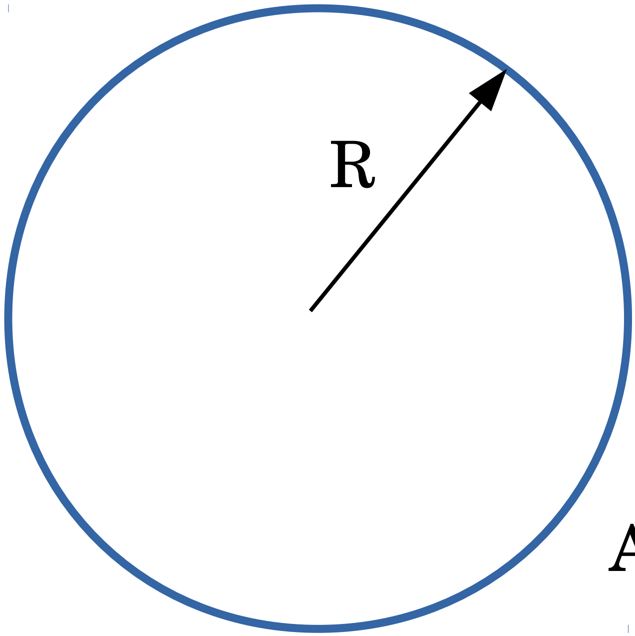
SEPARATION OF VARIABLES – 3D SPHERICAL



Hollow sphere of radius R

Potential on the surface $V_0(\theta)$

Find potential inside the sphere.

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$\text{At } r=R, \quad V(R,\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = V_0(\theta)$$

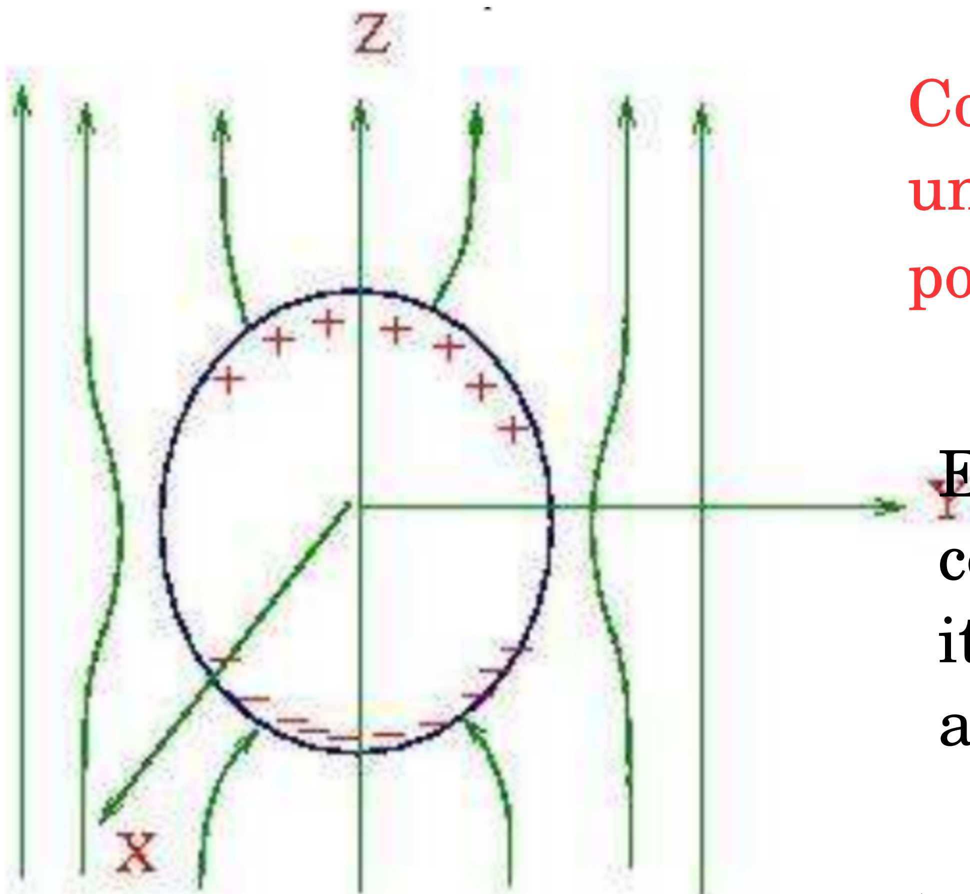
$$A_l = \frac{2l+1}{2R^l}\int_0^{\pi} V_0(\theta) P_l(\cos\theta)\sin\theta\, d\theta$$

$$\text{If, } V_0(\theta) = k\sin^2(\theta/2) = \frac{k}{2}[P_0(\cos\theta) - P_1(\cos\theta)]$$

$$A_0 = k/2, \qquad A_1 = -k/2R$$

$$V(r,\theta) = \frac{k}{2}\left(1 - \frac{r}{R}\cos\theta\right)$$

SPHERICAL SYMMETRY + SEPARATION



Conducting sphere of radius R in a uniform electric field **E**. What is the potential outside the sphere?

$$\boldsymbol{E} = E_0 \hat{\boldsymbol{z}}$$

Electric field induces charges on the conductor, which modifies the field itself near the sphere. The field far away from the sphere is unperturbed.

$$\boldsymbol{E} \,|_{r \to \infty} = E_0 \hat{\boldsymbol{z}}$$

$$V(r, \theta) \,|_{r \to \infty} = -E_0 z + C = -E_0 r \cos\theta + C$$

Sphere is an equipotential (set it zero) , $C = 0$

If we assume a separable form for the potential, the general solution is,

$$V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

SPHERICAL SYMMETRY + SEPARATION

The general solution must match the potential at infinity.

As $r \to \infty$, the B_l terms in the general solution go to zero.

$$V(r,\theta)\big|_{r\to\infty} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = -E_0 r\cos\theta = -E_0 r P_1$$

$$A_0 = 0, \quad A_1 = -E_0, \quad A_{2,3,\ldots} = 0$$

$$V(r,\theta) = -E_0 r\cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

Boundary condition on surface of conductor, $V(R,\theta) = 0$

$$B_0 = 0, \quad \frac{B_1}{R^2} = E_0 R, \quad B_{2,3,\ldots} = 0$$

$$V(r,\theta) = -E_0 r\cos\theta + \frac{E_0 R^3}{r^2}\cos\theta$$

SPHERICAL SYMMETRY + SEPARATION

$$V(r,\theta) = -E_0\left(1-\frac{R^3}{r^3}\right)r\cos\theta \qquad \text{if } V_0 = 0$$

$$\boldsymbol{E} = -\nabla . V = -\left(\hat{r}\frac{\partial}{\partial r}+\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\right)V(r,\theta)$$

$$\boldsymbol{E} = \hat{r}E_0\left(1+\frac{2R^3}{r^3}\right)\cos\theta-\hat{\theta}E_0\left(1-\frac{R^3}{r^3}\right)\sin\theta$$

$$\boldsymbol{E}.\hat{\boldsymbol{r}}\,|_{r=R} = \frac{\sigma}{\epsilon_0} \qquad \Rightarrow \qquad \sigma = 3\epsilon_0 E_0\cos\theta$$

$$Q_{\text{upper}} = 3\epsilon_0 E_0 R^2\int_0^{2\pi}d\phi\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta = 3\pi\epsilon_0 E_0 R^2$$

$$Q_{\text{lower}} = -Q_{\text{upper}} = -3\pi\epsilon_0 E_0 R^2$$