Fast Iterative Solvers
Project 2: Multigrid Solver

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W-Cycle

Table (1) presents the parameters used in the *Multigrid Solver* algorithm using W-cycles. In this case, the threshold of 10^{-10} is chosen as the convergence criteria for the relative residual. The parameter ν_1 for initial smoothing determines how many times the Gauss-Seidel function is applied to the solution, reducing high-frequency errors that contribute significantly to the total residual.

One of the interesting observations in this algorithm is the impact of the number of initial smoothing operations on the convergence process. It is found that as the number of smoothing operations increases, the number of iterations required to reach the specified convergence threshold decreases. In other words, more smoothing operations lead to faster convergence. This is demonstrated in Figure (1), where the plot shows the relative residual as a function of the iteration count. As the smoothing count rises, the convergence rate improves, bringing the solution closer to the desired accuracy level.

Furthermore, the impact of mesh refinement on the convergence process is also highlighted. When the mesh is refined, the nodal values are calculated based on nearby nodes, providing a smoother and more accurate approximation of the solution. In a fine mesh, the difference between the neighboring node's value and the node of interest (which affects the high-frequency errors) is reduced, resulting in a decrease in the residual at each iteration. This leads to a more efficient convergence process as compared to a coarse mesh, where the larger differences between nodes tend to hinder convergence and require more iterations to reach the desired solution accuracy.

Overall, the *Multigrid Solver* with *W-cycles*, as described by the parameters in Table (1), leverages initial smoothing and mesh refinement to achieve faster convergence and higher accuracy in solving the problem at hand. The interplay between these parameters plays a significant role in optimizing the performance of the solver.

Table 1: W-cycle input parameters

n	ν_1	ν_2	γ	m
4	1	1	2	13
4	2	1	2	11
7	1	1	2	12
7	2	1	2	9

V-Cycle

Table (2) presents the parameters used in the *Multigrid Solver* algorithm using V-cycles. In this case, the threshold of 10^{-10} is chosen as the convergence criteria for the relative residual.

Figure (2) displays the relative residual plotted against the iteration count.

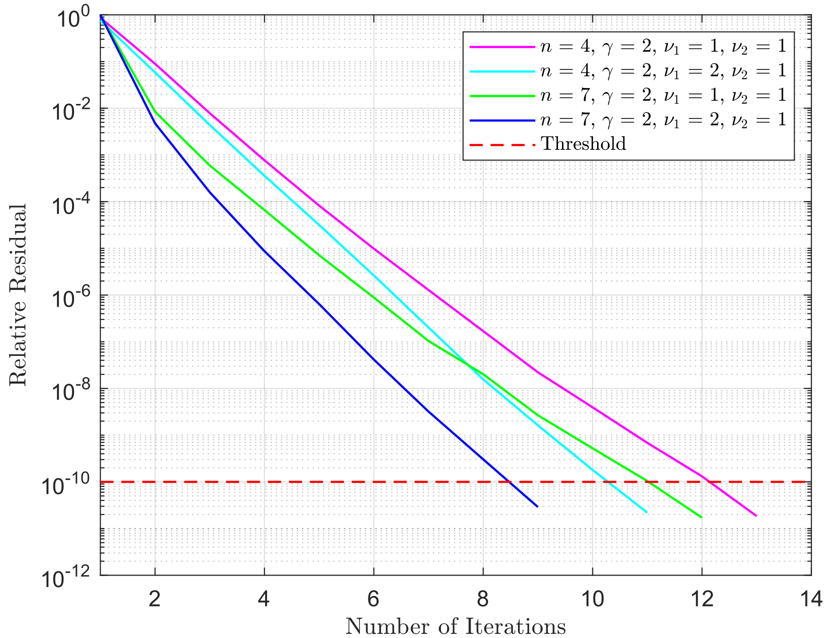

Figure 1: Relative residual vs. number of iterations for the W cycle

Same trend as W-cycle is observed where increasing the smoothing count results in a faster convergence rate. Smoothing refers to the iterative process of reducing errors in the solution by applying the Gauss-Seidel function multiple times. As the smoothing count is increased, high-frequency errors contributing significantly to the total residual are progressively reduced, leading to faster convergence.

Despite the similarity in the behavior of initial smoothing between the V-cycle and W-cycle, there is a notable difference in their convergence performance. It is observed that the V-cycle requires a higher number of iterations to achieve convergence compared to the W-cycle. This discrepancy can be attributed to the distinct approaches employed by the two cycles.

In the W-cycle, when solving the error equation at the coarse level, the process is repeated twice. This results in a more accurate and refined initial guess for the subsequent iteration, which subsequently reduces the overall number of iterations required to achieve convergence. The repetition of solving the error equation at the coarse level contributes to a better approximation of the solution and accelerates the convergence process.

On the other hand, the V-cycle follows a different approach, which leads to a higher number of iterations for convergence. While it also utilizes smoothing steps to reduce errors, it does not involve the double solving of the error equation at the coarse level like the W-cycle does. As a result, the initial guess for the next iteration may not be as refined, and it may take more iterations for the solution to converge.

In summary, the *Multigrid Solver* using *V-cycles*, as characterized by the parameters in Table (2), exhibits faster convergence with increasing smoothing count. However, it generally requires more iterations to achieve convergence compared to the W-cycle, primarily because the W-cycle benefits from the additional double solving the error equation at the coarse level. Understanding the differences and trade-offs between the V-cycle and W-cycle can aid in selecting the most appropriate solver strategy based on the specific problem and desired convergence efficiency.

Table 2: V-cycle input parameters

n	ν_1	ν_2	γ	m
4	1	1	1	14
4	2	1	1	12
7	1	1	1	16
7	2	1	1	14

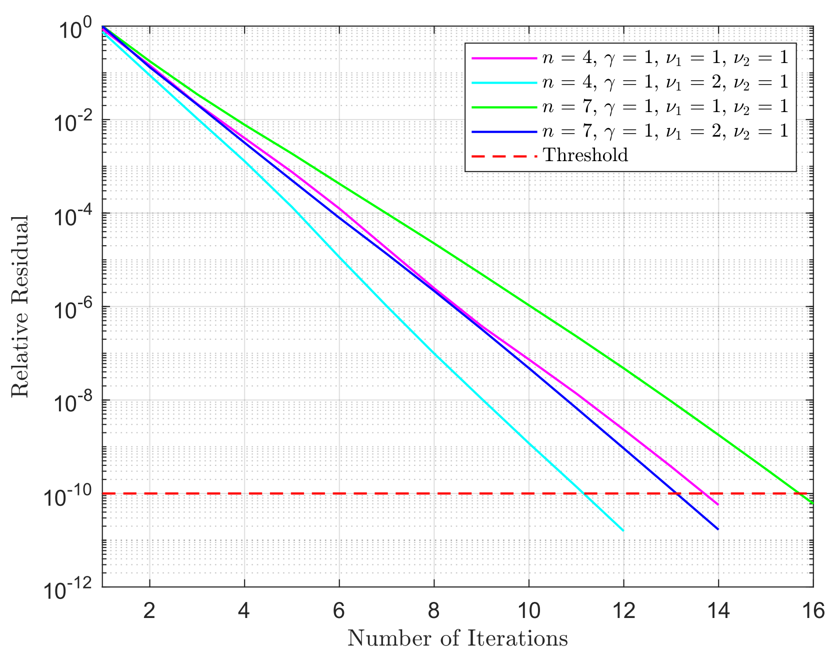

Figure 2: Relative residual vs. number of iterations for the V cycle

Runtime

Table (3) presents the parameters used in the *Multigrid Solver* algorithm for runtime comparison. In this case $n = 9$ is chosen to measure a more accurate runtime of the solver and an average of three evaluation is reported.

The relationship between the number of iterations required for convergence and the overall runtime of the solver involves a tradeoff. This tradeoff is demonstrated in Figure (3), where it becomes evident that increasing the number of smoothing steps beyond a certain point results in longer runtime despite the decrease in the number of iterations needed for convergence. The reason for this lies in the increased number of function evaluations required as the smoothing steps are increased, which contributes to the longer runtime.

Additionally, the W-Cycle consistently outperforms the V-Cycle in terms of runtime for all parameters. This observation holds true, reinforcing the finding that the W-Cycle not only requires fewer iterations but also exhibits shorter runtime when compared to the V-Cycle. As γ is set to higher, the number of time for which the error equation is solved at the coarse level increases, reducing the overall number of iterations needed to achieve convergence. However, the runtime increases in this case due to higher computational effort.

In summary, the tradeoff between the number of iterations and runtime in the solver becomes apparent when considering the impact of increasing the smoothing steps on both aspects. While increasing smoothing steps can lead to faster convergence in terms of iterations, it also comes with the cost of longer runtime due to increased function evaluations. The evaluation of both the W-Cycle and V-Cycle using the parameters from Table (3) and the subsequent comparison in Figure (3) confirms that the W-Cycle is more efficient in terms of both fewer iterations and shorter runtime, making it a preferable choice for solving the problem at hand.

Table 3: Runtime comparison input parameters

n	γ	ν_1	ν_2	m
9	1	1	1	17
9	1	2	1	14
9	1	3	1	13
9	1	4	1	11
9	2	1	1	11
9	2	2	1	8
9	2	3	1	8
9	2	4	1	7
9	3	1	1	10
9	3	2	1	8
9	3	3	1	7
9	3	4	1	7
9	4	1	1	9
9	4	2	1	7
9	4	3	1	7
9	4	4	1	7

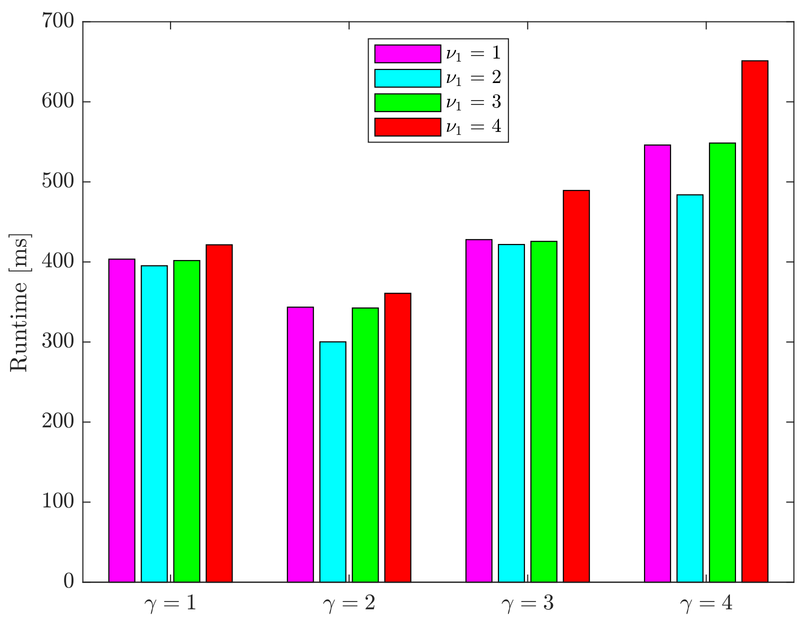

Figure 3: Runtime [ms] comparison for parameters in Table (3)

Numerical vs. Exact Solution

Figures (4), (5), and (6) present the exact solution, numerical solution, and the corresponding error for $n = 4$, $n = 7$, and $n = 10$, respectively. As shown, the error reduction is in the order of magnitude of 10^{-2} and 10^{-4} for $n = 7$ and $n = 10$ with respect to $n = 4$, respectively.

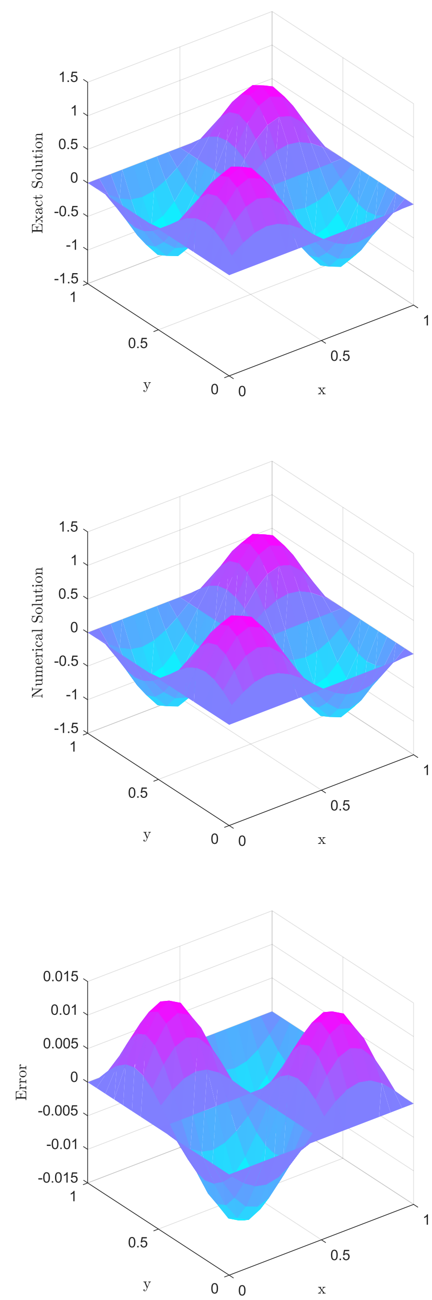

Figure 4: Exact solution, numerical solution, and error for $n = 4$

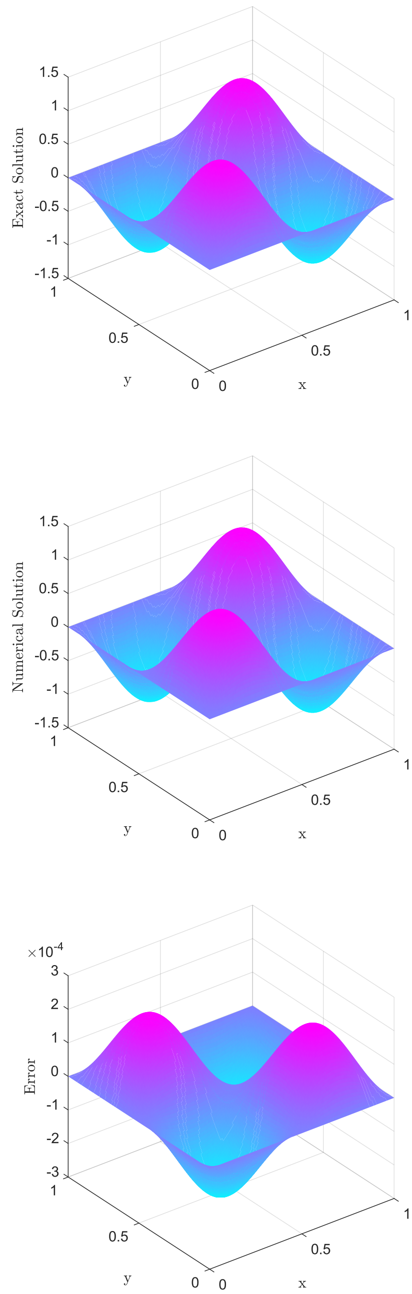

Figure 5: Exact solution, numerical solution, and error for $n = 7$

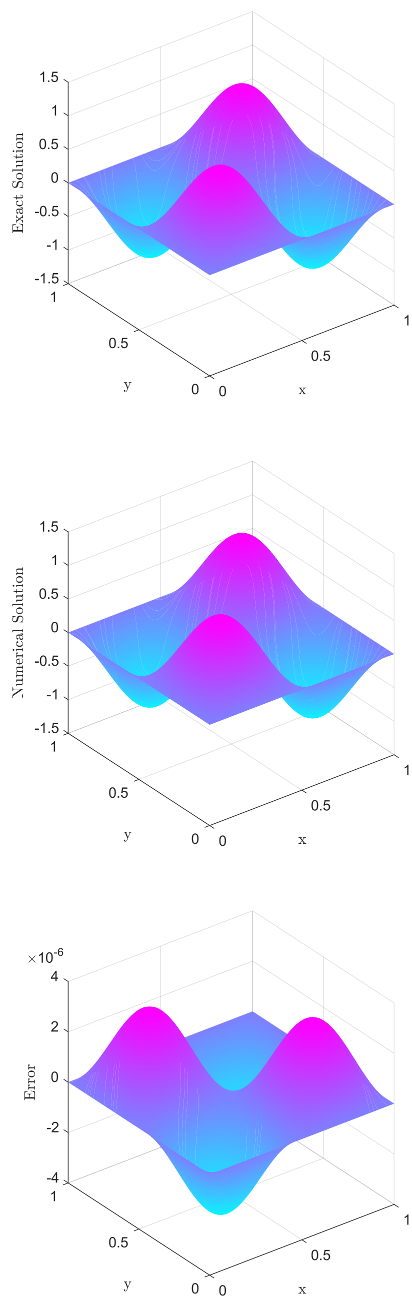

Figure 6: Exact solution, numerical solution, and error for $n = 10$