Dynamic Analysis of structures using FEM

Prepared in partial fulfillment of the

Study Project

Course No. - **MATH F266**

Submitted By

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Prepared for

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I extend my sincere thanks to **Dr. Sangita Yadav** for providing the opportunity to undertake this project. His guidance and project have helped me to understand Numerical Analysis better and to have in-depth understanding of programming of time history analysis of traverse vibrating beam. I also extend my sincere thanks to **Dr. Rajesh Kumar** for his constant guidance and help in understanding the structural analysis. I am deeply grateful to both of them for their guidance throughout the project and constant motivation in pursuing the study.

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Introduction

The project aims to develop a MATLAB code for **the time history analysis of forced vibrating cantilever beams**. The time-history analysis provides for linear or nonlinear evaluation of dynamic structural response under loading, which may vary according to the specified time function. This can be used to find the structure's durability and prevent damages and catastrophic failure.

The initial part of the report covers the fundamentals of dynamic analysis, including the brief theory about vibration, dynamic analysis of single DOF, and multi DOF. Any dynamic problem consists of governing differential equations with boundary conditions that can be solved analytically or numerical. To solve the time history analysis solution for transverse vibrating beam first, we derive the governing Euler Bernoulli equation for beam. Then the problem is solved using two different approaches–

1. **Modal Analysis**
2. **Direct Integration Method**

The direct integration method involves the result to PDE will be time history analysis for the given model. A computer works on binary and hence require numerical techniques to solve the partial differential equation.

The problem for the transverse vibrating beam is programmed on MATLAB using the Newmark method, which is a direct integration method. MATLAB is a programming language having a numeric computing environment. Using finite element method the beam is discretized and global stiffness and mass matrix is obtained which is required to obtain governing partial differential equations.

Vibration Theory

Vibration is a mechanical phenomenon where oscillations(back and forth motion) occur about an equilibrium point. The vibration of a system involves the alternating transfer of energy between its potential and kinetic forms.

In a damped system, some energy is dissipated at each cycle of vibration. The structures take on great quantity of energy generated by earthquakes. That energy can produce considerable deformations, even fracture of a structure. A structure must have capacity to dissipate that energy, either by permanent deformation or by dampers placed in the structure. The most frequently applied damper is one based on viscous fluids, or hydraulic dampers. Viscosity or internal friction is a friction force produced by rubbing a one fluid layer against the other while they are in motion. In such device a fluid is pushed by a piston from one into the other chamber. Viscous dampers are devices enabling displacements due to temperature changes, creepage and shrinking, without creation of significant force, but dissipating great quantities of energy during sudden occurrence of dynamic entrance of seismic energy.

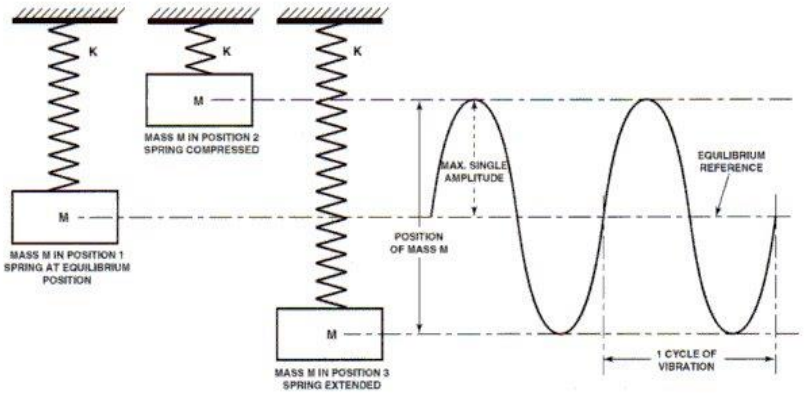


Fig: Free vibration(without damping) of lumped spring mass and its displacement graph.

The above fig is single degree of freedom(DOF) system. Degree of freedom is minimum number of independent coordinates require to specify the motion of a system at any instant. In case of uniform beam there is infinite degree of freedom, but we approximate using finite degree of freedom to reduce computational efforts. Also, in some case we use lumped mass parameter to reduce degree of freedom to reduce computational power.

Structure Modelled as single DOF system

1. Undamped

In dynamic analysis of structure, the simplest model is single DOF system in which we neglect the damping or restive frictional force. As shown in figure the mass m is on a frictionless surface and there is spring which opposes the displacement of mass from the equilibrium position. The governing partial differential equation can be obtain using energy method or force balance in horizontal direction.

The governing differential equation for the model is –

$$\boldsymbol{m\frac{\partial^2 u}{\partial t^2} + ku = 0}$$

or

$$\boldsymbol{m\ddot{u} + ku = 0}$$

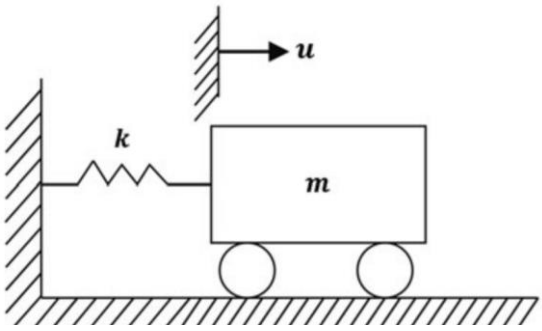


Fig: Single lumped spring mass system(Single DOF)

Solution -

$$\boldsymbol{u = A.sin(\omega_n t + \emptyset)\ , where\ \omega_n = \sqrt{k/m}}$$

u = Displacement from equilibrium position

A = Amplitude

ω_n = natural frequency of oscillation

2. Damped

In dynamic analysis of structure, the simplest model is single DOF system in which we consider the damping or restive frictional forces.

If the mass is given some initial velocity, then the mass will move in that direction till its kinetic energy reduces to zero and converted in potential energy in spring. Now the

The governing differential equation for the model is –

$$\boldsymbol{m\frac{\partial^2 u}{\partial t^2} + c\frac{\partial u}{\partial t} + ku = 0}$$

Solution -

$$\ddot{u} + 2.\xi.\omega_n.\dot{u} + {\omega_n}^2.u = 0, where\ \xi = \frac{c}{2\sqrt{k.m}}$$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

If $\xi = 1$, (**critical damped**)

$$\boldsymbol{u = (A + Bt)e^{-\xi\omega_n t}}$$

If $\xi < 1$, (**Underdamped**)

$$\boldsymbol{u = Ae^{-\xi\omega_n t}\sin(\omega_d t + \emptyset)}, \text{ where } \omega_d = \omega_n\sqrt{1-\xi^2}$$

If $\xi > 1$, (**Overdamped**)

$$\boldsymbol{u = c_1 e^{(-\xi+\sqrt{\xi^2+1})\omega_n t} + c_2 e^{(-\xi-\sqrt{\xi^2-1})\omega_n t}},$$

Where, $c_1 = \frac{x_0\omega_n\left(-\xi+\sqrt{\xi^2+1}\right)+x_0}{2\omega_n\sqrt{\xi^2-1}}$, $c_2 = \frac{-x_0\omega_n\left(\xi-\sqrt{\xi^2-1}\right)-x_0}{2\omega_n\sqrt{\xi^2-1}}$

Here,

ω_d = damped frequency,

ω_n = natural frequency,

u = displacement from equilibrium position,

ξ = damping ratio,

c = damping coefficient,

A,B = constants

Response to Dynamic Load

$$\boldsymbol{m\ddot{u} + c\dot{u} + ku = F(t)}$$

Complete Solution = Complementary function (Transient response) + Particular Integral (Steady state)

Structure Modelled as multiple DOF system

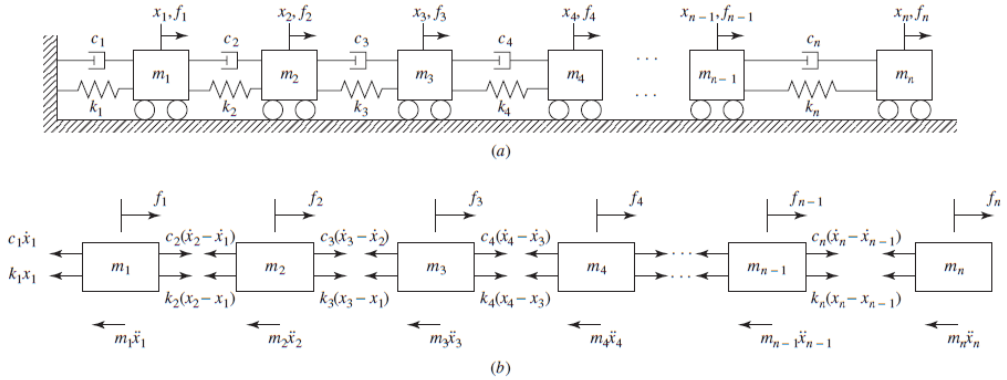


Figure 2.7 (*a*) An *n*-degree-of-freedom system; (*b*) free-body diagrams of the masses.

$$[\boldsymbol{m}]\ddot{\boldsymbol{x}} + [\boldsymbol{c}]\dot{\boldsymbol{x}} + [\boldsymbol{k}]\boldsymbol{x} = \boldsymbol{F}(\boldsymbol{t})$$

Where [m], [c], and [k] denote the mass, damping, and stiffness matrices, respectively:

$$[m] = \begin{bmatrix} m_1 & 0 & 0 & \cdots & 0 \\ 0 & m_2 & 0 & \cdots & 0 \\ 0 & 0 & m_3 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & m_n \end{bmatrix}$$

$$[c] = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 & \cdots & 0 & 0 \\ -c_2 & c_2 + c_3 & -c_3 & \cdots & 0 & 0 \\ 0 & -c_3 & c_3 + c_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -c_{n-1} & c_n \end{bmatrix}$$

$$[k] = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & \cdots & 0 & 0 \\ -k_2 & k_2 + k_3 & -k_3 & \cdots & 0 & 0 \\ 0 & -k_3 & k_3 + k_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -k_{n-1} & k_n \end{bmatrix}$$

$$\vec{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{Bmatrix}, \quad \dot{\vec{x}} = \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{Bmatrix}, \quad \ddot{\vec{x}} = \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \vdots \\ \ddot{x}_n \end{Bmatrix}, \quad \vec{f} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{Bmatrix}$$

For free vibration –

$$[M]\{\ddot{x}\} + [K]\{x\} = 0$$

This differential equation can be solved by assuming the following type of solution:

$$\{x\} = \{X\}e^{i\omega t}$$

$$(-\omega^2[M] + [K])e^{i\omega t} = 0$$

Since $e^{i\omega t}$ cannot equal zero the equation reduces to the following.

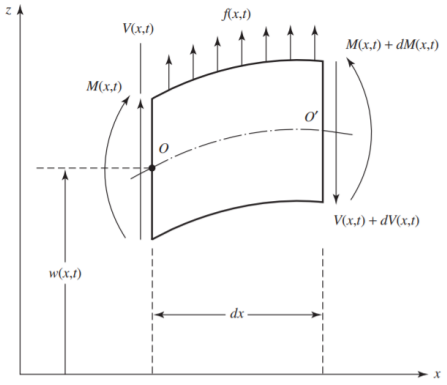
$$(-\omega^2[M] + [K]) = 0$$

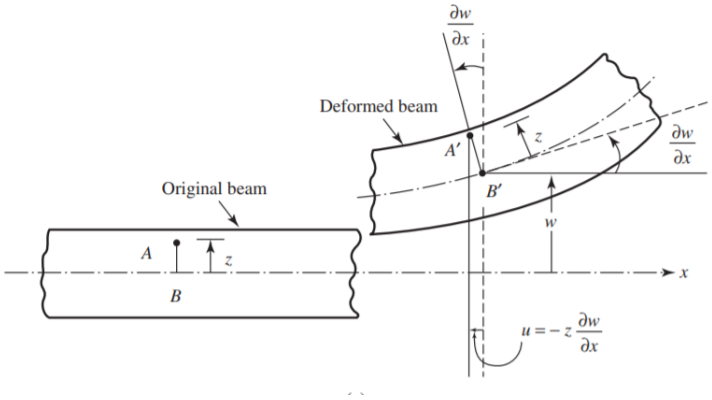
This problem becomes a eigen value problem, the solution to the problem results in N **eigenvalues** (i.e. $\omega_1, \omega_2, \dots, \omega_N$), where N corresponds to the number of degrees of freedom. The values of {X} that correspond to each eigenvalue are called the **eigenvectors**

Dynamic analysis of Continuum Model

Time history analysis for transverse vibration of beam

The governing partial differential equation for free transverse vibration of beam on the Euler-Bernoulli theory is given by –





$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} = F(t)$$

The above equation can be derived by taking differential element of beam and equating the forces and moments, as shown in figure.

To solve such problem, we consider Finite Element Method(FEM) in which we implement the discretisation and use numerical methods to get the solution for time history analysis. Also we require following method to solve the differential equation-

1. Modal Analysis
2. Direct Integration Method

Analytical solution of the equation of motion for a single-degree-of-freedom system is usually not possible if the excitation—applied force F(t) varies arbitrarily with time or if the system is nonlinear. We use numerical methods for such cases and to obtain solution. For the project we use Newmark method for numerical approximation of integration of differential equations and to tackle such problems.

Modal Analysis

Modal Analysis is used to evaluate and superimpose free-vibration mode shapes to find displacement pattern. Mode shape describe the vibration of beam in which the structure naturally vibrates. For structure with N degree of freedom will have N respective mode shapes. Each mode shapes are independent.

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} = F(t)$$

The vibration solution can be found using the method of separation of variables as

$$w(x,t) = W(x)T(t)$$

$$w(x,t) = Re[\widehat{w}(x)e^{-i\omega t}]$$

For **free vibration**

In the absence of a transverse load, F(t), we have free vibration. This equation can be solved using Fourier decomposition.

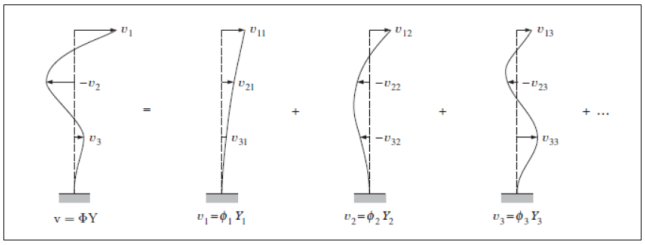


Figure 1 - Resultant displacement and modal components

As show in figure the free vibrating l s time history analysis is computed using the principle of superposition. On right there are different modes which are superimposed such that it satisfies the boundary conditions and adding their displacements at each node to equivalent displacement.

Periodic Function and Fourier Series

Any periodic function can be represented as infinite sum of sine and cosine terms using Fourier series. The process of representing a periodic function as sum of harmonic function is called harmonic analysis. If x(t) is a periodic function with period τ, its Fourier series given by –

$$x(t) = \frac{a_0}{2} + a_1\cos(\omega t) + a_2\cos(2\omega t) + \ldots + b_1\sin(\omega t) + b_2\sin(2\omega t) + \ldots$$

Where $a_0, a_{1,\ldots}, b_1, b_{2,\ldots}$ are constant and we do the following steps to determine a_n *and* b_n.

$$a_0 = \frac{\omega}{\pi}\int_0^{2\pi/\omega} x(t)\,dt$$

$$a_n = \int_0^{2\pi/\omega} x(t)\cos(n\omega t)\,dt$$

$$b_n = \int_0^{2\pi/\omega} x(t)\sin(n\omega t)\,dt$$

Example: Find the dynamic response of a uniform beam simply supported at both ends and subjected to a harmonically varying load:

$$f(x,t) = f_0 \sin\left(\frac{n\pi x}{l}\right) \sin(\omega t)$$

Where f_0 is constant, n is an integer, l is the length of beam, and ω is the frequency of variation of the load. Assume the initial displacement and the intial velocity of the beam to be zero.

SOLUTION The equation of motion of the beam is given by

$$EI\frac{\partial^4 w(x,t)}{\partial x^4} + \rho A\frac{\partial^2 w(x,t)}{\partial t^2} = f_0 \sin\left(\frac{n\pi x}{l}\right) \sin(\omega t) \ldots (1)$$

The homogeneous (or free vibration) solution of the above equation can be expressed as

$$w(x,t) = \sum_1^\infty \sin\left(\frac{i\pi x}{l}\right)(C_i \cos(\omega_i t) + D_i \cos(\omega_i t)) \ldots (2)$$

where ω_i is the natural frequency of the beam, given by

$$\omega_i = i^2\pi^2\sqrt{\frac{EI}{\rho A l^4}}$$

The particular integral of the equation can be expressed as

$$w(x,t) = a_n \sin\left(\frac{n\pi x}{l}\right) \sin(\omega t)$$

where the expression for a_n can be found by substituting the above equation in equation 1

$$a_n = \frac{f_0 l^4}{EL(n\pi)^4[1 - (\frac{\omega}{\omega_n})^2]}$$

The total solution of the equation is given by sum of its homogeneous solution and the particular integral:

$$w(x,t) = \sum_1^\infty \sin\left(\frac{i\pi x}{l}\right)(C_i \cos(\omega_i t) + D_i \cos(\omega_i t)) + a_n \sin\left(\frac{n\pi x}{l}\right) \sin(\omega t) \ldots (3)$$

The initial conditions of the beam are given by

$$\mathrm{w}(\mathrm{x}, 0) = 0$$

$$\frac{\partial w(x,0)}{\partial t} = 0$$

Substituting the above equation in equation 3. We obtain,

$$C_i = 0 \text{ for all i.}$$

$$D_i = 0 \; for \; i \neq n$$

$$D_i = -a_n \frac{\omega}{\omega_n} \; for \; i = n$$

Thus, the solution of the beam becomes

$$w(x,t) = \frac{f_0 l^4}{EL(n\pi)^4[1 - (\frac{\omega}{\omega_n})^2]} \sin\left(\frac{n\pi x}{l}\right)(\sin(\omega t) - \frac{\omega}{\omega_n}\sin(\omega_n t)$$

Direct Integration Method

Analysis involves the integration of structural properties and behaviours at a series of time steps which are small relative to loading duration.

Newmark Method

A time stepping method developed by N. M. Newmark based on the following equations:

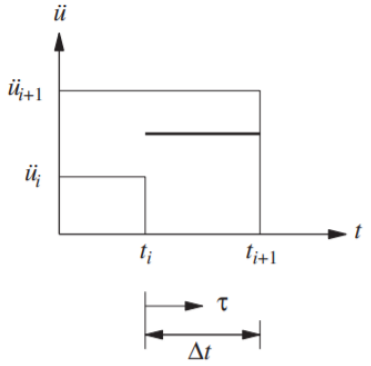
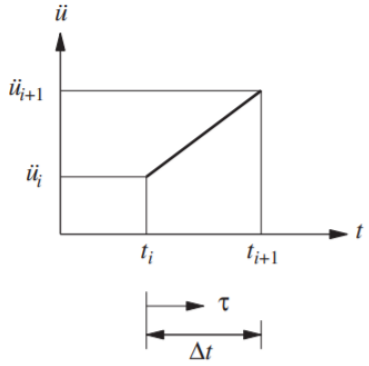
$$\dot{u}_{i+1} = \dot{u}_i + [(1-\gamma)\Delta t]\ddot{u}_i + (\gamma \Delta t)\ddot{u}_i$$

$$u_{i+1} = u_i + \dot{u}_i \Delta t + [(0.5-\beta)(\Delta t)^2]\ddot{u}_i + [(\beta)(\Delta t)^2]\ddot{u}_{i+1}$$

The parameters β and γ define the variation of acceleration over a time step and determine the stability and accuracy characteristics of the method. Typical selection for γ is $1/2$, and $1/6 \leq \beta \leq 1/4$ is satisfactory from all points of view, including that of accuracy. These two equations, combined with the equilibrium equation at the end of the time step, provide the basis for computing u_{i+1}, $\dot{u}_{i+1}$, and $\ddot{u}_{i+1}$ at time i + 1 from the known u_i, $\dot{u}_i$, and $\ddot{u}_i$ at time i. Iteration is required to implement these computations because the unknown $\ddot{u}_i$+1. There are two special cases of Newmark's method are the well-known

1. constant average acceleration
2. linear acceleration methods.

TABLE 1 AVERAGE ACCELERATION AND LINEAR ACCELERATION METHODS

Constant Average Acceleration	Linear Acceleration
	
$\ddot{u}(\tau) = \frac{1}{2}(\ddot{u}_{i+1} + \ddot{u}_i)$	$\ddot{u}(\tau) = \ddot{u}_i + \frac{\tau}{\Delta t}(\ddot{u}_{i+1} - \ddot{u}_i)$ .2)
$\dot{u}(\tau) = \dot{u}_i + \frac{\tau}{2}(\ddot{u}_{i+1} + \ddot{u}_i)$	$\dot{u}(\tau) = \dot{u}_i + \ddot{u}_i\tau + \frac{\tau^2}{2\Delta t}(\ddot{u}_{i+1} - \ddot{u}_i)$ 3)
$\dot{u}_{i+1} = \dot{u}_i + \frac{\Delta t}{2}(\ddot{u}_{i+1} + \ddot{u}_i)$	$\dot{u}_{i+1} = \dot{u}_i + \frac{\Delta t}{2}(\ddot{u}_{i+1} + \ddot{u}_i)$ 4)
$u(\tau) = u_i + \dot{u}_i\tau + \frac{\tau^2}{4}(\ddot{u}_{i+1} + \ddot{u}_i)$	$u(\tau) = u_i + \dot{u}_i\tau + \ddot{u}_i\frac{\tau^2}{2} + \frac{\tau^3}{6\Delta t}(\ddot{u}_{i+1} - \ddot{u}_i)$ 5)
$u_{i+1} = u_i + \dot{u}_i\,\Delta t + \frac{(\Delta t)^2}{4}(\ddot{u}_{i+1} + \ddot{u}_i)$	$u_{i+1} = u_i + \dot{u}_i\,\Delta t + (\Delta t)^2\left(\frac{1}{6}\ddot{u}_{i+1} + \frac{1}{3}\ddot{u}_i\right)$ 6)

Linear Systems

For linear systems, it is possible to modify Newmark's original formulation, to permit solution of equation (1) and (4) without iteration. Specialized for linear systems, Equation (4)

$$m\ddot{u}_{i+1} + c\dot{u}_{i+1} + ku_{i+1} = p_{i+1}$$

becomes ...(7)

From Eq. (1), $\ddot{u}_{i+1}$ can be expressed in terms of u_{i+1}:

$$\ddot{u}_{i+1} = \frac{1}{\beta(\Delta t)^2}(u_{i+1} - u_i) - \frac{1}{\beta \Delta t}\dot{u}_i - \left(\frac{1}{2\beta} - 1\right)\ddot{u}_i \qquad ...(8)$$

Substituting Eq. (8) into Eq. (1) gives

$$\dot{u}_{i+1} = \frac{\gamma}{\beta \Delta t}(u_{i+1} - u_i) + \left(1 - \frac{\gamma}{\beta}\right)\dot{u}_i + \Delta t\left(1 - \frac{\gamma}{2\beta}\right)\ddot{u}_i \quad ....(9)$$

Next, Equation (8) and (9) are substituted into the governing equation (5.4.7) at time i + 1. This substitution gives

$$\hat{k}u_{i+1} = \hat{p}_{i+1} \quad ...(10)$$

$$\hat{k} = k + \frac{\gamma}{\beta \Delta t}c + \frac{1}{\beta(\Delta t)^2}m \quad ...(11)$$

$$\hat{p}_{i+1} = p_{i+1} + \left[\frac{1}{\beta(\Delta t)^2}m + \frac{\gamma}{\beta \Delta t}c\right]u_i + \left[\frac{1}{\beta \Delta t}m + \left(\frac{\gamma}{\beta} - 1\right)c\right]\dot{u}_i + \left[\left(\frac{1}{2\beta} - 1\right)m + \Delta t\left(\frac{\gamma}{2\beta} - 1\right)c\right]\ddot{u}_i \quad ..(12)$$

With kˆ and pˆi+1 known from the system properties m, k, and c, algorithm parameters γ and β, and the state of the system at time i defined by u_i, $\dot{u}_i$, and $\ddot{u}_i$, the displacement at time i + 1 is computed from

$$u_{i+1} = \frac{\hat{p}_{i+1}}{\hat{k}} \quad ...(13)$$

Once u_{i+1} is known, the velocity $\dot{u}_{i+1}$and acceleration $\ddot{u}_{i+1}$ can be computed from Equation (9) and (8), respectively. The acceleration can also be obtained from the equation of motion at time i + 1:

$$\ddot{u}_{i+1} = \frac{p_{i+1} - c\dot{u}_{i+1} - ku_{i+1}}{m} \quad ...(14)$$

rather than by Eq. (8). Equation (14) is needed to obtain $\ddot{u}_0$ t o start the time stepping computations [see Eq. (10)]. In Newmark's method, the solution at time i + 1 is determined from Eq. (7), the equilibrium condition at time i + 1. Such methods are called implicit methods. Although the resisting force is an implicit function of the unknown u_{i+1}, it was easy to calculate

for linear systems. Table 2 summarizes the time-stepping solution using Newmark's method as it might be implemented on the computer.

Table 2 Newmark's Method: Linear Systems

Special cases

(1) Constant average acceleration method ($\gamma = \frac{1}{2}, \beta = \frac{1}{4}$)

(2) Linear acceleration method ($\gamma = \frac{1}{2}, \beta = \frac{1}{6}$)

1.0 *Initial calculations*

1.1 $\ddot{u}_0 = \dfrac{p_0 - c\dot{u}_0 - ku_0}{m}$.

1.2 Select Δt.

1.3 $a_1 = \dfrac{1}{\beta(\Delta t)^2}m + \dfrac{\gamma}{\beta \Delta t}c$; $a_2 = \dfrac{1}{\beta \Delta t}m + \left(\dfrac{\gamma}{\beta} - 1\right)c$; and

$$a_3 = \left(\frac{1}{2\beta} - 1\right)m + \Delta t\left(\frac{\gamma}{2\beta} - 1\right)c.$$

1.4 $\hat{k} = k + a_1$.

2.0 *Calculations for each time step*, $i = 0, 1, 2, \ldots$

2.1 $\hat{p}_{i+1} = p_{i+1} + a_1 u_i + a_2 \dot{u}_i + a_3 \ddot{u}_i$.

2.2 $u_{i+1} = \dfrac{\hat{p}_{i+1}}{\hat{k}}$.

2.3 $\dot{u}_{i+1} = \dfrac{\gamma}{\beta \Delta t}(u_{i+1} - u_i) + \left(1 - \dfrac{\gamma}{\beta}\right)\dot{u}_i + \Delta t\left(1 - \dfrac{\gamma}{2\beta}\right)\ddot{u}_i$.

2.4 $\ddot{u}_{i+1} = \dfrac{1}{\beta(\Delta t)^2}(u_{i+1} - u_i) - \dfrac{1}{\beta \Delta t}\dot{u}_i - \left(\dfrac{1}{2\beta} - 1\right)\ddot{u}_i$.

3.0 *Repetition for the next time step.* Replace i by $i + 1$ and implement steps 2.1 to 2.4 for the next time step.

Newmark's method is stable if

$$\frac{\Delta t}{T_n} \leq \frac{1}{\pi\sqrt{2}} \frac{1}{\sqrt{\gamma - 2\beta}} \qquad (15)$$

For $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{4}$ this condition becomes

$$\frac{\Delta t}{T_n} < \infty \qquad (16a)$$

This implies that the constant average acceleration method is stable for any Δt, no matter how large; however, it is accurate only if Δt is small enough, as discussed at the end of

Section 5.3. For $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{6}$, Eq. (15) indicates that the linear acceleration method is stable if

$$\frac{\Delta t}{T_n} \leq 0.551 \qquad 16b)$$

However, as in the case of the central difference method, this condition has little significance in the analysis of SDF systems because a much shorter time step than $0.551T_n$ must be used to obtain an accurate representation of the excitation and response.

Example 3

Solve Example 1 by the constant average acceleration method using $\Delta t = 0.1$ sec.

Solution

1.0 *Initial calculations*

$$m = 0.2533 \qquad k = 10 \qquad c = 0.1592$$
$$u_0 = 0 \qquad \dot{u}_0 = 0 \qquad p_0 = 0$$

1.1 $\ddot{u}_0 = \dfrac{p_0 - c\dot{u}_0 - ku_0}{m} = 0.$

1.2 $\Delta t = 0.1.$

1.3 $a_1 = \dfrac{4}{(\Delta t)^2}m + \dfrac{2}{\Delta t}c = 104.5; \quad a_2 = \dfrac{4}{\Delta t}m + c = 10.29; \quad$ and $a_3 = m = 0.2533.$

1.4 $\hat{k} = k + a_1 = 114.5.$

2.0 *Calculations for each time step*, $i = 0, 1, 2, \ldots$

2.1 $\hat{p}_{i+1} = p_{i+1} + a_1 u_i + a_2 \dot{u}_i + a_3 \ddot{u}_i = p_{i+1} + 104.5\, u_i + 10.29\, \dot{u}_i + 0.2533\, \ddot{u}_i.$

2.2 $u_{i+1} = \dfrac{\hat{p}_{i+1}}{\hat{k}} = \dfrac{\hat{p}_{i+1}}{114.5}.$

2.3 $\dot{u}_{i+1} = \dfrac{2}{\Delta t}(u_{i+1} - u_i) - \dot{u}_i.$

2.4 $\ddot{u}_{i+1} = \dfrac{4}{(\Delta t)^2}(u_{i+1} - u_i) - \dfrac{4}{\Delta t}\dot{u}_i - \ddot{u}_i.$

3.0 *Repetition for the next time step.* Steps 2.1 to 2.4 are repeated for successive time steps and are summarized in Table E5.3, where the theoretical result (from Table E5.1a) is also included.

Example 4

Solve Example 1 by the linear acceleration method using $\Delta t = 0.1$ sec.

Solution

1.0 *Initial calculations*

$$m = 0.2533 \qquad k = 10 \qquad c = 0.1592$$
$$u_0 = 0 \qquad \dot{u}_0 = 0 \qquad p_0 = 0$$

TABLE 3 NUMERICAL SOLUTION BY CONSTANT AVERAGE ACCELERATION METHOD

t_i	p_i	$\hat{p}_i$ (Step 2.1)	$\ddot{u}_i$ (Step 2.4)	$\dot{u}_i$ (Step 2.3)	u_i (Step 2.2)	Theoretical u_i
0.0	0.0000		0.0000	0.0000	0.0000	0.0000
0.1	5.0000	5.0000	17.4666	0.8733	0.0437	0.0328
0.2	8.6603	26.6355	23.1801	2.9057	0.2326	0.2332
0.3	10.0000	70.0837	12.3719	4.6833	0.6121	0.6487
0.4	8.6603	123.9535	−11.5175	4.7260	1.0825	1.1605
0.5	5.0000	163.8469	−38.1611	2.2421	1.4309	1.5241
0.6	0.0000	162.9448	−54.6722	−2.3996	1.4230	1.4814
0.7	0.0000	110.1710	−33.6997	−6.8182	0.9622	0.9245
0.8	0.0000	21.8458	−2.1211	−8.6092	0.1908	0.0593
0.9	0.0000	−69.1988	28.4423	−7.2932	−0.6043	−0.7751
1.0	0.0000	−131.0066	47.3701	−3.5026	−1.1441	−1.2718

1.1 $\ddot{u}_0 = \dfrac{p_0 - c\dot{u}_0 - ku_0}{m} = 0.$

1.2 $\Delta t = 0.1.$

1.3 $a_1 = \dfrac{6}{(\Delta t)^2}m + \dfrac{3}{\Delta t}c = 156.8; \quad a_2 = \dfrac{6}{\Delta t}m + 2c = 15.52;$ and

$a_3 = 2m + \dfrac{\Delta t}{2}c = 0.5146.$

1.4 $\hat{k} = k + a_1 = 166.8.$

2.0 *Calculations for each time step,* $i = 0, 1, 2, \ldots$

2.1 $\hat{p}_{i+1} = p_{i+1} + a_1 u_i + a_2 \dot{u}_i + a_3 \ddot{u}_i = p_{i+1} + 156.8u_i + 15.52\dot{u}_i + 0.5146\ddot{u}_i.$

2.2 $u_{i+1} = \dfrac{\hat{p}_{i+1}}{\hat{k}} = \dfrac{\hat{p}_{i+1}}{166.8}.$

2.3 $\dot{u}_{i+1} = \dfrac{3}{\Delta t}(u_{i+1} - u_i) - 2\dot{u}_i - \dfrac{\Delta t}{2}\ddot{u}_i.$

2.4 $\ddot{u}_{i+1} = \dfrac{6}{(\Delta t)^2}(u_{i+1} - u_i) - \dfrac{6}{\Delta t}\dot{u}_i - 2\ddot{u}_i.$

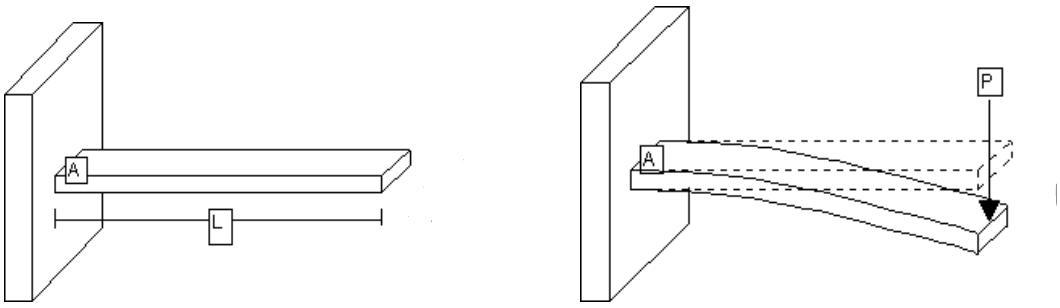
3.0 *Repetition for the next time step.* Steps 2.1 to 2.4 are repeated for successive time steps and are summarized in Table E5.4, where the theoretical result (from Table 1a) is also included.

TABLE 4 NUMERICAL SOLUTION BY LINEAR ACCELERATION METHOD

t_i	p_i	$\hat{p}_i$ (Step 2.1)	$\ddot{u}_i$ (Step 2.4)	$\dot{u}_i$ (Step 2.3)	u_i (Step 2.2)	Theoretical u_i
0.0	0.0000		0.0000	0.0000	0.0000	0.0000
0.1	5.0000	5.0000	17.9904	0.8995	0.0300	0.0328
0.2	8.6603	36.5748	23.6566	2.9819	0.2193	0.2332
0.3	10.0000	102.8221	12.1372	4.7716	0.6166	0.6487
0.4	8.6603	185.5991	−12.7305	4.7419	1.1130	1.1605
0.5	5.0000	246.4956	−39.9425	2.1082	1.4782	1.5241
0.6	0.0000	243.8733	−56.0447	−2.6911	1.4625	1.4814
0.7	0.0000	158.6538	−33.0689	−7.1468	0.9514	0.9245
0.8	0.0000	21.2311	0.4892	−8.7758	0.1273	0.0593
0.9	0.0000	−115.9590	31.9491	−7.1539	−0.6954	−0.7751
1.0	0.0000	−203.5678	50.1114	−3.0508	−1.2208	−1.2718

Results and Formulation

The forced vibration of damped cantilever beam time displacement graph is computed using Finite Element Method, and Newmark Method using MATLAB. To compute the numerical solution to governing partial differential equation for the transverse vibration of beam MATLAB is used. MATLAB is programming language used for manly for numeric computation using matrix based method, developed by MathWorks. The Newmark method is used for the numerical approximation, and to plot the time history analysis of vibrating beam.



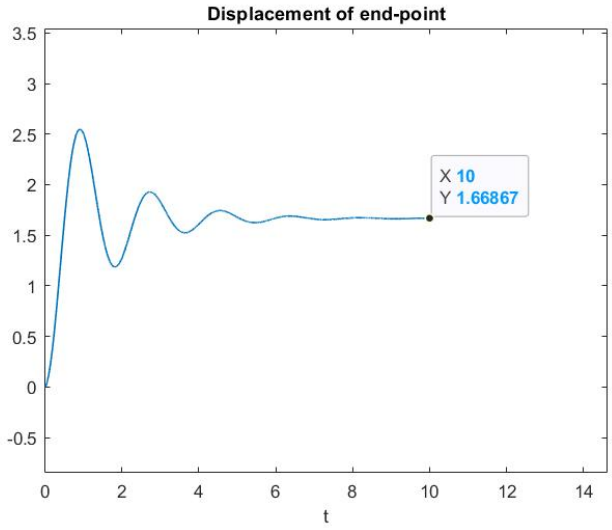
Rayleigh damping is introduced to the equilibrium equation by the component $C\dot{Y}$ where the damping matrix C is given by a portion of the mass matrix M and a potion of the stiffness matric K. The equation below presents the Rayleigh damping formulation.

$$\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}$$

Example 1. The solution for the cantilever beam with given values below:
Force at free end = P = 5 N

- rho = Density = 1 Kg/m^3
- A = Cross sectional Area = 1 m^2
- E = Modulus of elasticity = 1 Pa
- I = Moment of Inertia = 1 kg.m^2
- L = Length =1 m
- c1 = 0.1, c2 = 0.1 (Damping Coefficient)
- Ne = Number of elements = 100
- T = 10 second (Total time)
- dt = 10^{-3} second (Time interval)

Numerical Solution: The displacement of free end after vibration frequency decrease to zero (let say at 10 sec) comes out to be 1.6687 as shown in figure below.



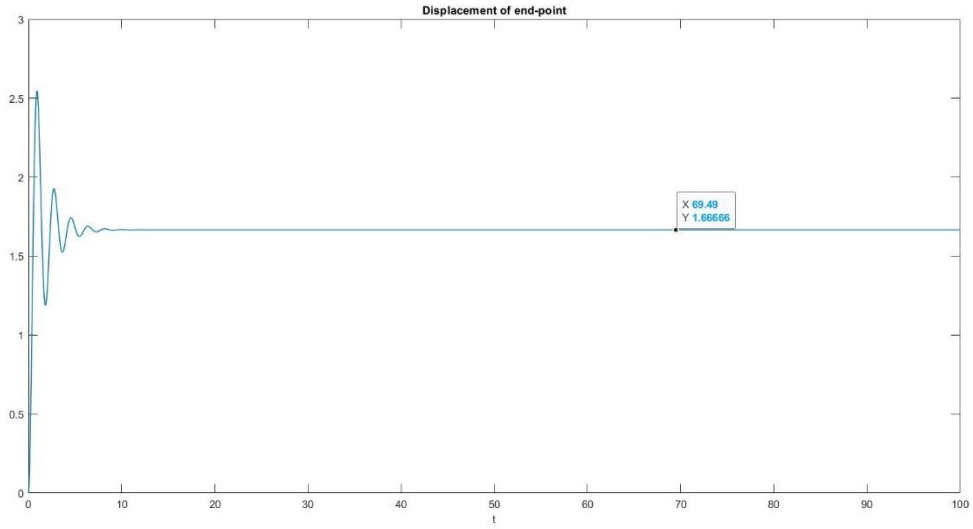
Analytical Solution:
The displacement of free end at equilibrium can be calculated using the formula used below for the cantilever beam.

$$\delta = FL^3/(3EI) = 5/3 = 1.66666666666666667$$

Value computed from the code = 1.668671073288804

Percentage error = 0.2004407%

Example 2: If the time taken to be 100 seconds then for the same initial conditions the time displacement graphs comes out to be.



Analytical Solution:
The displacement of free end at equilibrium can be calculated using the formula used below for the cantilever beam.

$$\delta = FL^3/(3EI) = 5/3 = 1.66666666666666667$$

Value computed from the code = 1.666658129205772

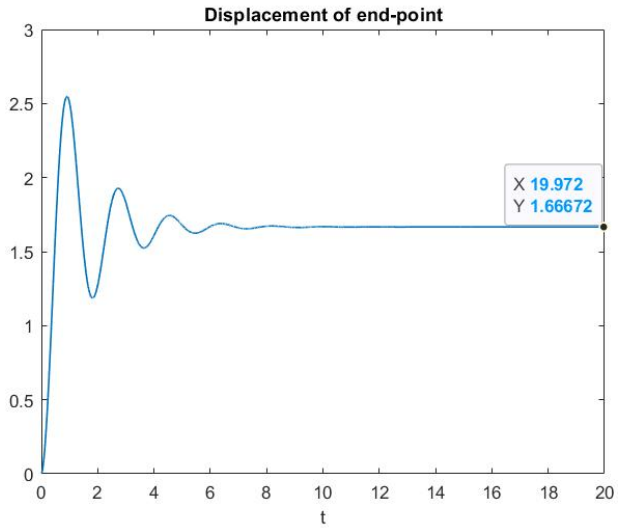
Percentage error = 0.00085374608%

Example 3: If the time taken be 20 seconds and the number of nodes 200**.** The result with this value comes out to be

$$\delta = FL^3/(3EI) = 5/3 = 1.66666666666666667$$

Value computed from the code = 1.666719718541577

Percentage error = 0.0053051875%

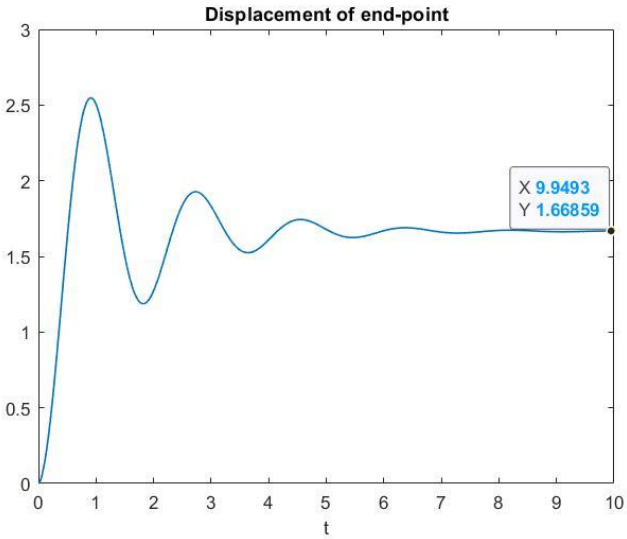


Example 4: When time increment for the Newmark method is decreased to 10^{-4} then result comes out to be better than previous solution.

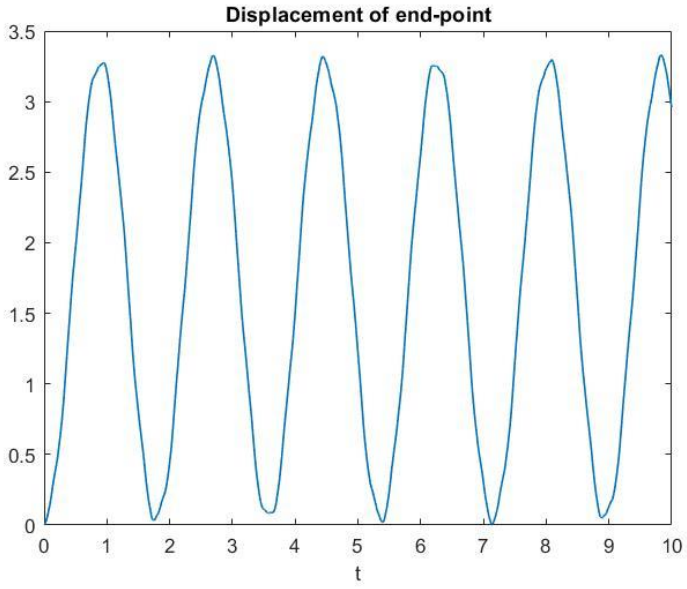
$$\delta = FL^3/(3EI) = 5/3 = 1.66666666666666667$$

Value computed from the code = 1.668633831577067

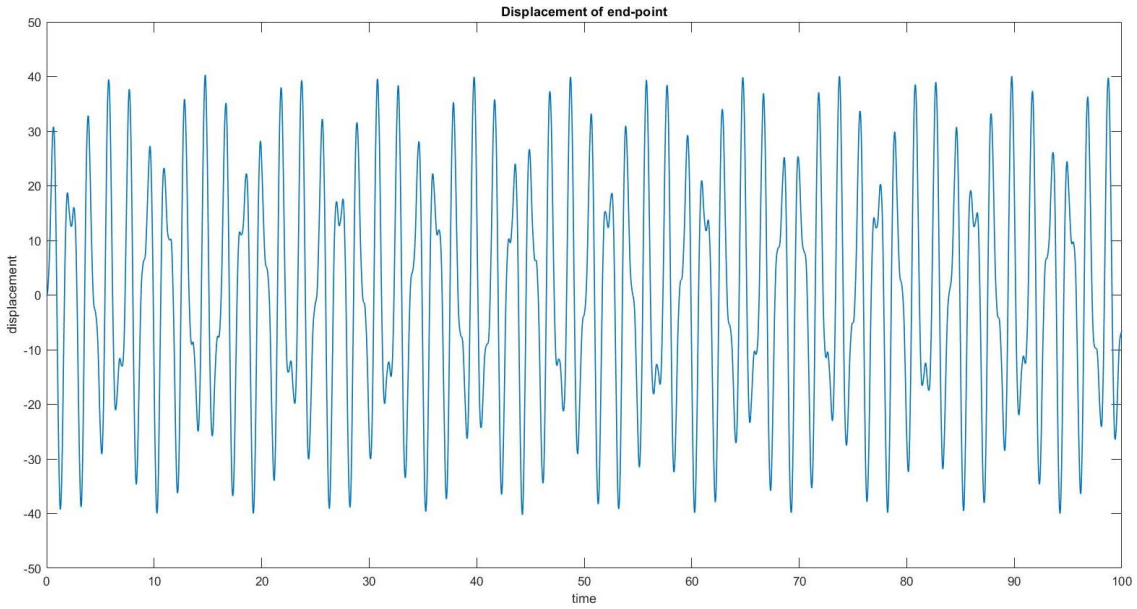
Percentage error = 0.1967165%



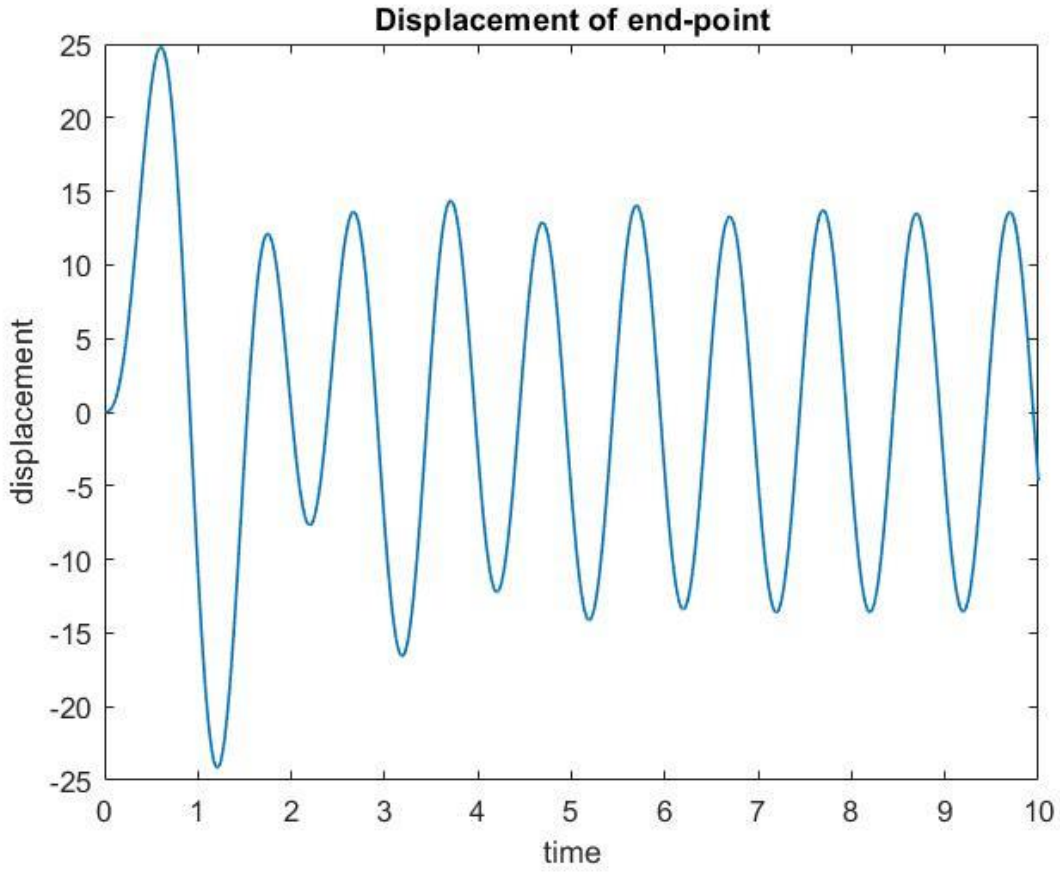
Example 5: Free vibration of free end of cantilever beam under same as previous condition without damping.



Example 6: Time varying force applied at free end = 100sin(2.pi.t) without damping and same value initial value as example 1.



Example 7: Damping in the previous example 5.



Summary

Understanding Vibration is crucial for preventing structures from catastrophic failure. The report brief structure vibration by taking the case of force vibrating beam under transverse loading and building a mathematical model to simulate the model using MATLAB. The Newmark method is used to solve the problem as it can also be used for nonlinear forces. Initially, fundaments of vibrations are covered, and later, the modal analysis and direct integration method are introduced. Fourier analysis is required to understand modal analysis in which superposition is used.

The transverse vibrating beam can be solved using both methods, but for MATLAB, the Newmark method is used because it is a numerical method that can apply to all similar problems with slight modifications. The results for the steady-state as per example 1 in the results section coincide with the analytical result with an accuracy of 99.79956%, which can be more accurate if the number of elements is more, time increment smaller with more damping, and after a long time to achieve the complete, steady-state. As in example 2, the steady-state is achieved, and hence the results are more accurate compared to the previous result. In example 3, the percentage error comes out to be 0.0053051875% when the number of nodes is increased, and the time is taken to be 20 seconds. In example 4, the time increment for the Newmark method is decreased to evaluate the acceleration at more intervals. Hence, the result for this condition comes out to be better than example 1.

Then more results were checked with time-varying force with or without damping. The Finite element method plays a vital role in discretization and computing the stiffness and mass matrix. The Newmark method is used as a time-stepping method to compute acceleration at different intervals.

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