Estimating derivative w/ 4 points

First 2 pts. → Taylor expand

$$f(x+\delta) = f + \delta f' + \frac{\delta^2}{2} f'' + \frac{\delta^3}{6} f''' + \frac{\delta^4}{24} f^{(4)} + \frac{\delta^5}{120} f^{(5)} + \dots \quad ①$$

$$f(x-\delta) = f - \delta f' + \frac{\delta^2}{2} f'' - \frac{\delta^3}{6} f''' + \frac{\delta^4}{24} f^{(4)} - \frac{\delta^5}{120} f^{(5)} + \dots \quad ②$$

subtract ② from ① & divide by 2δ to isolate f'

$$\frac{f(x+\delta) - f(x-\delta)}{2\delta} \approx f' + \frac{\delta^3}{6} f''' + \frac{\delta^{5}}{120} f^{(5)} \quad ⑤$$

add next 2 pts. to cancel out f''' term

$$f(x+2\delta) = f + 2\delta f' + \frac{4\delta^2}{2} f'' + \frac{8\delta^3}{6} f''' + \frac{16\delta^4}{24} f^{(4)} + \frac{32\delta^5}{120} f^{(5)} \dots \quad ③$$

$$f(x-2\delta) = f - 2\delta f' + \frac{4\delta^2}{2} f'' - \frac{8\delta^3}{6} f''' + \frac{16\delta^4}{24} f^{(4)} - \frac{32\delta^5}{120} f^{(5)} \quad ④$$

subtract ④ from ③ & divide by 4δ

$$\frac{-f(x+2\delta) + f(x-2\delta)}{4\delta} \approx f' + \frac{4}{6}\delta^3 f''' + \frac{2}{15}\delta^5 f^{(5)} \quad ⑥$$

subtract 4 × ⑤ from ⑥

$$\frac{-f(x+2\delta) + f(x-2\delta) - 8f(x+\delta) + 8f(x-\delta)}{4\delta}$$

$$= 3f' + \frac{12}{120}\delta^5 f^{(5)}$$

$$\Rightarrow f' \approx \frac{-f(x+2\delta) + f(x-2\delta) - 8f(x+\delta) + 8f(x-\delta)}{12\delta} + \underbrace{\frac{4}{120}\delta^4 f^{(5)}}_{\text{truncation error}}$$

Error calculations for 4 pt. derivative

- from Jon's slides:

leading expansion error $\sim \underbrace{\delta^4 f^{(5)}}_{e_t} + \underbrace{\frac{f\varepsilon_f}{\delta}}_{e_r}$

truncation error

~ machine precision

roundoff error

differentiate this & set to zero to minimize

$$\hookrightarrow \quad -\frac{f\varepsilon_f}{\delta^2} + f^{(5)}\delta^3 = 0$$

$$\Rightarrow \frac{f\varepsilon_f}{\delta^2} = f^{(5)}\delta^3$$

$$\Rightarrow \delta \sim \left(\frac{f\varepsilon_f}{f^{(5)}}\right)^{1/5} \equiv x_c\,\varepsilon_f^{1/5}$$

↳ can set x_c to x except near zero

- fractional accuracy

$$\frac{e_r + e_t}{|f'|} \sim \frac{\delta^4 f^{(5)} + (f\varepsilon_f)/\delta}{|f'|}$$

plug in $\delta = x_c\,\varepsilon_f^{1/5} = \left(\frac{f\varepsilon_f}{f^{(5)}}\right)^{1/5}$

$$\Rightarrow \frac{e_r + e_t}{|f'|} \sim \frac{\left(\frac{f\varepsilon_f}{f^{(5)}}\right)^{4/5} f^{(5)} + \frac{f\varepsilon_f}{\left(\frac{f\varepsilon_f}{f^{(5)}}\right)^{1/5}}}{|f'|}$$

assume $f, f', f^{(5)}$ all share the same characteristic length scale

$$\Rightarrow \frac{e_r + e_t}{|f'|} \sim \varepsilon_f^{4/5}$$