Imaginary parts of non-trivial Riemann zeroes as eigenvalues of 2×2 matrices with prime entries.

March 13, 2023

Samarjeet Singh Sidhu

Abstract

In this paper we calculate the eigenvalues of 2×2 matrices with prime entries and find that that many of these eigenvalues correspond with the imaginary parts of the non-trivial Riemann zeroes, in some cases the difference between the Zeroes and Eigenvalues being $\approx 8 * 10^{-6}$. We then heuristically study this correspondence between them. The paper is broadly divided into four sections. First section consists of the introduction and the motivation of the approach used. Second section consists of some definitions and basic proofs about the matrix series. The third section consists of heuristic data and computations which show the main results and section four consists of some approaches made for this project and observations related to them but were not discussed in detail in section three.

1 Introduction

The Riemann Zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s}$$

for complex values of s, where s can be represented by $\sigma + it$. While converging only for complex numbers s with $\Re(s) > 1$, this function can be analytically continued to the whole complex plane (with a single pole at s = 1) and can be represented by the functional equation [2]

$$\zeta(s) = 2^s \pi^{s-1} \sin(\pi s/2) \Gamma(1-s) \zeta(1-s) \tag{1}$$

The Riemann Zeta-function was first introduced by Euler and he proved the identity[5]

$$\sum_{n=1}^{N} \frac{1}{n^s} = \prod_{p\ prime} \frac{1}{1-p^{-s}}$$

Which relates prime numbers to the Zeta function, but it was Riemann who, in the 1850's, generalized its use and showed that the distribution of primes is related to the location of the zeros of Zeta function. Riemann conjectured that the non trivial zeros of $\zeta(s)$ are located on the critical line $\sigma = \frac{1}{2}$. This conjecture, known as the Riemann Hypothesis (RH), has never been proved or disproved, and is probably the most important unsolved problem in mathematics. One of the conjectures related to RH is the Hilbert-Polya Conjecture which states that the non-trivial zeros of the Riemann zeta function correspond to eigenvalues of a self-adjoint operator, and whose proof can lead to the proof of the RH. Research into this connection has intensified following the observation that the spacings of the zeroes of the Zeta function on the line $\sigma = \frac{1}{2}$ and the spacings of the eigenvalues of a Gaussian Unitary Ensemble(GUE) of Hermitian random matrices have the same distribution[5][3]. Thus if distribution of prime numbers is related to the Riemann zeroes which are in turn related to eigenvalues of an operator or matrices, one is inclined to think if it is possible to "combine" prime number and eigenvalues of matrices and relate them together to zeroes of $\zeta(s)$. Thus as one possible approach we define a sequence of matrices with prime entries and find their eigenvalues. We proceed as below

Definition 1.1 A sequence P is the sequence of all odd prime numbers $\Rightarrow$ $P = (3, 5, 7, 11, 13, 17, 19...)$ where P_i represents the i^{th} odd prime number and $i \in N$

Definition 1.2 A prime matrix M_i is defined as follow

$$M_i = \begin{bmatrix} P_i & P_{i+2} \\ P_{i+1} & P_{i+3} \end{bmatrix}$$

where $i \in N$ and P_i is the i^{th} prime number in P.

Definition 1.3 A sequence of prime matrices $\mathcal{M}$ is defined as $\mathcal{M} = (M_1, M_2, M_3, M_4, M_5, M_6, ...M_i...)$ where M_i represents the i^{th} prime matrix

2 Basic properties of $\mathcal{M}$

We will prove two basic results regarding $\mathcal{M}$, namely the existence of two distinct eigenvalues λ_a and λ_b for each matrix $M_i \in \mathcal{M}$ where $\lambda_a \neq \lambda_b \neq 0$. The other result we will prove is that the spectral radius of each matrix $M_i \in \mathcal{M}$ is unique and distinct $\Rightarrow \lambda_\alpha = \lambda_\beta$ iff $\alpha = \beta$ where λ_α and λ_β are the largest eigenvalues of M_α and M_β respectively.

Lemma 2.1 Every matrix $M_i \in \mathcal{M}$ is non-singular.

Proof Let us take an arbitrary matrix

$$K = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where $K \in \mathcal{M}$. $Det(K) = ad - bc$ and we observe that by **Definition 1.2** ad and bc are products of two distinct prime numbers and by Fundamental Theorem of Arithmetic $ad \neq bc$ and thus their difference is $\neq 0$. Hence Lemma 2.1 is proved.

Theorem-2.1 For every $M_i \in \mathcal{M}$ there exists two distinct, non-zero eigenvalues λ_a and $\lambda_b \Rightarrow \lambda_a \neq \lambda_b \neq 0$ where $a \neq b$.

Proof existence of two eigenvalues $\lambda_a, \lambda_b \neq 0$ is shown via **Lemma 2.1**. We again take the arbitrary matrix K. We know that eigenvalues of a 2×2 matrix can be represented by a quadratic equation in the form of

$$\lambda^2 - Tr(K)\lambda + \det(K) = 0$$

Discriminant of above quadratic equation is equal to $(a+d)^2 - 4ad + 4bc$. On rewriting this we get $(a-d)^2 + 4bc$ which is always positive and non-zero. Hence Theorem 2.1 is proved.

Corollary-2.1 The largest eigenvalue of any $M_i \in \mathcal{M}$ is always positive. This can be easily verified by using quadratic formula and noting that Trace and square root of discriminant are always positive and so is their sum.

Lemma-2.2 If we have two eigenvalues λ_a, λ_b of any matrix $M_i \in \mathcal{M}$ such that $\lambda_b > \lambda_a$ then $\lambda_b > |\lambda_a|$

$Proof:-$ This can be easily verified by taking the ratio

$$\frac{\lambda_b}{\lambda_a}$$

$$\frac{Tr(M_i) + \sqrt{Tr^2(M_i) - 4\left|M_i\right|}}{\left|Tr(M_i) - \sqrt{Tr^2(M_i) - 4\left|M_i\right|}\right|}$$

$$\frac{Tr(M_i) + Tr(M_i)\sqrt{1 - \frac{4|M_i|}{Tr^2(M_i)}}}{\left|Tr(M_i) - Tr(M_i)\sqrt{1 - \frac{4|M_i|}{Tr^2(M_i)}}\right|}$$

It is easily observed that whether $|M_i| > 0$ or $|M_i| < 0$ the ratio is always greater than 1 and hence $\lambda_b > \lambda_a$ and Lemma 2.2 is proved

We are only interested in the larger eigenvalue of the two eigenvalues thus we will disregard the smaller one and from now refer to λ_b as simply λ and λ_i would refer to the larger eigenvalue of the M_i unless stated otherwise.

Theorem-2.2 The spectral radius of all $M_i \in \mathcal{M}$ is distinct and unique $\Rightarrow$ $\rho(M_\alpha) = \rho(M_\beta) \Leftrightarrow \alpha = \beta$.

Proof

Take 2 arbitrary matrices $M_\alpha, M_\beta \in \mathcal{M}$ where $\alpha < \beta$ such that $\rho(M_\alpha) = \rho(M_\beta)$.

$$M_\alpha = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}, M_\beta = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix}$$

By Definition 1.2, for every matrix index x,y each element $\alpha_{x,y} < \beta_{x,y}$. If we square both these matrices then

$$M_\alpha^2 = \begin{bmatrix} \alpha_{11}^2 + \alpha_{12}\alpha_{21} & \alpha_{11}\alpha_{12} + \alpha_{12}\alpha_{22} \\ \alpha_{11}\alpha_{21} + \alpha_{21}\alpha_{22} & \alpha_{22}^2 + \alpha_{12}\alpha_{21} \end{bmatrix}$$

$$M_\beta^2 = \begin{bmatrix} \beta_{11}^2 + \beta_{12}\beta_{21} & \beta_{11}\beta_{12} + \beta_{12}\beta_{22} \\ \beta_{11}\beta_{21} + \beta_{21}\beta_{22} & \beta_{22}^2 + \beta_{12}\beta_{21} \end{bmatrix}$$

Similarly if we raise each both these matrices to any positive integer power k, for each index x,y the element of ${M_\beta}^k$ would still be larger than the corresponding element of ${M_\alpha}^k$.

The frobenius norm $\|.\|_F$on any matrix $N = \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix}$ is defined as $\|N\|_F = \sqrt{n_{11}^2 + n_{12}^2 + n_{21}^2 + n_{22}^2}$.$^{pg-115}$[6]

Thus if we take the frobenius norm of both matrices then

$$\left\|{M_\alpha}^k\right\|_F < \left\|{M_\beta}^k\right\|_F$$

taking the K^{th} root on both sides of the equation

$$\left\|{M_\alpha}^k\right\|_F^{1/k} < \left\|{M_\beta}^k\right\|_F^{1/k}$$

and taking the limit $k \to \infty$, from the Gelfand formula$^{Theorem 4.4}$[6] we get

$$\rho(M_\alpha) < \rho(M_\beta)$$

Which is a contradiction, hence Theorem 2.2 is proved.

Corollary-2.2 From Lemma 2.2 and Corollary 2.1 it can be easily observed that the largest eigenvalue of any $M_i \in \mathcal{M}$ is equal to the spectral radius of that matrix and from Theorem 2.2 it can be easily verified that all the elements in the sequence $\mathscr{E}_\lambda$ of the largest eigenvalues λ_i for all matrices $M_i \in \mathcal{M}$ are in ascending order $\Rightarrow \lambda_i < \lambda_j$ for $i < j$ where $\lambda_i, \lambda_j \in \mathscr{E}_\lambda$.

3 Finding eigenvalues and comparing them with zeta zeroes

Next we use computer code to calculate eigenvalues for our matrices and put them alongside the imaginary part of non-trivial zeroes of Riemann zeta function. For the purpose of these calculations, a pre-made table of the first $2 * 10^6$ imaginary parts of non-trivial Riemann zeroes was downloaded from [1].

We consider an eigenvalue as a Riemann zero if $|t - \lambda| < \epsilon$ for any arbitrarily small value of ϵ.We use these eigenvalues in the Riemann zeta function $\zeta(1/2 + i\lambda)$ and tabulate the values in Table 1 alongside $\zeta(1/2+it)$ where and t represent the eigenvalues and the imaginary part of the non-trivial zeroes of Riemann zeta function. From Table 1 one assumes that there is some type of correlation between the eigenvalues of these matrices and the imaginary part of non-trivial Riemann zeroes. And as we calculate further, this relationship does indeed seem to hold (Refer to table 2) but with certain caveats. The first and most obvious one is that not all the eigenvalues are Riemann zeroes but in fact they seem to occur after a certain gap in the set of eigenvalues and we will examine this

Table 1: Eigenvalues λ_i,$\zeta(\frac{1}{2}+\lambda_i t)$, corresponding imaginary part of non-trivial zeroes(t) and $\zeta(\frac{1}{2}+it)$

i	$\boldsymbol{\lambda_i}$	$\boldsymbol{\zeta(\frac{1}{2}+i\lambda_i)}$	t	$\boldsymbol{\zeta(\frac{1}{2}+it)}$
1	14.14142	-8.22e-4 +i5.25e-3	14.134 72	3.13e-6 -i1.96e-5
2	18.64365			
3	24.96148	0.024 -i0.062	25.010 85	3.41e-6 -i9.82e-6
4	30.39480	-0.019 +i0.034	30.424 87	-4.08e-6 -i6.86e-6
5	36.65475			
6	44.74856			
7	51.51414			
8	60.78960	0.059 -i0.032	60.831 77	1.2e-5 -i7.29e-6
9	69.39476	0.13 -i0.29	69.546 40	6.63e-10 -i2.47e-9
8	76.40812			
9	84.28474			
10	92.35416	-0.23 -i0.36	92.491 89	-1.91e-5 -i2.02e-5
11	101.54700	0.38 +i0.58	101.317 85	-8.29e-7 +i3.06e-6
13	110.35601			
14	120.39867			
15	129.21059	0.032 -i0.90	129.578 70	-4.9e-6 +i8.53e-6
16	136.23149			
17	145.24264			
18	153.17742	-0.29 +i0.60	153.024 69	1.03e-5 -i1.23e-5
19	162.36952			
20	174.41759	0.27 +i0.66	174.754 19	-6.08e-7 +i3.53e-6

further in the paper. The second is that even though these eigenvalues seem to be close to the Riemann zeroes, they are not strictly equal to them. And the third is that not every Riemann zero is encoded by these eigenvalues.

ϵ	Mean
0.29	0.51 + 3.27e-3
0.01	2.82e-3 -i6.49e-6
0.001	-3.20e-5 -i7.45e-5

Table 2: Mean values for few values of ϵ where Mean=$\frac{\sum_{n=1}^{N}\zeta(\frac{1}{2}+i\lambda_n)}{N}$ where N is the number of eigenvalues that are Riemann zeroes.

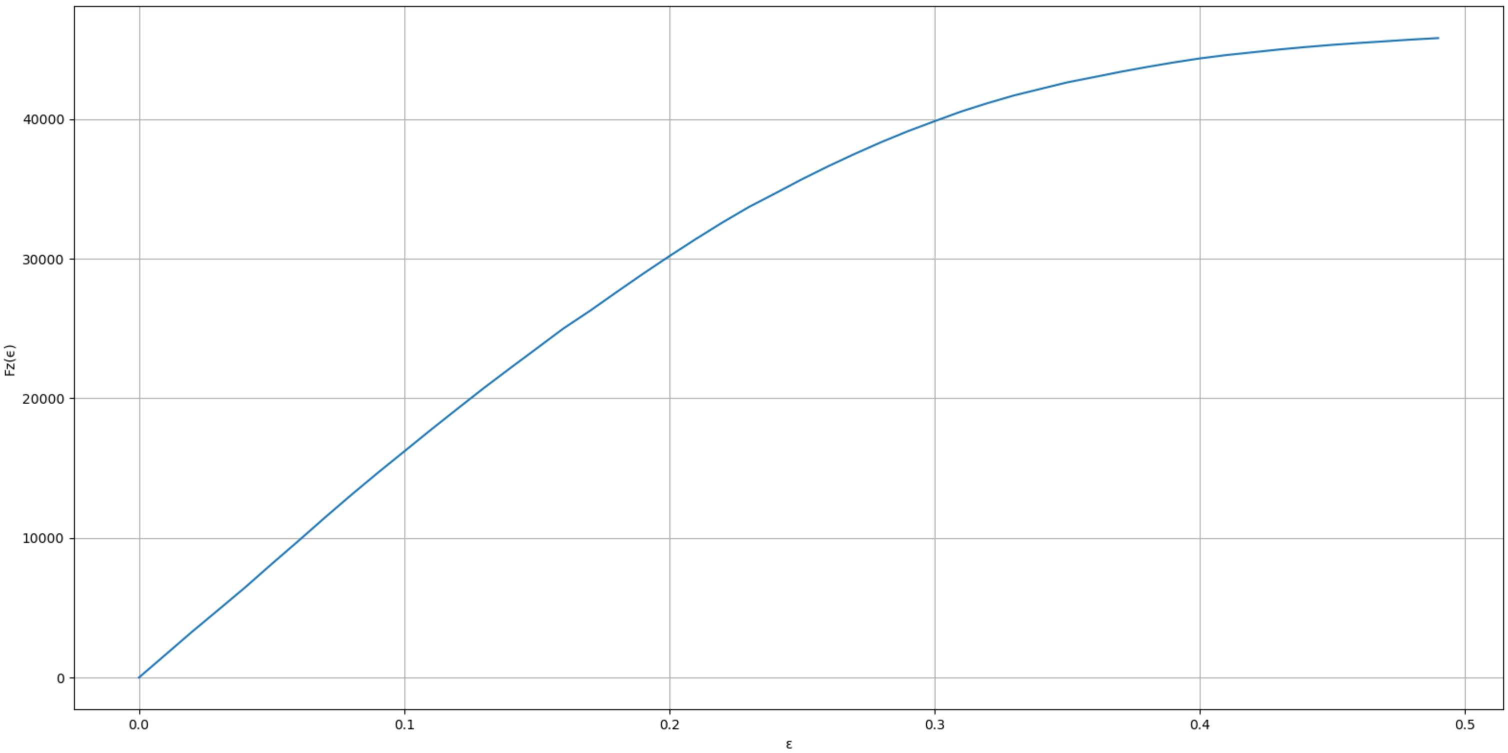


Figure 1: Number of eigenvalues that are considered as zeroes for the particular value of ϵ. Notice how the graph starts to plateau as we reach $\epsilon = 0.5$

3.1 Number of Riemann zeroes in sequence of eigenvalues as a function of ϵ.

The number of Riemann zeroes we get in these eigenvalues is a function $F_z(\epsilon)$ in terms of ϵ. We take different values of ϵ on the X-axis and the number of zeroes they give on Y-axis and plot its graph [Fig 1].

For the first nearly $2 * 10^6$ values of t we can find 2 Riemann zeroes in the sequence of eigenvalues for $\epsilon = 10^{-5}$, 19 Riemann zeroes for $\epsilon = 10^{-4}$, 170 Riemann zeroes for $\epsilon = 10^{-3}$, 1630 Riemann zeroes for $\epsilon = 10^{-2}$, 15994 Riemann zeroes for $\epsilon = 10^{-1}$, 30000 Riemann zeroes for $\epsilon = 0.2$ and 38660 Riemann zeroes for $\epsilon = 0.29$ (most of the initial calculations were done for $\epsilon = 0.29$ for no particular other than that it is approximately half of the Euler–Mascheroni constant)

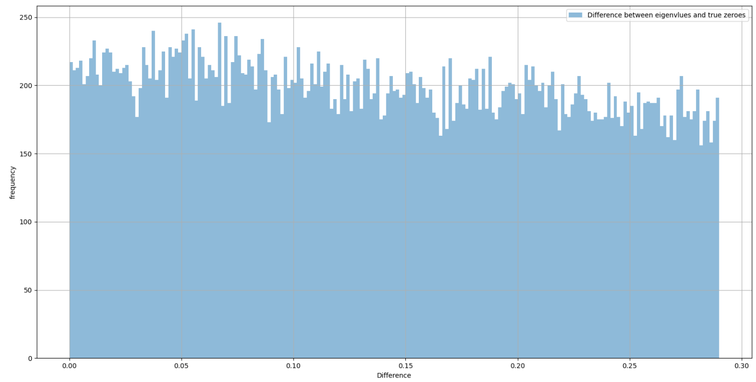


Figure 2: Difference between eigenvalues that are Riemann zeroes and true Riemann zeroes. Notice as we move away from zero the height of peaks gradually decreases

3.2 Difference between Eigenvalues and Riemann zeroes.

Next we will study the difference between true value of t and λ for this we use python to take two arrays which contain the imaginary part of the non-trivial zeroes and the the corresponding eigenvalues for different values of ϵ and plot there histograms [Fig 2].

From these histograms we observe that the difference between these eigenvalues and true value of the non trivial zeroes can range anywhere from arbitrarily close to zero (in the magnitude of 10^{-6}) to ϵ.

3.3 Eigenvalues that are Riemann zeroes and their relation with the trace of the matrix.

Upon close inspection we find that all the Riemann zeroes that occur in the sequence of eigenvalues are close to the trace of the matrix M_i. For example, the first Riemann zero occurring in the set of eigenvalues i.e- 14.14142843 belongs to the matrix $M_1 = \begin{bmatrix} 3 & 5 \\ 7 & 11 \end{bmatrix}$ whose trace is 14. Similarly the next eigenvalue that corresponds to a Riemann zero 25.01085758 belongs to the matrix $M_3 = \begin{bmatrix} 7 & 11 \\ 13 & 17 \end{bmatrix}$ whose trace is 24. To study the contribution of the trace to the actual value of Riemann zeroes we take the expression

$$Tr(M) + \delta_r = \frac{Tr(M)}{2}\left(1 + \sqrt{1 - \frac{4Det(M)}{Tr(M)^2}}\right)$$

where δ_r represents $|\lambda - Tr(M)|$ and on taking the trace from L.H.S to R.H.S get an expression for δ_r in terms of Trace and Determinant

$$\Rightarrow \delta_r = \frac{Tr(M)}{2}\left(\sqrt{1-\frac{4Det(M)}{Tr(M)^2}}-1\right)$$

Next we plot a histogram for δ_r[Fig 3].

We find (At least for the set of eigenvalues we have) that the probability of finding bigger δ_r decreases rapidly as we move away from zero, with δ_r mostly staying under 20 for the majority of time. We take this moment to also observe that the two matrices whose eigenvalues are extremely accurate (difference between eigenvalues and True value of Riemann zeroes being $8.86 * 10^{-6}$ and $8.78 * 10^{-6}$ and $\zeta(\frac{1}{2} + i\lambda)$ being -9.44e-5 + i1.06e-4 and 9.52e-5 -i2.78e-5 respectively) have $\delta_r = 3$ or more accurately δ_r is 3.00016 for the first matrix and 3.00011. These two matrices are $\begin{bmatrix} 423109 & 423121 \\ 423127 & 423133 \end{bmatrix}$ and $\begin{bmatrix} 550267 & 550279 \\ 550283 & 550289 \end{bmatrix}$. Another thing we can notice about these two matrices is that the primes in these matrices do not have have extremely large gaps and are relatively close to each other. Although the data is too small to make any conclusive and solid observations, it can act as a possible starting point in finding Riemann zeroes with extreme accuracy in the sequence of eigenvalues but this requires further analysis on a much larger set of Riemann zeroes.

Even near zero, the probability of finding δ_r oscillates between suddenly surges for a particular range of values and then decreasing rapidly for the next. This shows even if the trace of the matrix grown arbitrarily large as the eigenvalues tend to infinity, $\frac{\left(\sqrt{1-\frac{4Det(M)}{Tr(M)^2}}-1\right)}{2}$ grows arbitrarily small such that $\delta_r << Tr(M)$ and λ. The biggest value of δ_r that we encounter till n=38660 is around 56 for a single eigenvalue.

The Riemann-Siegel Z function Z(t) is related to the Riemann zeta function^{Pg-16}[7]

$$\zeta(\frac{1}{2}+it)e^{i\vartheta(t)} = Z(t) = 2\sum_{n=1}^{N}\frac{\cos[\vartheta(t)-t\log(n)]}{\sqrt{n}} + R_t$$

where $\vartheta(t)$ is the Riemann–Siegel theta function represented by[7]

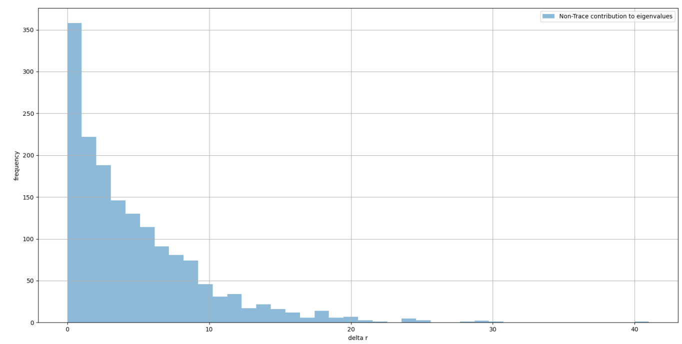
$$\arg\left(\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)\right)-\frac{\log\pi}{2}t$$

which via asymptotic expansion is

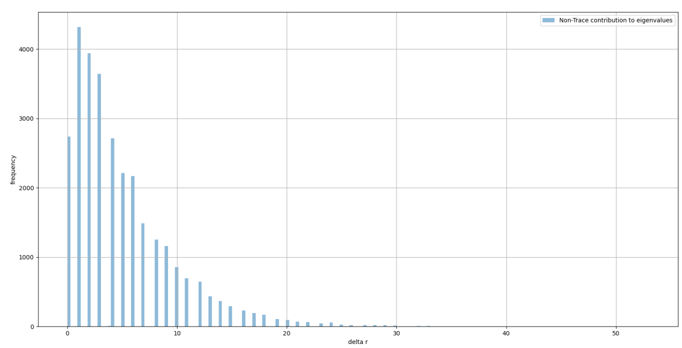
$$\vartheta(t) \sim \frac{t}{2}\log(\frac{t}{2\pi})-\frac{t}{2}-\frac{\pi}{8}+\frac{1}{48t}+\frac{7}{5760t^3}+...$$

and

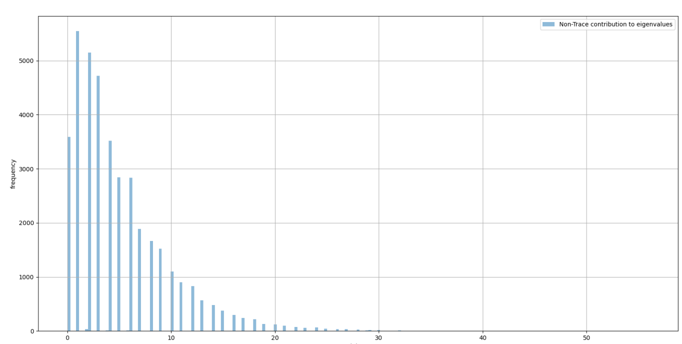
$$R_n \approx (-1)^{N-1}\left(\frac{t}{2\pi}\right)^{-1/4}\frac{\cos[2\pi(p^2-p-\frac{1}{16})]}{\cos(2\pi p)}$$



(a) $\epsilon = 0.01$



(b) $\epsilon = 0.2$



(c) $\epsilon = 0.29$

Figure 3: Notice the distinct bars that make up the histogram, suggesting only certain values of δ_r are allowed. The shape of histogram doesn't change much with ϵ as shown by figure 3(b) and 3(c). The occurrence of continuous histogram in 3(a) is because of comparatively small data set.

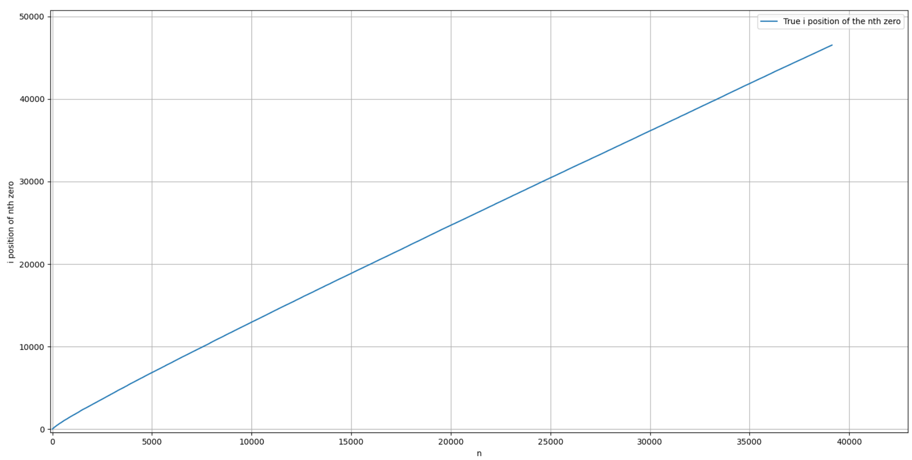


Figure 4: Position i in the sequence of eigenvalues of the n^{th} eigenvalue that is also a Riemann zero

Where $N = \left\lfloor \left(\frac{t}{2\pi}\right)^{1/2} \right\rfloor$ and $p = \left(\frac{t}{2\pi}\right)^{1/2} - N$ [7]. Finding Riemann zeroes via Z(t) turns into a problem of finding values of t for which $Z(t)$ changes sign and Pg-119,Pg-157 of [2] discusses this in detail. We know that $Tr(M) = P_i + P_{i+3}$ and for small enough values of δ_r ($\delta_r \approx 0$), it becomes a problem of finding $P_i + P_{i+3}$ or $i\log(i) + (i+3)\log(i+3)$ such that it follows the patterns mentioned in [2]. Although larger values of δ_r become rarer and rarer they still occur and thus for finding eigenvalues that are also Riemann zeroes and have a larger δ_r need a further understanding of the occurrence of different values of δ_r.

3.4 Position of eigenvalues that are Riemann zeroes in the sequence of all eigenvalues of M

Next we turn our attention towards the positioning of these zeroes within the eigenvalues. We first first plot a graph [Fig 4] with X-axis representing n^{th} eigenvalue that is also a Riemann zero, where $n \in N$ and Y-axis representing position i in the sequence of eigenvalues of the n^{th} Riemann zero occurring in the eigenvalues. For the purpose of this graph we have taken $\epsilon = 0.29$

We observe that the graph grows non-linearly and to find the approximate gaps after which a Riemann zero can occur in the eigenvalue series, we try to find an approximate equation for this graph. Without using any curve fitting tool, we can arrive at one of the possible candidates using some observations and minimal trial and error :

$$\log^2(n) + 2\sqrt{n}\log(n) + Li(n) + n - \sqrt{\epsilon.n\log(n)} \tag{2}$$

where Li(n) is the logarithmic integral. We plot the above equation alongside the error[Fig 5].

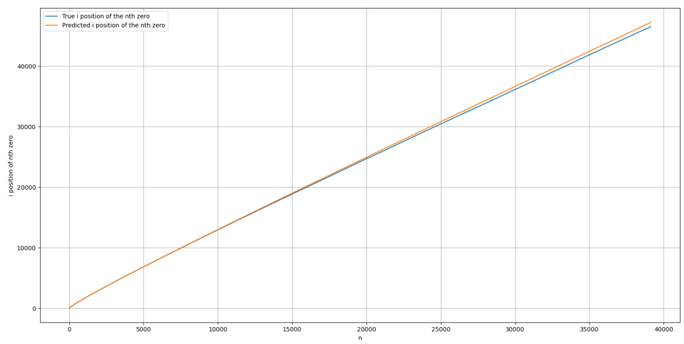
This approximate equation was built around the positions of first 2220 Riemann zeroes that occur in the eigenvalue series and was then tested on larger data, and it held extremely well till n=10^4. The absolute error is relatively small till n=10^4 oscillating from 0 to 50 till this point, and then it slowly starts to diverge with the absolute error reaching a maximum of 703 near n=39116.

A problem with this equation is that although is gives a very good approximation for $\epsilon = 0.29$, it fails for other values of ϵ. Although we can still find an approximate equation for different values of ϵ using equation 1 with appropriate modifications in terms of ϵ. For example, the equation for $\epsilon = 0.2$ is

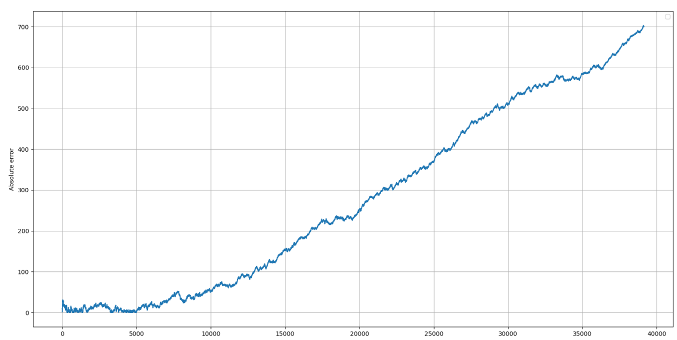
$$\frac{1}{\epsilon}(\log^2(n) + Li(N) - \sqrt{n\log(n)}) + 2\sqrt{n}\log(n) + n \tag{3}$$

Which is a slight modification of equation 2 can provide an a decent approximation for $\epsilon = 0.2$ and we plot its graph and error in Fig 6.

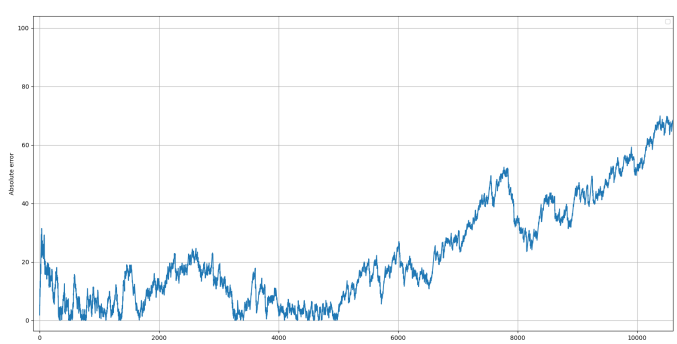
And similarly for equation 2 this equation was first first estimated for an array containing first 1620 Riemann zeroes that occur in the eigenvalue series and then tested till n=30180 and it holds well till n=12340 with absolute error coming around at a maximum of 120 in few places and otherwise much less than that. After n=12340 the series begins to diverge.



(a) The predicted line was constructed using equation 2

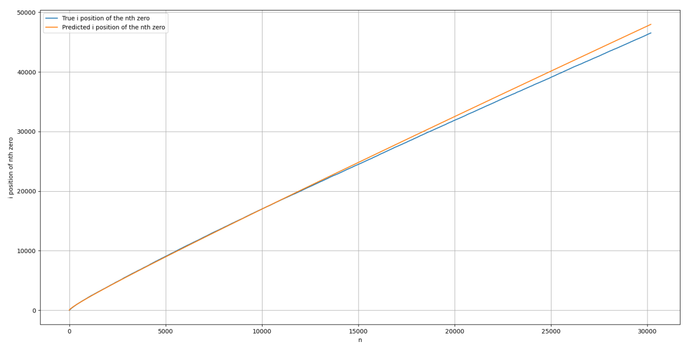


(b) error plot between actual and predicted values

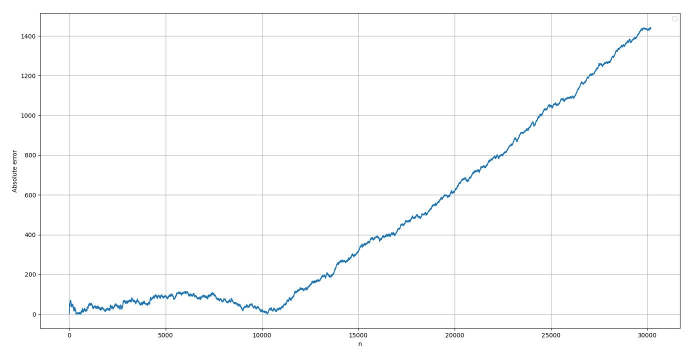


(c) error plot from n=0 till around n=10^4

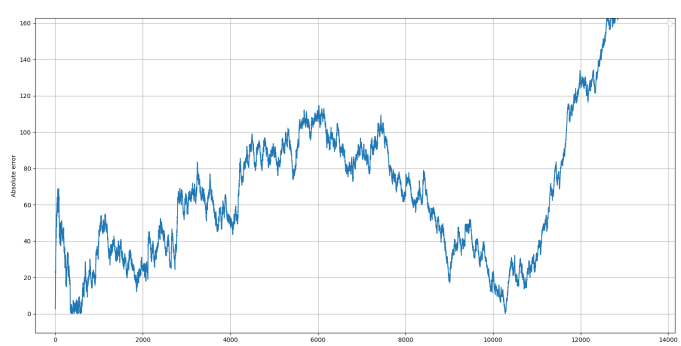
Figure 5: Plot of actual and predicted positions i in sequence of eigenvalues for the n^{th} eigenvalue that is a Riemann zero and the error in it for $\epsilon = 0.29$



(a) The predicted line was constructed using equation 3

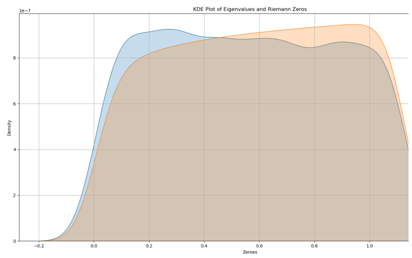


(b) error plot between actual and predicted values

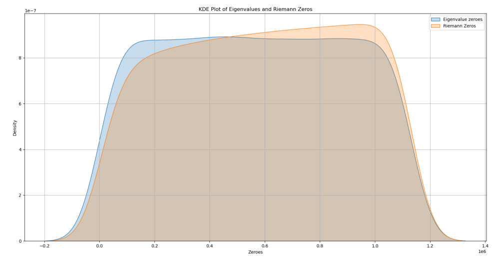


(c) error plot from n=0 till around n=1.210^4

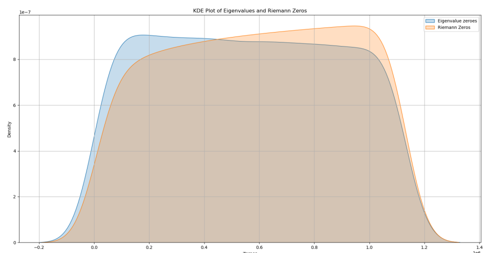
Figure 6: Plot of actual and predicted positions i in sequence of eigenvalues for the n^{th} eigenvalue that is a Riemann zero and the error in it for $\epsilon = 0.2$



(a) $\epsilon = 0.01$



(b) $\epsilon = 0.2$



(c) $\epsilon = 0.29$

Figure 7: KDE plots for different values of ϵ

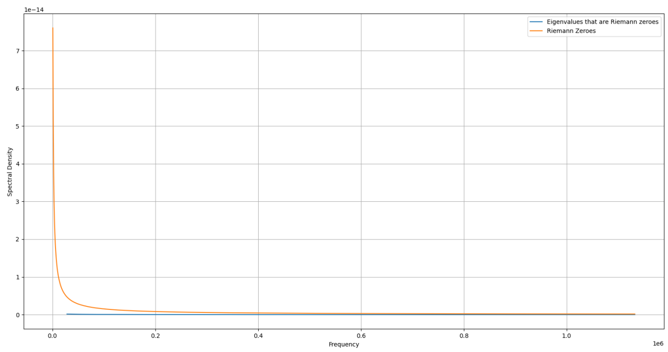
3.5 KDE graphs and Spectral Density Plots of Eigenvalues and True Riemann zeroes

Next we calculate the Kernel Density Estimate of the imaginary part of non-trivial zeroes and on our eigenvalues which correspond to Riemann zeroes [Fig 7].

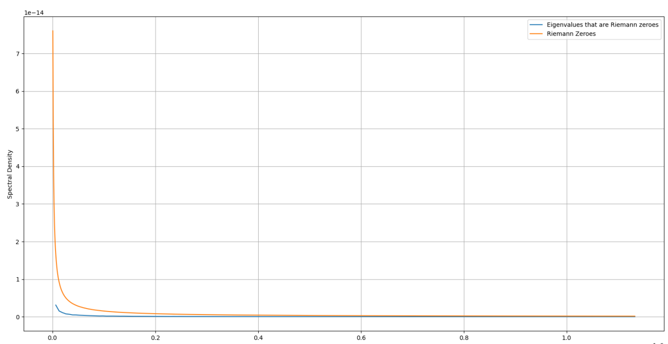
These different KDE graphs overlap significantly, and show an amount of correlation between our two data sets despite the difference in their sizes and missing values.

We also plot the Spectral Density of both t and λ [Fig 8] Both these sets have an inverse log graph as their spectral densities, and although both of these graphs do not coincide it does suggest some underlying similarities in their structure.

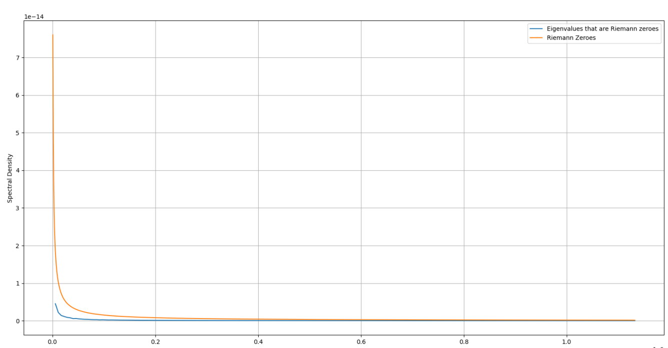
Note that for these graphs the maximum number of data points for eigenvalues that are also Riemann zeroes is around 30,000 while for true Riemann zeroes we have $2 * 10^6$ data points.



(a) $\epsilon = 0.01$



(b) $\epsilon = 0.2$

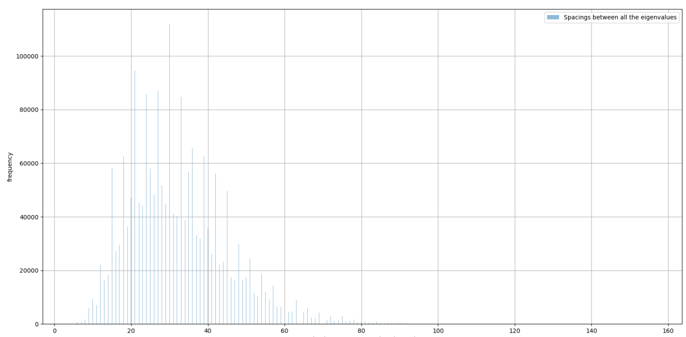


(c) $\epsilon = 0.29$

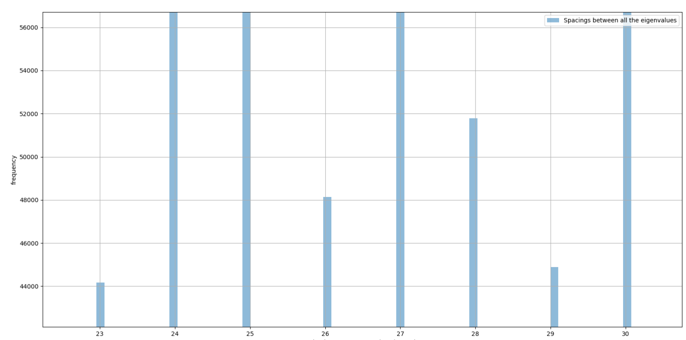
Figure 8: Spectral Density plots for different values of ϵ

3.6 Spacing between consecutive eigenvalues and also between consecutive eigenvalues that are also Riemann zeroes.

It makes sense to study the normalized gaps between our eigenvalues and compare them to that of t, but because not every zero is encoded by the eigenvalues a reasonable comparison cannot be done by the author on these gaps and we just plot a histogram of the difference between consecutive eigenvalues [Fig 9], non-normalized Riemann zeroes [Fig 10] and the difference between consecutive eigenvalues which are also Riemann zeroes for different values of ϵ[Fig 11].



(a) Histogram of eigenvalue spacings



(b) The histogram has distinct bars which have gaps in between suggesting that the difference between these eigenvalues have a recurring values within a certain range for each bar

Figure 9: Spacing between consecutive eigenvalues

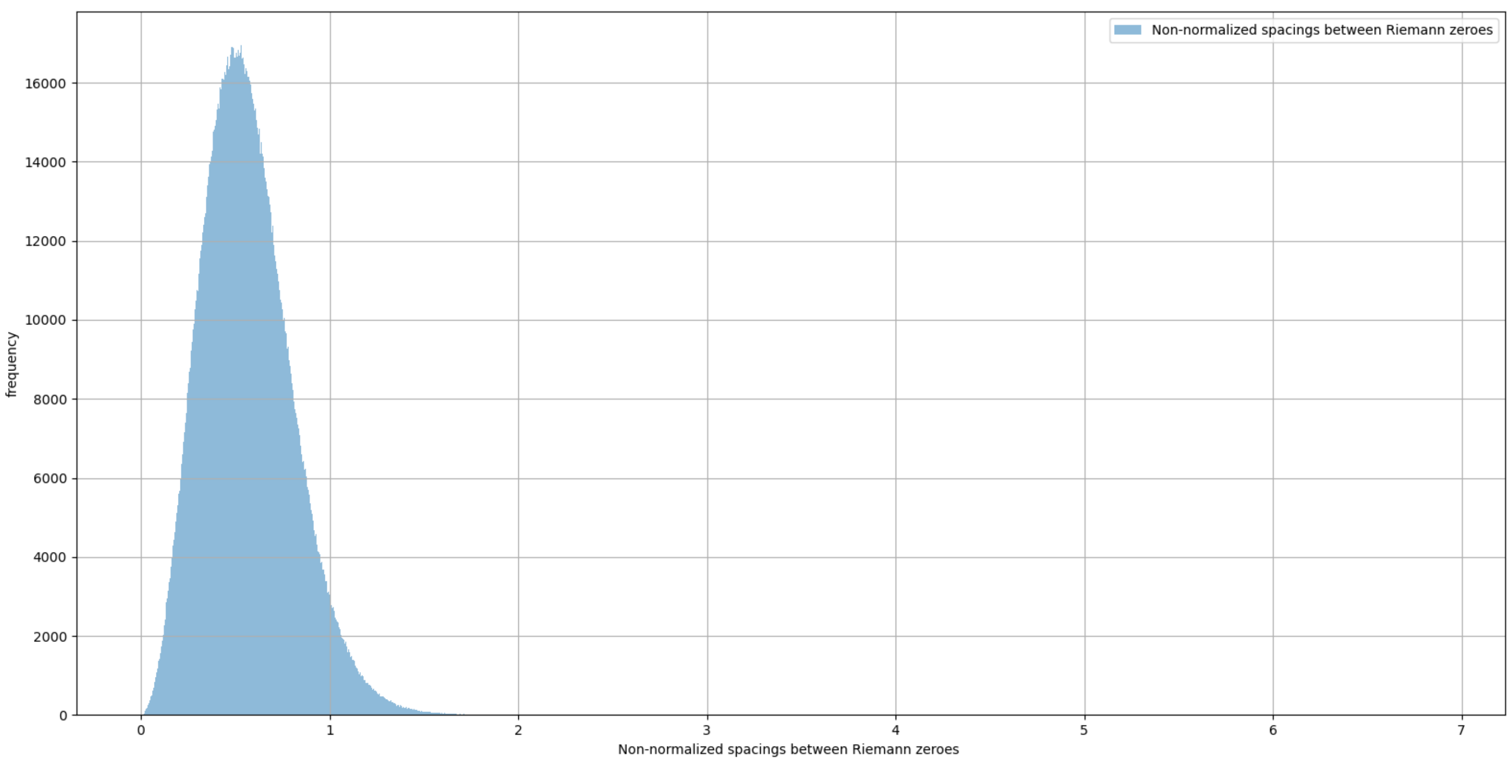
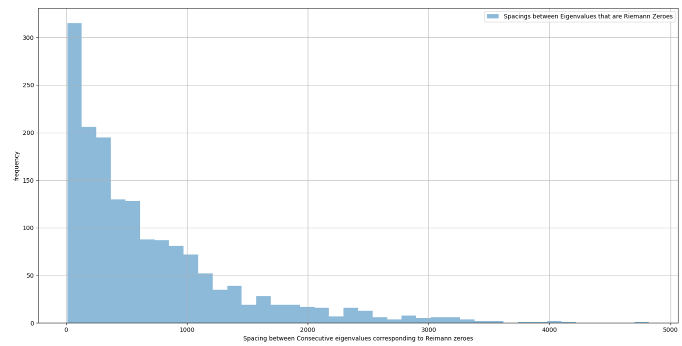
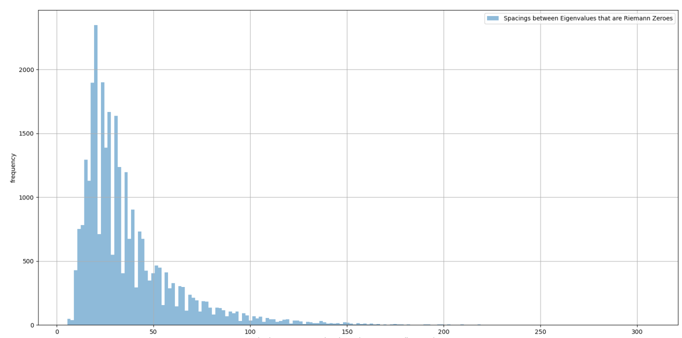


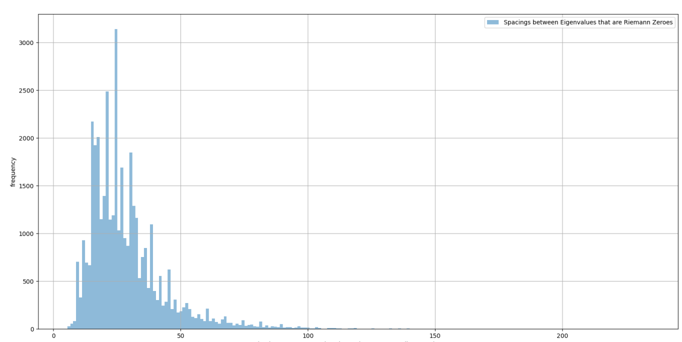
Figure 10: Spacing between consecutive eigenvalues



(a) $\epsilon = 0.01$

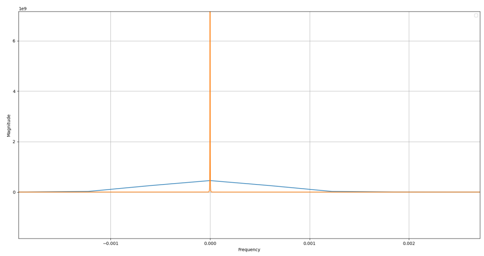


(b) $\epsilon = 0.2$

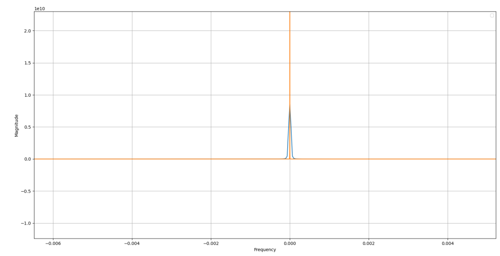


(c) $\epsilon = 0.29$

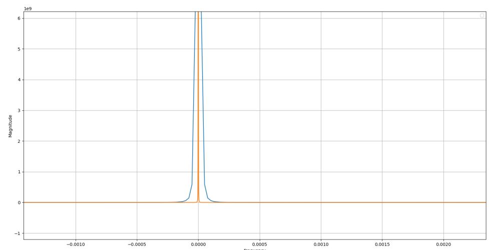
Figure 11: Spacings between consecutive eigenvalues corresponding to Riemann zeroes for different values of ϵ



(a) $\epsilon = 0.01$



(b) $\epsilon = 0.2$



(c) $\epsilon = 0.29$

Figure 12: Fourier transform comparison between eigenvalues corresponding to Riemann zeroes for different values of ϵ and true Riemann zeroes.

4 Other approaches and observations not discussed in detail

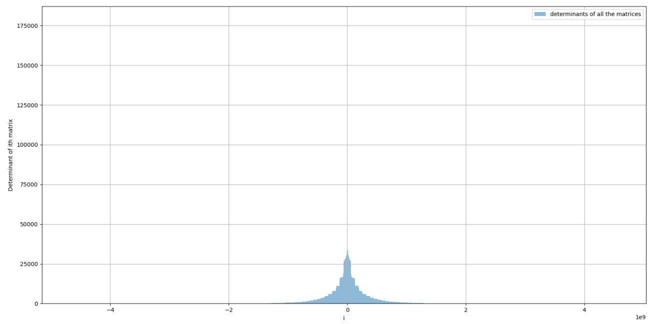
This section contains of computational results and other observations that are not discussed in this paper but were approaches taken by the author during the course of this project:

4.1 Fourier Transform

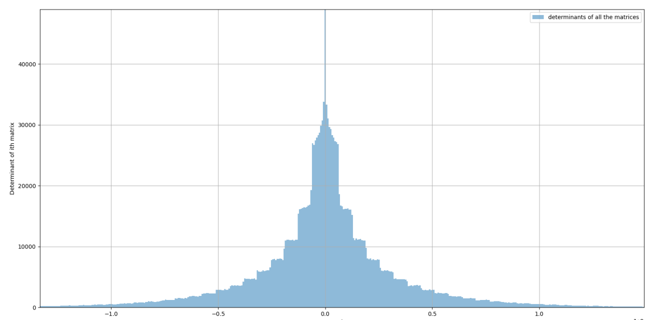
We apply the Fourier transform on arrays containing Riemann zeroes and eigenvalues corresponding to Riemann zeroes for different values of ϵ[Fig 12].

4.2 Histogram of Determinants

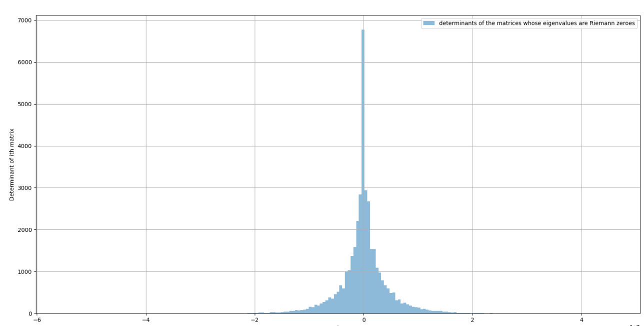
We plot a histogram of the determinants of all our prime matrices in Fig 13.



(a) The histogram appears to be symmetric along the line Y-axis, with an extremely tall and distinctive peak centered around x=0.



(b) While the "Step-like" features of the histogram appear to be on the same level as each other a closer inspection reveals to the contrary.



(c) Histogram of determinants of matrices whose eigenvalues are Riemann zeroes.

Figure 13: Histogram of determinants of our prime matrices. A further study of their distributions can help pinpoint on how exactly do Riemann zeroes occur as eigenvalues of prime matrices

References

[1] https://www-users.cse.umn.edu/ odlyzko/zeta$_{t}$*ables/index.html*

[2] Edwards, Harold M.Riemann's zeta function. 111 Fifth Avenue, New York, New York 10003, ACADEMIC PRESS, INC. ISBN 0-12-232750-0-0

[3] Carl M. Bender, Dorje C. Brody and Markus P. Muller: Hamiltonian for the Zeroes of the Riemann Zeta Function. Physical Review Letters(30 March 2017). DOI:10.1103/PhysRevLett.118.130201

[4] P.J. Forrester , A.M. Odlyzko: Gaussian unitary ensemble eigenvalues and Riemann ζ function zeros: A nonlinear equation for a new statistic.(1996)

[5] A.M. Odlyzko: On the distribution of spacings between zeros of the zeta function.

[6] Zhang, Fuzhen. Matrix Theory Basic Results and Techniques, Second edition, Springer. DOI 10.1007/978-1-4614-1099-7 . ISBN 978-1-4614-1098-0 . e-IS BN 978-1 -4614-1099-7

[7] Glendon Ralph Pugh: The Riemann-Siegel formula and large scale computations of the Riemann zeta function.