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Page No. 1

# COMPLEX NUMBER SYSTEM

Complex no. is a no. that can be expressed in the form of $a+ib$ (or $x+iy$) where $a$ and $b$ are real numbers and $i$ is the root of $x^2+1=0$. (No real number satisfies this equation)

## Fundamental THEOREM

All polynomial eq^n's with real or complex coefficients in a single variable have a solution in complex number system.

## Properties:

① $z_1+z_2 = z_2+z_1$ , $z_1z_2 = z_2z_1$

② $(z_1+z_2)+z_3 = z_1+(z_2+z_3)$ , $(z_1z_2)z_3 = z_1(z_2z_3)$

③ $z_1(z_2+z_3) = z_1z_2 + z_1z_3$

⇒ Complex Conjugate

∴ A complex no. is real if and only if it equal to its conjugate.

If $z = x+iy$ then, $\bar{z} = x-iy$.

$\therefore \bar{\bar{z}} = z$

Prove: $\overline{z_1z_2} = \bar{z}_1 \cdot \bar{z}_2$

Let; $z_1 = x+iy$ & $z_2 = a+ib$.

$$z_1z_2 = (x+iy)(a+ib)$$
$$= ax + ibx + iay + \cancel{i^2}by \quad -by.$$
$$= (ax-by) + i(bx+ay)$$
$$\overline{z_1z_2} = (ax-by) - i(bx+ay).$$
$$\bar{z}_1 \cdot \bar{z}_2 = (x-iy)(a-ib)$$
$$= ax - ibx - iay + \cancel{i^2}by \quad -by$$
$$= (ax-by) - i(bx+ay)$$

Hence, LHS = RHS

$$\overline{z_1z_2} = \bar{z}_1 \cdot \bar{z}_2$$

Hence, proved.

$$z^n = (x+iy)^n = r^n(\cos\theta + i\sin\theta)^n$$

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

# Function of COMPLEX VARIABLE

Let $z = x+iy$, in which $x$ & $y$ are independent real variables, is known as a COMPLEX VARIABLE then the map $f: C \to C$ defined as

$$w = f(z)$$

is called the function of complex variable.

For example; $f(z) = \bar{z}$, $f(z) = \frac{\bar{z}}{z}$

$$f(z) = \underbrace{x}_{u(x,y)} + i\underbrace{y}_{v(x,y)}$$

$$f(z) = u(x,y) + i\,v(x,y)$$

$\Rightarrow$ $f(z) = \bar{z}.\,Re(z) + z^2 + Im(z)$

$$= (x-iy)x + (x+iy)^2 + y$$

$$= x^2 - ixy + x^2 - y^2 + i2xy + y.$$

$$= (2x^2 - y^2 + y) + i\,xy$$

$$\underset{(x,y)}{u} = 2x^2 - y^2 + y \qquad \underset{(x,y)}{v} = xy$$

$\Rightarrow$ $f(z) = |z|$

$$u = \sqrt{x^2+y^2} \qquad v = 0.$$

* LIMIT

Let $w = f(z)$ be a function of $z$ and let $u(x,y)$ & $v(x,y)$ denote the real & imaginary parts of $w$.
Let $w_0 = u_0 + iv_0$ be a fix complex number.
We say that, the limits of $f(z)$ as $z \to z_0$ (tends to)

$$\lim_{z \to z_0} f(z) = w_0.$$

when, $u(x,y) \to u_0$ & $v(x,y) \to v_0$.

$$\Rightarrow \lim_{z \to 0} \frac{z}{|z|} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x+iy}{\sqrt{x^2+y^2}} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x+imx}{\sqrt{x^2+m^2x^2}}$$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{1+im}{\sqrt{1+m^2}}$$

$\boxed{\text{Two Path Test}}$

$[y = mx]$

LIMIT DOES NOT EXIST
(as is depending on 'm').

$1x$, $2x$ (different values from two different paths)

* CONTINUITY

A function $w = f(z)$ is said to be continuous at $z = z_0$ if $\lim_{z \to z_0} f(z)$ exists and is equal to $f(z_0)$.

i.e. $\lim_{z \to z_0} f(z) = f(z_0)$.

value of $f(z)$ at $z = z_0$.

* DIFFERENTIABILITY

Let $w = f(z)$, be a function of complex variable $z$ and let $z_0 + \Delta z_0$ denote a point in the neighbourhood of a fixed point $z_0$.
Then, the derivative $f'(z)$ is defined by the relation

$$f'(z_0) = \lim_{\Delta z_0 \to 0} \frac{f(z_0 + \Delta z_0) - f(z_0)}{\Delta z_0}$$

$$\frac{dw}{dz} = \lim_{\Delta z_0 \to 0} \frac{\Delta w}{\Delta z_0}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y}.$$

According to two path test;
when $\Delta z \to 0$, along a path parallel to real axis
i.e. $\Delta y = 0$ and $\Delta x \to 0$

$$\frac{dw}{dz} = \lim_{\Delta x \to 0} \frac{\Delta u + i\Delta v}{\Delta x}.$$

$$\frac{dw}{dz} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + i \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}$$

$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \rightarrow (1).$$

Now; $\Delta z \to 0$, along a path parallel to imaginary axis, $\Delta x = 0$, $\Delta y \to 0$

$$\frac{dw}{dz} = \lim_{\Delta y \to 0} \frac{\Delta u + i\Delta v}{i\Delta y}$$

$$= -i \lim_{\Delta y \to 0} \frac{\Delta u}{\Delta y} + \lim_{\Delta y \to 0} \frac{\Delta v}{\Delta y}$$

$$\frac{dw}{dz} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \quad — (2)$$

As; $f'(z)$ exist, then

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \text{and} \quad \boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

$$\boxed{u_x = v_y} \qquad \boxed{v_x = -u_y}$$

**Cauchy Reiman (CR) Equations**

* If function is differentiable, then CR eqn must satisfy.

* Differentiability imply continuity but converse is not true.

**Que** Show that the function $f(z) = \bar{z}$ is continuous at the point $z = 0$ but not differentiable at $z = 0$.

$$f(z) = \bar{z}$$
$$f(z) = x - iy$$

$$\lim_{z \to 0} f(z) = \lim_{\substack{x \to 0 \\ y \to 0}} (x - iy)$$
$$= 0$$
$$f(0) = \bar{0} = 0$$

Hence; the function is continuous at $z = 0$.

Now;
$$f'(z_0) = \lim_{z \to 0} \frac{f(z_0 + \Delta z_0) - f(z_0)}{\Delta z_0}$$

$$f'(0) = \lim_{\Delta z \to 0} \frac{f(0+\Delta z_0) - f(0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{f(\Delta z_0) - 0}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta \bar{z}}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

$y = mx$

$\Delta y = m\Delta x$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - im\Delta x}{\Delta x + im\Delta x}$$

$$f'(0) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{1-im}{1+im}$$ (depends on 'm')

Hence; $f'(z)$ does not exist at $z = 0$.

# ANALYTIC FUNCTION

The function, $w = f(z)$ is said to be analytic at a point $z_0$, if its derivative $f'(z)$ exist at $z = z_0$ and at every point in some neighbourhood of $z_0$.

ANALYTICITY $\Rightarrow$ DIFFERENTIABILITY

Necessary and sufficient condition for analyticity

Necessary & sufficient condition for a function $f(z) = u(x,y) + i\,v(x,y)$ to be analytic in a region R in the z plane is that u, v and their first order partial derivatives

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \text{ and } \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \text{ are continuous in } R$$

and their Cauchy Reiman (CR) eqns are satisfied at every point in R.

Sufficient condition for a function to be analytic.

Suppose that real & imaginary parts $u(x,y)$ & $v(x,y)$ of the function $f(z) = u(x,y) + i\,v(x,y)$ are continuous ~~and ha~~ for and have continious first order partial derivative in domain & if u and v satisfies the CR eqn at all points in the domain D, then the function is analytical.

and;

$$f'(z) = u_x + i v_x = v_y - i u_y$$

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Analyticity implies differentiability but not conversely.

→ $f(z) = |z|^2$ is differentiable but not analytical. (at $z=0$)

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z_0) - f(z_0)}{\Delta z}$$

$\therefore \boxed{|z|^2 = z.\bar{z}}$

Date / /
Page No. 8

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z).\overline{(z+\Delta z)} - z\bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)(\bar{z}+\Delta \bar{z}) - z\bar{z}}{\Delta z}.$$

$$f(z) = \lim_{\Delta z \to 0} \left( \frac{z\Delta \bar{z}}{\Delta z} + \bar{z} + \Delta \bar{z} \right)$$

$\boxed{f'(0) = 0}$

function is differentiable at $z=0$.

Now, for $z=0$, $\lim_{\Delta z \to 0} \left( \frac{\Delta \bar{z}}{\Delta z} \right)$ does not exist.

& is not differentiable at $z \neq 0$.

Que $f(z) = u(x,y) + iv(x,y)$.

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}.$$

Solⁿ $f(z) = u(x,y) + iv(x,y)$.

then;

$\Rightarrow$ $f'(z) = u_x + iv_x = u_y - iv_y$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} - i\frac{\partial u}{\partial y}.$$

CR eqⁿs:-

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$$\left[ f'(z) \Rightarrow \cancel{\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \right]$$

Hence, proved.

Que: $f(z) = x^2 + iy^2$ ; does $f'(z)$ exist at any point?

Sol$^n$: Here;

$$u(x,y) = x^2 \quad , \quad v(x,y) = y^2$$

$$\frac{\partial u}{\partial x} = 2x \quad , \quad \frac{\partial v}{\partial y} = 2y \quad , \quad \frac{\partial u}{\partial y} = 0 \quad , \quad \frac{\partial v}{\partial x} = 0 .$$

From;

$$f'(z) \Rightarrow \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

$$2x + 0 = 2y - 0$$

$$\underline{(x = y)}.$$

Function is differentiable for $\boxed{x = y}$

Que: $f(z) = x^4 - i(y+1)^3$. Determine the point where the $fn^n$ is differentiable and find its differentiation.

Sol$^n$: $fn^n$ is differentiable for points satisfying —

$$4x^3 + 3(y+1)^2 = 0 . \quad \text{—(1)}$$

$$f'(z) = u_x + i\,v_x .$$
$$= 4x^3$$

OR

$$f'(z) = v_y - i u_y .$$
$$= -3(y+1)^2$$
$$= 4x^3 \quad (\text{from (1)}).$$

**Que** $f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & \text{for } z \neq 0 \\ 0, & \text{for } z = 0 \end{cases}$

Show that the function satisfies CR $eq^n$ at $z=0$ but differentiation does not exist.

**Sol^n**

CR $eq^ns$ :-

$$u_x = v_y \quad , \quad u_y = -v_x .$$

$$u(x,y) = \frac{x^3 - y^3}{x^2+y^2} \quad , \quad v(x,y) = \frac{x^3+y^3}{x^2+y^2} \quad , (x,y) \neq (0,0)$$

$$u(0,0) = v(0,0) = f(0) = 0 .$$

$$\therefore u_x = \lim_{x \to 0} \left( \frac{u(x,0) - u(0,0)}{x} \right)$$

$$= \lim_{x \to 0} \frac{x - 0}{x} = 1 .$$

$$\therefore u_y = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y}$$

$$= \lim_{y \to 0} \frac{-y-0}{y} = -1$$

$$\therefore v_x = \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x}$$

$$= \lim_{x \to 0} \frac{x-0}{x} = 1$$

$$\therefore v_y = 1 .$$

Hence, CR $eq^ns$ satisfy at $z=0$.

$$f'(z_0) = \lim_{z \to 0} \frac{f(z+z_0) - f(z_0)}{z}$$

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z}$$

$$= \lim_{z \to 0} \frac{\frac{x^3(1+i) + y^3(1-i)}{x^2+y^2} - 0}{x+iy}$$

$$= \lim_{z \to 0} \frac{x^3(1+i) - y^3(1-i)}{(x^2+y^2)(x+iy)}.$$

$y = mx \Rightarrow$

$$= \lim_{x \to 0} \frac{x^3(1+i) - m^3x^3(1-i)}{(x^2+m^2x^2)(x+imx)}.$$

$$= \lim_{x \to 0} \frac{(1+i) - m^3(1-i)}{(1+m^2)(1+im)}. \quad \{\text{depends on } m\}$$

Thus; $f'(0)$ does not exist.

Hence; derivatave does not exist.

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# CR eq$^n$ in POLAR ~~COORDINATES~~ FORM

$f(z) = u(x,y) + iv(x,y) = u(r,\theta) + iv(r,\theta)$.

$re^{i\theta} = z = x+iy$

$= r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta}$

Que Show that the polar form of Cauchy-Reimann eq$^n$ is : $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ & $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

Use the result to show that $\log z$ is analytic.

Sol$^n$ Let $f(z) = u(r,\theta) + iv(r,\theta)$ , $z = re^{i\theta}$

where; $x = r\cos\theta$ , $y = r\sin\theta$, $r = \sqrt{x^2+y^2}$,

$\theta = \tan^{-1}\frac{y}{x}$.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}(\cos\theta) + \frac{\partial u}{\partial y}(\sin\theta)$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)$$

Also; $$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{\partial v}{\partial x}(\cos\theta) + \frac{\partial v}{\partial y}(\sin\theta).$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \theta} = \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y}(r\cos\theta).$$

Then;

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}(\cos\theta) + \frac{\partial v}{\partial y}(\sin\theta)$$

CR eq$^n$ in CARTESIAN ; $u_x = v_y$ & $u_y = -v_x$.

$$\frac{\partial v}{\partial r} = \left(-\frac{\partial u}{\partial y}\right)\cos\theta + \frac{\partial u}{\partial x}\sin\theta$$

$$= -\frac{1}{r}\left[r\cos\theta\frac{\partial u}{\partial y} - (r\sin\theta)\frac{\partial u}{\partial x}\right]$$

$$= -\frac{1}{r}\left[\frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)\right]$$

$$\boxed{\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}}$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}(\cos\theta) + \frac{\partial u}{\partial y}(\sin\theta)$$

$$= \frac{\partial v}{\partial y}(\cos\theta) + \left(-\frac{\partial v}{\partial x}\right)\sin\theta$$

$$= \frac{1}{r}\left[\frac{\partial v}{\partial y}(r\cos\theta) - \frac{\partial v}{\partial x}(r\sin\theta)\right]$$

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}}$$

$$\log z = \log(re^{i\theta}) = \log r + \log e^{i\theta}$$

$$= \log r + i\theta$$

$$u = \log r, \quad v = \theta$$

$$u_r = \frac{1}{r}, \quad v_\theta = 1, \quad v_r = 0, \quad u_\theta = 0$$

So; log z is analytic.

# HARMONIC FUNCTION

$u_{xx} + v_{yy} = 0$ — v is harmonic conjugate of u.

$u_x = v_y, \quad u_y = -v_x$

$u_{xx} = v_{xy}, \quad u_{yy} = -v_{xy}$

$u_{xx} + v_{yy} = 0$

A real valued function $\phi(x,y)$ of 2 variables x, y that has continuous second order continuous partial derivative in a domain D and satisfies the Laplace eqn. is said to be harmonic in D.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

THEOREM

If $f(z) = u(x,y) + iv(x,y)$ is analytic in domain D. Then, the real valued function $u(x,y)$ & $v(x,y)$ satisfy the laplace eq$^n$ i.e. $u_{xx} + u_{yy} = 0$ & $v_{xx} + v_{yy} = 0$ respectively in D. i.e. $u_{xx}$ & $v_{xx}$ are harmonic in D

REMARK: converse of above is true.

If $f(z) = u + iv$ analytic, then both $u$ & $v$ $\rightarrow$ HARMONIC

NOTE: If $f(z) = u(x,y) + iv(x,y)$ is analytic in Domain D, then its real & imaginary parts $u(x,y)$ & $v(x,y)$ are harmonic functions. The function $v(x,y)$ is called conjugate harmonic function of $u(x,y)$ in D.

- The conjugate harmonic function $v(x,y)$ for a given $u(x,y)$ can be ~~written~~ obtained by using the CR eq$^n$s.
- The conjugate harmonic function is unique except for an additive constant.

# MILNE-THOMSON METHOD (to find complex harmonic conjugate)

For the function $f(z) = u + iv$, let $u$ is given

As ~~its~~ function is ANALYTIC $\Rightarrow$ function is DIFFERENTIA

$$f'(z) = u_x + iv_x$$

Using CR-eq$^n$; $u_x = v_y$, $u_y = -v_x$

$$f'(z) = u_x - iu_y$$

Replace; $\boxed{x \rightarrow z}$ & $\boxed{y \rightarrow 0}$; $f'(z) = \phi(z)$

by taking integration, we get the function $f(z)$ upto integrating constant.

**Question**: Show that the function is harmonic and determine its conjugate function.

$u = 2x(1-y)$

**Sol^n**

$u = 2x - 2xy$

$u_x = 2 - 2(y) = 2(1-y)$ , $u_y = -2x$

$u_{xx} = 0$ , $u_{yy} = 0$

Using CR eq^ns ; $u_x = v_y$ & $u_y = -v_x$

$v_y = u_x = 2(1-y)$

Integrating partially w.r.t $y$, we get :-

$$v = 2\left(y - \frac{y^2}{2}\right) + \phi(x) = 2y - y^2 + \phi(x) \quad \text{— (A)}.$$

Again ; $v_x = -u_y = \boxed{2x = v_x}$ — (B)

Differentiating (A) partially w.r.t. $x$ :-

$$v_x = \phi'(x) \quad \text{— (C)}$$

From, (B) & (C)

$\phi'(x) = 2x$ , $\boxed{\phi(x) = x^2 + C}$

$\boxed{v = 2y - y^2 + x^2 + C}$ from (A).

— x — x — x — x — x —

$$\int u \cdot v \cdot dx = u \cdot \int v \cdot dx - \int \left( \frac{du}{dx} \int v\,dx \right) dx$$

⇒ (Integration by parts).

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**Que.** If $u = 2x(1-y)$, then find $v = ??$

**Sol^n** $u_x = 2(1-y)$, $u_y = -2x$

Using CR eq^ns, & Milne Thomson, –

$f'(z) = u_x - iu_y.$

$f'(z) = 2(1-y) + 2ix$

$x \to z$, $y \to 0$

$f'(z) = 2 + 2iz$

From integration;

$$\boxed{f(z) = 2z + iz^2 + c}$$

$$f(z) = 2(x+iy) + i(x+iy)^2 + c$$
$$= 2x + 2iy + i(x^2 - y^2 + 2ixy) + c$$
$$= (2x - 2xy) + i(x^2 - y^2 + 2y) + c$$
$$= 2x(1-y) + i\underbrace{(x^2 - y^2 + 2y + c)}_{v}$$

**Que.** $u = e^{2x}(x\cos 2y - y\sin 2y)$.

Find $v$ and $f(z)$ ??

**Sol^n** $u_x = e^{2x}(\cos 2y) + 2e^{2x}(x\cos 2y - y\sin 2y)$

Using CR eq^ns & Milne Thomson Theorem;

$f'(z) = u_x - iu_y.$

$u_y = e^{2x}(-2x\sin 2y - y\,2\cos 2y - \sin 2y)$

$f'(z) = u_x - iu_y.$

$f'(z) = e^{2x}\cos 2y + 2e^{2x}x\cos 2y - 2e^{2x}y\sin 2y$

$-i(-2e^{2x}x\sin 2y - 2ye^{2x}\cos 2y - e^{2x}\sin 2y$

$$f'(z) = (e^{2x}\cos 2y + 2e^{2x}x\cos 2y - 2e^{2x}y\sin 2y)$$
$$+ i(2e^{2x}x\sin 2y + 2ye^{2x}\cos 2y + e^{2x}\sin 2y)$$

$x \to z$, $y \to 0$.

$$f'(z) = (e^{2z} + 2ze^{2z}) + i(0)$$
$$f'(z) = e^{2z} + 2ze^{2z}$$

From integration ;

$$f(z) = \frac{e^{2z}}{2} + 2\left[\frac{z\cdot e^{2z}}{2} - \int \frac{e^{2z}}{2}\right].$$

$$f(z) = \frac{e^{2z}}{2} + 2\left[\frac{z\cdot e^{2z}}{2} - \frac{e^{2z}}{4}\right]. + c$$

$$f(z) = \cancel{\frac{e^{2z}}{2}} + z\cdot e^{2z} - \cancel{\frac{e^{2z}}{2}} + c$$

$$\boxed{f(z) = z\cdot e^{2z} + c}$$

Put $z = x + iy$.

$$f(z) = (x+iy)\cdot e^{2(x+iy)} + c$$
$$f(z) = (x+iy)\cdot e^{2x}\cdot e^{2iy} + c.$$
$$f(z) = (x+iy)\cdot e^{2x}\cdot(\cos 2y + i\sin 2y) + c.$$
$$f(z) = e^{2x}[x\cos 2y - y\sin 2y] + ie^{2x}\underbrace{[x\sin 2y + y\cos 2y + c]}_{v}$$

Method-②

$u_x = e^{2x}(\cos 2y) + (x\cos 2y - y\sin 2y)\cdot 2e^{2x}$

$u_y = e^{2x}[-2x\sin 2y - (y\cdot 2\cos 2y + \sin 2y]$.

$= e^{2x}[-2x\sin 2y - 2y\cos 2y - \sin 2y]$.

CR eqns $u_x = v_y$, $u_y = -v_x$.

$v_y = e^{2x}\cos 2y + (x\cos 2y - y\sin 2y)\cdot 2e^{2x}$.

Integrating w.r.t. 'y' :-

$$v = e^{2x}\frac{\sin 2y}{2} + 2e^{2x}\left[x\frac{\sin 2y}{2} - \left\{y \cdot -\frac{\cos 2y}{2} - \int 1 \cdot -\frac{\cos 2y}{2}dy\right\}\right] + \phi(x)$$

$$= e^{2x}\frac{\sin 2y}{2} + 2e^{2x}\left[\frac{x\sin 2y}{2} + \frac{y\cos 2y}{2} - \frac{\sin 2y}{4}\right] + \phi(x)$$

$$\boxed{v = xe^{2x}\sin 2y + e^{2x}\cdot y\cos 2y + \phi(x)} \longrightarrow ①.$$

$v_x = -u_y$

$v_x = e^{2x}[2x\sin 2y + 2y\cos 2y + \sin 2y] - ②$.

Diff. ① w.r.t. $x$ :-

$v_x = \sin 2y[2xe^{2x} + e^{2x}] + 2\cdot e^{2x}\cdot y\cdot\cos 2y + \phi'(x)$ - ③

From ② and ③ :-

$\cancel{x\cdot 2e^{2x}\sin 2y} + \cancel{2e^{2x}y\cos 2y} + \cancel{e^{2x}\sin 2y} = \cancel{2xe^{2x}\sin 2y} + \cancel{\sin 2y\, e^{2x}} + \cancel{2e^{2x}\cdot y\cdot\cos 2y} + \phi'(x)$

$\phi'(x) = 0$

$\boxed{\phi(x) = C}$ — ④

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Page No. 19.
Shivalal

From ① and ④ :-
$v = xe^{2x}\sin 2y + e^{2x}\cdot y\cos 2y + C$.

$f(z) = u + iv$
$= e^{2x}(x\cos 2y - y\sin 2y) + i(x.e^{2x}\sin 2y + ye^{2x}\cos 2y + C)$
$= e^{2x}[x\cos 2y - y\sin 2y + i x\sin 2y + i y\cos 2y] + iC$
$= e^{2x}[x(\cos 2y + i\sin 2y) + iy(\cos 2y + i\sin 2y)] + iC$
$= e^{2x}[x.e^{2iy} + iy.e^{2iy}] + iC$

$= e^{2x}.e^{2iy}[x + iy] + iC$
$= e^{2(x+iy)}.(x+iy) + iC$
$\boxed{f(z) = e^{2z}.z + iC}$

Date: 07/08/23

Que. $u - v = (x-y)(x^2 + 4xy + y^2)$.
Determine $f(z)$.

Sol. Differentiate partially w.r.t. $x$ :-
$u_x - v_x = (x-y)(2x+4y) + (x^2+4xy+y^2)$ — ①
Differentiate partially w.r.t $y$ :-
$u_y - v_y = 3x^2 - 3y^2 - 6xy$.

$\therefore u_x = v_y$ & & $u_y = -v_x$.
$-v_x - u_x = 3x^2 - 3y^2 - 6xy$ (using CR eq^ns).
$u_x + v_x = -3x^2 + 3y^2 + 6xy$ — ②
Adding ① & ② :-
$\boxed{u_x = 6xy}$

Putting $u_x$ in ② :-
$v_x = -3x^2 + 3y^2$

Another method → MILNE THOMSON !!

Integrating $u_x$ & $v_x$ partially w.r.t. $x$ :-
$u = 3x^2y + \phi_1(x)$ — (A)
$v = -x^3 + 3xy^2 + \phi_2(y)$ — (B)

Now, differentiate (A) & (B) w.r.t. $y$ :-

$$\left.\begin{aligned} u_y &= 3x^2 + \phi_1'(y) \\ v_y &= 6xy + \phi_2'(y) \end{aligned}\right\} (C)$$

From CR eq^ns :-

$$\left.\begin{aligned} u_y &= -v_x = 3x^2 - 3y^2 \\ v_y &= u_x = 6xy \end{aligned}\right\} (D)$$

On comparing (C) & (D) :-

$$\phi_1'(y) = -3y^2 \quad ; \quad \phi_2'(y) = 0$$

On integrating :-

$$\phi_1(y) = -y^3 + C_1 \quad ; \quad \phi_2(y) = C_2$$

From (A) & (B) :-

$$\boxed{u = 3x^2y - y^3 + C_1}$$

&

$$\boxed{v = -x^3 + 3xy^2 + C_2}$$

$\therefore$ Hence ; $f(z) = u + iv$

$$\begin{aligned} &= (3x^2y - y^3 + C_1) + i(-x^3 + 3xy^2 + C_2) \\ &= 3x^2y - y^3 - ix^3 + 3ixy^2 + (C_1 + iC_2) \\ &= 3x^2y - y^3 + (ix)^3 + 3ixy^2 + C \\ &= (ix)^3 - y^3 + 3\cdot ix \cdot y^2 - 3\cdot(ix)^2\cdot y + C \\ &= (ix - y)^3 + C \\ &= i^3(x + iy)^3 + C \end{aligned}$$

$$\boxed{f(z) = -iz^3 + C}$$

Que. If $f(z) = \phi + i\psi$ represents the complex potential for an electrical field and $\psi = x^2 - y^2 + \dfrac{x}{x^2+y^2}$;

Determine the conjugate fn, $\phi$ and $f(z)$.

Soln

Differentiate $\psi$ w.r.t. $x$ :-

$$\psi_x = 2x + \frac{(x^2+y^2)(\cancel{1}) - x(2x)}{(x^2+y^2)^2}.$$

$$\psi_x = 2x + \frac{\cancel{2x^3} + \cancel{2xy^2} - 2x^2}{\cancel{(x^2+y^2)^2}} \quad \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\psi_x = 2x - \frac{x^2-y^2}{(x^2+y^2)^2}$$

Differentiate $\psi$ w.r.t. $y$ :-

$$\psi_y = -2y + \frac{(0) - 2xy}{(x^2+y^2)^2} = -2y - \frac{2xy}{(x^2+y^2)^2}$$

Using CR eqns :-

$$\phi_x = \psi_y.$$

$$\phi_x = \cancel{2x} + \cancel{\frac{2x^3 + 2xy^2 - 2x^2}{(x^2+y^2)^2}} - 2y - \frac{2xy}{(x^2+y^2)^2}$$

$$\phi_y = -\psi_x$$

$$\cancel{\phi_y} = \cancel{-2x} - \cancel{\frac{2x^3 + 2xy^2 - 2x^2}{(x^2+y^2)^2}}$$

$$\phi_y = -2x + \frac{x^2-y^2}{(x^2+y^2)^2} \quad \text{—(1)}$$

Integrating (1) w.r.t. y ;

$$\phi = -2xy + \frac{y}{(x^2+y^2)} + g(y)$$

Using Milne - Thomson Theorem :-

$$\therefore \quad \psi_x = 2x - \frac{\cancel{y^2} x^2 - y^2}{(x^2+y^2)^2} \quad ; \quad \psi_y = -2y - \frac{2xy}{(x^2+y^2)^2}$$

$$f'(z) = \psi_y - i\psi_x = -2y - \frac{2xy}{(x^2+y^2)^2} - i\left[2x - \frac{\cancel{y^2} x^2 - y^2}{(x^2+y^2)^2}\right].$$

Put $x \to z$ & $y \to 0$ .

$$f'(z) = -i\left[2z - \frac{z^2}{z^4} + c\right] = -i\left[2z - \frac{1}{z^2} + c\right].$$

Integrating :-

function,

$$\boxed{f(z) = -i\left[z^2 + \frac{1}{z} + c\right]}$$

Put $z = x+iy$ .

$$f(z) = -i\left[(x+iy)^2 + \frac{1}{(x+iy)} + c\right]$$

$$f(z) = -i\left[x^2 - y^2 + \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2} + i2xy + c\right].$$

$$f(z) = \underbrace{\left(2xy - \frac{y}{x^2+y^2} + c\right)}_{\phi} - i\underbrace{\left(x^2 - y^2 + \frac{x}{x^2+y^2}\right)}_{\psi}.$$

Date: 10/08/2023

# COMPLEX INTEGRATION

The definite integrals for the functions of real variables is defined as $\int_a^b f(x)\,dx$ where path of integration is along a straight line $x=a$ to $x=b$. i.e straight line.
But, in complex integration, the path could be along any curve from $z=a$ to $z=b$.

* Curves in the complex plane.

Let $x(t)$ and $y(t)$ be two continuous functions of a real variable $t$, $a \le t \le b$, then $z = z(t)$
$= x(t) + iy(t)$

- The curve is said to be closed if the $z(a) = z(b)$.

- A curve is simple if it does not intersect itself.
$z(t_1) \neq z(t_2)$, for $t_1$ and $t_2$

- A piecewise-continuous curve is continuous on ~~closed interval~~ $[a,b]$ except for atmost ~~for~~ at finite no. of jumps in the interval.

- The curve $z = z(t)$ is said to be SMOOTH, if $z(t)$ is continuously differentiable. and $z'(t) \neq 0$; $a \le t \le b$.

- The curve $z = z(t)$ is called CONTOUR if it is smooth or piecewise-continuous.

- LINE INTEGRAL / CONTOUR INTEGRAL.
The line integral / contour integral of $f(z)$ over the contour $C$ is denoted by $\int_C f(z)\,dz$

and defined as –

$$\int_C f(z)dz = \int_C (u+iv)d(x+iy)$$

$$= \int_C (udx - vdy) - i\int_C (vdx + udy).$$

Line integral / contour integral depends upon the path.

PROPERTIES

(1) If the contour (curve) is divided into two paths $C_1$ and $C_2$, then:

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$

(2) If the sense of integration is reversed, then the sign of integration changes.

$$\int_{z_0}^{z} f(z)dz = -\int_{z}^{z_0} f(z)dz.$$

where; $z_0$ and $z$ are the two ends of the contour $C$.

(3) $$\int_C (f_1(z) + f_2(z))dz = \int_C f_1(z)dz + \int_C f_2(z)dz.$$

(4) $$\int_C K f(z)dz = K\int_C f(z)dz.$$

**REMARK**

Since, $|z-z_0|$ is the distance b/w the points $z$ and $z_0$, the eq$^n$ $|z-z_0| = a$ gives the locus of all points at a distance 'a' from $z_0$.

Hence, $|z-z_0| = a$ represents a CIRCLE of RADIUS 'a' and centre at $z_0$.

Similarly, $|z-z_0| < a$ represents the INTERIOR of circle C which holds for every point inside C.

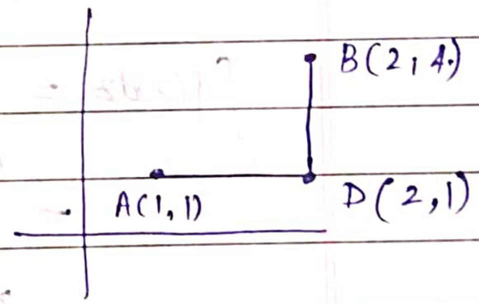
Que (C.P.) Integrate $f(z) = z^2$ from $A(1,1)$ to $B(2,4)$ along

① the line segment parallel to x-axis and DB parallel to y-axis.

② the straight line AB joining the two points.

③ the curve C, $x = t$, $y = t^2$.

Sol$^n$ ① $\int_C f(z)dz = \int_C z^2 \cdot dz$




$= \int_{AD} z^2 dz + \int_{DB} z^2 dz$

$= \int_{AD} (x+iy)^2 \, d(x+iy) + \int_{DB} (x+iy)^2 \, d(x+iy)$

$= \int_{x=1}^{x=2} (x+i \cdot 1)^2 dx + \int_{y=1}^{y=4} (2+iy)^2 \cdot i\,dy$

$= \left[\frac{(x+i)^3}{3}\right]_1^2 + \left[\frac{(2+iy)^3}{3}\right]_1^4 \frac{\cancel{i}}{\cancel{i}}$

$= \frac{\cancel{(2+i)^3} - (1+i)^3}{3} + \frac{(2+i4)^3 - \cancel{(2+i)^3}}{3}$

$= \frac{\cancel{8+i^3+12i+6i^2} - 1 - i^3 - 3i - 3i^2 + 8 + 64i^3 + 48i + 96i^2 - \cancel{8 - i^3 - 12i - 6i^2}}{3}$

$$= \frac{63i^3 + 93i^2 + 45i + 7}{3} = \frac{-63i - 93 + 45i + 7}{3}$$

$$= \frac{-86}{3} - \frac{18i}{3} = \boxed{\frac{-86}{3} - 6i}$$ Answer.

② $\int_C f(z)dz = \int_C z^2 . dz$

$\because y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} . (x - x_1)$

The eq$^n$ of the line AB;

$$y - 1 = \frac{4-1}{2-1}(x-1)$$

$$\boxed{y = 3x - 2}$$

$$\int_C f(z)dz = \int_C (x+iy)^2 d(x+iy)$$

$$= \int_{x=1}^{2} [x + i(3x-2)]^2 . d[x + i(3x-2)].$$

$$= \left[\frac{[x + i(3x-2)]^3}{3}\right]_1^2$$

$$= \frac{(2+4i)^3 - (1+i)^3}{3}$$

$$= \frac{8 + 64i^3 + 48i + 96i^2 - 1 - i^3 - 3i - 3i^2}{3}$$

$$= \frac{63i^3 + 93i^2 + 45i + 7}{3}$$

$$= \frac{-63i - 93 + 45i + 7}{3} = \frac{-18i - 86}{3}$$

$$= \boxed{\frac{-86}{3} - 6i}$$ Ans

(3) Along, $x = t$, $y = t^2$

$$\int_C f(z)\,dz = \int_C z^2\,dz$$

Put; $z = x(t) + iy(t)$

$$\Rightarrow \int_C [x(t) + iy(t)]^2\, d[x(t) + iy(t)]$$

$$\Rightarrow \begin{matrix} 1 \le t \le 2 \\ 1 \le t^2 \le 4 \end{matrix} > 1 \le t \le 2$$

$$\int_1^2 [(t + it^2)]^2\, d(t + it^2)$$

$$\left[\frac{(t+it^2)^3}{3}\right]_1^2 \Rightarrow \frac{(2+4i)^3 - (1+i)^3}{3}$$

$$\Rightarrow \boxed{-\frac{86}{3} - 6i}$$ Answer

CONCLUSION :

$\boxed{\text{NOTE}}$ : In all the three cases, the integral is same
" If $f(z)$ is analytic then integral does not depend upon the contour."

Question Evaluate : $\displaystyle\int_{1-i}^{2+i} (2x + 2iy + 3)\,dz$

(1) Along the path $x = t + 1$, $y = 2t^2 - 1$
(2) Along the straight line, joining $1-i$ to $2+i$.

Soln (1)

$$t = x - 1 \qquad t = \sqrt{\frac{y+1}{2}} \qquad -1 \le y \le 1$$

$$1 \le x \le 2$$

$$0 \le t \le 1 \qquad 0 \le t \le 1$$

~~→ Along $x = t$~~,

$$\int_{1-i}^{2+i} 2(x+iy) + 3d(x+iy)$$

Put $x = t+1$ , $y = 2t^2-1$

$$\Rightarrow \int_0^1 2[t+1+i(2t^2-1)]\,d[t+1+i(2t^2-1)] + 3\int_0^1 d[t+1+i(2t^2-1)]$$

$$\Rightarrow 2\left\{\frac{[t+1+i(2t^2-1)]^2}{2}\right\}_0^1 + 3\left[t+1+i(2t^2-1)\right]_0^1$$

$$\Rightarrow [(2+i(7)^2 - (1-i)^2] + 3[2+i-1+i]$$

$6+12i$

② Eq$^n$ of straight line: $(1-i) \rightarrow (2+i)$

$(1,-1) \rightarrow (2,1)$

$$y - y_1 = \frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

$$y-1 = \frac{1+1}{2-1}(x-2)$$

$$y-1 = +2(x-2)$$

$$y-1 = +2x-4$$

$$\boxed{y = 2x-3}$$

$$\int_{1-i}^{2+i} 2(x+iy) + 3\,d(x+iy)$$

Put $y = 2x-3$

$$\Rightarrow \int_{1-i}^{2+i} 2[x+i(2x-3)] + 3\,d[x+i(2x-3)].$$

$$\Rightarrow \int_{1}$$

$$\Rightarrow \int_C f(z)dz = \int_C (2x+2iy+3)dz = \int_C (2x+2iy+3)\,d(x+iy).$$

$$= \int_1^2 [2x+2i(2x-3)+3)\,d(x+i(2x-3))$$

$$= (1+2i)\left[(2+4i)\frac{x^2}{2} - 6ix + 3x\right]_1^2$$

$$= (1+2i)\left[(2+4i)\times\frac{4}{2} - 12i + 6 - \frac{(2+4i)}{2} + 6i - 3\right]$$

$$= (1+2i)[(1+2i)3 - 6i + 3]$$

$$= (1+2i)[3+6i-6i+3) = 6(1+2i)$$

$$= \underline{6+12i}.$$

① $1 \le x \le 2 \qquad 1 \le t+1 \le 2 \Rightarrow 0 \le t \le 1$

$-1 \le y \le +1 \quad -1 \le 2t^2-1 \le +1 \Rightarrow 0 \le t \le 1 \Bigg\} \Rightarrow 0 \le t \le 1$

$$\int_C f(z)dz = \int_{1-i}^{2+i} [2x(t) + 2iy(t) + 3]\,d[x(t)+iy(t)]$$

$$\Rightarrow \int_{1-i}^{2+i} [2(t+1)+2i(2t^2-1)+3]\,d[(t+1)+i(2t^2-1)]$$

$$= \int_{1-i}^{2+i} [2t + 4it^2 + 5 - 2i](dt + 0 + i4t\,dt - 0)$$

$$= \int_0^1 (2t + 8it^2 + 4it^2 - 16t^3 + 5 + 20it - 2i + 8t)\,dt$$

$$= \left[\frac{2t^2}{2} + \frac{8it^3}{3} + \frac{4it^3}{3} - \frac{16t^4}{4} + 5t + \frac{20it^2}{2} - 2it + \frac{8t^2}{2}\right]_0^1$$

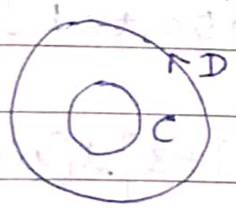
$$= 1 + \frac{8i}{3} + \frac{4i}{3} - \frac{16}{4} + 5 + \frac{20i}{2} - 2i + \frac{8}{2}$$

$$= \underline{\underline{12i + 6}}.$$

# SIMPLY CONNECTED DOMAIN

A domain of region is said to be simply connected if every closed curve inside d closes only points of d or any simple closed curve which lies inside d can be shrunk to a point inside d without learning d.

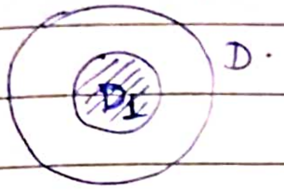
For example; A plane sheet of paper with no holes, in it is a simply connected domain. If any closed curve C shrunk then it shrinks to a point lying inside d.




# MULTIPLE CONNECTED DOMAIN

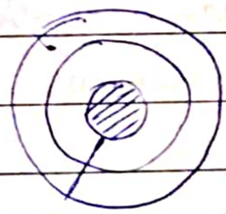
A domain which is not simply connected is known as multiple connected domain.

Example: A plane sheet of paper from which some interior parts are removed is a multiple connected domain. Let $D_1$ be removed from $D$. If a closed curve $C_1$ is shrunk, then it cannot be shrunked to a point as $D_1$ is not a part of $D$.




NOTE: If it has one hole → doubly connected
If it has two holes → triply connected.

① Any multiple connected domain can be converted to a simply connected domain by introducing sufficient numbers of cuts in the domain.



# CAUCHY's INTEGRAL THEOREM

Let $f(z)$ be analytic and $f'(z)$ be continuous in a simply connected domain $D$, then –

$$\boxed{\oint_C f(z)\,dz = 0}$$

along every simple, smooth (piecewise too), closed curve $C$, contained in $D$.

The condition that $D$ is simply connected is necessary condition.

$$\oint \frac{dz}{z-z_0} = 2\pi i \text{ (for } z_0 \text{ inside } C)$$



The theorem cannot be applied for multiple connected domain. However, the condition that $f(z)$ is analytic and $f'(z)$ is continuous in D are sufficient.

Example: $\oint_C \frac{dz}{(z-z_0)^n} = 0$, $n \neq 1$ for $C: |z-z_0| = r$

then; $f(z) = \frac{1}{(z-z_0)^n}$

Here; $f(z)$ is not differentiable at $z = z_0$ and hence not analytic at $z = z_0$

NOTE: The requirement of the condition that $f'(z)$ is continuous in D in a Cauchy's Integral Theorem can be relaxed. Then we have another theorem known as Cauchy Goursat Theorem.

All que

Que Evaluate $\int \frac{dz}{z-z_0}$, where C is any simple closed curve and $z_0$ is:-

(a) Outside C (b) Inside C

Solution Let $|z-z_0| = r$

(b) $z - z_0 = re^{i\theta}$

$z = z_0 + re^{i\theta}$

$dz = ire^{i\theta}$

$$\int_0^{2\pi} \frac{rie^{i\theta}\, d\theta}{re^{i\theta}} = 2\pi i$$

(c) $e^{(1-n)2\pi i} = 1$.

(b) When $z_0$ is inside the integral will be $2\pi i$ and when $z_0$ is outside the integral will be 0.

Que. $\oint_C \frac{dz}{(z-z_0)^n}$, once the simple closed curve C, where n is an integer $\geq 2$ and $z_0$ is inside C.

Soln $|z-z_0|^n = r$

$|z-z_0| = r^{1/n}$

$z - z_0 = r^{1/n} \cdot e^{i\theta}$

$z = z_0 + r^{1/n} \cdot e^{i\theta}$

$dz = i r^{1/n} e^{i\theta}$

$$\int_0^{2\pi} \frac{i r^{1/n} e^{i\theta} d\theta}{r \cdot e^{i\theta n}} = r^{\frac{1}{n}-1} i \int_0^{2\pi} e^{(1-n)i\theta} d\theta .$$

$$= \frac{r^{\frac{1}{n}-1} i}{(1-n)i} \left[ e^{(1-n)i\theta} \right]_0^{2\pi} = \frac{r^{\frac{1}{n}-1}}{(1-n)} [\underbrace{e^{(1-n)2\pi i}}_{1} - \underbrace{e^0}_{1}]$$

$= \boxed{0}$.

Que $\int_C e^{\sin z^2} dz$, $C: |z| = 1$

Soln $f(z) = e^{\sin z^2}$ (Analytic function for $e^z$)

$f'(z)$ is continuous

Therefore; $\int_C e^{\sin z^2} dz = 0$ {By Cauchy Integral Theorem}.

Que $\oint_C \tan z \, dz$, $C: |z| = 1$

Soln $f(z) = \tan z$ (not analytic)

$f(z) = \cot z$. (not continuous).

Therefore;

$f(z) = \frac{\sin z}{\cos z}$ when $\cos z = 0$ for $\frac{n\pi}{2}$

Put $n = 1, 3, 5, \ldots \left( \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots \right)$

Therefore, it lies outside the simple curve

Hence;

$$\oint_C \tan z \, dz = 0.$$

Que

$$\oint_C \frac{3z+5 \cdot dz}{z^2+2z} \quad ; \; C: |z| = 1.$$

Soln

$$f(z) = \frac{3z+5}{z^2+2z}$$

$$\cancel{f'(z) = \frac{(z^2+2z)(3) - (3z+5)(2z+1)}{(z^2+2z)^2}}$$

$$\cancel{= \frac{3z^2+6z-6z^2-3z-10z-5}{(z^2+2z)^2}}$$

$$\cancel{= \frac{-3z^2-7z-5}{(z^2+2z)^2}}$$

$$f(z) = \frac{3z+5}{z^2+2z} = \frac{3z+5}{z(z+2)} = \frac{A}{z} + \frac{B}{z+2}$$

$$= \frac{5}{2z} + \frac{1}{2(z+2)}$$

$$\oint_C \frac{3z+5}{z^2+2z} dz = \frac{5}{2} \oint_C \frac{1}{z} dz + \frac{1}{2} \oint_C \frac{1}{z+2} dz$$

$$= \frac{5}{2} \times 2\pi i + \frac{1}{2} \times 0 = \boxed{5\pi i}$$

**Formula** CAUCHY INTEGRAL FORMULA

Let $f(z)$ be analytic in a simple connected domain D.

Let $z_0$ be any point in D and C be any simple closed curve in D and closing the point $z = z_0$.

Then;

$$\boxed{f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)} dz}$$

where C is traversed in the anti-clockwise direction.

**Que** $\oint_C \frac{e^z}{z+1} dz$ ; C: $\left| z + \frac{1}{2} \right| = 1$

**Soln**

$f(z) = e^z$.

$z_0 = -1$.

$$\Rightarrow \oint_C \frac{e^z}{z - (-1)} dz = 2\pi i \cdot f(z_0).$$

$$= 2\pi i \cdot e^{-1} = \boxed{\frac{2\pi i}{e}}$$

* $\cos \pi = -1$
$\cos 2\pi = +1$

# CAUCHY INTEGRAL formula for DERIVATIVE

Let, $f(z)$ be analytic in a simple connected domain D. Let, $z_0$ be any point in D and C be any simple closed curve in D and closing the point $z = z_0$.
Let $f(z)$ has derivatives of all order in D which are also analytic in D and further –

$$\boxed{f^{n}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz}, \quad n = 1, 2, 3, 4, \ldots$$

Date: 12/08/2023.

→ Cauchy Integral Theorem :-

$$\oint_C f(z)\,dz = 0.$$

→ Cauchy Integral Formula :-

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz$$

→ Cauchy Integral Formula for derivative :-

$$f^{n}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz.$$

Question Evaluate the integral.

$$\int_C \frac{\cos \pi z^2}{(z-1)(z-2)} \cdot dz \quad ; \quad C : |z| = 3.$$

Solⁿ $\oint_C \cos \pi z^2 \left[\frac{1}{z-2} - \frac{1}{z-1}\right] dz$ (using partial fraction)

$$= \oint_C \left(\frac{\cos \pi z^2}{z-2} - \frac{\cos \pi z^2}{z-1}\right) dz$$

$$= \oint_C \frac{\cos \pi z^2}{z-2} dz - \oint_C \frac{\cos \pi z^2}{z-1} dz$$

$$= 2\pi i \cdot f(z_0) - 2\pi i \cdot f(z_0)$$

$$= 2\pi i \cdot [\cos \pi (2)^2] - 2\pi i \cdot [\cos \pi (1)^2]$$

$$= 2\pi i \cdot \cos 4\pi - 2\pi i \cos \pi$$

$$= 2\pi i + 2\pi i = \boxed{4\pi i}$$

Que $\oint_C \frac{e^z \, dz}{z+1}$, $c: \left|z+\frac{1}{2}\right| = 1$.

Solⁿ $2\pi i \cdot f(z_0)$

$\Rightarrow 2\pi i \cdot e^{-1} \Rightarrow \boxed{\frac{2\pi i}{e}}$

Que $\oint_C \frac{dz}{2-\bar{z}}$; $c: |z|=1$

Solⁿ $2-\bar{z} = 2 - \frac{z\bar{z}}{z} = 2 - \frac{|z|^2}{z} = 2 - \frac{1}{z} = \frac{2z-1}{z}$.

$$\frac{1}{2-\bar{z}} = \frac{z}{2z-1}$$

Then;

$$\therefore \oint_C \frac{z \, dz}{2z-1} = \frac{1}{2} \oint_C \frac{z \, dz}{z-\frac{1}{2}}$$

$\Rightarrow \left(\frac{1}{2}\right) 2\pi i \cdot f(z_0)$

$\Rightarrow \left(\frac{1}{2}\right) 2\pi i \cdot \frac{1}{2} = \boxed{\frac{\pi i}{2}}$

**Que** $\oint_C \frac{z^2+1}{z(2z-1)} dz$ ; $C : |z| = 1$

**Sol$^n$**

$$\frac{1}{z(2z-1)} = \frac{A}{z} + \frac{B}{2z-1}$$

$$1 = A(2z-1) + Bz.$$

$$A = -1$$

$$B = 2.$$

$$\frac{1}{z(2z-1)} = \frac{-1}{z} + \frac{2}{2z-1}.$$

$$\Rightarrow \oint_C (z^2+1)\left[\frac{2}{2z-1} - \frac{1}{z}\right] dz.$$

$$\Rightarrow \frac{2}{2}\oint_C \frac{z^2+1}{z-\frac{1}{2}} dz - \oint_C \frac{z^2+1}{z} dz.$$

$$\Rightarrow 2\pi i . f(z_0) - 2\pi i f(z_0)$$

$$\Rightarrow 2\pi i . \left(\frac{1}{4}+1\right) - 2\pi i (1)$$

$$\rightarrow 2\pi i . \frac{5}{4_2} - 2\pi i \Rightarrow \frac{5\pi i}{2} - 2\pi i$$

$$\rightarrow \frac{5\pi i - 4\pi i}{2} = \boxed{\frac{\pi i}{2}}$$

**Que** $\oint_C \frac{3z^2+z}{z^2-1} dz$ : $c : |z| = 2$

**Sol$^n$**

$$\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{A}{z-1} + \frac{B}{z+1}.$$

$$1 = A(z+1) + B(z-1)$$

$$A = \frac{1}{2}, \quad B = -\frac{1}{2}$$

$$\Rightarrow \oint_C 3z^2+z\left[\frac{1}{2(z-1)}-\frac{1}{2(z+1)}\right]dz.$$

$$\Rightarrow \frac{1}{2}\oint_C \frac{3z^2+z}{z-1}dz - \frac{1}{2}\oint_C \frac{3z^2+z}{z+1}dz.$$

$$\Rightarrow \frac{1}{2}\cdot 2\pi i\cdot f(z_0) - \frac{1}{2}\cdot 2\pi i\cdot f(z_0)$$

$$\Rightarrow \frac{1}{\cancel{2}}\cdot \cancel{2}\pi i\cdot 4 - \frac{1}{\cancel{2}}\cdot 2\pi i\cdot \cancel{2}. = \boxed{2\pi i}$$

Que: $\oint_C \frac{e^{-z}}{z^2}dz$, $c: |z|=1.$

Soln $\Rightarrow \oint_C \frac{e^{-z}}{(z-0)^2}dz$

$f(z) = e^{-z}$

$f'(z) = -e^{-z}$

$f'(0) = -1$

$$\Rightarrow \frac{2\pi i}{1!}\cdot f'(z_0)$$

$$\Rightarrow \frac{2\pi i}{1.}(-1) = \boxed{-2\pi i}$$

Que $\oint_C \frac{e^{2z}}{(z+1)^4}dz$, $c: |z|=2$

Soln $\frac{2\pi i}{3!}\cdot f^3(z_0)$

$f(z) = e^{2z}$

$f'(z) = 2e^{2z}$

~~$f'(-1) = 2e^{-2} = \frac{2}{e^2}$~~

$f^2(z) = 4e^{2z}$

$f^3(z) = 8e^{2z}$

$f^3(-1) = \frac{8}{e^2} = 8e^{-2}$

$$\Rightarrow \frac{\cancel{2}\pi i}{\cancel{6}\,3}\cdot\frac{8}{e^2}$$

$$= \frac{8\pi i}{3e^2} = \boxed{\frac{8\pi i e^{-2}}{3}}$$

Date / /
Page No 40

Date : 17/08/2023

Que $\oint_C \frac{dz}{(z^2+4)^2}$ ; $C : |z-i| = 2$

Soln $= \oint_C \frac{dz}{(z^2+4)^2}$

$\therefore \frac{1}{(z^2+4)^2} = \frac{1}{(z^2-4i^2)^2} = \frac{1}{(z-2i)^2(z+2i)^2}$

$z = \pm 2i$

$z = 2i$ $\quad (z = -2i) \rightarrow$ Analytic in domain

$\oint_C \frac{1/(z+2i)^2}{(z-2i)^2} dz \rightarrow f(z)$ (ANALYTIC) (inside C)

$= \frac{1}{(z+2i)^2}$

$$f^n(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz$$

$$\Rightarrow \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i \cdot f^n(z_0)}{n!}$$

$$\Rightarrow \oint_C \frac{1/(z+2i)^2}{(z-2i)^2} dz = \frac{2\pi i \cdot 2}{1! \cdot 64 i} = \boxed{\frac{\pi}{16}} \text{ Answer}$$

$\frac{\pi}{16}$ — ($\frac{2\pi i \cdot 2}{64 i}$ reduced: $\frac{\cancel{2}\pi}{\cancel{16}}$)

$f^n(z) = \frac{-2}{(z+2i)^3}$

$f'(+2i) = \frac{-2}{64 i^3}$

$= \frac{2}{64 i}$

Qu $\oint_C \frac{dz}{(z^2+4)}$ ; $C: |z| = 4$

Sol^n From previous question;

$$\oint_C \frac{1}{(z^2+4)} = \frac{1}{(z-2i)(z+2i)} = \frac{A}{(z-2i)} + \frac{B}{(z+2i)}$$

Using partial fraction –

$1 = A(z+2i) + B(z-2i)$

$A = \frac{1}{4i}$

$B = \frac{1}{-4i}$

$$\frac{1}{4i}\oint_C \frac{dz}{(z-2i)} - \frac{1}{4i}\oint \frac{dz}{(z+2i)}$$

$$\Rightarrow \frac{1}{4i}(2\pi i)\cdot \cancel{f(z_0)}^{1} - \frac{1}{4i}(2\pi i) \cancel{f(z_0)}^{1}.$$

$\Rightarrow \boxed{0}$ Ans

Done

Que Evaluate the integral :-

$$\oint_C \frac{3z+5}{z^2+2z}\,dz \quad ; \quad C : |z|=1$$

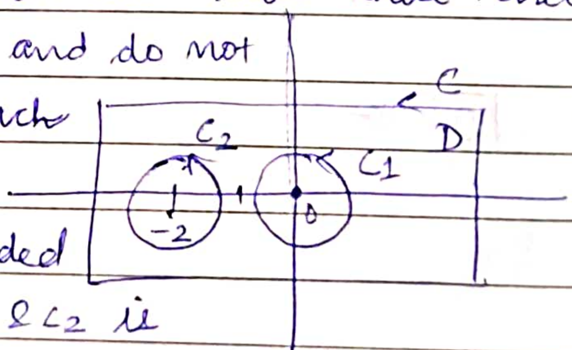
Question Using Cauchy Integral theorem & its extension, Evaluate :-

$$\oint_C \frac{dz}{z(z+2)}$$

where $C$ is any rectangle containing the points. $z=0$ & $z=-2$ inside it,

Solution The integrand $\frac{1}{z(z+2)}$ is not analytic at $z=0$ and $z=-2$.

Both the points lie inside the curve $C$.
Enclose the points $z=0$ and $z=-2$ by circles $C_1$ and $C_2$ of radii $r_1$ and $r_2$ respectively. such that both these circles lie inside $C$ and do not intersect with each other.




The domain D bounded by curve $C$, $C_1$ & $C_2$ is a multipli connected domain. Hence, by the extension of Cauchy integral Theorem, for multipli connected domain —

$$\oint_C \frac{dz}{z(z+2)} = \oint_{C_1} \frac{dz}{z(z+2)} + \oint_{C_2} \frac{dz}{z(z+2)}$$

$$= \frac{1}{2}\oint_{C_1}\left(\frac{1}{z}-\frac{1}{z+2}\right)dz + \frac{1}{2}\oint_{C_2}\left(\frac{1}{z}-\frac{1}{z+2}\right)dz$$

$$= \frac{1}{2}\left[\underbrace{\oint_{C_1}\frac{1}{z}dz}_{2\pi i} - \underbrace{\oint_{C_1}\frac{1}{z+2}dz}_{0} + \underbrace{\oint_{C_2}\frac{1}{z}dz}_{0} - \underbrace{\oint_{C_2}\frac{1}{z+2}dz}_{2\pi i}\right]$$

Out of domain of respective circles

$$= \boxed{0}$$ Answer