UNIVERSITY of WEST FLORIDA

Binary Logistic Regression

Statistics for Data Science II

Introduction

Suppose we now have outcomes that are binary (only two possible responses).

For example, suppose Y_i was "the student passes the class,'' then:

$$Y_i = \begin{cases} 1 & \text{if student passes} \\ 0 & \text{if student does not pass} \end{cases}$$

Binary variables do not always take on yes/no answers!

e.g., "Do you prefer cats or dogs?"

e.g., "Is the pug fawn or black?"

Introduction

We model binary outcomes using logistic regression.

$$\ln\left(\frac{\pi}{1-\pi}\right) = \beta_0 + \beta_1 X_1 + \ldots + \beta_k X_k,$$

where $\pi = P[Y = 1]$ = the probability of the outcome.

How is this different from linear regression?

$$Y = \beta_0 + \beta_1 X_1 + \ldots + \beta_k X_k$$

Introduction

Why can't we use OLS estimation?

1. The residuals are not normally distributed.
2. The residuals do not have constant variance.
3. The predicted values (probabilities) do not always fall between 0 and 1, the only possible values for the probability of success.

Modeling

Recall the binary logistic regression model,

$$\ln\left(\frac{\pi}{1-\pi}\right) = \beta_0 + \beta_1 X_1 + \ldots + \beta_k X_k,$$

We will specify this in R using the `glm()` function, specifying `family = "binomial"`.

e.g., `glm(outcome ~ predictor1 + predictor2 + ..., data = dataset, family = "binomial")`

The binomial distribution is used for 0/1 outcomes, thus, is why we specify it here.

Modeling

Example:

A researcher is interested in how the GRE, college GPA, and prestige of the undergraduate institution affect admission into graduate school. The response variable, admit/don't admit, is a binary variable. Let's model graduate school admission as a function of GRE, college GPA, and prestige of the undergraduate institution.

```
m1 <- glm(admit ~ gre + gpa + rank, data = data, family = "binomial")
```

Modeling

Example:

```
summary(m1)[12]
```

```
## $coefficients
##                Estimate  Std. Error   z value     Pr(>|z|)
## (Intercept) -3.44954840 1.132846009 -3.045029 2.326583e-03
## gre          0.00229396 0.001091839  2.101005 3.564052e-02
## gpa          0.77701357 0.327483878  2.372677 1.765968e-02
## rank        -0.56003139 0.127136989 -4.404945 1.058109e-05
```

The resulting model is

$$\ln\left(\frac{\hat{\pi}}{1-\hat{\pi}}\right) = -3.45 + 0.002\ \text{GRE} + 0.78\ \text{GPA} - 0.56\ \text{rank},$$

where $\hat{\pi}$ is the probability of admittance to graduate school.

Interpretations

Recall the binary logistic regression model,

$$\ln\left(\frac{\pi}{1-\pi}\right) = \beta_0 + \beta_1 X_1 + \ldots + \beta_k X_k,$$

We are modeling the log odds, which are not intuitive with interpretations.

To be able to discuss the odds, we will "undo" the natural log by exponentiation.

i.e., if we want to interpret the slope for X_i, we will look at $e^{\hat{\beta}_i}$.

When interpreting $\hat{\beta}_i$, it is an additive effect on the log odds.

When interpreting $e^{\hat{\beta}_i}$, it is a multiplicative effect on the odds.

Interpretations

Why is it a multiplicative effect?

$$\ln\left(\frac{\pi}{1-\pi}\right) = \beta_0 + \beta_1 X_1 + \ldots + \beta_k X_k$$

$$\exp\left\{\ln\left(\frac{\pi}{1-\pi}\right)\right\} = \exp\left\{\beta_0 + \beta_1 X_1 + \ldots + \beta_k X_k\right\}$$

$$\frac{\pi}{1-\pi} = e^{\beta_0} e^{\beta_1 X_1} \ldots e^{\beta_k X_k}$$

Interpretations

In linear regression, we interpret $\hat{\beta}_i$:

> For a 1 [unit of predictor] increase in [predictor name], we expect [outcome] to [increase or decrease] by $\left[|\hat{\beta}_i|\right]$ [unit of outcome].

In logistic regression, we interpret $e^{\hat{\beta}_i}$ (the odds ratio):

> For a 1 [unit of predictor] increase in [predictor name], we expect the odds of [outcome] to be multiplied by $[e^{\hat{\beta}_i}]$.
>
> For a 1 [unit of predictor] increase in [predictor name], we expect the odds of [outcome] to [increase or decrease] by $\left[100 \times \left(1 - e^{\hat{\beta}_i}\right)\right]$ %.

Interpretations

Example:

Convert all $\hat{\beta}_i$ to odds ratios and provide brief interpretations for the graduate school admissions data.

```
round(exp(coefficients(m1)), 4)
```

```
## (Intercept)         gre         gpa        rank
##      0.0318      1.0023      2.1750      0.5712
```

Interpretations

Example:

```
## (Intercept)         gre         gpa        rank
##      0.0318      1.0023      2.1750      0.5712
```

For a 1 point increase in GRE score, the odds of admission increase by .23%.

For a 1 point increase in GPA, the odds of admission increase by 118%.

For a 1 point increase in rank of undergraduate institution, the odds of admission decrease by 43%.

Interpretations

Example:

Suppose we turn the rank of undergraduate institution into a factor variable and use that as the only predictor of admission to graduate school. Let a top tier (rank $= 1$) undergraduate institution be the reference.

```
data <- dummy_cols(data, select_columns = "rank")
m2 <- glm(admit ~ rank_2 + rank_3 + rank_4, data = data, family = "binomial")
round(exp(coefficients(m2)), 4)
```

```
## (Intercept)      rank_2      rank_3      rank_4
##      1.1786      0.4724      0.2555      0.1851
```

The odds of someone from a 2nd tier undergraduate institution being admitted to graduate school are 0.47 times that of someone from a top tier undergraduate institution.

Inference

Statistical Test for β_i

Hypotheses

$$H_0: \ \beta_i = \beta_i^{(0)} \mid \beta_i \geq \beta_i^{(0)} \mid \beta_i \leq \beta_i^{(0)}$$
$$H_1: \ \beta_i = \beta_i^{(0)} \mid \beta_i < \beta_i^{(0)} \mid \beta_i > \beta_i^{(0)}$$

Test Statistic

$$z_0 = \frac{\hat{\beta}_i - \beta_i^{(0)}}{\mathrm{SE}_{\hat{\beta}_i}}$$

Rejection Region

Reject H_0 if $p < \alpha$.

Inference

Example:

```
summary(m1)[12]
```

```
## $coefficients
##                Estimate  Std. Error   z value     Pr(>|z|)
## (Intercept) -3.44954840 1.132846009 -3.045029 2.326583e-03
## gre          0.00229396 0.001091839  2.101005 3.564052e-02
## gpa          0.77701357 0.327483878  2.372677 1.765968e-02
## rank        -0.56003139 0.127136989 -4.404945 1.058109e-05
```

Thus, all are significant predictors of admission to graduate school.

Inference

Confidence Interval for β_i

$$\hat{\beta}_i \pm z_{1-\alpha/2}\mathsf{SE}_{\hat{\beta}_i}$$

Example:

```
confint(m1)
```

```
##                     2.5 %       97.5 %
## (Intercept) -5.7109591680 -1.260314066
## gre          0.0001715446  0.004461385
## gpa          0.1415710585  1.428341503
## rank        -0.8149612229 -0.315479733
```

Inference

We can also find the CI for OR_i by exponentiating the lower and upper bounds.

Example:

```
round(exp(confint(m1)),3)
```

```
##                2.5 % 97.5 %
## (Intercept) 0.003  0.284
## gre         1.000  1.004
## gpa         1.152  4.172
## rank        0.443  0.729
```

Predictions

Recall the logistic regression model,

$$\ln\left(\frac{\pi_i}{1-\pi_i}\right) = \beta_0 + \beta_1 X_{1i} + \ldots + \beta_k X_{ki}$$

We can solve for the probability, which allows us to predict the probability that $Y_i = 1$ given the specified model:

$$\pi_i = \frac{\exp\{\beta_0 + \beta_1 X_{1i} + \ldots + \beta_k X_{ki}\}}{1 + \exp\{\beta_0 + \beta_1 X_{1i} + \ldots + \beta_k X_{ki}\}}$$

Predictions

Example:

```
data$p_hat <- predict(m1, type="response")
head(data)
```

```
## # A tibble: 6 x 5
##   admit   gre   gpa  rank  p_hat
##   <dbl> <dbl> <dbl> <dbl>  <dbl>
## 1     0   380  3.61     3 0.190
## 2     1   660  3.67     3 0.318
## 3     1   800  4        1 0.718
## 4     1   640  3.19     4 0.149
## 5     0   520  2.93     4 0.0980
## 6     1   760  3        2 0.379
```

Diagnostics

We generally are not worried about residuals in logistic regression.

We can still look at Cook's distance.

Recall that we look for "spikes" on the graph.

This allows us to determine any leverage/influence points.

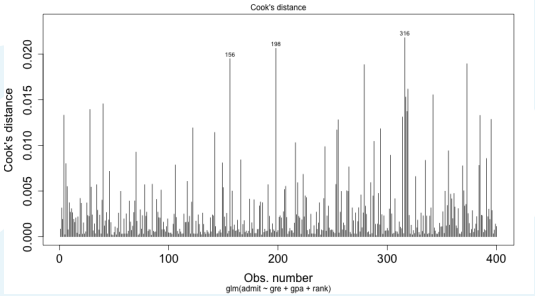
Leverage/influence points are ones that have an effect on the regression model.

If we detect leverage/influence points, we can perform sensitivity analysis to determine how “different” the model is.

Diagnostics

Example:

```
plot(m1, which = 4)
```



Diagnostics

We can also check for multicollinearity using the VIF.

Recall that VIF > 10 indicates multicollinearity.

Example:

```
vif(m1)
```

```
##      gre      gpa     rank
## 1.135657 1.134708 1.004034
```

Visualization

Example:

Let the probability of admittance to graduate school be on the y-axis and GPA be on the x-axis. We will hold the GRE score and rank of undergraduate institution constant by plugging in their median values.

```
c1 <- coefficients(m1)
data$pred_med <- exp(c1[1] + c1[2]*median(data$gre) + c1[3]*data$gpa +
                     c1[4]*median(data$rank))/(1+exp(c1[1] +
                     c1[2]*median(data$gre) + c1[3]*data$gpa +
                     c1[4]*median(data$rank)))
```

Visualization

Example:

```
head(data)
```

```
## # A tibble: 6 x 6
##   admit   gre   gpa  rank  p_hat pred_med
##   <dbl> <dbl> <dbl> <dbl>  <dbl>    <dbl>
## 1     0   380  3.61     3 0.190     0.393
## 2     1   660  3.67     3 0.318     0.404
## 3     1   800  4        1 0.718     0.467
## 4     1   640  3.19     4 0.149     0.319
## 5     0   520  2.93     4 0.0980    0.276
## 6     1   760  3        2 0.379     0.287
```

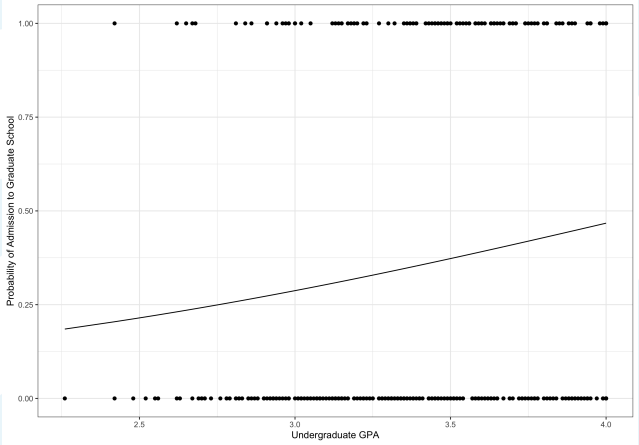
Visualization

Example:

```
p2 <- data %>% ggplot(aes(x = gpa, y = admit)) +
  geom_point() +
  geom_line(aes(y = pred_med))+
  xlab("Undergraduate GPA") +
  ylab("Probability of Admission to Graduate School") +
  theme_bw()
```

Visualization

Example:



Visualization

Example:

Instead of looking at median values, let's plug in the best possible values for GRE (the maximum) and rank of undergraduate institution (the minimum).

```
data$pred_best <- exp(c1[1] + c1[2]*max(data$gre) + c1[3]*data$gpa +
                      c1[4]*min(data$rank))/(1+exp(c1[1] +
                      c1[2]*max(data$gre) + c1[3]*data$gpa +
                      c1[4]*min(data$rank)))
```

Visualization

Example:

```
head(data)
```

```
## # A tibble: 6 x 7
##   admit   gre   gpa  rank  p_hat pred_med pred_best
##   <dbl> <dbl> <dbl> <dbl>  <dbl>    <dbl>     <dbl>
## 1     0   380  3.61     3 0.190     0.393     0.653
## 2     1   660  3.67     3 0.318     0.404     0.663
## 3     1   800  4        1 0.718     0.467     0.718
## 4     1   640  3.19     4 0.149     0.319     0.575
## 5     0   520  2.93     4 0.0980    0.276     0.526
## 6     1   760  3        2 0.379     0.287     0.539
```

Visualization

Example:

```
p3 <- data %>% ggplot(aes(x = gpa, y = admit)) +
  geom_point() +
  geom_line(aes(y = pred_best))+
  xlab("Undergraduate GPA") +
  ylab("Probability of Admission to Graduate School") +
  theme_bw()
```

Visualization

Example: