Poisson Rates

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Definition

The Poisson Distribution is discrete and used for count data, where you determine the probability that a given number of events occur within a time period. This distribution takes one parameter, often denoted with the symbol λ which is a known fixed rate which represents the number of events expected to occur in a certain time interval. We would say

$$N \sim Poisson(\lambda)$$

Probability Mass Function

The probability of observing k events occur within a time interval is given by the equation:

$$\text{P}(k \text{ events occur}) = \frac{e^{-\lambda}\lambda^k}{k!}$$

where $k = 0,\ 1,\ 2\ldots$

This pmf can be adapted so that the parameter is a time rate, $\lambda = rt$. In this case, r is 1 unit per time period. The equation then changes to the following:

$$\text{P}(k \text{ events occur in time } t) = \frac{e^{-rt}(rt)^k}{k!}$$

Poisson vs Binomial Distribution

Use the Poisson distribution when you are trying to determine the probability of a certain number of defined events happening in a given time period.

Use the Binomial distribution when you are trying to determine the probability of a certain number of defined events happening out of a determined number of trials.

The binomial distribution converges to the poisson for a large sample size. (i.e. $np \to \lambda$ for large n)

Poisson Process

To explain the poisson process, I will first define the following variables:

- N_t = count of the number of events that are registered at time t
- υ = the mean number of events per unit time (*intensity*)
- υ_t = mean number of events in t time units

Here, $N_t \sim Poisson(\lambda)$ where $\lambda = \upsilon t$

Definition

The collection $N_t : t \geq 0$ is called the poisson process if it satisfies the following:

1. $N_0 = 0$
2. If $s < t$, N_s and $N_t - N_s$ are independent (*memoryless*)
3. $N_{s+t} - N_t$ and N_s have the same distribution (*stationary*)
4. $lim_{t \to 0} = \frac{P(N_t=1)}{t} = \upsilon$
5. $lim_{t \to 0} = \frac{P(N_t>1)}{t} = 0$ (can't have two events happen at the same time)

Descriptive Statistics

Based on the pmf, let's derive the mean and variance of the poisson distribution:

$$\begin{aligned}
\mu = E[N] &= \sum_{k=0}^{\infty} k \frac{e^{-\lambda}\lambda^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{(k-1)!} \\
&= \lambda \sum_{k=1}^{\infty} \frac{e^{-\lambda}\lambda^{k-1}}{(k-1)!} && \text{Let } m = k-1 \\
&= \lambda \sum_{m=0}^{\infty} \frac{e^{-\lambda}\lambda^m}{m!} \\
&= \lambda \cdot 1 && \text{Since } \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} = 0 \\
&= \lambda
\end{aligned}$$

We will calculate variance using the formula $\sigma^2 = E[N(N-1)] + E[N] - E[N]^2$.

We will first derive $E[N(N-1)]$

$$\begin{aligned}
E[N(N-1)] &= \sum_{k=0}^{\infty} k(k-1) \frac{e^{-\lambda}\lambda^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{(k-2)!} \\
&= \lambda^2 \sum_{k=2}^{\infty} \frac{e^{-\lambda}\lambda^{k-2}}{(k-2)!} && \text{Let } m = k-2 \\
&= \lambda^2 \sum_{m=0}^{\infty} \frac{e^{-\lambda}\lambda^m}{m!} \\
&= \lambda^2 \cdot 1 && \text{Since } \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} = 0 \\
&= \lambda^2
\end{aligned}$$

Now, we can solve for the variance:

$$\sigma^2 = Var[N] = E[N(N-1)] + E[N] - E[N]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

So, for the poisson distribution, $\mu = \sigma^2 = \boldsymbol{\lambda}$

Example

An example of using the poisson process to model the number of lightening strikes can be found in the following article:

Stochastic Modeling of Lightning Occurrence by Nonhomogeneous Poisson Process