a function of p, for Watts–Strogatz networks of size $N = 10^5$ and various values of the average degree $\langle k \rangle$. Data from Zanette (2001).

fraction of the population. The average over all realizations of r_∞ is therefore positive but in fact hides a bimodal distribution where the fraction of realizations with finite r_∞ naturally grows as p increases.

In order to consider how the model of Maki and Thompson (1973) is affected by the possible degree heterogeneity of the network on which the spreading takes place, the densities of ignorants, spreaders, and stiflers in each class of degree k have to be introduced, namely i_k, s_k and r_k. The evolution equations for these densities can be written as

$$\begin{aligned}
\frac{\mathrm{d}i_k}{\mathrm{d}t} &= -\lambda k i_k(t) \sum_{k'} s_{k'}(t) \frac{k'-1}{k'} P(k'|k) \\
\frac{\mathrm{d}s_k}{\mathrm{d}t} &= +\lambda k i_k(t) \sum_{k'} s_{k'}(t) \frac{k'-1}{k'} P(k'|k) \\
&\quad - \alpha k s_k(t) \sum_{k'} [s_{k'}(t) + r_{k'}(t)] P(k'|k) \\
\frac{\mathrm{d}r_k}{\mathrm{d}t} &= \alpha k s_k(t) \sum_{k'} [s_{k'}(t) + r_{k'}(t)] P(k'|k),
\end{aligned} \tag{10.12}$$

where $P(k'|k)$ is the conditional probability that an edge departing from a vertex of degree k arrives at a vertex of degree k' and the terms $(k'-1)/k'$ take into account that each spreader must have one link connected to another spreader, from which it received the information. Even in the case of uncorrelated networks with $P(k'|k) = k'P(k')/\langle k \rangle$, the non-linear term in the evolution of the stifler density makes these equations much more involved than their counterparts for the SIR model and an analytical solution has not yet been derived.[3] Numerical simulations (Moreno, Nekovee and Vespignani, 2004b; Moreno, Nekovee and Pacheco, 2004a) nonetheless allow the spread of information to be characterized in this model. The final density of individuals who are aware of the rumor, i.e. the global reliability r_∞ of the information/rumor diffusion process, increases as expected if α decreases. More interestingly, homogeneous networks allow for larger levels of reliability than heterogeneous ones, for the same value of the average degree and of the parameters λ and α (Liu, Lai and Ye, 2003; Moreno, Nekovee and Vespignani, 2004b). This result may seem surprising as epidemic propagation is usually enhanced by the presence of hubs. For information spreading, however, hubs produce conflicting effects: they potentially allow a large number of nodes to be reached, but lead as well to many spreader–spreader and spreader–stifler interactions, thus turning themselves into stiflers before being able to inform all their neighbors. Owing to

[3] An additional linear term corresponding to the fact that spreaders could spontaneously become stiflers can be added (Nekovee *et al.*, 2007), leading to the appearance of an epidemic threshold in homogeneous networks, but this modification does not allow for an analytical solution of Equations (10.12).

the percolation properties of heterogeneous networks, the inhibition of propagation for a small fraction of the hubs will fragment the network in non-communicating components, isolating many nodes that will therefore never be informed. Numerical simulations also allow the final density of ignorants in each degree class to be measured, showing that it decays exponentially with the degree. This result implies that hubs acquire the rumor or information very efficiently: from the point of view of the hubs, high reliability is obtained even at large α.

While the reliability of the information spreading process is clearly important in technological applications, its scalability represents a crucial issue as well (Vogels *et al.*, 2003). Information systems and infrastructures have become very large networks and, while the number of messages may not be a concern in direct social contacts, the load L (defined as the average number of messages per node sent during the spreading process) imposed on a technological network when transmitting information should be kept at a minimum. The trade-off between a large reliability r_∞ and a small load L is measured through the *efficiency* $E = r_\infty/L$. From this perspective, scale-free networks lead to slightly more efficient processes than homogeneous random networks (see Figure 10.2). Additionally, epidemic-like spreading process achieves a better efficiency, for a broad range of α-values, than

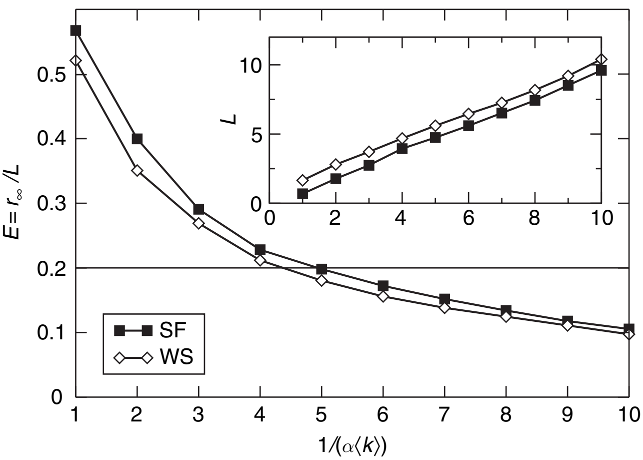

Fig. 10.2. Efficiency of the rumor spreading process as a function of $1/(\alpha\langle k\rangle)$, for networks of size $N = 10^4$ and average degree $\langle k\rangle = 6$. The homogeneous Watts–Strogatz network with rewiring probability $p = 1$ (diamonds) yields a lower efficiency than the heterogeneous one (SF, black squares). Larger α corresponds to smaller load and smaller reliability but larger efficiency. The straight horizontal line corresponds to the efficiency of a broadcast algorithm, for which each node passes the message deterministically to all its neighbors (except the one from which it received the information): in this case $r_\infty = 1$ and $L = \langle k\rangle - 1$. The inset shows the generated load as a function of $1/(\alpha\langle k\rangle)$. Data from Moreno, Nekovee and Vespignani (2004b).

the simplest broadcast algorithm in which each node transmits the information deterministically to all its neighbors. Finally, the hubs also play a role if they are taken as seeds, i.e. as the initial source of information. The degree of the initially informed node does not affect the final reliability of the process, but a larger degree yields a faster growth of $r(t)$. For intermediate stages of the process, larger densities of stiflers are therefore obtained. These results highlight the importance of well-connected hubs in the efficiency of rumor/information spreading processes: if the process aims at a given level of reliability, starting from well-connected nodes allows this level to be reached quickly, i.e. at smaller costs or load.

10.3 Opinion formation and the Voter model

As mentioned earlier in this chapter, numerous models have been devised to describe the evolution of opinions and cultural traits in a population of interacting agents. A classic example of collective behavior is given by the emergence of consensus in a population of individuals who can a priori have contradictory opinions and interact pairwise. While several models have been put forward to study this phenomenon, the Voter model represents the simplest modeling framework for the study of the onset of consensus due to opinion spreading.

The Voter model is defined on a population of size N in which each individual i has an opinion characterized by a binary variable $s_i = \pm 1$: only two opposite opinions are here allowed (for example a political choice between two parties). The dynamical evolution, starting from a random configuration of opinions, is the following: at each elementary step, an agent i is randomly selected, chooses one of his neighbors j at random and adopts his opinion, i.e. s_i is set equal to s_j (one time step corresponds to N such updates). This process therefore mimics the homogenization of opinions but, since interactions are binary and random, does not guarantee the convergence to a uniform state. When the connectivity pattern of individuals can be modeled as a regular D-dimensional lattice, the update rules lead to a slow coarsening process, i.e. to the growth of spatially ordered regions formed by individuals sharing the same opinion: large regions tend to expand while small ones tend to be "convinced" by the larger neighboring groups of homogeneous agents (see Figure 10.3). The dynamics is therefore defined by the evolution of the frontiers or "interfaces" between these ordered regions. For $D < 2$, the density of interfaces decays with time t (measured as the number of interactions per individual) as $t^{(D-2)/2}$ (Krapivsky, 1992; Frachebourg and Krapivsky, 1996). A logarithmic decay is observed for $D = 2$ (Dornic *et al.*, 2001), while a stationary active state with coexisting domains is reached for $D > 2$. In this last case, consensus (i.e. a global homogeneous state) is, however, obtained for finite systems; in a population of size N, fluctuations lead to a consensus after a typical time $\tau \propto N$.

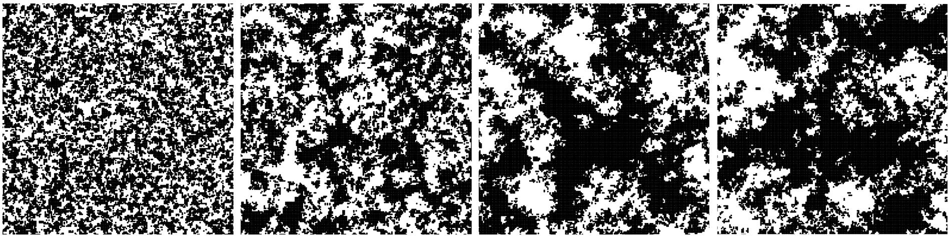

Fig. 10.3. Evolution of the Voter model for agents located on the sites of a two-dimensional square lattice of linear size $L = 200$. Black dots represent opinions $+1$, while empty spaces are left where opinions take the value -1. From left to right the system, starting from a disordered configuration, is represented at times $t = 5, 50, 500, 1000$. Adapted from Dornic *et al.* (2001).

As most interactions in the social context do not take place on regular lattices, recent studies have investigated the dynamics of the Voter model for agents interacting according to various complex connectivity patterns (Castellano *et al.*, 2003; Vilone and Castellano, 2004; Suchecki, Eguíluz and San Miguel, 2005a; Wu and Huberman, 2004; Sood and Redner, 2005; Castellano *et al.*, 2005; Suchecki, Eguíluz and San Miguel, 2005b; Castellano, 2005; Antal, Redner and Sood, 2006). Because of the lack of a Euclidean distance defining spatial domains and interfaces, coarsening phenomena are not well defined on general networks, in contrast with the case of regular lattices. The evolution of the system can, however, be measured by the fraction $n_A(t)$ of active bonds at time t, i.e. of bonds connecting sites with opposite values of the variable s_i, and by the survival probability $P_S(t)$ of a process, which is the probability that the fully ordered state has not been reached up to time t. The evolution of the time to reach a completely ordered state (time to consensus) with the size of the population is also clearly of great interest.

In this context, Castellano *et al.* (2003) have investigated the Voter model for individuals interacting along a Watts–Strogatz small-world network. After an initial transient, $n_A(t)$ exhibits a plateau whose length increases with the system size N, and which is followed for any finite N by an exponential decrease to 0. In the end the net result is an ordering time scale smaller than in the one-dimensional lattice, for which $n_A(t)$ decreases steadily as a power-law. Moreover, the height of the plateau increases as the parameter p of the Watts–Strogatz model increases, i.e. as the randomness of the network is increased. This dynamical process can be understood along the lines of Section 5.3.1. In the one-dimensional lattice, ordering is obtained through random diffusion of active bonds, which annihilate when they meet, resulting in $n_A(t) \sim 1/\sqrt{t}$. At finite p, the shortcuts inhibit this random diffusion through the influence of distant nodes with different opinions. This creates a "pinning" phenomenon, where the shortcuts act like obstacles to the homogenization process. The crossover is reached when the typical size of an ordered domain, namely $1/n_A(t)$, reaches the typical distance between shortcuts $\mathcal{O}(1/p)$ (see Chapter 3). The crossover time is thus given by $t^* \sim p^{-2}$, with $n_A(t^*) \sim p$. In addition, the various curves giving $n_A(t, p)$ can be rescaled by

$$n_A(t, p) = p\mathcal{G}(tp^2), \tag{10.13}$$

where $\mathcal{G}$ is a scaling function behaving as $\mathcal{G}(x) \sim \sqrt{x}$ for $x \ll 1$ and $\mathcal{G}(x) = \text{const}$ for large x (see Figure 10.4).

Further insight is given by the study of the survival probability $P_S(t)$, which decreases as $\exp[-t/\tau(N)]$, with $\tau(N) \propto N$ (Castellano *et al.*, 2003), as shown also analytically in the annealed version of the small-world networks (Vilone and Castellano, 2004).[4] On the other hand, the fraction of active bonds *averaged only over surviving runs*, $n_A^S(t)$, reaches a finite value at large times. In fact, the long-time decay of $n_A(t) = P_S(t)n_A^S(t)$ is solely due to the exponential decay of $P_S(t)$, showing that the system never orders in the thermodynamic limit, but retains a finite density of active links. The picture is therefore very different from the usual coarsening occurring in finite dimensions for which $n_A^S(t)$ steadily decays as a power-law of time, showing that the small-world effect created by shortcuts strongly affects the behavior of the Voter model.

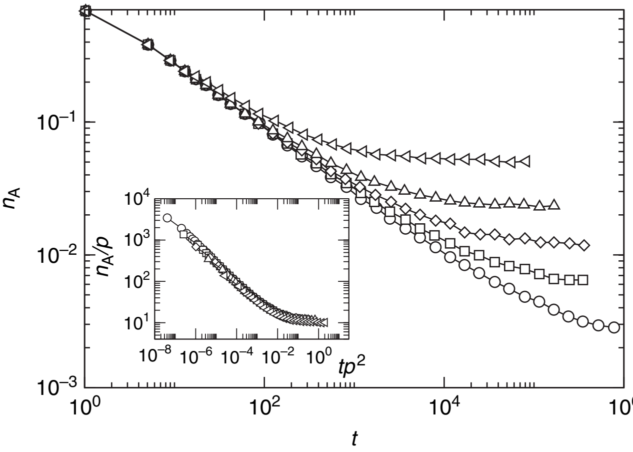

Fig. 10.4. Density of active links as a function of time for Watts–Strogatz networks of size $N = 10^5$ and various rewiring probabilities p: $p = 0.0002$ (circles), $p = 0.0005$ (squares), $p = 0.001$ (diamonds), $p = 0.002$ (triangles up) and $p = 0.005$ (triangles left). The inset displays $n_A(t, p)/p$ vs tp^2, showing the validity of Equation (10.13). Data from Castellano *et al.* (2003).

[4] In the annealed small-world graphs, the long-range links are rewired randomly at each time step.

Various studies, following these first investigations, have focused on the effect of the network heterogeneity and on the behavior of the time needed to reach consensus as a function of the network size N. As noted by Suchecki *et al.* (2005a) and Castellano (2005), the definition of the Voter model has to be made more precise if the individuals are situated on the nodes of a heterogeneous network. A randomly chosen individual has a degree distribution given by the network's distribution $P(k)$, while the *neighbor* of a randomly chosen node has degree distribution $kP(k)/\langle k\rangle$, and hence a higher probability of being a hub. Since, in the update rule, the two interacting nodes do not play symmetric roles, such a difference can be relevant, and leads to the following possible rules: (i) in the original definition, a randomly selected node picks at random one of its neighbors and adopts its state or opinion; (ii) in the link update rule, a link is chosen at random and the update occurs in a random direction along this link; (iii) in the *reverse* Voter model, a randomly chosen *neighbor* j of a randomly chosen node i adopts the opinion of i. These three definitions are equivalent on a lattice, but may induce relevant changes in heterogeneous networks since the probability for a hub to update its state will vary strongly from one rule to the other (Suchecki *et al.*, 2005a; Castellano, 2005).

As shown in Figure 10.5, the essential features of the dynamical process are similar on heterogeneous and homogeneous networks: $P_{\rm S}(t)$ decays exponentially with a characteristic time scale $\tau(N)$ depending on the system size, while $n_{\rm A}^{\rm S}(t)$ reaches a plateau: the system shows incomplete ordering, in contrast with the case of finite-dimensional lattices.[5] Various behaviors for $\tau(N)$ are, however, obtained depending on the updating rules. For agents interacting on Barabási–Albert networks, numerical simulations yield $\tau(N) \sim N^{\nu}$ with $\nu \approx 0.88$ for the original Voter model, and $\tau(N) \sim N$ for the link-update rule (Suchecki *et al.*, 2005a; Castellano *et al.*, 2005; Castellano, 2005).

For arbitrary (uncorrelated) scale-free networks with degree distribution $P(k) \sim k^{-\gamma}$, it is possible to obtain analytically the behavior of $\tau(N)$ through a mean-field approach which groups the nodes in degree classes (Sood and Redner, 2005), in the same way as seen in previous chapters. We start by defining ρ_k as the fraction of nodes having opinion $+1$ among the nodes of degree k. This density evolves because of the probabilities $P(k; - \to +)$ and $P(k; + \to -)$ that a node of degree k changes state, respectively from -1 to $+1$ or the opposite. In the original Voter model, the change $- \to +$ occurs in an update if the first randomly selected node has degree k and opinion -1 (which has probability $P(k)(1-\rho_k)$) and if its randomly chosen neighbor has opinion 1. In an uncorrelated network, this

[5] Suchecki *et al.* (2005b) have also observed that the Voter model can even display a coarsening dynamics on a particular scale-free structured network, which however is known to have an effective dimension equal to 1 (Klemm and Eguíluz, 2002b).

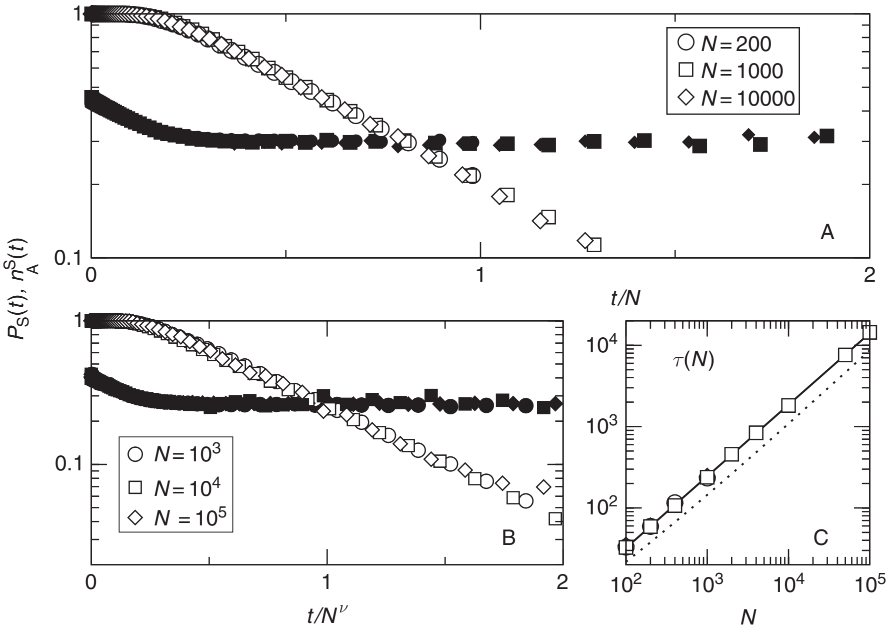

Fig. 10.5. Survival probability $P_S(t)$ (empty symbols) and fraction $n_A^S(t)$ of active bonds in surviving runs (filled symbols) for the voter model on (A) a random homogeneous graph with $\langle k \rangle = 10$ and (B) a Barabási–Albert (BA) network with $\langle k \rangle = 6$, for various sizes. Time is rescaled (A) by N and (B) by N^ν with $\nu \approx 0.88$. The plot (C) shows $\tau(N)$ vs N for the BA network; the continuous line corresponds to N^ν, the dotted one to $N/\ln(N)$: both scalings are compatible with the data over three orders of magnitude. Data from Castellano *et al.* (2005).

probability reads

$$\sum_{k'} \frac{k' P(k')}{\langle k \rangle} \rho_{k'}, \tag{10.14}$$

since the probability for the neighbor to have degree k' is $k' P(k')/\langle k \rangle$, and we sum over all possible degrees k'. We thus obtain

$$P(k; - \to +) = P(k)(1 - \rho_k) \sum_{k'} \frac{k' P(k')}{\langle k \rangle} \rho_{k'}, \tag{10.15}$$

and similarly

$$P(k; + \to -) = P(k)\rho_k \sum_{k'} \frac{k' P(k')}{\langle k \rangle} (1 - \rho_{k'}). \tag{10.16}$$

The number N_k^+ of nodes of degree k and state $+1$ thus evolves according to $dN_k^+/dt = N\,(P(k; - \to +) - P(k; + \to -))$. Since $\rho_k = N_k^+/N_k$, we obtain the evolution equation for ρ_k in the mean-field approximation and for uncorrelated networks

$$\frac{\mathrm{d}\rho_k}{\mathrm{d}t} = \frac{N}{N_k}\left[P(k; - \to +) - P(k; + \to -)\right], \tag{10.17}$$

which can be rewritten as

$$\frac{\mathrm{d}\rho_k}{\mathrm{d}t} = \sum_{k'} \frac{k' P(k')}{\langle k \rangle}(\rho_{k'} - \rho_k). \tag{10.18}$$

This shows that the quantity $\omega = \sum_k k\rho_k P(k)/\langle k \rangle$ is conserved by the dynamics, and that a stationary state with $\rho_k = \omega$ is obtained. In contrast with the study of epidemic spreading detailed in Chapter 9, the long time limit of ρ cannot be obtained self-consistently, but a recursion equation for the consensus time $\tau(N)$ can be written as a function of the densities ρ_k (Sood and Redner, 2005):

$$\begin{aligned}\tau(\{\rho_k\}) = &\sum_k P(k; - \to +)\left[\tau(\rho_k + \delta_k) + \delta t\right] \\ &+ \sum_k P(k; + \to -)\left[\tau(\rho_k - \delta_k) + \delta t\right] \\ &+ Q(\{\rho_k\})\left[\tau(\{\rho_k\}) + \delta t\right],\end{aligned} \tag{10.19}$$

where $Q(\{\rho_k\}) = 1 - \sum_k P(k; - \to +) - \sum_k P(k; + \to -)$ is the probability that no update takes place during δt, and $\delta_k = 1/(NP(k))$ is the change in ρ_k when a node of degree k changes opinion. Using the expressions of $P(k; - \to +)$, $P(k; + \to -)$ and expanding to second order in δ_k yields

$$\sum_k (\rho_k - \omega)\partial_k \tau - \frac{1}{2N}\sum_k \frac{1}{P(k)}(\rho_k + \omega - 2\omega\rho_k)\partial_k^2 \tau = 1, \tag{10.20}$$

where we have $\delta t = 1/N$ as the elementary time-step, and $\partial_k \equiv \partial/\partial\rho_k$. The first term vanishes at long times, thanks to the convergence of ρ_k to ω. Using $\partial_k = kP(k)/\langle k \rangle \partial_\omega$, Equation (10.20) becomes

$$\frac{\langle k^2 \rangle}{\langle k \rangle^2}\omega(1 - \omega)\partial_\omega^2 \tau = -N, \tag{10.21}$$

which can be integrated with the conditions that τ vanishes for both $\omega = 0$ and $\omega = 1$ (these values of ω indeed correspond to the two possible consensus states, from which the system cannot evolve). We finally obtain

$$\tau(N) = -N\frac{\langle k \rangle^2}{\langle k^2 \rangle}\left[\omega \ln \omega + (1 - \omega)\ln(1 - \omega)\right]. \tag{10.22}$$

Table 10.1 Scaling of the time to consensus $\tau(N)$ for uncorrelated scale-free networks with distribution $P(k) \sim k^{-\gamma}$, as a function of the exponent γ.

γ	$\gamma > 3$	$\gamma = 3$	$3 > \gamma > 2$	$\gamma = 2$	$\gamma < 2$
$\tau(N)$	N	$N/\ln N$	$N^{(2\gamma-4)/(\gamma-1)}$	$(\ln N)^2$	$\mathcal{O}(1)$

For an initial random uncorrelated configuration $\rho_k(0) = \rho(0)$, $\omega = \rho(0)$ and

$$\tau(N) = -N\frac{\langle k\rangle^2}{\langle k^2\rangle}\left[\rho(0)\ln\rho(0) + (1-\rho(0))\ln(1-\rho(0))\right]. \tag{10.23}$$

The dependence of the convergence time on the system size N therefore depends on the fluctuations of the degree distribution and can be computed by using Equation (2.3) and the results of Appendix 1. In Table 10.1 we report the scaling of the convergence time as a function of the system size N in random scale-free networks with power-law exponent γ. The result obtained for $\gamma = 3$ indicates a logarithmic scaling $\tau(N) \sim N/\ln N$, apparently in contradiction to the power-law best fit $\tau(N) \sim N^{\nu}$ with $\nu \approx 0.88$ obtained by Suchecki *et al.* (2005a) and Castellano *et al.* (2005). On the other hand, the logarithmic fit is still compatible and the small logarithmic corrections are hard to validate on the three orders of magnitude accessible from simulations (see Figure 10.5).

A similar analysis can be carried out for the reverse Voter model, confirming the relevance of the updating rule in the case of heterogeneous networks (Castellano, 2005). The analytical mean-field approach leads indeed to

$$\tau(N) = -N\langle k\rangle\left\langle\frac{1}{k}\right\rangle\left[\rho(0)\ln\rho(0)(1-\rho(0))\ln(1-\rho(0))\right], \tag{10.24}$$

and thus to a completely different scaling with respect to the exponents of the network degree distribution: $\tau(N) \sim N$ for $\gamma > 2$, $\tau(N) \sim N\ln N$ for $\gamma = 2$, and $\tau(N) \sim N^{1/(\gamma-1)}$ for $1 < \gamma < 2$ (Castellano, 2005).

In summary, the dynamics of the Voter model is strongly different for agents interacting on the nodes of a network with respect to the case of regular lattices. Interestingly, the possible heterogeneous structure of the interaction network only marginally affects the dynamical process: the presence of hubs does not modify the absence of ordering in the thermodynamic limit but only the scaling of the ordering time for finite sizes. In this respect, the Voter model is very different from processes such as epidemics spreading for which the divergence of $\langle k^2\rangle$ has strong consequences, as explained in Chapter 9.

Let us finally note that interesting extensions of the Voter model have recently been studied. Castelló, Eguíluz and San Miguel (2006) introduce the possibility of an intermediate state '± 1', through which an individual must pass when changing opinion from $+1$ to -1 or the opposite.[6] This modification is in the spirit of the *Naming Game* model, in which an agent can have an arbitrary number of possible opinions at the same time, and which has been recently extensively studied for agents interacting on networks with various topologies (Steels, 1996; Lenaerts *et al.*, 2005; Baronchelli *et al.*, 2006; Dall'Asta and Baronchelli, 2006). It turns out that such a modification of the Voter model leads to a faster convergence to consensus for agents interacting on a small-world network, with $\tau \sim \ln N$, essentially by avoiding the "pinning" phenomenon due to shortcuts.

10.4 The Axelrod model

In the Voter model, each agent possesses a unique opinion which can take two discrete values only. While an interesting extension consists of considering opinions as continuous variables (Deffuant *et al.*, 2000; Ben-Naim *et al.*, 2003), it is clear that social influence and interaction do not act on a single dimensional space. In general, social influence applies to a wide range of cultural attributes such as beliefs, attitudes and behavior, which cannot be considered in isolation. Social influence is more likely when more than one of these attributes are shared by the interacting individuals, and it may act on more than a single attribute at once. It is in this spirit that Axelrod (1997b) has proposed a simple but ambitious model of social influence that studies the convergence to global polarization in a multi-dimensional space for the individual's attributes.

In the Axelrod model each agent is endowed with a certain number F of cultural features defining the individual's attributes, each of those assuming any one of q possible values. The Voter model is thus a particular case of Axelrod's model, with $F = 1$ and $q = 2$. The model takes into account the fact that agents are likely to interact with others only if they already share cultural attributes, and that the interaction then tends to reinforce the similarity. The precise rules of the model are therefore the following: at each time step, two neighboring agents are selected. They interact with probability proportional to the number of features (or attributes) for which they share the same value (among the q possible ones).[7] In this case, one of the features for which they differ is chosen, and one of the agents selected at random adopts the value of the other. The F different attributes are therefore not

[6] See also Dall'Asta and Castellano (2007) for a slightly different mechanism which also takes into account a memory effect.

[7] In some versions of the model, the probability of interaction is simply 1 if at least one feature is shared, and 0 otherwise.

completely independent in that the evolution of each feature depends on the values of the others (and in particular the possible agreement with other agents).

Numerical simulations performed for agents connected as in a regular two-dimensional grid show the influence of the number of features F per agent and of the cultural variability q on the final state of the system. The local convergence rules can lead either to global polarization or to a culturally fragmented state with coexistence of different homogeneous regions. As F increases, the likelihood of convergence towards a globally homogeneous state is enhanced, while this likelihood decreases when q increases (Axelrod, 1997b). Indeed, when more features are present (at fixed q), there is a greater chance that two neighboring agents share at least one of them, and therefore the probability of interaction is enhanced. At larger q instead, it is less probable for each feature to have a common value in two agents, and they interact with smaller probability. Castellano, Marsili and Vespignani (2000) have studied in detail the phase diagram of the model, uncovering a non-equilibrium phase transition between the ordered (homogeneous opinions) and disordered (culturally fragmented) phases. The dynamics evolves through a competition between the initially disordered situation and the tendency of the interaction to homogenize the agents' attributes, leading to a coarsening phenomenon of regions in which the agents have uniform features. The system finally reaches an absorbing state in which all links are inactive: each link connects agents who either have all features equal or all different, so that no further evolution is possible. At small q, the absorbing state reached by the system is expected to be homogeneous, while at large q one expects a highly fragmented state. The order parameter which determines the transition is defined by the average size of the largest homogeneous region in the final state, $\langle S_{\max}\rangle$. At fixed F, a clear transition is observed as q crosses a critical value q_c: for $q < q_c$, $\langle S_{\max}\rangle$ increases with the system size L and tends to L^2 (for a two-dimensional lattice of linear size L, the number of sites or agents is equal to $N = L^2$), while $\langle S_{\max}\rangle/L^2 \to 0$ for $q > q_c$. The transition is continuous for $F = 2$, discontinuous for $F > 2$, and can be analyzed through mean-field approaches (Castellano *et al.*, 2000). For agents interacting on a one-dimensional chain, a mapping to a reaction–diffusion process shows that the critical value q_c is equal to F if the initial distribution of values taken by the features is uniform, and to F^2 for an initial Poisson distribution (Vilone, Vespignani and Castellano, 2002).

Interestingly, cultural drift and global interactions can also be introduced into the model in a simple way. Random changes in the attributes' values at a certain rate r, or the addition of an interaction of the agents with a global field, account for these phenomena (Klemm *et al.*, 2003a; Klemm *et al.*, 2005; González-Avella *et al.*, 2006). An interesting result in this context is that even a very small rate of random changes drives the population of agents to an homogeneous state for any value of F and q by potentially allowing two neighboring agents who did not share any feature to interact again.

The possible effect of long-distance interactions perturbing the grid ordering of agents was also briefly discussed in the original work of Axelrod (1997b). The evolution of Axelrod's model for agents interacting on complex networks with various topological characteristics has subsequently been investigated in detail through numerical simulations by Klemm *et al.* (2003b). In the case of Watts–Strogatz (two-dimensional) small-world networks, the disorder is shown to favor the ordered (homogeneous) state: the critical value q_c grows as the disorder p is increased (see Figure 10.6). On random scale-free networks, the hubs enhance the spreading of cultural traits so much that the value of q_c diverges. At any finite size, an effective transition from the ordered to the fragmented state is observed at a pseudo-critical value $q_c(N)$, which diverges as N^{β} (with $\beta \sim 0.4$ for Barabási–Albert networks, see Figure 10.7) with the size N of the network (Klemm *et al.*, 2003b). In other words, the existence of a culturally fragmented phase is hindered by the presence of hubs and is no longer possible in the thermodynamic limit. Once again, this behavior is reminiscent of the results obtained for the Ising model for which the critical temperature diverges for scale-free networks (see Section 5.3.2), owing to the strong polarization role of the hubs. Finally, we refer to Centola *et al.* (2007) and to Section 10.6 for the interesting case in which the network of interaction itself may evolve on the same time scale as the agents' features.

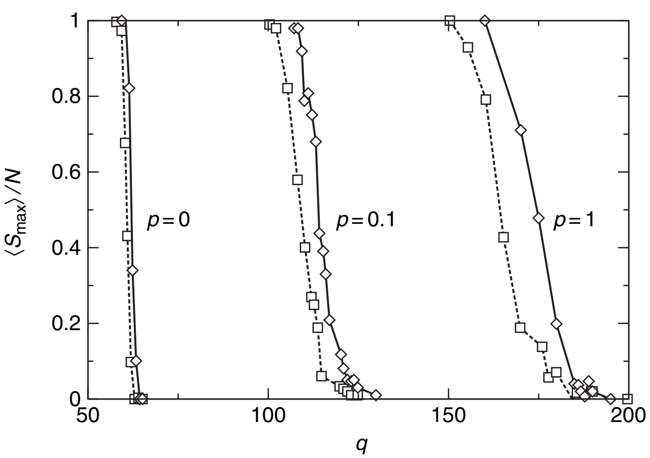

Fig. 10.6. Average order parameter $\langle S_{\max}\rangle/N$ as a function of q for three different values of the small-world parameter p. System sizes are $N = 500^2$ (squares) and $N = 1000^2$ (diamonds), number of features $F = 10$. Data from Klemm *et al.* (2003b).

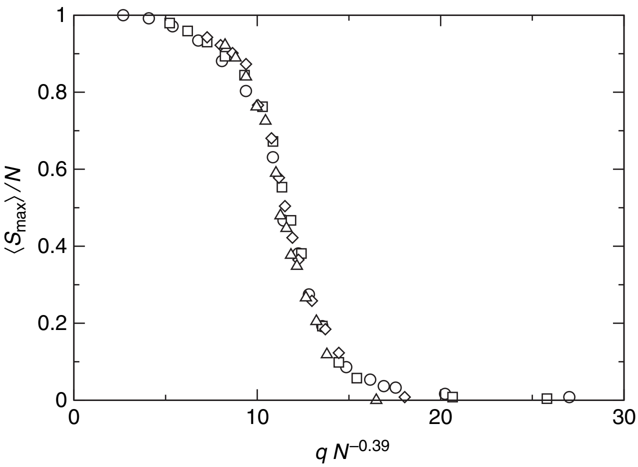

Fig. 10.7. Plot of $\langle S_{\max}\rangle/N$ versus $qN^{-\beta}$ in random scale-free networks for $F = 10$, for different system sizes: 1000 (circles), 2000 (squares), 5000 (diamonds), and 10000 (triangles). Data from Klemm *et al.* (2003b).

10.5 Prisoner's dilemma

Another interesting and often puzzling social behavior that has attracted the attention of scientists across different research areas lies in the emergence of cooperation between a priori selfish interacting agents. One of the best-known paradigms of a system able to display both cooperative and competitive behaviors is given by the prisoner's dilemma game (Rapoport and Chammah, 1965; Axelrod and Hamilton, 1981). Indeed, the basic two-players iterated prisoners dilemma has been described as the *Escherichia coli* of the social sciences and it is one of the basic framework for the study of evolution and adaptation of social behavior (Axelrod, 1997a). In this model, agents (players) interact pairwise, and have at each interaction (or round of the game) the option either to cooperate (C strategy) or to defect (D strategy). They obtain various payoffs according to these choices. If both players cooperate, they earn a quantity R ("reward"), if they both defect they earn P ("punishment") while if they use different strategies, the defector earns T ("temptation" to defect) and the cooperator S ("sucker's payoff"), as sketched in Figure 10.8.

The language and payoff matrix have their inspiration in the following situation from which the name of the model derives. Two suspects A and B are arrested by the police. The police do not have sufficient evidence for a conviction and separate them in the hope that each one of them may testify for the prosecution of the other. If both A and B remain silent (they cooperate with each other) they both receive a sentence for minor charges. If one of the two betrays the other, the betrayer goes

Player A / Player B	Cooperation	Defection
Cooperation	R / R	T / S
Defection	S / T	P / P

Fig. 10.8. Sketch of the prisoner's dilemma rules for two players A and B. If the two players cooperate (C strategy), they both receive the reward R. If they both defect (D strategy), they are both punished and receive only P. If one betrays the other by defecting while the other cooperates, the traitor receives T (temptation to defect) and the cooperator gets S.

free while the other receives a heavy sentence. This is the case in which one of the two agents has defected the other while the second one is still cooperating. Finally both of them could betray and get a sentence intermediate between the two previous cases. The main question is therefore what would be the best strategy for the two suspects. If $T > R > P > S$, an interesting paradox arises. In any given round, it is in each agent's interest to defect, since the D strategy is better than the C one, regardless of the strategy chosen by the other player. On the other hand, on average, if both players systematically use the D strategy, they earn only P while they would receive R if they both chose to cooperate: in the long run, cooperation is therefore favored.

While the initial definition involves two players engaged in successive encounters, Nowak and May (1992) have considered the evolution of a population of players situated on the nodes of a two-dimensional square lattice, thus introducing local effects. For simplicity, the various payoffs are taken as $P = S = 0$, $R = 1$, and the only parameter is $T > 1$, which characterizes the advantage of defectors against cooperators. Moreover, at each round (time) t agents play with their neighbors, and each agent tries to maximize its payoff by adopting in the successive round $t + 1$ the strategy of the neighbor who scored best at t. Agents have otherwise no memory of past encounters. A large variety of patterns is generated by such rules (Nowak and May, 1992; 1993). In particular, for $T < 1.8$ a cluster of defectors in a background of cooperators shrinks, while it grows for $T > 1.8$. A cluster of cooperators instead grows for $T < 2$ and shrinks for $T > 2$. In the parameter region $1.8 < T < 2$, many chaotically evolving patterns of C and D regions are observed.

If agents interact on more realistic networks, a plethora of rich and interesting behaviors emerges. On Watts–Strogatz small-world networks, the fraction of defectors rises from 0 at small values of T to 1 at large T, but with intriguing features such as a non-monotonic evolution of this fraction (at fixed T) when the small-world parameter p changes (Abramson and Kuperman, 2001). The introduction of an "influential node" which has links to a large number of other agents, in an otherwise small-world topology, leads moreover to sudden drops (breakdowns) of the cooperation when this node becomes a defector, followed by gradual recovery of the cooperation (Kim *et al.*, 2002a).

Santos and Pacheco (2005) (see also Santos, Rodrigues and Pacheco [2005]; Santos, Pacheco and Lenaerts [2006a]; [2006b]; Santos and Pacheco [2006]) show on the other hand that cooperation is more easily achieved in heterogeneous topologies than in homogeneous networks (Figure 10.9). Gómez-Gardeñes *et al.* (2007a) analyze the transition, as T is increased, from cooperation to defection, in heterogeneous and homogeneous networks. Interestingly, the paths are quite different in the two topologies. In homogeneous networks, all individuals are cooperators for $T = 1$. As T increases, some agents remain "pure cooperators", i.e. always cooperate, while others fluctuate. Pure defectors arise as T is further increased, and finally take over at large T. In heterogeneous networks, the hubs remain pure cooperators until the cooperators' density vanishes; thanks to their percolation properties, the subgraph of all pure cooperators therefore remains connected, and the pure defectors start by forming various small clusters which finally merge and invade the whole network. For homogeneous networks, on the other hand, the cooperator core splits easily (as T grows) in several clusters surrounded by a sea of fluctuating agents. The comparison of these two behaviors has the following outcome: for T close to 1, a larger fraction of cooperators is observed in homogeneous topologies, but heterogeneous networks yield larger levels of cooperation as the temptation to defect, T, is increased to intermediate or large values.

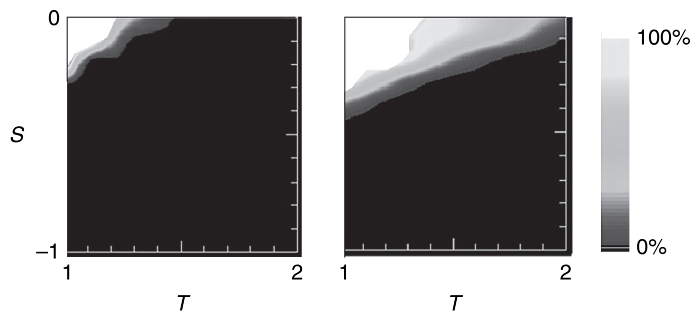

Fig. 10.9. Evolution of the density of cooperators in the prisoner's dilemma for individuals with different networks of contact, as a function of the parameters T and S. Left: homogeneous network; right: scale-free network. Adapted from Santos *et al.* (2006b).

Cooperation on *empirical* social networks has also been studied: the density of cooperators then exhibits a complex non-stationary behavior corresponding to transitions between various quasi-stable states (Holme *et al.*, 2003). Lozano, Arenas and Sanchez (2006) also investigate the role of networks' community structures by comparing the prisoner's dilemma model on empirical social networks with very different community properties: the possible presence of local hubs in communities has a strong stabilizing role and favors cooperation, while more homogeneous communities can more easily change strategy, with all agents becoming defectors. Strong community structures (with sparse connectivity between communities) also contribute to the stability of a community composed of cooperators.

The various static topological features of real social networks thus have a very strong impact on the emergence of cooperative phenomena between selfish and competing agents. Interestingly, taking into account the *plasticity* of the network, i.e. the possibility for (social) links to disappear, be created, or rewired, leads to further interesting effects, as described in the next section.

10.6 Coevolution of opinions and network

Most studies of the dynamical phenomena taking place on complex networks have focused on the influence of the network's topological features on the dynamics. Many networks, however, have a dynamical nature, and their evolution time scales may have an impact on the dynamical processes occurring between the nodes. Such considerations are particularly relevant for social networks which continuously evolve a priori on various time scales (both fast and slow).

Let us first consider the case in which the dynamical evolution of the nodes' (or agents') states is very fast with respect to the network topological evolution. This is the point of view adopted by Ehrhardt, Marsili and Vega-Redondo (2006a) (see also Ehrhardt, Marsili and Vega-Redondo [2006b]). In this approach, each agent carries an internal variable (which can describe an opinion, a level of knowledge, etc), which is updated through interaction with its neighbors, for example through diffusion processes (mimicking knowledge diffusion phenomena) or opinion exchanges. Each link between agents decays spontaneously at a certain rate λ, and new links are created at rate ξ *only between agents whose internal variables are close enough*. The topology thus has an impact on the evolution of the agents' states, which in its turn determines how the topology can be modified. When the knowledge or opinion update (rate ν) is fast with respect to the link's update process, the competition between link decay and creation rates leads to a phase transition between a very sparse phase in which the population is divided into many small clusters, and a denser globally connected network with much larger average degree. This transition, which can be studied through mean-field approaches in the limit $\nu \gg 1$, turns out to be sharp, and to display hysteresis phenomena (Ehrhardt *et al.*, 2006a; 2006b).

More generally, however, the time scale on which opinions and interactions evolve can be of the same order. Links can be broken more easily if the two interacting agents differ in their opinions but new contacts do generally appear owing to random events in the social life of the individuals. A model integrating the breaking of social relations is given by a population of agents interacting through the Voter model paradigm (Zanette and Gil, 2006). The interaction between two agents who do not share the same opinion can lead either to a local consensus (one agent adopts the opinion of the other), or to a breaking of the link if the agents fail to reach an agreement (this occurs with a certain probability p). Starting from a fully connected population, and depending on the model's parameters, the final state can be formed of one or more separated communities of agents sharing the same opinion. In order to mimic the introduction of new social relations, another natural hypothesis consists in considering that links do not "disappear" but are simply rewired by the agent who decides to change interaction partner. Holme and Newman (2006) uncover an interesting out-of-equilibrium phase transition in the coevolution of the network of contacts of agents interacting through a Voter-like model. In this model, starting from a random network of agents with randomly chosen opinions, an agent is selected at each time step; with probability p, he adopts the opinion of one randomly selected neighbor, and with probability $1 - p$ one of his links is rewired towards another agent who shares the same opinion. The total number of edges is thus conserved during the evolution. When p is smaller than a certain critical value, the system evolves towards a set of small communities of agents who share the same opinion. At large p on the other hand, opinions change faster than the topology and consensus is obtained in a giant connected cluster of agents.[8]

The coevolution of agents, opinions, or strategies and of their interaction network also has a strong impact on the Axelrod model. For example, it is possible to consider that the links between agents who do not share any attributes can be broken. If such links are then rewired at random, the critical value of q (number of possible values of each attribute) above which the system becomes disordered is largely increased (Centola *et al.*, 2007). In other words, the parameter range leading to global consensus is strongly enlarged. The structure of the network itself is also affected, breaking into small components of agents with homogeneous opinions at

[8] Strong effects of the network adaptive character are also observed for a model of agents with continuous opinions and bounded confidence who can break the contacts with other agents who have too diverse an opinion (Kozma and Barrat, 2008).

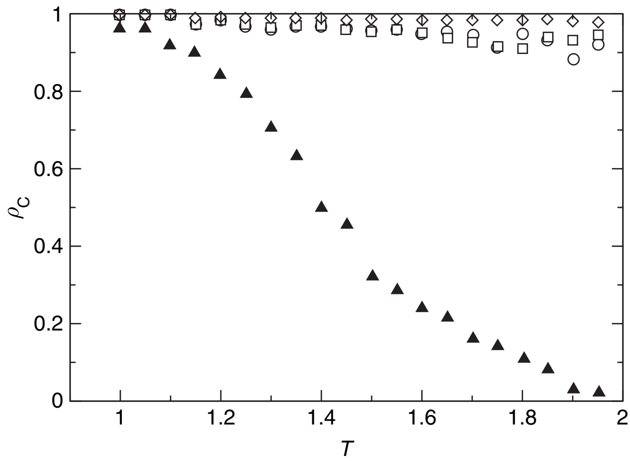

Fig. 10.10. Average fraction of cooperators ρ_C in the steady state as a function of the temptation to defect T, for various rewiring probabilities p: $p = 0$ (triangles), $p = 0.01$ (circles), $p = 0.1$ (squares) and $p = 1$ (diamonds). Data from Zimmermann *et al.* (2004).

large q. Various studies have also shown that cooperation is favored in the prisoner's dilemma case when agents can change their neighborhood. More precisely, links between cooperators are naturally maintained; in the case of a link between a cooperator and a defector, the cooperator may want to break the interaction, but not the defector, so that these reactions are balanced, and for simplicity only links between defectors are assumed to be rewired, with a certain probability p at each time step (Zimmermann, Eguíluz and San Miguel, 2001; 2004; Zimmermann and Eguíluz, 2005; Eguíluz *et al.*, 2005b). Defectors are thus effectively competitive, since they rewire links at random until they find a cooperative neighbor. A rich phenomenology follows from these dynamical rules, yielding a stationary state with a large number of cooperators as soon as $p > 0$ (see Figure 10.10), exploited by a smaller number of defectors, who have on average a larger payoff. Random perturbations to this stationary state can either increase the number of cooperators, or on the contrary trigger large-scale avalanches towards the absorbing state in which all agents are defectors (Zimmermann and Eguíluz, 2005). The rate of topological changes plays also an important role: faster responses of individuals to the nature of their ties yield easier cooperation (Santos *et al.*, 2006a).

While the interactions and feedback between dynamical processes *on* networks and *of* their topologies have been described here from the point of view of socially motivated models, continuous topological reshaping of networks by the dynamics which they support has a broad range of applications. The evolution of the World Wide Web, for example, is certainly strongly influenced by the performance and popularity of search engines. Infrastructure networks may also be affected by rapid evolutions during cascading failures (see Chapter 6), or may have to be reshaped because of the traffic they carry (Chapter 11). While the coevolution of networks and dynamical behavior according to the feedback between dynamical processes and topology is clearly a key issue in the understanding of many systems, its study in the context of complex networks is still at an early stage.