Fast Iterative Solvers
Project 3: Eigen Solvers

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Power Iteration

Figure (1) shows the results of implementing power iteration to find the largest eigenvalue of the symmetric matrix `power_test_msr`. In this case, the tolerance of the convergence is 10^{-8}. Final eigenvalue of 7650603.31390997 is obtained after 776 iterations to reach to the convergence criteria. As shown in Figure (1), $|\lambda_k - \lambda_{k-1}|$ oscillates in the first iterations and decreases linearly afterwards until convergence.

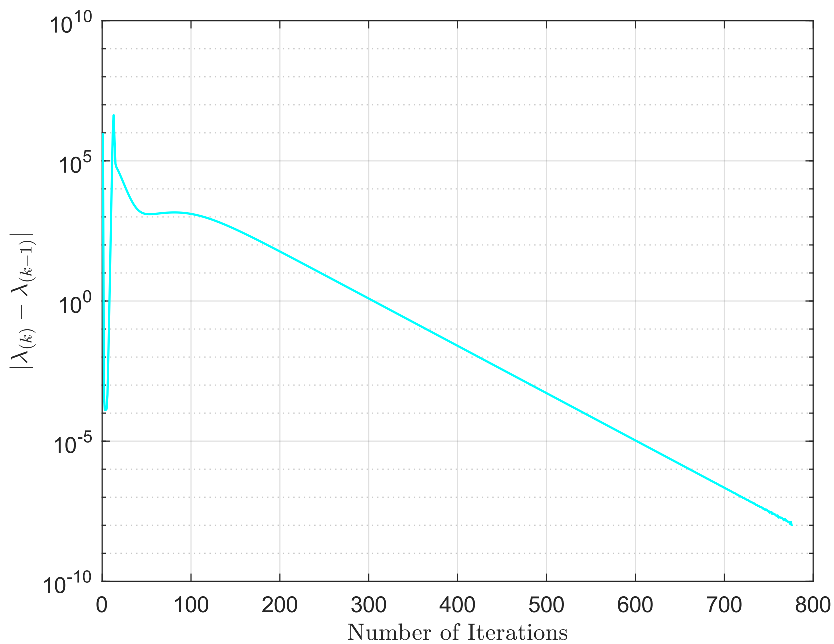

Figure 1: Power Iteration Convergence for `power_test_msr` (Tolerance = 10^{-8})

Figure (2) shows the results for the convergence tolerance of 10^{-10}. Final eigenvalue of 7650603.31391022 is obtained after 828 iterations to reach to the convergence criteria. As shown in Figure (2), $|\lambda_k - \lambda_{k-1}|$ oscillates in the first iterations and decreases linearly afterwards, but oscillates again close to convergence. It can be concluded that decreasing the convergence criteria increases the number of iterations and does not change the approximated value of the largest eigenvalue significantly.

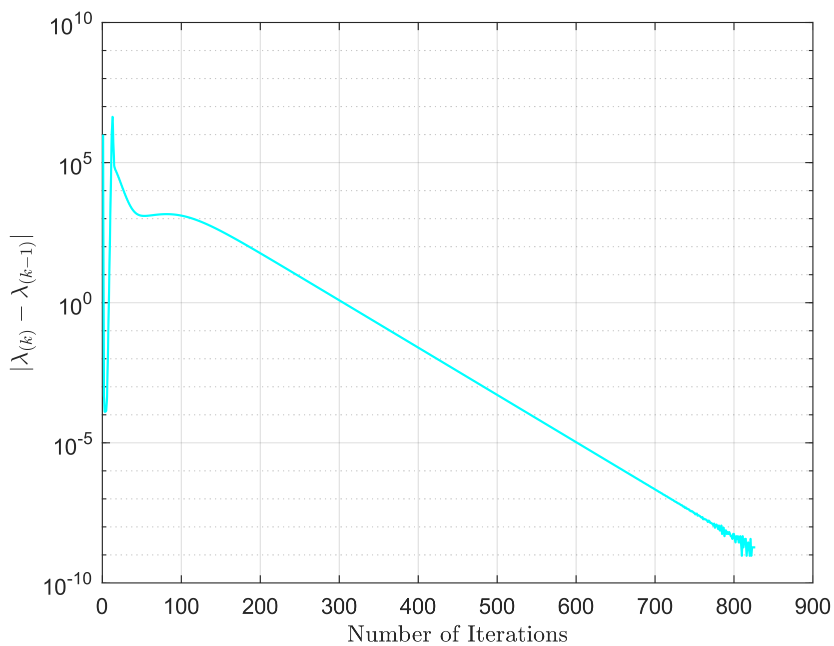

Figure 2: Power Iteration Convergence for `power_test_msr` (Tolerance $= 10^{-10}$)

Lanczos Algorithm vs. Power Iteration

Figure (3) shows the results of implementing power iteration to find the largest eigenvalue of the symmetric matrix `cg_test_msr`. In this case, the tolerance of the convergence is 10^{-8}. Final eigenvalue of 9598.60808796397 is obtained after 2146 iterations to reach to the convergence criteria. As shown in Figure (1), $|\lambda_k - \lambda_{k-1}|$ decreases until convergence.

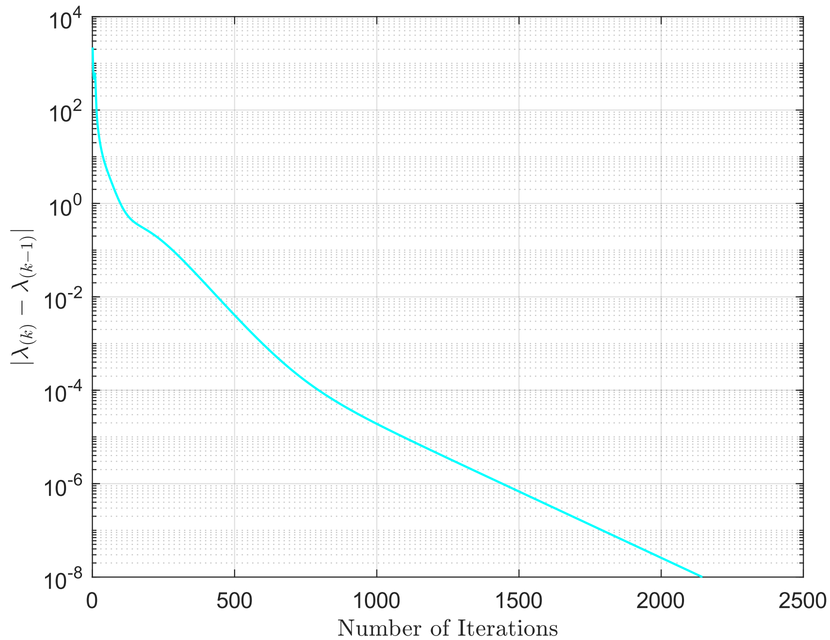

Figure 3: Power Iteration Convergence for `cg_test_msr`

The settings for running the Lanczos algorithm are shown in Table(1).

Table 1: Settings of Lanczos Algorithm

m	**Tolerance**
30	10^{-2}
50	10^{-4}
75	10^{-6}
100	10^{-10}

Figures (4), (5), (6), and (7) show the convergence of Lanczos algorithm for $m = 30$, $m = 50$, $m = 75$ and $m = 100$, respectively. In this case, $|\lambda_k - \lambda_{k-1}|$ decreases until convergence.

Tables (2) and (3) show the comparison of accuracy and runtime of Power Iteration and Lanczos algorithms for settings in Table (1), respectively. In this case, accuracy is defined as the difference between the obtained value from the algorithm and the largest eigenvalue for the symmetric matrix `cg_test_msr` equal to 9598.6080894852857. As shown, increasing the number of Krylov space (m) leads to a better approximation of the largest eigenvalue. However, increasing the number of Krylov space (m) increases the number of iterations and runtime. It is important to mention that although the Power Iteration converges in less iterations than Lanczos algorithm for $m = 100$, it takes a longer time to converge. It can be concluded that Lanczos algorithm is faster than the Power Iteration algorithm and gives a better approximation of the largest eigenvalue for higher number of Krylov space (m).

Table 2: Accuracy comaprison between Lanczos and Power Iteration Algorithms

Method	**Eigenvalue**	**Difference**
Lanczos Algorithm ($m = 30$)	9582.91974574371	15.6883
Lanczos Algorithm ($m = 50$)	9598.53122138019	0.0769
Lanczos Algorithm ($m = 75$)	9598.60793744962	1.5204e-04
Lanczos Algorithm ($m = 100$)	9598.60808947007	1.5216e-08
Power Iteration	9598.60808796397	1.5213e-06

Table 3: Runtime comaprison between Lanczos and Power Iteration Algorithms

Method	**Iterations**	**Time [s]**
Lanczos Algorithm ($m = 30$)	91	0.049068
Lanczos Algorithm ($m = 50$)	495	0.081996
Lanczos Algorithm ($m = 75$)	1798	0.137648
Lanczos Algorithm ($m = 100$)	3206	0.190452
Power Iteration	2146	5.877876

For a better comparison of the two algorithms, the tolerance for all the Krylov spaces is set to 10^{-8}. Tables (4) and (5) show the comparison of accuracy and runtime of Power Iteration and Lanczos algorithms with tolerance of 10^{-8}, respectively. As shown, increasing the tolerance increases the number of iterations and runtime for $m = 30$, $m = 50$, and $m = 75$. The accuracy is not improved significantly for $m = 30$ and $m = 50$, however, the accuracy has improved by 2 orders of magnitude for $m = 75$. It can be concluded that Lanczos algorithm is faster than the Power Iteration algorithm for the same tolerance, however, the accuracy does not improve significantly for higher number of Krylov space (m).

Table 4: Accuracy comaprison between Lanczos and Power Iteration Algorithms for Tolerance of 10^{-8}

Method	**Eigenvalue**	**Difference**
Lanczos Algorithm ($m = 30$)	9583.17869817638	15.4294
Lanczos Algorithm ($m = 50$)	9598.53676578095	0.0713
Lanczos Algorithm ($m = 75$)	9598.60808786054	1.6247e-06
Lanczos Algorithm ($m = 100$)	9598.60808796451	1.5208e-06
Power Iteration	9598.60808796397	1.5213e-06

Table 5: Runtime comaprison between Lanczos and Power Iteration Algorithms for Tolerance of 10^{-8}

Method	**Iterations**	**Time [s]**
Lanczos Algorithm ($m = 30$)	764	0.055799
Lanczos Algorithm ($m = 50$)	1032	0.088511
Lanczos Algorithm ($m = 75$)	2502	0.132511
Lanczos Algorithm ($m = 100$)	2502	0.181152
Power Iteration	2146	5.863293

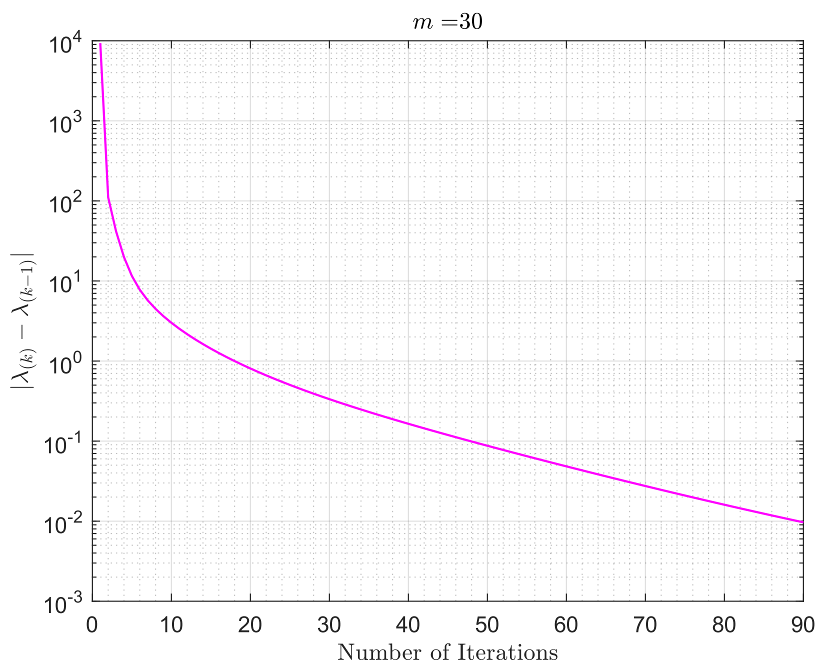

Figure 4: Lanczos Algorithm Convergence for $m = 30$

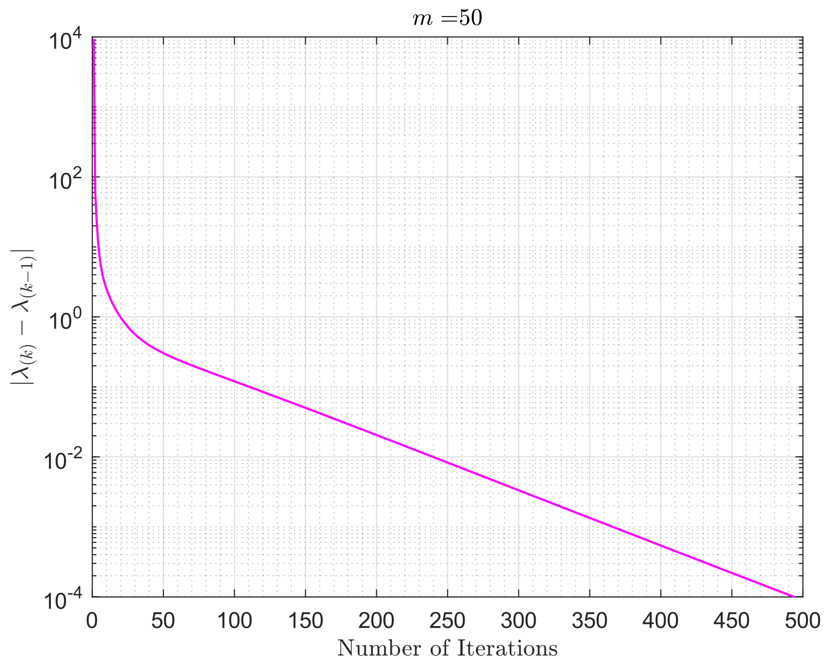

Figure 5: Lanczos Algorithm Convergence for $m = 50$

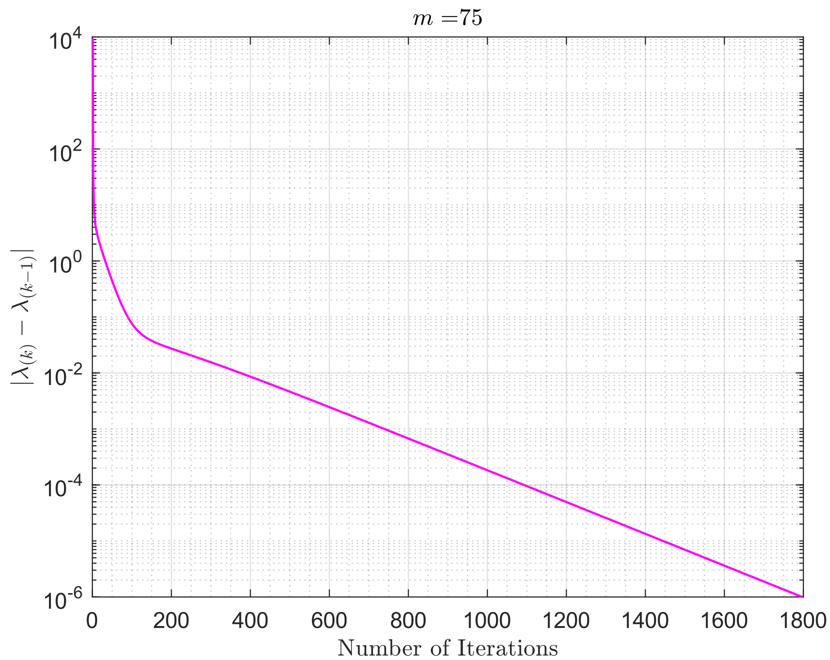

Figure 6: Lanczos Algorithm Convergence for $m = 75$

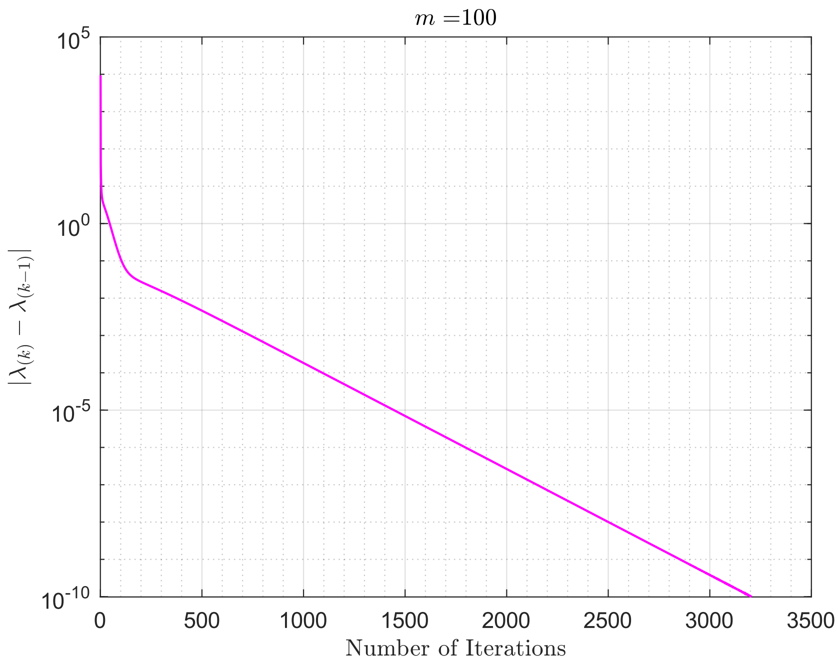

Figure 7: Lanczos Algorithm Convergence for $m = 100$