MATH 113 – Honors Calculus

• SHEET 1: INTRODUCING THE CONTINUUM •

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This sheet introduces **the continuum** $\mathcal{C}$ through a series of axioms.

Axiom 1. *The continuum is a nonempty set* $\mathcal{C}$.

We often refer to elements of $\mathcal{C}$ as *points*.

Definition 1.1. Let X be a set. An *ordering* on the set X is a subset of $X \times X$, denoted by $<$, with elements $(x, y) \in <$ written as $x < y$, satisfying the following properties:

(a) For all $x, y \in X$ such that $x \neq y$, either $x < y$ or $y < x$.

(b) For all $x, y \in X$, if $x < y$ then $x \neq y$.

(c) For all $x, y, z \in X$, if $x < y$ and $y < z$ then $x < z$.

Axiom 2. *There exists an ordering* $<$ *on* $\mathcal{C}$.

Theorem 1.2. *If* x *and* y *are points of* $\mathcal{C}$, *then* $x < y$ *and* $y < x$ *are not both true.*

Definition 1.3. If $A \subset \mathcal{C}$ is a subset of $\mathcal{C}$, then a point $a \in A$ is a *first* point of A if, for every element $x \in A$, either $a < x$ or $a = x$. Similarly, a point $b \in A$ is called a *last* point of A if, for every $x \in A$, either $x < b$ or $x = b$.

Lemma 1.4. *If* A *is a nonempty, finite subset of* $\mathcal{C}$, *then* A *has a first and last point.*

Theorem 1.5. *Suppose that* A *is a set of* n *distinct points in* $\mathcal{C}$, *or, in other words,* $A \subset \mathcal{C}$ *has cardinality* n. *Then symbols* $a_1, \ldots, a_n$ *may be assigned to each point of* A *so that* $a_1 < a_2 < \cdots < a_n$, *i.e.* $a_i < a_{i+1}$ *for* $1 \leq i \leq n - 1$.

Definition 1.6. If $x, y, z \in \mathcal{C}$ and both $x < y$ and $y < z$, then we say that y is *between* x and z.

Corollary 1.7. *Of three distinct points, one must be between the other two.*

Axiom 3. $\mathcal{C}$ *has no first or last point.*

Definition 1.8. If $a < b$, then the set of all points between a and b is called a *region*, denoted by $\underline{ab}$.

Warning 1.9. One often sees the notation (a, b) for regions. We will reserve the notation (a, b) for ordered pairs in a product $A \times B$. These are very different things.

Theorem 1.10. *If x is a point of $\mathcal{C}$, then there exists a region $\underline{ab}$ such that $x \in \underline{ab}$.*

We now come to one of the most important definitions of this course:

Definition 1.11. Let A be a nonempty subset of $\mathcal{C}$. A point p of $\mathcal{C}$ is called a *limit point* of A if every region R containing p has nonempty intersection with $A \setminus \{p\}$. Explicitly, this means:

$$\text{for every region } R \text{ with } p \in R, \text{ we have } R \cap (A \setminus \{p\}) \neq \varnothing.$$

Notice that we do not require that a limit point p of A be an element of A.

Theorem 1.12. *If p is a limit point of A and $A \subset B$, then p is a limit point of B.*

Lemma 1.13. *If $\underline{ab}$ is a region in $\mathcal{C}$, then:*

$$\mathcal{C} = \{x \mid x < a\} \cup \{a\} \cup \underline{ab} \cup \{b\} \cup \{x \mid b < x\}.$$

Definition 1.14. If $\underline{ab}$ is a region in $\mathcal{C}$, then $\mathcal{C} \setminus (\{a\} \cup \underline{ab} \cup \{b\})$ is called the *exterior* of $\underline{ab}$ and is denoted by $\operatorname{ext} ab$.

Lemma 1.15. *If $\underline{ab}$ is a region in $\mathcal{C}$, then:*

$$\mathcal{C} = \operatorname{ext} ab \cup \{a\} \cup \{b\} \cup \underline{ab}.$$

Lemma 1.16. *No point in the exterior of a region is a limit point of that region. No point of a region is a limit point of the exterior of that region.*

Theorem 1.17. *If two regions have a point x in common, their intersection is a region containing x.*

Corollary 1.18. *If n regions $R_1, \ldots, R_n$ have a point x in common, then their intersection $R_1 \cap \cdots \cap R_n$ is a region containing x.*

Theorem 1.19. *Let $A, B \subset \mathcal{C}$. If p is a limit point of $A \cup B$, then p is a limit point of A or B.*

Corollary 1.20. *Let $A_1, \ldots, A_n$ be n subsets of $\mathcal{C}$. If p is a limit point of $A_1 \cup \cdots \cup A_n$, then p is a limit point of at least one of the sets A_k.*

The converse is also true, so we have both directions:

Corollary 1.21. *Let $A_1, \ldots, A_n$ be n subsets of $\mathcal{C}$. Then p is a limit point of $A_1 \cup \cdots \cup A_n$ if and only if p is a limit point of at least one of the sets A_k.*

Definition 1.22. Two sets A and B are *disjoint* if $A \cap B = \varnothing$.

Theorem 1.23. *If p and q are distinct points of $\mathcal{C}$, then there exist disjoint regions R and S containing p and q, respectively.*

Corollary 1.24. *A subset of $\mathcal{C}$ consisting of one point has no limit points.*

Theorem 1.25. *A finite subset $A \subset \mathcal{C}$ has no limit points.*

Corollary 1.26. *If $A \subset \mathcal{C}$ is finite and $x \in A$, then there exists a region R such that $A \cap R = \{x\}$.*

Definition 1.27. A set is *infinite* if it is not finite.

Theorem 1.28. *If p is a limit point of A and R is a region containing p, then the set $R \cap A$ is infinite.*

Exercise 1.29. Find realizations of the continuum $(\mathcal{C}, <)$. That is, find examples of sets $\mathcal{C}$ endowed with a relation $<$ satisfying all of the axioms (so far). Are they the same? What does “the same” mean here?