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W2 Lesson 1

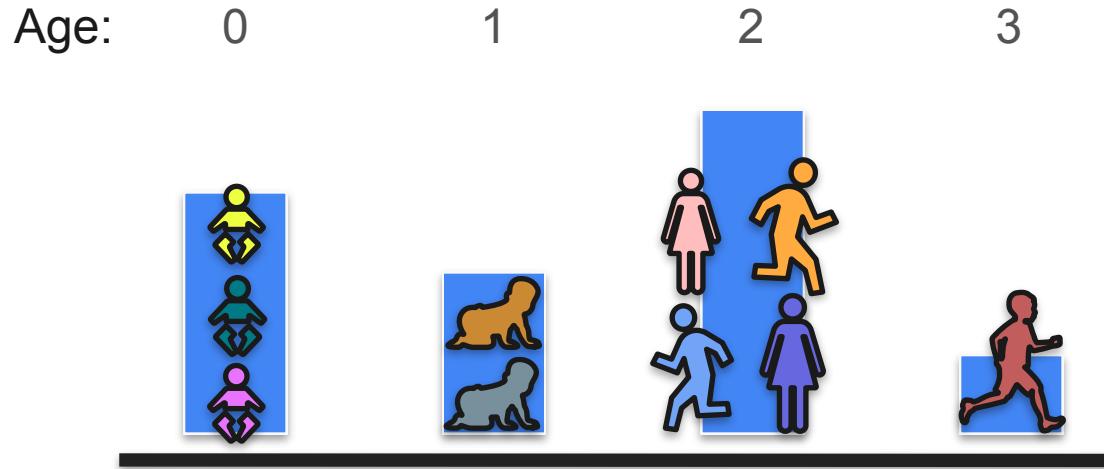


DeepLearning.AI

Describing Distributions

Expected value

Mean: Example



Mean: Example

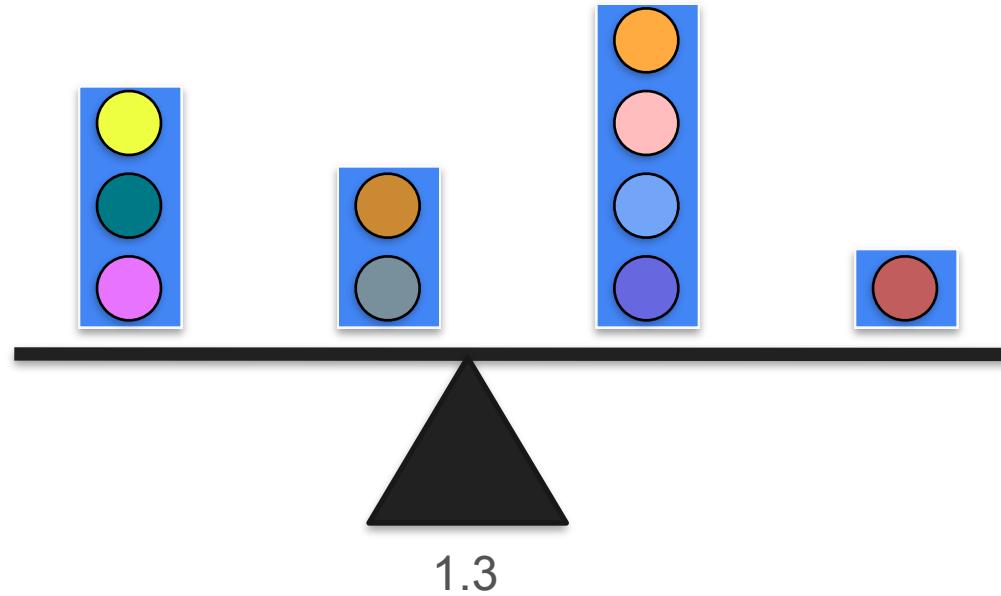
Age:

0

1

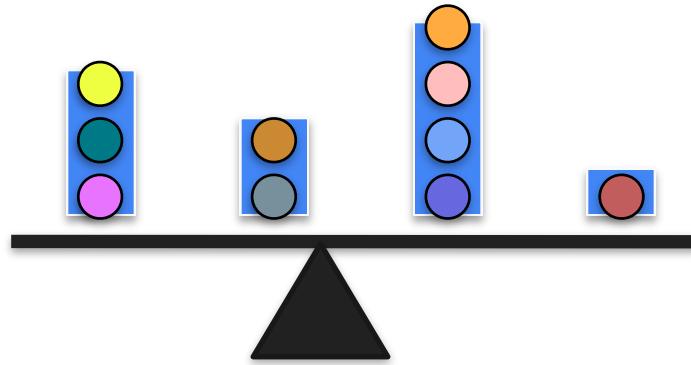
2

3



Mean: Example

Age: 0 1 2 3



$$\begin{array}{r} 0 + 0 + 0 + 1 + 1 + 2 + 2 + 2 + 3 \\ \hline 10 \end{array}$$

$$= \frac{13}{10}$$

$$= 1.3$$

Children in a Room

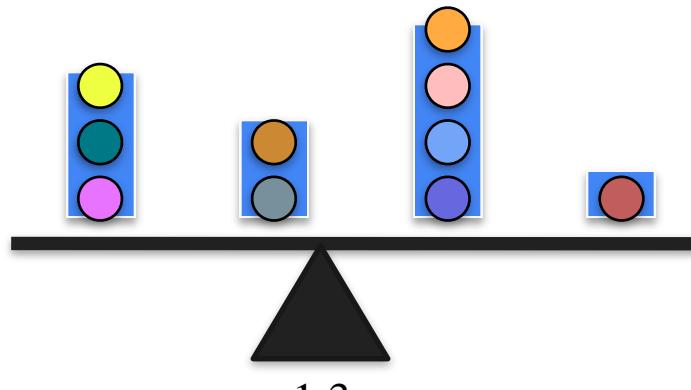
Age:

0

1

2

3



$$\frac{0 + 0 + 0 + 1 + 1 + 2 + 2 + 2 + 3}{10}$$

$$= \frac{13}{10} = 1.3$$

$$= \frac{3 \cdot 0 + 2 \cdot 1 + 4 \cdot 2 + 1 \cdot 3}{10}$$

Weighted average

$$= \frac{3}{10} \cdot 0 + \frac{2}{10} \cdot 1 + \frac{4}{10} \cdot 2 + \frac{1}{10} \cdot 3 = 1.3$$

Expected Value: Motivation Example 1

You play a game with a friend



You win 10 dollars



You win nothing

Game cost:



Do you play the game?

What is the maximum amount of money you would pay to play this game?

Expected Value: Motivation Example 1

You play a game with a friend



You win 10 dollars



You win nothing

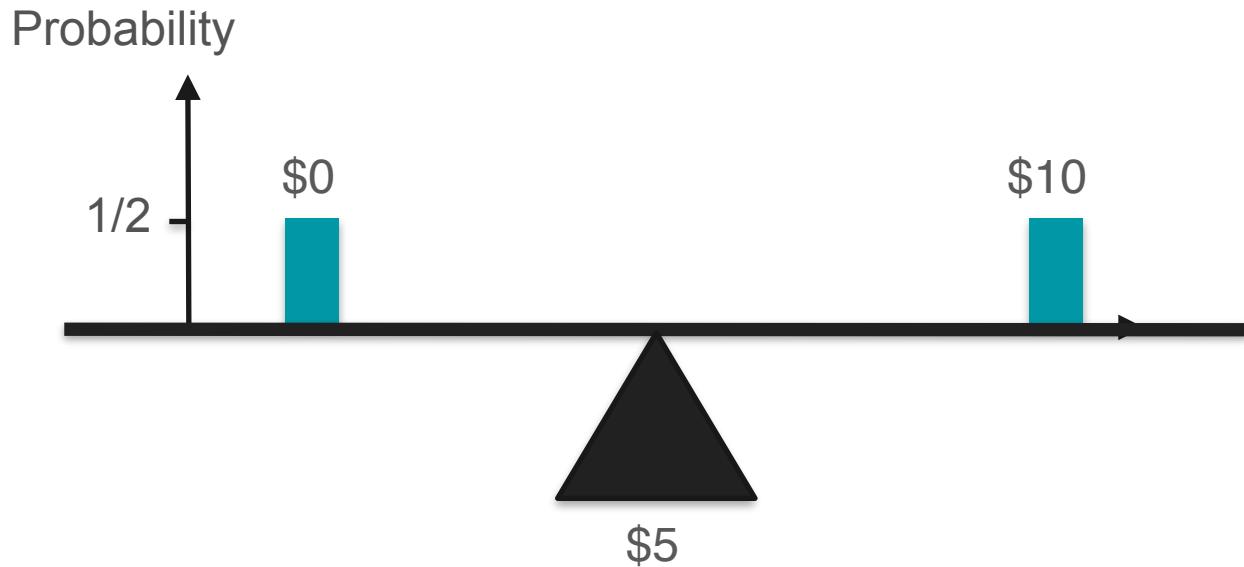
Game cost:

\$5

Long term: $0.5 \cdot \$10 + 0.5 \cdot \$0 = \$5$ →

You expect to win \$5 on average
 $E[X] = 5$

Expected Value: Motivation Example 1



Expected Value: Motivation Example 2

Play another game



Flip three coins. For each heads you win \$1

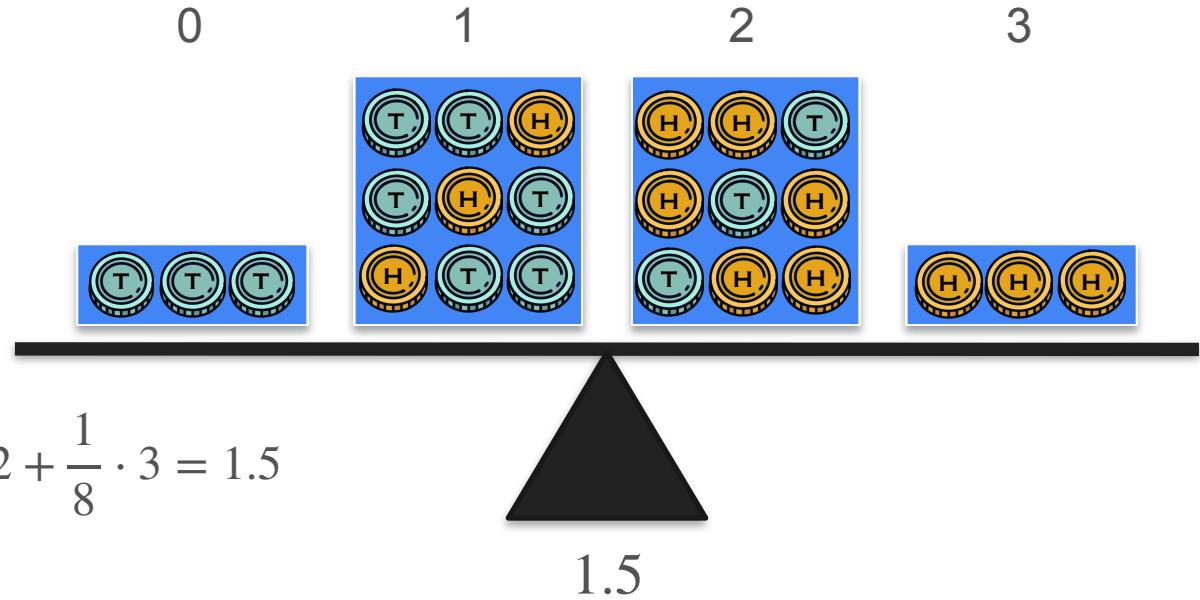
What is the maximum amount of money you would pay to play this game?

Expected Value: Motivation Example 2

X : Number of heads

$$\mathbb{E}[X] = 1.5$$

$$\mathbb{E}[X] = \frac{1}{8} \cdot 0 + \frac{3}{8} \cdot 1 + \frac{3}{8} \cdot 2 + \frac{1}{8} \cdot 3 = 1.5$$



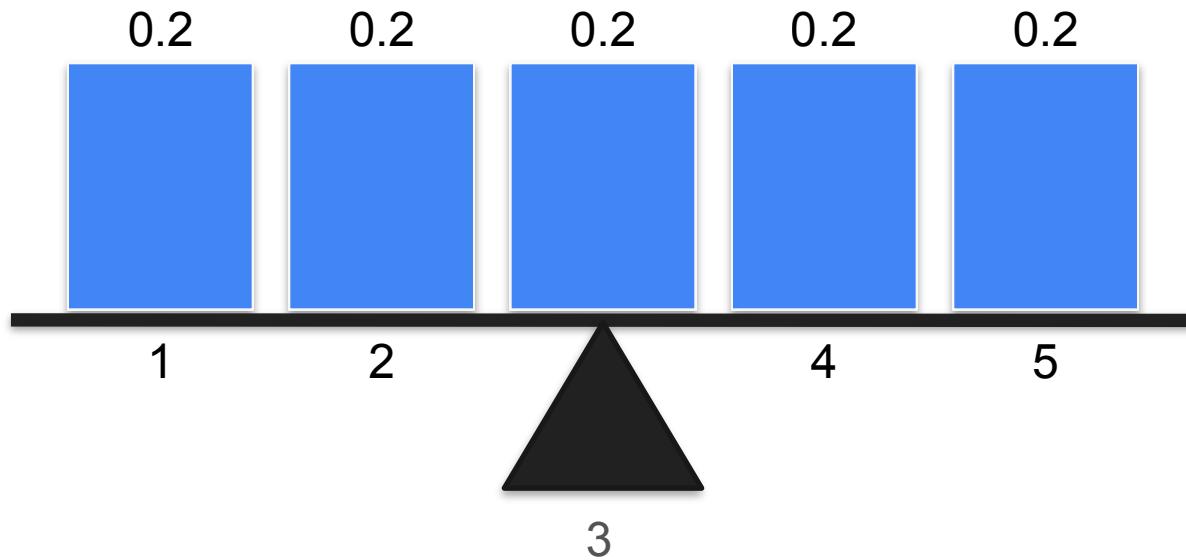
Expected Value: Discrete Case

X a discrete
random variable

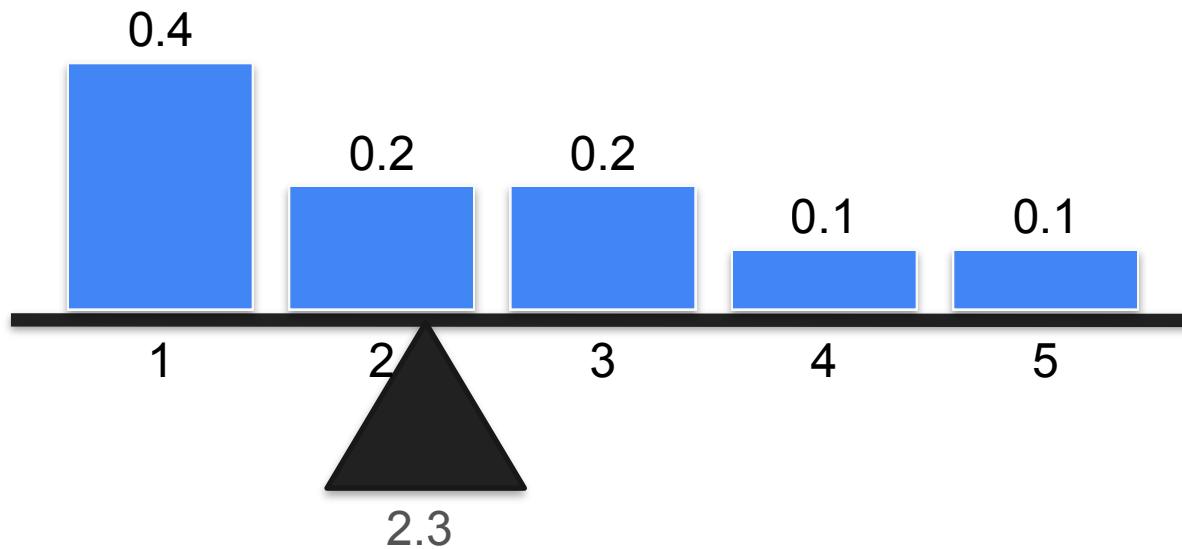
PMF of X
 $p_X(x) = \mathbf{P}(X = x)$

$$\mathbb{E}[X] = \sum_x x p_X(x)$$

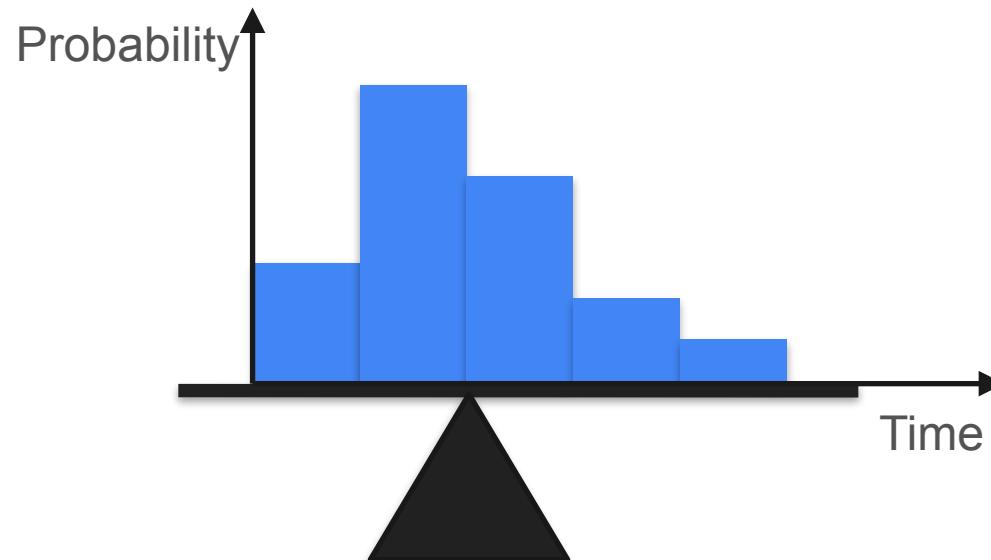
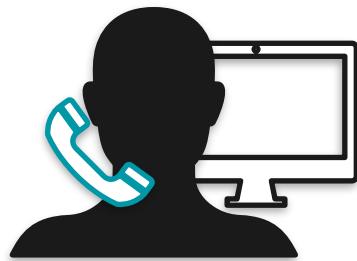
Expected Value



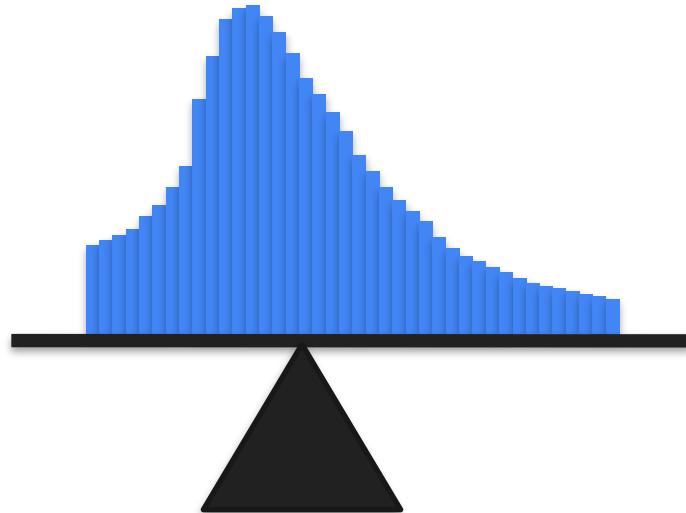
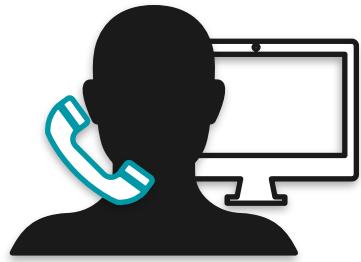
Expected Value



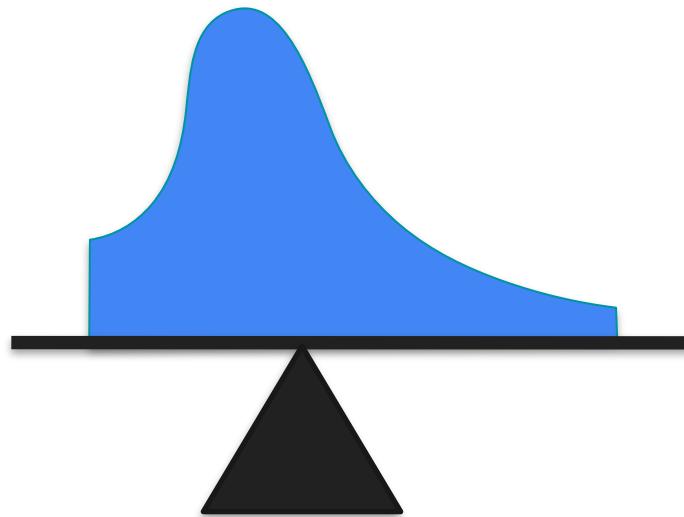
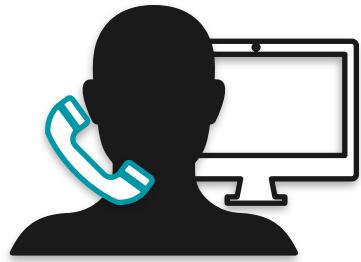
Expected Value - Continuous



Expected Value - Continuous



Expected Value - Continuous



Expected Value - Continuous

Discrete random variables

$$\mathbb{E}[X] = \sum_x x p_X(x)$$

Weighted using PMF

Continuous random variables

$$\int_{-\infty}^{\infty} x \cdot f_X(x) dx$$

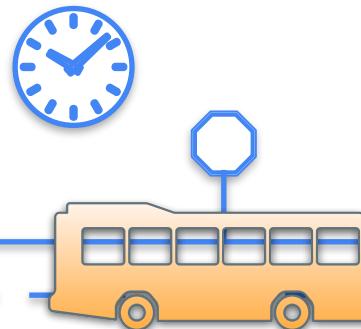
Integrals

Weighted using PDF

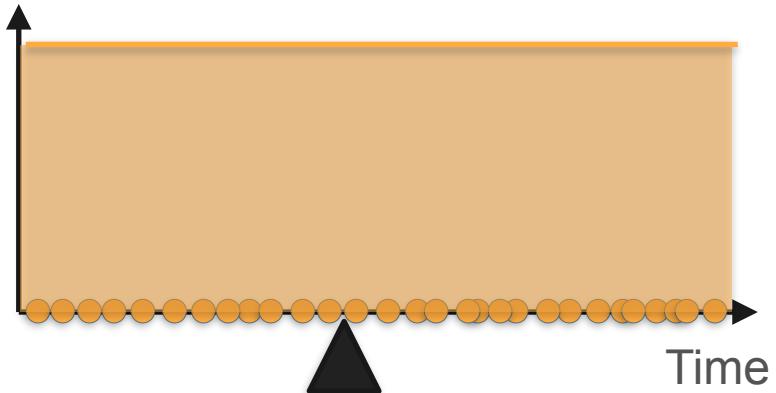
Expected Value

Waiting Time
15 min
32 min
58 min
1 min
47 min
14 min
37 min
8 min
29 min
55 min
...

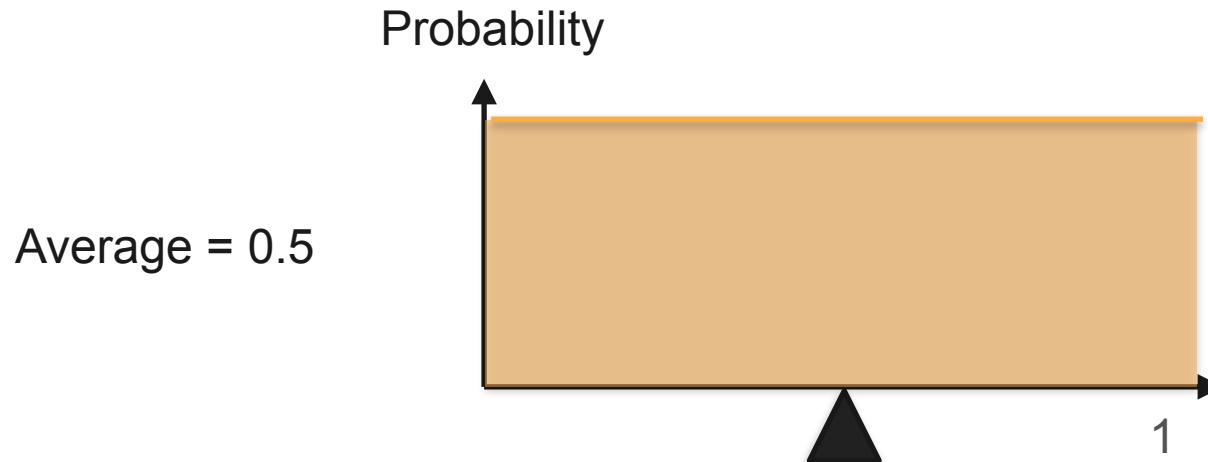
Average = 20.833



Probability

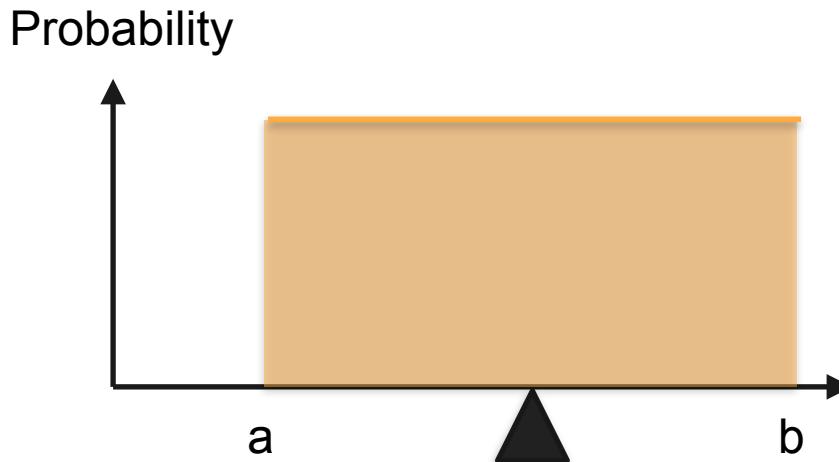


Expected Value: Uniform Distribution

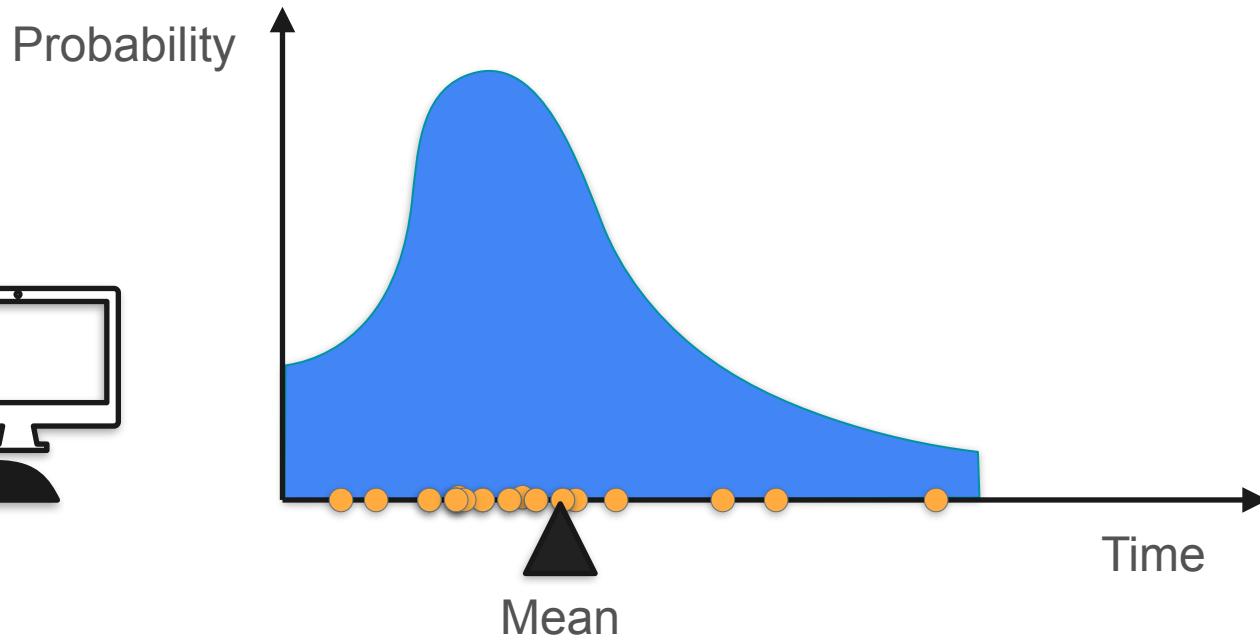


Expected Value: Uniform Distribution

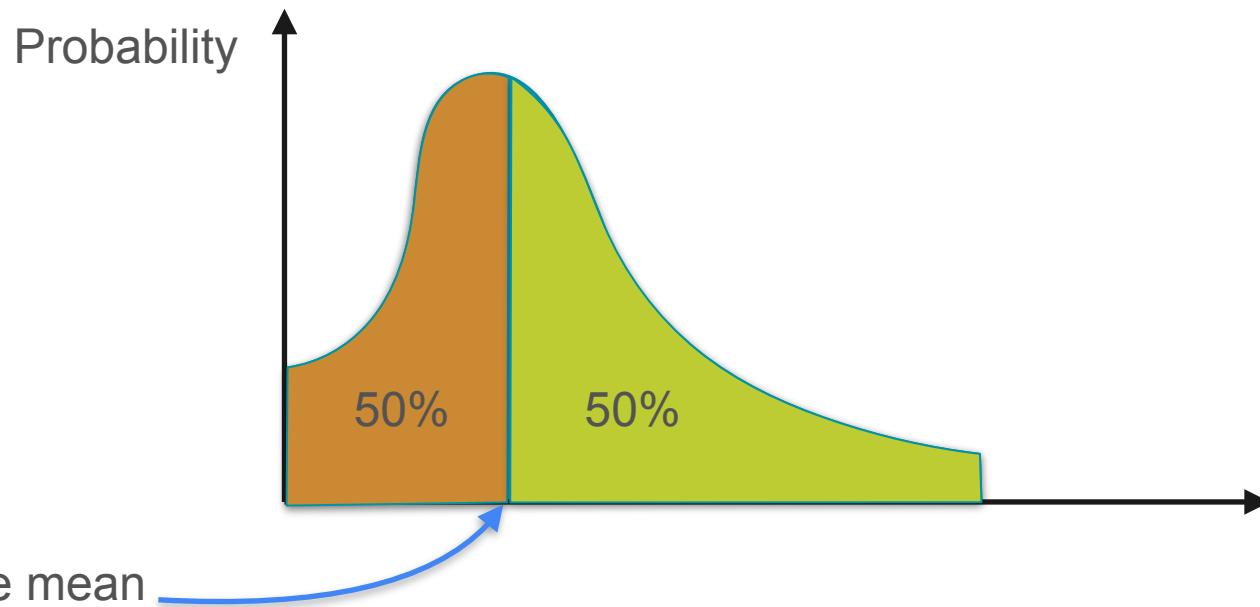
$$\text{Average} = \frac{a + b}{2}$$



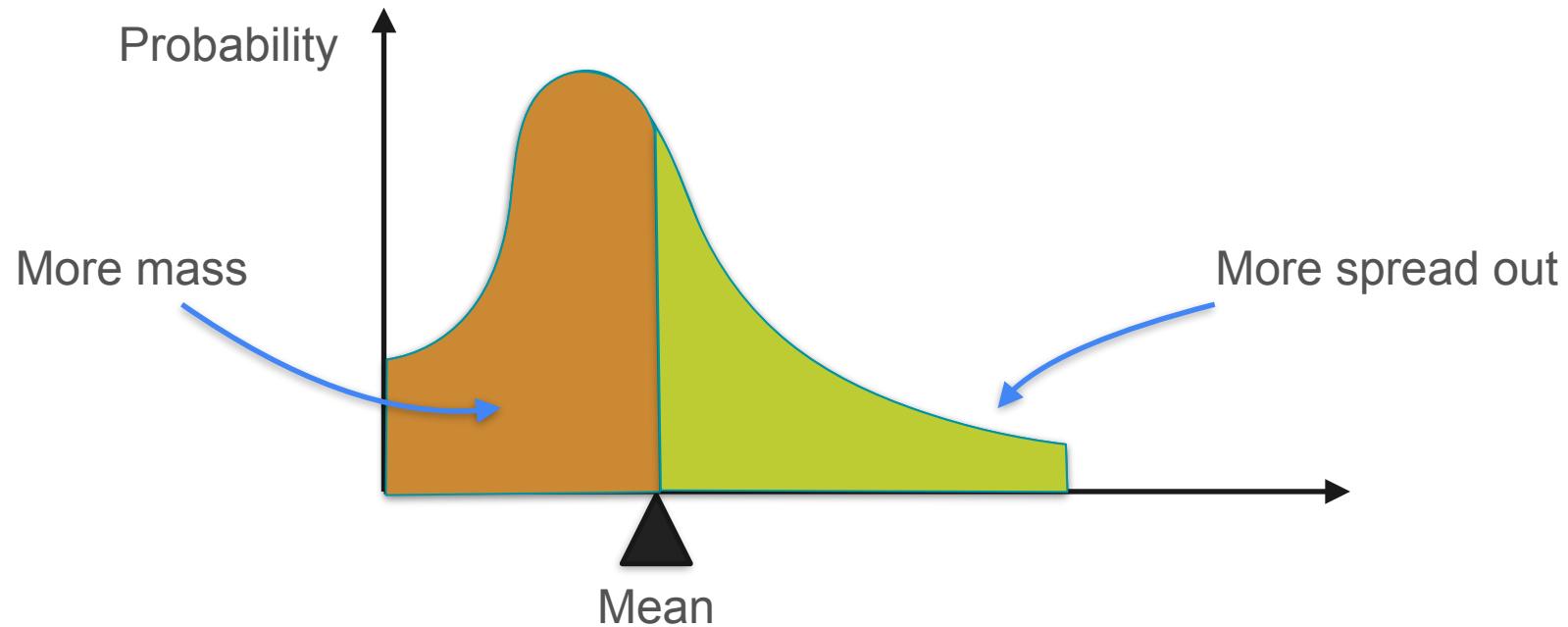
Expected Value



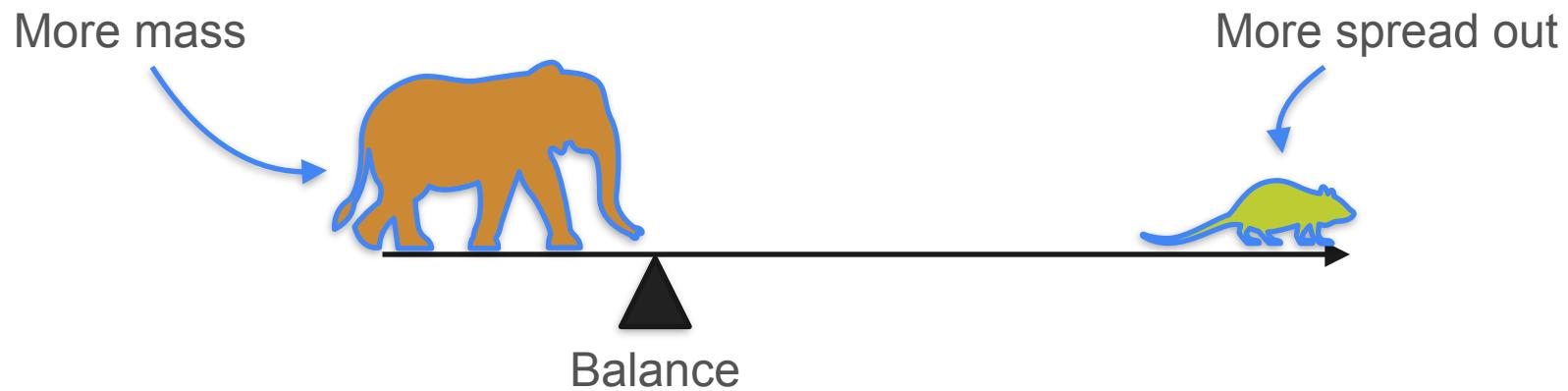
Expected Value: Common Misconception



Expected Value: Common Misconception



Expected Value: Common Misconception



Expected Value

- $\mathbb{E}[X]$
- Mean / Balancing point
- Defined for discrete and continuous random variables
- Weighted average of the PMF / PDF



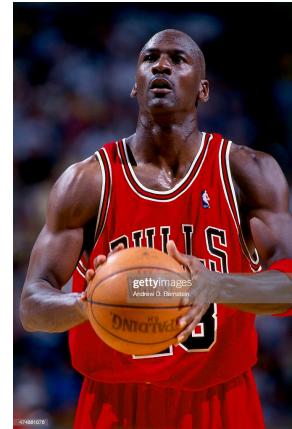
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Describing Distributions

**Other measures of central
tendency**

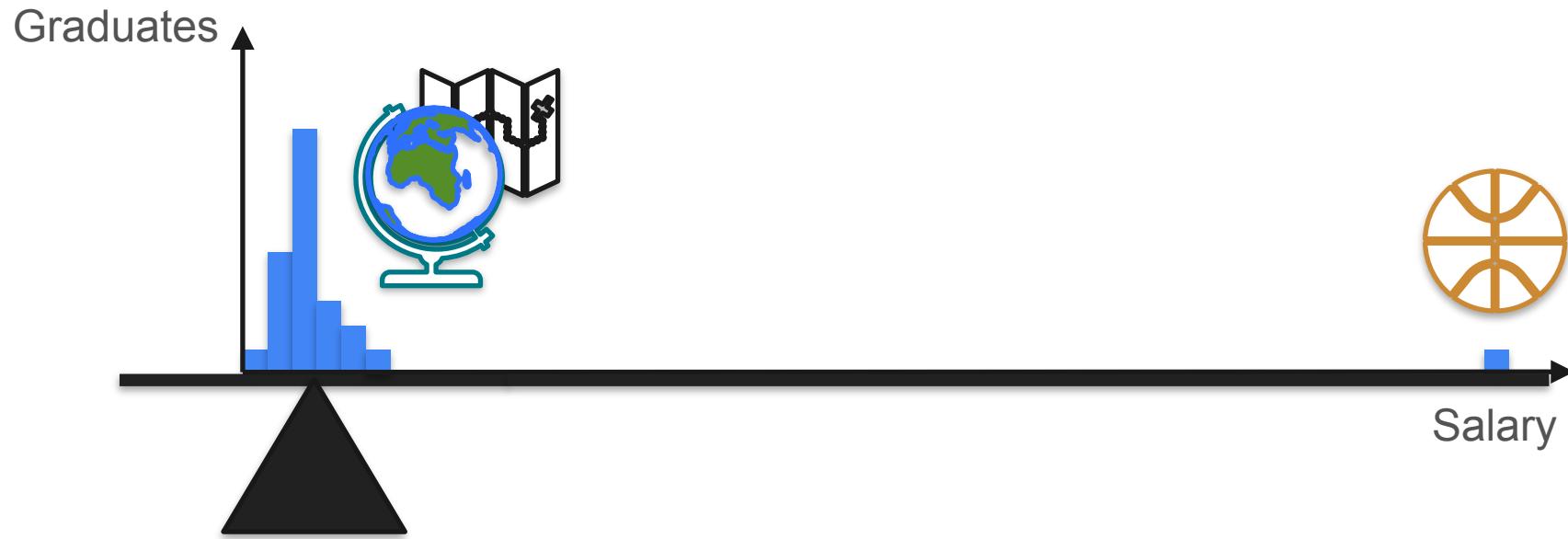
Median: Motivation

- In the 1980s, the starting salary for a geography graduate at the University of North Carolina was \$250,000.
 - For the rest of the country, the starting salary for a geography graduate was \$22,000.
 - Why?
 1. The program was really good
 2. The university had great connections
 3. One student made lots of money
- 



Michael Jordan

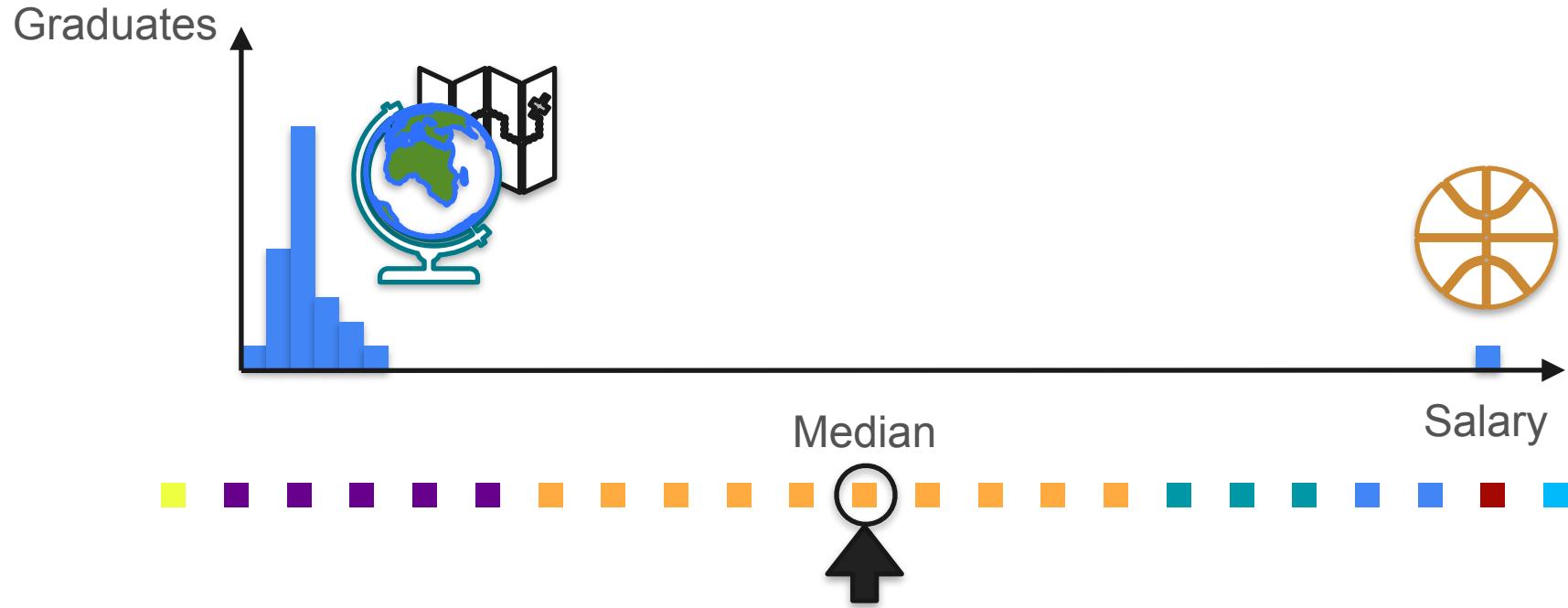
Outliers



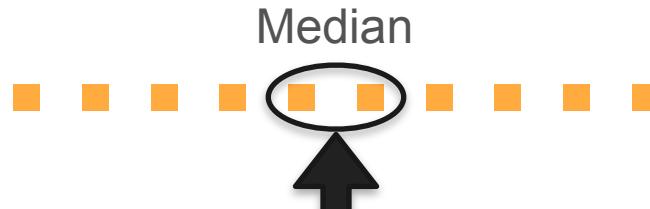
Outliers



Median

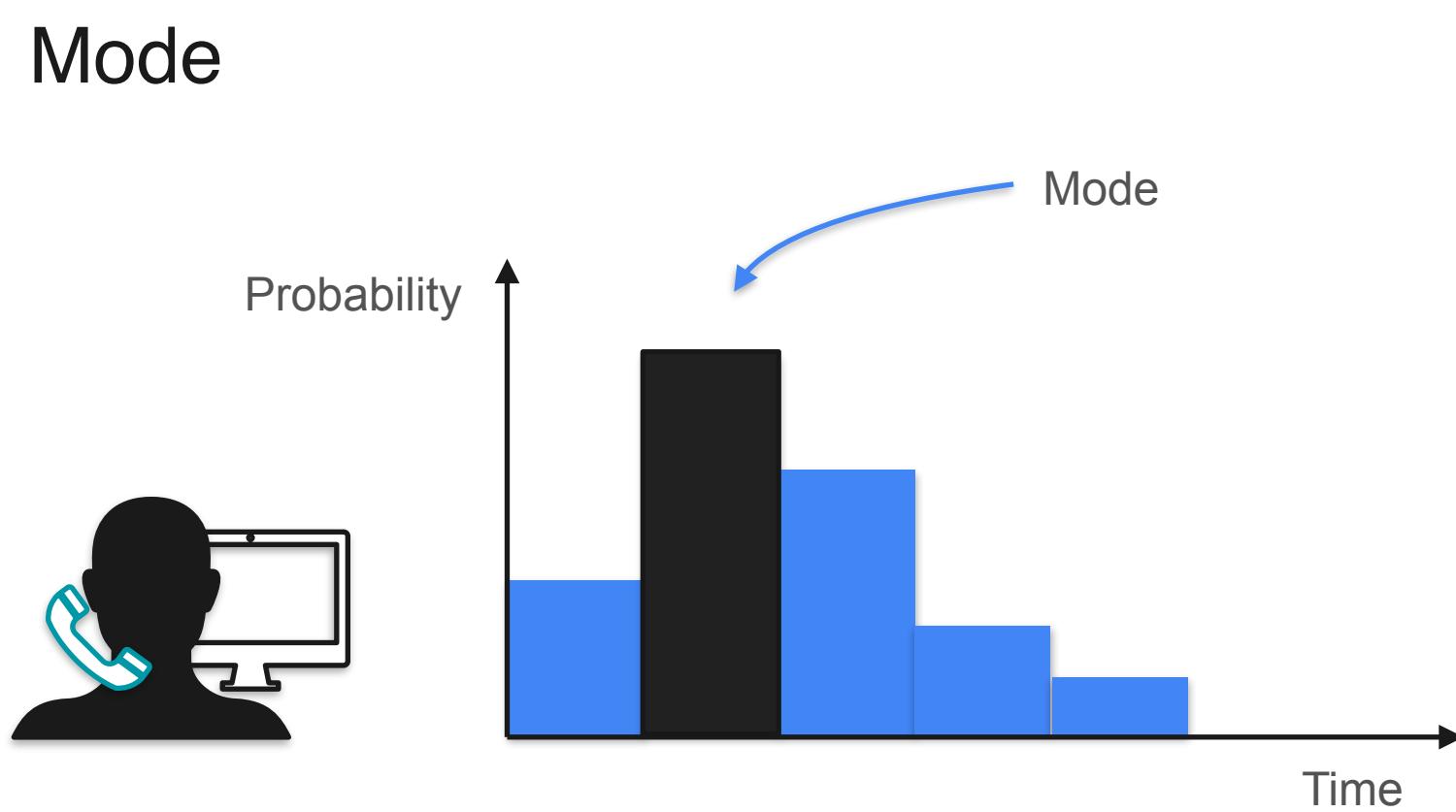


Median

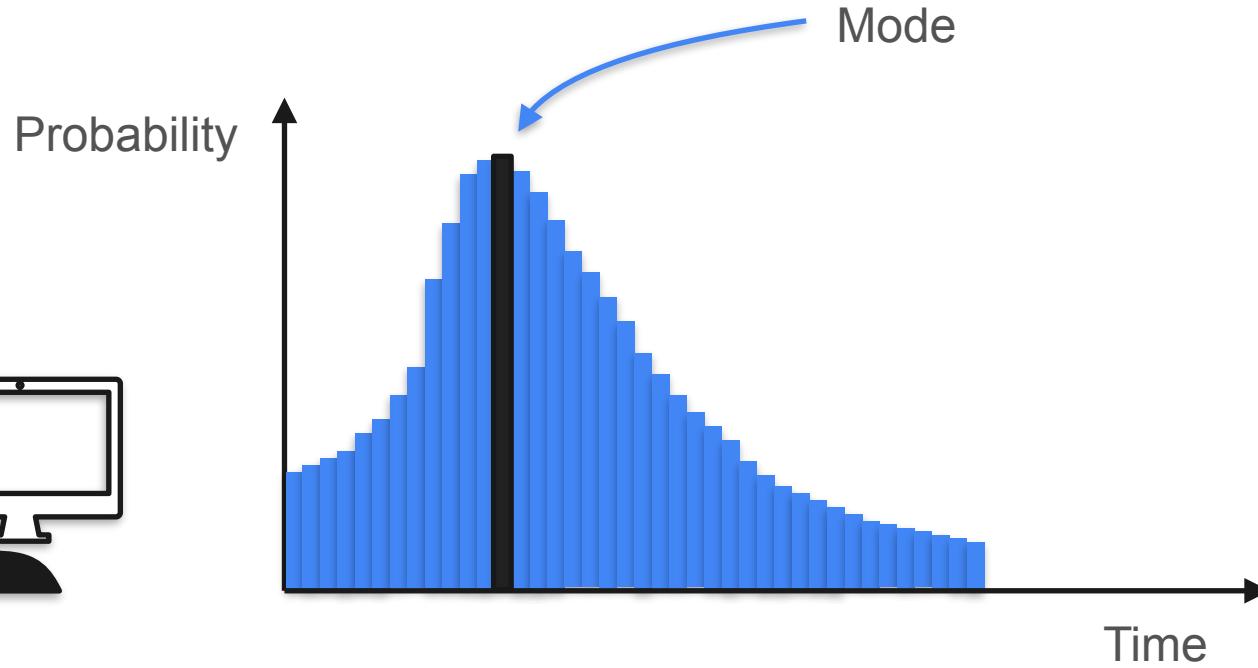
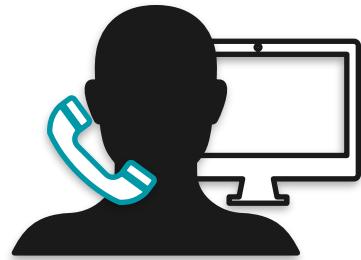


Median
Average of the
two middle ones

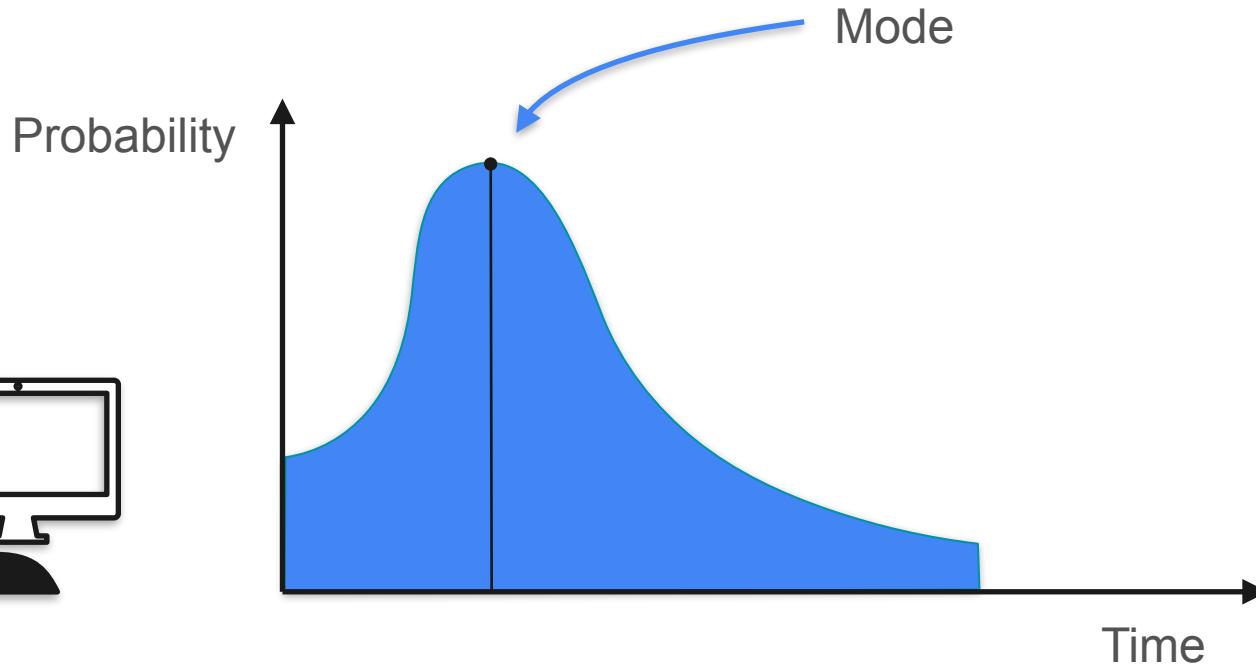
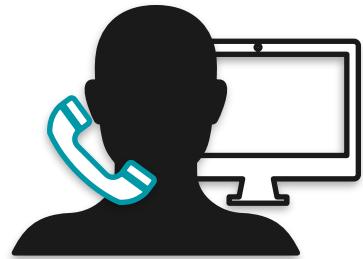
Mode



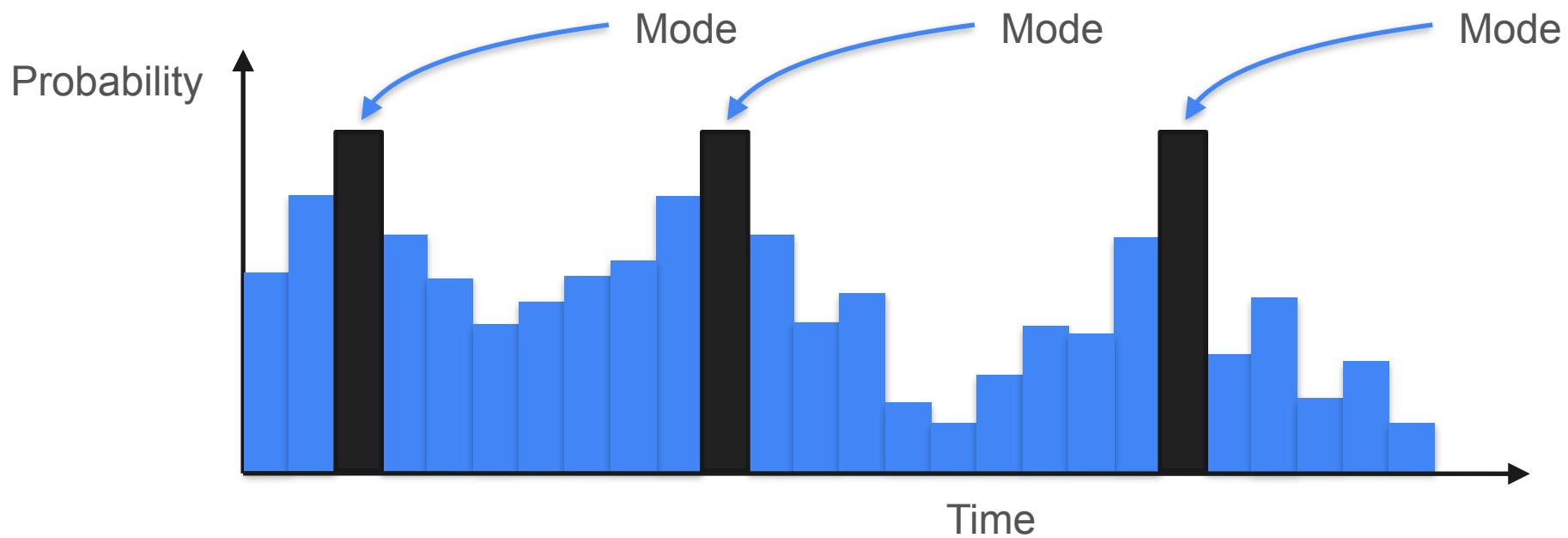
Mode



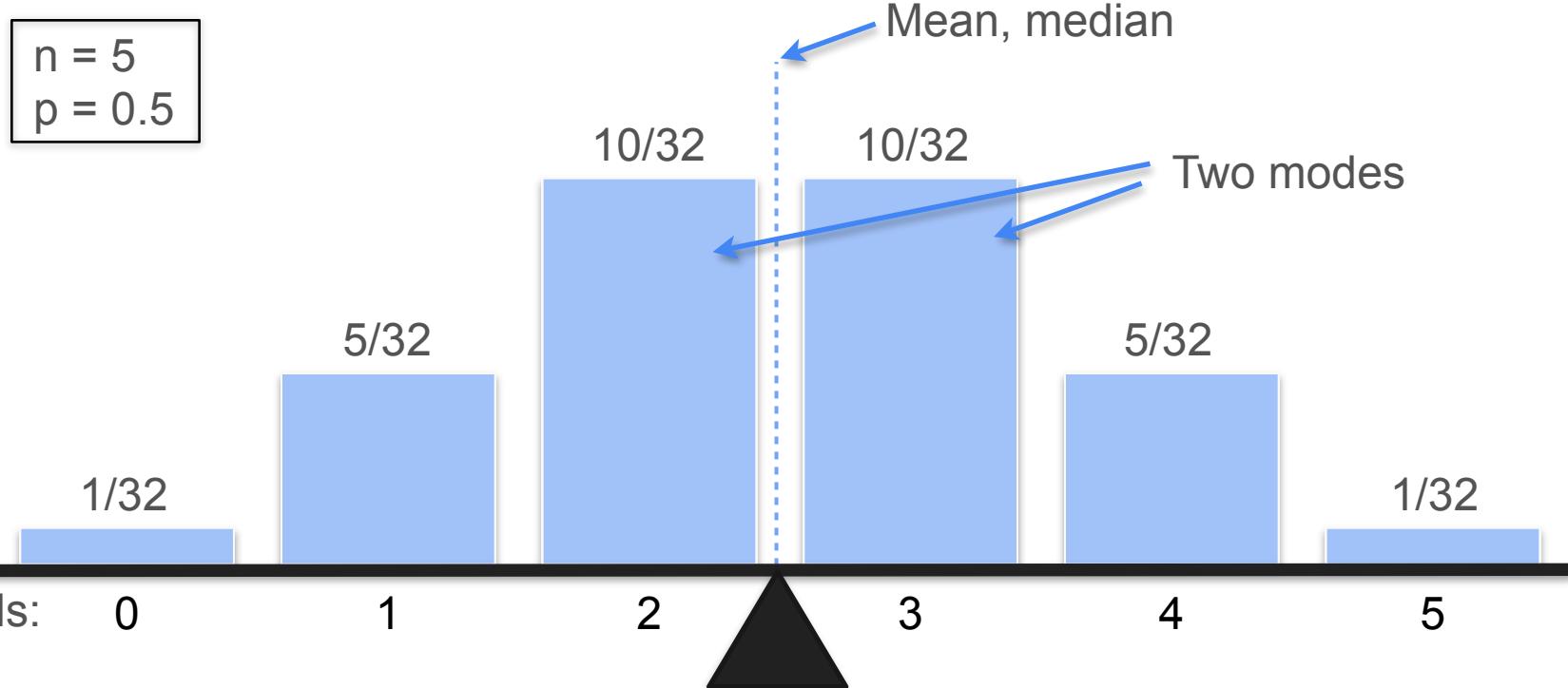
Mode



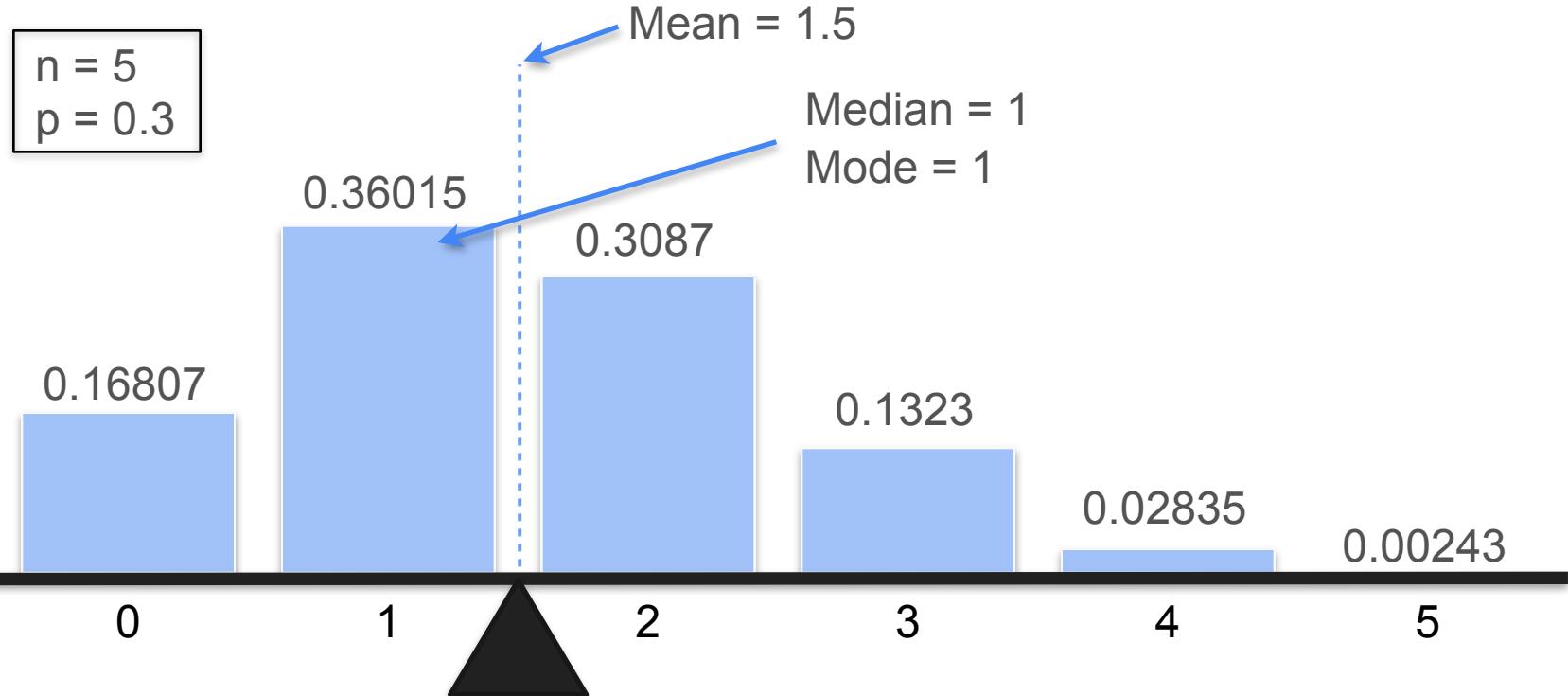
Mode: Multimodal Distribution



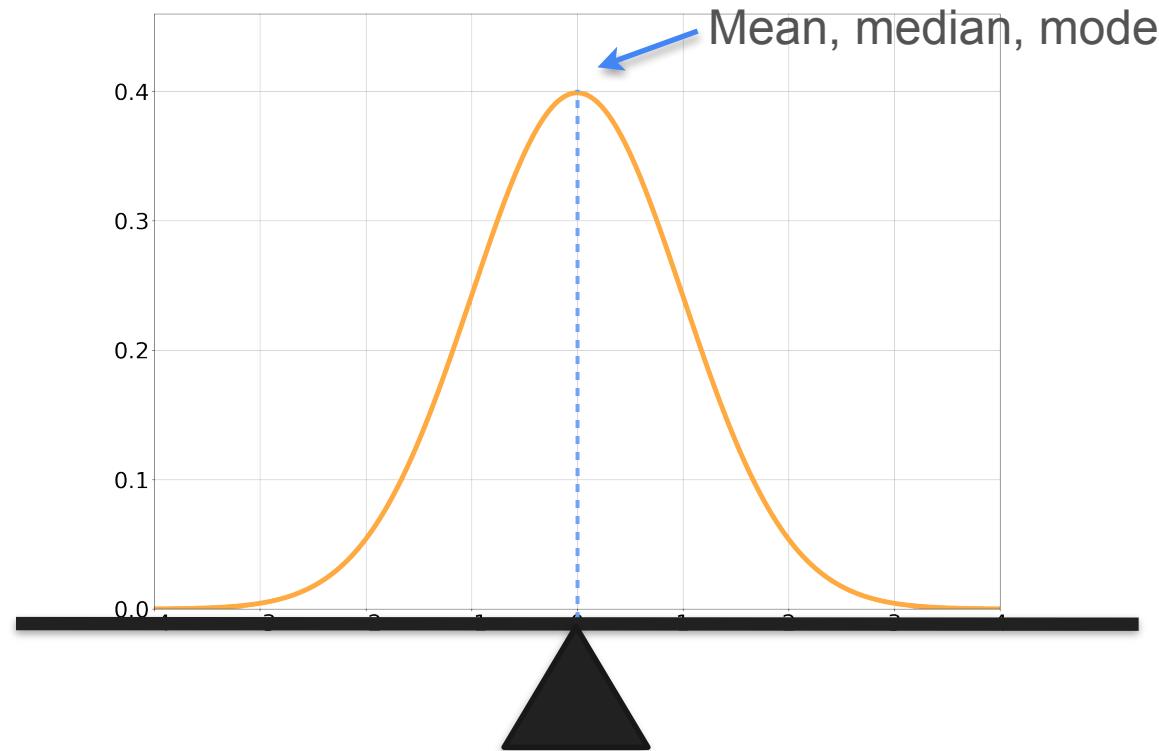
Mean, Median and Mode in Binomial Distribution



Mean, Median and Mode in Binomial Distribution



Mean, Median and Mode in Normal Distribution



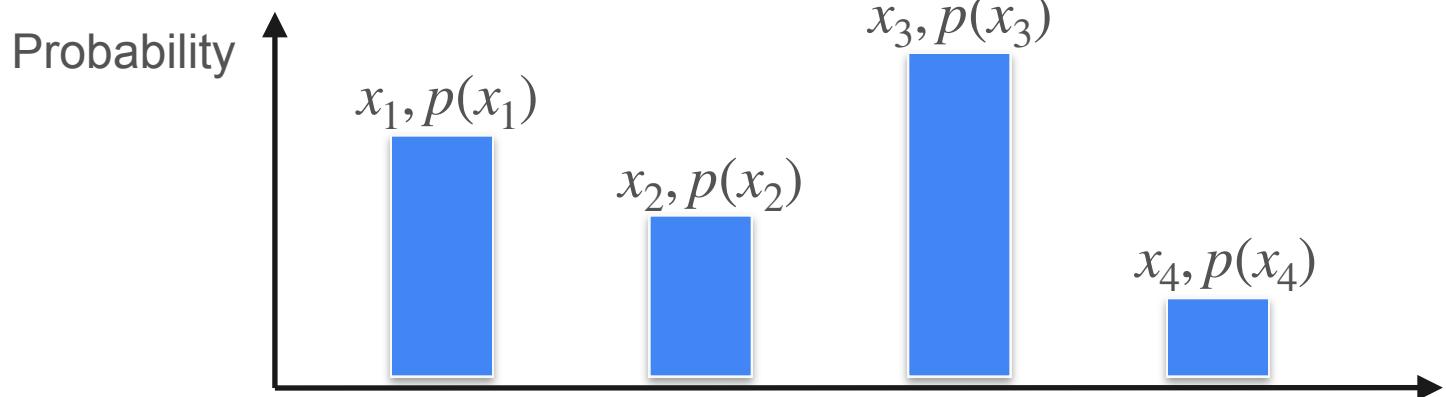


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Describing Distributions

Expected value of a function

Expected Value of a Function



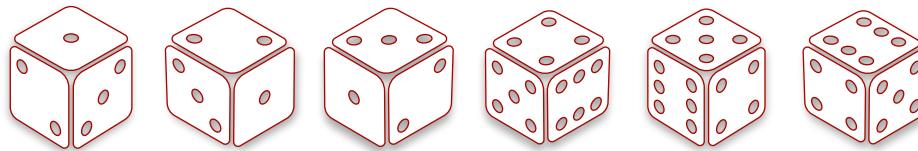
$$\mathbb{E}[X] = x_1p(x_1) + x_2p(x_2) + x_3p(x_3) + x_4p(x_4)$$

$$E[g(X)] = g(x_1)p(x_1) + g(x_2)p(x_2) + g(x_3)p(x_3) + g(x_4)p(x_4)$$

Expected Value of a Function

Probability: $1/6$ $1/6$ $1/6$ $1/6$ $1/6$ $1/6$

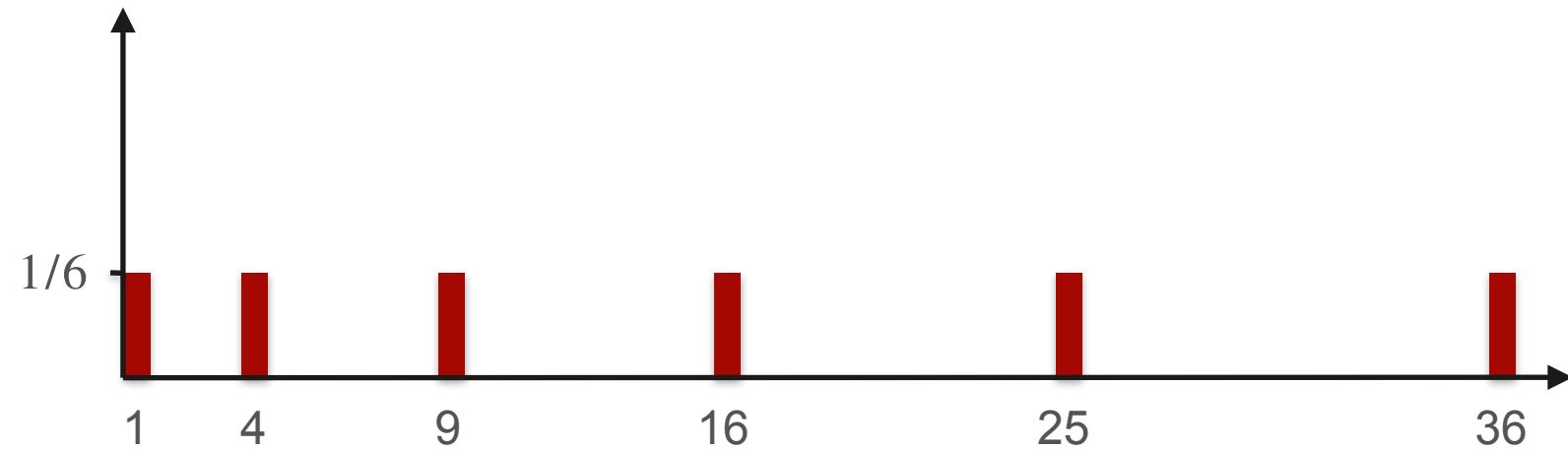
Roll: 1 2 3 4 5 6



Square: 1 4 9 16 25 36

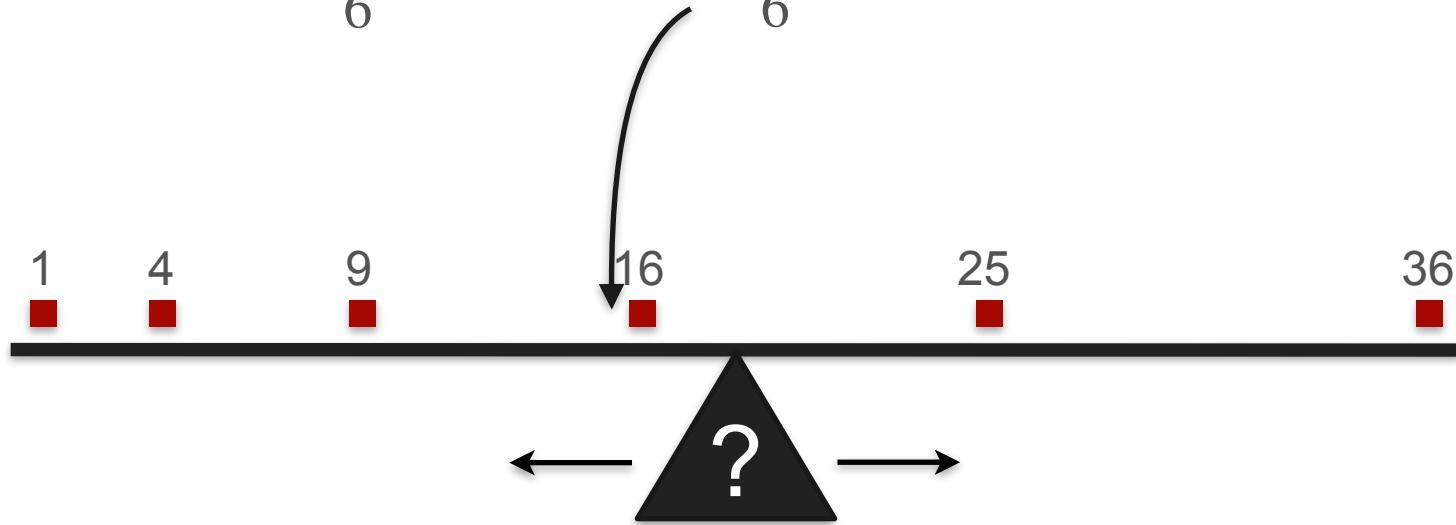
Expected Value of a Function

Probability



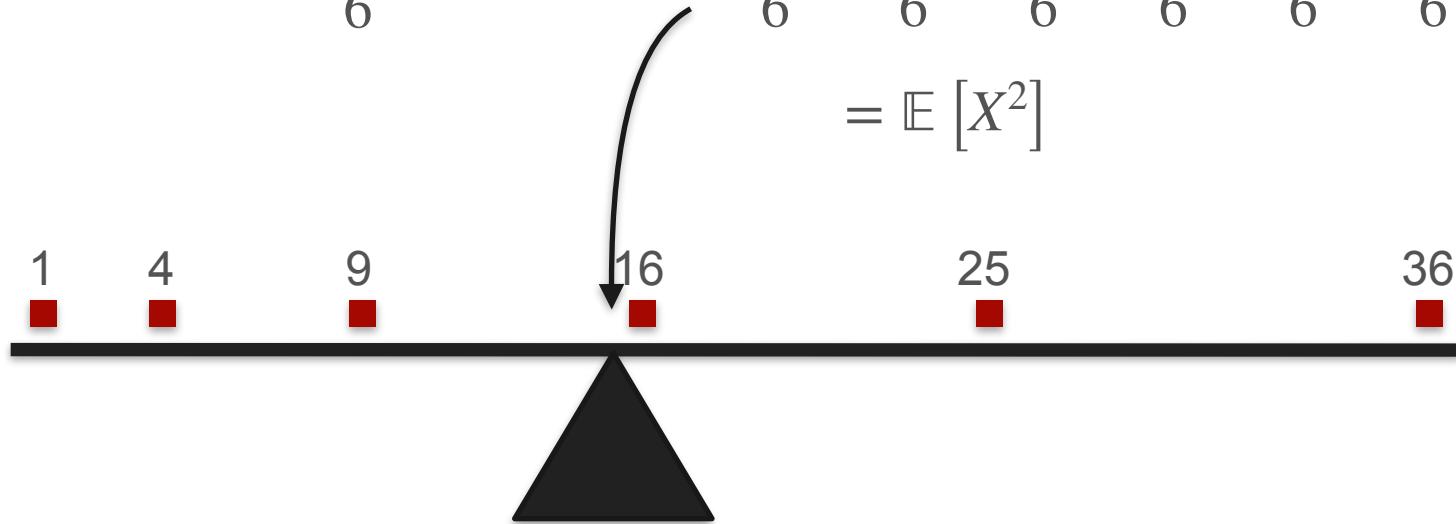
Expected Value of a Function

$$\frac{1 + 4 + 9 + 16 + 25 + 36}{6} = \frac{91}{6}$$



Expected Value of a Function

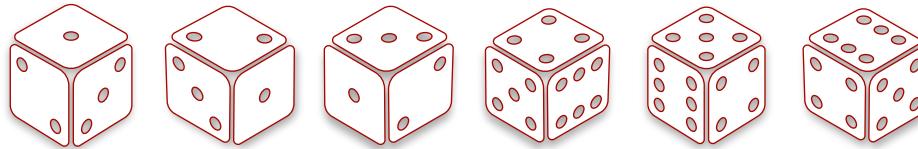
$$\frac{1 + 4 + 9 + 16 + 25 + 36}{6} = \frac{91}{6} = \frac{1^2}{6} + \frac{2^2}{6} + \frac{3^2}{6} + \frac{4^2}{6} + \frac{5^2}{6} + \frac{6^2}{6}$$
$$= \mathbb{E}[X^2]$$



Expectation of Linear Function

Probability: $1/6$ $1/6$ $1/6$ $1/6$ $1/6$ $1/6$

Roll: 1 2 3 4 5 6



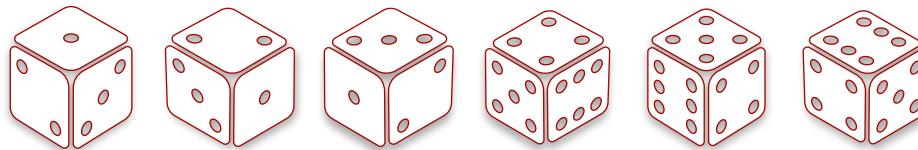
Double: 2 4 6 8 10 12

Wins 2 - 5 4 - 5 6 - 5 8 - 5 10 - 5 12 - 5

Expectation of Linear Function

Probability: $1/6$ $1/6$ $1/6$ $1/6$ $1/6$ $1/6$

Roll: 1 2 3 4 5 6

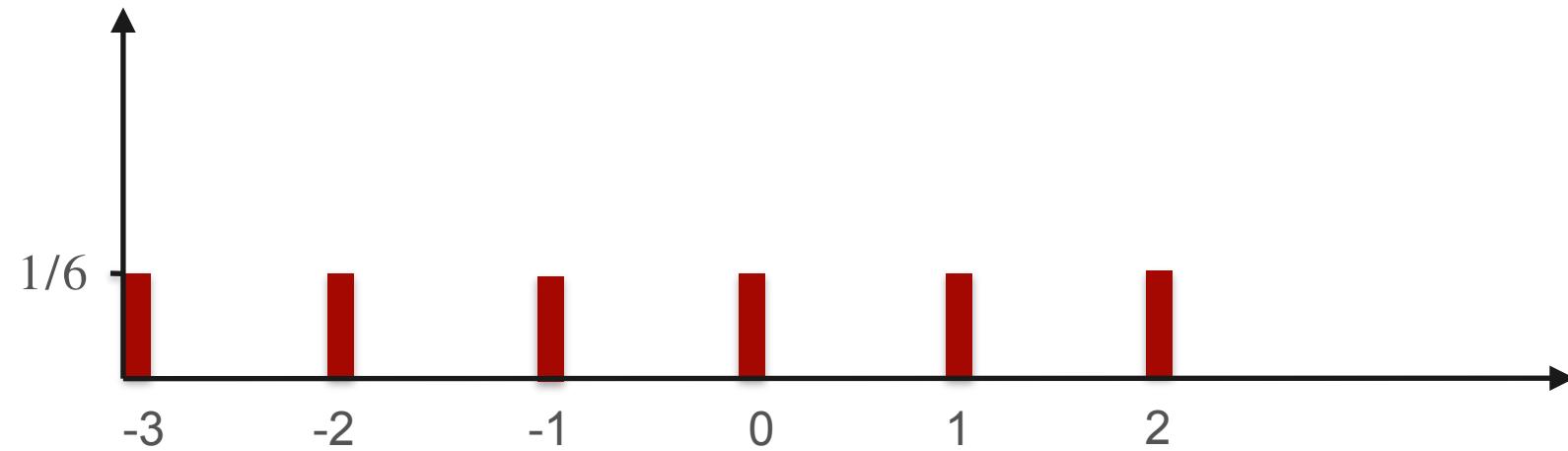


Double: 2 4 6 8 10 12

Wins -3 -2 -1 0 1 2

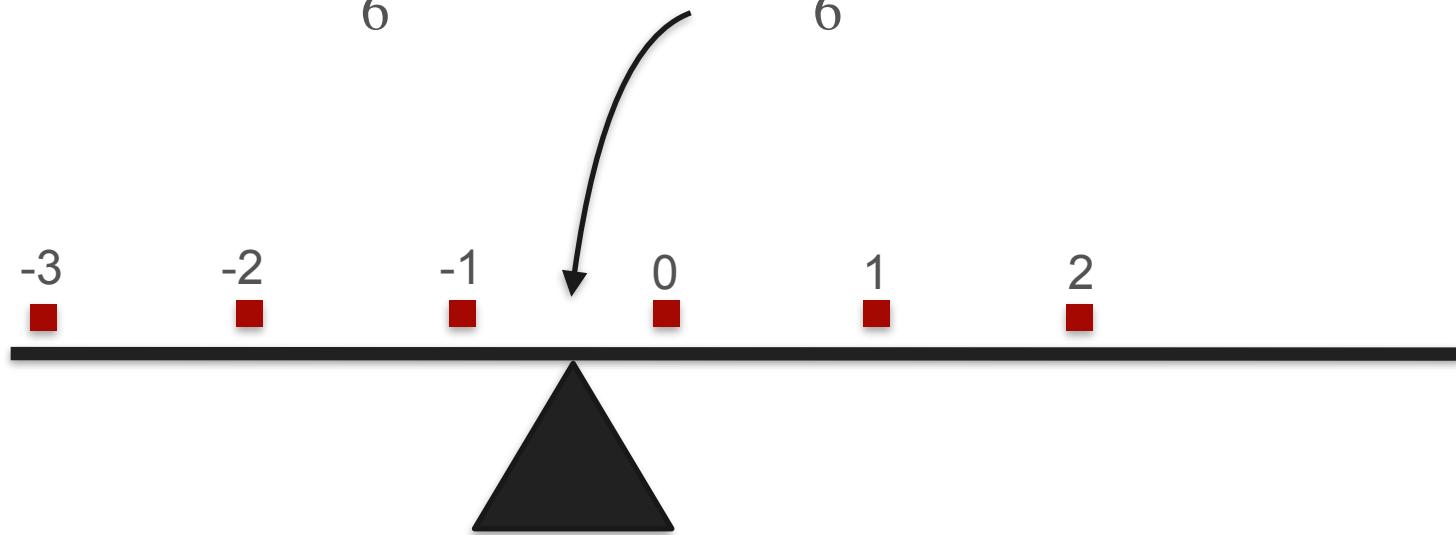
Expected Value of a Function

Probability



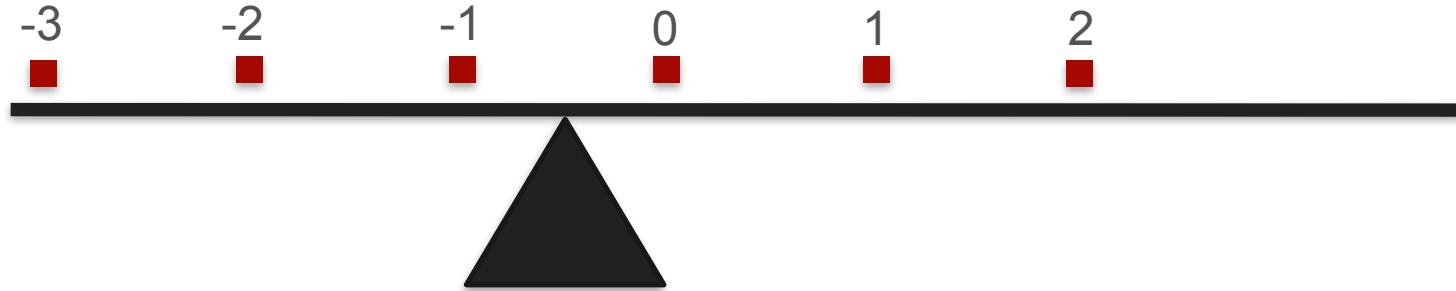
Expected Value of a Function

$$\frac{-3 + -2 + -1 + 0 + 1 + 2}{6} = \frac{-3}{6} = -0.5$$



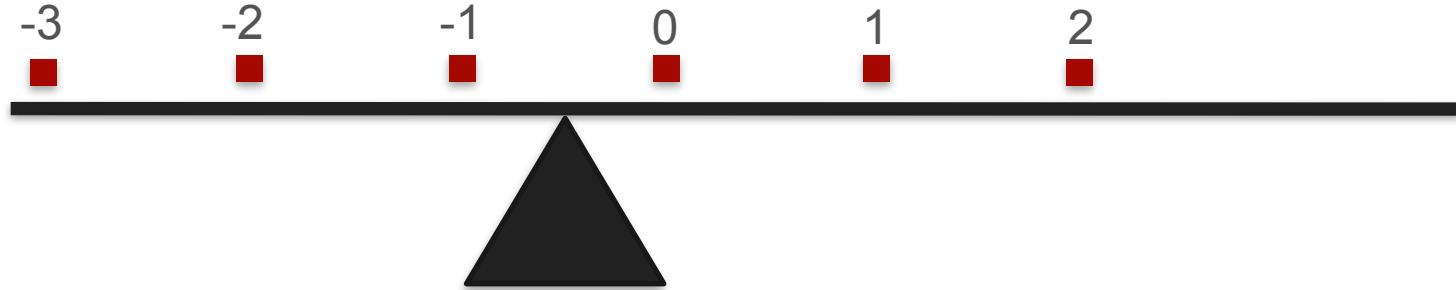
Expected Value of a Function

$$\frac{(2 \cdot 1 - 5) + (2 \cdot 2 - 5) + (2 \cdot 3 - 5) + (2 \cdot 4 - 5) + (2 \cdot 5 - 5) + (2 \cdot 6 - 5)}{6} = \frac{-3}{6} = -0.5$$



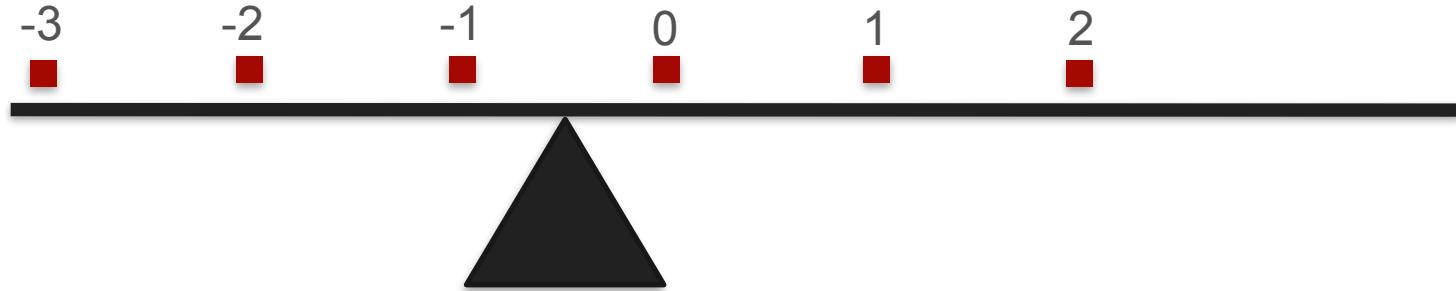
Expected Value of a Function

$$\frac{(2 \cdot 1) + (2 \cdot 2) + (2 \cdot 3) + (2 \cdot 4) + (2 \cdot 5) + (2 \cdot 6) + 6 \cdot (-5)}{6} = \frac{-3}{6} = -0.5$$



Expected Value of a Function

$$\frac{(2 \cdot 1) + (2 \cdot 2) + (2 \cdot 3) + (2 \cdot 4) + (2 \cdot 5) + (2 \cdot 6)}{6} + (-5) = \frac{-3}{6} = -0.5$$



Expected Value of a Function

$$\mathbb{E}[2 \cdot X + (-5)] = \frac{2 \cdot (1 + 2 + 3 + 4 + 5 + 6)}{6} + (-5) = \frac{-3}{6} = -0.5$$

$= 2 \cdot \mathbb{E}[X] + (-5)$



In general:

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$$

$$\mathbb{E}[aX] = a\mathbb{E}[X]$$

$$\mathbb{E}[b] = b$$



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Describing Distributions

Sum of expectations

Sum of Expectations

You play a game:

Flip a coin. If heads you win \$ 1, else you win nothing.

Then roll a die. You win the amount you roll.

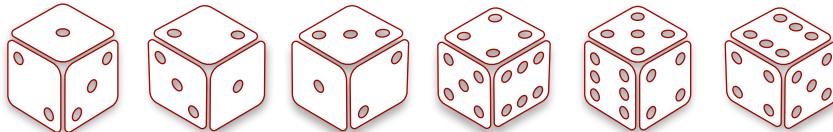


Win \$1



Win nothing

Win \$1 \$2 \$3 \$4 \$5 \$6



What are your expected winnings for the game?

Sum of Expectations

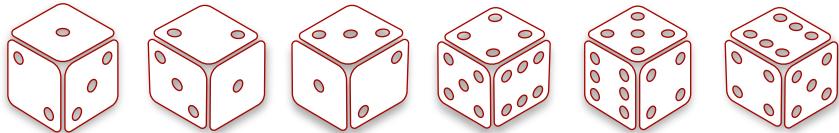


Win \$1



Win nothing

Win \$1 \$2 \$3 \$4 \$5 \$6



$$\mathbb{E}[X_{coin}] = \$0.5$$

$$\mathbb{E}[X_{dice}] = \$3.5$$

$$\mathbb{E}[X] = \mathbb{E}[X_{coin}] + \mathbb{E}[X_{dice}] = \$0.5 + \$3.5 = \$4$$

In general: $\mathbb{E}[X_1 + X_2] = \mathbb{E}[X_1] + \mathbb{E}[X_2]$

Sum of Expectations

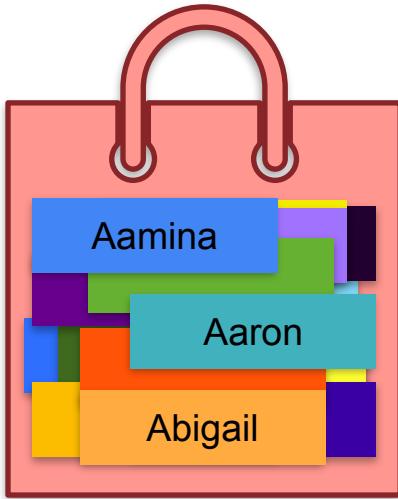


Expected number of
correct assignments?



8 billion people

Sum of Expectations



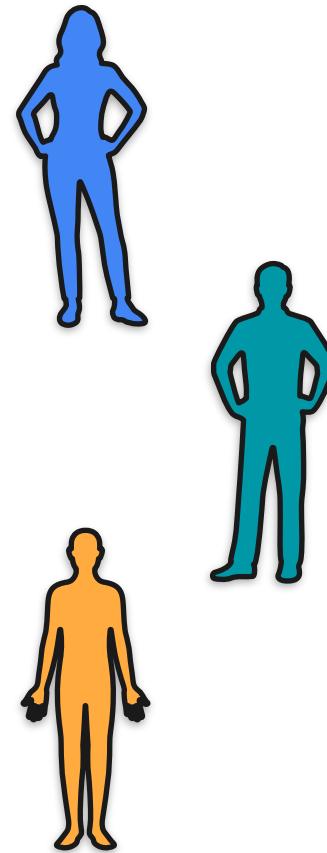
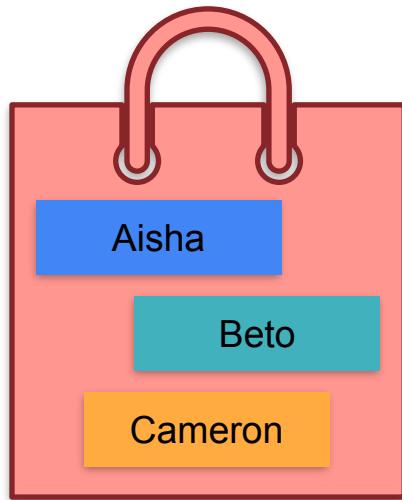
1

Expected number of
correct assignments?

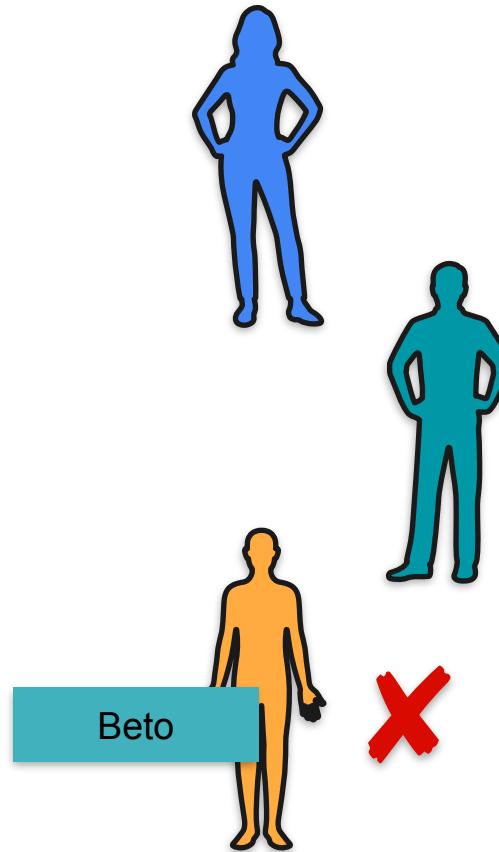
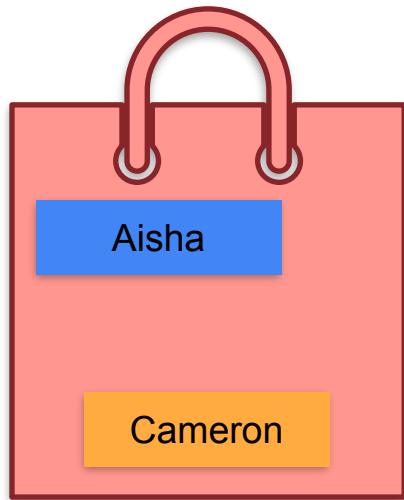


8 billion people

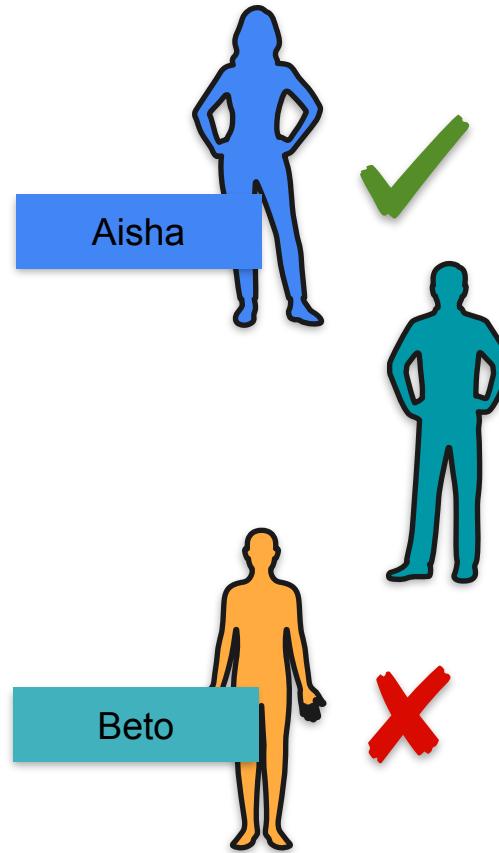
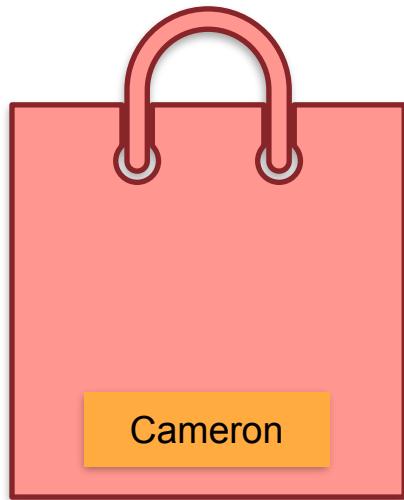
Sum of Expectations



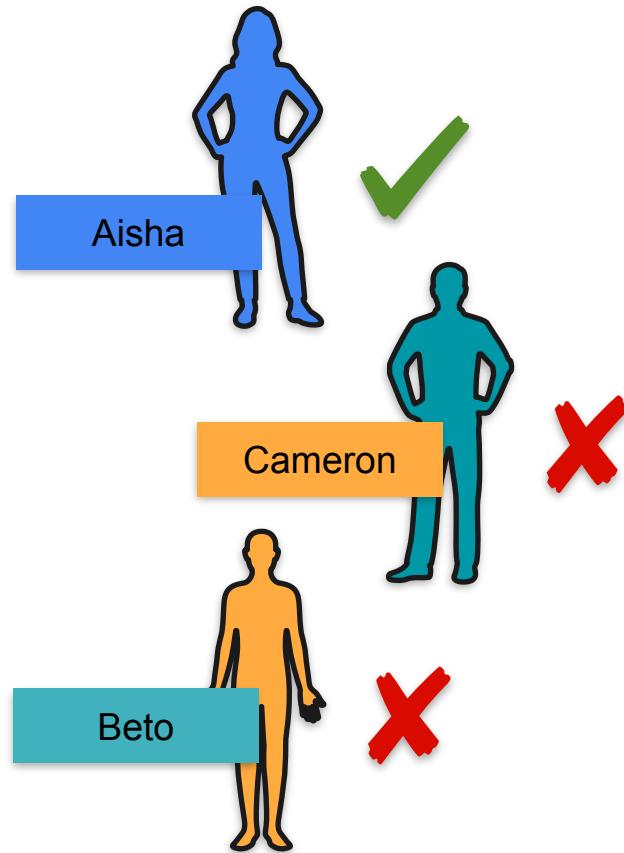
Sum of Expectations



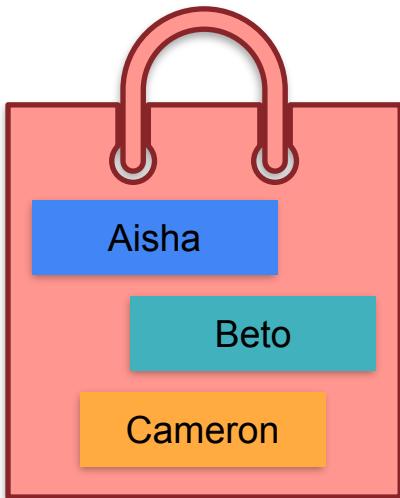
Sum of Expectations



Sum of Expectations



Sum of Expectations



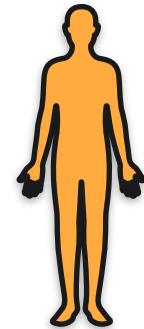
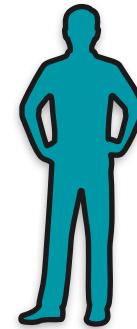
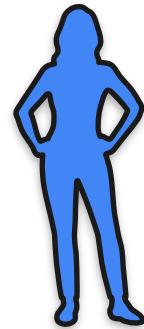
Average
1

Correct

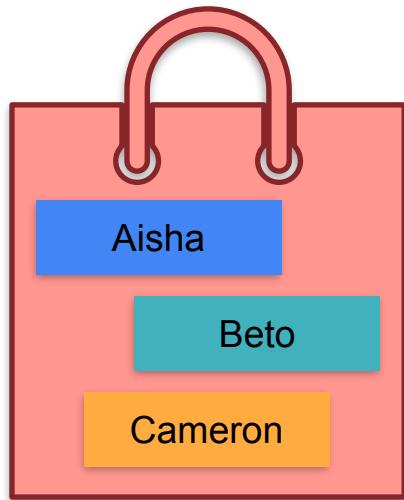
3
1
1
0
0
1

6

Aisha	Beto	Cameron
Aisha	Cameron	Beto
Beto	Aisha	Cameron
Beto	Cameron	Aisha
Cameron	Aisha	Beto
Cameron	Beto	Aisha

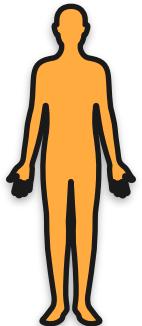
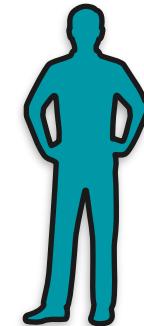


Sum of Expectations



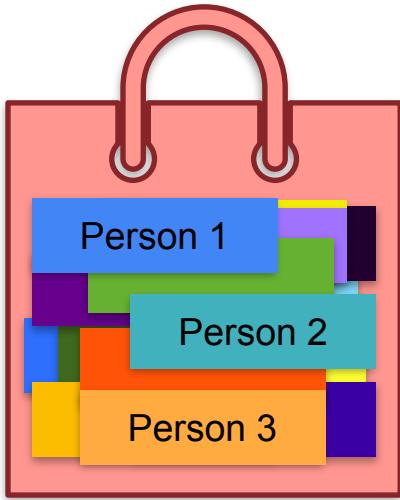
$\frac{1}{3}$
 $\frac{1}{3}$
 $\frac{1}{3}$

$$\begin{aligned}\mathbb{E}[\text{Matches}] &= \mathbb{E}[A] + \mathbb{E}[B] + \mathbb{E}[C] \\ &= \frac{1}{3} + \frac{1}{3} + \frac{1}{3} \\ &= 1\end{aligned}$$



Average
1

Sum of Expectations



Expected number = ?



8 billion people

Sum of Expectations



$$\mathbb{E} [\text{Matches}] = \mathbb{E} [X_{\text{person}_1}] + \mathbb{E} [X_{\text{person}_2}] + \dots + \mathbb{E} [X_{\text{person}_n}]$$

n people ($n = 8$ billion)

$$= \frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n}$$

In general:

$$\mathbb{E} [X_1 + X_2 + \dots + X_n] = \mathbb{E} [X_1] + \mathbb{E} [X_2] + \dots + \mathbb{E} [X_n]$$

$$= n \cdot \frac{1}{n} = 1$$



DeepLearning.AI

Describing Distributions

Variance

Variance Motivation: Fair Price To Play the Game

You play a game with a friend



You win 1 dollar



You lose 1 dollar

What is the fair amount of money to pay to play this game?

Variance Motivation: Fair Price To Play the Game

You play a game with a friend



You win 1 dollar



You lose 1 dollar

Game cost:

\$0

What is the fair amount of money to pay to play this game?

Variance Motivation: Fair Price To Play the Game

You play a game with a friend



You win 100 dollars



You lose 100 dollars

What is the fair amount of money to pay to play this game?

Variance Motivation: Fair Price To Play the Game

You play a game with a friend



You win 100 dollars



You lose 10

Game cost:

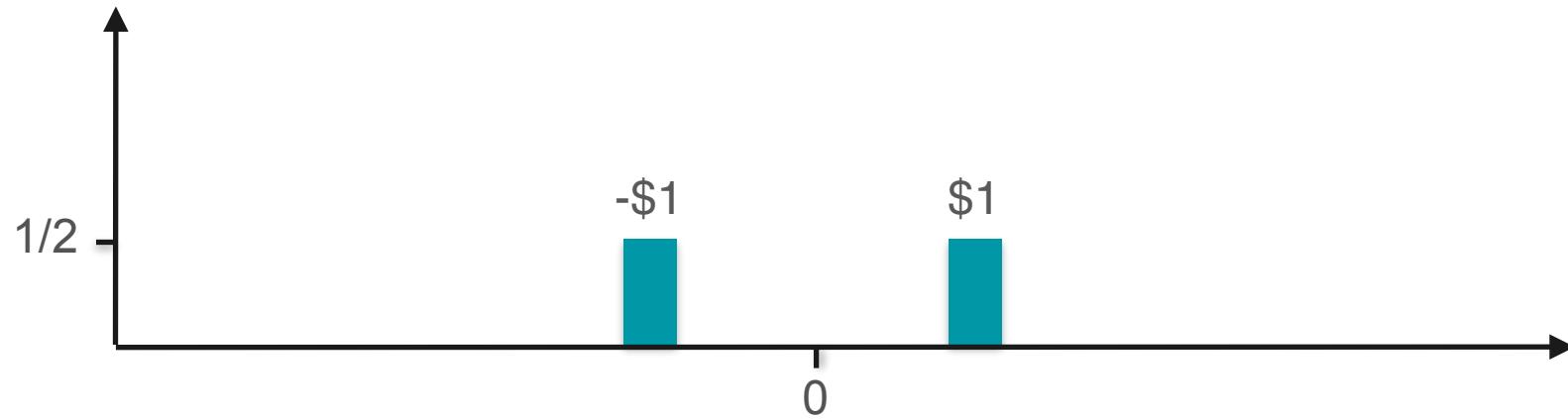
Variance!

0

What is the fair amount of money to pay to play this game?

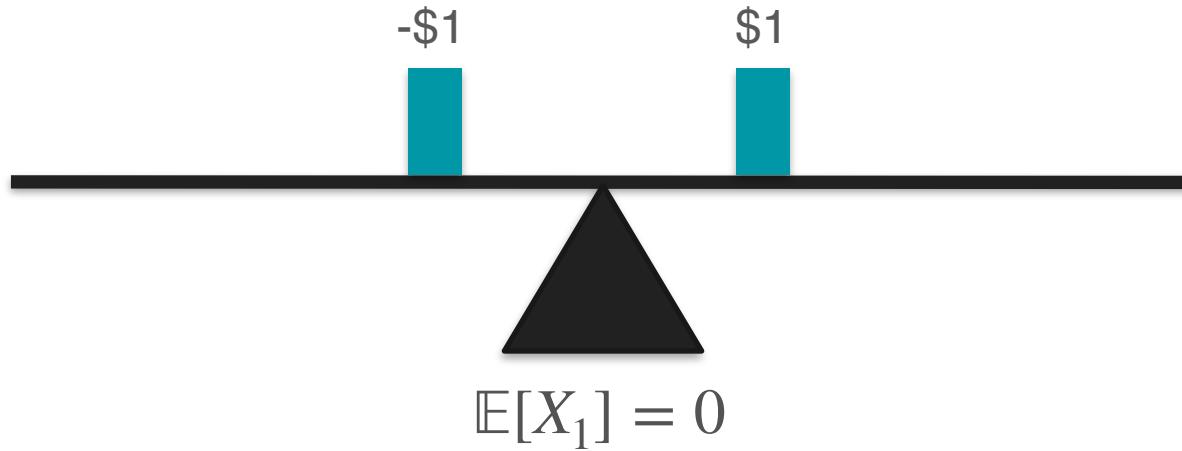
Variance Motivation: Measuring Spread

Probability



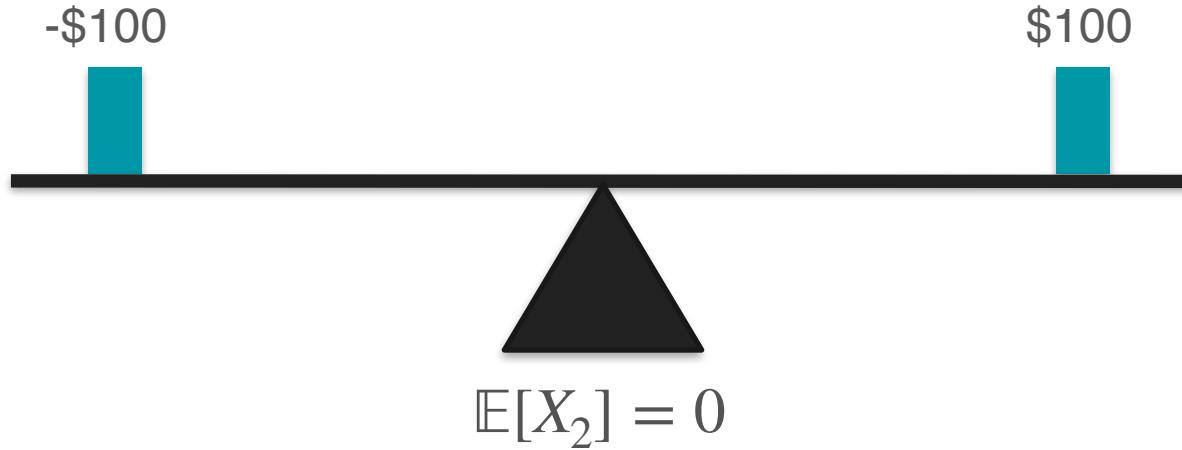
Variance Motivation: Measuring Spread

X_1 = amount of money gained in game 1

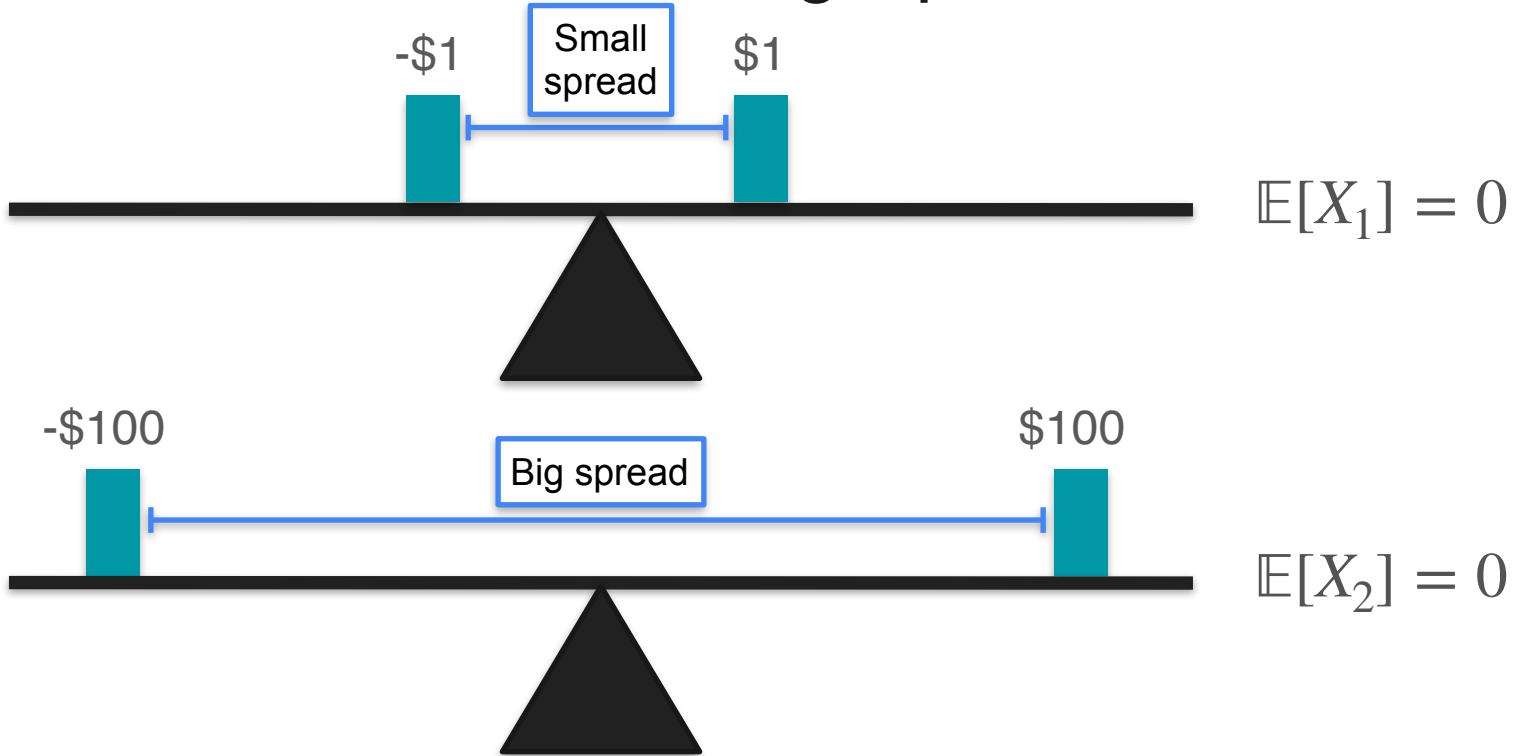


Variance Motivation: Measuring Spread

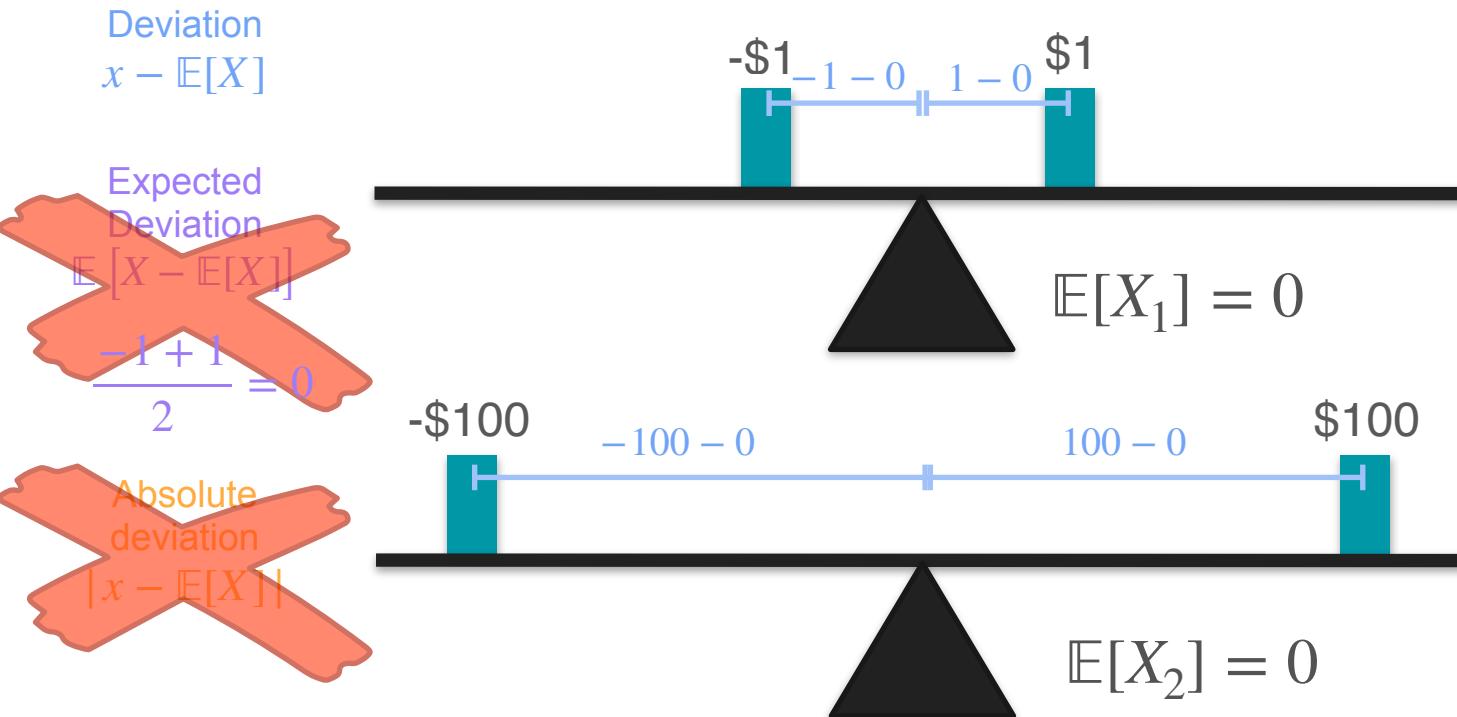
X_2 = amount of money gained in game 2



Variance Motivation: Measuring Spread

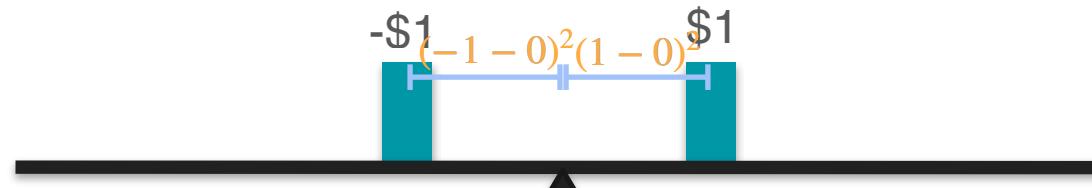


Variance Motivation: Measuring Spread



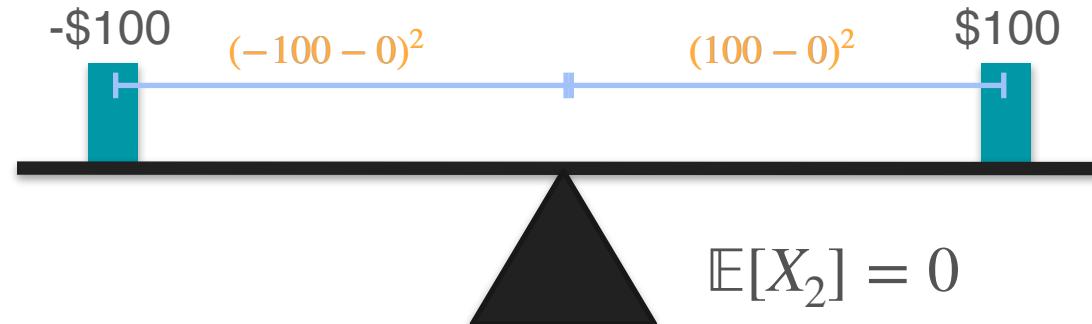
Variance Motivation: Measuring Spread

Deviation
 $x - \mathbb{E}[X]$



Squared
deviation
 $(x - \mathbb{E}[X])^2$

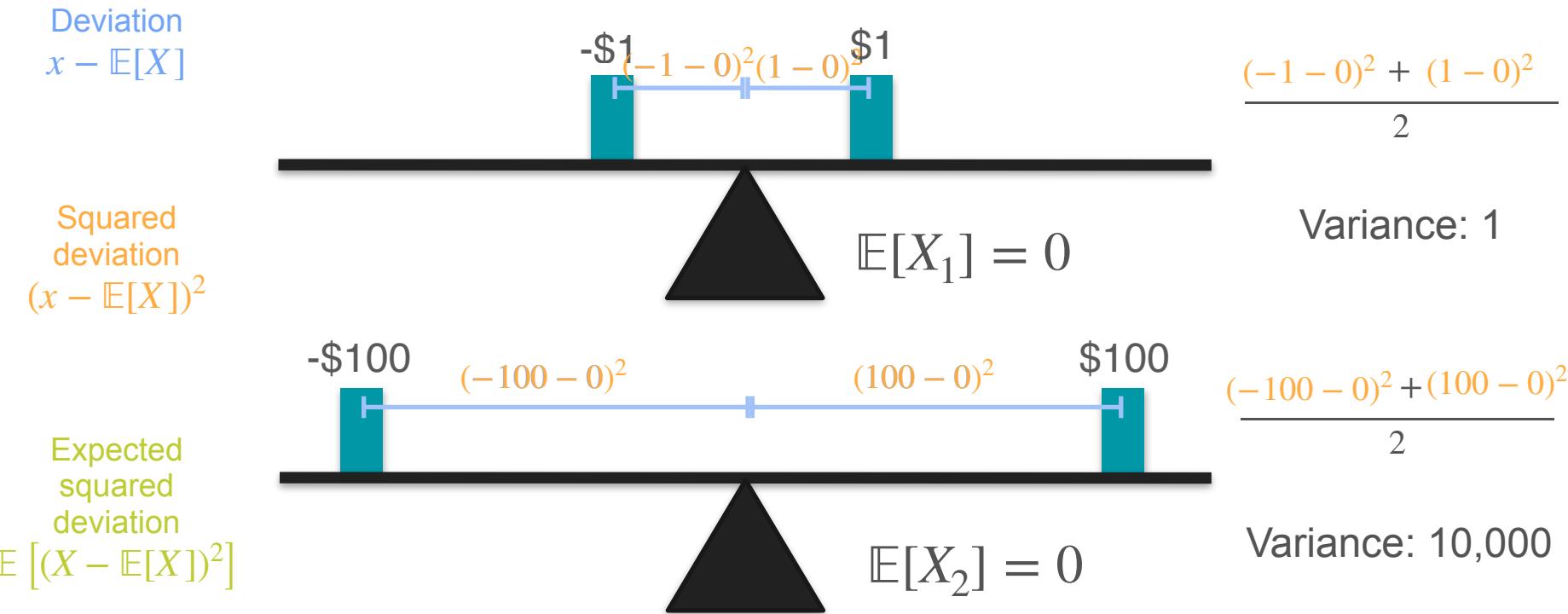
$$\mathbb{E}[X_1] = 0$$



Expected
squared
deviation
 $\mathbb{E}[(X - \mathbb{E}[X])^2]$

$$\mathbb{E}[X_2] = 0$$

Variance Motivation: Measuring Spread



Variance Formula

$$\text{Variance} = \mathbb{E} [(X - \mathbb{E}[X])^2]$$

1. Find X's mean
2. Find the deviation from that mean for every value of X
3. Square those deviations
4. Average those squared deviations

“Average squared deviation”

Variance Motivation: Centering With Mean

Game 1



You win 2 dollars



You lose 2 dollars

Game 2



You win 3 dollars



You lose 1 dollar

Which of these games has greater variance?

Hint: Think of the spread

Variance Motivation: Centering With Mean

Game 1



You win 2 dollars



You lose 2 dollars

Game 2



You win 3 dollars



You lose 1 dollar

They have the same variance

Variance Motivation: Centering With Mean

Game 1



You win 2 dollars

$$\mathbb{E}[X_1] = \frac{1}{2} \cdot (-2) + \frac{1}{2} \cdot 2 = 0$$

You lose 2 dollars

Game 2



You win 3 dollars

$$\mathbb{E}[X_2] = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 3 = 1$$

You lose 1 dollar

Variance Motivation: Centering With Mean

Game 1



You win 2 dollars

$$\mathbb{E}[X_1] = 0$$

$$\frac{1}{2}(-2 - \mathbb{E}[X_1])^2 + \frac{1}{2}(2 - \mathbb{E}[X_1])^2 = 4$$

You lose 2 dollars

Game 2



You win 3 dollars

Different price, but same spread

$$\mathbb{E}[X_2] = 1$$

$$\frac{1}{2}(-1 - \mathbb{E}[X_2])^2 + \frac{1}{2}(3 - \mathbb{E}[X_2])^2 = 4$$

You lose 1 dollar

Variance Formula

$$\begin{aligned}Var(X) &= \mathbb{E}[(X - \mathbb{E}[X])^2] \\&= \mathbb{E}[X^2] - \mathbb{E}[X]^2\end{aligned}$$

Variance Formula

$$\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2 - 2\mathbb{E}[X]X + \mathbb{E}[X]^2]$$

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

$$= \mathbb{E}[X^2] - \mathbb{E}[2\mathbb{E}[X]X] + \mathbb{E}[\mathbb{E}[X]^2]$$

$$= \mathbb{E}[X^2] - 2\mathbb{E}[X]\mathbb{E}[X] + \mathbb{E}[X]^2$$

$$= \mathbb{E}[X^2] - 2\mathbb{E}[X]^2 + \mathbb{E}[X]^2$$

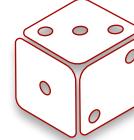
$$= \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$\mathbb{E}[\text{constant} \cdot X] = \text{constant} \cdot \mathbb{E}[X]$

$\mathbb{E}[X]$ is a constant

$\mathbb{E}[\text{constant}] = \text{constant}$

Properties of the Variance

Probability:	1/6	1/6	1/6	1/6	1/6	1/6
Roll:	1	2	3	4	5	6
						
Win Double:	\$2	\$4	\$6	\$8	\$10	\$12
Net Amount:	-\$3	-\$1	\$1	\$3	\$5	\$7

Properties of the Variance

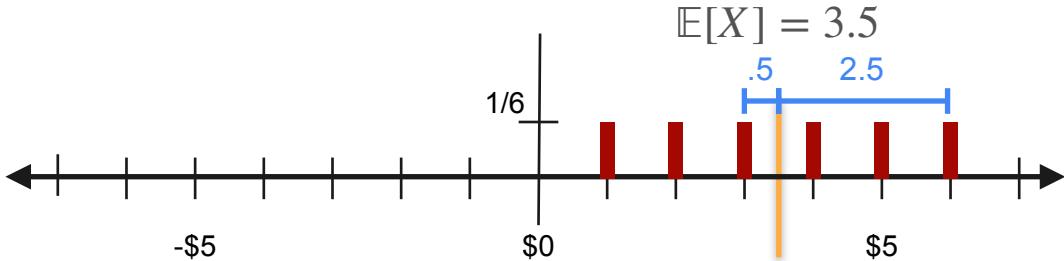
changes the spread

$$\text{Var}(aX + b) = a^2 \text{Var}(X)$$

doesn't change the spread

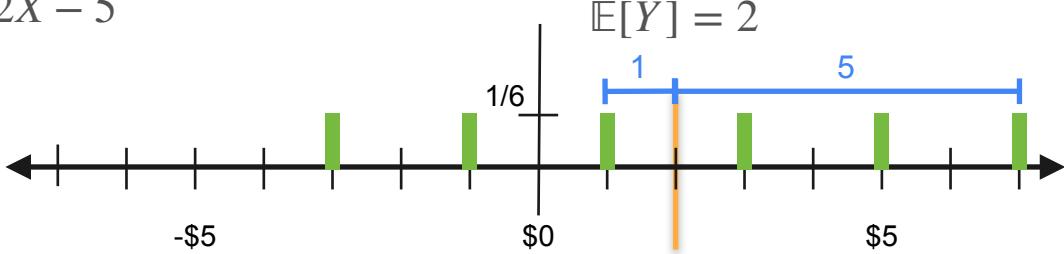
Variance: "Average squared deviation"

Dice roll is random variable: X



Net winnings is random variable: $Y = 2X - 5$

$$\text{Var}(Y) = \text{Var}(2X - 5) = 4\text{Var}(X)$$





DeepLearning.AI

Describing Distributions

Standard deviation

Standard Deviation

$$\text{Var}(X) = \mathbb{E}[(X - \mu)^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

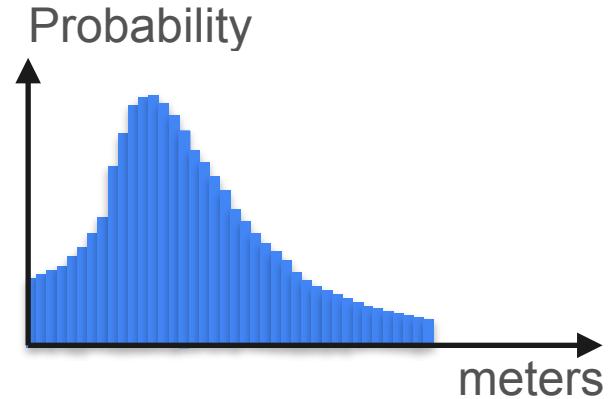
Say X is measured in meters.

Then $\mathbb{E}[X]$ is measured in meters.

Then $\text{Var}(X)$ is measured in meters².

Then $\sqrt{\text{Var}(X)}$ is measured in meters.

Let's call $std(X) = \sqrt{\text{Var}(X)}$, the *standard deviation* of X

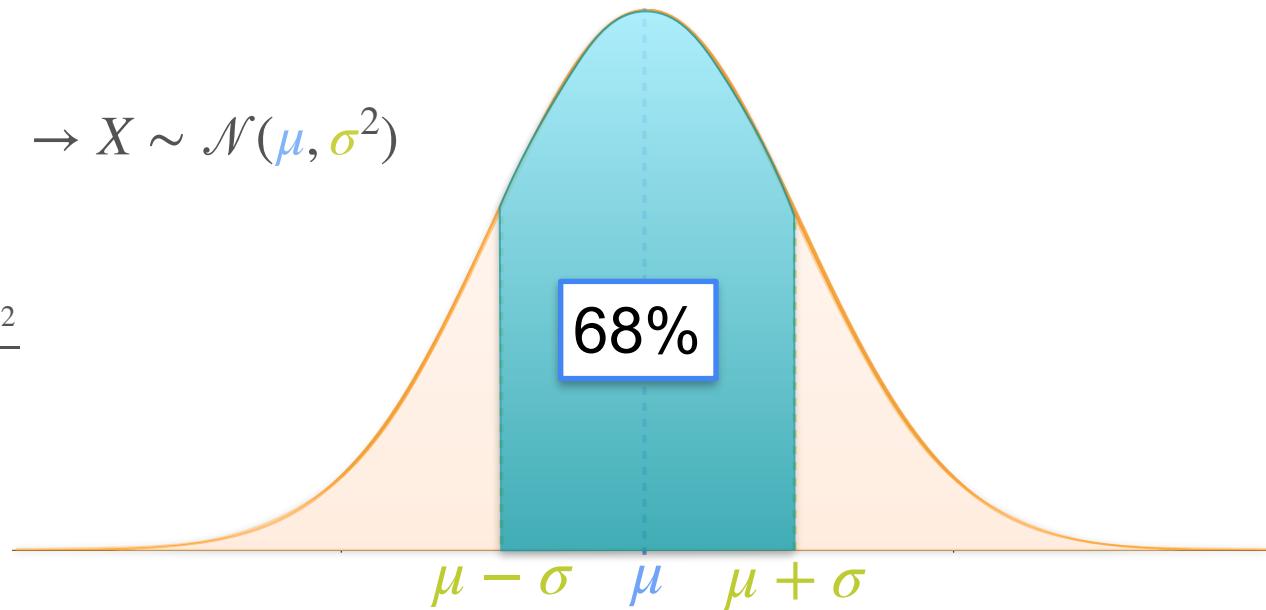


Normal Distribution: 68-95-99.7 Rule

Parameters:

- μ : center of the bell
 - σ : spread of the bell
- $$\rightarrow X \sim \mathcal{N}(\mu, \sigma^2)$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

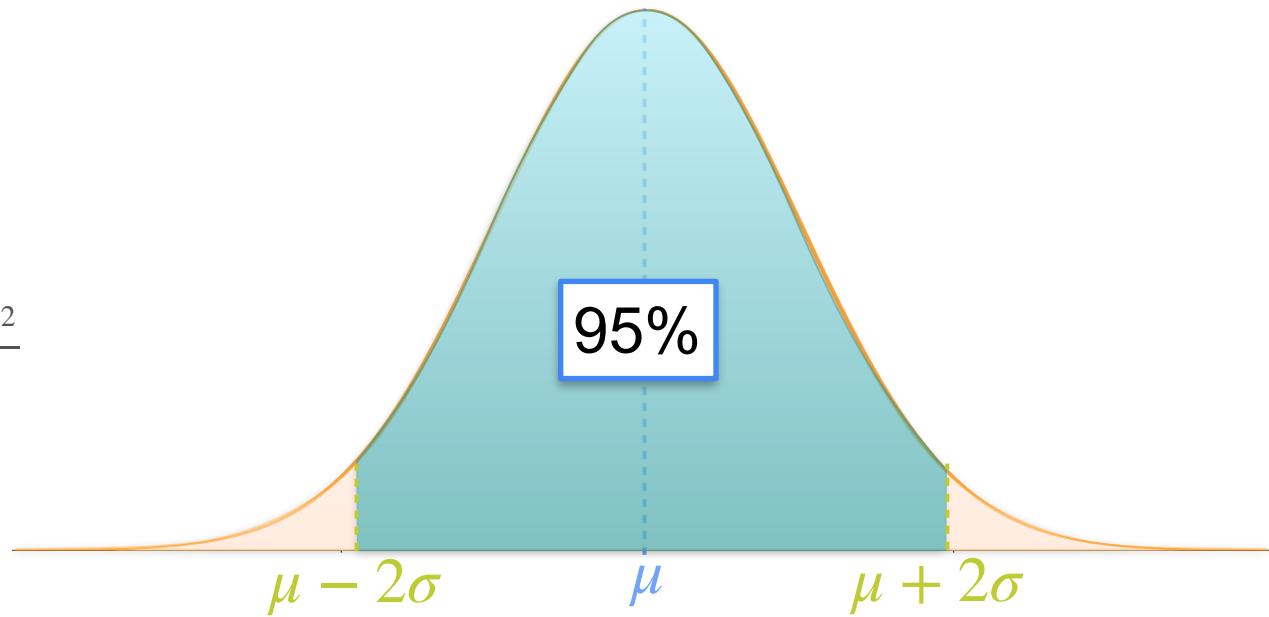


Normal Distribution: 68-95-99.7 Rule

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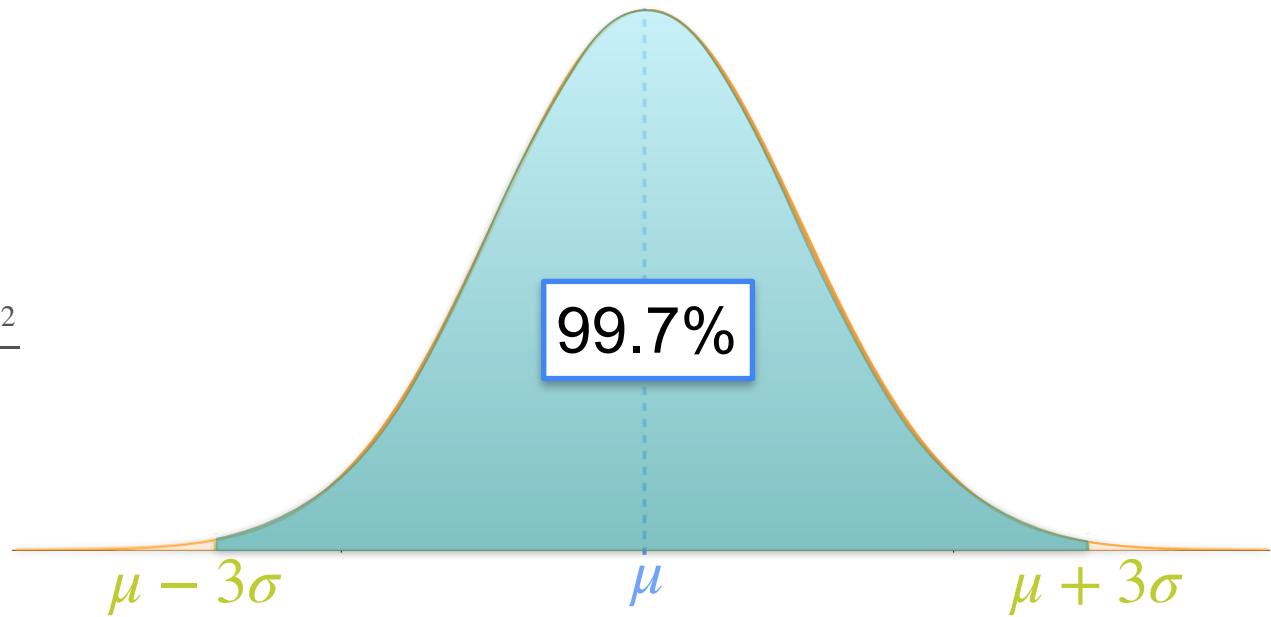


Normal Distribution: 68-95-99.7 Rule

Parameters:

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- σ : spread of the bell

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

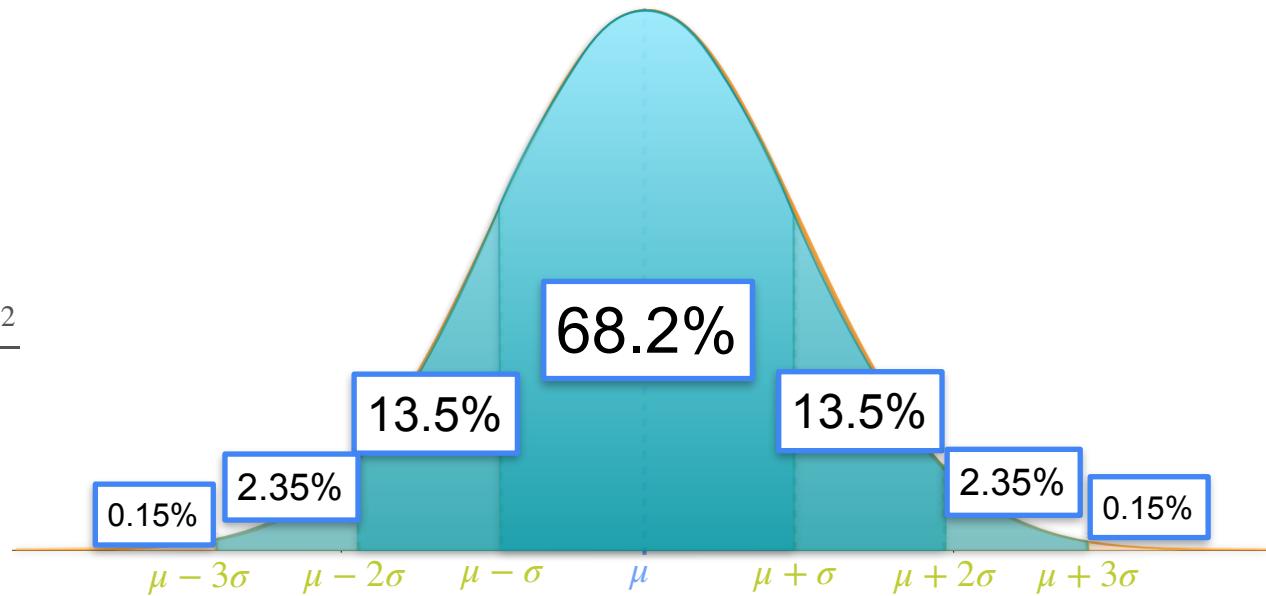


Normal Distribution: 68-95-99.7 Rule

Parameters:

- μ : center of the bell
- σ : spread of the bell

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$



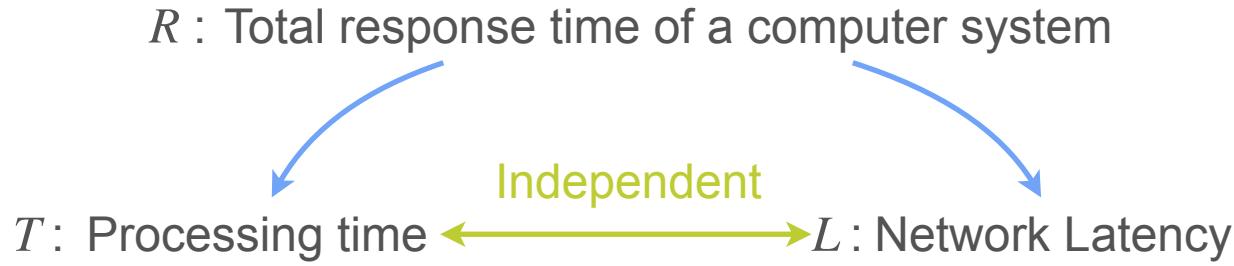


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Describing Distributions

Sum of Gaussians

Sum of Gaussians: an Example

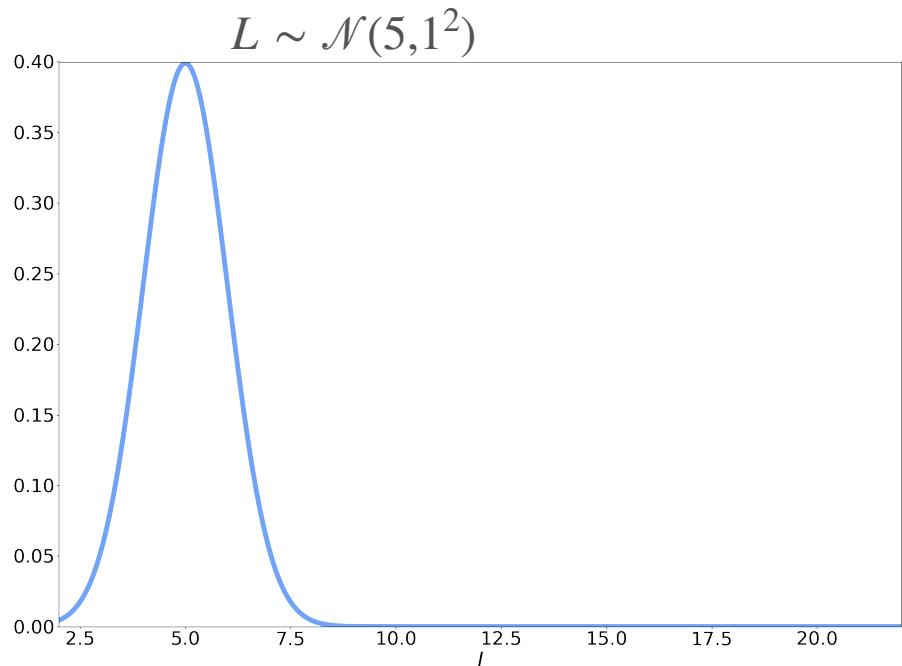
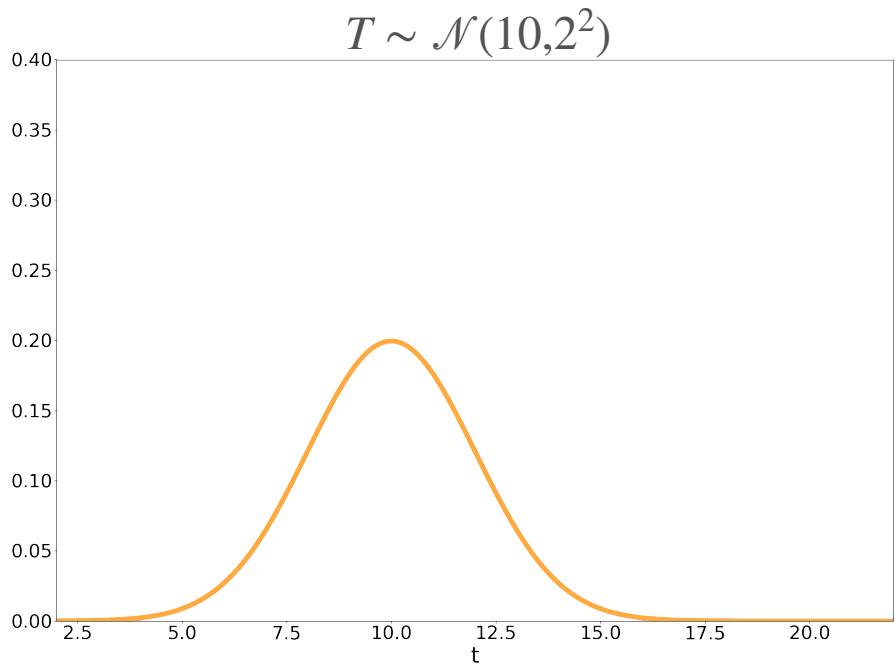


$$T \sim \mathcal{N}(10, 2^2)$$

$$L \sim \mathcal{N}(5, 1^2)$$

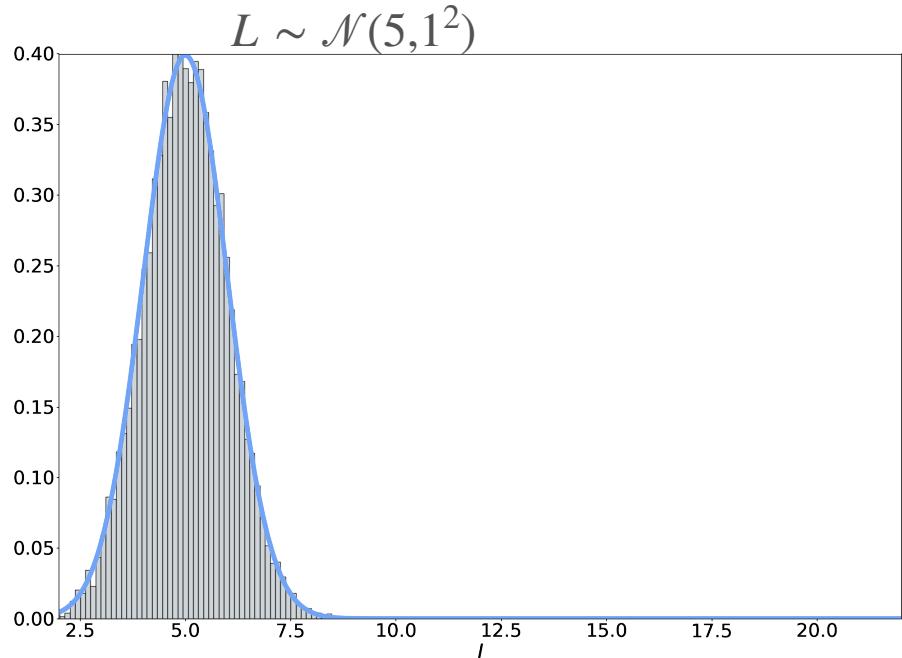
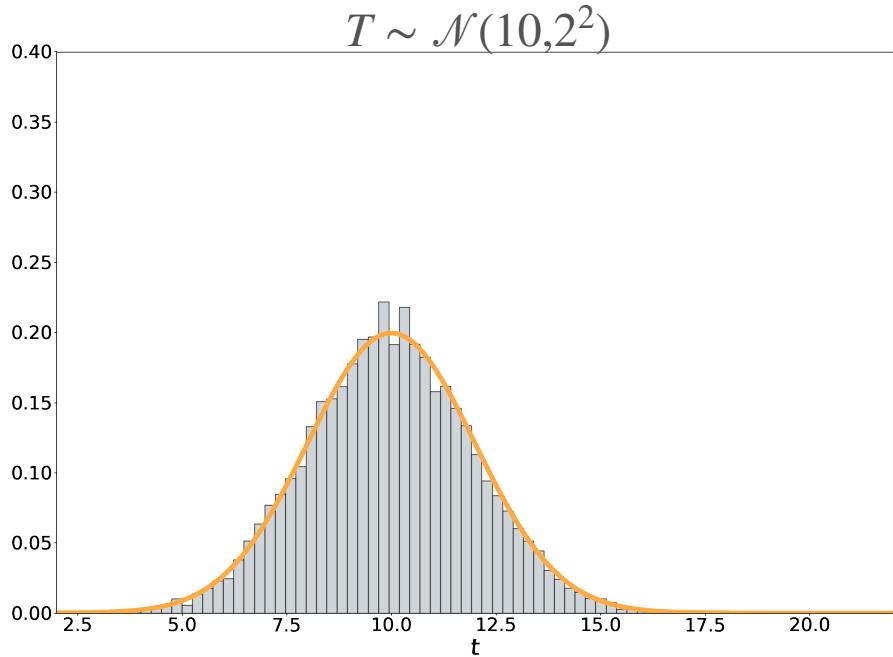
$$R = T + L$$

Sum of Gaussians



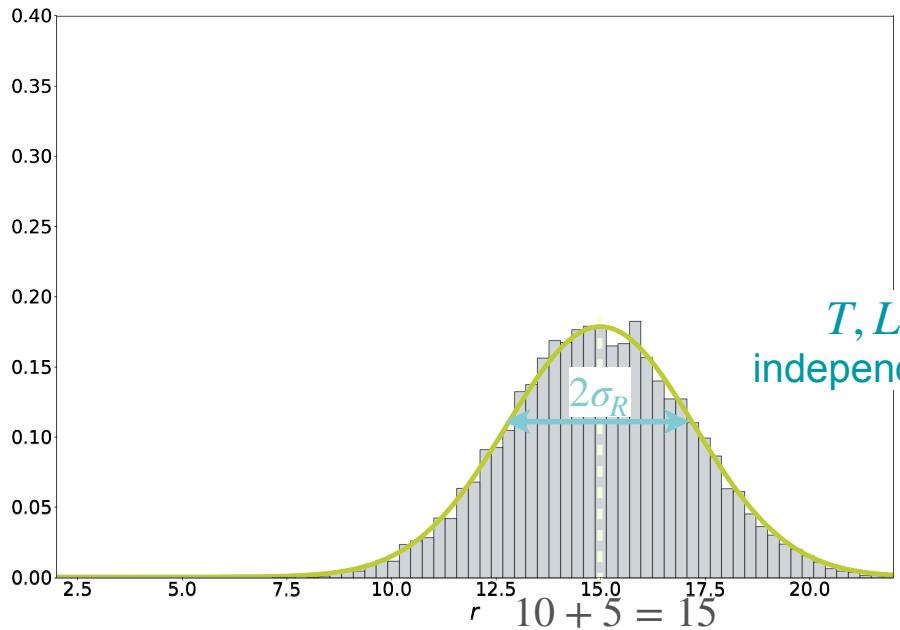
Sum of Gaussians

Sample each variable 10000 times



Sum of Gaussians

$$R = T + L$$



R is still Gaussian!

Expectation is linear

$$\begin{aligned}\mu_R &= \mathbb{E}[R] = \mathbb{E}[T + L] = \mathbb{E}[T] + \mathbb{E}[L] \\ &= \mu_T + \mu_L = 10 + 5 = 15\end{aligned}$$

$$\begin{aligned}\sigma_R^2 &= \text{Var}(R) = \text{Var}(T + L) \\ &= \text{Var}(T) + \text{Var}(L) = \sigma_T^2 + \sigma_L^2\end{aligned}$$

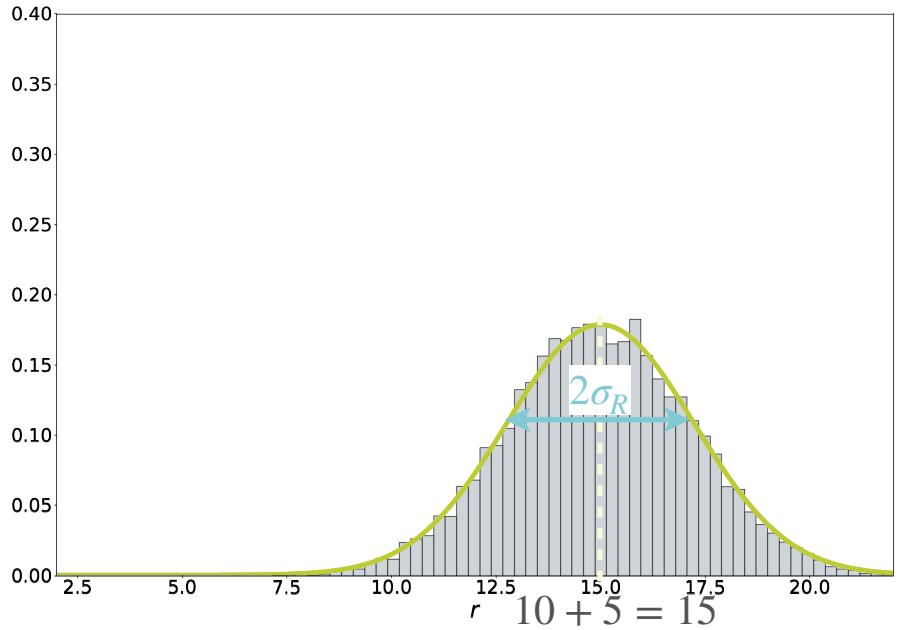
$$= 4 + 1 = 5$$

Sum of Gaussians

$$R = T + L$$

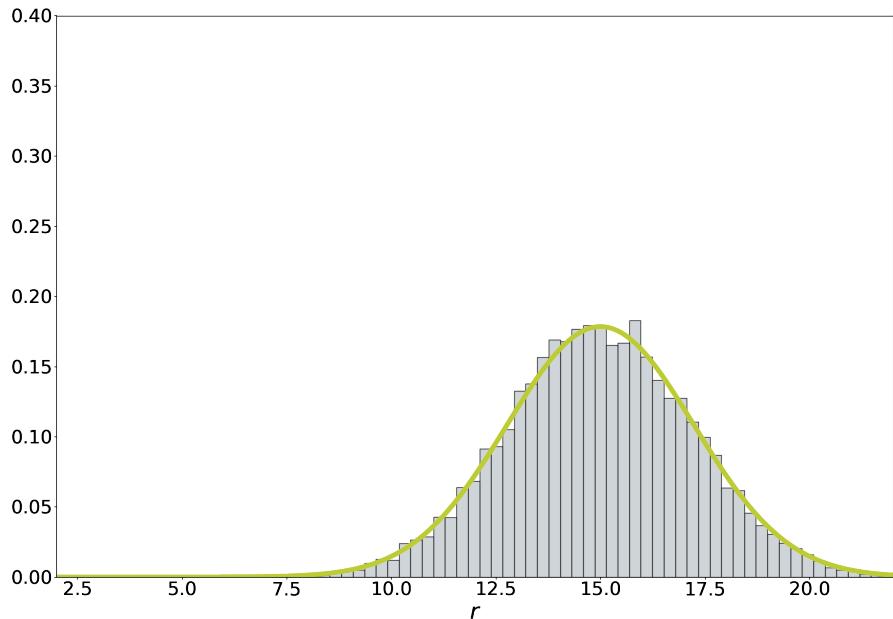
R is still Gaussian!

$$R = (T + L) \sim \mathcal{N}(10 + 5, 4 + 1)$$



Sum of Gaussians

$$R = T + L$$



In general: $W = aX + bY$

Independent $\begin{cases} X \sim \mathcal{N}(\mu_X, \sigma_X^2) \\ Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2) \end{cases}$

$$\rightarrow W \sim \mathcal{N} \left(a\mu_x + b\mu_y, a^2\sigma_x^2 + b^2\sigma_y^2 \right)$$

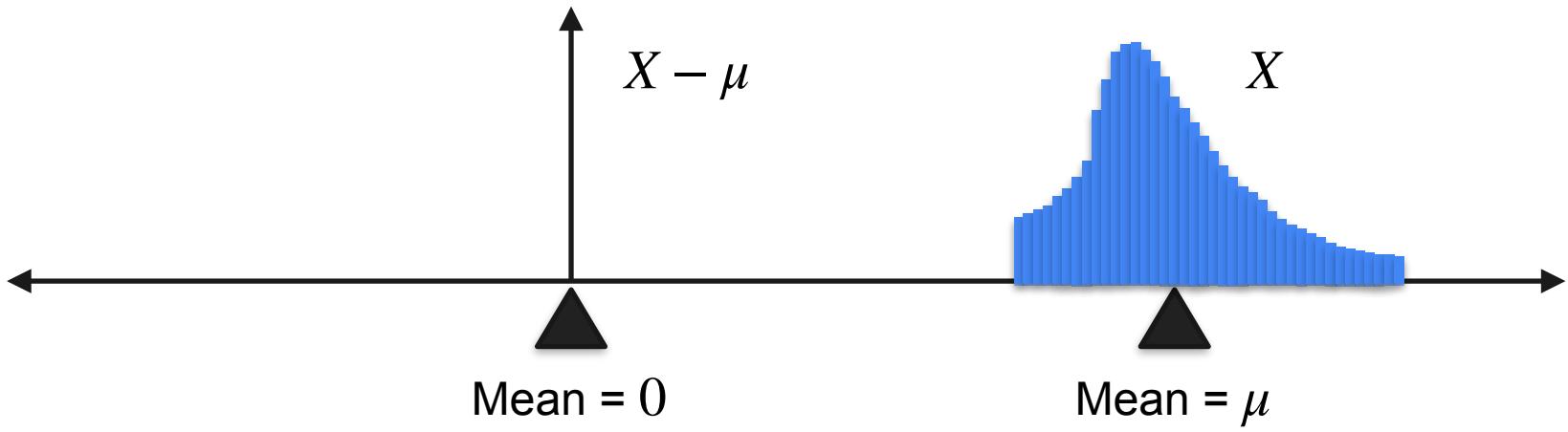


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Describing Distributions

Standardizing a Distribution

Everything Is Nicer When the Mean Is 0



$$X \rightarrow X - \mu$$

Everything Is Nicer When the Mean Is 0

Why?

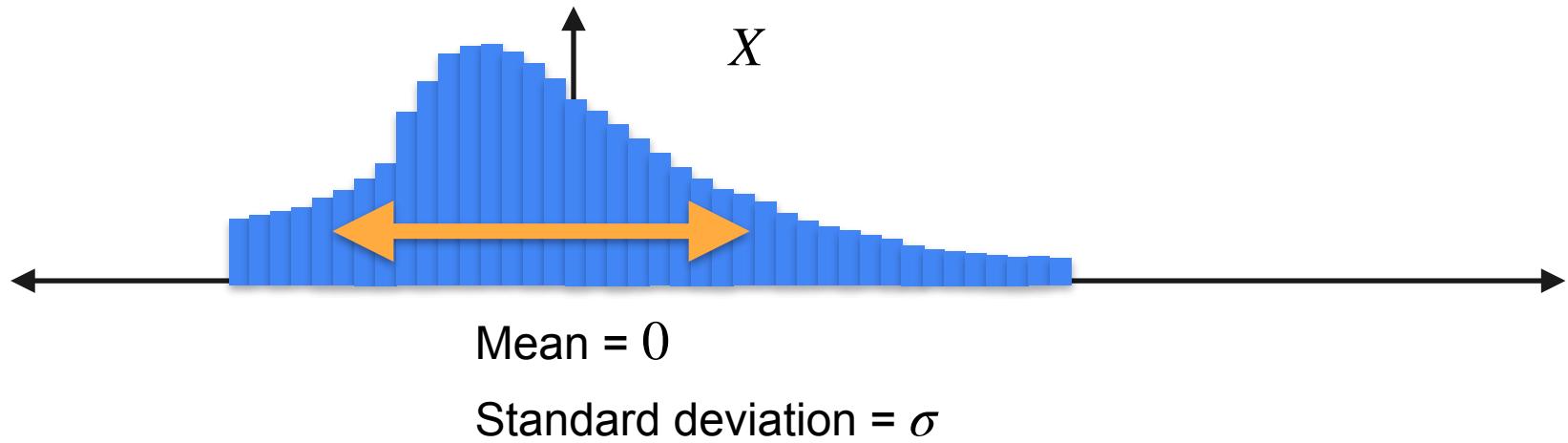
$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

$$\mathbb{E}[X - \mu] = \mathbb{E}[X] - \mathbb{E}[\mu]$$

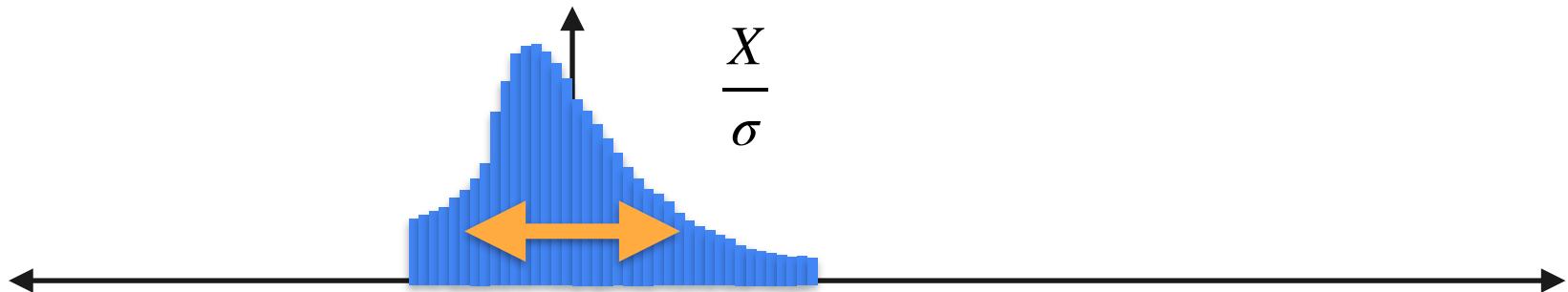
$$= \mathbb{E}[X] - \mu$$

$$= 0$$

Everything Is Nicer When the Standard Deviation Is 1



Everything Is Nicer When the Standard Deviation Is 1



Mean = 0

Standard deviation = 1

$$X \rightarrow \frac{X}{\sigma}$$

Everything Is Nicer When the Standard Deviation Is 1

Why?

$$Var(cX) = \mathbb{E}[(cX)^2] - \mathbb{E}[cX]^2$$

$$= \mathbb{E}[c^2X^2] - (c\mathbb{E}[X])^2$$

$$= c^2\mathbb{E}[X^2] - c^2\mathbb{E}[X]^2$$

$$= c^2(\mathbb{E}[X^2] - \mathbb{E}[X]^2)$$

$$= c^2Var(X)$$

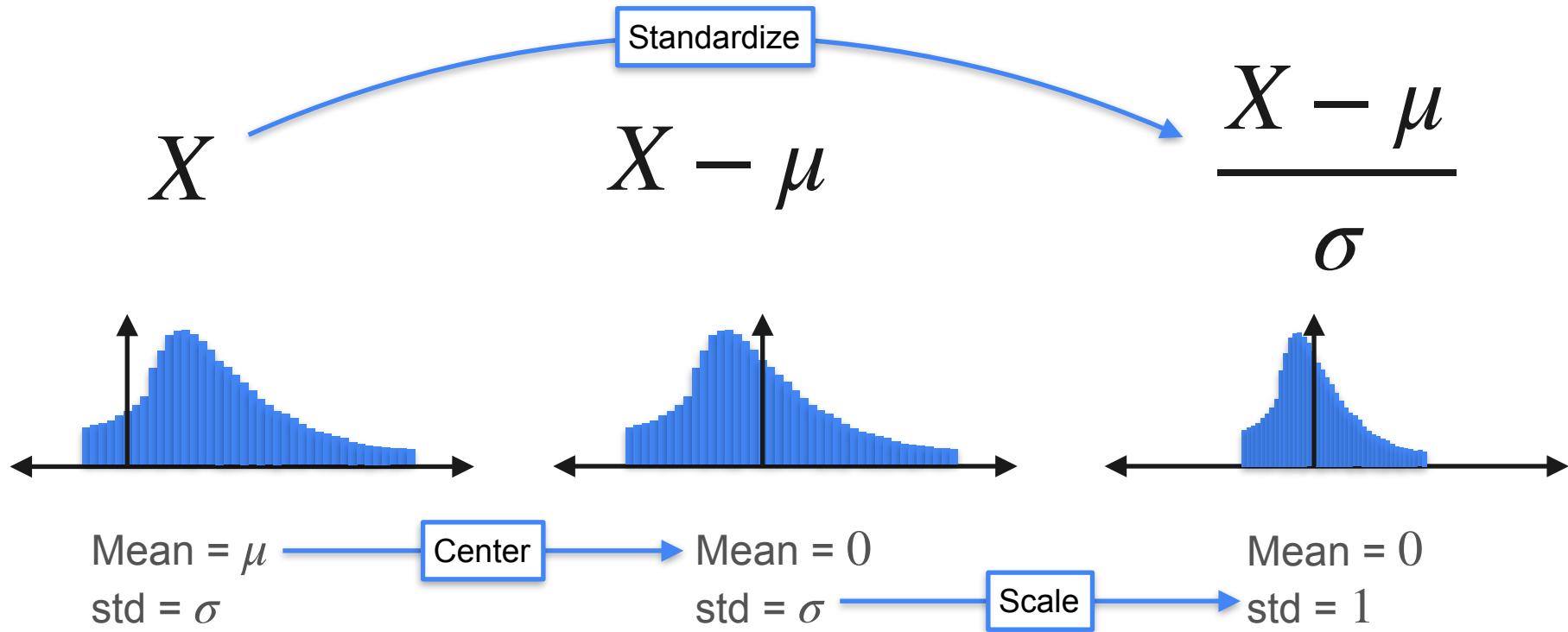
$$Var\left(\frac{X}{\sigma}\right) = \frac{1}{\sigma^2}Var(X)$$

$$std\left(\frac{X}{\sigma}\right) = \frac{1}{\sigma}std(X)$$

$$= \frac{\sigma}{\sigma}$$

$$= 1$$

Standardize a Distribution





DeepLearning.AI

Describing Distributions

**Skewness and Kurtosis:
Moments of a Distribution**

Moments of a Distribution

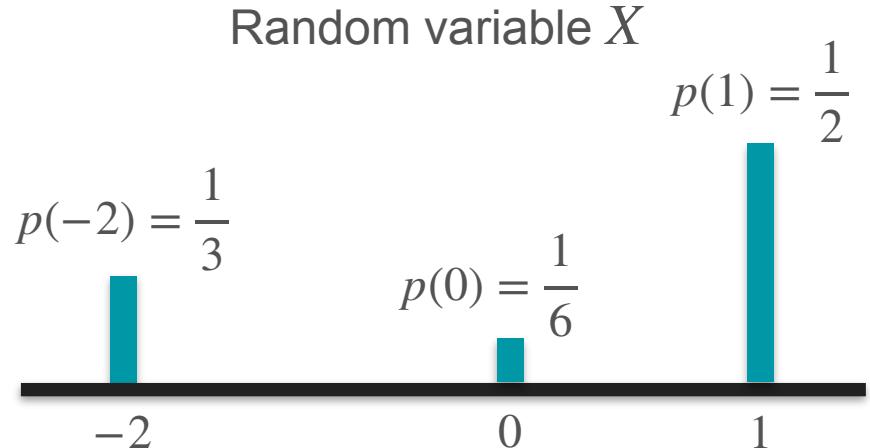
$$\mathbb{E}[X] = \frac{1}{3}(-2) + \frac{1}{6}(0) + \frac{1}{2}(1)$$

$$\mathbb{E}[X^2] = \frac{1}{3}(-2)^2 + \frac{1}{6}(0)^2 + \frac{1}{2}(1)^2$$

$$\mathbb{E}[X^3] = \frac{1}{3}(-2)^3 + \frac{1}{6}(0)^3 + \frac{1}{2}(1)^3$$

...

$$\mathbb{E}[X^k] = \frac{1}{3}(-2)^k + \frac{1}{6}(0)^k + \frac{1}{2}(1)^k$$



Moments of a Distribution

$$\mathbb{E}[X] = p_1x_1 + p_2x_2 + \cdots + p_nx_n$$

$$\mathbb{E}[X^2] = p_1x_1^2 + p_2x_2^2 + \cdots + p_nx_n^2$$

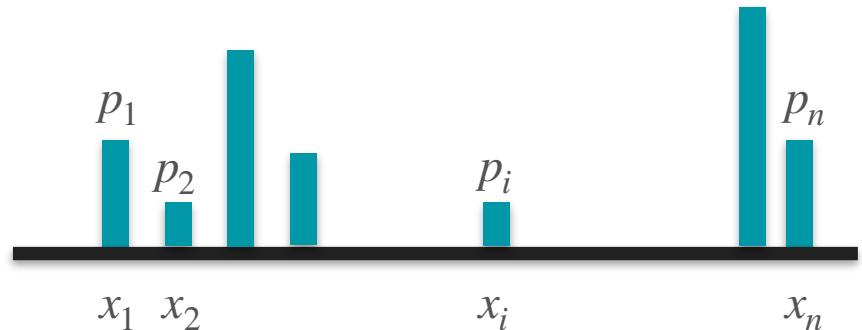
$$\mathbb{E}[X^3] = p_1x_1^3 + p_2x_2^3 + \cdots + p_nx_n^3$$

$$\mathbb{E}[X^4] = p_1x_1^4 + p_2x_2^4 + \cdots + p_nx_n^4$$

...

$$\mathbb{E}[X^k] = p_1x_1^k + p_2x_2^k + \cdots + p_nx_n^k$$

Random variable X





DeepLearning.AI

Describing Distributions

**Skewness and Kurtosis:
Skewness**

Lottery vs Insurance



Ticket: \$1
Jackpot: \$100

Lottery

You **win** \$99 with 1% probability

You **lose** \$1 with 99% probability



Cost: \$1
Crash Reparation: \$100

Car insurance

You **win** \$1 with 99% probability

You **lose** \$99 with 1% probability

Lottery vs Insurance



Ticket: \$1
Jackpot: \$100

Lottery



Cost: \$1
Crash Reparation: \$100

Car insurance





Ticket: \$1
Jackpot: \$100

Lottery

$$\mathbb{E}[X_1] = -1 \cdot 0.99 + 99 \cdot 0.01 = 0$$

Lose 1
With probability 0.99

Win 99
With probability 0.01



Cost: \$1
Crash Reparation: \$100

Car insurance

$$\mathbb{E}[X_2] = -99 \cdot 0.01 + 1 \cdot 0.99 = 0$$

Lose 99
With probability 0.01

Win 1
With probability 0.99



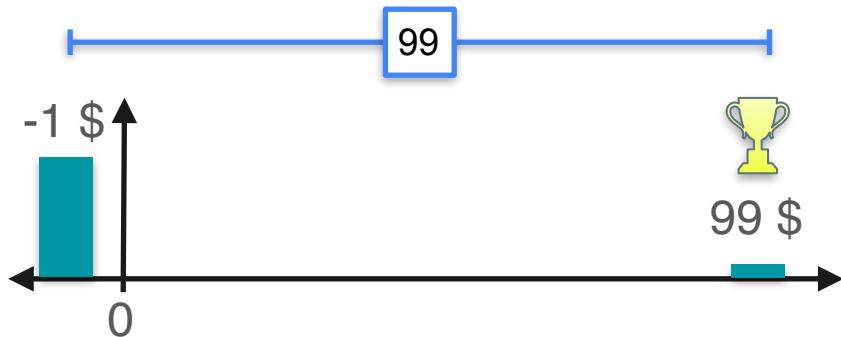


Ticket: \$1
Jackpot: \$100

Lottery

$$\mathbb{E}[X_1] = -1 \cdot 0.99 + 99 \cdot 0.01 = 0$$

$$Var(X_1) = (-1)^2 \cdot 0.99 + (99)^2 \cdot 0.01 = 99$$

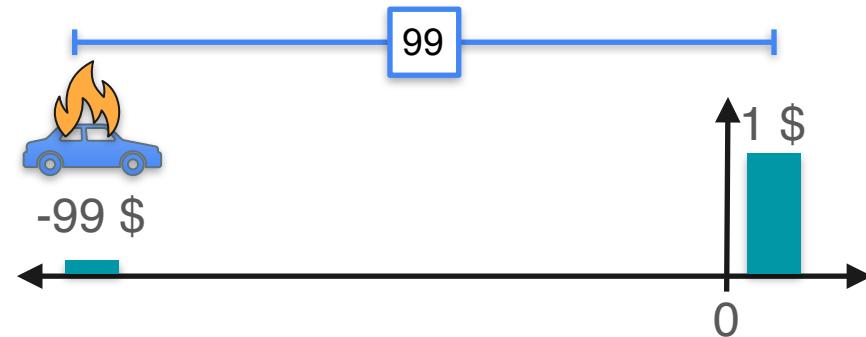


Cost: \$1
Crash Reparation: \$100

Car insurance

$$\mathbb{E}[X_2] = -99 \cdot 0.01 + 1 \cdot 0.99 = 0$$

$$Var(X_2) = (-99)^2 \cdot 0.01 + (1)^2 \cdot 0.99 = 99$$





Ticket: \$1
Jackpot: \$100

Lottery

$$\mathbb{E}[X_1] = 0$$

$$Var(X_1) = 99$$



Cost: \$1
Crash Reparation: \$100

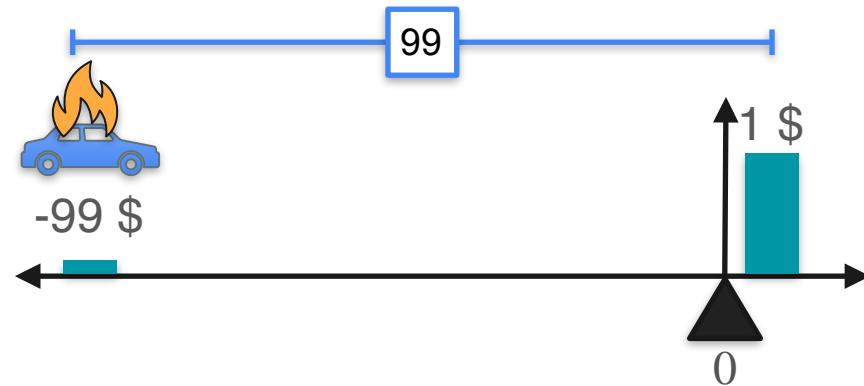
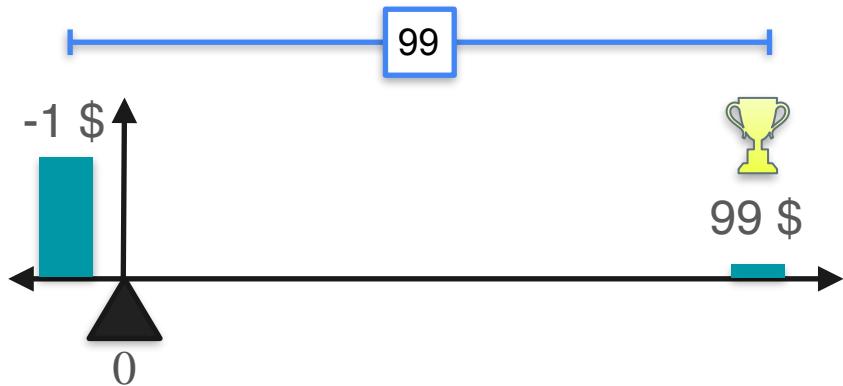
Car insurance

Same expectation
Same variance

How to tell them apart?

$$\mathbb{E}[X_2] = 0$$

$$Var(X_2) = 99$$





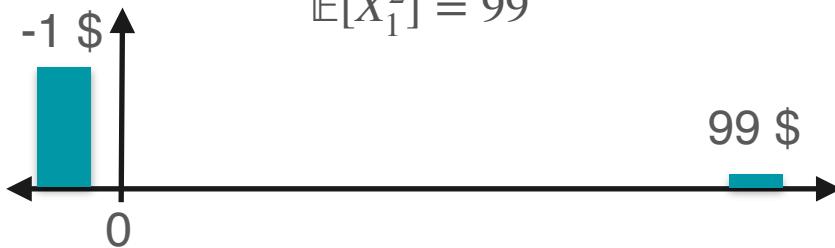
Ticket: \$1
Jackpot: \$99

Lottery

$$\mathbb{E}[X_1] = 0$$

$$Var(X_1) = 99$$

$$\mathbb{E}[X_1^2] - \mathbb{E}[X_1]^2 = 99$$



Cost: \$1
Crash Reparation: \$99

Car insurance

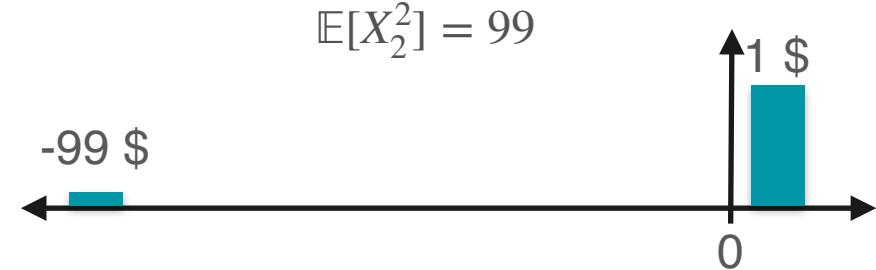
Same expectation
Same variance

How to tell them apart?

$$\mathbb{E}[X_2] = 0$$

$$Var(X_2) = 99$$

$$\mathbb{E}[X_2^2] - \mathbb{E}[X_2]^2 = 99$$





Ticket: \$1
Jackpot: \$99

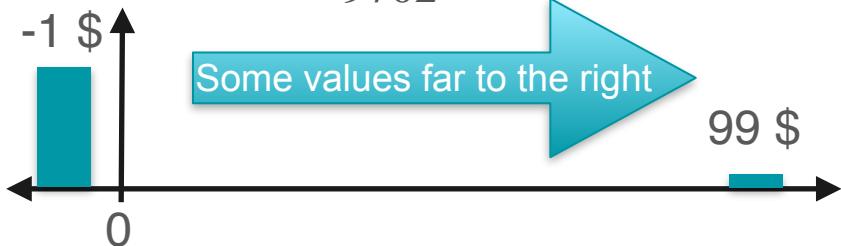
Lottery

$$\mathbb{E}[X_1] = 0$$

$$\mathbb{E}[X_1^2] = 99$$

$$\mathbb{E}[X_1^3] = (-1)^3 \cdot 0.99 + (99)^3 \cdot 0.01$$

$$= 9702$$



Cost: \$1
Crash Reparation: \$99

Car insurance

Same expectation
Same variance

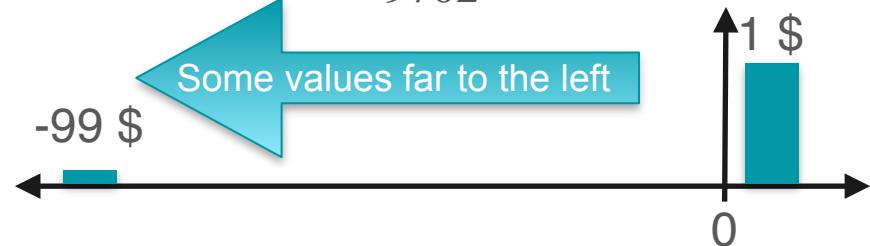
How to tell them apart?

$$\mathbb{E}[X_2] = 0$$

$$\mathbb{E}[X_2^2] = 99$$

$$\mathbb{E}[X_2^3] = (1)^3 \cdot 0.99 + (-99)^3 \cdot 0.01$$

$$= -9702$$





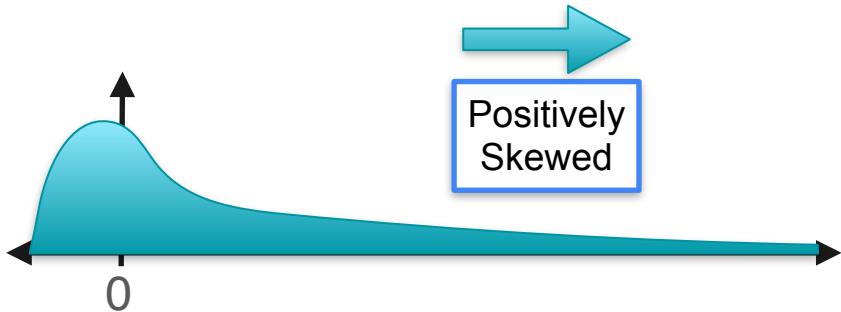
Ticket: \$1
Jackpot: \$99

Lottery

$$\mathbb{E}[X_1] = 0$$

$$\mathbb{E}[X_1^2] = 99$$

$$\mathbb{E}[X_1^3] = \text{Large positive value}$$



Cost: \$1
Crash Reparation: \$99

Car insurance

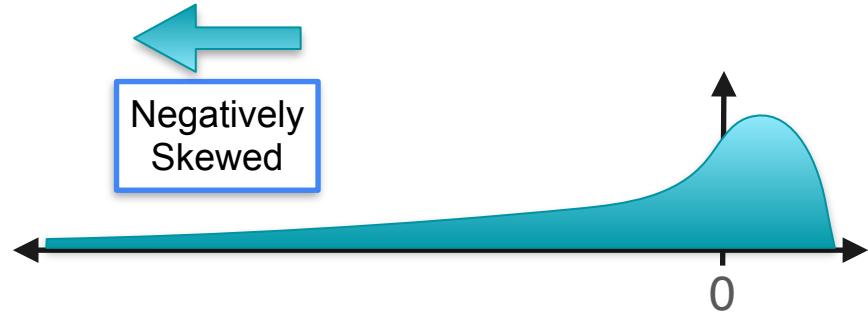
Same expectation
Same variance

How to tell them apart?

$$\mathbb{E}[X_2] = 0$$

$$\mathbb{E}[X_2^2] = 99$$

$$\mathbb{E}[X_2^3] = \text{Large negative value}$$



Skewness

$$\mathbb{E}[X^3]$$

Almost...

Need to standardize...

Skewness

$$\text{Skewness} = \mathbb{E} \left[\left(\frac{X - \mu}{\sigma} \right)^3 \right]$$

Skewness



Positively
Skewed



Not
Skewed



Negatively
Skewed



$$E \left[\left(\frac{X - \mu}{\sigma} \right)^3 \right] > 0$$

$$E \left[\left(\frac{X - \mu}{\sigma} \right)^3 \right] = 0$$

$$E \left[\left(\frac{X - \mu}{\sigma} \right)^3 \right] < 0$$



DeepLearning.AI

Describing Distributions

**Skewness and Kurtosis:
Kurtosis**

Kurtosis: Example

Game 1

Which one
is riskier?

probability $\frac{1}{2}$: You win 1 dollar

probability $\frac{1}{2}$: You lose 1 dollar

Game 2

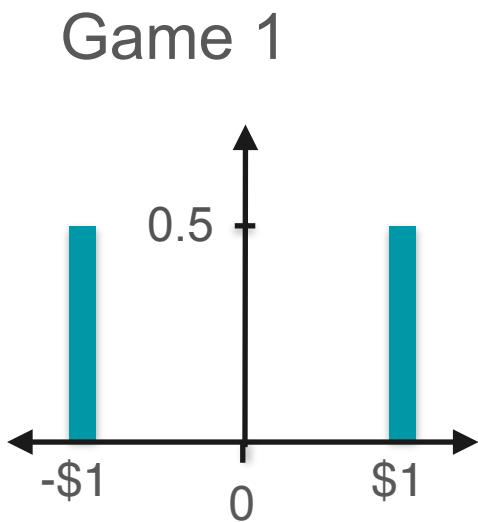
probability $\frac{100}{202}$: You win 10 cents

probability $\frac{100}{202}$: You lose 10 cents

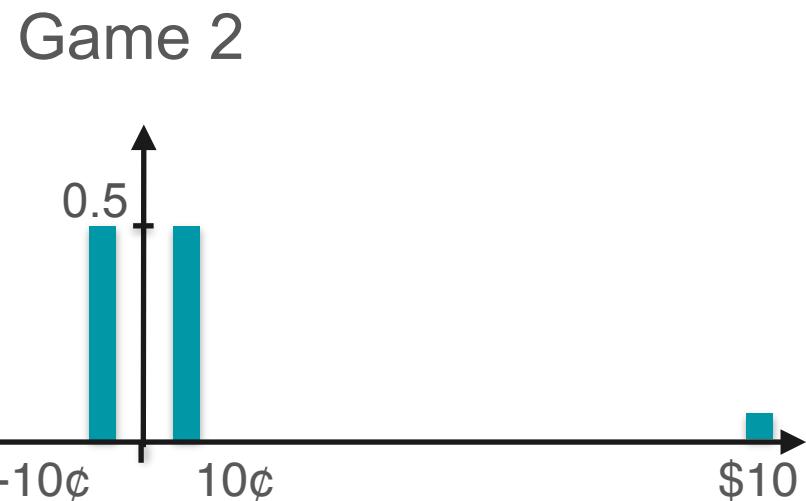
probability $\frac{1}{202}$: You win 10 dollars

probability $\frac{1}{202}$: You lose 10 dollars

Kurtosis: Example



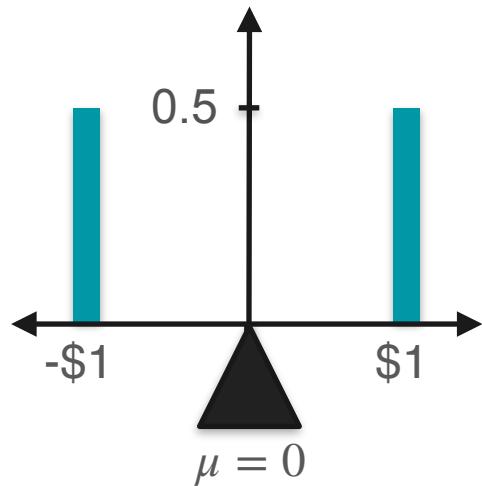
Expected value?
Standard deviation?
Skewness?



Kurtosis: Example Expected Value

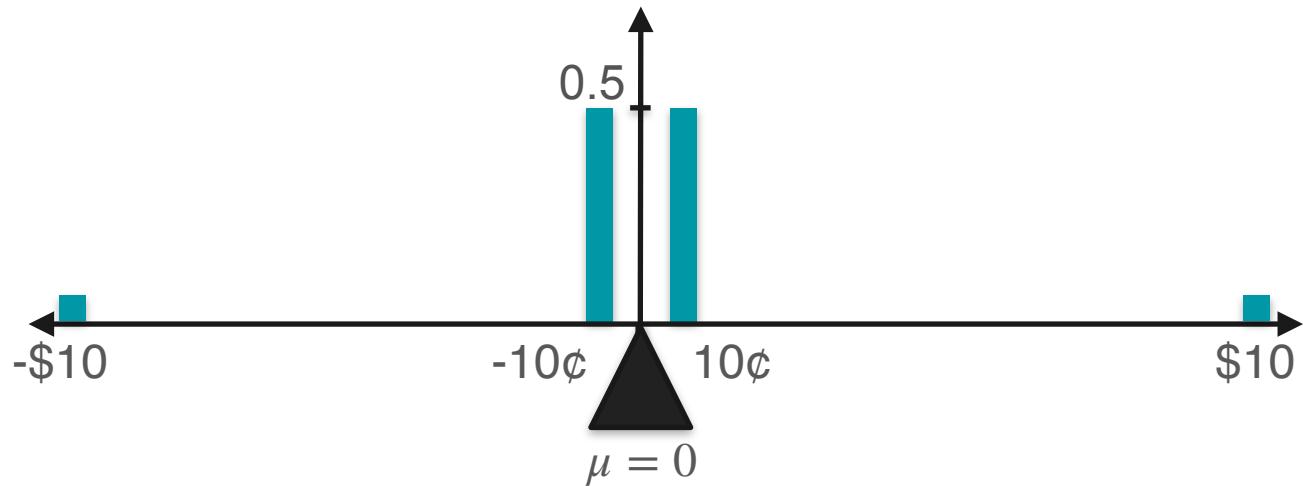
Game 1

$$\mathbb{E}[X_1] = 0$$



Game 2

$$\mathbb{E}[X_2] = 0$$



Kurtosis: Example Variance

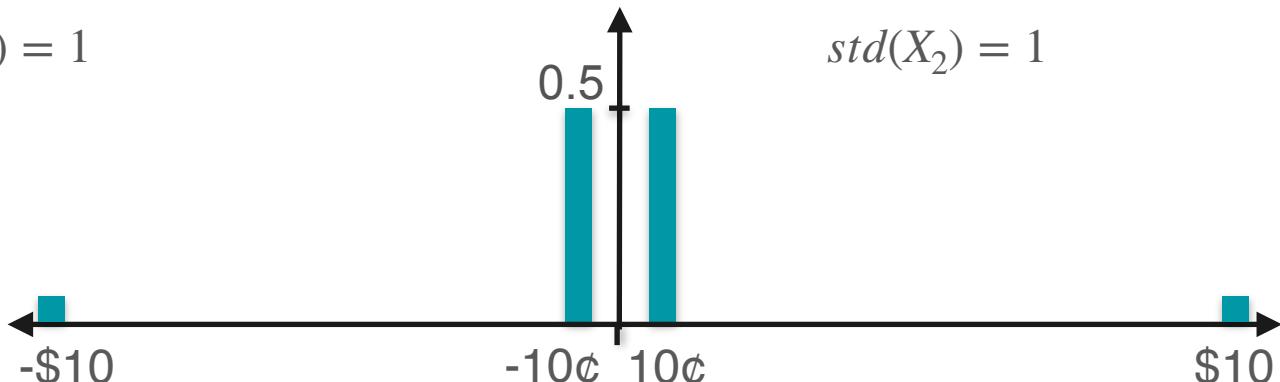
Game 1



$$Var(X_1) = 1$$

$$std(X_1) = 1$$

Game 2



$$Var(X_2) = 1$$

$$std(X_2) = 1$$

$$\begin{aligned}\mathbb{E}[X_1^2] &= \frac{1}{2}(-1)^2 + \frac{1}{2}(1)^2 \\ &= 1\end{aligned}$$

$$\begin{aligned}\mathbb{E}[X_2^2] &= \frac{100}{202}(-0.1)^2 + \frac{100}{202}(0.1)^2 + \frac{1}{202}(-10)^2 + \frac{1}{202}(10)^2 \\ &= \frac{100}{202} \cdot \frac{1}{100} + \frac{100}{202} \cdot \frac{1}{100} + \frac{1}{202} \cdot 100 + \frac{1}{202} \cdot 100 \\ &= \frac{1}{202} + \frac{1}{202} + \frac{100}{202} + \frac{100}{202} = 1\end{aligned}$$

Kurtosis: Example Skewness

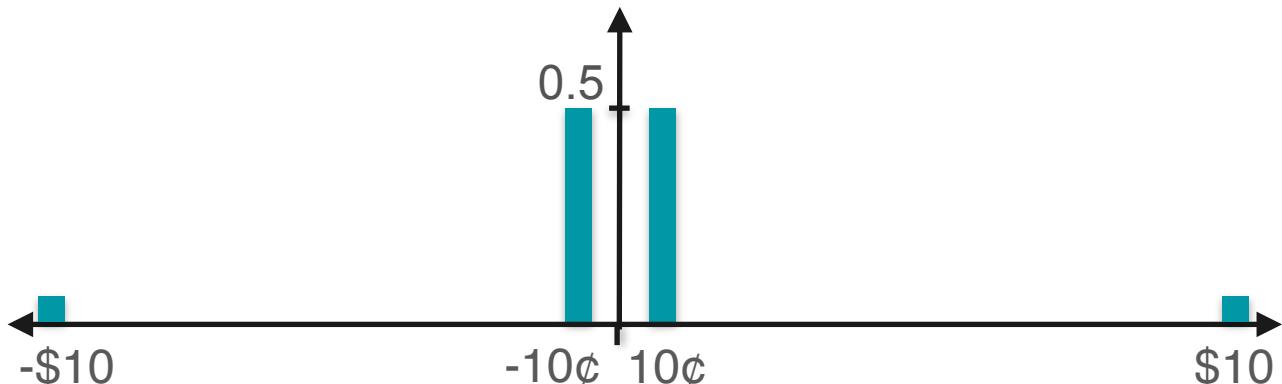
Game 1

$$Skew(X_1) = 0$$

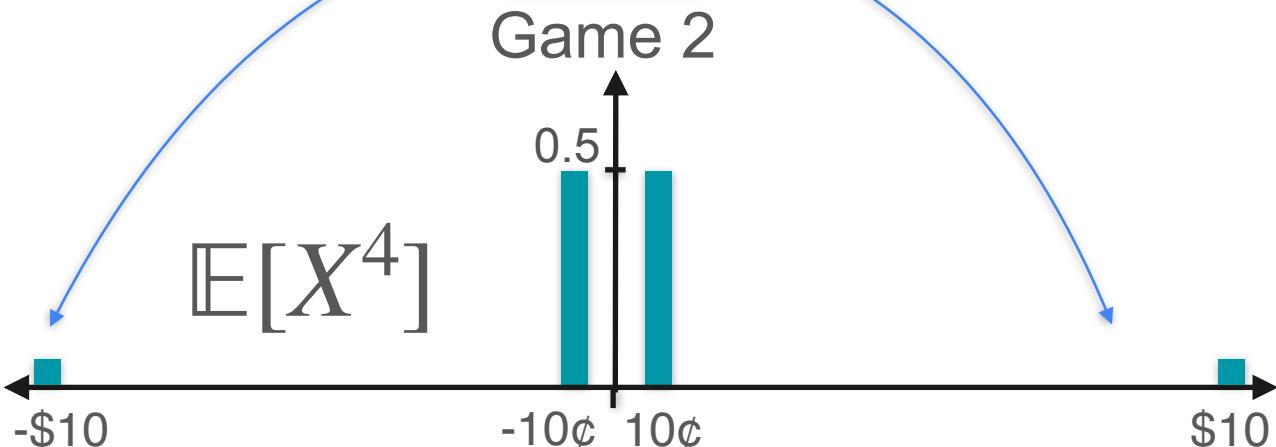
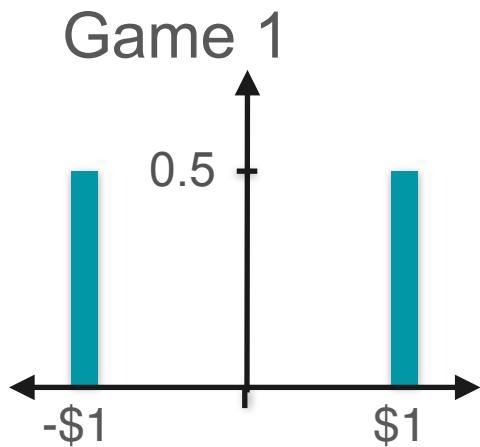


Game 2

$$Skew(X_2) = 0$$



Kurtosis



$$E[X_1] = 0 \quad E[X_2] = 0$$

$$Var(X_1) = 1 \quad E[X_1^2] = 1$$

$$Skew(X_1) = 0 \quad E[X_1^3] = 0$$

$$E[X_2] = 0 \quad E[X_2] = 0$$

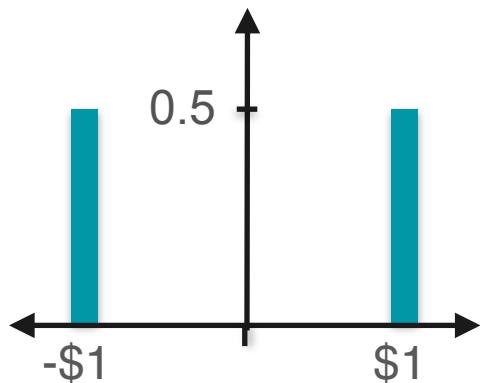
$$Var(X_2) = 1 \quad E[X_2^2] = 1$$

$$Skew(X_2) = 0 \quad E[X_2^3] = 0$$

Has values way
farther from 0

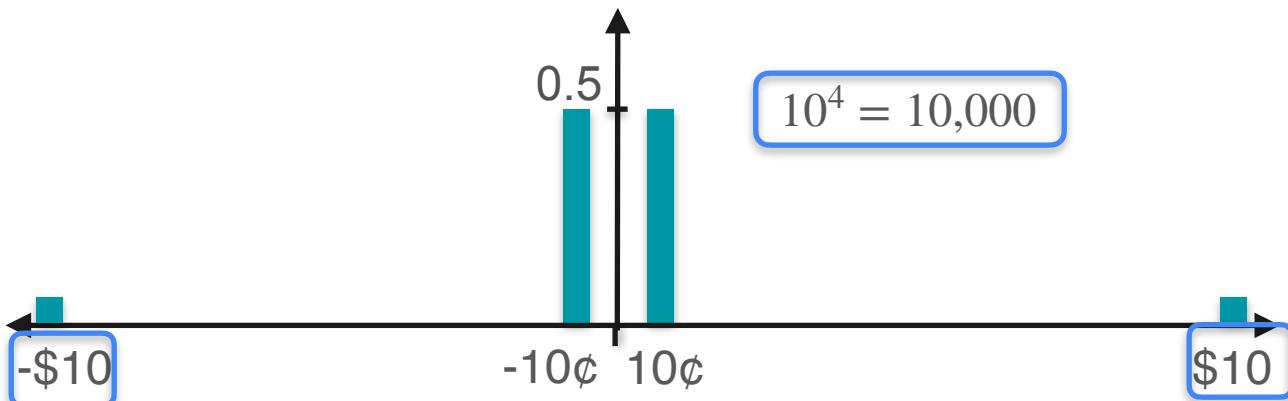
Kurtosis

Game 1



$$\begin{aligned}\mathbb{E}[X_1^4] &= \frac{1}{2}(-1)^4 + \frac{1}{2}(1)^4 \\ &= 1\end{aligned}$$

Game 2



$$\begin{aligned}\mathbb{E}[X_2^4] &= \frac{100}{202}(-0.1)^4 + \frac{100}{202}(0.1)^4 + \frac{1}{202}(-10)^4 + \frac{1}{202}(10)^4 \\ &= 99.01\end{aligned}$$

Kurtosis

$$\mathbb{E}[X^4]$$

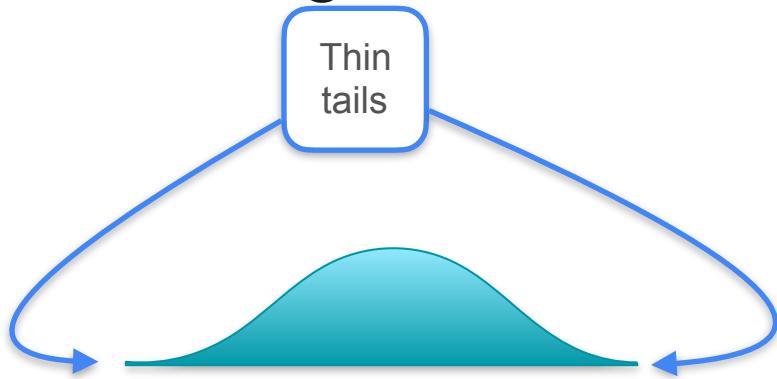
Almost...

Need to standardize...

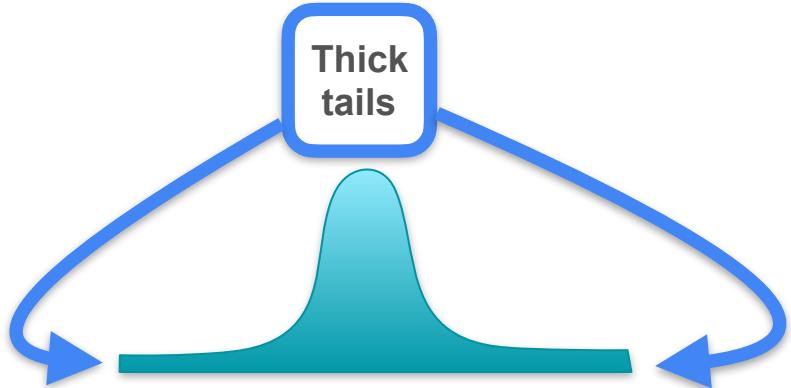
Kurtosis

$$\text{Kurtosis} = E \left[\left(\frac{X - \mu}{\sigma} \right)^4 \right]$$

Kurtosis: High and Low



$$E \left[\left(\frac{X - \mu}{\sigma} \right)^4 \right] = \text{small}$$



$$E \left[\left(\frac{X - \mu}{\sigma} \right)^4 \right] = \text{large}$$

Even if they have the same variance!

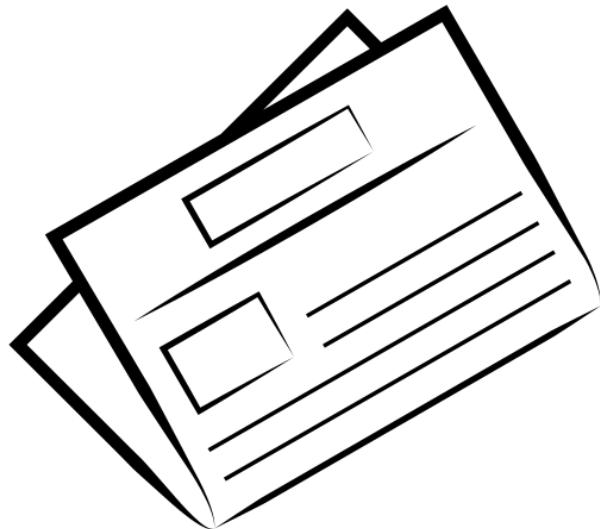


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Describing Distributions

Quantiles and Box-Plots

Quantiles: Example



Newspaper advertisement

Newspaper Advertisement (X)			
18.3	18.4	23.2	51.2
35.2	29.7	75	8.7
65.9	14.2	54.7	25.9

Quantiles: Example

What is the median here?

The point that splits your data in half

Newspaper Advertisement (X)			
18.3	18.4	23.2	51.2
35.2	29.7	75	8.7
65.9	14.2	54.7	25.9

Quantiles: Example

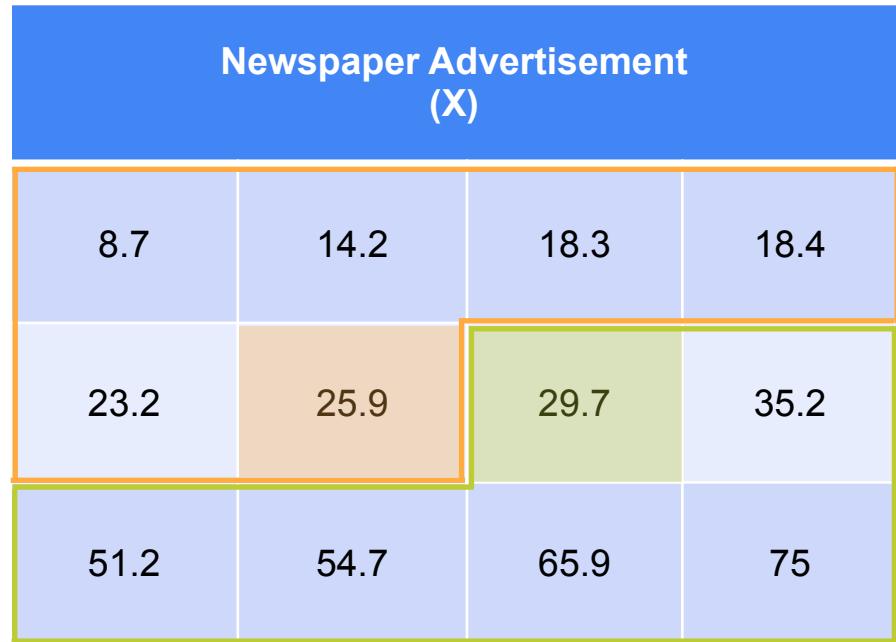
What is the median here?

The point that splits your data in half

$$\text{Median} = \frac{25.9 + 29.7}{2} = 27.8$$

50% quantile

Second quartile



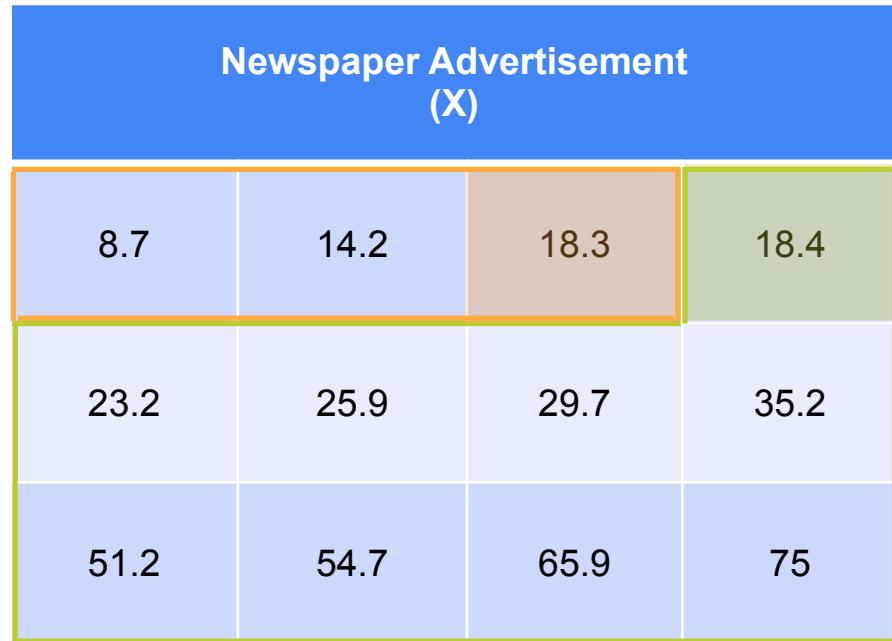
Quantiles: Example

What about the point that leaves 1/4 of your data to the left and 3/4 to the right?

$$q_{0.25} = Q1 = \frac{18.3 + 18.4}{2} = 18.35$$

25% quantile

First quartile



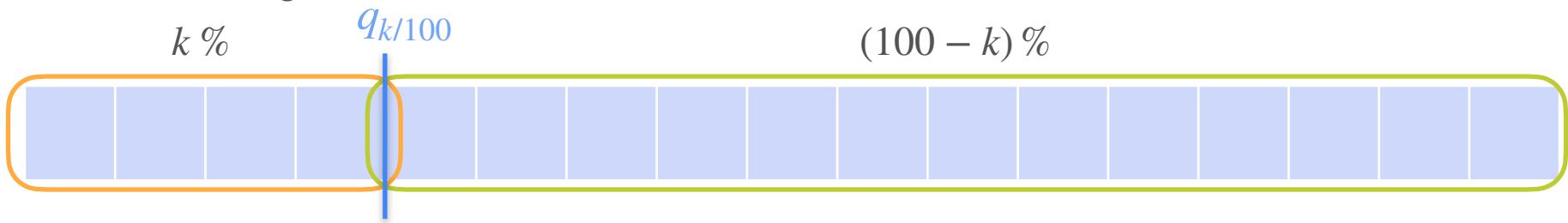
Quantiles

In general:

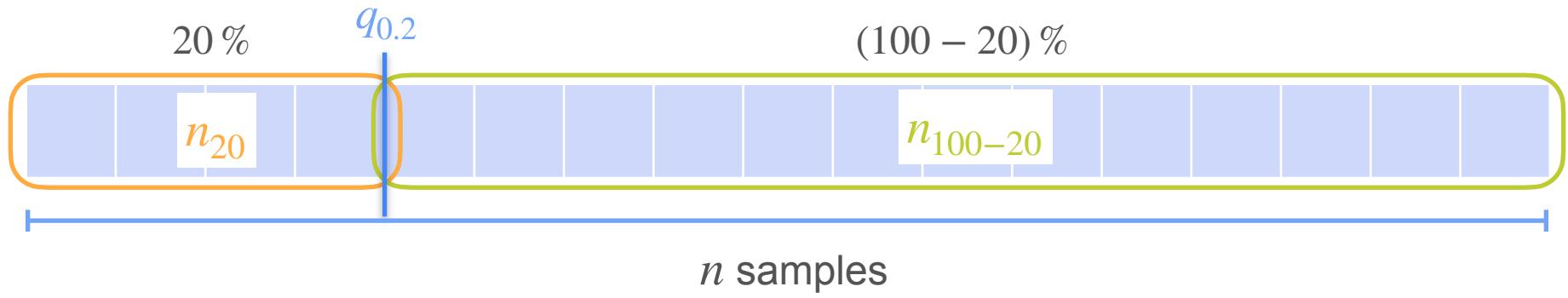
The **k%** quantile ($q_{k/100}$) is the value that leaves k% of your data to the left and $(100-k)\%$ of your data to the right

Some common quantiles:

- 25% quantile (first quartile - Q1)
- 50% quantile (median - Q2)
- 75% quantile (third quartile - Q3)

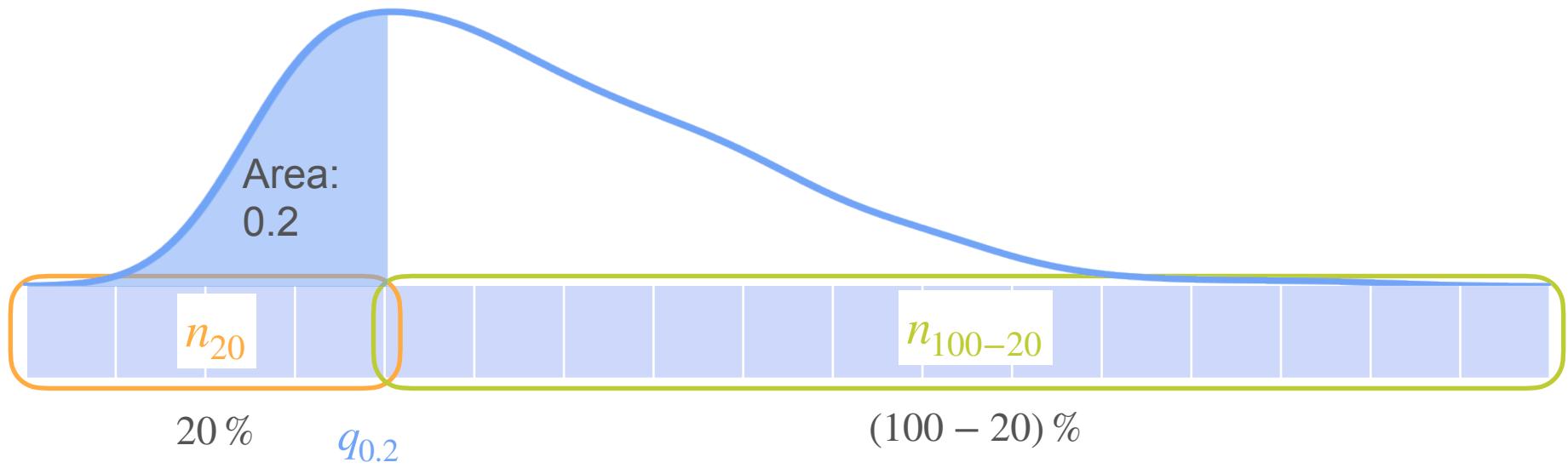


Quantiles



$$\frac{20}{100} = \frac{n_{20}}{n} \approx \mathbf{P}(X \leq q_{0.2})$$

Quantiles



k% quantile ($q_{k/100}$) is the value such that $\mathbf{P}(X \leq q_{k/100}) = \frac{k}{100}$

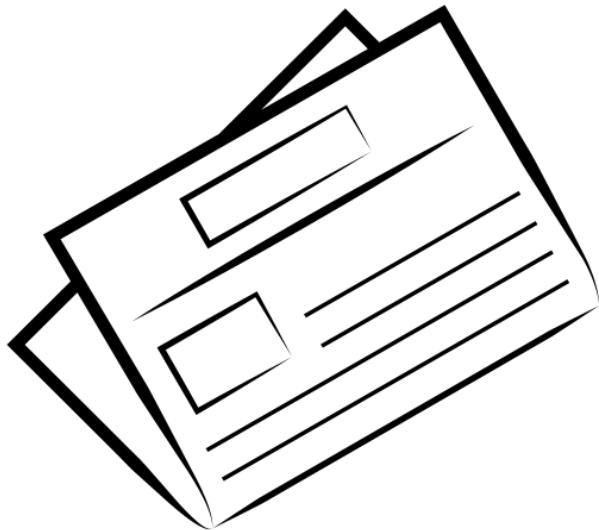


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Describing Distributions

**Visualizing data:
Box-Plots**

Box-Plots



Newspaper advertisement

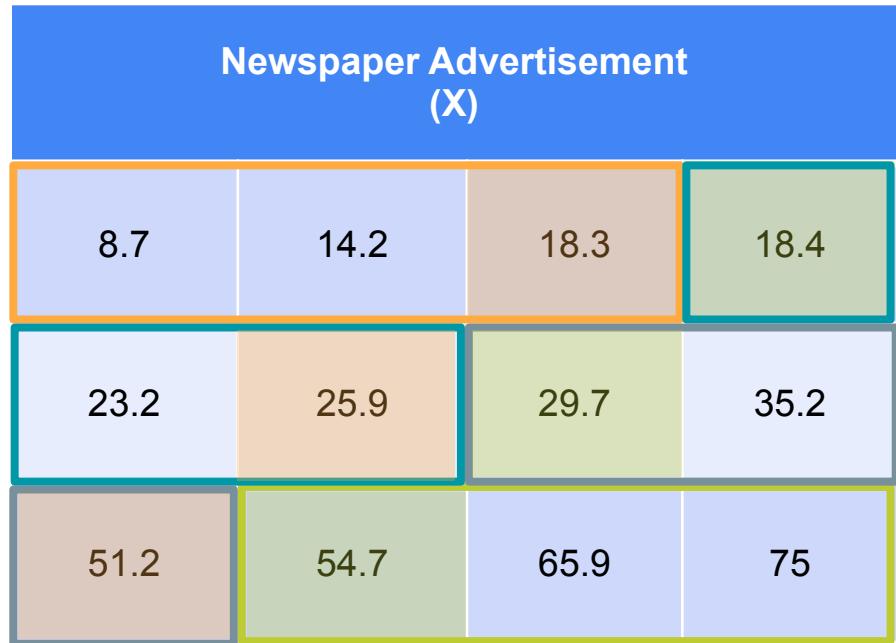
Newspaper Advertisement (X)			
8.7	14.2	18.3	18.4
23.2	25.9	29.7	35.2
51.2	54.7	65.9	75

Box-Plots

$$Q1 = q_{0.25} = \frac{18.3 + 18.4}{2} = 18.35$$

$$Q2 = q_{0.50} = \frac{25.9 + 29.7}{2} = 27.8$$

$$Q3 = q_{0.75} = \frac{51.2 + 54.7}{2} = 52.95$$



Box-Plots

$$Q1 = q_{0.25} = \frac{18.3 + 18.4}{2} = 18.35$$

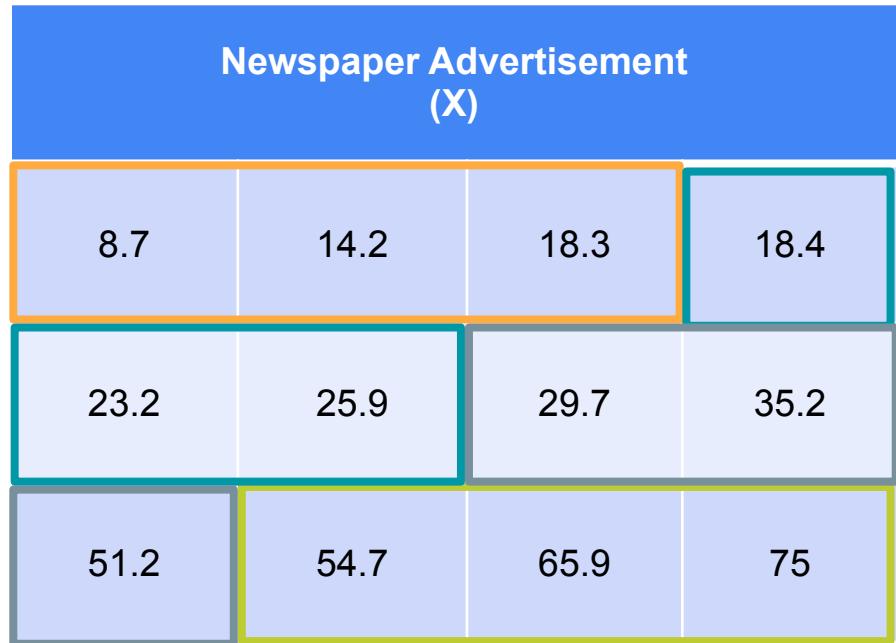
$$Q2 = q_{0.50} = \frac{25.9 + 29.7}{2} = 27.8$$

$$Q3 = q_{0.75} = \frac{51.2 + 54.7}{2} = 52.95$$

Interquartile range (IQR)

$$IQR = Q3 - Q1 = 52.95 - 18.35 = 34.6$$

$$\min = 8.7 \quad \max = 75$$



Box-Plots

$$Q1 = q_{0.25} = \frac{18.3 + 18.4}{2} = 18.35$$

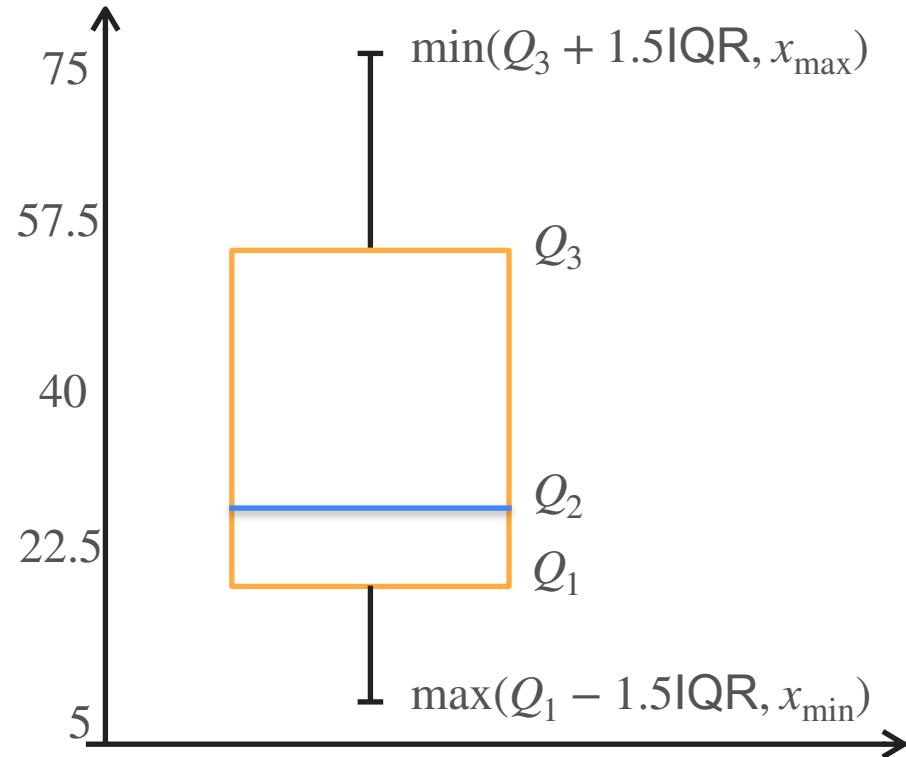
$$Q2 = q_{0.5} = \frac{25.9 + 29.7}{2} = 27.8$$

$$Q3 = q_{0.75} = \frac{51.2 + 54.7}{2} = 52.95$$

Interquartile range (IQR)

$$IQR = Q3 - Q1 = 52.95 - 18.35 = 34.6$$

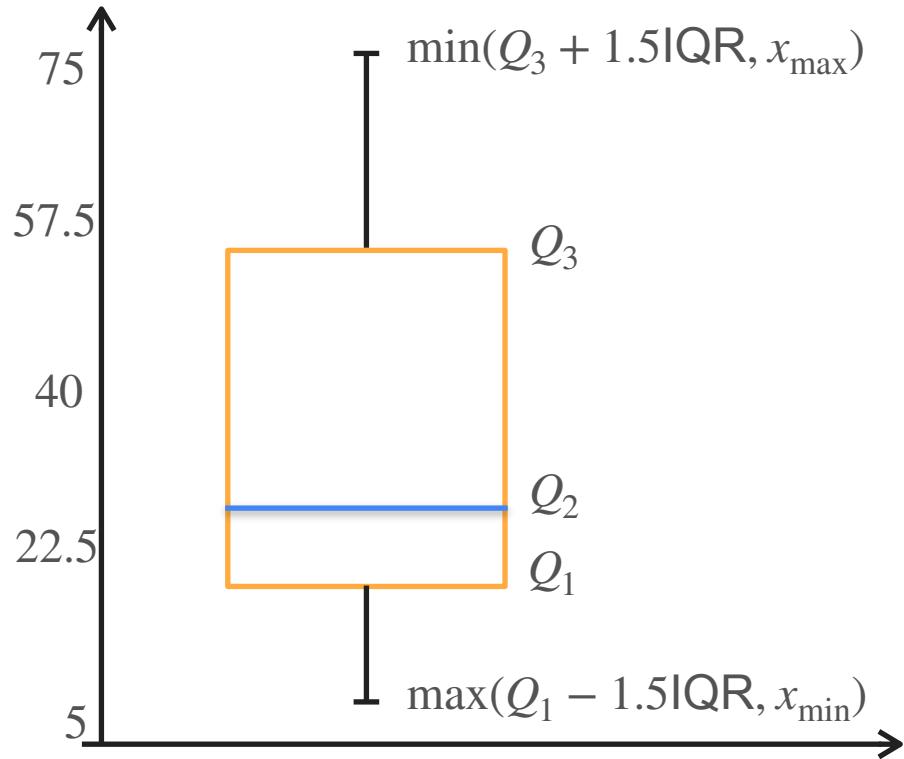
$$x_{\min} = 8.7 \quad x_{\max} = 75$$



Box-Plots

What can you tell from this plot?

- Data is skewed
- No outliers (whiskers were cut at max and min value)
- Analyze dispersion



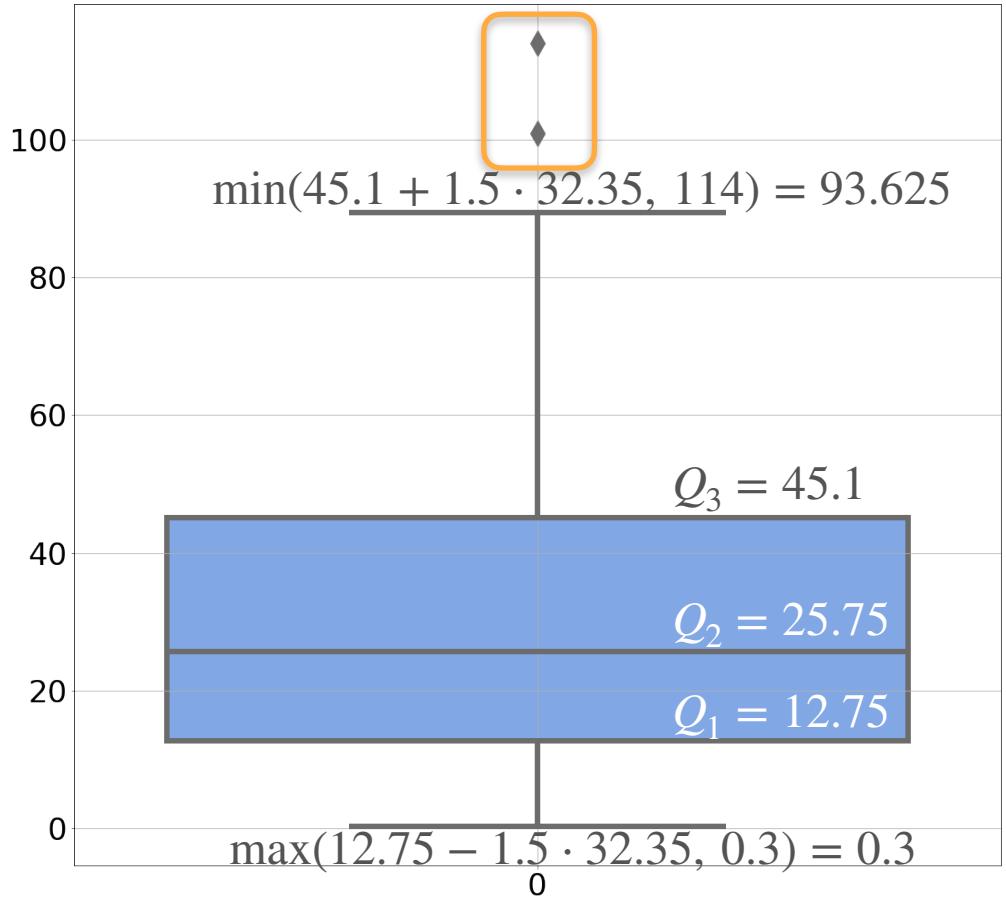
Box-Plots

Let's see how the box-plot looks for the whole dataset

$$Q_1 = 12.75 \quad Q_2 = 25.75 \quad Q_3 = 45.1$$

$$\text{IQR} = 32.35 \quad x_{\min} = 0.3 \quad x_{\max} = 114$$

Now you can see two outliers



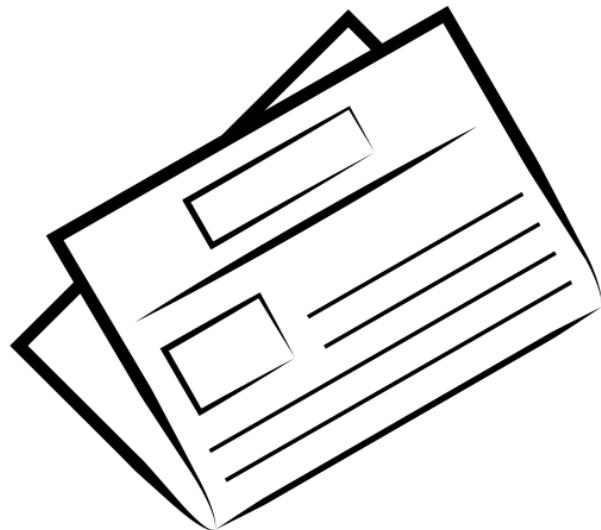


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Describing Distributions

**Visualizing data:
Kernel density estimation**

Density Estimation



Newspaper advertisement

Newspaper Advertisement
(X)

8.7	14.2	18.3	18.4
23.2	25.9	29.7	35.2
51.2	54.7	65.9	75

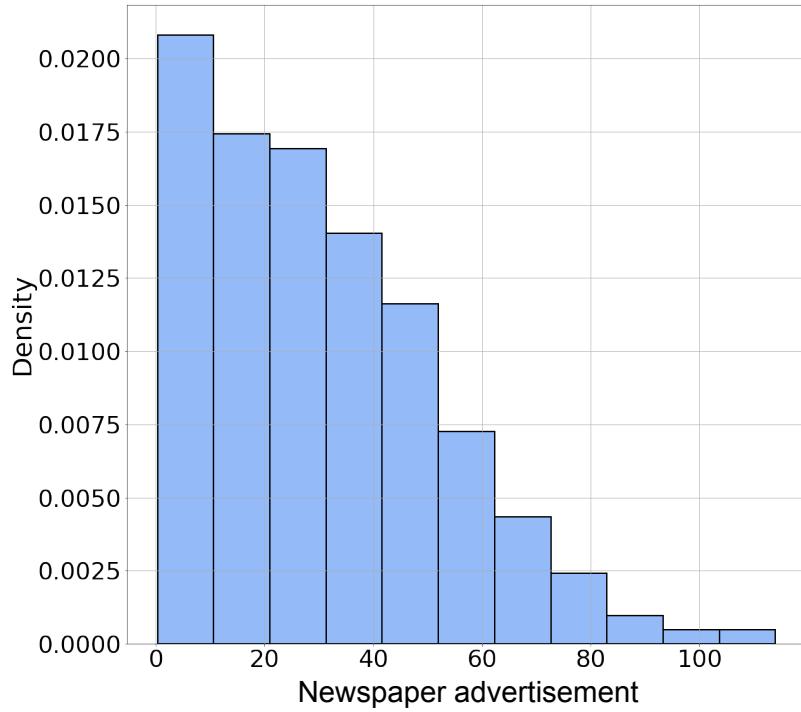
Histograms

It represents a density function

- It is positive
- Area under the curve is 1

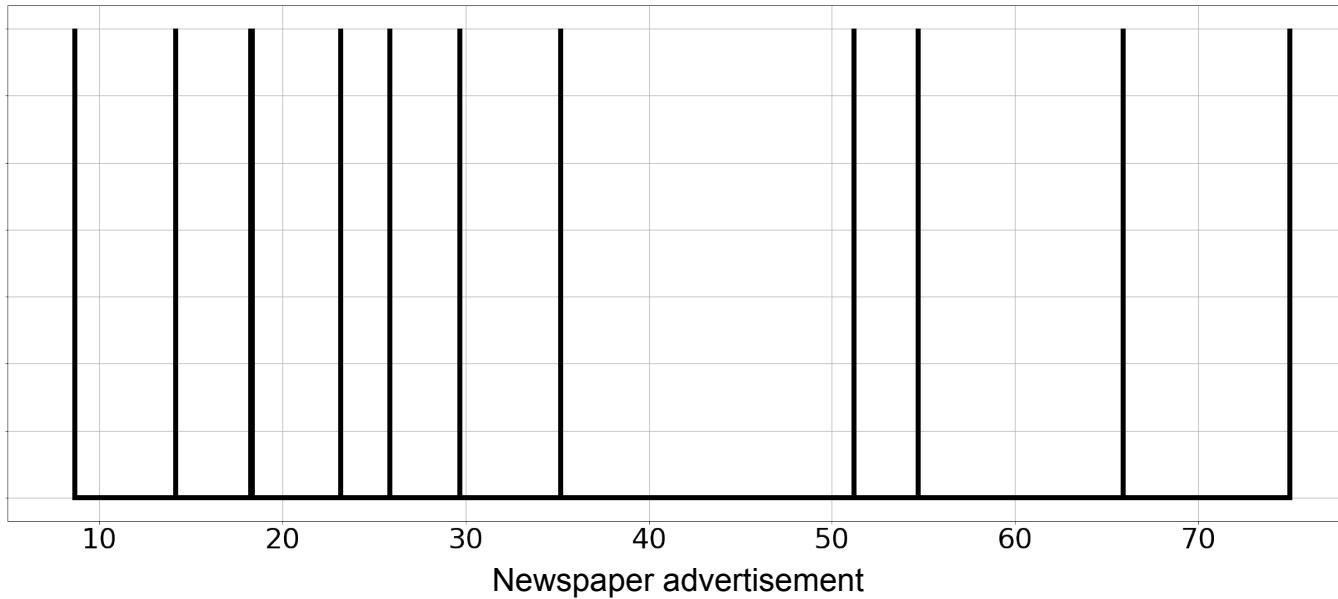
But...

- PDFs are usually smooth function
- The discontinuities come from the method and not the data



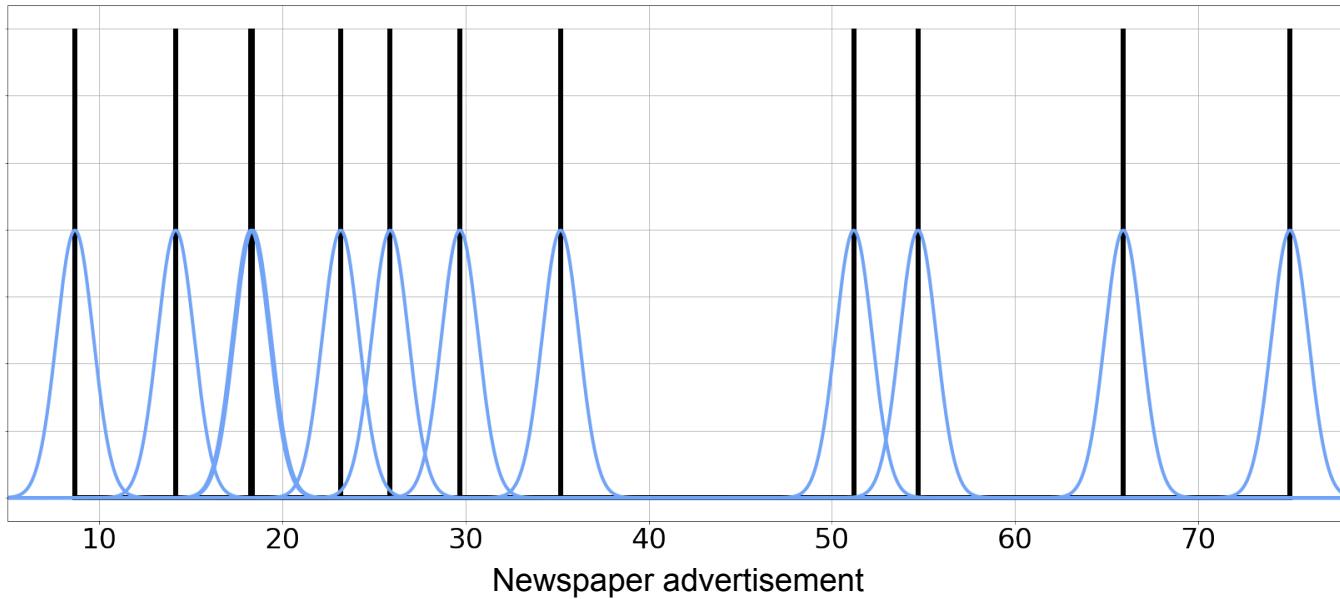
Kernel Density Estimation

First: draw your observations along the x axis



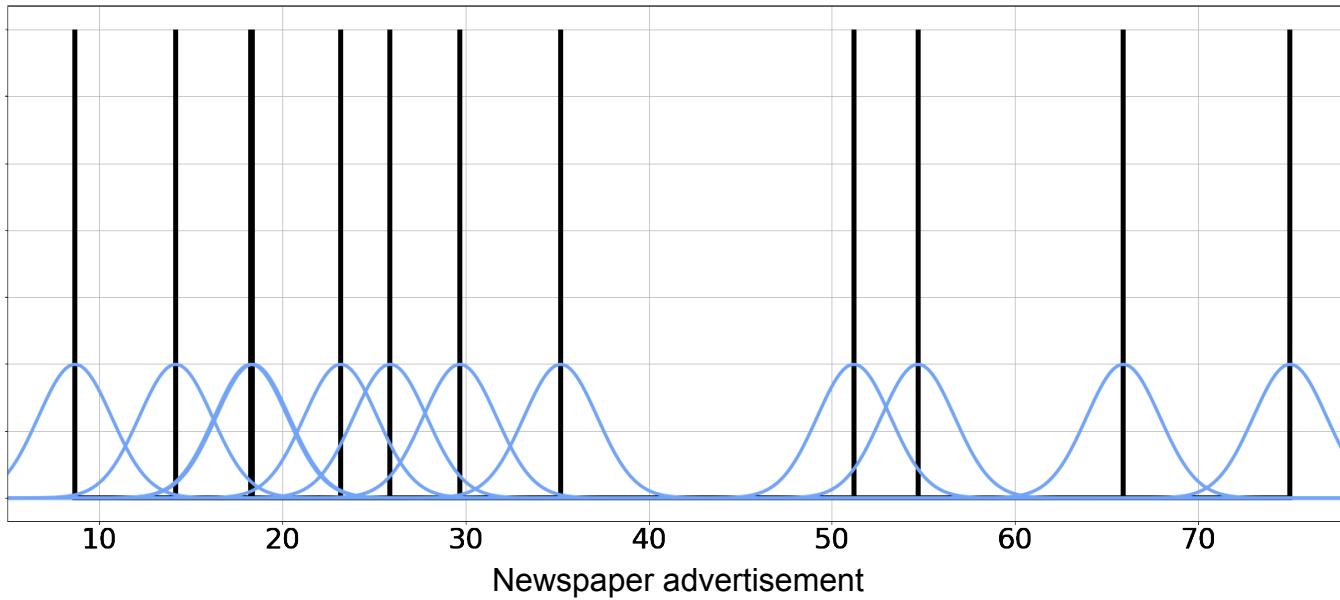
Kernel Density Estimation

Second: draw a gaussian centered at each observation



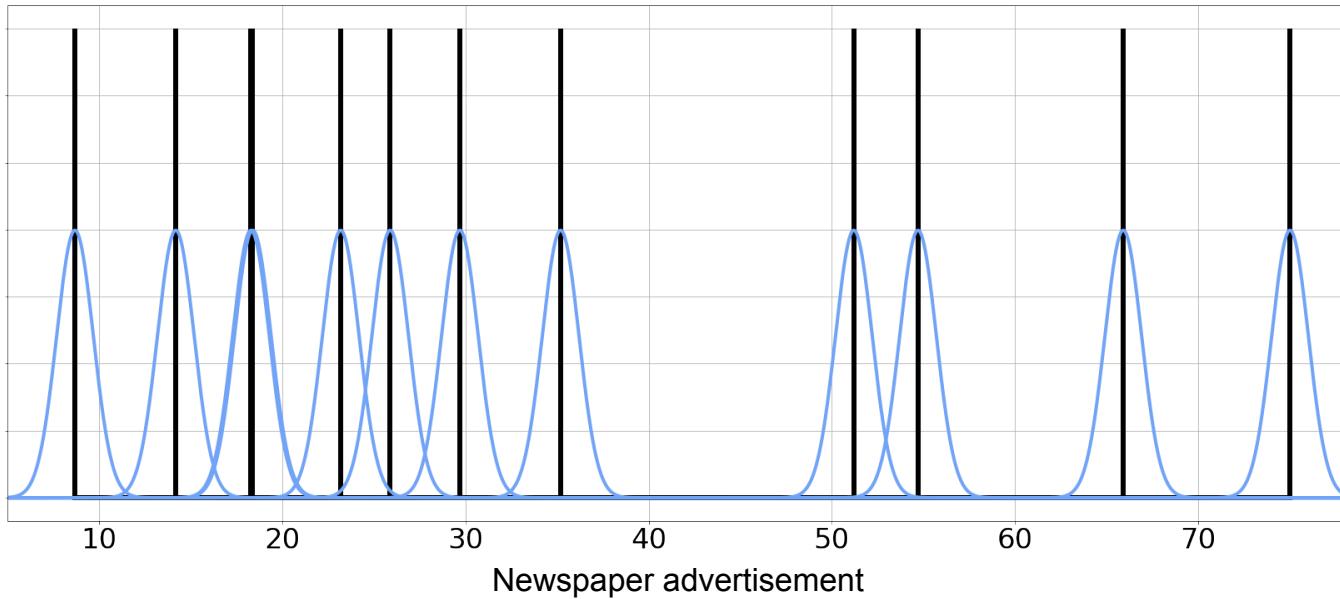
Kernel Density Estimation

Second: draw a gaussian centered at each observation



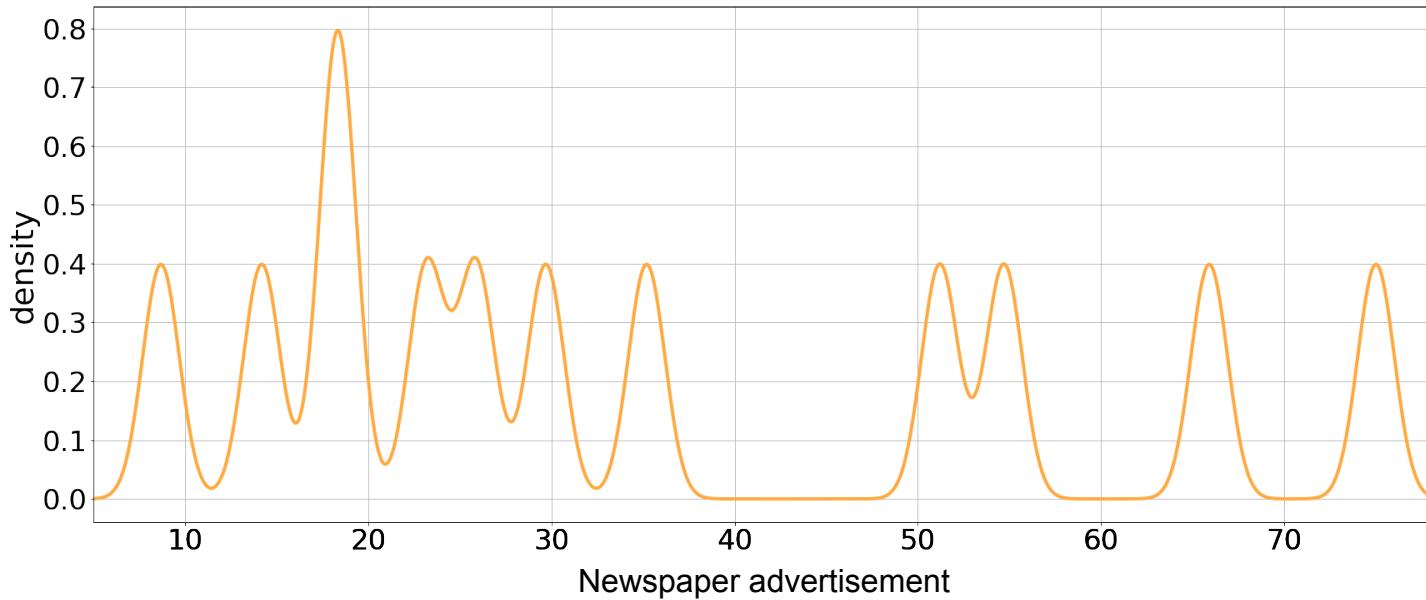
Kernel Density Estimation

Second: draw a gaussian centered at each observation



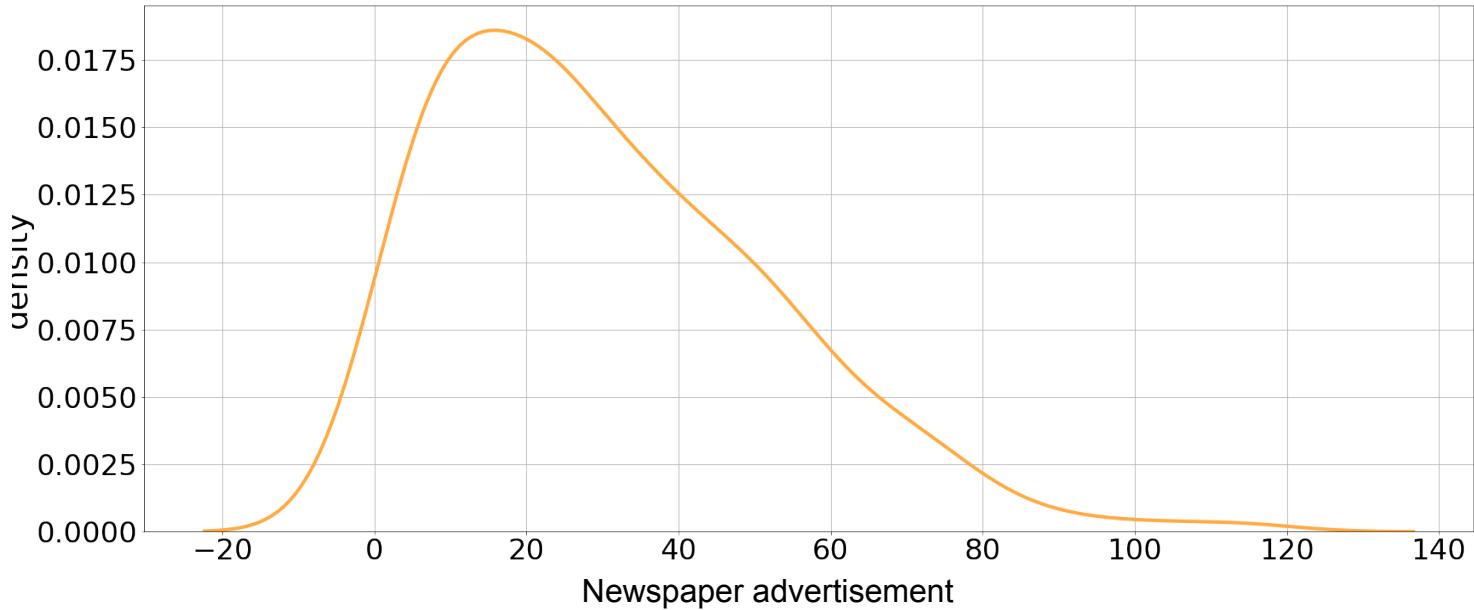
Kernel Density Estimation

Third: multiply everything by $1/n$ and sum the curves



Kernel Density Estimation

What if
you used
all the
dataset?



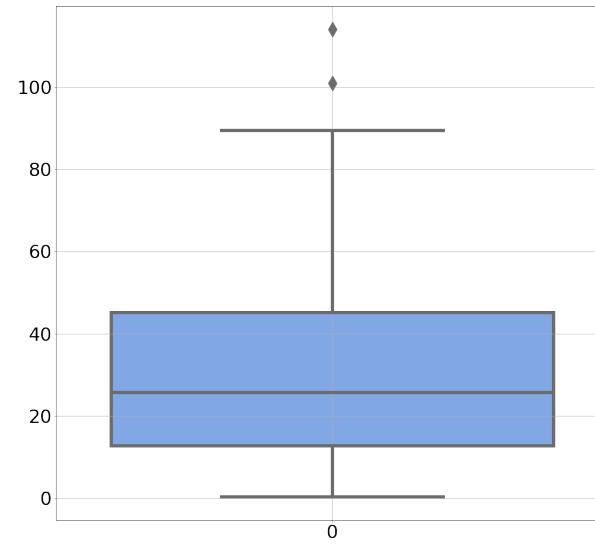
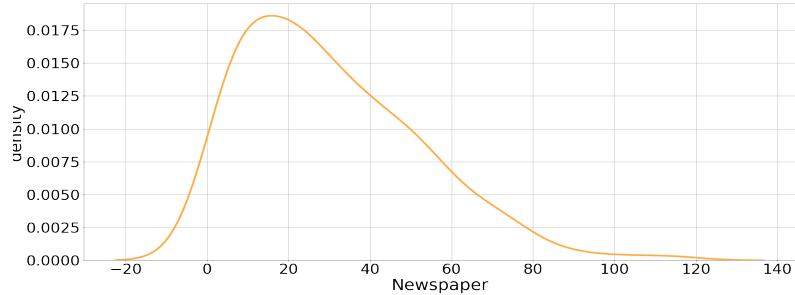


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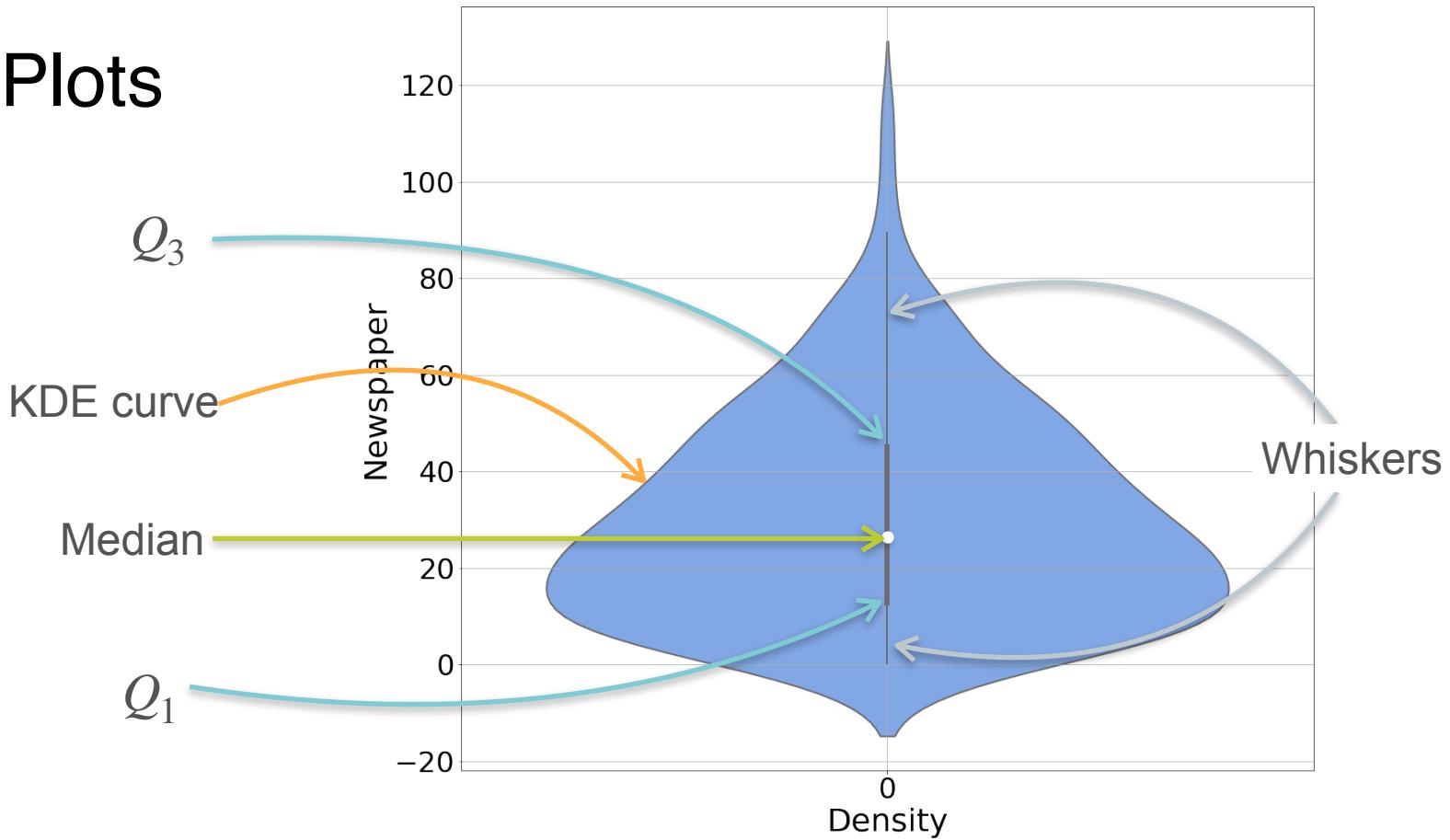
Describing Distributions

**Visualizing data:
Violin Plots**

Violin Plots



Violin Plots





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Describing Distributions

**Visualizing data:
QQ plots**

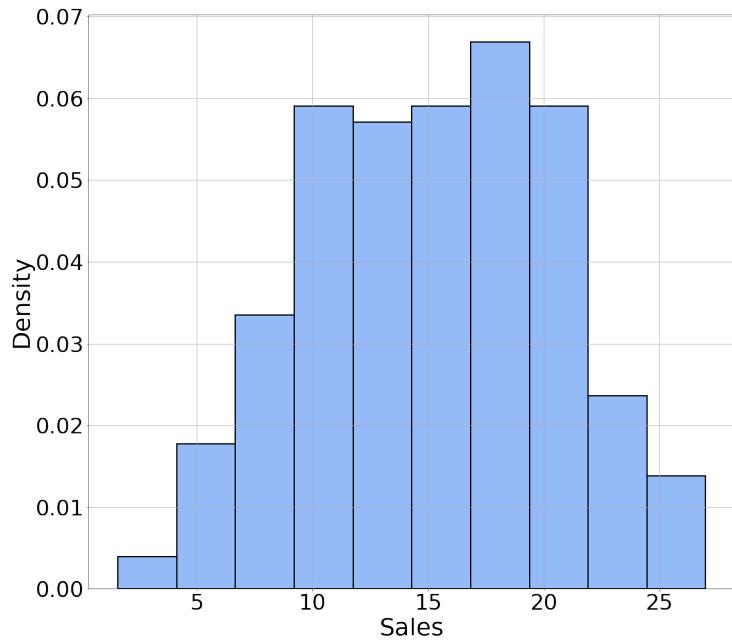
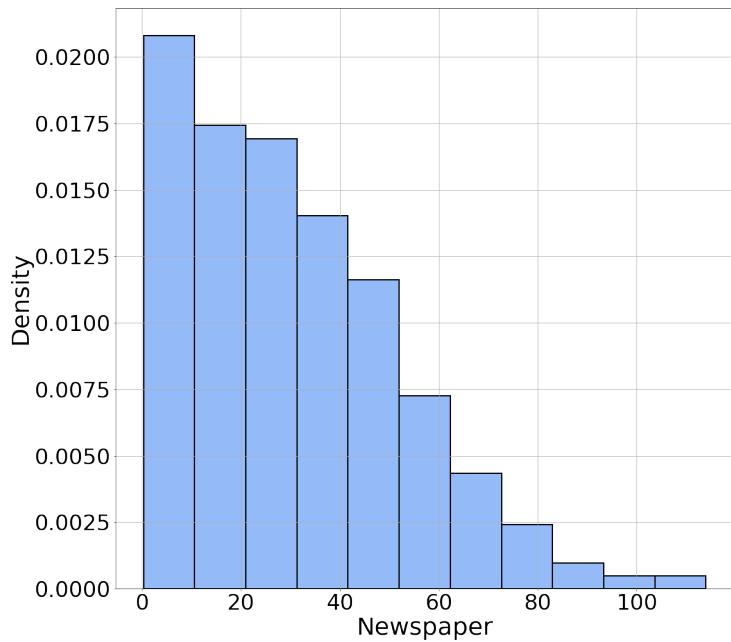
Assessing Normality of Data

Some models assume normally distributed data

- Linear regression
- Logistic regression
- Gaussian Naive Bayes
- Others

Some tests used in Data Science also assume normality.

Assessing Normality of Data



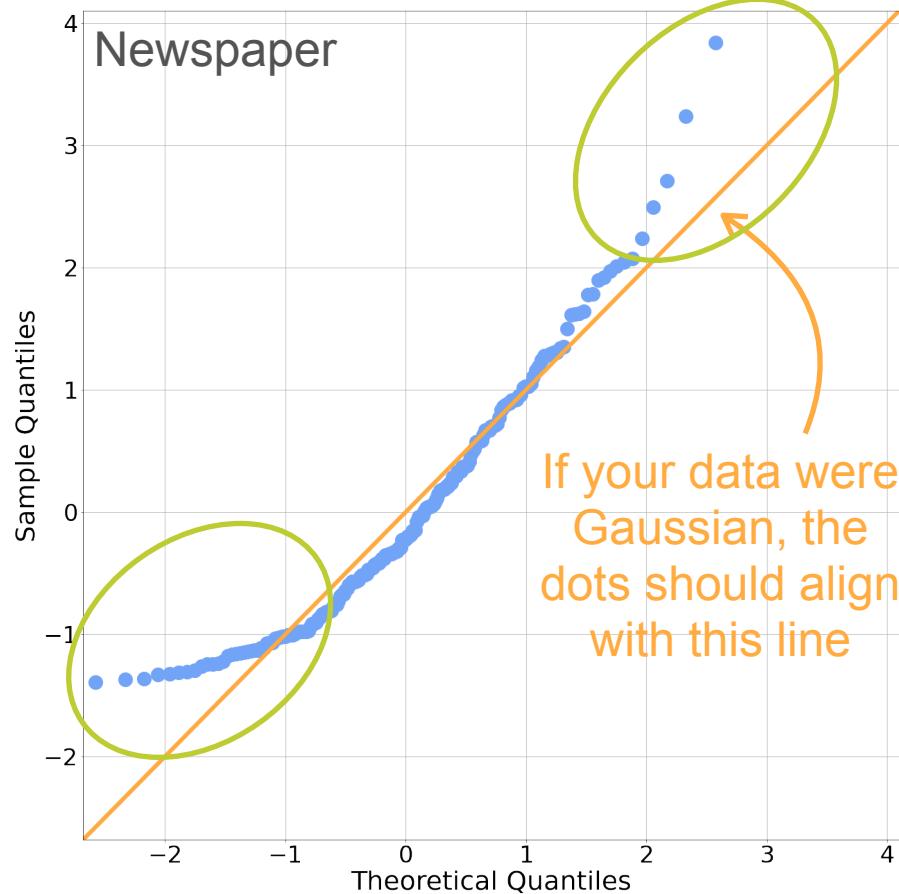
QQ Plots

Quantile-Quantile plots (QQ Plots) compare quantiles

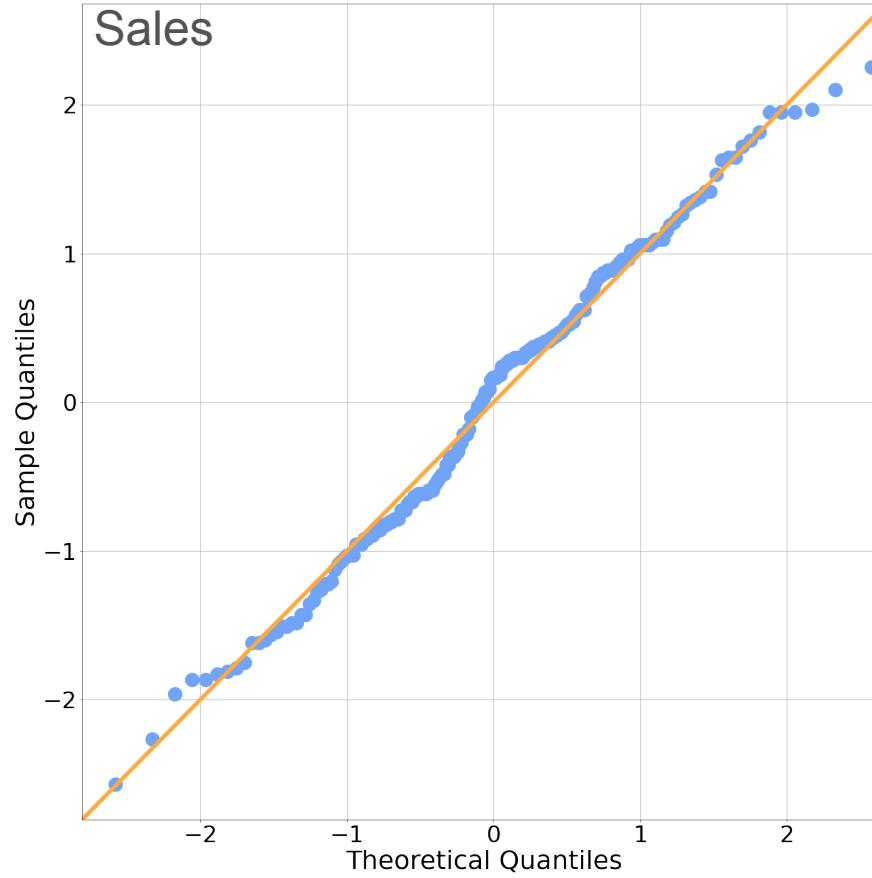
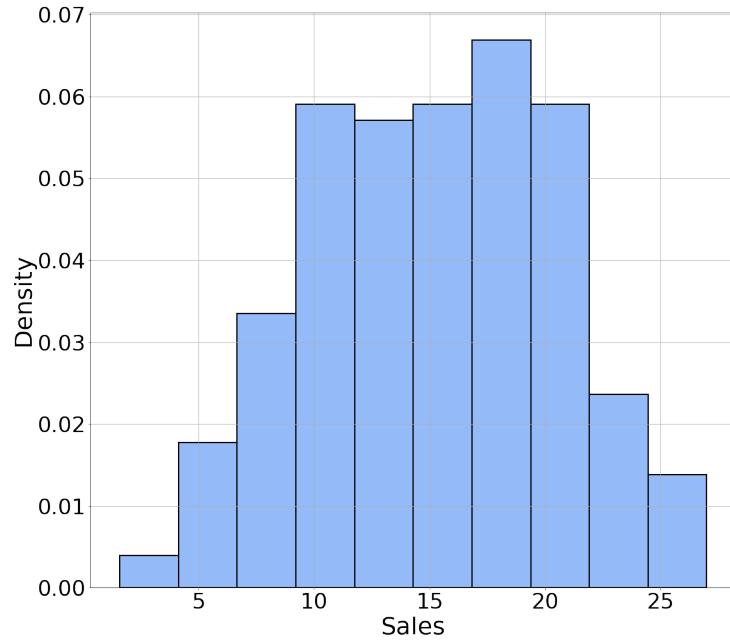
- Standardize your data:

$$\left(\frac{x - \mu}{\sigma} \right)$$

- Compute quantiles
- Compare to gaussian quantiles



QQ Plots



W2 Lesson 2



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Probability Distributions with Multiple Variables

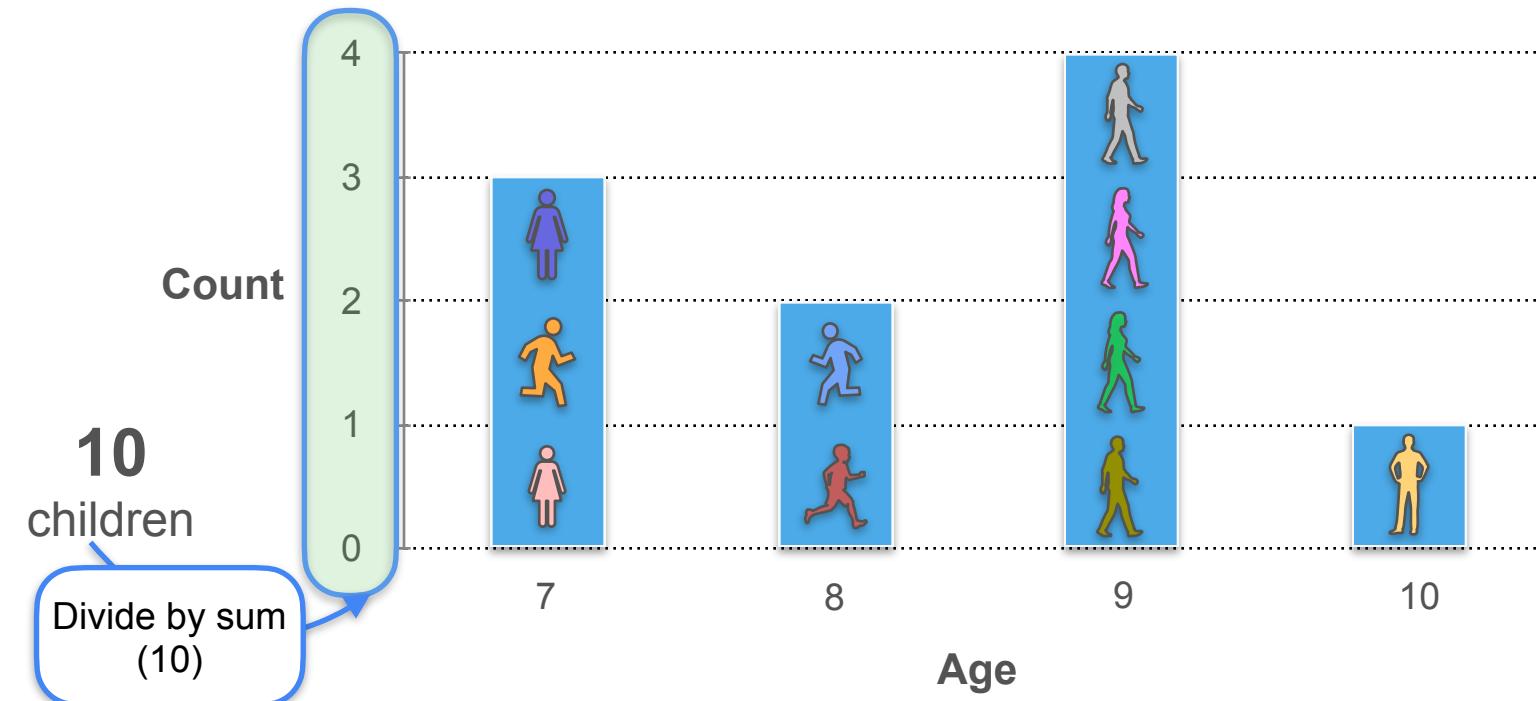
**Joint Distribution
(Discrete)
Part 1**

Joint Distributions (Discrete): Example 1

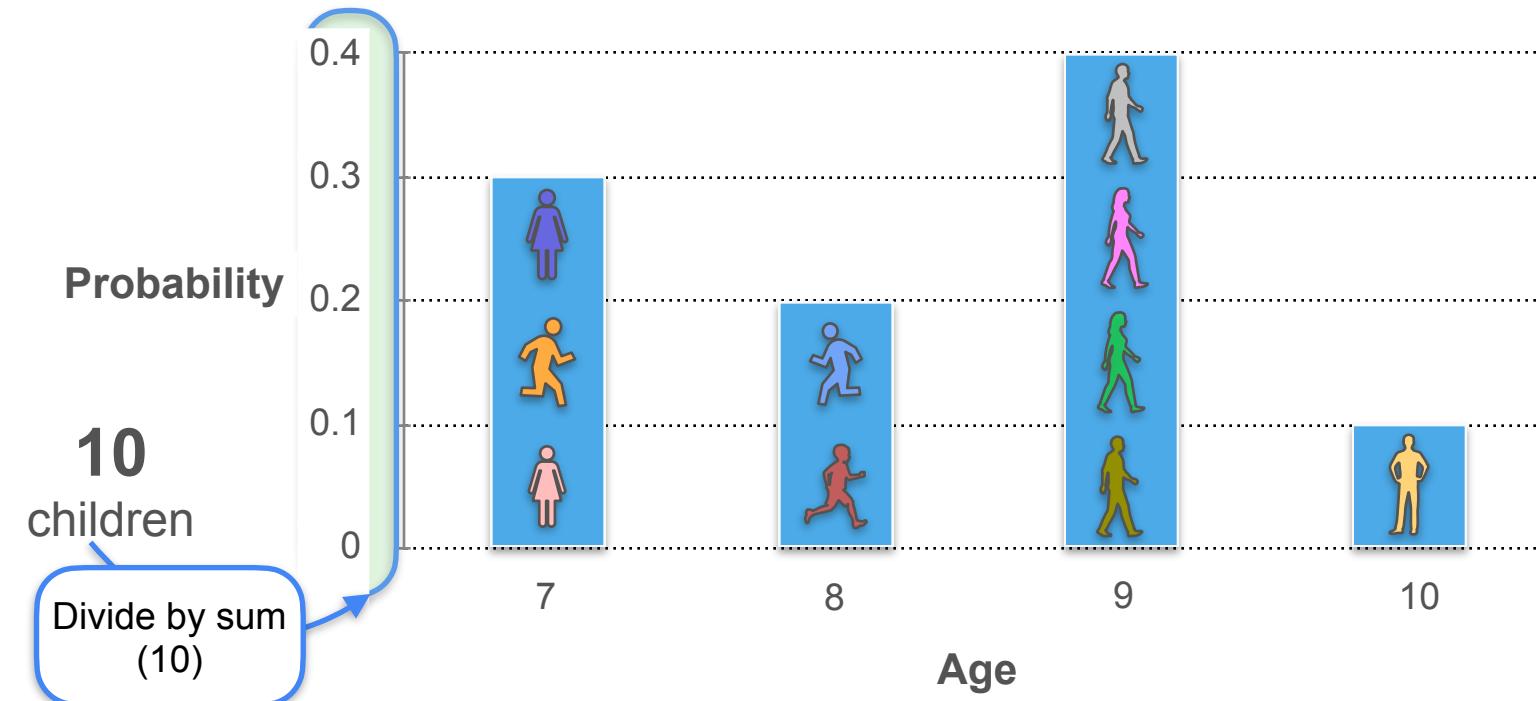
Age (Year)	Count	
7	3	
8	2	
9	4	
10	1	

10
children

Joint Distributions (Discrete): Example 1



Joint Distributions (Discrete): Example 1



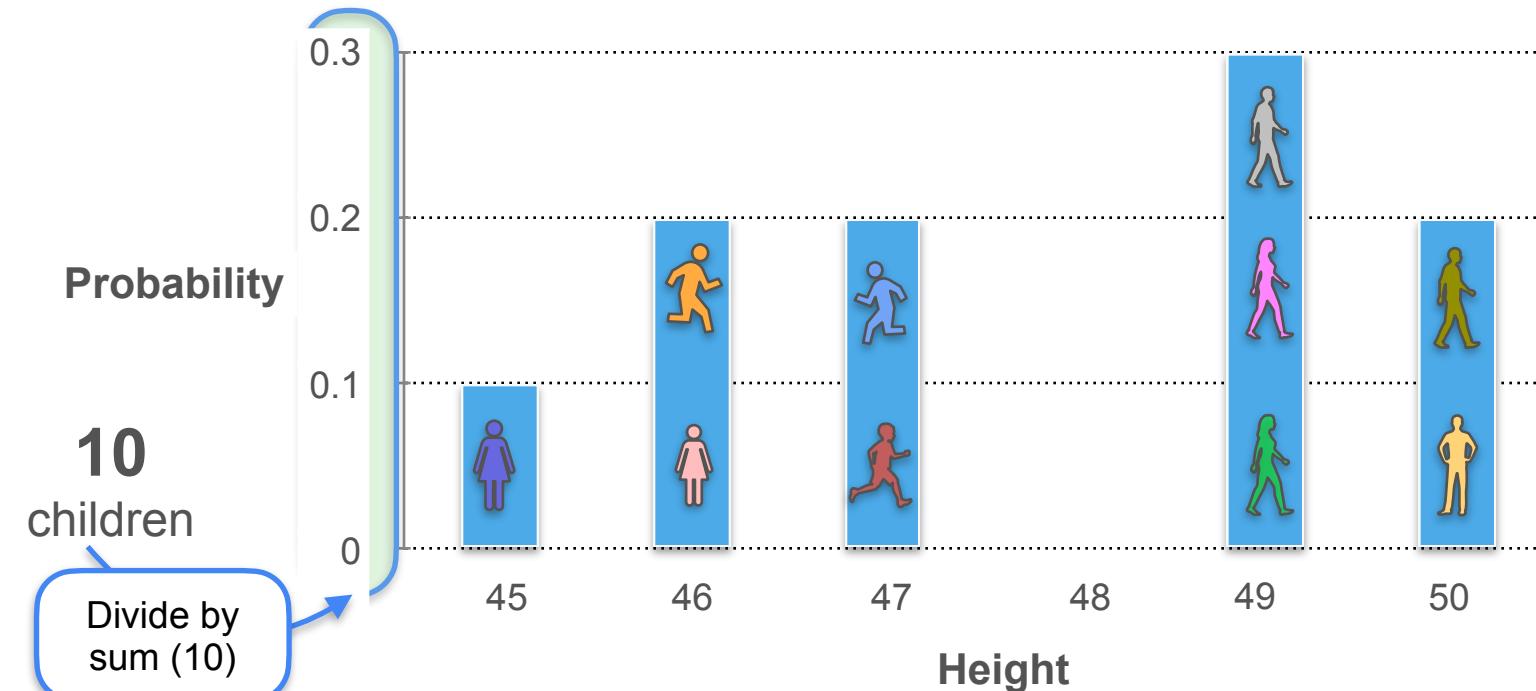
Joint Distributions (Discrete): Example 1

Age (years): 7 7 7 8 8 9 9 9 9 10

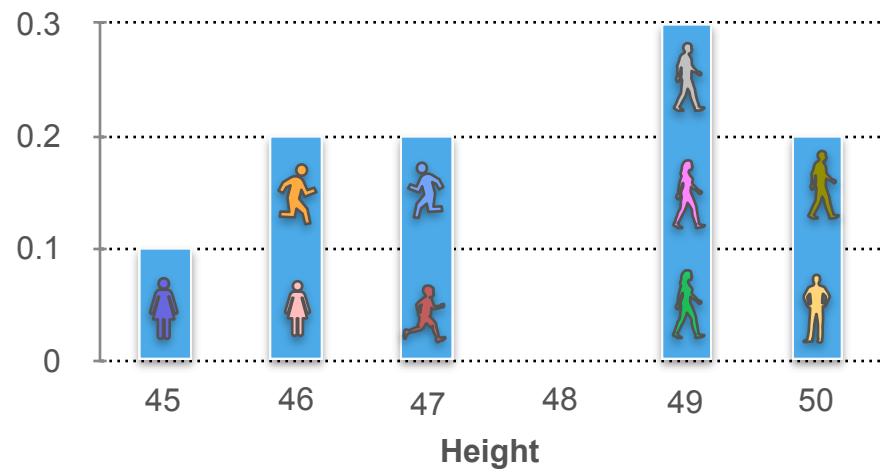
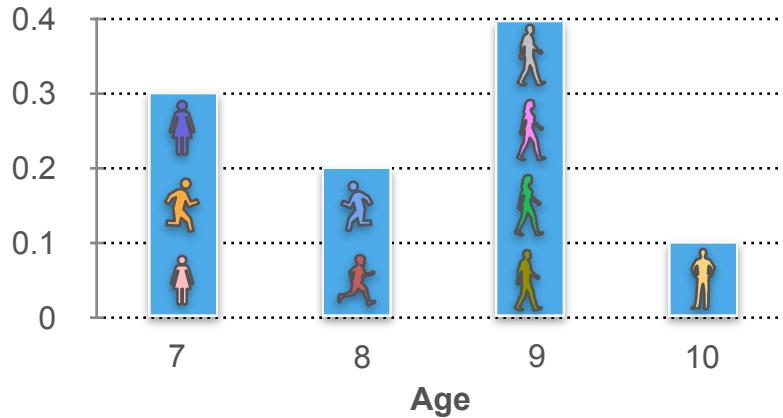
Height (in): 45 46 46 47 47 49 49 49 50 50

Height (in)	Count
45	1
46	2
47	2
48	0
49	3
50	2

Joint Distributions (Discrete): Example 1

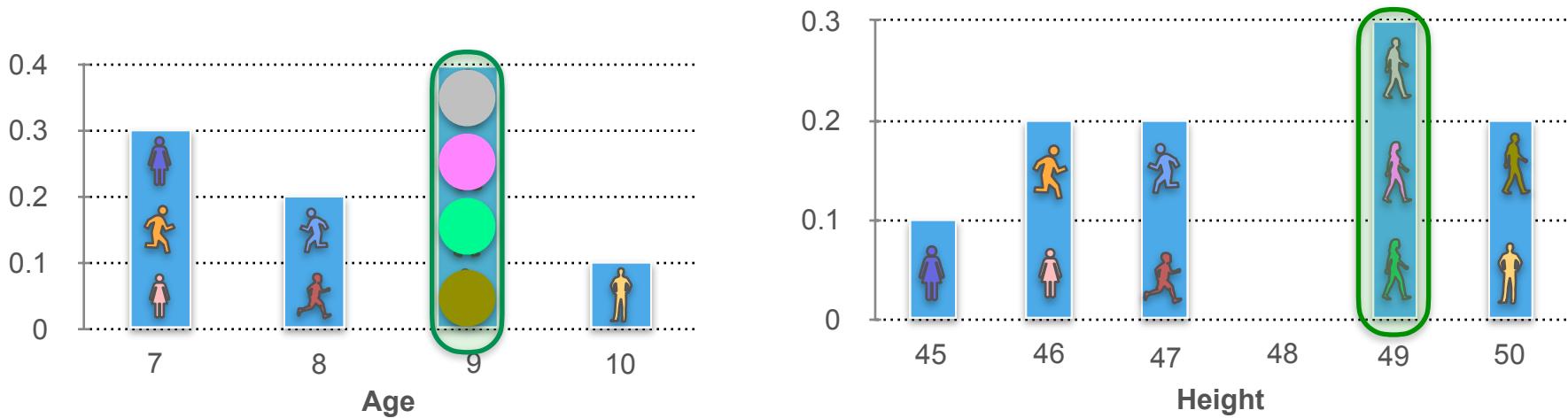


Joint Distributions (Discrete): Example 1



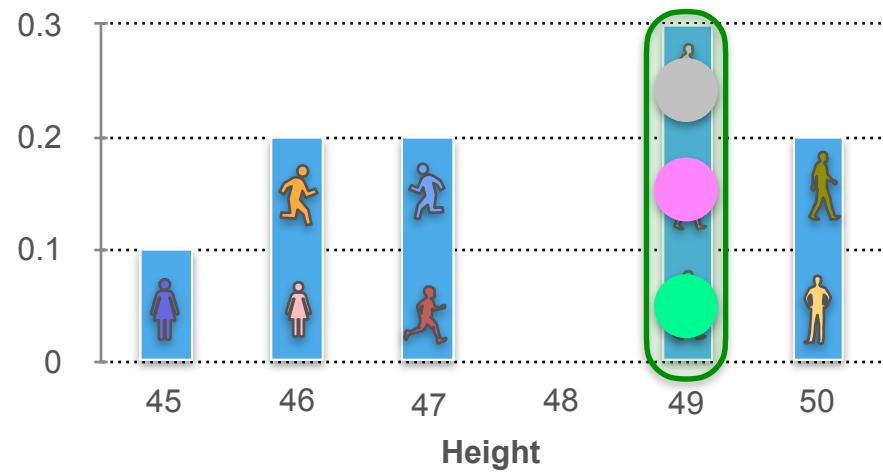
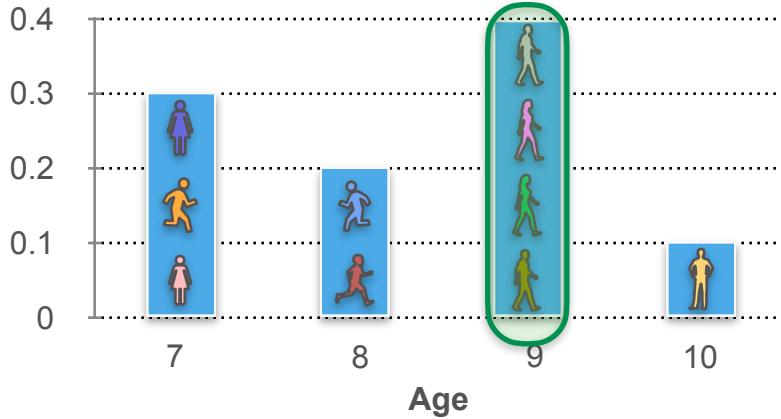
What is the probability that a child is **9 years** old and **49 inches** tall?

Joint Distributions (Discrete): Example 1



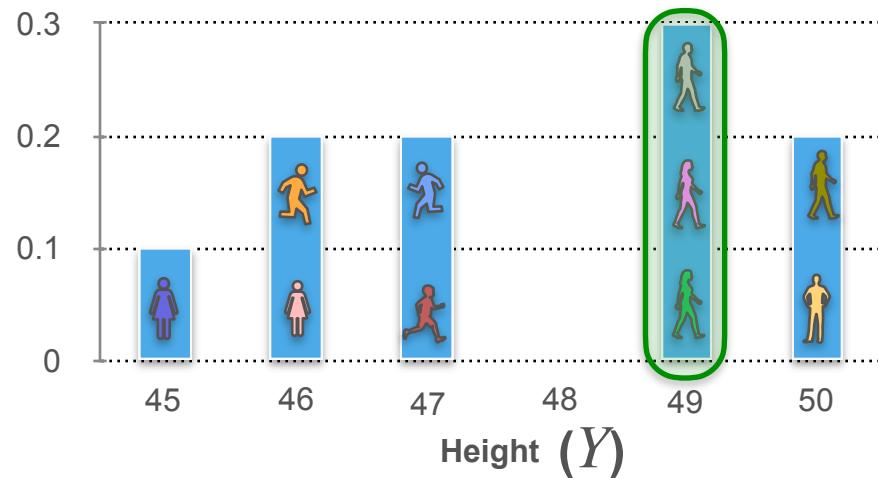
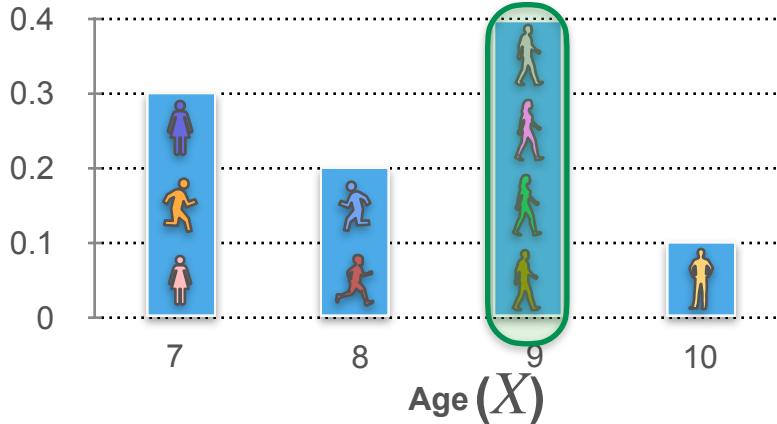
What is the probability that a child is **9 years** old and **49 inches** tall?

Joint Distributions (Discrete): Example 1



What is the probability that a child is **9 years** old and **49 inches** tall?

Joint Distributions (Discrete): Example 1



What is the probability that a child is 9 years old and 49 inches tall?

$$p_{XY}(9, 49) = \mathbf{P}(X = 9, Y = 49) = \frac{\text{Grey circle}}{10} = \frac{3}{10}$$

Joint Distributions (Discrete): Example 1

What is the probability that a child is **9 years** old and **49 inches** tall?

$$p_{XY}(9, 49) = \mathbf{P}(X = 9, Y = 49) = \frac{\text{_____}}{10} = \frac{3}{10}$$

$$p_{XY}(x, y) = \mathbf{P}(X = x, Y = y)$$

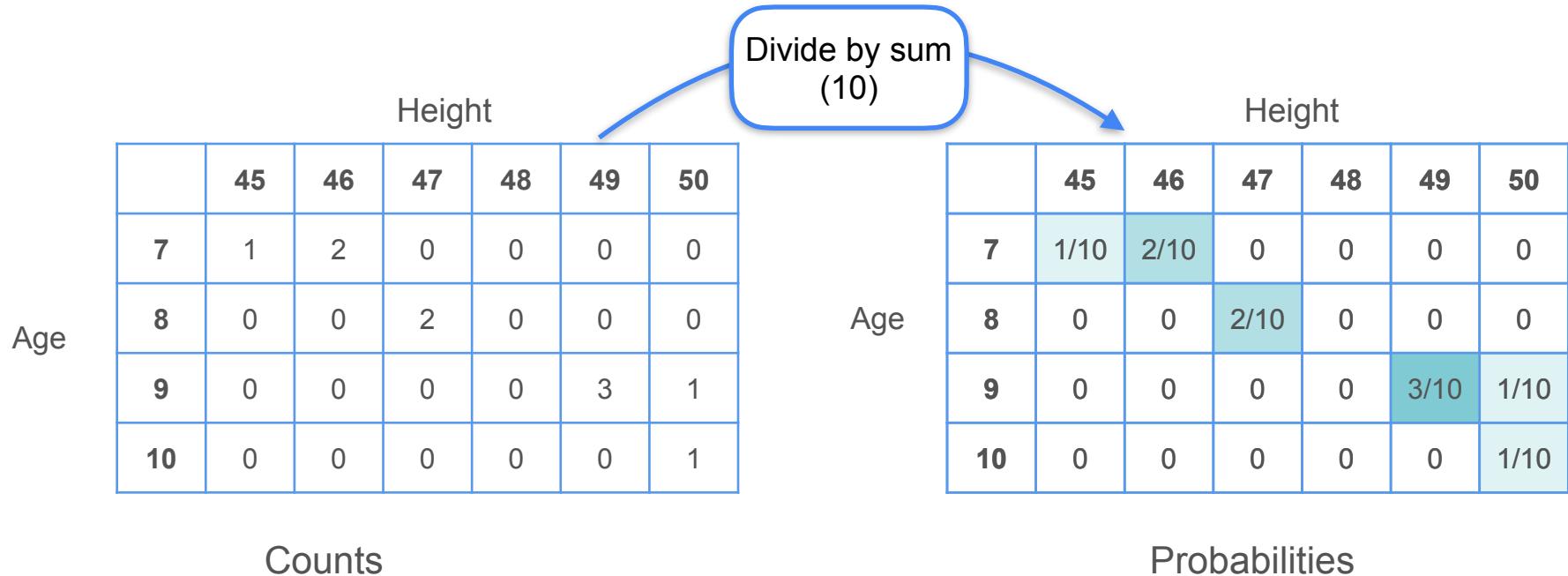
Joint Distributions: Example 1

		 	 	  	 	
Age (years):	7	7	8	9	9	10
Height (in):	45	46	47	49	49	50

Height

	45	46	47	48	49	50
7	1	2	0	0	0	0
8	0	0	2	0	0	0
9	0	0	0	0	3	1
10	0	0	0	0	0	1

Joint Distributions: Example 1



Joint Distributions: Example 1

Joint Distribution

All probabilities for all possible combinations of X and Y

		Height (Y)					
		45	46	47	48	49	50
Age (X)	7	1/10	2/10	0	0	0	0
	8	0	0	2/10	0	0	0
	9	0	0	0	0	3/10	1/10
	10	0	0	0	0	0	1/10
		Probabilities					

Joint Distributions: Example 1

What is the probability that a child is 8 and 48 inches tall?

$$p_{XY}(x, y) = \mathbf{P}(X = x, Y = y)$$

$$p_{XY}(8, 48) = \mathbf{P}(X = 8, Y = 48)$$

$$p_{XY}(8, 48) = 0$$

$$p_{XY}(7, 46) = \frac{2}{10}$$

	Height (Y)					
	45	46	47	48	49	50
Age (X)	7	1/10	2/10	0	0	0
7	0	0	2/10	0	0	0
8	0	0	0	0	3/10	1/10
9	0	0	0	0	0	0
10	0	0	0	0	0	0

Probabilities



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Probability Distributions with Multiple Variables

**Joint Distribution
(Discrete)
Part 2**

Joint Distributions (Discrete): Example 2

X

the number rolled on the 1st dice



$$\frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6}$$

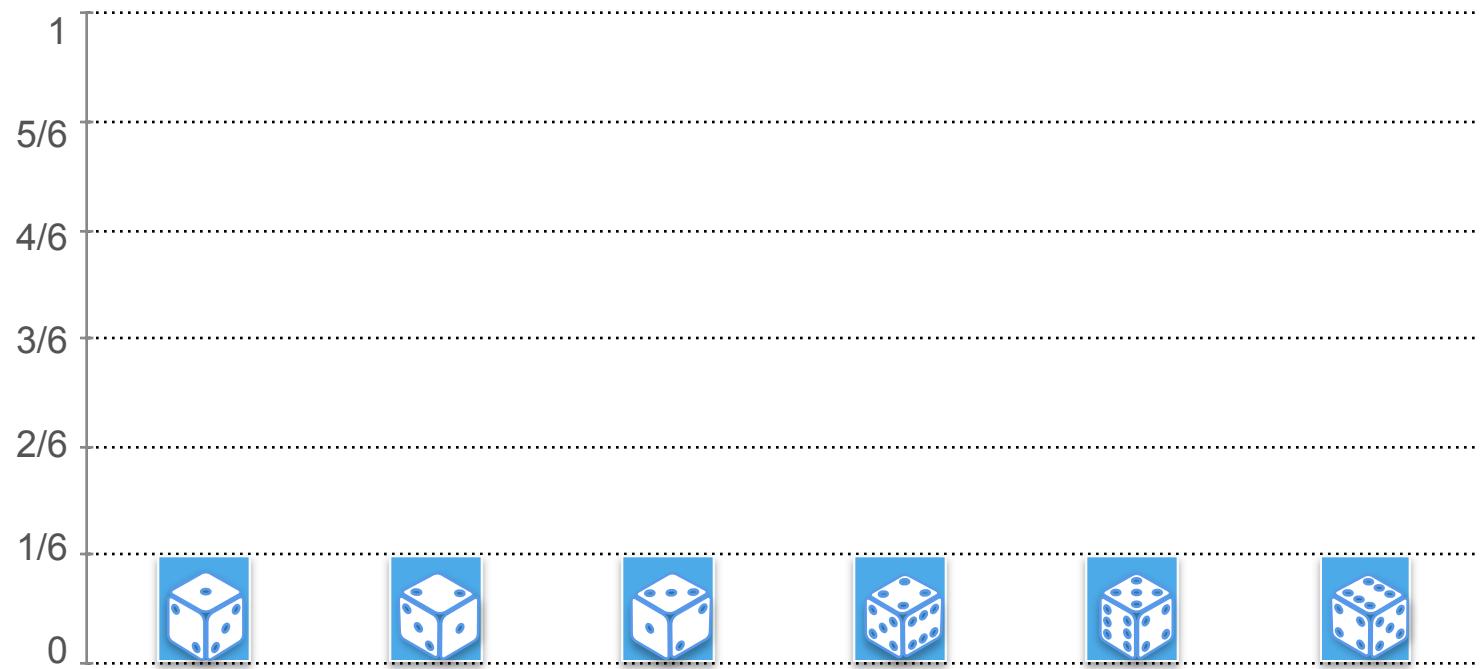
Y

the number rolled on the 2nd dice



$$\frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6}$$

Joint Distributions: Example 2



Joint Distributions: Example 2

X : the number rolled on the 1st dice

Y : the number rolled on the 2nd dice

X and Y are independent



$$P(x) \quad 1/6 \quad 1/6 \quad 1/6 \quad 1/6 \quad 1/6 \quad 1/6$$

$$P(y) \quad 1/6 \quad 1/6 \quad 1/6 \quad 1/6 \quad 1/6 \quad 1/6$$

$$p_{XY}(\text{dice } x, \text{dice } y) = P(\text{dice } x) \cdot P(\text{dice } y) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

$$p_{XY}(x, y) = P(X = x, Y = y) = \frac{1}{36}$$

	1	2	3	4	5	6
1	1/36	1/36	1/36	1/36	1/36	1/36
2	1/36	1/36	1/36	1/36	1/36	1/36
3	1/36	1/36	1/36	1/36	1/36	1/36
4	1/36	1/36	1/36	1/36	1/36	1/36
5	1/36	1/36	1/36	1/36	1/36	1/36
6	1/36	1/36	1/36	1/36	1/36	1/36

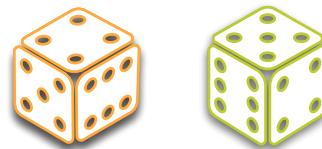
Joint Distributions: Example 2

Thus for independent discrete random variables:

$$p_{XY}(x, y) = \mathbf{P}(X = x, Y = y) = \mathbf{P}(x) \cdot \mathbf{P}(y)$$

Joint Distributions (Discrete): Example 2

X



Y

the number rolled on the 1st dice

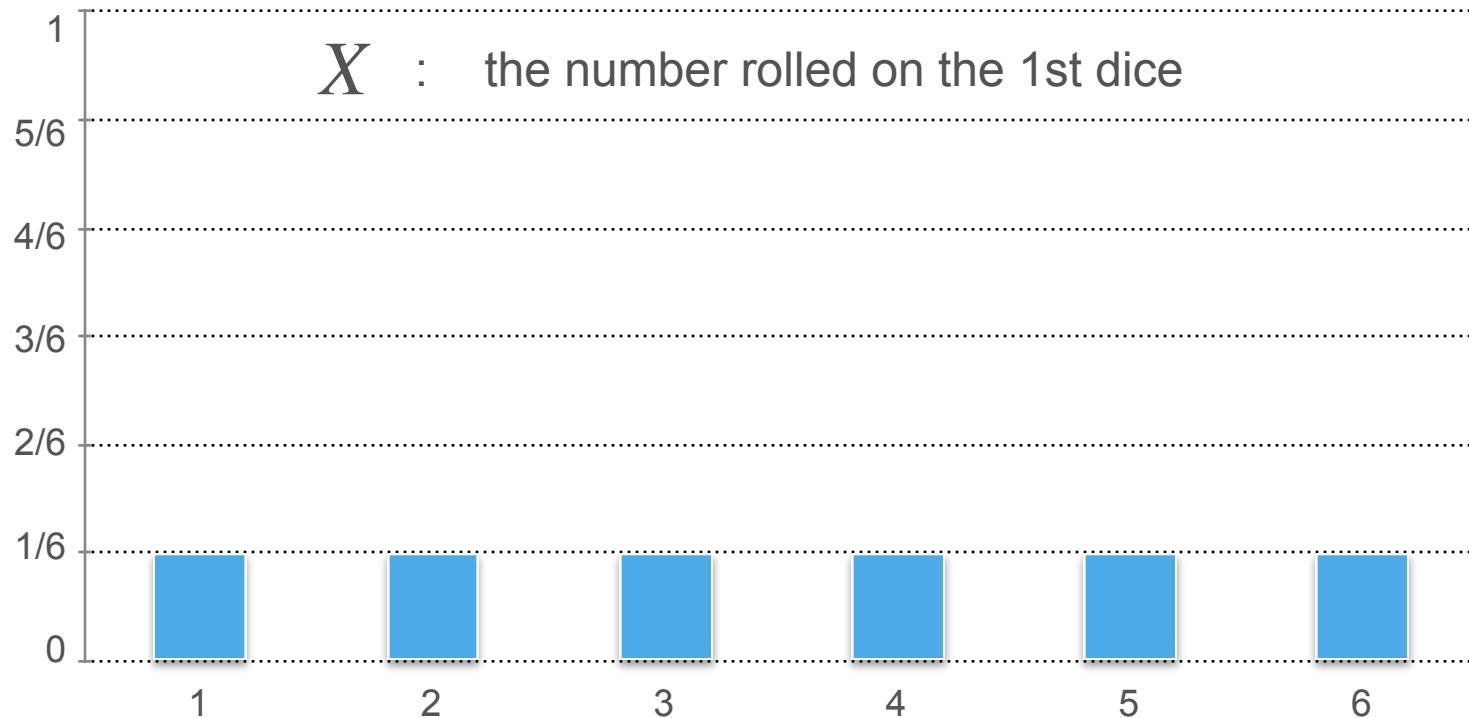
sum of the two dice

+

$$X = 4$$

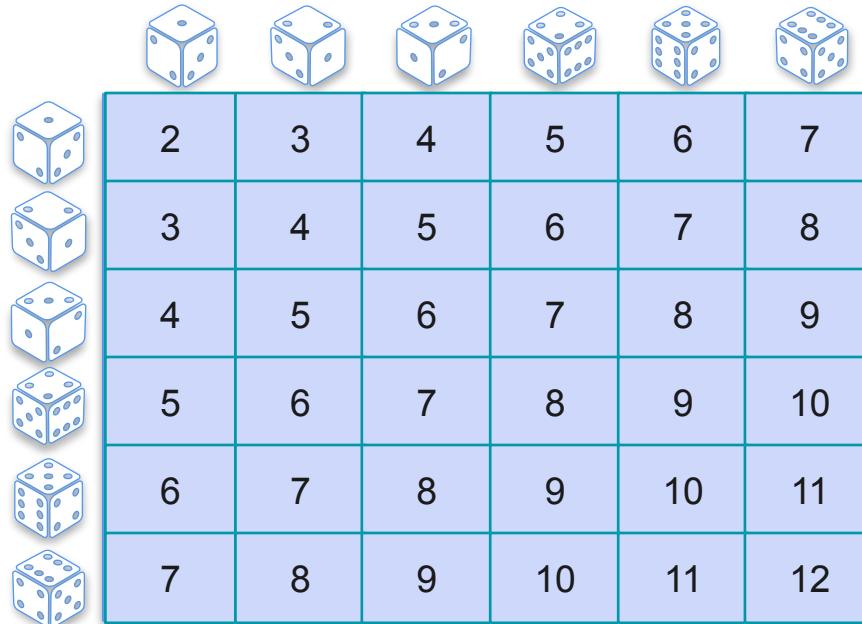
$$Y = 4 + 5$$

Joint Distributions: Example 2



Joint Distributions - Example 3

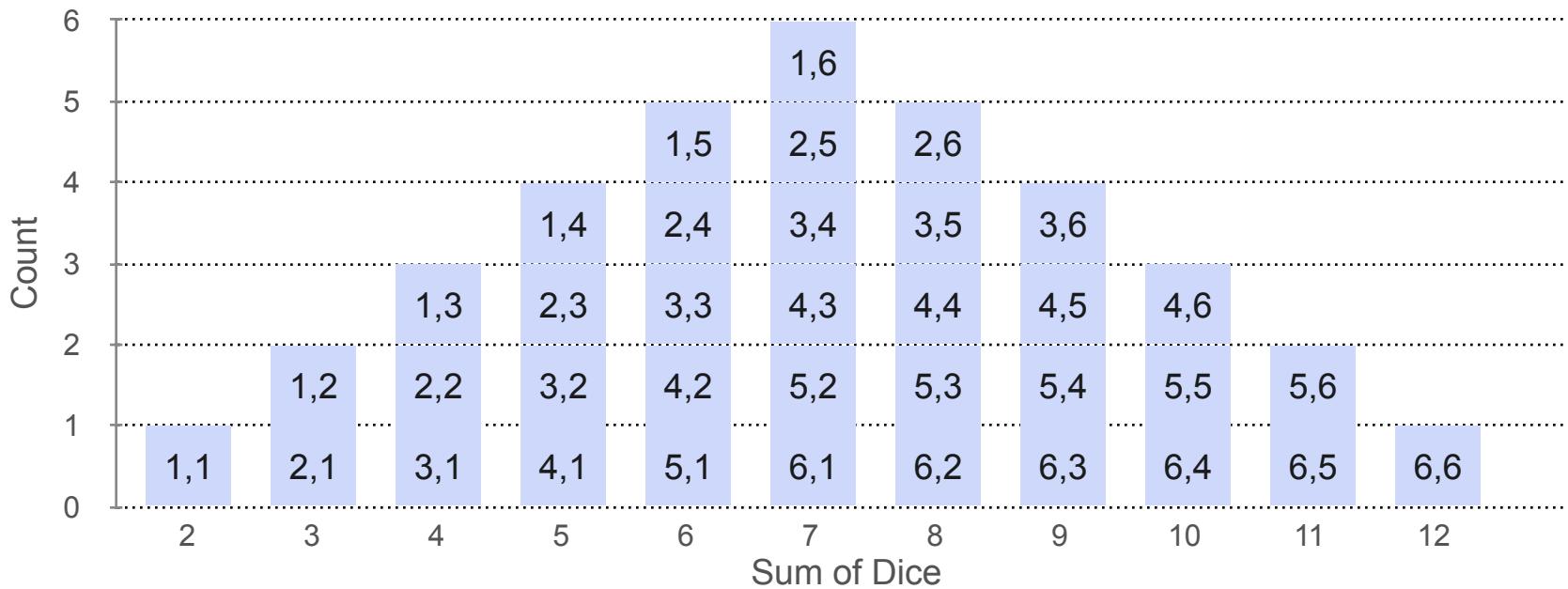
Y : Sum of both dice



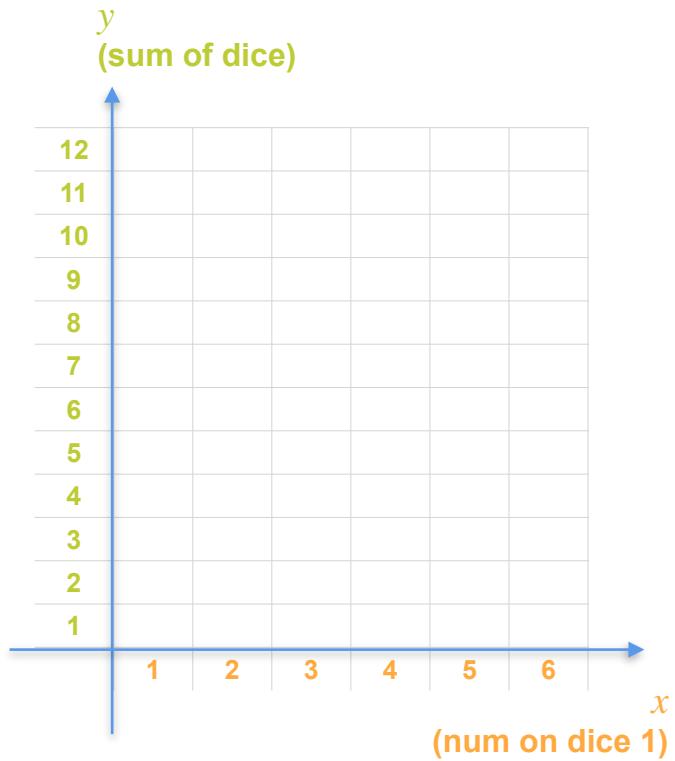
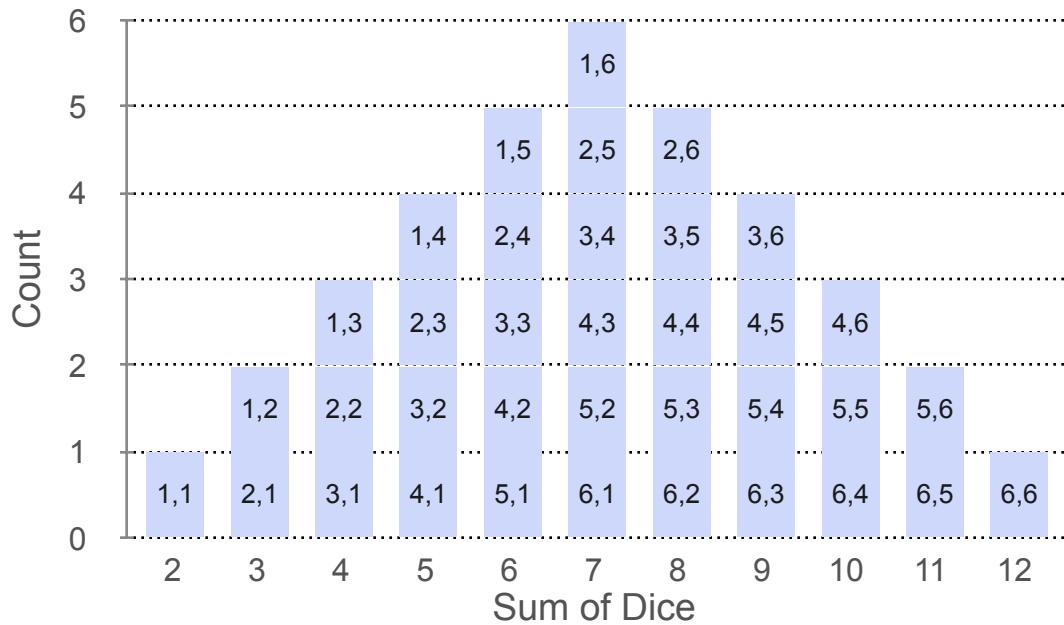
Joint Distributions: Example 3



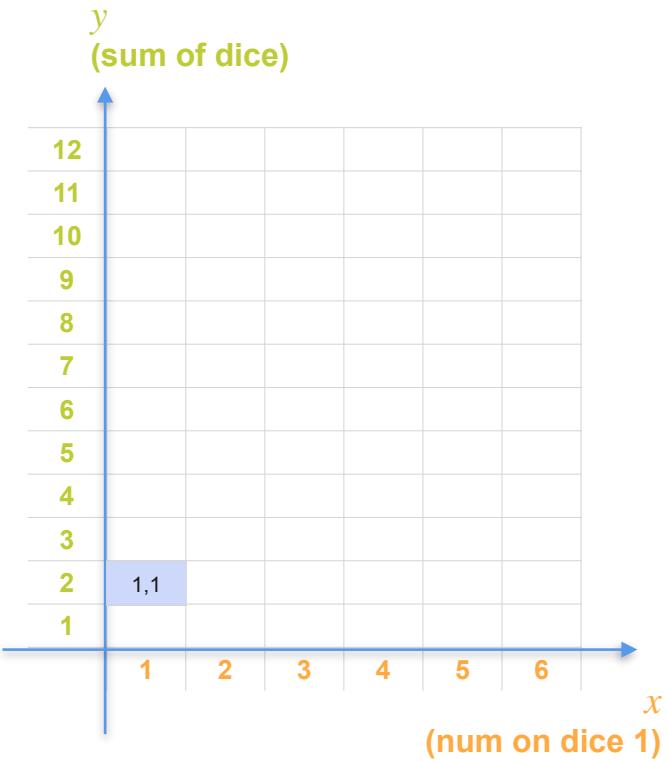
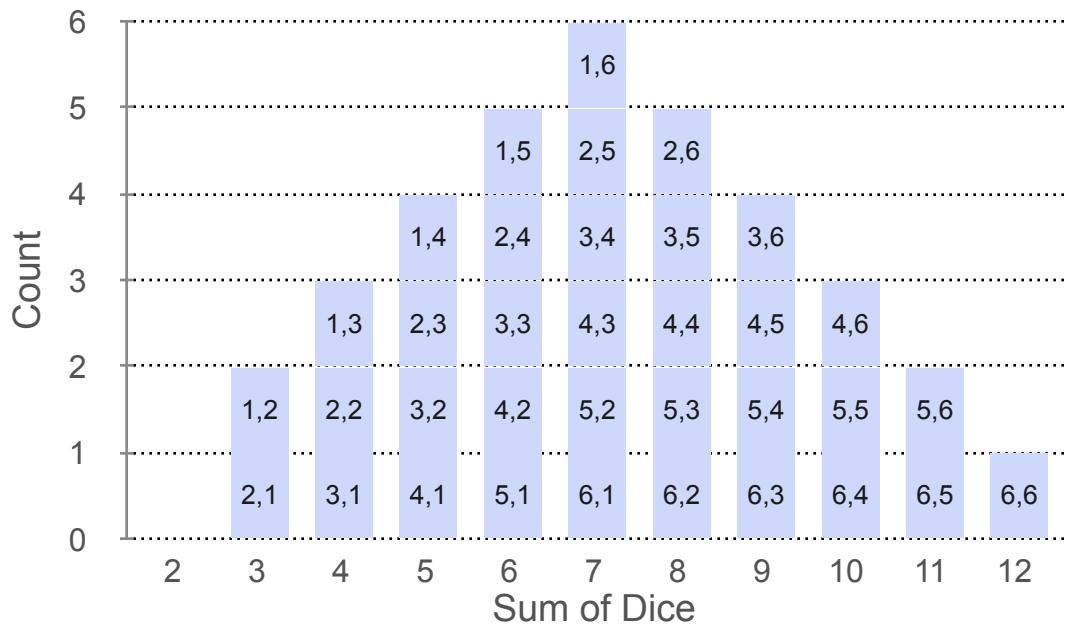
Joint Distributions: Example 3



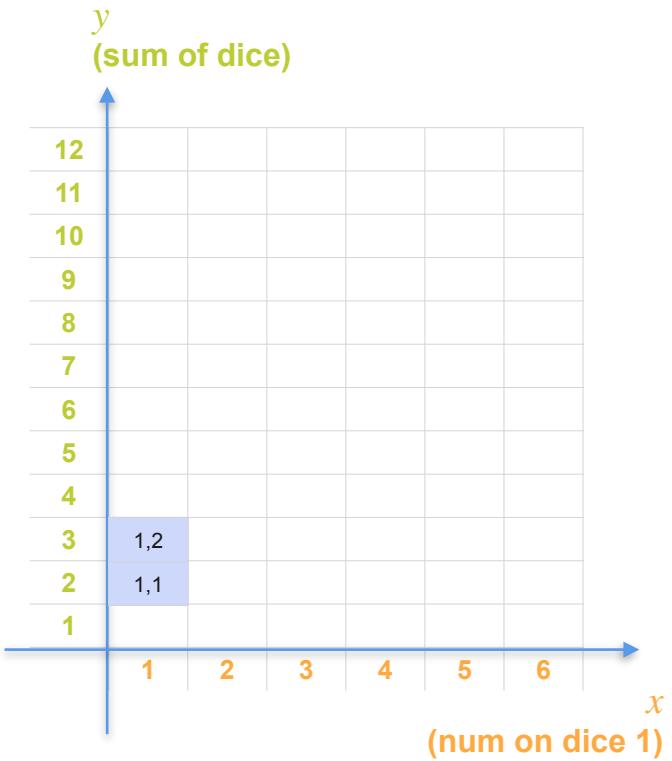
Joint Distributions: Example 3



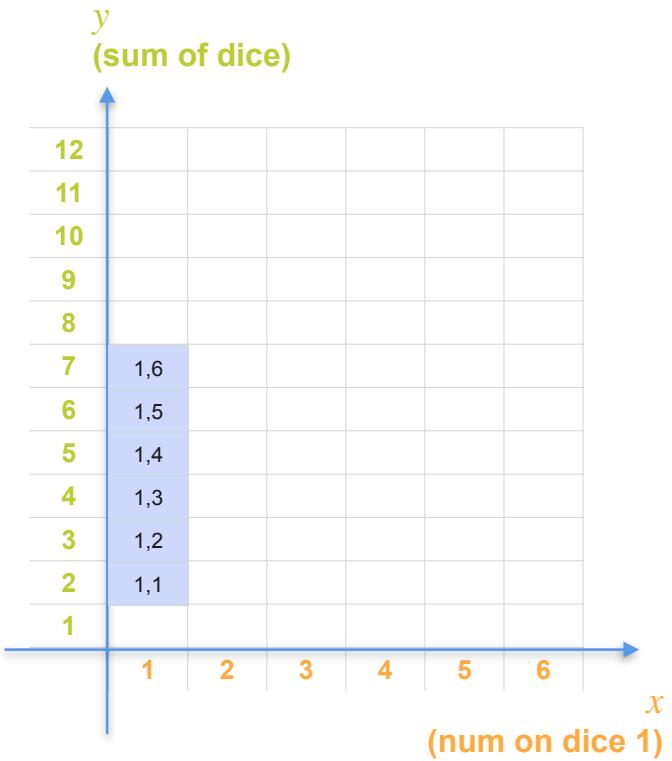
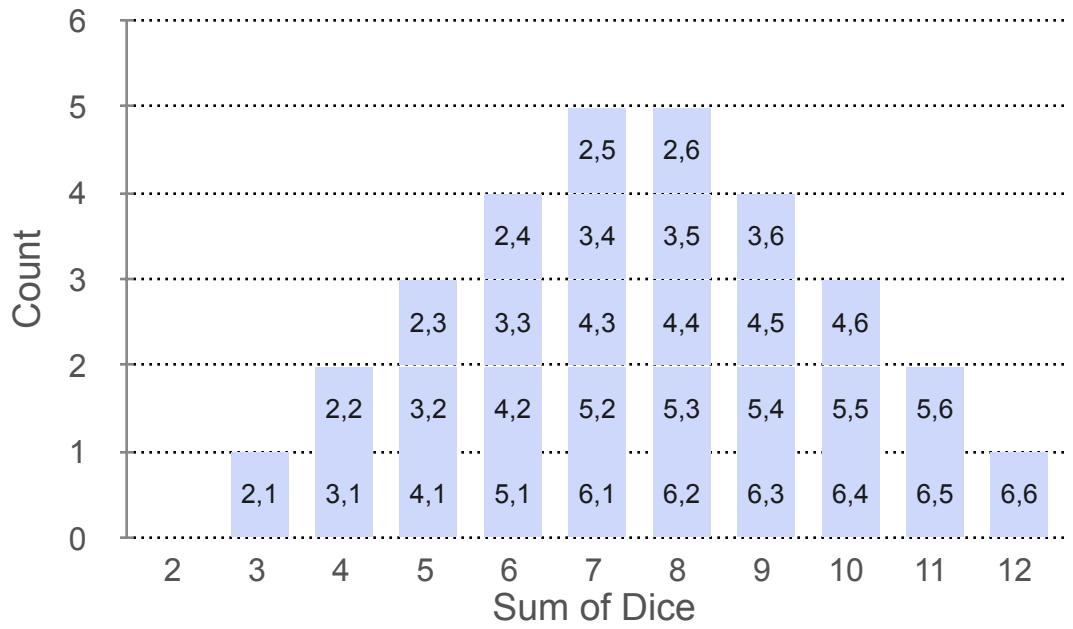
Joint Distributions: Example 3



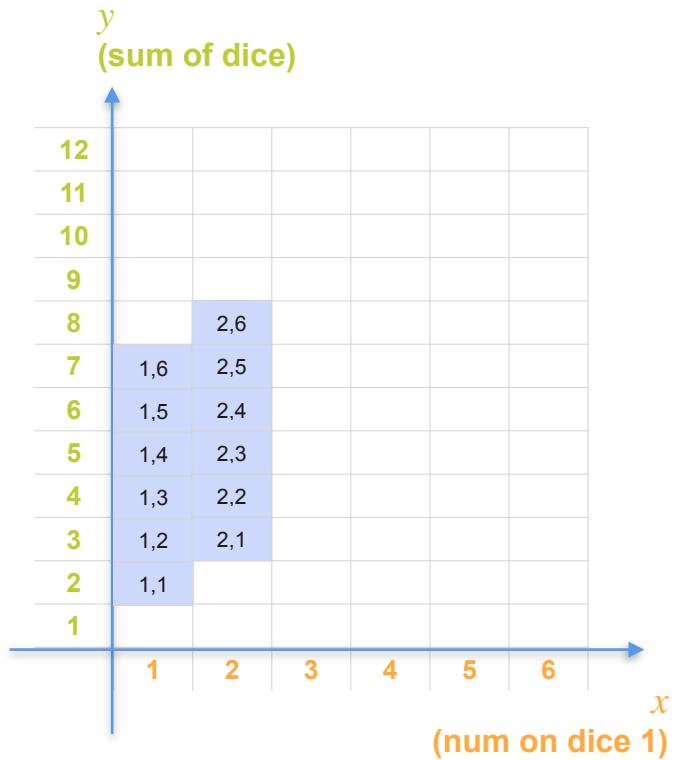
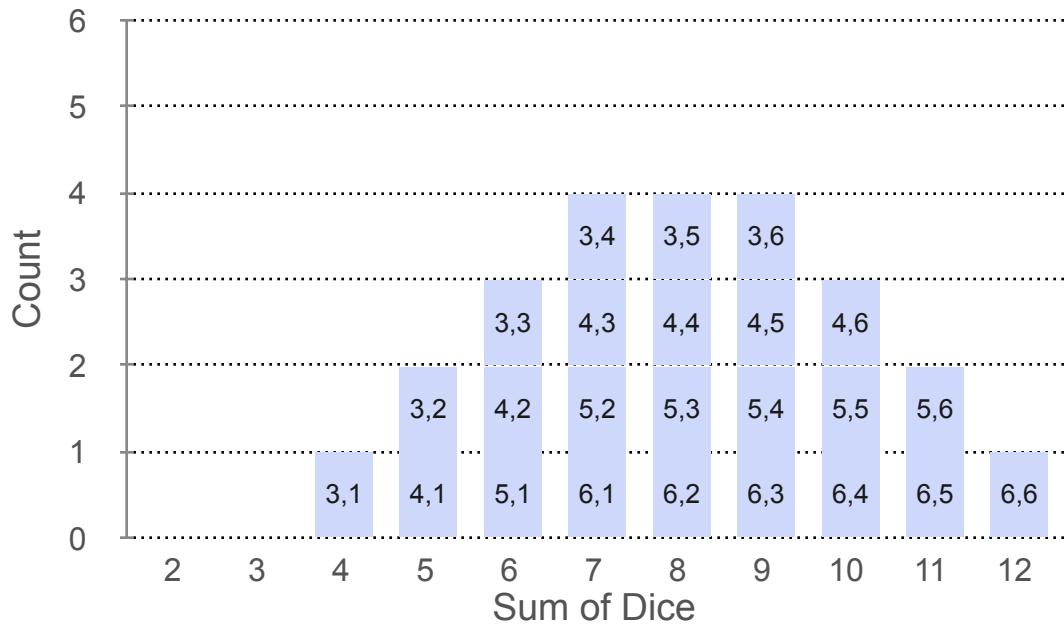
Joint Distributions: Example 3



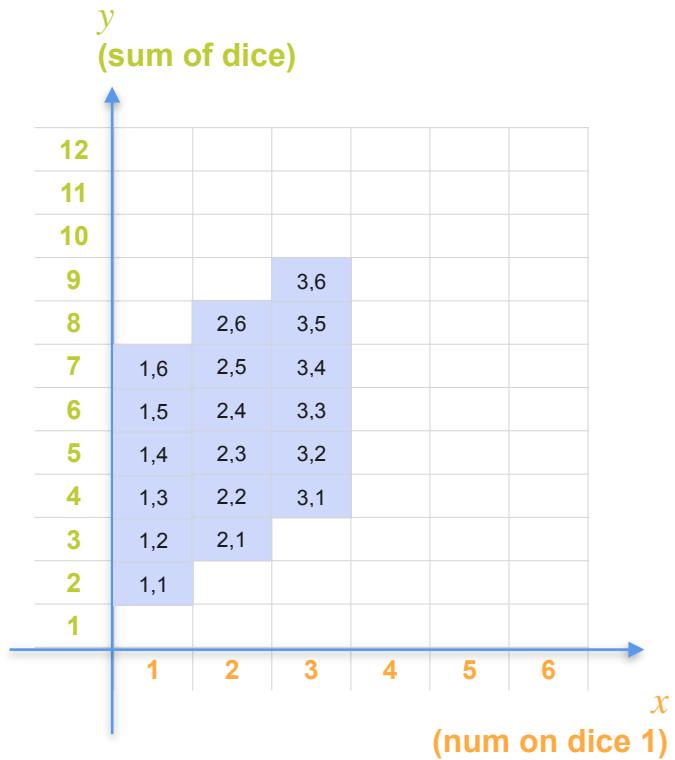
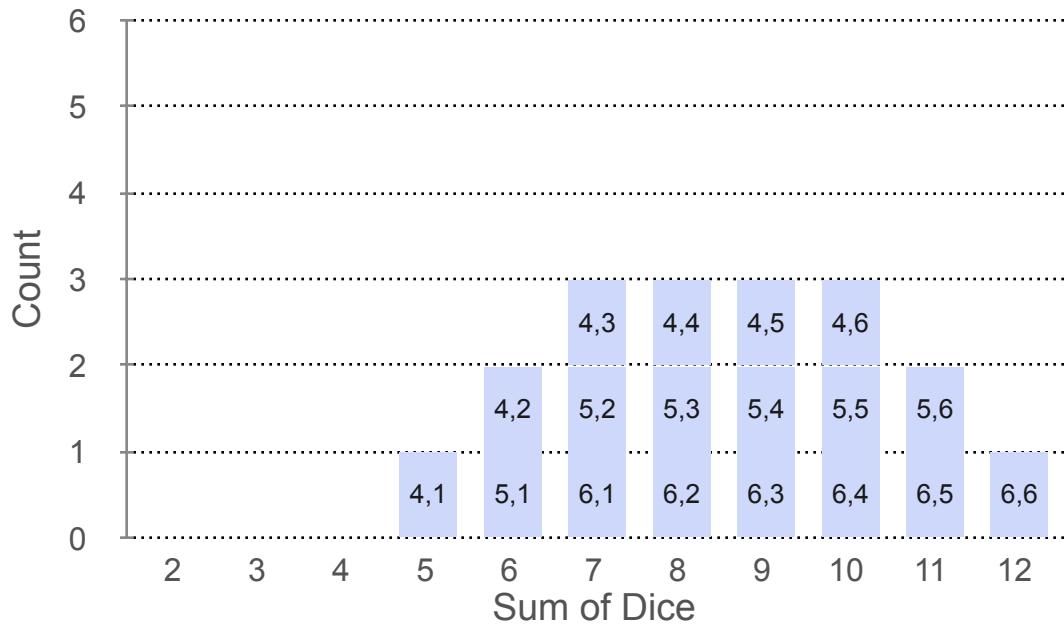
Joint Distributions: Example 3



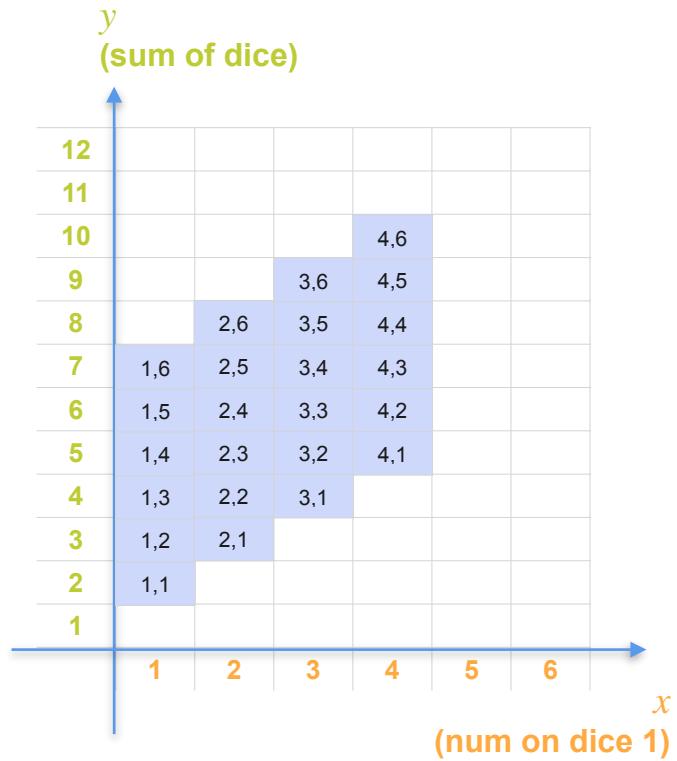
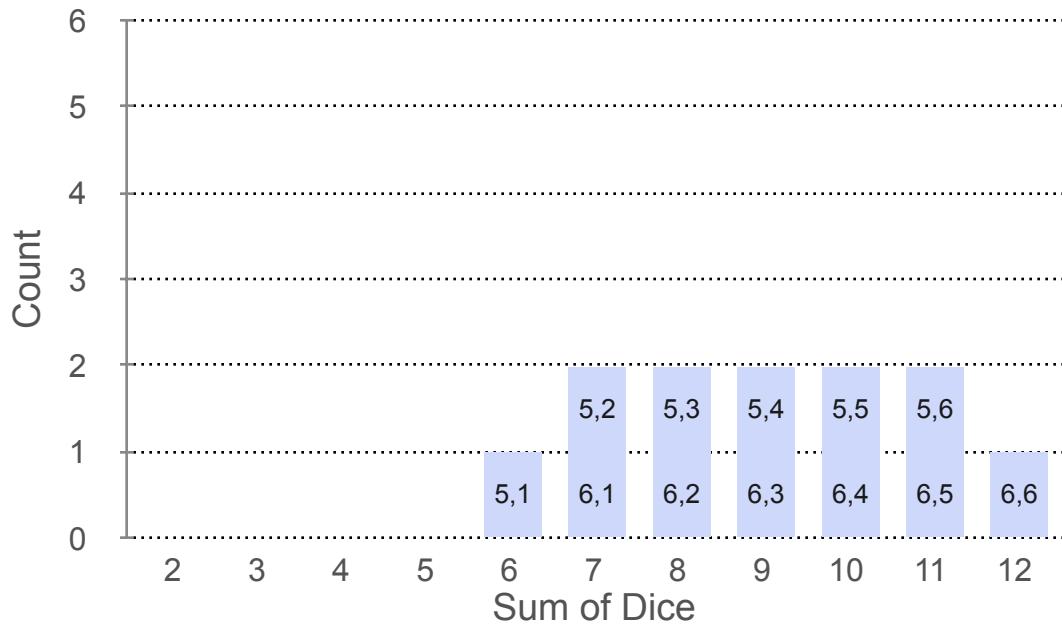
Joint Distributions: Example 3



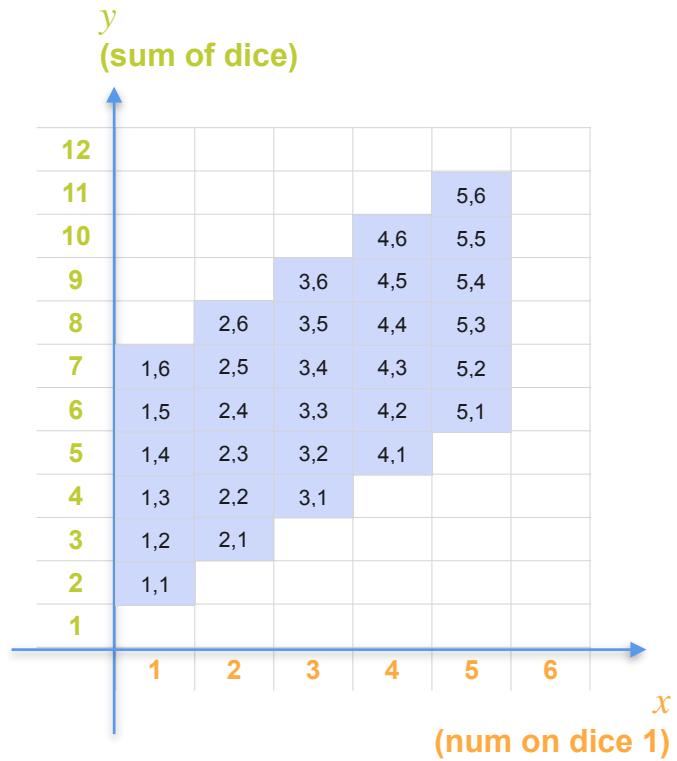
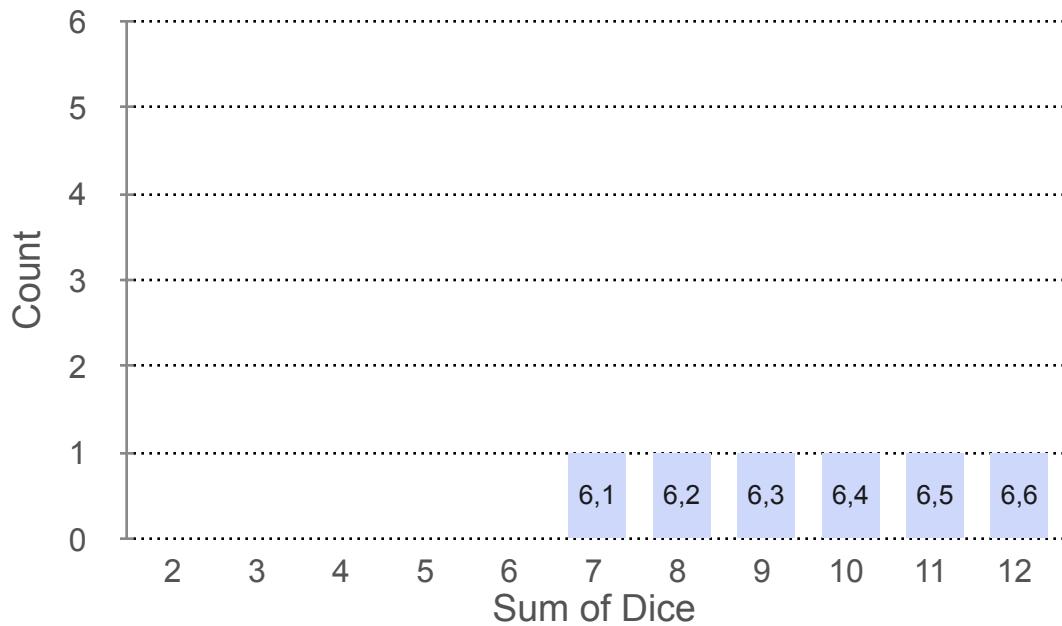
Joint Distributions: Example 3



Joint Distributions: Example 3



Joint Distributions: Example 3

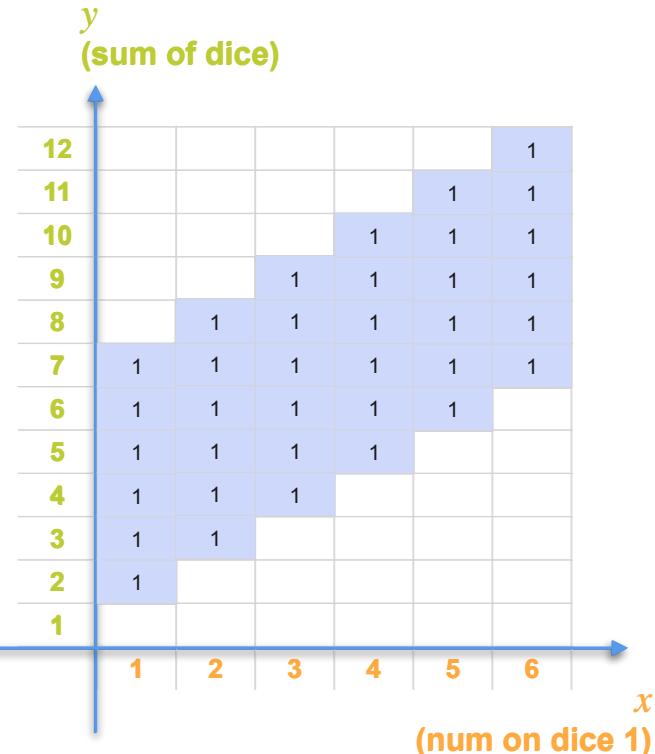
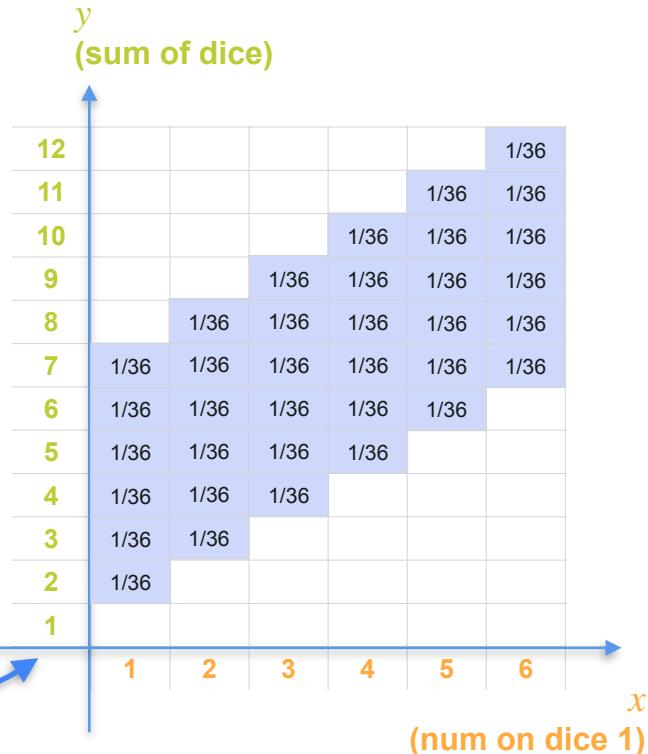


Joint Distributions: Example 3

Joint Distribution for
 X and Y

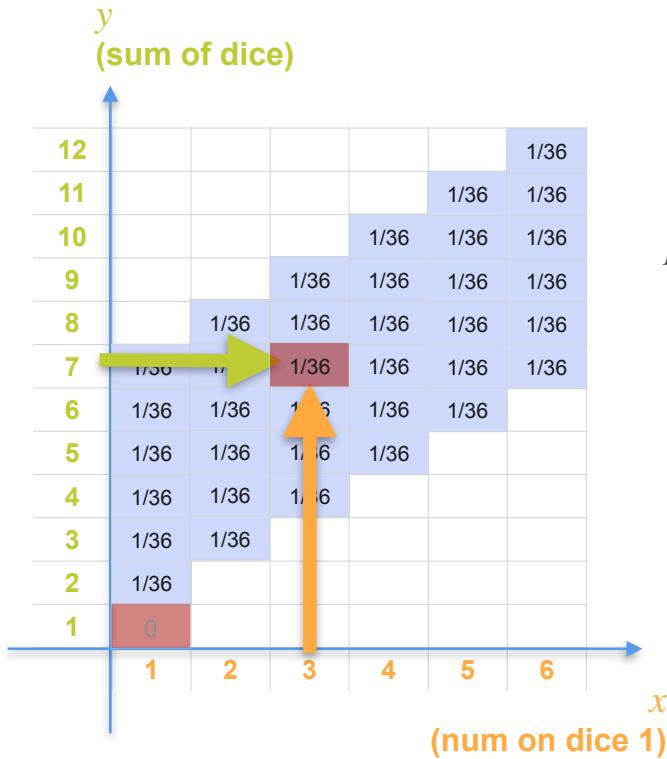
36
possible
outcomes

Divide by sum
(36)



Joint Distributions: Example 3

Joint Distribution for
X and Y



$$p_{XY}(3, 7) = \mathbf{P}(X = 3, Y = 7) = \frac{1}{36}$$

$$p_{XY}(1, 1) = \mathbf{P}(X = 1, Y = 1) = 0$$



DeepLearning.AI

Probability Distributions with Multiple Variables

**Joint Distribution
(Continuous)**

Joint Continuous Distributions

X : age of a child in year

Y : discrete values of height of child in inches

X : the number rolled on the 1st dice

Y : the number rolled on the 2nd dice

X : the number rolled on the 1st dice

Y : sum of both dice

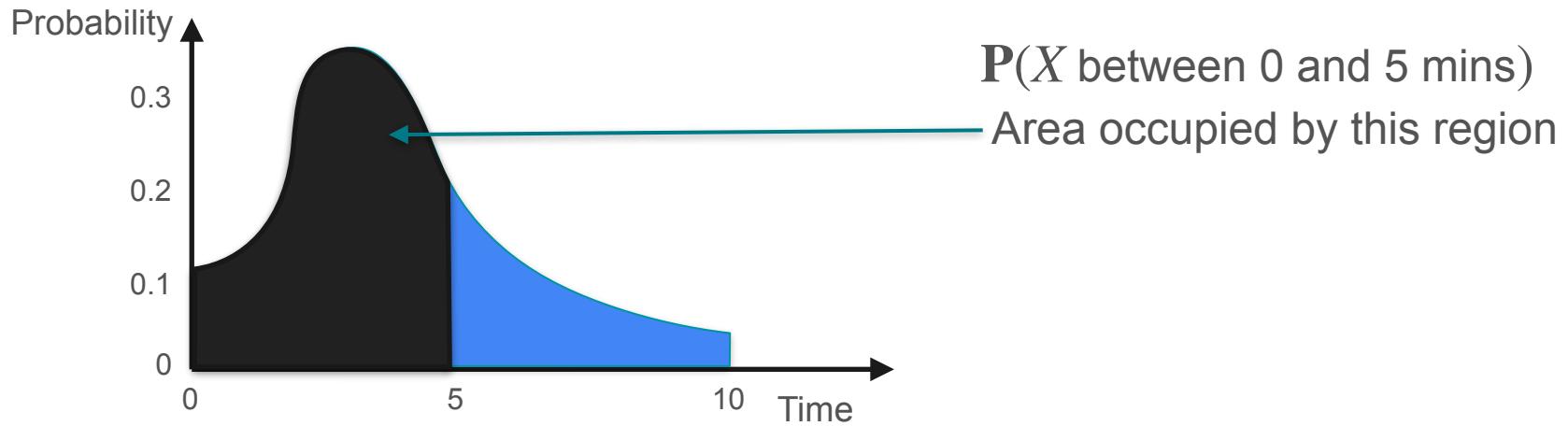
X and Y are
Discrete Random Variables

What about when X and Y are
Continuous Random Variables?

Joint Continuous Distributions



X variable: Waiting time



Joint Continuous Distributions

X

Waiting time
before a call is picked up
[0 - 10 minutes]



2.4 minutes

1.5 minutes

Y

Customer
satisfaction rating
[0 - 10]



0.0

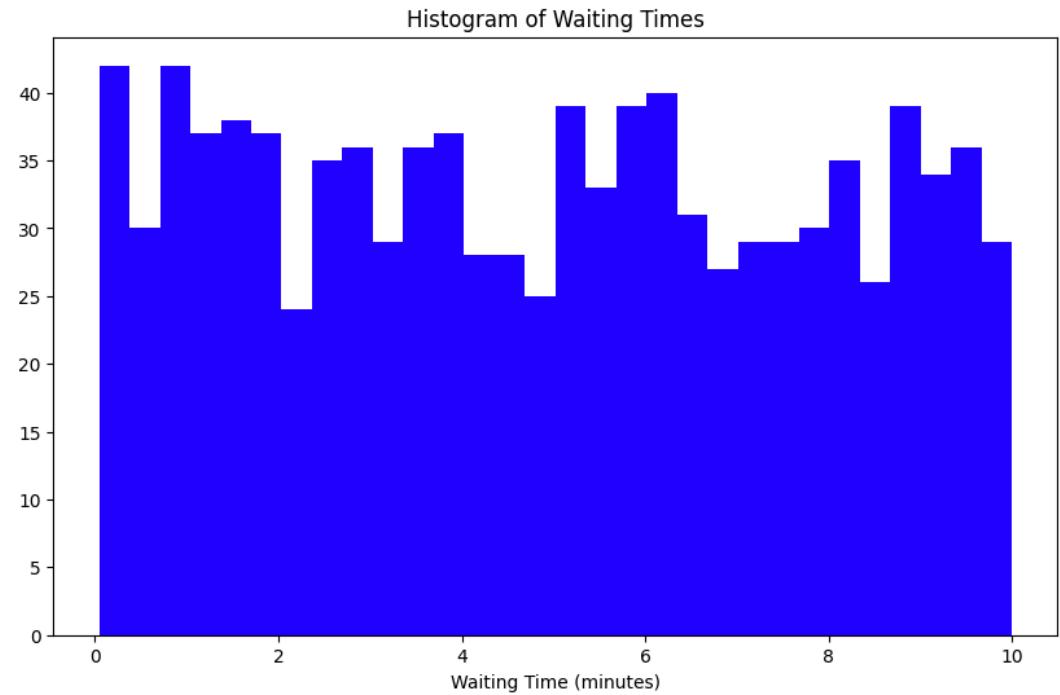
5.7

Both variables are
continuous

Joint Continuous Distributions: Dataset

X variable: Waiting time (mins)
0 - 10 mins

1000 customers

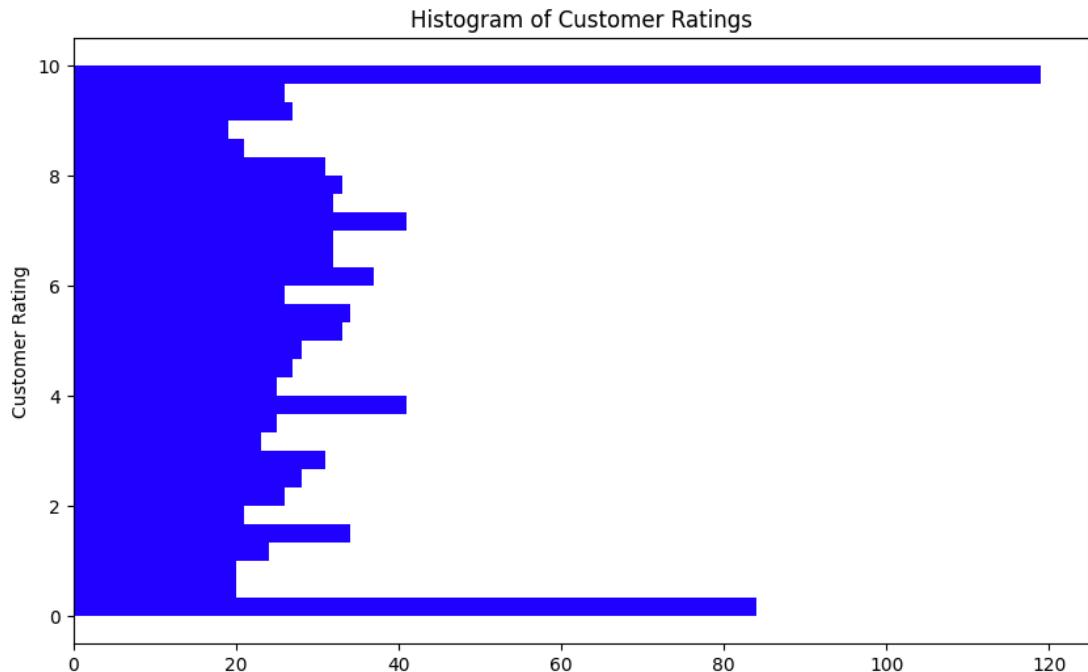


Joint Continuous Distributions: Dataset

Y variable: Satisfaction rating

0 - 10

1000 customers



Joint Continuous Distributions: Dataset

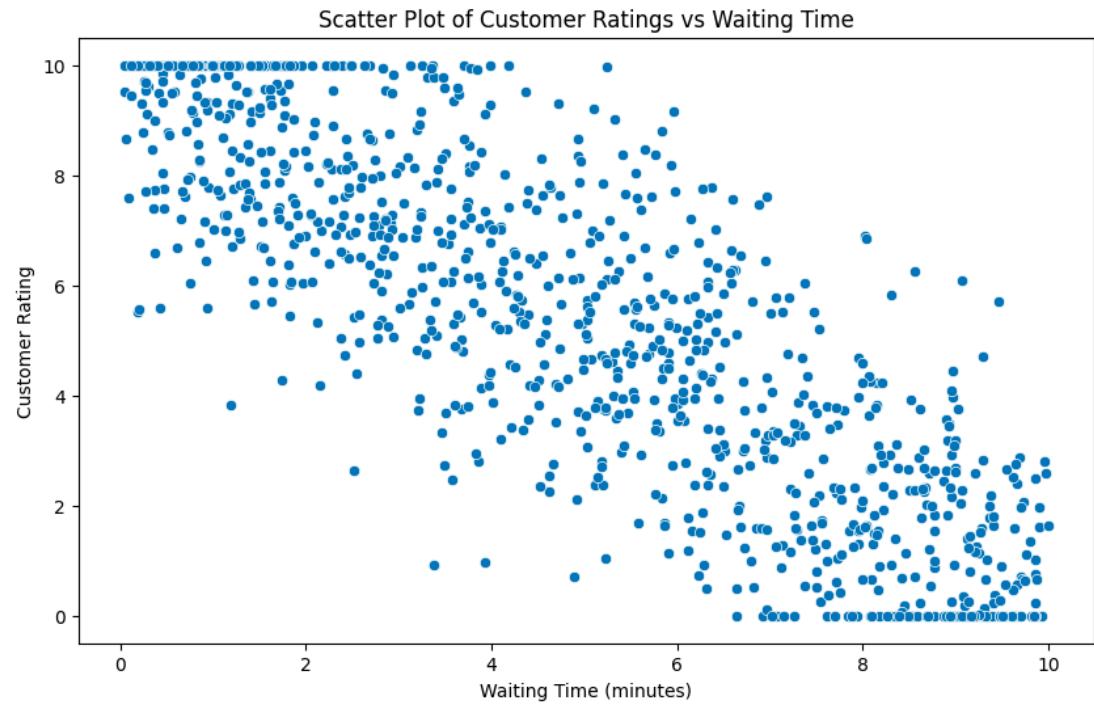
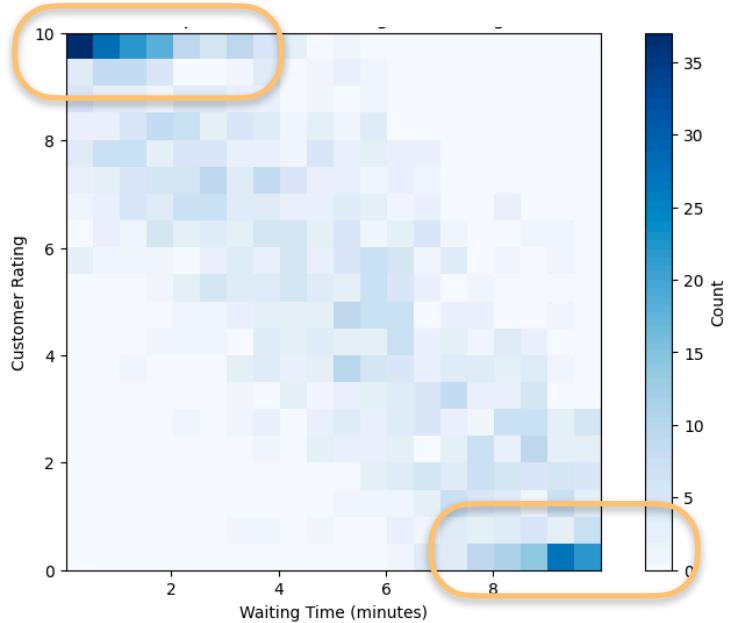
X variable: Waiting time (mins)
0 - 10 mins

Y variable: Satisfaction rating
0 - 10

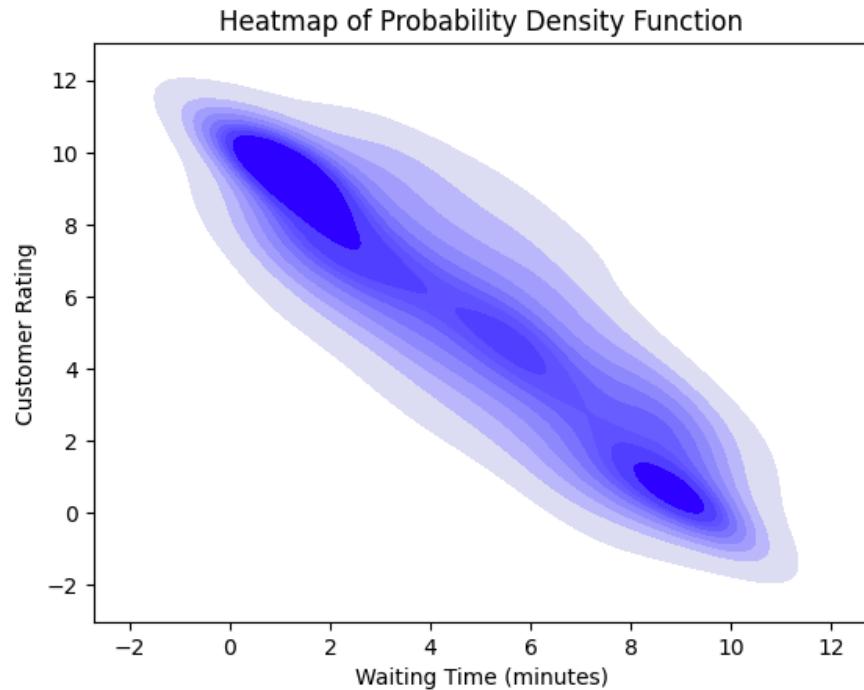
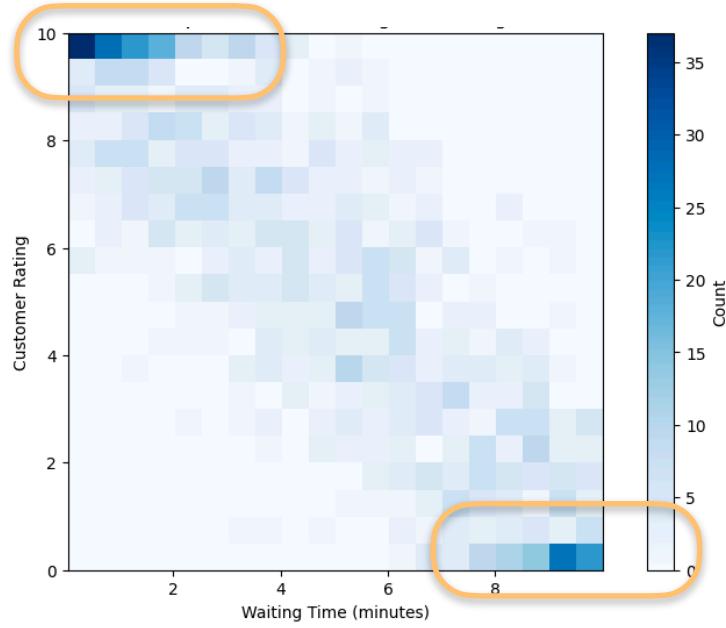
1000 customers



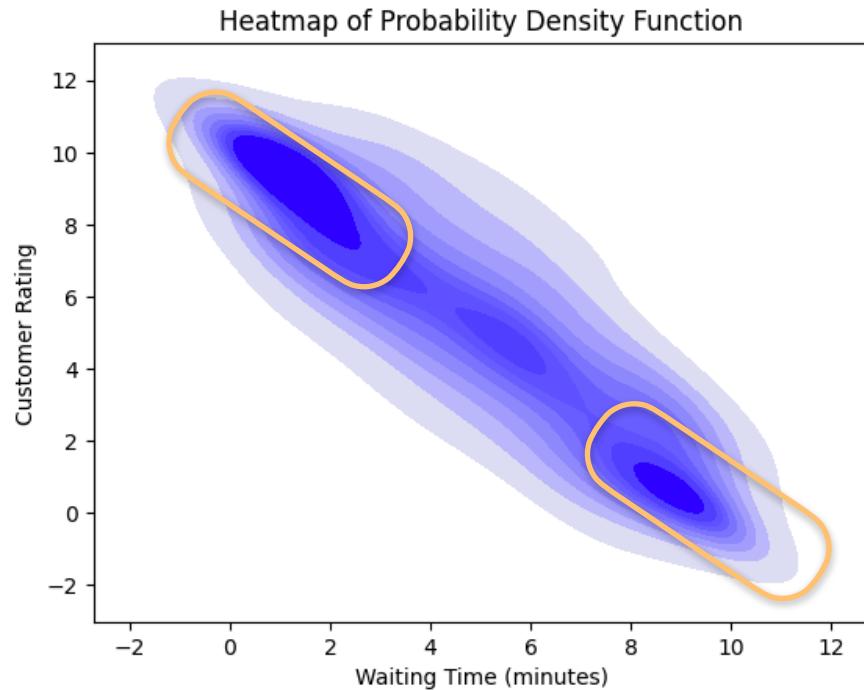
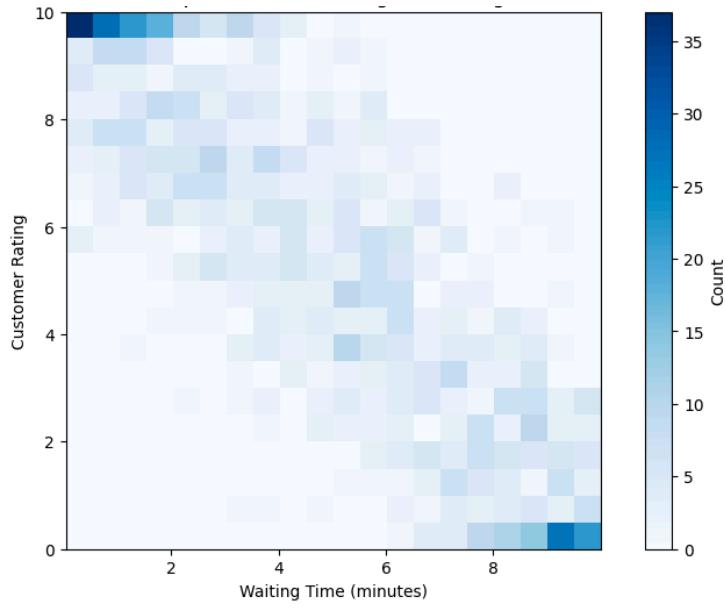
Joint Continuous Distributions: Dataset



Joint Continuous Distributions: Dataset

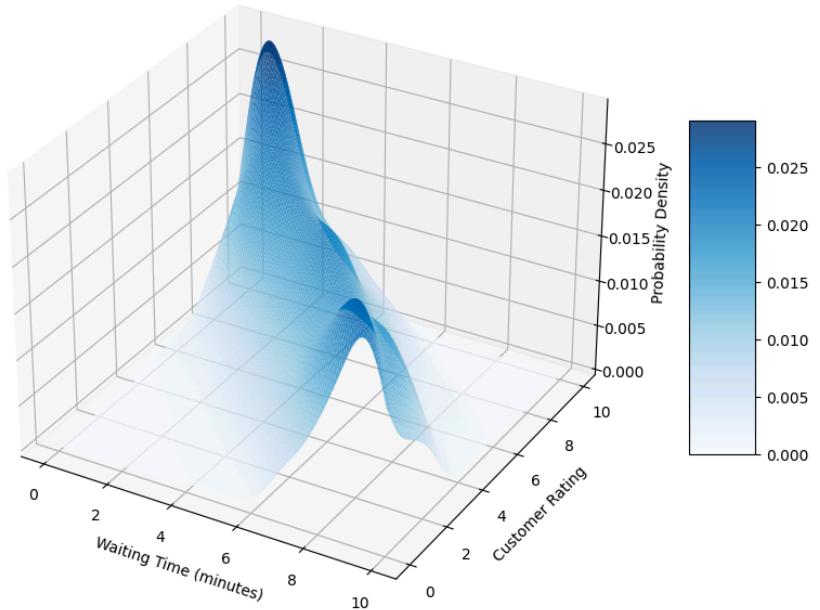


Joint Continuous Distributions: Dataset

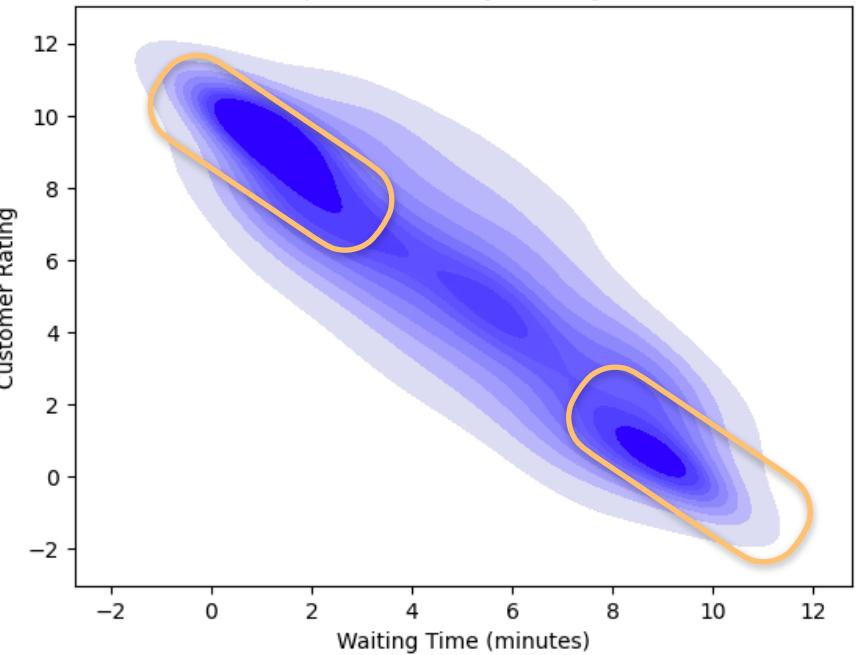


Joint Continuous Distributions: Dataset

3D Probability Density Distribution for Customer Ratings vs Waiting Time

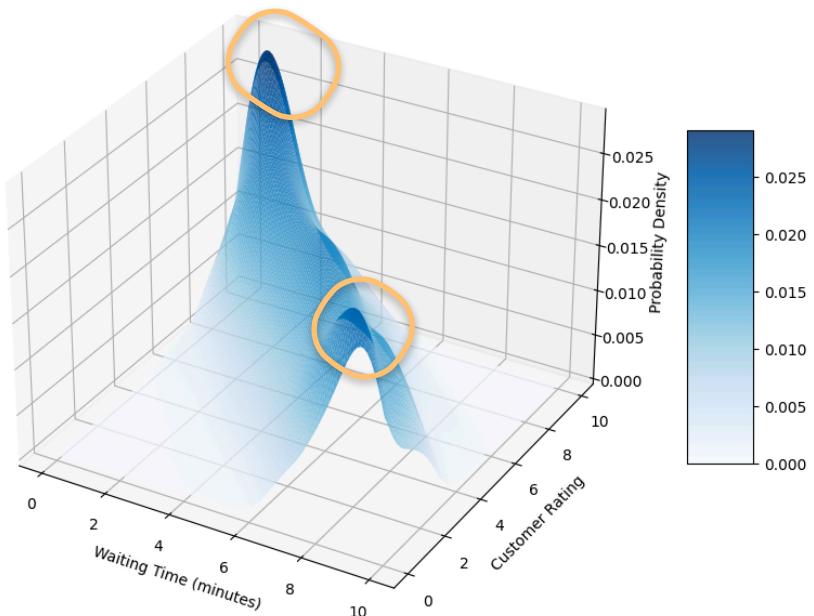


Heatmap of Probability Density Function

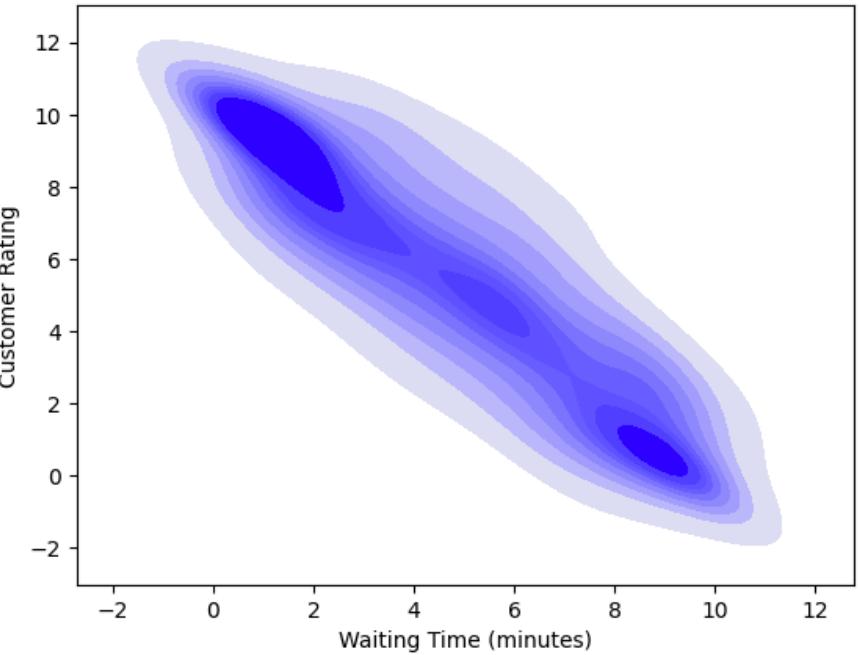


Joint Continuous Distributions: Dataset

3D Probability Density Distribution for Customer Ratings vs Waiting Time



Heatmap of Probability Density Function

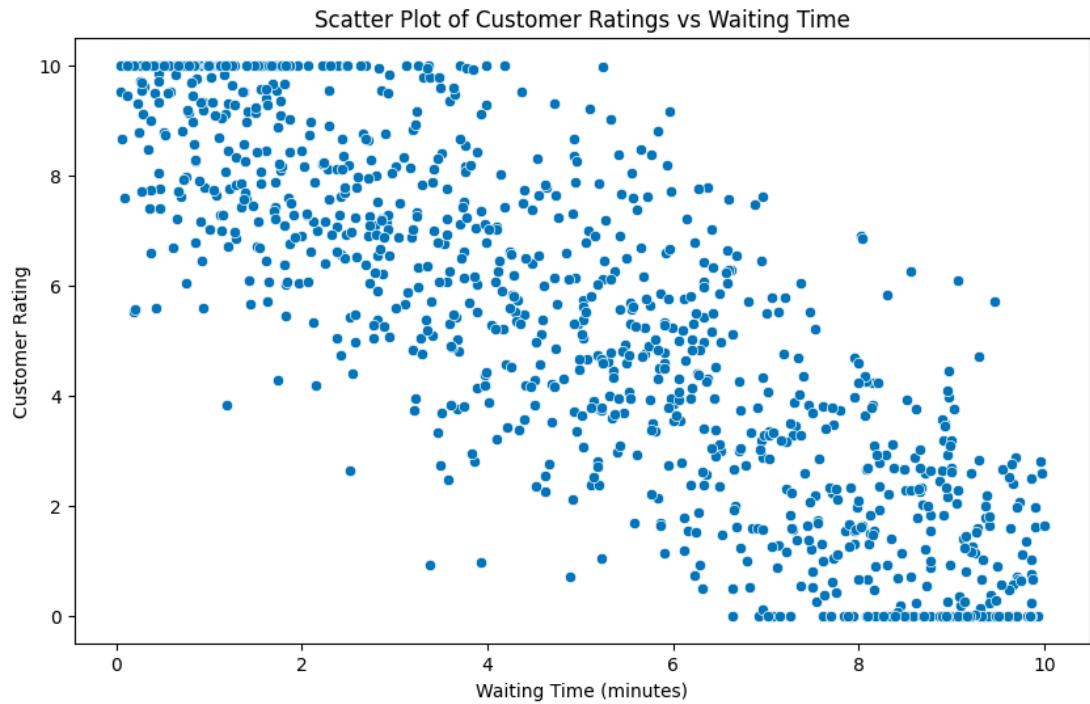


Joint Continuous Distributions: Dataset

X variable: Waiting time (mins)
0 - 10 mins

Y variable: Satisfaction rating
0 - 10

1000 customers



Expected Value

X variable: Waiting time (mins)

0 - 10 mins

Y variable: Satisfaction rating

0 - 10

1000 customers

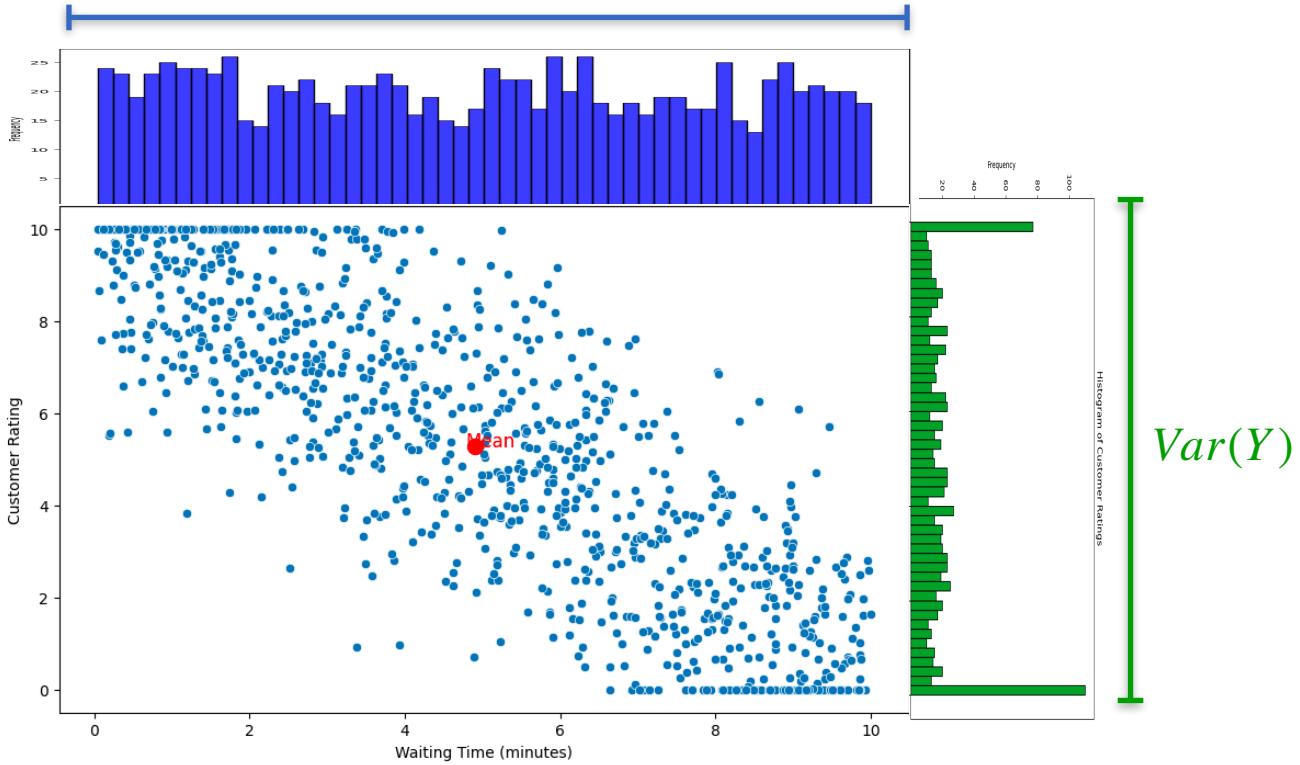
$$\mathbb{E}[X] = 4.903 \text{ minutes}$$

$$\mathbb{E}[Y] = 5.280$$



Variances

$$Var(X)$$

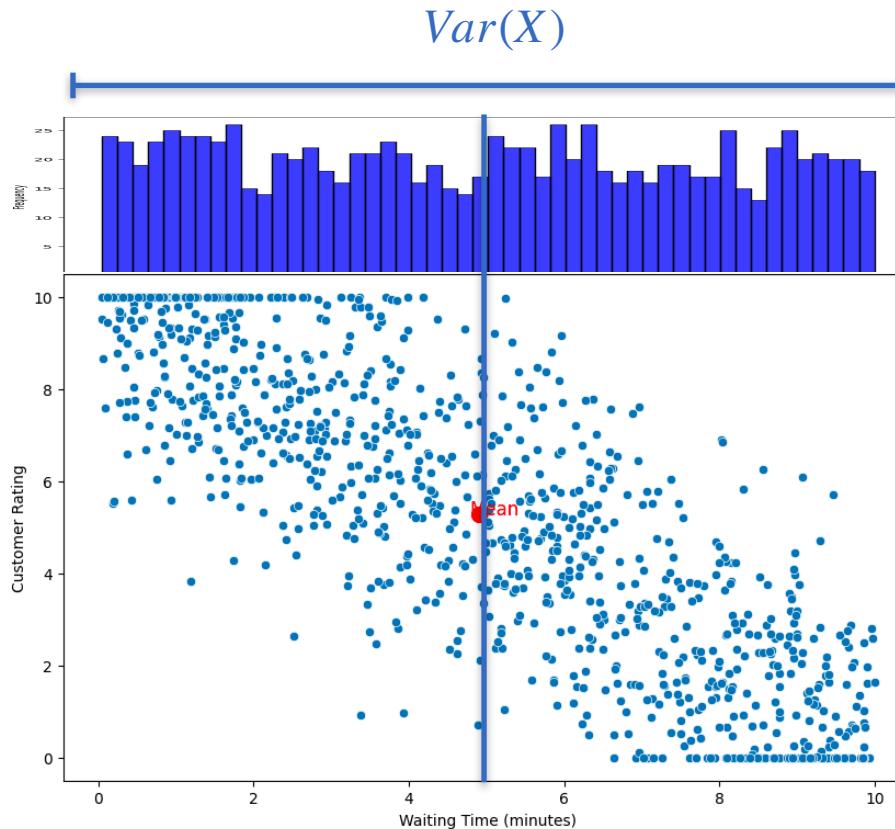


Variances

$$\mathbb{E}[X] = 4.903 \text{ minutes}$$

$$\mathbb{E}[X^2] = 32.561$$

$$\begin{aligned} \text{Var}(X) &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\ &= 32.561 - 4.903^2 \\ &= 8.526 \end{aligned}$$

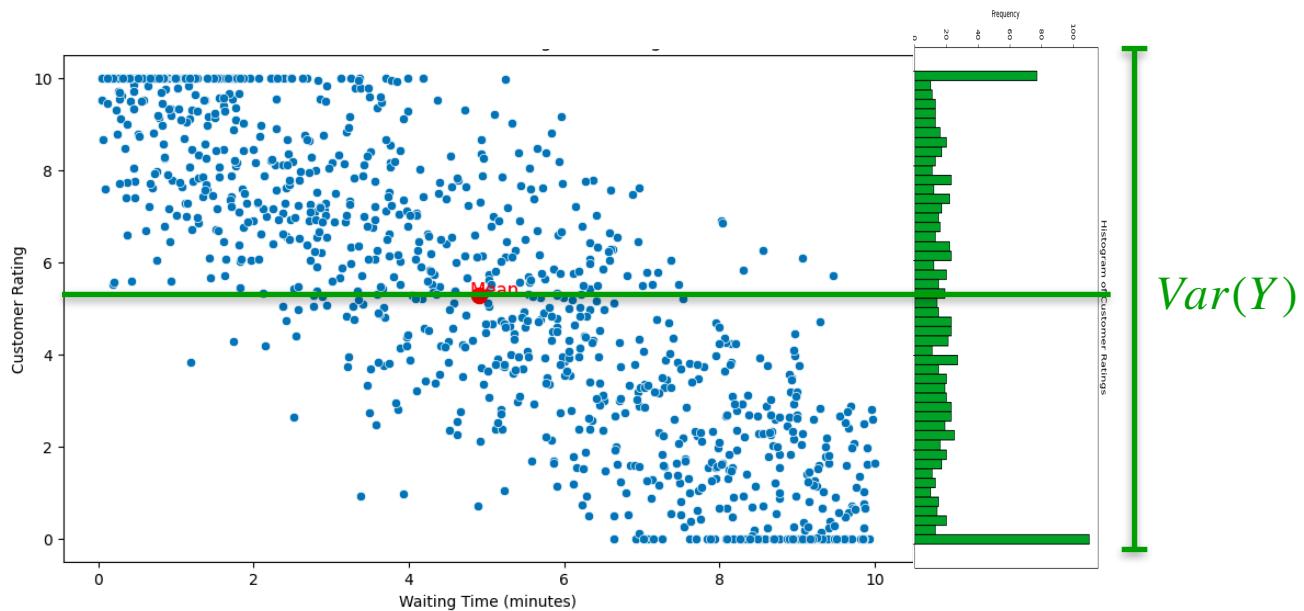


Variances

$$\mathbb{E}[Y] = 5.280$$

$$\mathbb{E}[Y^2] = 38.037$$

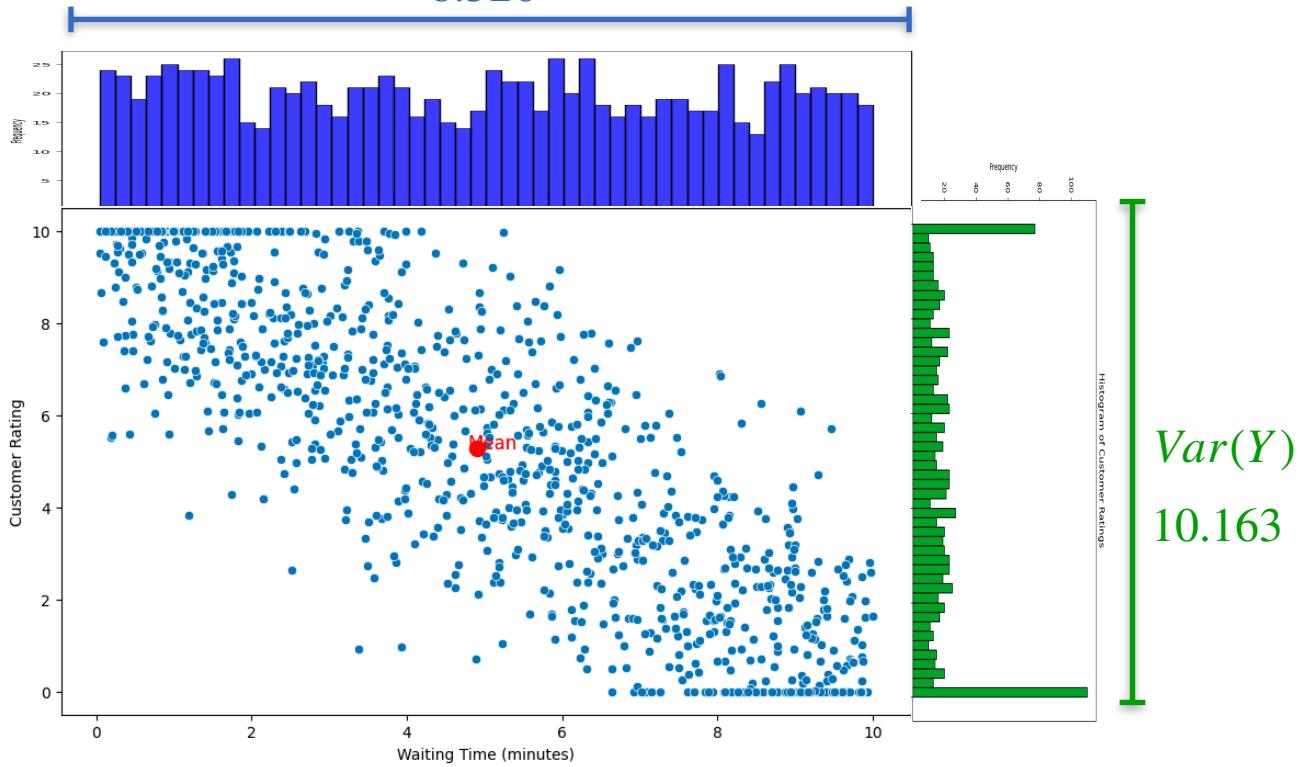
$$\begin{aligned}Var(Y) &= \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 \\&= 38.037 - 5.280^2 \\&= 10.163\end{aligned}$$



Variances

$$Var(X)$$

$$8.526$$





DeepLearning.AI

Probability Distributions with Multiple Variables

Marginal and Conditional Distribution

Marginal Distribution: Example 1

		Height (Y)					
		45	46	47	48	49	50
Age (X)	7	1/10	2/10	0	0	0	0
	8	0	0	2/10	0	0	0
	9	0	0	0	0	3/10	1/10
	10	0	0	0	0	0	1/10



Marginal Distribution

Distribution of one variable while ignoring the other

Marginal Distribution: Example 1

		Height (Y)					
		45	46	47	48	49	50
Age (X)	7	1/10	2/10	0	0	0	0
	8	0	0	2/10	0	0	0
	9	0	0	0	0	3/10	1/10
	10	0	0	0	0	0	1/10



To find the marginal distribution of height:

sum the joint probability distribution over all values of age

Marginal Distribution: Example 1

		Height (Y)					
		45	46	47	48	49	50
Age (X)	7	1/10	2/10	0	0	0	0
	8	0	0	2/10	0	0	0
	9	0	0	0	0	3/10	1/10
	10	0	0	0	0	0	1/10
		1/10	2/10	2/10	0	3/10	2/10



$$p_Y(y_j) = \sum_i p_{XY}(x_i, y_j)$$

Marginal Distribution: Example 1

		Height (Y)					
		45	46	47	48	49	50
Age (X)	7	1/10	2/10	0	0	0	0
	8	0	0	2/10	0	0	0
	9	0	0	0	0	3/10	1/10
	10	0	0	0	0	0	1/10
		1/10	2/10	2/10	0	3/10	2/10



$$p_Y(y_j) = \sum_i p_{XY}(x_i, y_j)$$

$$p_Y(50) =$$

Marginal Distribution: Example 1

		Height (Y)					
		45	46	47	48	49	50
Age (X)	7	1/10	2/10	0	0	0	0
	8	0	0	2/10	0	0	0
	9	0	0	0	0	3/10	1/10
	10	0	0	0	0	0	1/10
		1/10	2/10	2/10	0	3/10	2/10



$$p_Y(y_j) = \sum_i p_{XY}(x_i, y_j)$$

$$p_Y(50) = \sum_i p_{XY}(x_i, 50)$$

$$p_Y(50) = \frac{2}{10}$$

Marginal Distribution: Example 1

		Height (Y)					
		45	46	47	48	49	50
Age (X)	7	1/10	2/10	0	0	0	0
	8	0	0	2/10	0	0	0
	9	0	0	0	0	3/10	1/10
	10	0	0	0	0	0	1/10
		1/10	2/10	2/10	0	3/10	2/10

Age (years):	7	7	7	8	8	9	9	9	9	10
Height (in):	45	46	46	47	47	49	49	49	50	50

$$p_X(x_i) = \sum_j p_{XY}(x_i, y_j)$$

Marginal Distribution: Example 1

		Height (Y)					
		45	46	47	48	49	50
Age (X)	7	1/10	2/10	0	0	0	0
	8	0	0	2/10	0	0	0
	9	0	0	0	0	3/10	1/10
	10	0	0	0	0	0	1/10
		1/10	2/10	2/10	0	3/10	2/10



$$p_X(x_i) = \sum_j p_{XY}(x_i, y_j)$$

$$p_X(7) = \sum_j p_{XY}(7, y_j)$$

$$p_X(7) = \frac{3}{10}$$

Marginal Distribution: Example 1

		Height (Y)						Age (years): 7 7 7 8 8 9 9 9 9 9 9 9 10									
		45	46	47	48	49	50	Height (in): 45 46 46 47 47 47 49 49 49 49 50 50									
Age (X)	7	1/10	2/10	0	0	0	0	3/10									
	8	0	0	2/10	0	0	0	2/10									
	9	0	0	0	0	3/10	1/10	4/10									
	10	0	0	0	0	0	1/10	1/10									
			1/10	2/10	2/10	0	3/10	2/10									

Marginal Distribution of Age

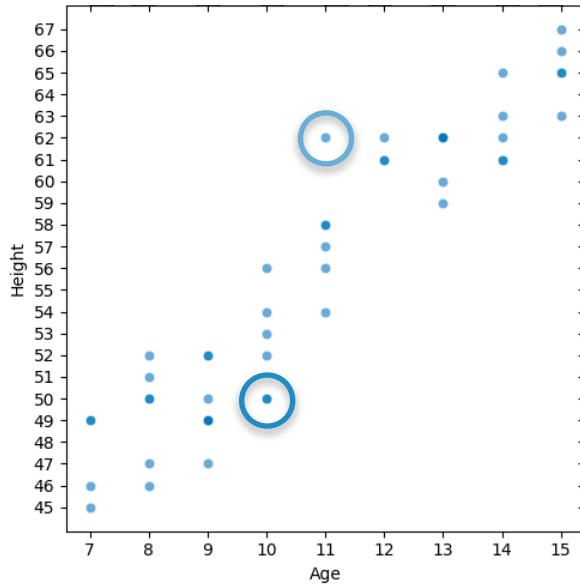


Marginal Distribution of Height



Marginal Distribution: Example 1

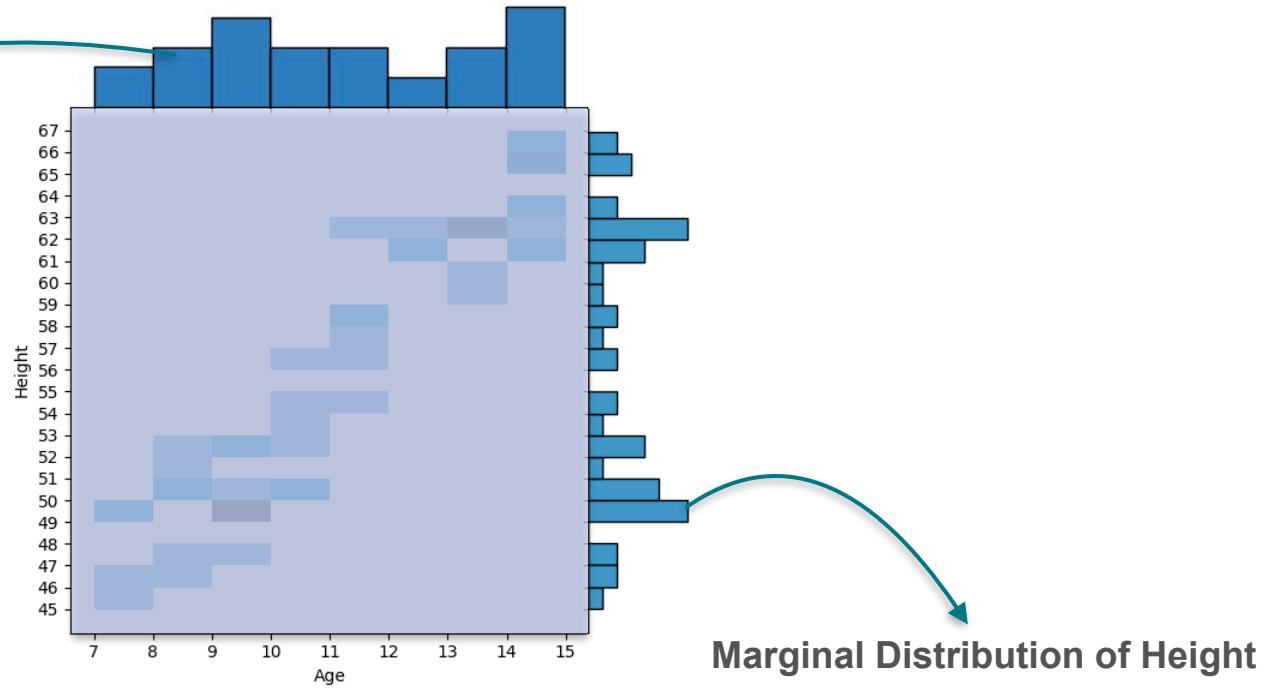
Age and Height Dataset
for 50 children



Marginal Distribution: Example 1

Marginal Distribution of Age

Age and Height Dataset
for 50 children



Marginal Distribution of Height

Marginal Distributions: Example 2

X : the number rolled on the 1st dice



$$\frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6}$$

Y : the number rolled on the 2nd dice



$$\frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6}$$

Marginal Distributions: Example 2

		Y							
		1	2	3	4	5	6		
X		1	1/36	1/36	1/36	1/36	1/36	1/36	1/6
1		1/36	1/36	1/36	1/36	1/36	1/36	1/36	1/6
2		1/36	1/36	1/36	1/36	1/36	1/36	1/36	1/6
3		1/36	1/36	1/36	1/36	1/36	1/36	1/36	1/6
4		1/36	1/36	1/36	1/36	1/36	1/36	1/36	1/6
5		1/36	1/36	1/36	1/36	1/36	1/36	1/36	1/6
6		1/36	1/36	1/36	1/36	1/36	1/36	1/36	1/6
		1/6	1/6	1/6	1/6	1/6	1/6		

X : the number rolled on the 1st dice

Y : the number rolled on the 2nd dice

$p_X(x_i) = \frac{1}{6}$

$p_Y(y_j) = \frac{1}{6}$

Marginal Distributions: Example 3



X : the number rolled on the 1st dice

Y : sum of the two dice

Marginal Distributions: Example 3



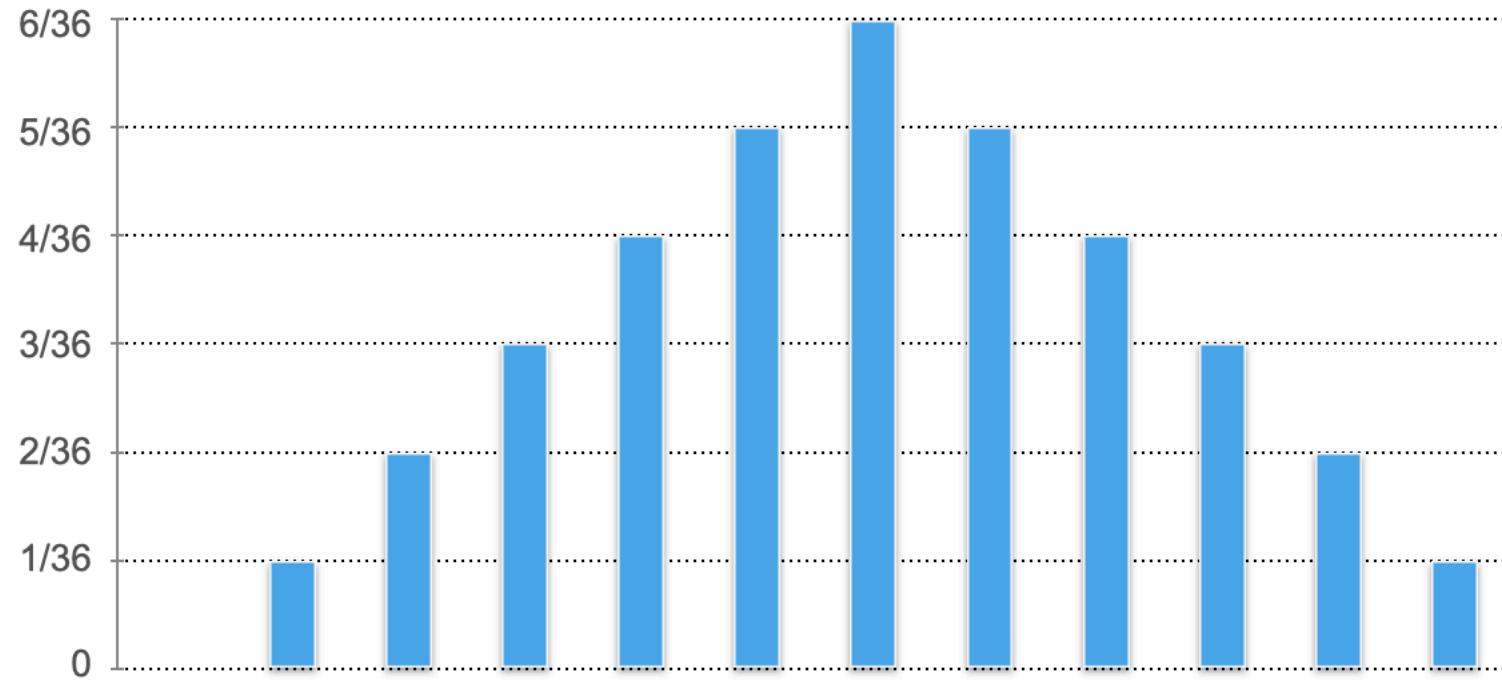
X : the number rolled on the 1st dice

Y : sum of the two dice

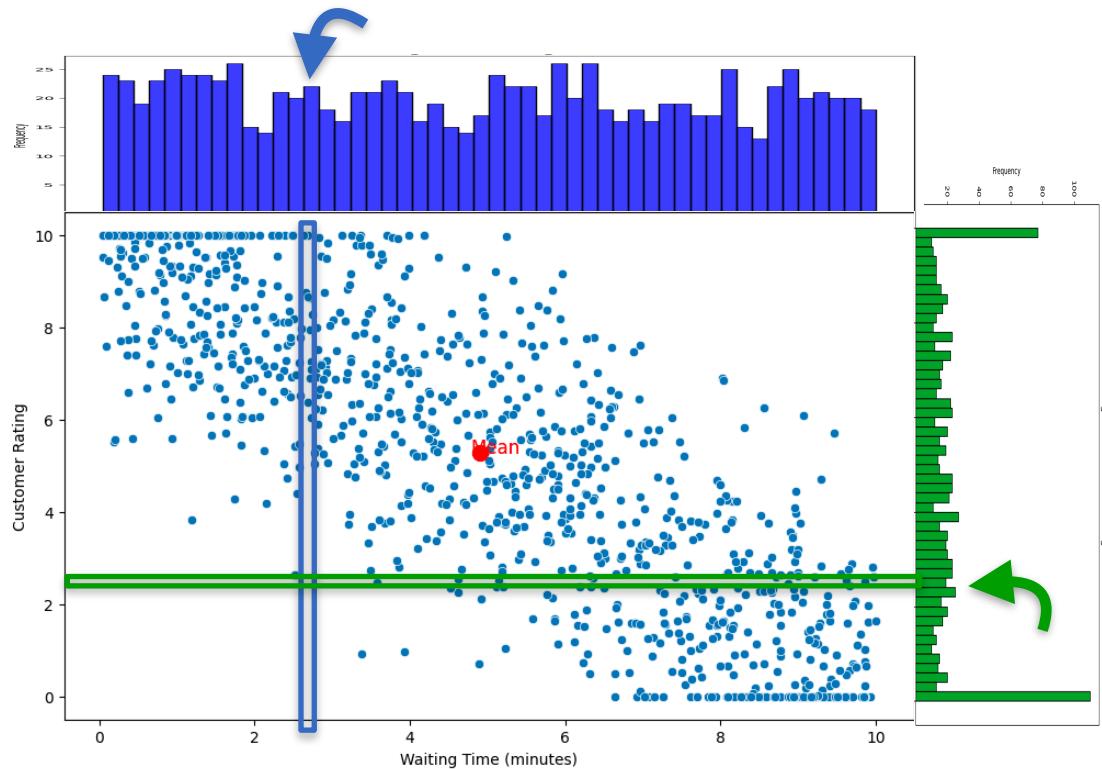
$$p_Y(y_j) = \sum_i p_{XY}(x_i, y_j) = ?$$

$$p_Y(10) = ? = \frac{3}{36}$$

Marginal Distributions: Example 2

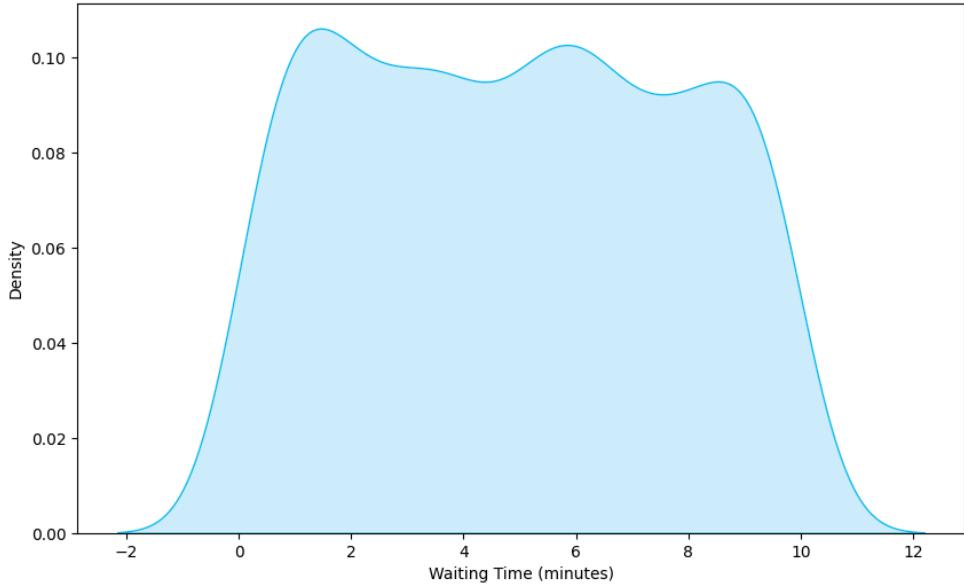
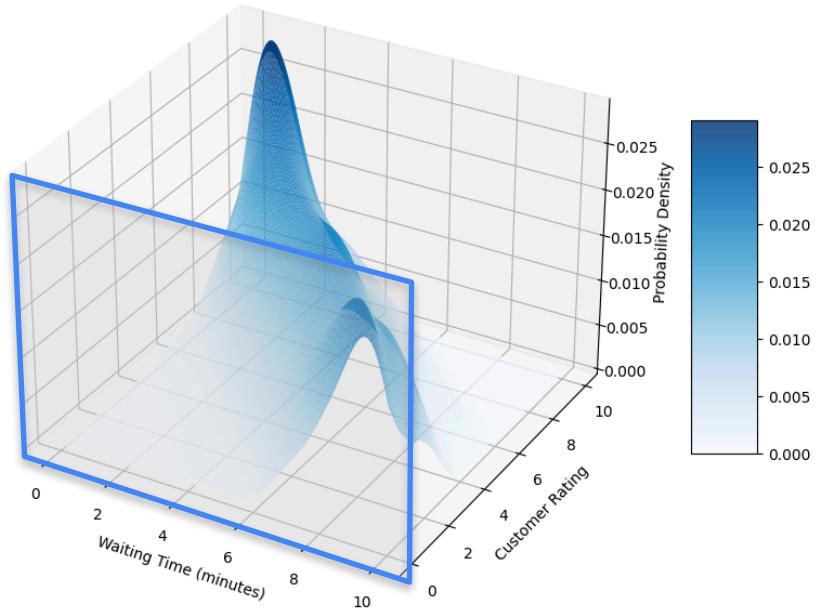


Marginal Distributions



Continuous Marginal Distribution

3D Probability Density Distribution for Customer Ratings vs Waiting Time



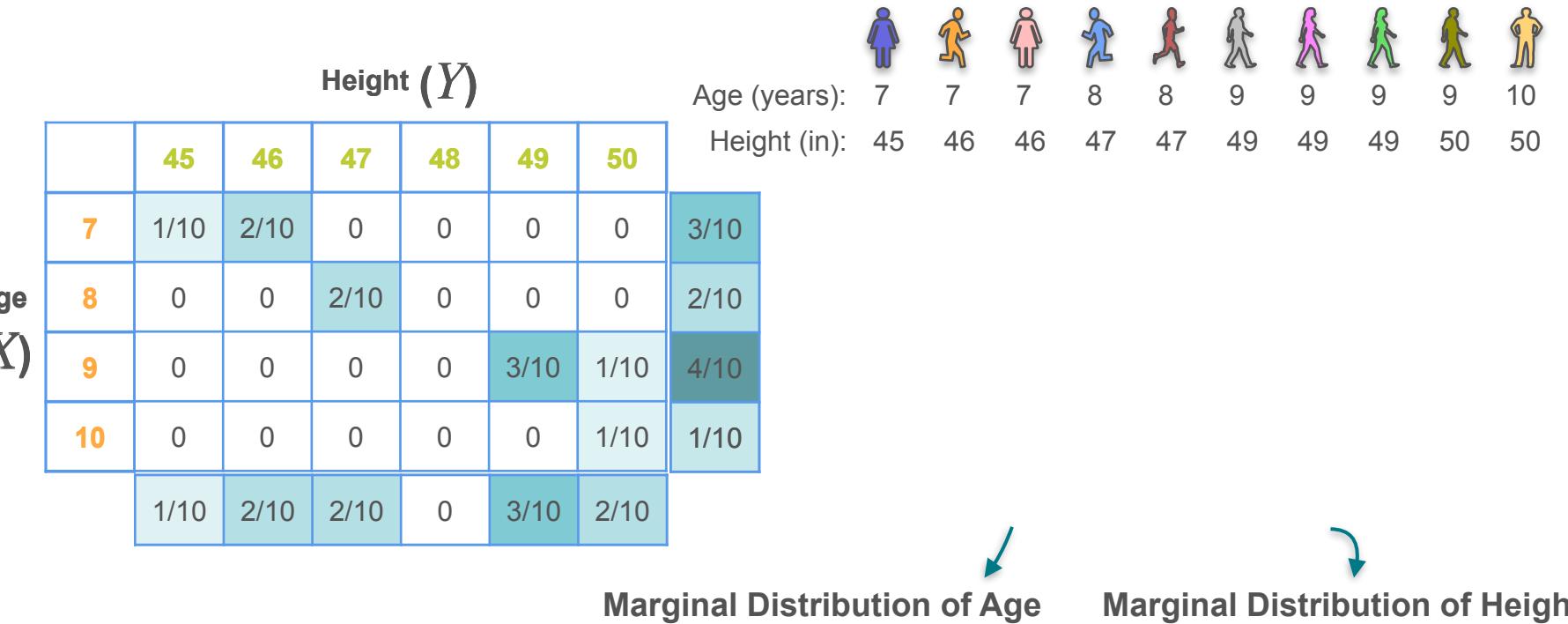


DeepLearning.AI

Probability Distributions with Multiple Variables

Conditional Distribution

Conditional Distribution: Example 1



Conditional Distribution: Example 1

		Height (Y)					
		45	46	47	48	49	50
Age (X)	7	1/10	2/10	0	0	0	0
	8	0	0	2/10	0	0	0
	9	0	0	0	0	3/10	1/10
	10	0	0	0	0	0	1/10



If age = 9, what is the distribution across the height variable?

Conditional Distribution

Conditional Distribution: Example 1

		Height (Y)					
		45	46	47	48	49	50
Age (X)	7	1/10	2/10	0	0	0	0
	8	0	0	2/10	0	0	0
	9	0	0	0	0	3/10	1/10
	10	0	0	0	0	0	1/10



If age = 9, what is the distribution across the height variable?

$$p_{Y|X=9}(y) = \mathbf{P}(Y = y | X = 9)$$

Conditional Distribution: Example 1

Age (X)	Height (Y)					P(X = 9)
	45	46	47	48	49	
9	0	0	0	0	3/10	1/10
Sum	0	0	0	0	3/10	1/10

Normalize

Divide by row sum

9	0	0	0	0	3/4	1/4
---	---	---	---	---	-----	-----

$$\mathbf{P}(Y = 49 | X = 9) = \frac{3}{4}$$
$$\mathbf{P}(Y = 49 | X = 9) = \frac{3/10}{4/10} = \frac{3}{4}$$

$$p_{Y|X=9}(y) = \mathbf{P}(Y = y | X = 9)$$

$$\mathbf{P}(A, B) = \mathbf{P}(A) \cdot \mathbf{P}(B | A)$$

$$\mathbf{P}(X = 9, Y = 49) = \mathbf{P}(X = 9) \cdot \mathbf{P}(Y = 49 | X = 9)$$

$$P(Y = 49 | X = 9) = \frac{\mathbf{P}(X = 9, Y = 49)}{\mathbf{P}(X = 9)}$$

Row sum

Conditional Distribution: Example 1

(X)	Height (Y)					P(X = 9)	Sum
	45	46	47	48	49		
9	0	0	0	0	3/10	1/10	

$$p_{Y|X=x}(y) = \frac{p_{XY}(x, y)}{p_X(x)}$$

$$p_{Y|X=9}(y) = \mathbf{P}(Y = y | X = 9)$$

$$\mathbf{P}(A, B) = \mathbf{P}(A) \cdot \mathbf{P}(B | A)$$

$$\mathbf{P}(X = 9, Y = 49) = P(X = 9) \cdot P(Y = 49 | X = 9)$$

$$P(Y = 49 | X = 9) = \frac{\mathbf{P}(X = 9, Y = 49)}{P(X = 9)}$$

Row sum

$$\mathbf{P}(Y = 49 | X = 9) = \frac{3/10}{4/10} = \frac{3}{4}$$

Discrete Conditional Distribution: Formula

$$p_{Y|X=x}(y) = \frac{p_{XY}(x, y)}{p_X(x)}$$

Conditional PDF of Y

Joint PDF of X and Y

Marginal distribution of X

The diagram illustrates the formula for the conditional probability of Y given $X=x$. It features a central equation $p_{Y|X=x}(y) = \frac{p_{XY}(x, y)}{p_X(x)}$. Four curved arrows originate from text labels placed around the equation and point to specific parts of it: one arrow from 'Conditional PDF of Y ' points to the term $p_{XY}(x, y)$; another from 'Joint PDF of X and Y ' points to the numerator $p_{XY}(x, y)$; a third from 'Marginal distribution of X ' points to the denominator $p_X(x)$; and a fourth arrow points directly to the entire equation.

Conditional Distributions: Example 2



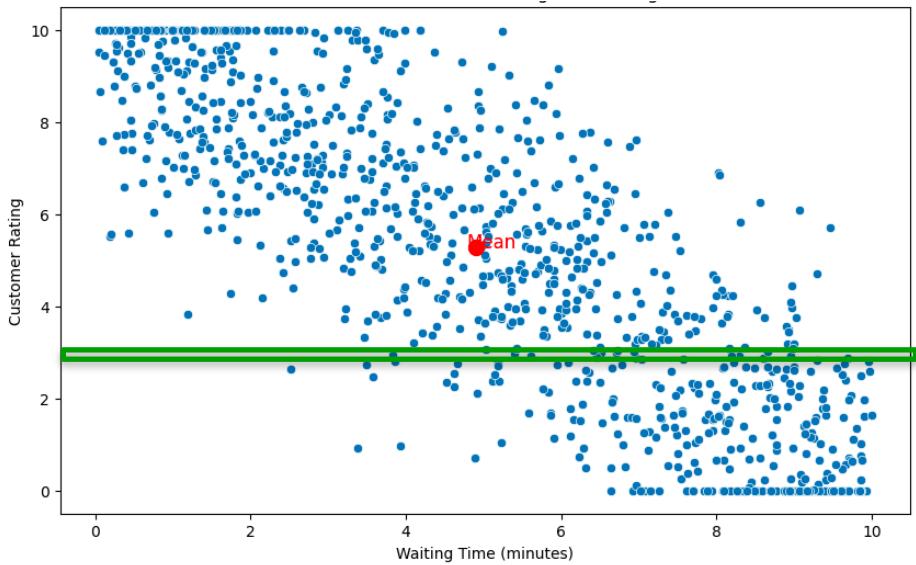
Dice 1: 1/6 1/6 1/6 1/6 1/6 1/6

Dice 2: 1/6 1/6 1/6 1/6 1/6 1/6

$$\begin{aligned} p_{Y|X=4}(y=1) &= \frac{p_{XY}(x=4, y=1)}{p_X(x=4)} \\ &= \frac{1/36}{1/6} \\ &= \frac{1}{6} \end{aligned}$$

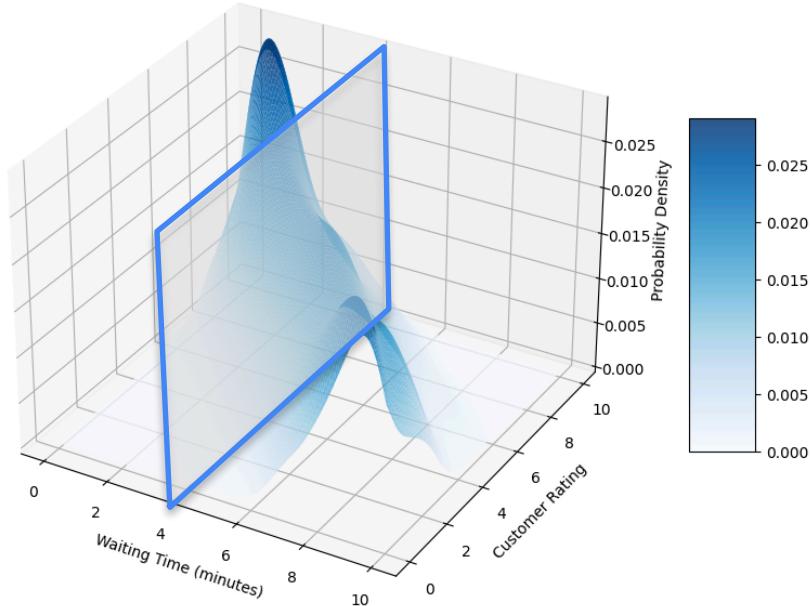
	Y							
	1	2	3	4	5	6	Sum	
X	1	1/36	1/36	1/36	1/36	1/36	1/36	1/6
	2	1/36	1/36	1/36	1/36	1/36	1/36	1/6
	3	1/36	1/36	1/36	1/36	1/36	1/36	1/6
	4	1/36	1/36	1/36	1/36	1/36	1/36	1/6
	5	1/36	1/36	1/36	1/36	1/36	1/36	1/6
	6	1/36	1/36	1/36	1/36	1/36	1/36	1/6

Conditional Distributions: Example 4

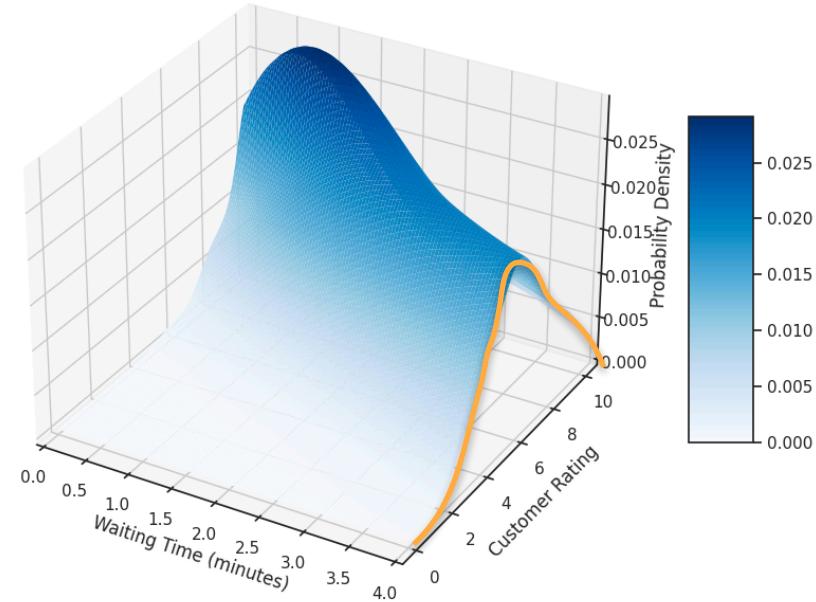


Continuous Conditional Distribution

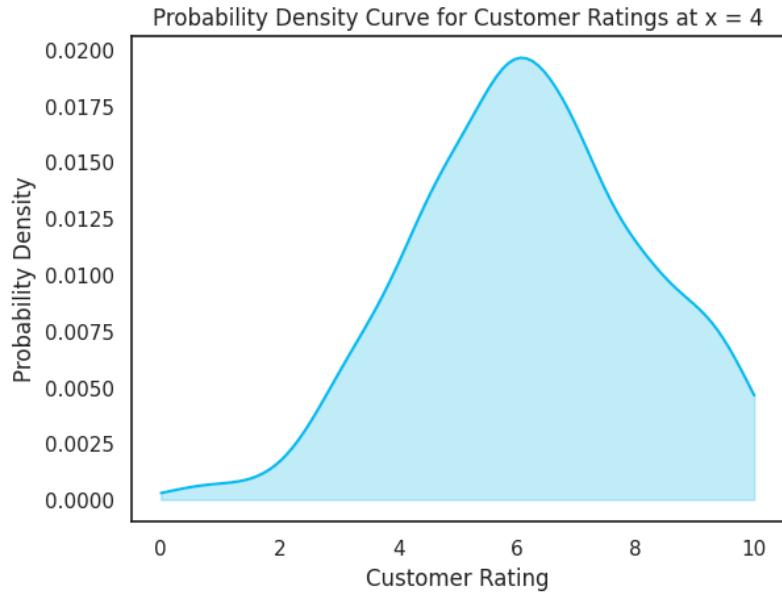
3D Probability Density Distribution for Customer Ratings vs Waiting Time



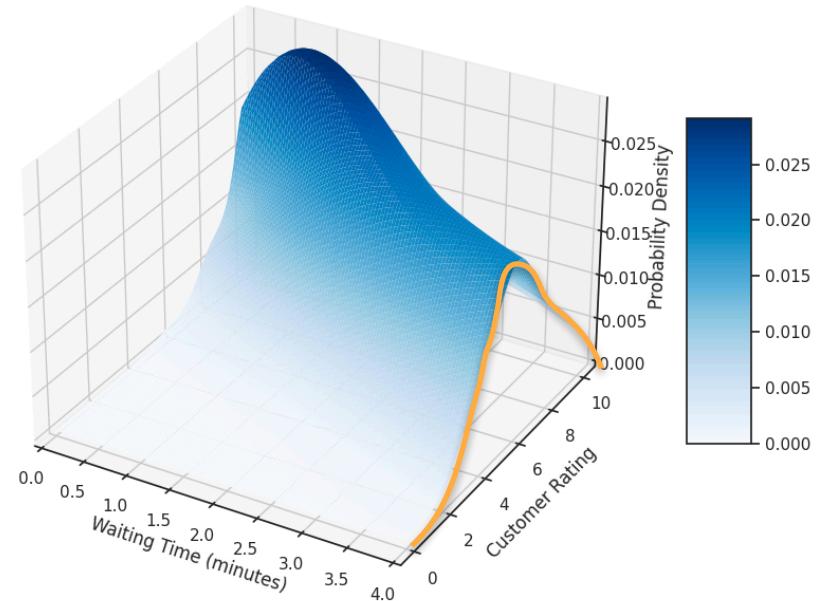
Probability distribution for rating given that waiting time was 4 minutes



Continuous Conditional Distribution



Conditional PDF of y given $x = 4$



Discrete Conditional Distribution

$$p_{Y|X=x}(y) = \frac{p_{XY}(x, y)}{p_X(x)}$$

Conditional PDF of Y

Joint PDF of X and Y

Marginal distribution of X

The diagram illustrates the derivation of the conditional probability formula. It shows three components: 'Conditional PDF of Y ' pointing to the numerator $p_{XY}(x, y)$; 'Marginal distribution of X ' pointing to the denominator $p_X(x)$; and 'Joint PDF of X and Y ' pointing to the overall fraction.

Continuous Conditional Distribution: Formula

$$f_{Y|X=x}(y) = \frac{f_{XY}(x, y)}{f_X(x)}$$

Conditional PDF of Y

Joint PDF of X and Y

Marginal distribution of X

The diagram illustrates the formula for the conditional probability density function (PDF) of Y given $X=x$. The formula is shown as:

$$f_{Y|X=x}(y) = \frac{f_{XY}(x, y)}{f_X(x)}$$

Three curved arrows originate from the components of the formula and point to their corresponding labels:

- A blue arrow points from $f_{Y|X=x}(y)$ to the label "Conditional PDF of Y ".
- A blue arrow points from $f_{XY}(x, y)$ to the label "Joint PDF of X and Y ".
- A blue arrow points from $f_X(x)$ to the label "Marginal distribution of X ".



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Probability Distributions with Multiple Variables

Covariance of a Dataset

Introduction to Covariance

Y_1 : height of the child (in)

Age (X)	Height (Y_1)
6	50
7	52
8	55
9	57
10	60
11	62
12	64
13	65
14	67
15	68

X : age of a child

Y_2 : grades in a test

Age (X)	Grades (Y_2)
6	5
7	7
8	8
9	3
10	1
11	1
12	6
13	10
14	2
15	7

- How is variable X related to each of the Y variables?
- How do you compare these relations?

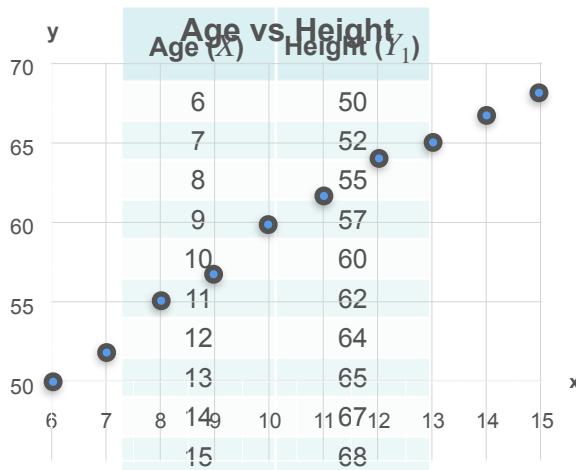
Y_3 : number of naps per day

Age (X)	Naps per day (Y_3)
6	8
7	7
8	6
9	5
10	4
11	3
12	2
13	1
14	1
15	0

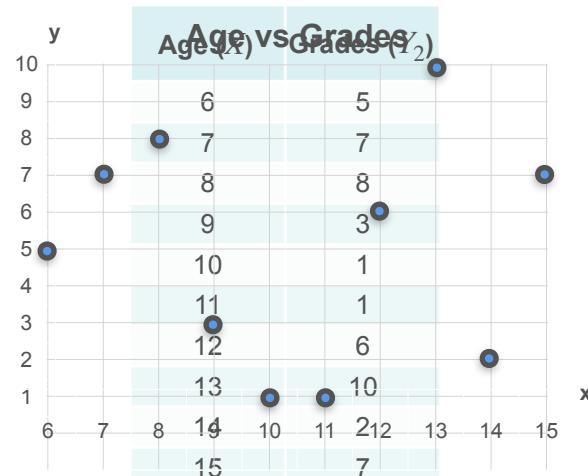
Introduction to Covariance

X : age of a child

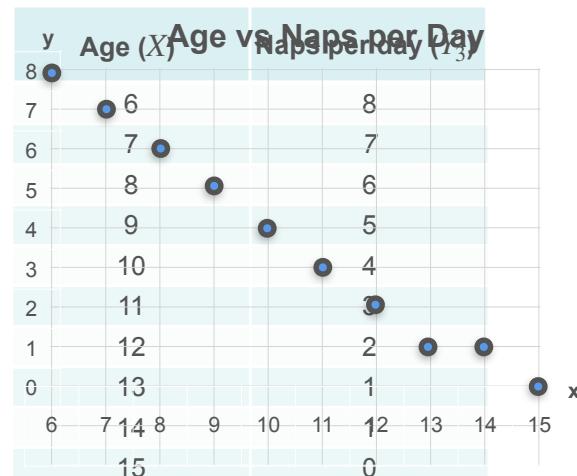
Y_1 : height of the child (in)



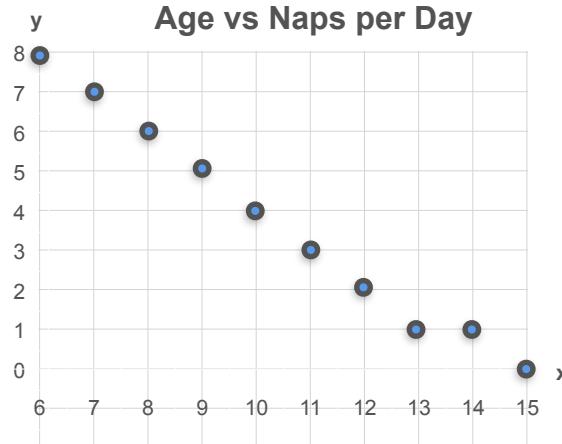
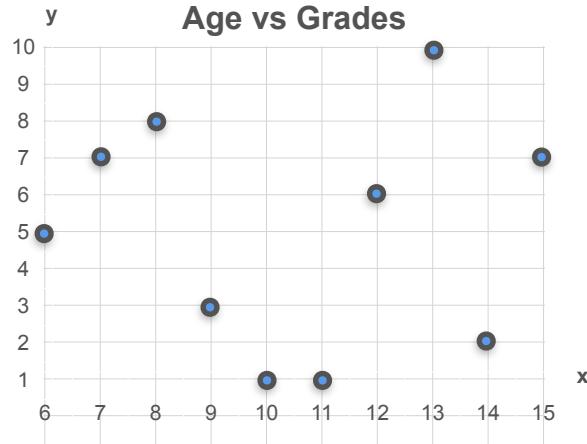
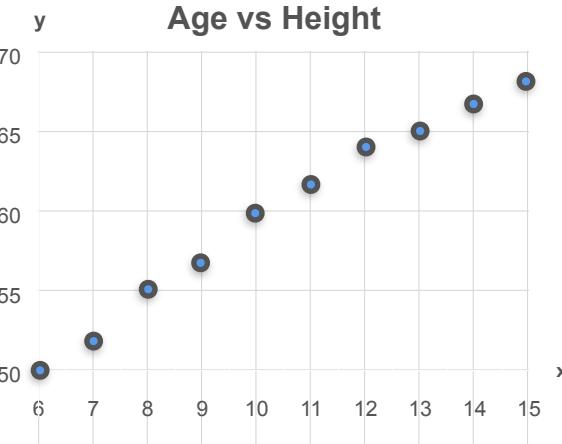
Y_2 : grades in a test



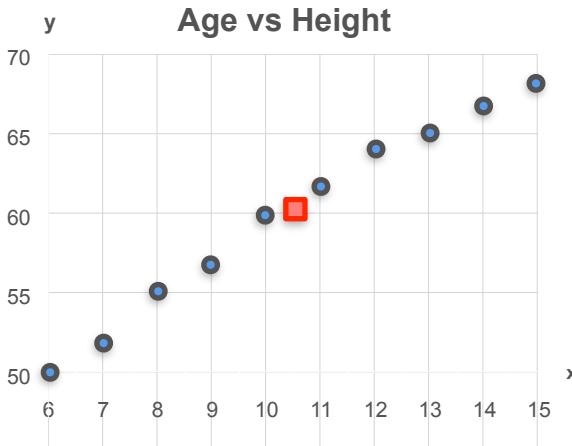
Y_3 : number of naps per day



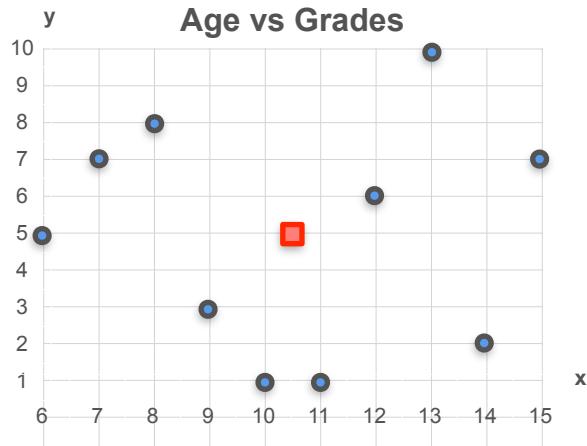
How To Compare These?



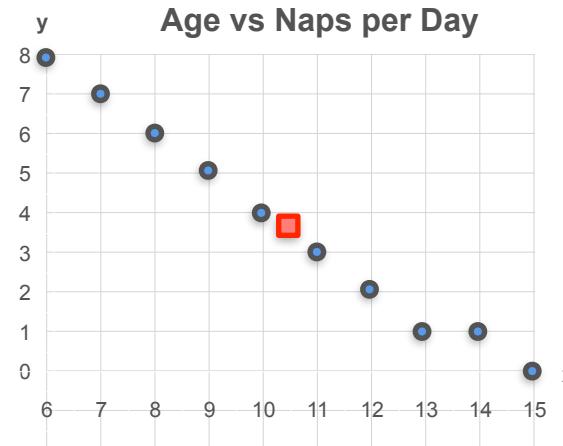
Mean?



$$\mu_x = 10.5 \quad \mu_y = 60$$

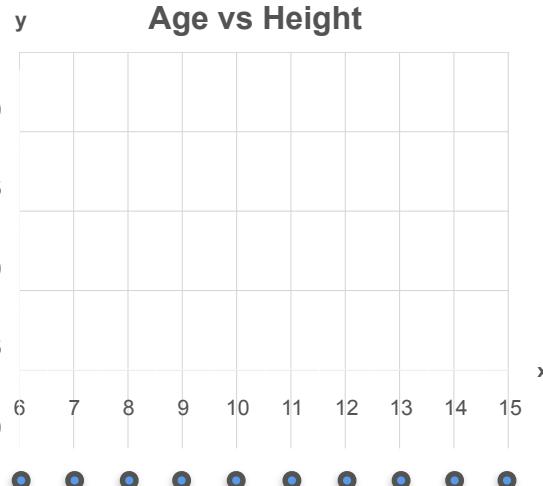


$$\mu_x = 10.5 \quad \mu_y = 5$$

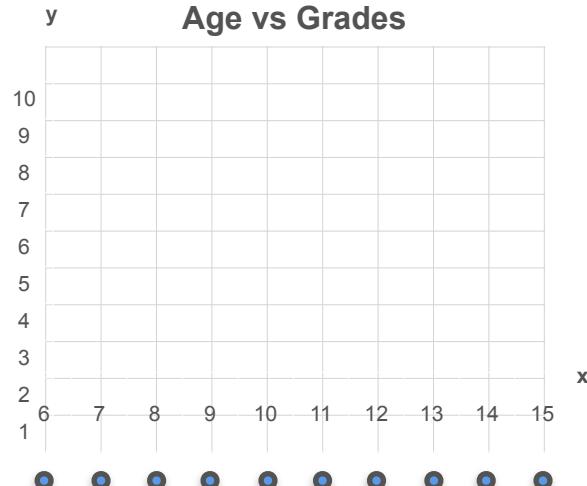


$$\mu_x = 10.5 \quad \mu_y = 3.7$$

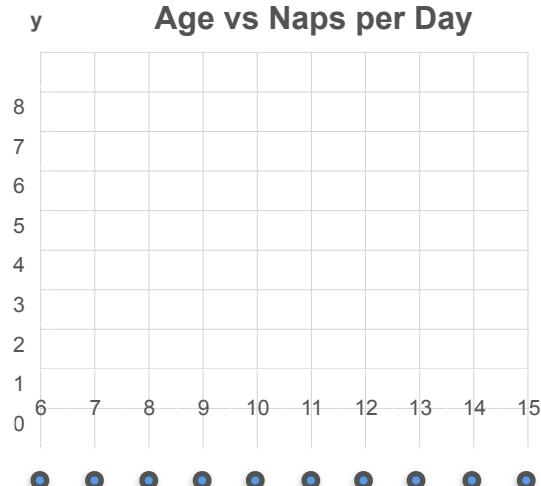
Horizontal (X) Variance



$$Var(X) = 9.17$$

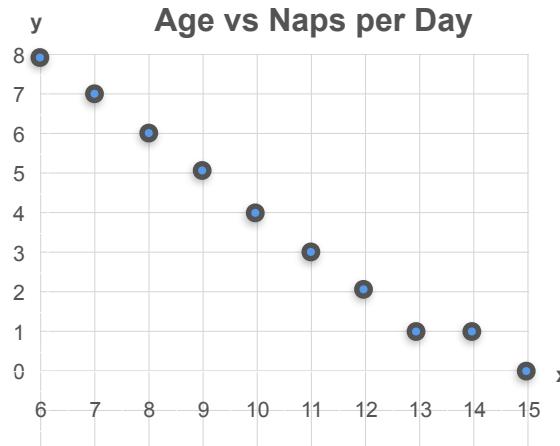
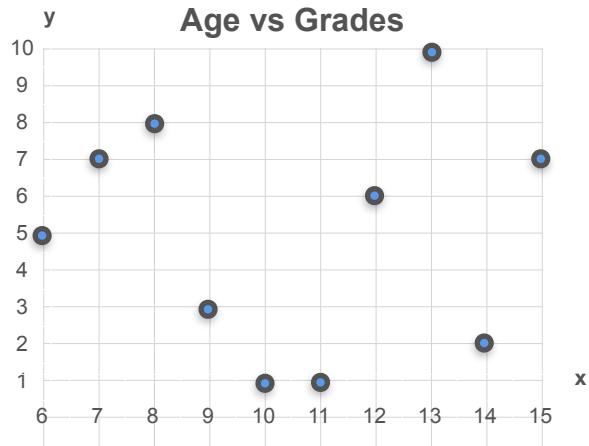
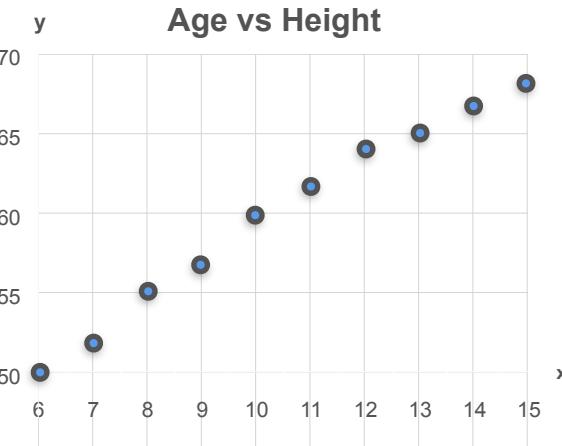


$$Var(X) = 9.17$$

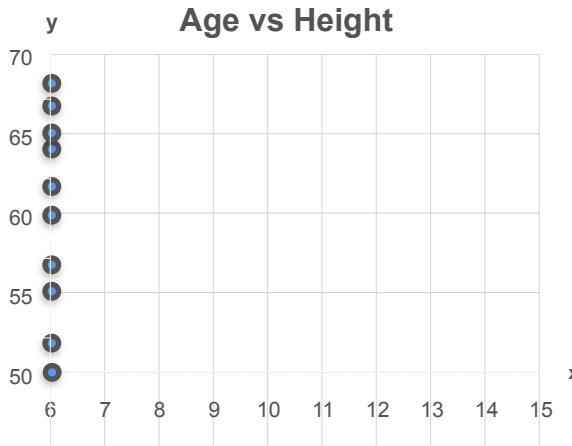


$$Var(X) = 9.17$$

Anything Else?



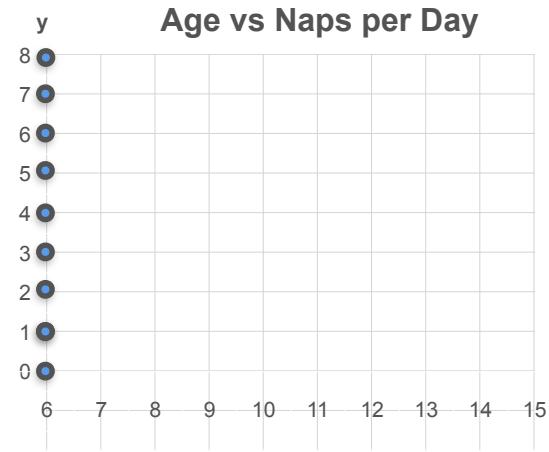
Vertical (Y) Variance



$$Var(Y) = 39.56$$

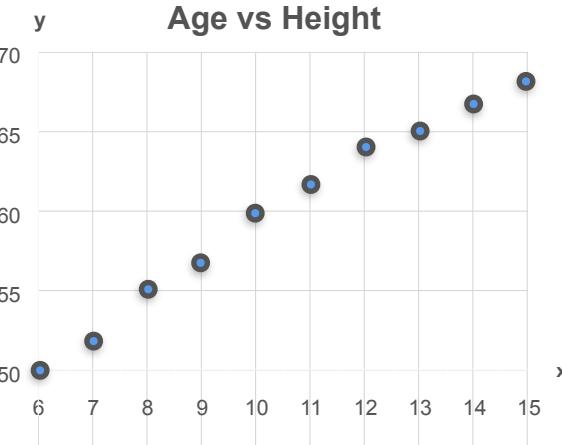


$$Var(Y) = 9.78$$

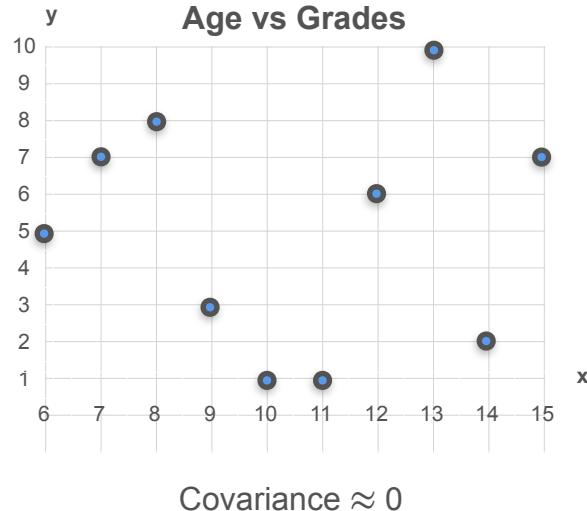


$$Var(Y) = 7.57$$

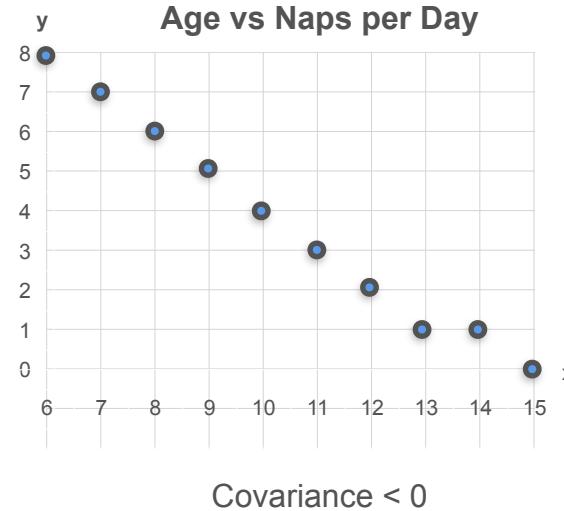
Still no Way To Compare Them



Covariance > 0

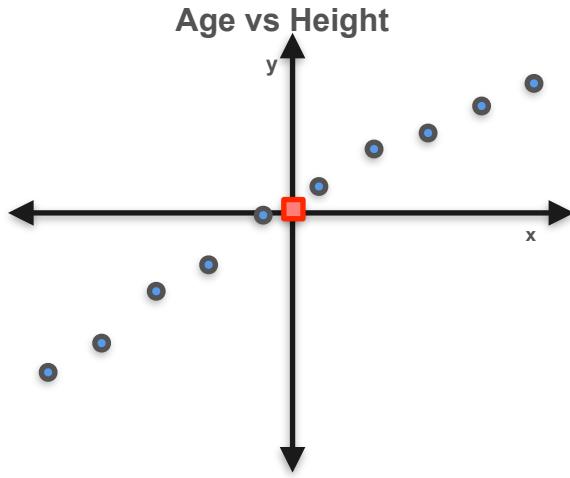


Covariance ≈ 0



Covariance < 0

First Step: Center Them

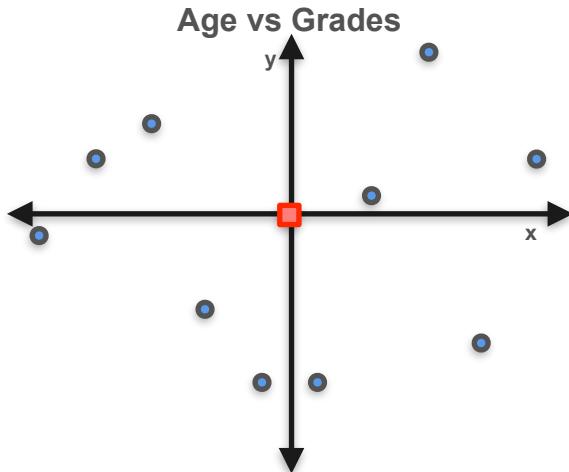


$$\mu_x = 0$$

$$\mu_y = 0$$

$$Var(X) = 1$$

$$Var(Y) = 1$$

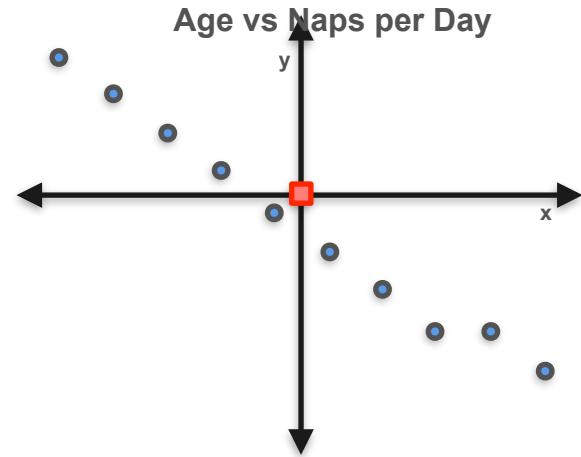


$$\mu_x = 0$$

$$\mu_y = 0$$

$$Var(X) = 1$$

$$Var(Y) = 1$$



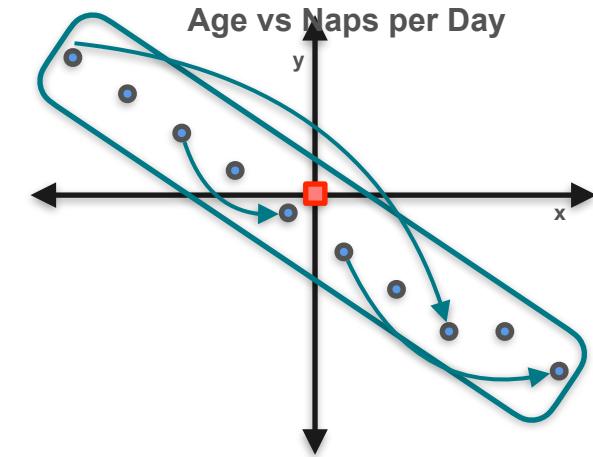
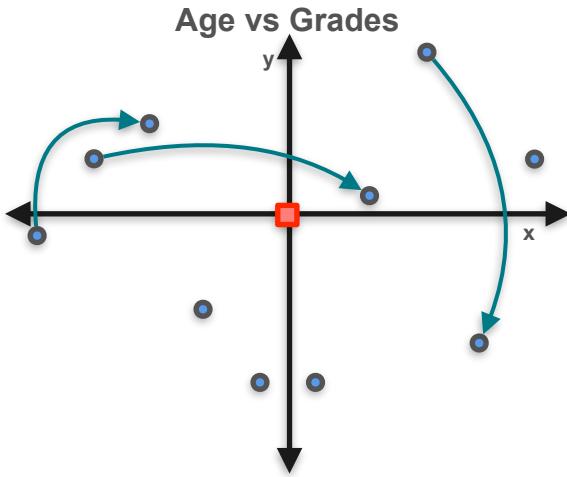
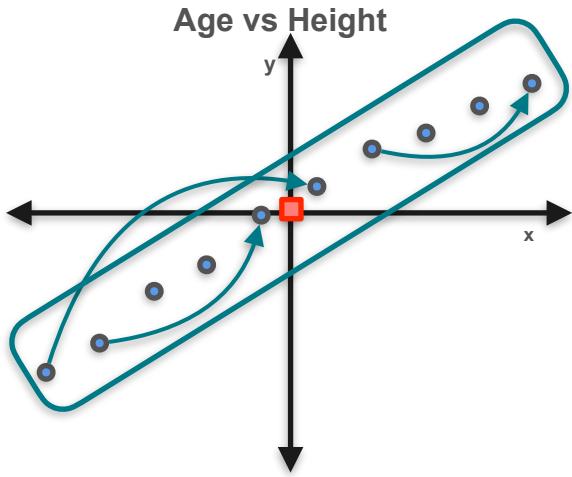
$$\mu_x = 0$$

$$\mu_y = 0$$

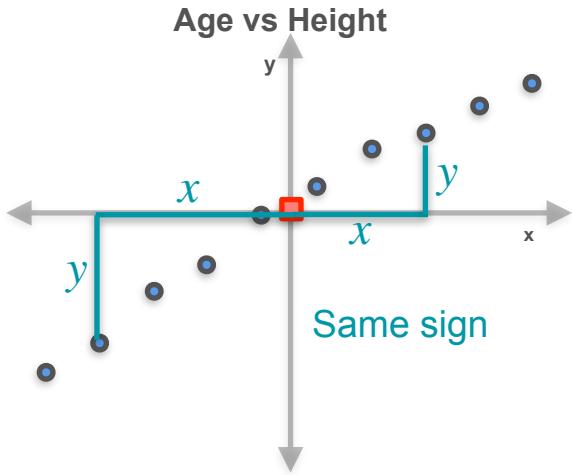
$$Var(X) = 1$$

$$Var(Y) = 1$$

Second Step: Notice Trend

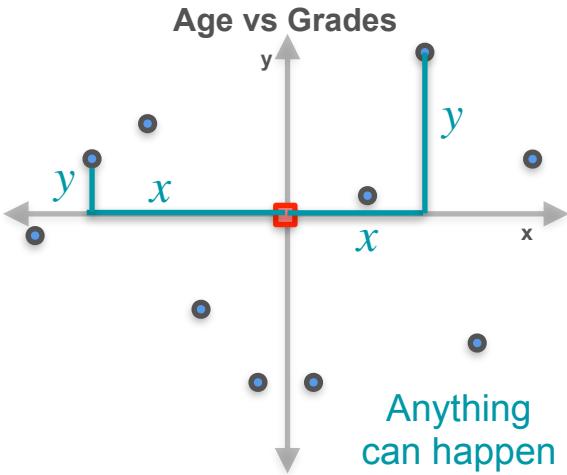


Positives and Negatives



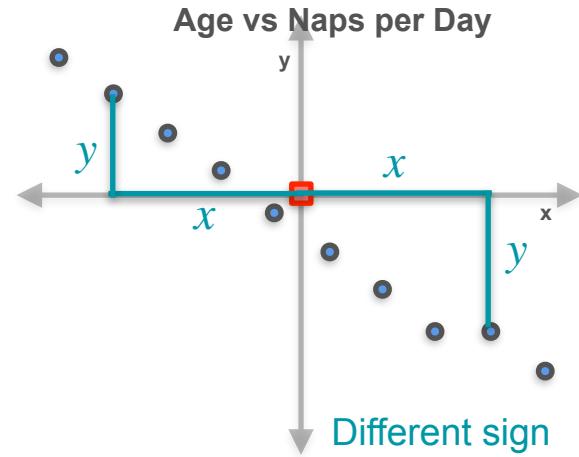
$$\sum xy > 0$$

Positive



$$\sum xy \approx 0$$

Both positive
and negative



$$\sum xy < 0$$

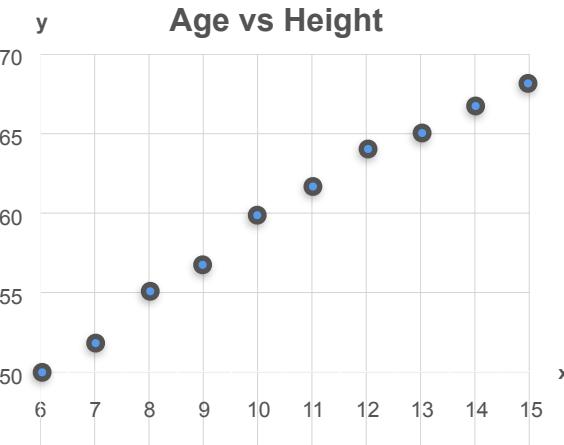
Negative

Covariance

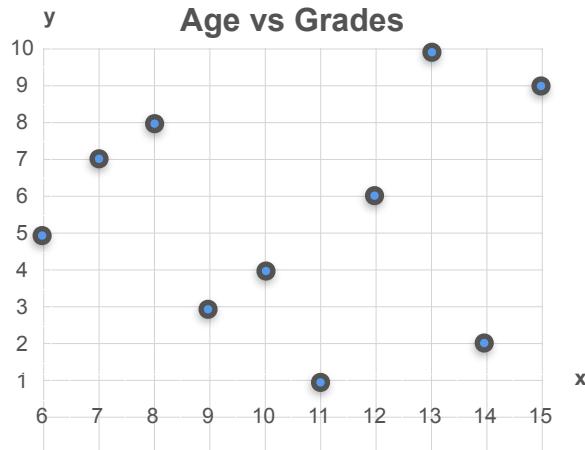
$$Cov(X, Y) = \sum xy \quad \text{Almost...}$$

$$Cov(X, Y) = \frac{\sum (x_i - \mu_x)(y_i - \mu_y)}{n}$$

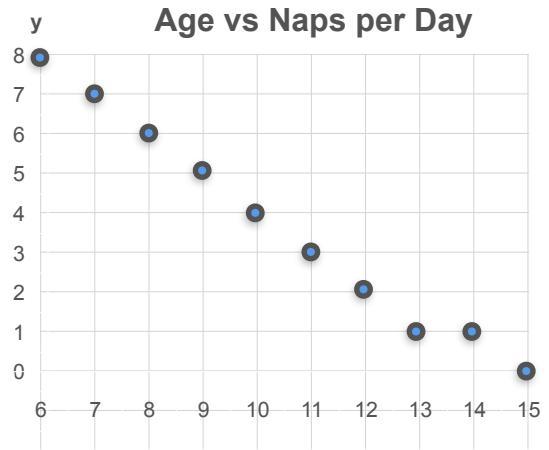
Covariance



$$\text{Cov}(X, Y) > 0$$



$$\text{Cov}(X, Y) \approx 0$$



$$\text{Cov}(X, Y) < 0$$

Covariance Formula

Age vs Height

Covariance > 0

$$\mu_x = 10.5 \quad \mu_y = 60$$

$$Cov(X, Y) = \frac{\sum (x_i - \mu_x)(y_i - \mu_y)}{n}$$

$$Cov(X, Y) = \frac{170}{10} = 17 > 0$$

Age (x)	Height (y)	$(x_i - \mu_x)$	$(y_i - \mu_y)$	$(x_i - \mu_x)(y_i - \mu_y)$
6	50	-4.5	-10	45
7	52	-3.5	-8	28
8	55	-2.5	-5	12.5
9	57	-1.5	-3	4.5
10	60	-0.5	0	0
11	62	0.5	2	1
12	64	1.5	4	6
13	65	2.5	5	12.5
14	67	3.5	7	24.5
15	68	4.5	8	36

$$\sum = 170$$

Covariance Formula

Age vs Naps per Day

Covariance < 0

$$\mu_x = 10.5 \quad \mu_y = 3.7$$

$$Cov(X, Y) = \frac{\sum (x_i - \mu_x)(y_i - \mu_y)}{n}$$

$$Cov(X, Y) = \frac{-74.5}{10} = -7.45 < 0$$

Age (x)	Naps (y)	$(x_i - \mu_x)$	$(y_i - \mu_y)$	$(x_i - \mu_x)(y_i - \mu_y)$
6	8	-4.5	4.3	-19.35
7	7	-3.5	3.3	-11.55
8	6	-2.5	2.3	-5.75
9	5	-1.5	1.3	-1.95
10	4	-0.5	0.3	-0.15
11	3	0.5	-0.7	-0.35
12	2	1.5	-1.7	-2.55
13	1	2.5	-2.7	-6.75
14	1	3.5	-2.7	-9.45
15	0	4.5	-3.7	-16.65

$$\sum = -74.5$$

Covariance Formula

Age vs Grades

Covariance ≈ 0

$$\mu_x = 10.5 \quad \mu_y = 5$$

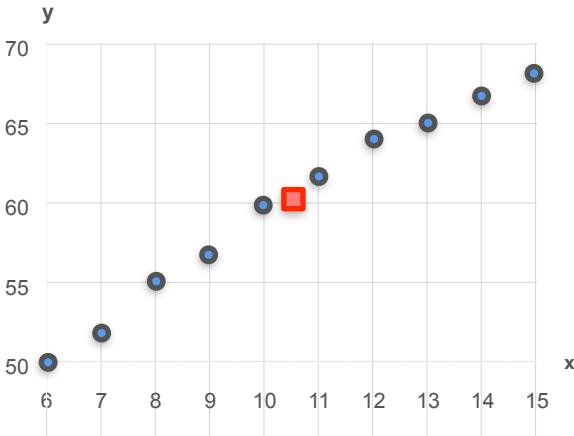
$$Cov(X, Y) = \frac{\sum (x_i - \mu_x)(y_i - \mu_y)}{n}$$

$$Cov(X, Y) = \frac{1}{10} = 0.1 \quad \approx 0$$

Age (x)	Grades (y)	$(x_i - \mu_x)$	$(y_i - \mu_y)$	$(x_i - \mu_x)(y_i - \mu_y)$
6	5	-4.5	0	0
7	7	-3.5	2	-7
8	8	-2.5	3	-7.5
9	3	-1.5	-2	3
10	1	-0.5	-4	2
11	1	0.5	-4	-2
12	6	1.5	1	1.5
13	10	2.5	5	12.5
14	2	3.5	-3	-10.5
15	7	4.5	2	9

$$\sum = 1$$

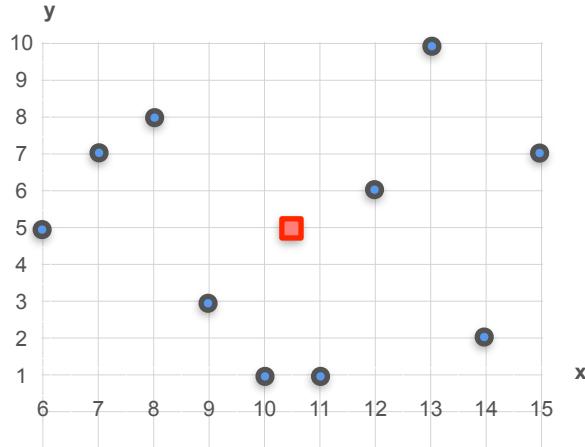
Comparing Correlations



Age vs Height

Covariance > 0

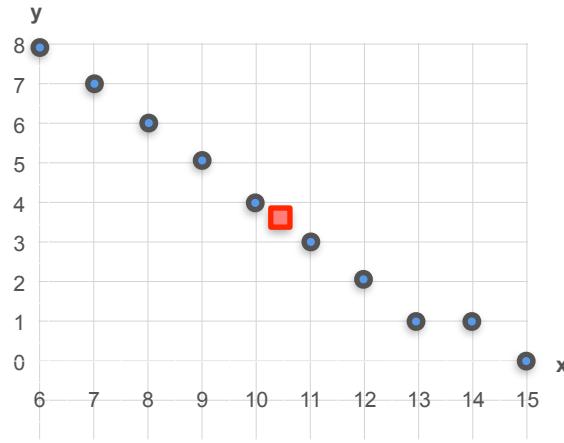
$$\text{Cov}(x, y) = 17$$



Age vs Grades

Covariance ≈ 0

$$\text{Cov}(x, y) = 0.1$$



Age vs Naps per Day

Covariance < 0

$$\text{Cov}(x, y) = -7.45$$



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Probability Distributions with Multiple Variables

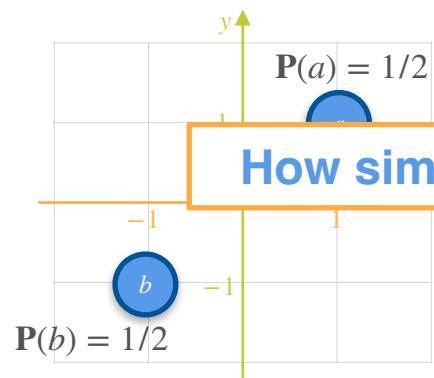
Covariance of a Probability Distribution

Covariance of a Probability Distribution: Motivation

X and Y are playing 3 games to either win or lose a dollar

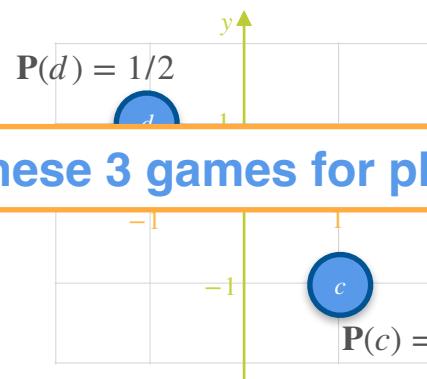
GAME 1

- a : Both players win \$1 each
 b : Both players lose \$1 each



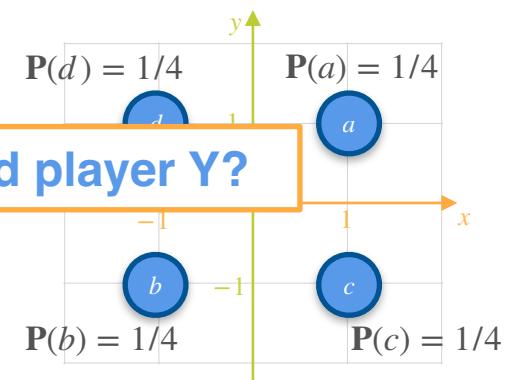
GAME 2

- c : X wins \$1 and Y loses \$1
 d : X loses \$1 and Y win \$1



GAME 3

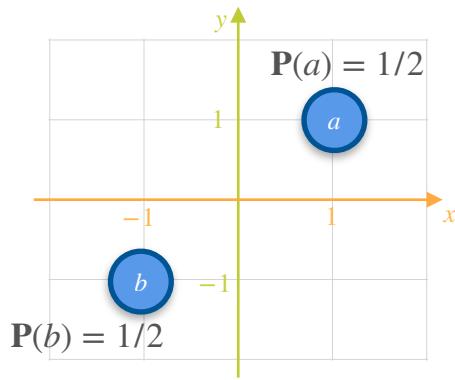
- a, b, c or d



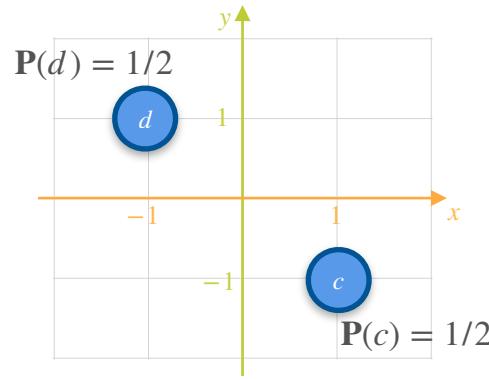
How similar are these 3 games for player X and player Y?

Covariance of a Probability Distribution: Motivation

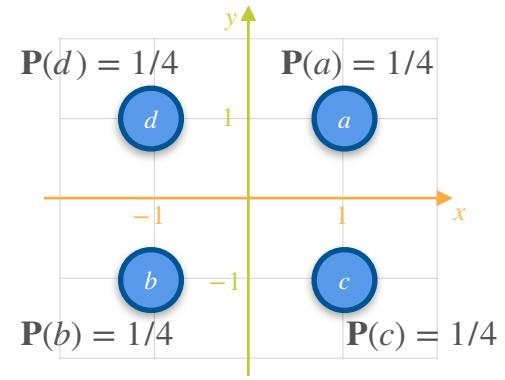
GAME 1



GAME 2



GAME 3



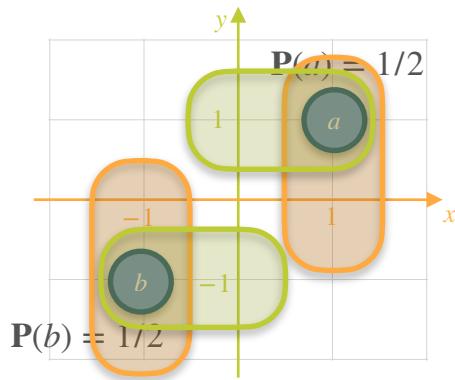
How similar are these 3 games for player X and player Y?

X : how much money in dollars player X wins

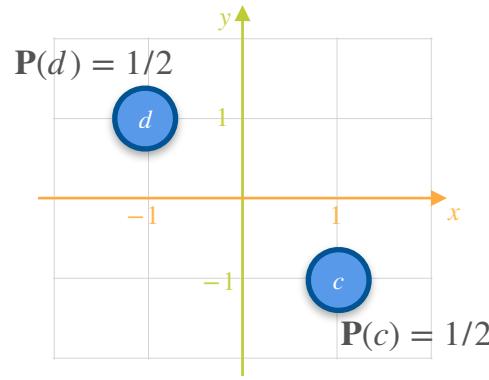
Y : how much money in dollars player Y wins

Covariance of a Probability Distribution: Motivation

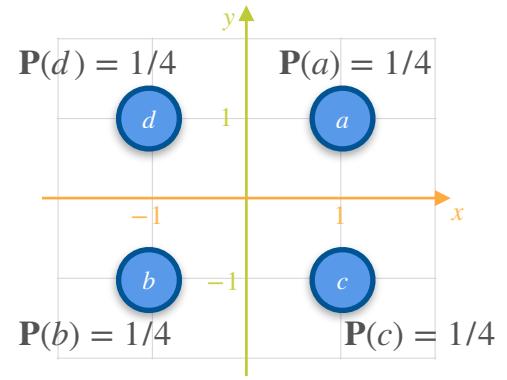
GAME 1



GAME 2



GAME 3

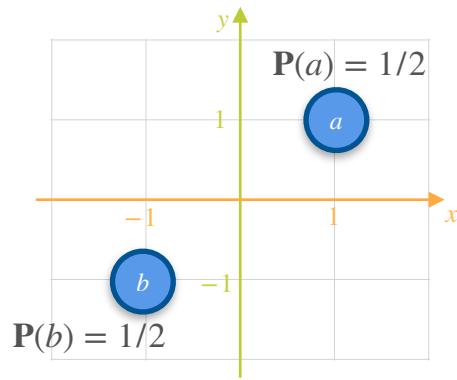


$$\mathbb{E}[X_1] = \frac{1}{2}(1) + \frac{1}{2}(-1) = 0$$

$$\mathbb{E}[Y_1] = \frac{1}{2}(1) + \frac{1}{2}(-1) = 0$$

Covariance of a Probability Distribution: Motivation

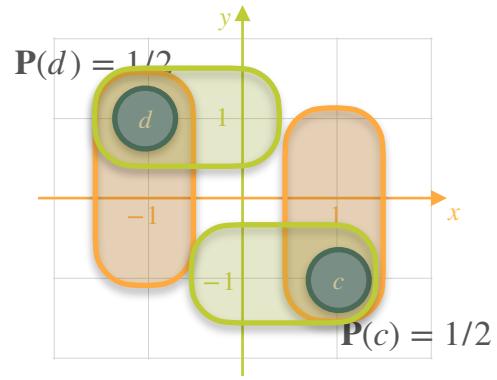
GAME 1



$$\mathbb{E}[X_1] = 0$$

$$\mathbb{E}[Y_1] = 0$$

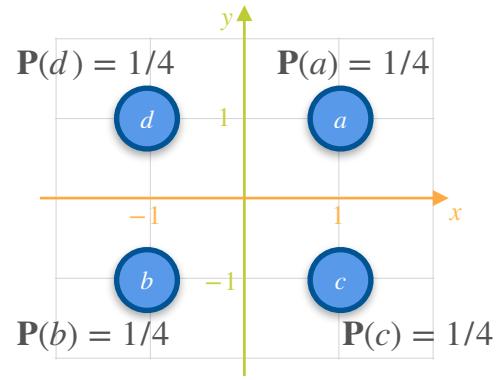
GAME 2



$$\mathbb{E}[X_2] = \frac{1}{2}(1) + \frac{1}{2}(-1) = 0$$

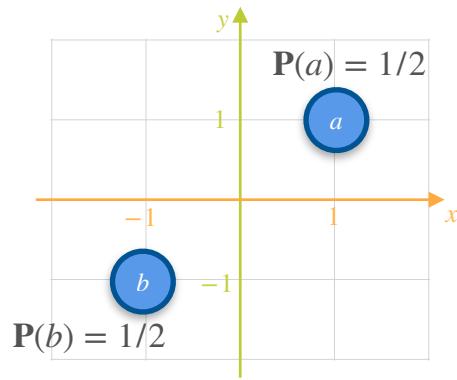
$$\mathbb{E}[Y_2] = \frac{1}{2}(1) + \frac{1}{2}(-1) = 0$$

GAME 3



Covariance of a Probability Distribution: Motivation

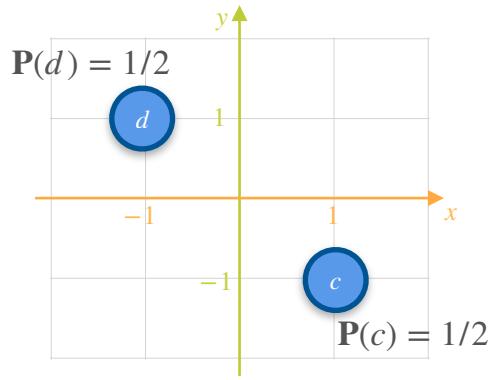
GAME 1



$$\mathbb{E}[X_1] = 0$$

$$\mathbb{E}[Y_1] = 0$$

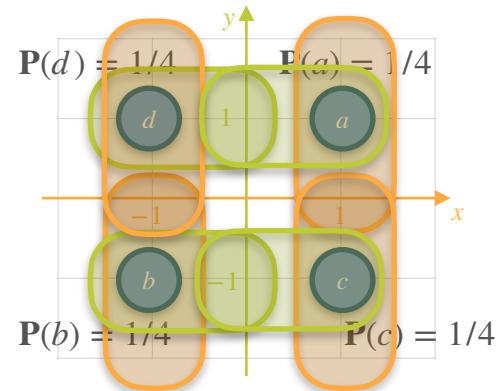
GAME 2



$$\mathbb{E}[X_2] = 0$$

$$\mathbb{E}[Y_2] = 0$$

GAME 3

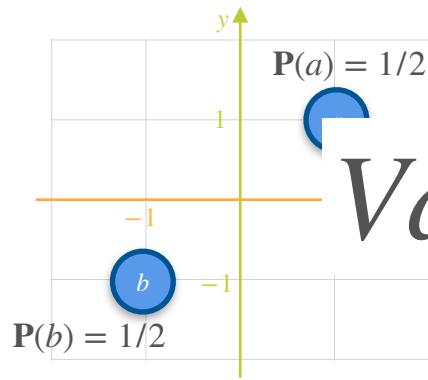


$$\mathbb{E}[X_3] = 2\left(\frac{1}{4}(1)\right) + 2\left(\frac{1}{4}(-1)\right) = 0$$

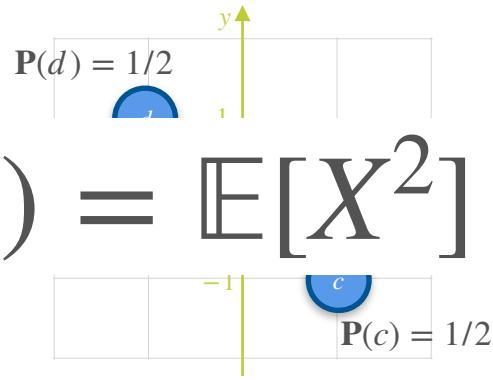
$$\mathbb{E}[Y_3] = 2\left(\frac{1}{4}(1)\right) + 2\left(\frac{1}{4}(-1)\right) = 0$$

Covariance of a Probability Distribution: Motivation

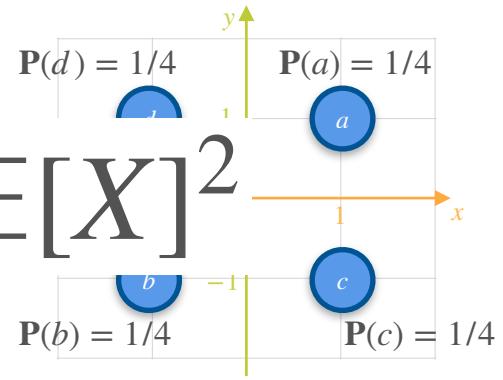
GAME 1



GAME 2



GAME 3



$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$\mathbb{E}[X_1] = 0$$

$$\mathbb{E}[X_2] = 0$$

$$\mathbb{E}[X_3] = 0$$

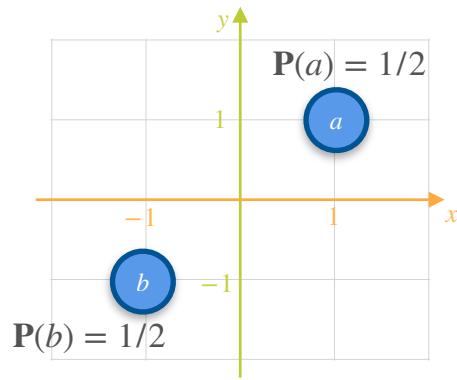
$$\mathbb{E}[Y_1] = 0$$

$$\mathbb{E}[Y_2] = 0$$

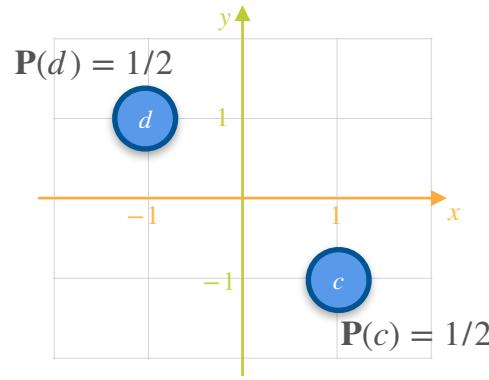
$$\mathbb{E}[Y_3] = 0$$

Covariance of a Probability Distribution: Motivation

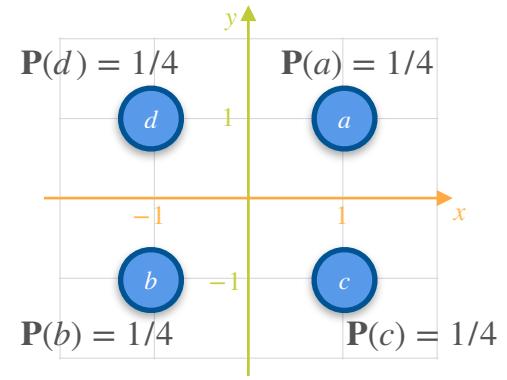
GAME 1



GAME 2



GAME 3



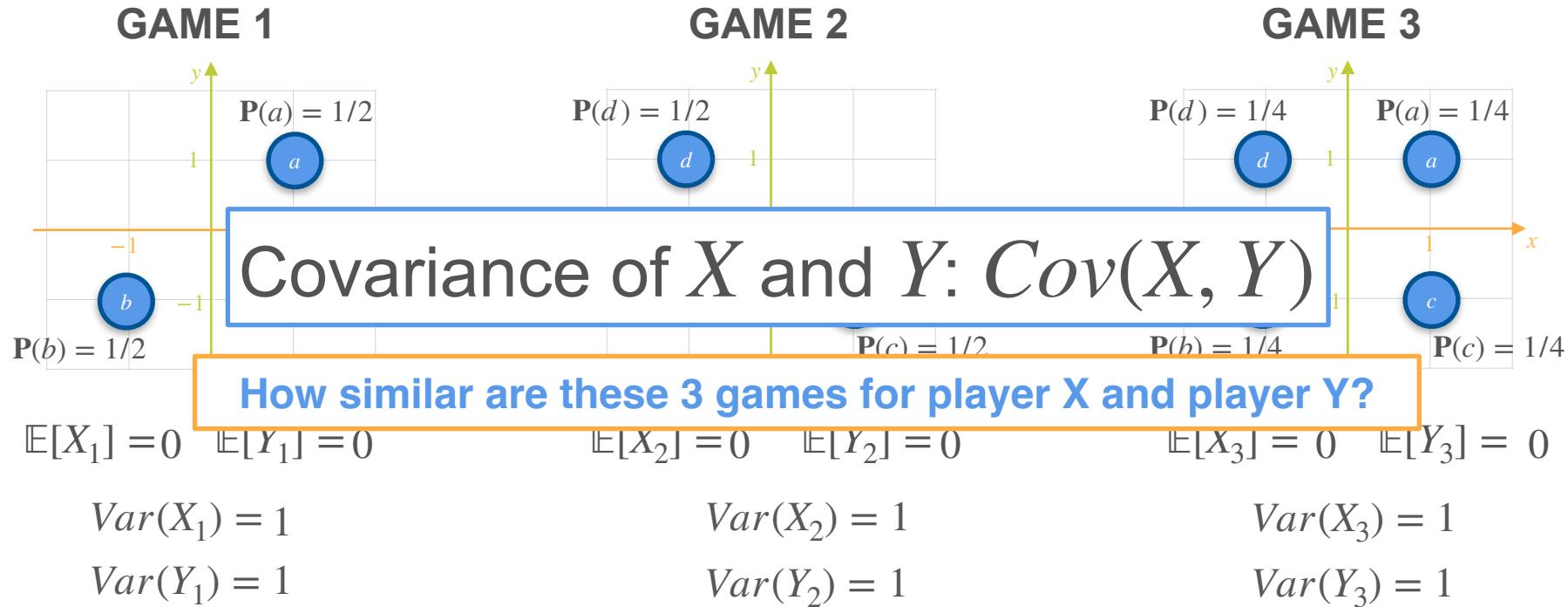
$$\mathbb{E}[X_1] = 0 \quad \mathbb{E}[Y_1] = 0$$

$$\mathbb{E}[X_2] = 0 \quad \mathbb{E}[Y_2] = 0$$

$$\mathbb{E}[X_3] = 0 \quad \mathbb{E}[Y_3] = 0$$

$$Var(X_1) = \mathbb{E}[X_1^2] - E[X_1]^2 = \frac{1}{2}(-1)^2 + \frac{1}{2}(1)^2 - 0^2 = 1$$

Covariance of a Probability Distribution: Motivation

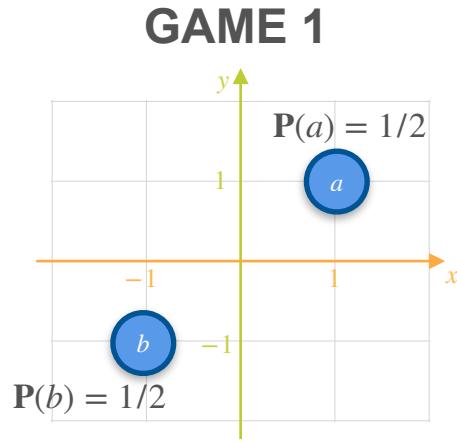


Covariance of a Probability Distribution: Motivation

Covariance of X and Y : $Cov(X, Y)$

$$Cov(X, Y) = \frac{\sum (x_i - \mu_x)(y_i - \mu_y)}{n}$$

Covariance of a Probability Distribution: Motivation

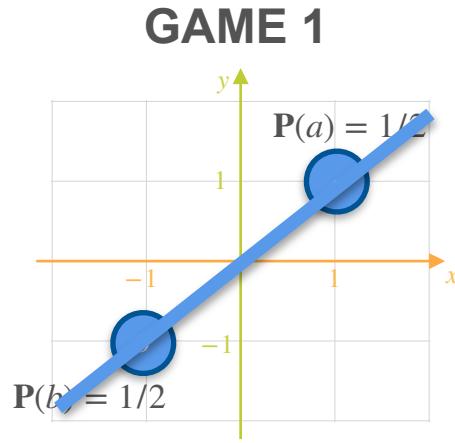


$$Cov(X, Y) = \frac{\sum (x_i - \mu_x)(y_i - \mu_y)}{n}$$

x	y	$(x_i - \mu_x)$	$(y_i - \mu_y)$	$(x_i - \mu_x)(y_i - \mu_y)$
1	1	1	1	1
-1	-1	-1	-1	1

$$Cov(X, Y) = \frac{\sum (x_i - \mu_x)(y_i - \mu_y)}{n} = \frac{2}{2} = 1$$

Covariance of a Probability Distribution: Motivation



$$\mu_x = \mathbb{E}[X_1] = 0$$

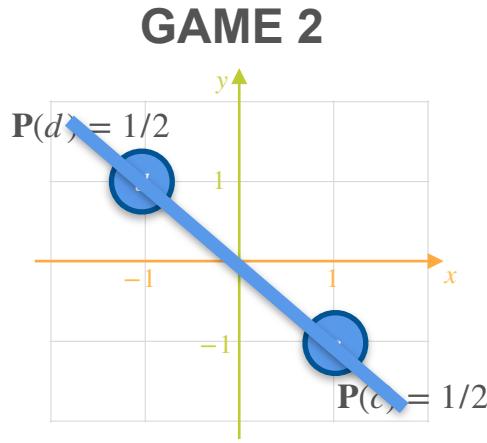
$$\mu_y = \mathbb{E}[Y_1] = 0$$

$$Cov(X, Y) = \frac{\sum (x_i - \mu_x)(y_i - \mu_y)}{n}$$

x	y	$(x_i - \mu_x)$	$(y_i - \mu_y)$	$(x_i - \mu_x)(y_i - \mu_y)$
1	1	1	1	1
-1	-1	-1	-1	1

$$Cov(X, Y) = \frac{\sum (x_i - \mu_x)(y_i - \mu_y)}{n} = \frac{2}{2} = 1$$

Covariance of a Probability Distribution: Motivation



$$\mu_x = \mathbb{E}[X_2] = 0$$

$$\mu_y = \mathbb{E}[Y_2] = 0$$

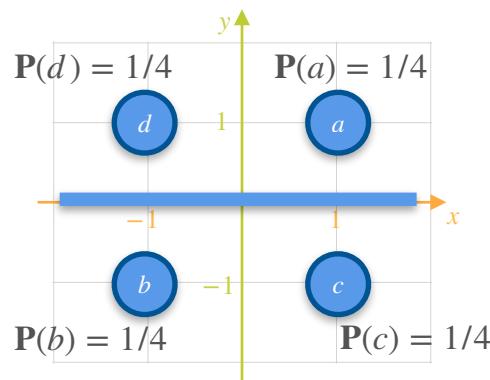
$$Cov(X, Y) = \frac{\sum (x_i - \mu_x)(y_i - \mu_y)}{n}$$

x	y	$(x_i - \mu_x)$	$(y_i - \mu_y)$	$(x_i - \mu_x)(y_i - \mu_y)$
1	-1	1	-1	-1
-1	1	-1	1	-1

$$Cov(X, Y) = \frac{\sum (x_i - \mu_x)(y_i - \mu_y)}{n} = \frac{-2}{2} = -1$$

Covariance of a Probability Distribution: Motivation

GAME 3



$$\mu_x = \mathbb{E}[X_3] = 0$$

$$\mu_y = \mathbb{E}[Y_3] = 0$$

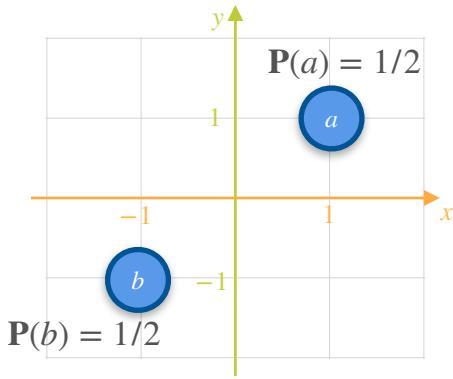
$$Cov(X, Y) = \frac{\sum (x_i - \mu_x)(y_i - \mu_y)}{n}$$

x	y	$(x_i - \mu_x)$	$(y_i - \mu_y)$	$(x_i - \mu_x)(y_i - \mu_y)$
1	1	1	1	1
1	-1	-1	1	-1
-1	1	1	-1	-1
-1	-1	-1	-1	1

$$Cov(X, Y) = \frac{\sum (x_i - \mu_x)(y_i - \mu_y)}{n} = \frac{0}{4} = 0$$

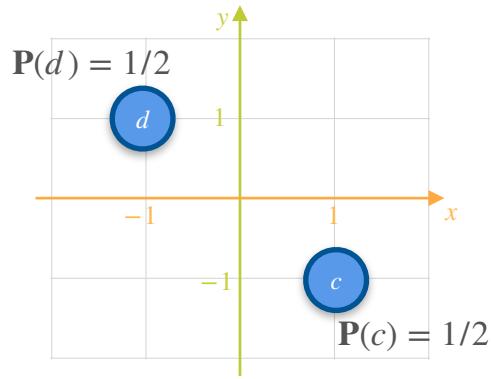
Covariance of a Probability Distribution: Motivation

GAME 1



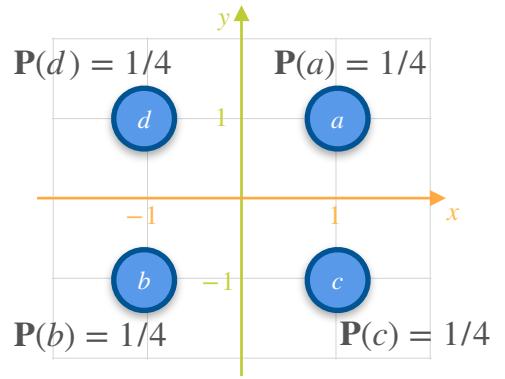
$$Cov(X, Y) = 1$$

GAME 2



$$Cov(X, Y) = -1$$

GAME 3



$$Cov(X, Y) = 0$$

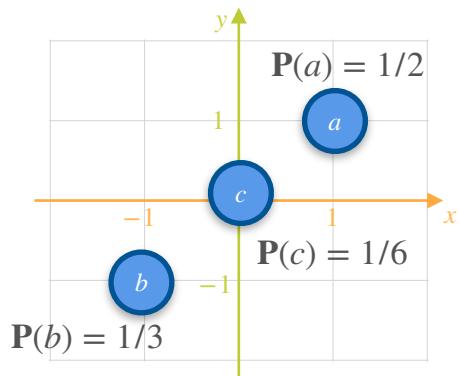
Covariance of a Probability Distribution: Motivation

GAME 4

a: Both players win \$1 each $P(a) = 1/2$

b: Both players lose \$1 each $P(b) = 1/3$

c: Neither players wins nor lose anything $P(c) = 1/6$



Unequal Probabilities

$$\mathbb{E}[X_4] = \frac{1}{2}(1) + \frac{1}{6}(0) + \frac{1}{3}(-1) = \frac{1}{6}$$

$$\mathbb{E}[Y_4] = \frac{1}{2}(1) + \frac{1}{6}(0) + \frac{1}{3}(-1) = \frac{1}{6}$$

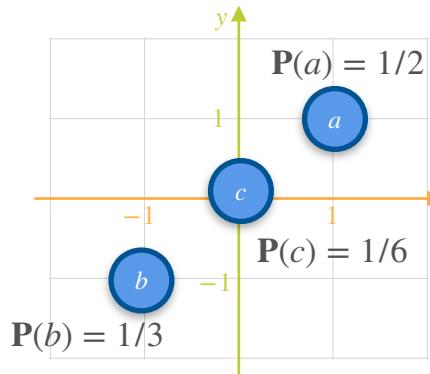
Covariance of a Probability Distribution: Motivation

GAME 4

a: Both players win \$1 each $\mathbf{P}(a) = 1/2$

b: Both players lose \$1 each $\mathbf{P}(b) = 1/3$

c: Neither players wins nor lose anything $\mathbf{P}(c) = 1/6$



Unequal Probabilities

$$\text{Var}(X_4) = \sum_{i=1}^N (x_i - \mu_x)^2 \cdot \mathbf{P}(x_i)$$

$$\mathbb{E}[X_4] = \frac{1}{6}$$

$$\text{Var}(X_4) = \frac{1}{2}\left(1 - \frac{1}{6}\right)^2 + \frac{1}{6}\left(0 - \frac{1}{6}\right)^2 + \frac{1}{3}\left(-1 - \frac{1}{6}\right)^2$$

$$\mathbb{E}[Y_4] = \frac{1}{6}$$

$$= 0.806$$

Covariance of a Probability Distribution: Motivation

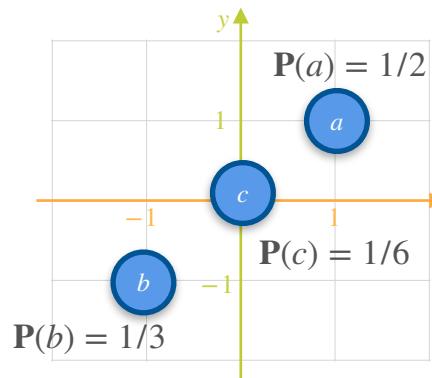
GAME 4

a: Both players win \$1 each $P(a) = 1/2$

b: Both players lose \$1 each $P(b) = 1/3$

c: Neither players wins nor lose anything $P(c) = 1/6$

Unequal Probabilities



$$\mathbb{E}[X_4] = \frac{1}{6} \quad \text{Var}(X_4) = 0.806$$

$$\mathbb{E}[Y_4] = \frac{1}{6} \quad \text{Var}(Y_4) = 0.806$$

$\text{Cov}(X, Y) = ?$

Covariance of Probability Distributions

$$Cov(X, Y) = \frac{\sum (x_i - \mu_x)(y_i - \mu_y)}{n} = \frac{1}{n} \sum (x_i - \mu_x)(y_i - \mu_y)$$

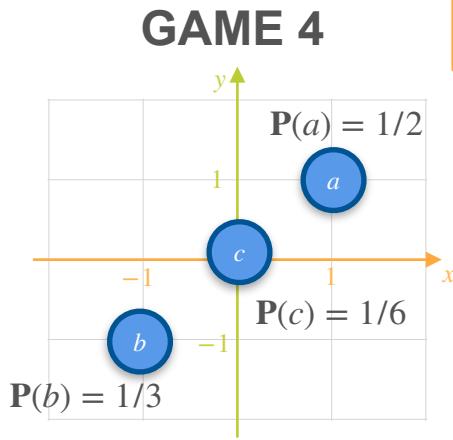
equal probabilities

$$Cov(X, Y) = \sum p_{XY}(x_i, y_i)(x_i - \mu_x)(y_i - \mu_y)$$

unequal probabilities

$$Cov(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

Covariance of a Probability Distribution: Motivation



Unequal Probabilities

$$\begin{aligned} \text{Cov}(X, Y) &= \sum p_{XY}(x_i, y_i)(x_i - \mu_x)(y_i - \mu_y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \\ &= \frac{1}{2}\left(1 - \frac{1}{6}\right)^2 + \frac{1}{6}\left(0 - \frac{1}{6}\right)^2 + \frac{1}{3}\left(-1 - \frac{1}{6}\right)^2 \end{aligned}$$

$$\mu_x = \mu_y = \mathbb{E}[X_4] = \mathbb{E}[Y_4] = 1/6$$

$$\text{Var}(X_4) = \text{Var}(Y_4) = 0.806$$

$$\text{Cov}(X, Y) = 0.806$$

Covariance?

$$\mathbb{E}(X) = 4.903$$

$$\mathbb{E}(Y) = 5.280$$

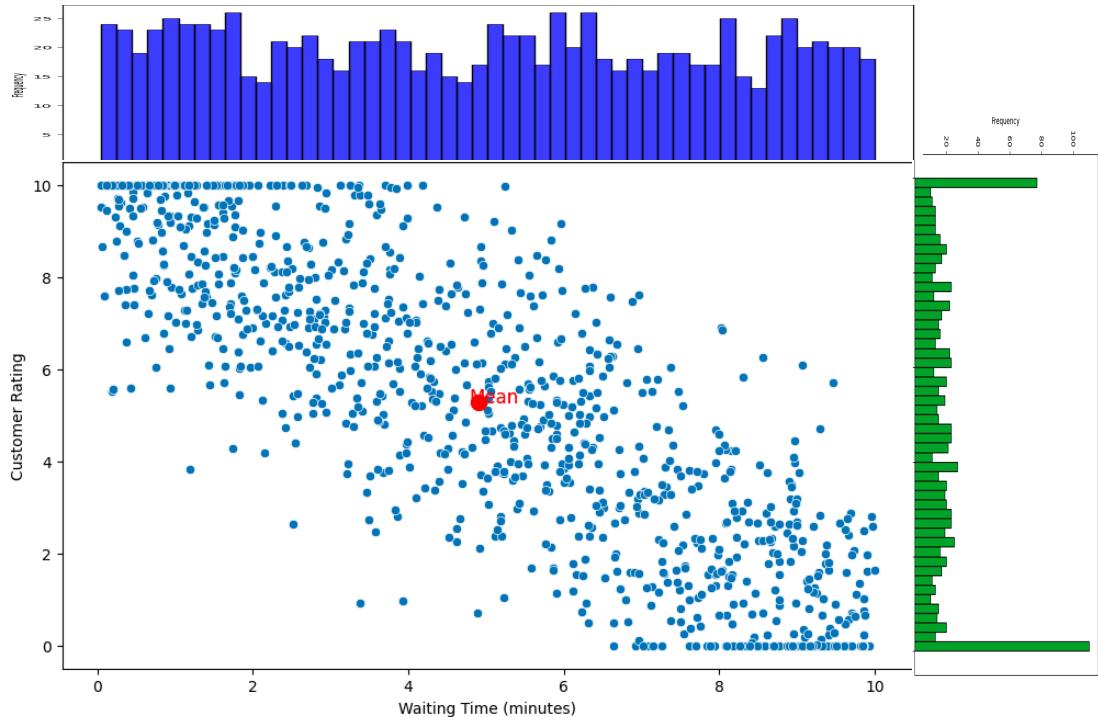
$$\mathbb{E}(XY) = 18.014$$

$$Cov(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

$$Cov(X, Y) = 18.014 - (4.903)(5.280)$$

$$Cov(X, Y) = 18.014 - (4.903)(5.280)$$

$$Cov(X, Y) = -7.878$$



Covariance of a Joint Continuous Distribution

$$Cov(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

$$\mathbb{E}(X) = 4.903$$

$$Cov(X, Y) = 18.014 - (4.903)(5.280)$$

$$\mathbb{E}(Y) = 5.280$$

$$Cov(X, Y) = 18.014 - (4.903)(5.280)$$

$$\mathbb{E}(XY) = 18.014$$

$$Cov(X, Y) = -7.878$$

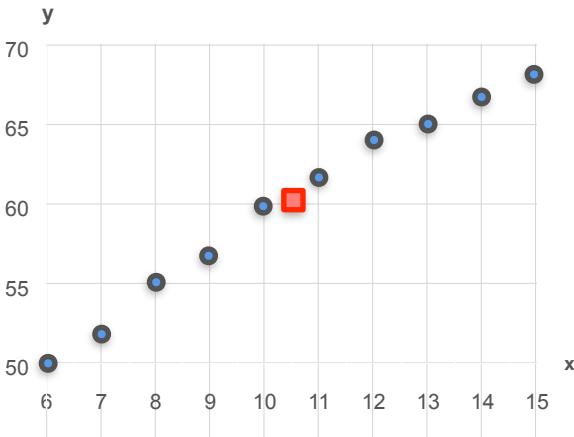


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Probability Distributions with Multiple Variables

Covariance Matrix

Covariance Matrix

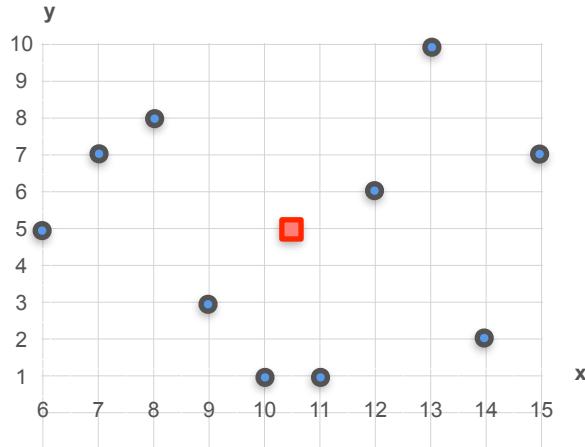


Age vs Height

$$\text{Var}(X) = 9.17$$

$$\text{Var}(Y) = 39.56$$

$$\text{Cov}(X, Y) = 17$$

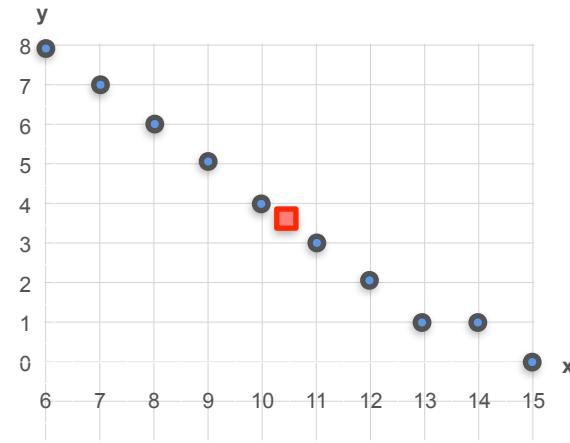


Age vs Grades

$$\text{Var}(X) = 9.17$$

$$\text{Var}(Y) = 9.78$$

$$\text{Cov}(X, Y) = 0.1$$



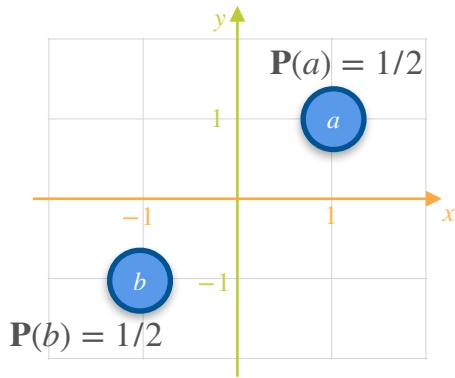
Age vs Naps per Day

$$\text{Var}(X) = 9.17$$

$$\text{Var}(Y) = 7.57$$

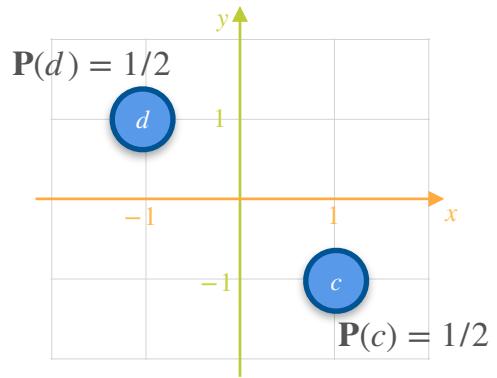
$$\text{Cov}(X, Y) = -7.45$$

Covariance Matrix



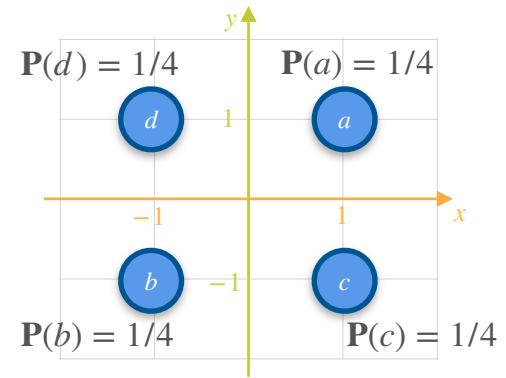
$$Var(X) = Var(Y) = 1$$

$$Cov(X, Y) = 1$$



$$Var(X) = Var(Y) = 1$$

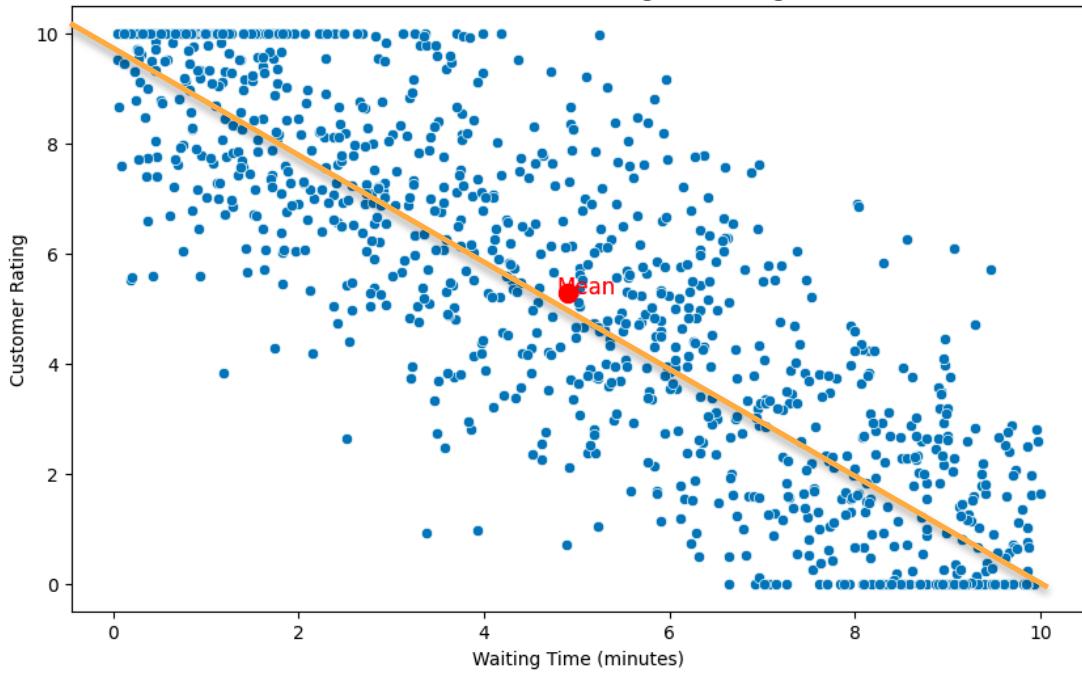
$$Cov(X, Y) = -1$$



$$Var(X) = Var(Y) = 1$$

$$Cov(X, Y) = 0$$

Covariance Matrix

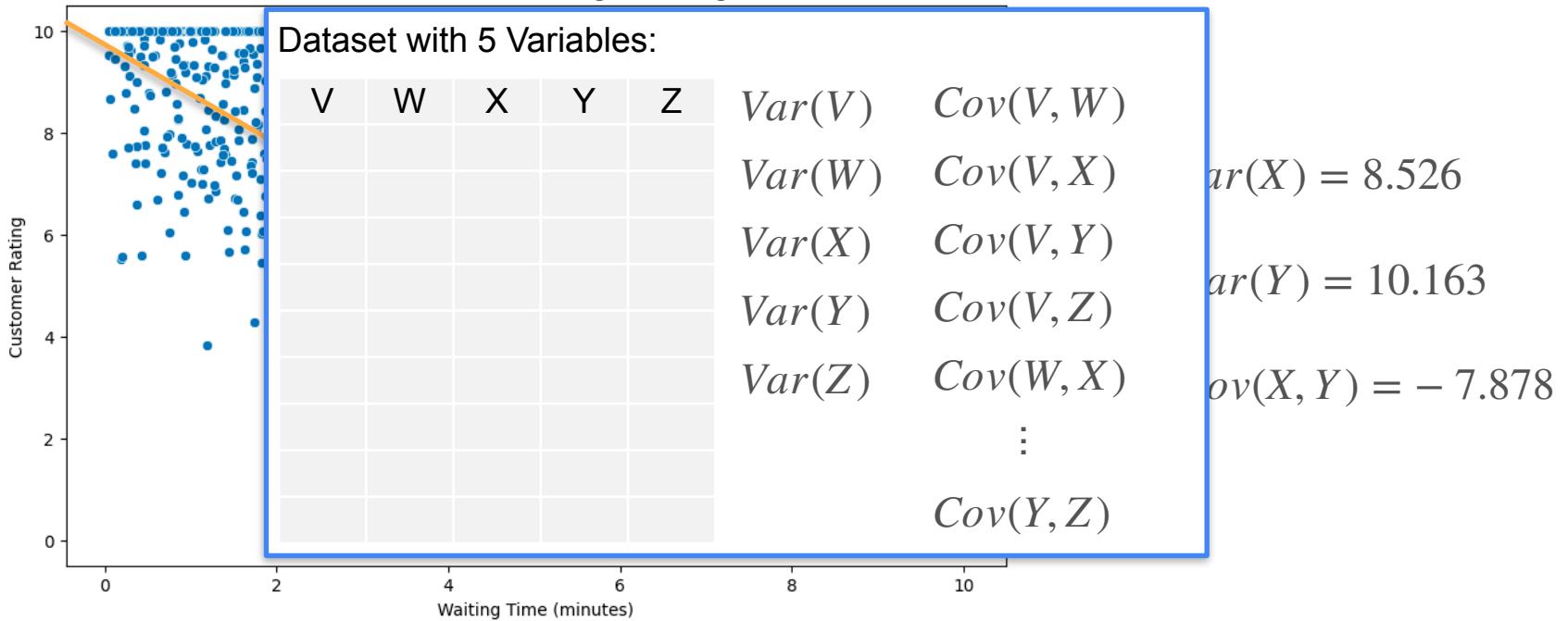


$$\text{Var}(X) = 8.526$$

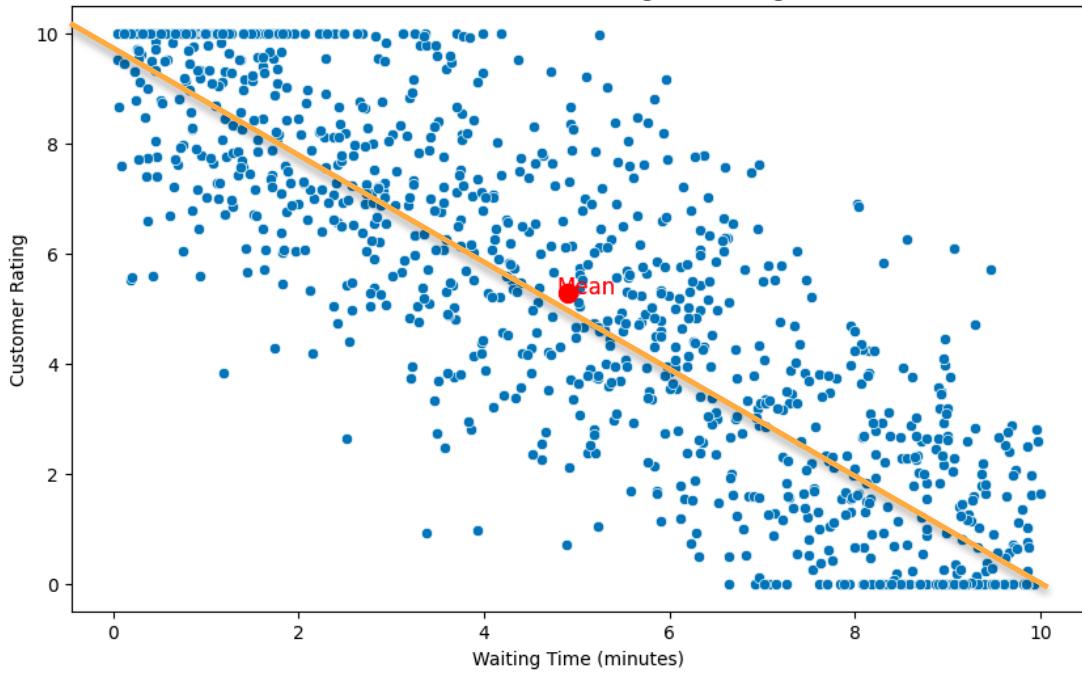
$$\text{Var}(Y) = 10.163$$

$$\text{Cov}(X, Y) = -7.878$$

Covariance Matrix



Covariance Matrix



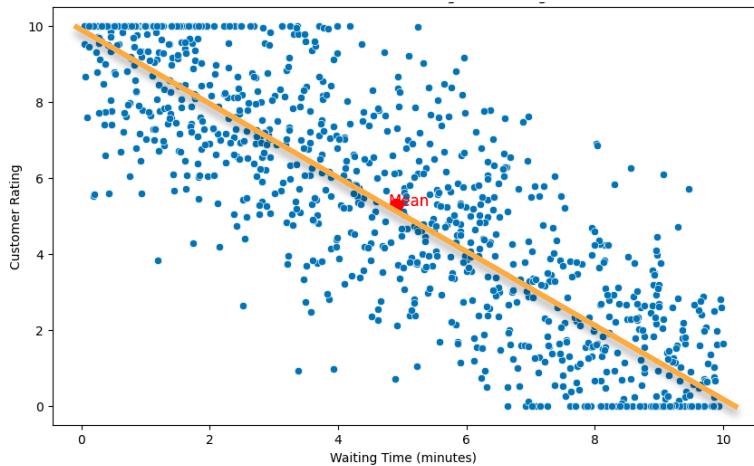
$$\text{Var}(X) = 8.526$$

$$\text{Var}(Y) = 10.163$$

$$\text{Cov}(X, Y) = -7.878$$

Covariance Matrix

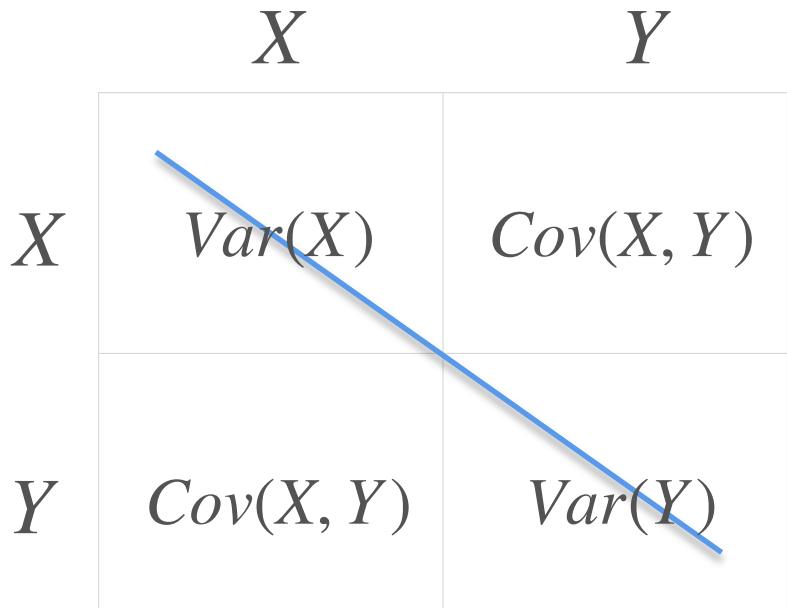
Covariance Matrix



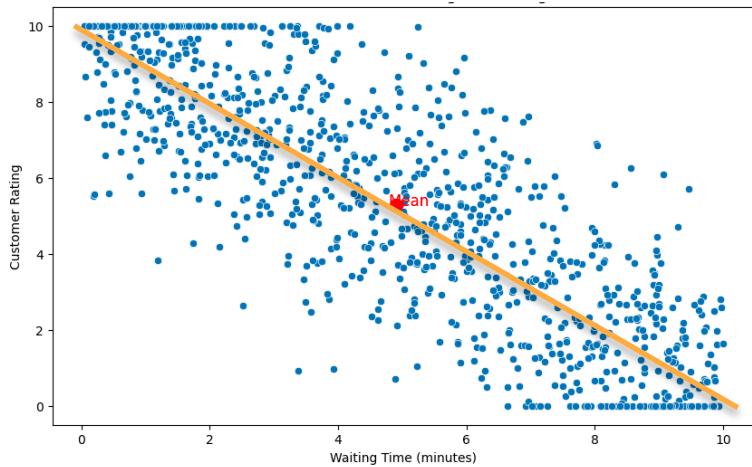
$$Var(X) = 8.526$$

$$Var(Y) = 10.163$$

$$Cov(X, Y) = -7.878$$



Covariance Matrix



$$Var(X) = 8.526$$

$$Var(Y) = 10.163$$

$$Cov(X, Y) = -7.878$$

	X	Y
X	$Var(X)$	$Cov(X, Y)$
Y	$Cov(X, Y)$	$Var(Y)$

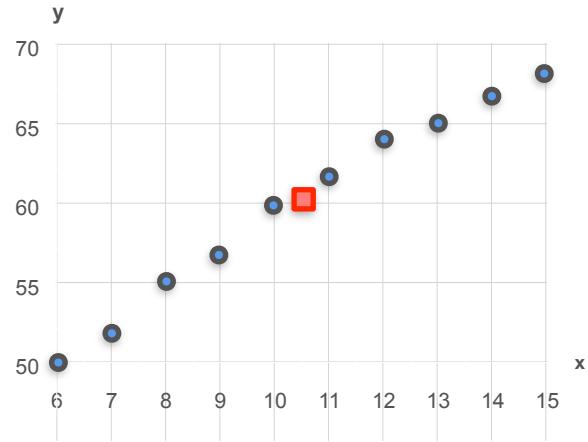
$$\begin{bmatrix} Var(X) & Cov(X, Y) \\ Cov(X, Y) & Var(Y) \end{bmatrix}$$

8.526	-7.878
-7.878	10.163

$$\begin{bmatrix} 8.534 & -7.878 \\ -7.878 & 10.173 \end{bmatrix}$$

Covariance Matrix

Covariance Matrix



Age vs Height

$$Var(X) = 9.17$$

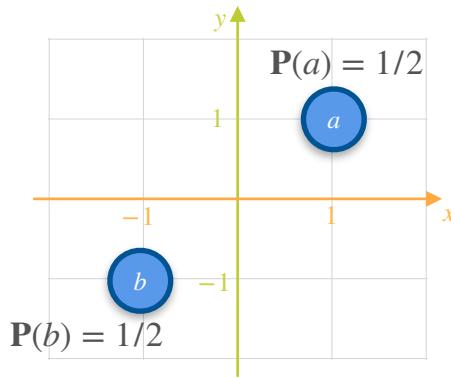
$$Var(Y) = 39.56$$

$$Cov(X, Y) = 17$$

$$\begin{bmatrix} Var(X) & Cov(X, Y) \\ Cov(X, Y) & Var(Y) \end{bmatrix}$$

$$\begin{bmatrix} 9.17 & 17 \\ 17 & 39.56 \end{bmatrix}$$

Covariance Matrix



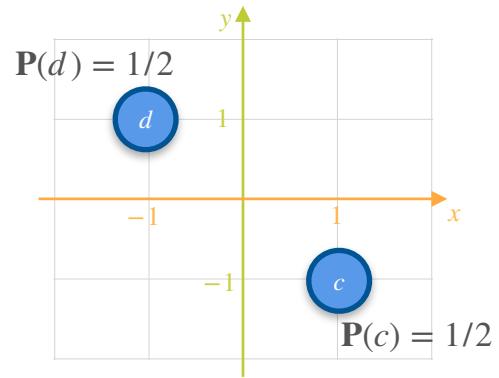
$$\begin{bmatrix} Var(X) & Cov(X, Y) \\ Cov(X, Y) & Var(Y) \end{bmatrix}$$

$$Var(X) = 1$$

$$Var(Y) = 1$$

$$Cov(X, Y) = 1$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$



$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$Var(X) = 1$$

$$Var(Y) = 1$$

$$Cov(X, Y) = -1$$

Covariance of a Joint Continuous Distribution

Dataset with 3 Variables:

X	Y	Z	$Var(X)$	$Cov(X, Y)$
X	Y	Z	$Var(Y)$	$Cov(X, Z)$
X	Y	Z	$Var(Z)$	$Cov(Y, Z)$

X	$Var(X)$	$Cov(X, Y)$	$Cov(X, Z)$
Y	$Cov(X, Y)$	$Var(Y)$	$Cov(Y, Z)$
Z	$Cov(X, Z)$	$Cov(Y, Z)$	$Var(Z)$

Covariance of a Joint Continuous Distribution

Dataset with 5 Variables:

V	W	X	Y	Z
			$Var(V)$	$Cov(V, W)$
			$Var(W)$	$Cov(V, X)$
			$Var(X)$	$Cov(V, Y)$
			$Var(Y)$	$Cov(V, Z)$
			$Var(Z)$	$Cov(W, X)$
				\vdots
				$Cov(Y, Z)$

$Var(V) \quad Cov(V, W)$
 $Var(W) \quad Cov(V, X)$
 $Var(X) \quad Cov(V, Y)$
 $Var(Y) \quad Cov(V, Z)$
 $Var(Z) \quad Cov(W, X)$
 \vdots
 $Cov(Y, Z)$

	V	W	X	Y	Z
V	$Var(V)$	$Cov(V, W)$	$Cov(V, X)$	$Cov(V, Y)$	$Cov(V, Z)$
W	$Cov(V, W)$	$Var(W)$	$Cov(W, X)$	$Cov(W, Y)$	$Cov(W, Z)$
X	$Cov(V, X)$	$Cov(W, X)$	$Var(X)$	$Cov(X, Y)$	$Cov(X, Z)$
Y	$Cov(V, Y)$	$Cov(W, Y)$	$Cov(X, Y)$	$Var(Y)$	$Cov(Y, Z)$
Z	$Cov(V, Z)$	$Cov(W, Z)$	$Cov(X, Z)$	$Cov(Y, Z)$	$Var(Z)$

Covariance of a Joint Continuous Distribution

$\sum =$

Covariance Matrix

	V	W	X	Y	Z
V	$Var(V)$	$Cov(V, W)$	$Cov(V, X)$	$Cov(V, Y)$	$Cov(V, Z)$
W	$Cov(V, W)$	$Var(W)$	$Cov(W, X)$	$Cov(W, Y)$	$Cov(W, Z)$
X	$Cov(V, X)$	$Cov(W, X)$	$Var(X)$	$Cov(X, Y)$	$Cov(X, Z)$
Y	$Cov(V, Y)$	$Cov(W, Y)$	$Cov(X, Y)$	$Var(Y)$	$Cov(Y, Z)$
Z	$Cov(V, Z)$	$Cov(W, Z)$	$Cov(X, Z)$	$Cov(Y, Z)$	$Var(Z)$

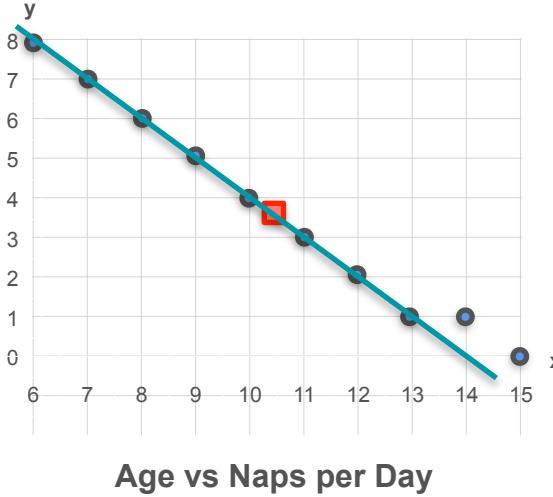


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Probability Distributions with Multiple Variables

Correlation Coefficient

Correlation Coefficient



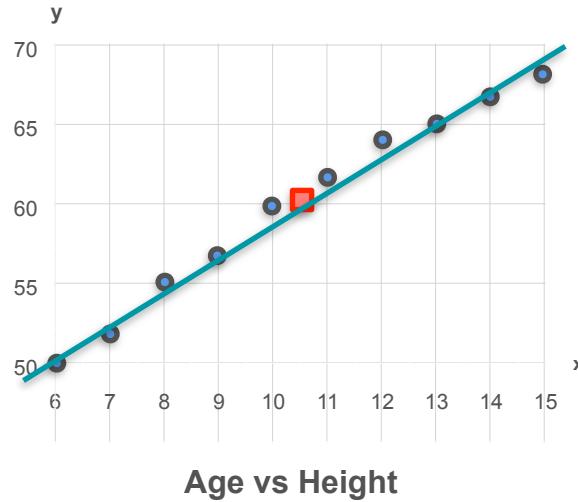
Age vs Naps per Day

$$Var(X) = 9.17$$

$$Var(Y) = 7.57$$

$$Cov(X, Y) = -7.45$$

Is the correlation here strong?



Age vs Height

$$Var(X) = 9.17$$

$$Var(Y) = 39.56$$

$$Cov(X, Y) = 17$$

Correlation Coefficient

Age vs Naps per Day

$$Var(X) = 9.17$$

$$Var(Y) = 7.57$$

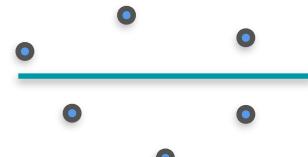
$$Cov(X, Y) = -7.45$$

-1



Correlation Coefficient

0



Age vs Height

$$Var(X) = 9.17$$

$$Var(Y) = 39.56$$

$$Cov(X, Y) = 17$$

1



Correlation Coefficient

Age vs Naps per Day

$$Var(X) = 9.17$$

$$Var(Y) = 7.57$$

$$Cov(X, Y) = -7.45$$

-1

0

1

Age vs Height

$$Var(X) = 9.17$$

$$Var(Y) = 39.56$$

$$Cov(X, Y) = 17$$

$$\text{Correlation Coefficient} = \frac{Cov(X, Y)}{\sigma_x \cdot \sigma_y} = \frac{Cov(X, Y)}{\sqrt{Var(X)} \cdot \sqrt{Var(Y)}}$$

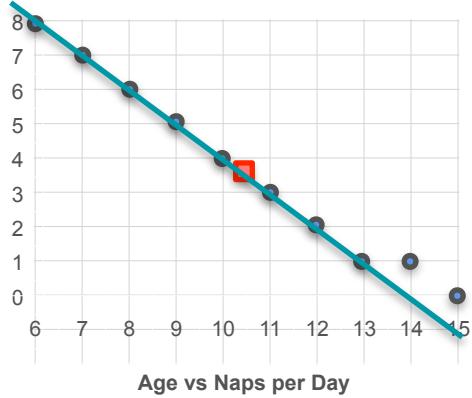
Correlation Coefficient

Age vs Naps per Day

$$Var(X) = 9.17$$

$$Var(Y) = 7.57$$

$$Cov(X, Y) = -7.45$$



$$\text{Correlation Coefficient} = \frac{Cov(X, Y)}{\sqrt{Var(X)} \cdot \sqrt{Var(Y)}}$$

$$= \frac{-7.45}{\sqrt{9.17} \cdot \sqrt{7.57}}$$

$$\approx -0.894$$

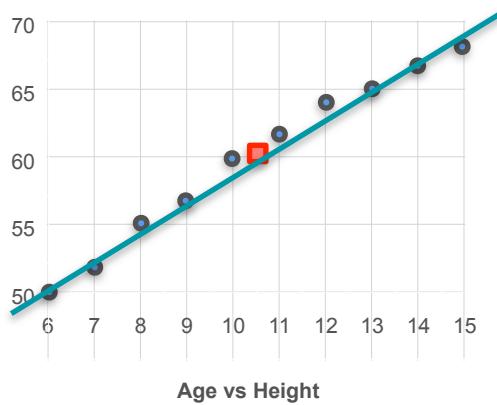
Correlation Coefficient

Age vs Height

$$Var(X) = 9.17$$

$$Var(Y) = 39.56$$

$$Cov(X, Y) = 17$$



Correlation
Coefficient

$$= \frac{Cov(X, Y)}{\sqrt{Var(X)} \cdot \sqrt{Var(Y)}}$$

$$= \frac{17}{\sqrt{9.17} \cdot \sqrt{39.56}}$$

≈ 0.893

Correlation Coefficient

Age vs Naps per Day

$$Var(X) = 9.17$$

$$Var(Y) = 7.57$$

$$Cov(X, Y) = -7.45$$

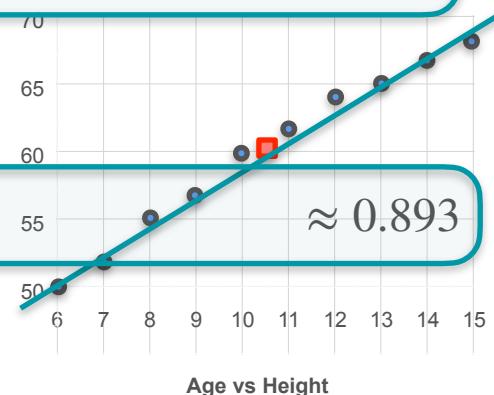


Age vs Height

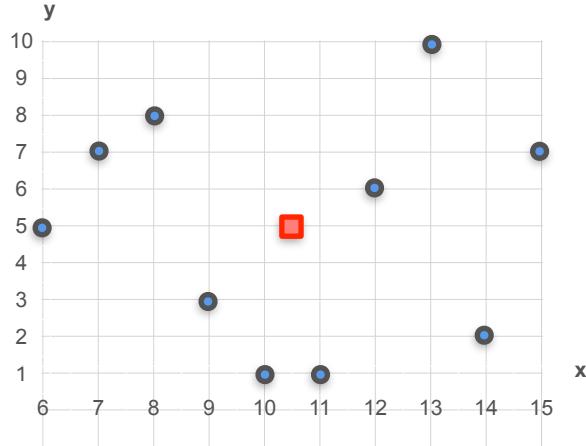
$$Var(X) = 9.17$$

$$Var(Y) = 39.56$$

$$Cov(X, Y) = 17$$



Correlation Coefficient



Age vs Grades

$$Var(X) = 9.17$$

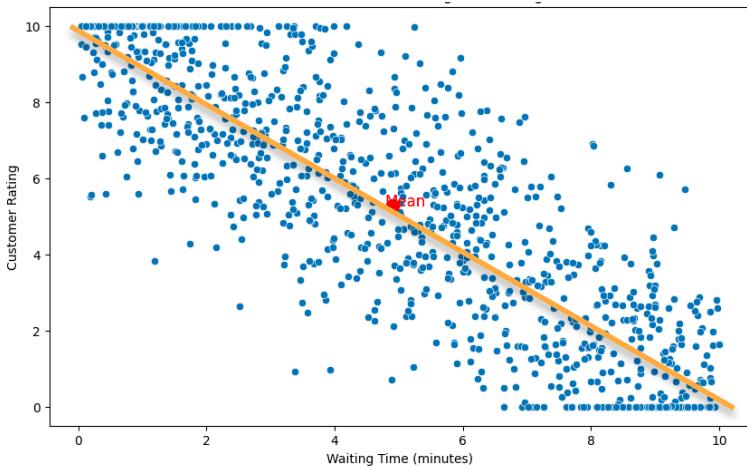
$$Var(Y) = 9.78$$

$$Cov(X, Y) = 0.1$$

$$\begin{aligned}\text{Correlation Coefficient} &= \frac{Cov(X, Y)}{\sqrt{Var(X)} \cdot \sqrt{Var(Y)}} \\ &= \frac{0.1}{\sqrt{9.17} \cdot \sqrt{9.78}}\end{aligned}$$

$$\approx 0.01$$

Correlation Coefficient



$$Var(X) = 8.526$$

$$Var(Y) = 10.163$$

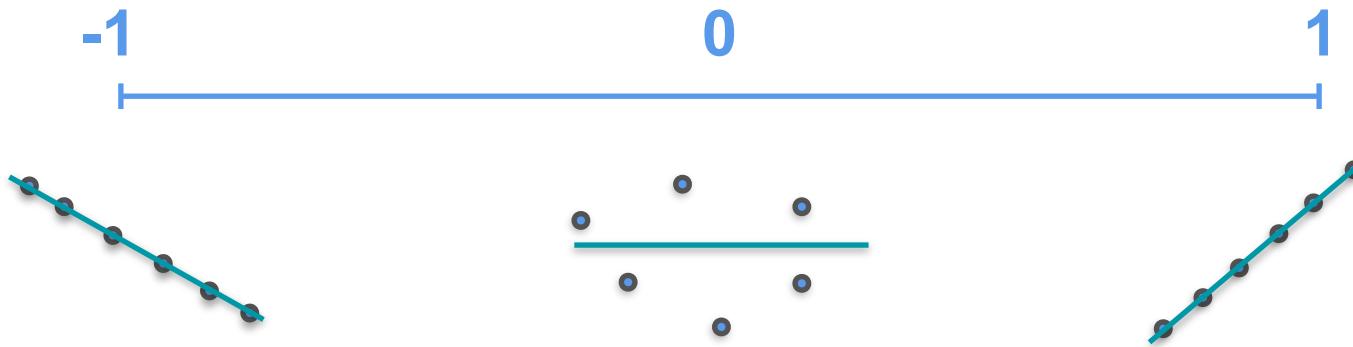
$$Cov(X, Y) = -7.878$$

$$\begin{aligned}\text{Correlation Coefficient} &= \frac{Cov(X, Y)}{\sqrt{Var(X)} \cdot \sqrt{Var(Y)}} \\ &= \frac{-7.878}{\sqrt{8.562} \cdot \sqrt{10.163}}\end{aligned}$$

≈ -0.845

Correlation Coefficient

Correlation Coefficient = $\frac{Cov(X, Y)}{\sigma_x \cdot \sigma_y}$ = $\frac{Cov(X, Y)}{\sqrt{Var(X)} \cdot \sqrt{Var(Y)}}$





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Probability Distributions with Multiple Variables

Multivariate Gaussian Distribution

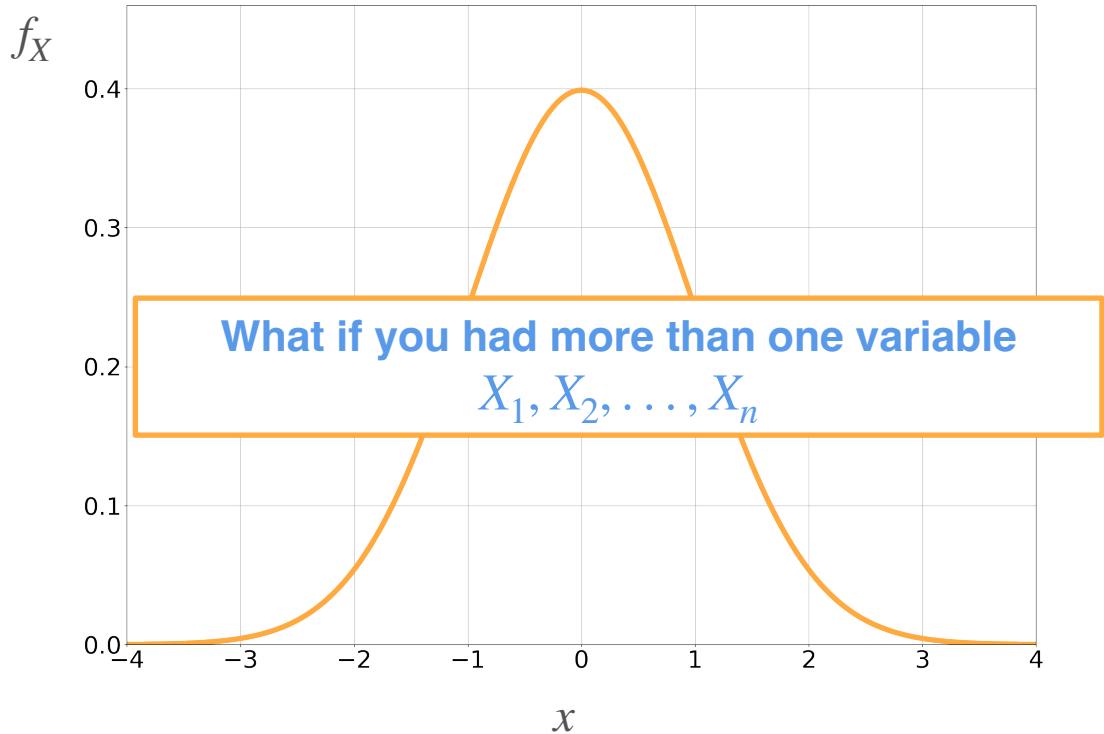
Multivariate Gaussian Distribution

For a single variable, X

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

Parameters:

- μ : center of the bell
- σ : spread of the bell



Multivariate Gaussian Distribution: an Example

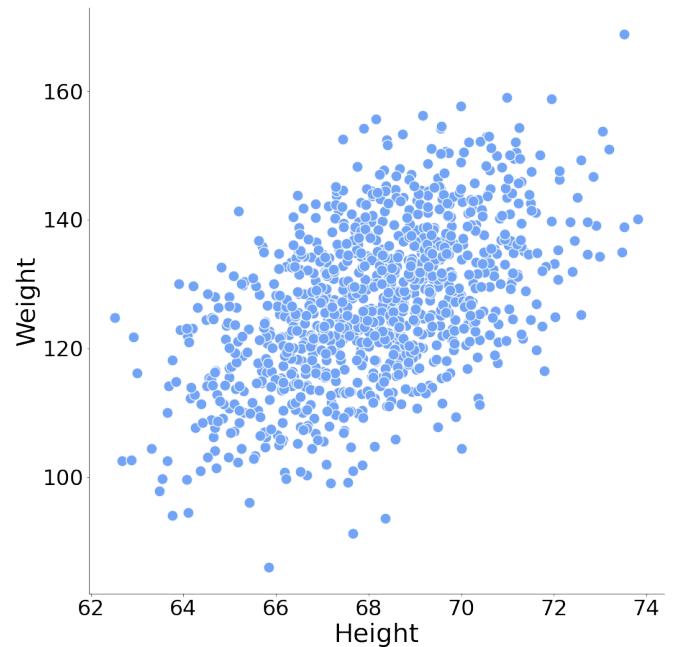
Two variables

H : Height of an adult in inches

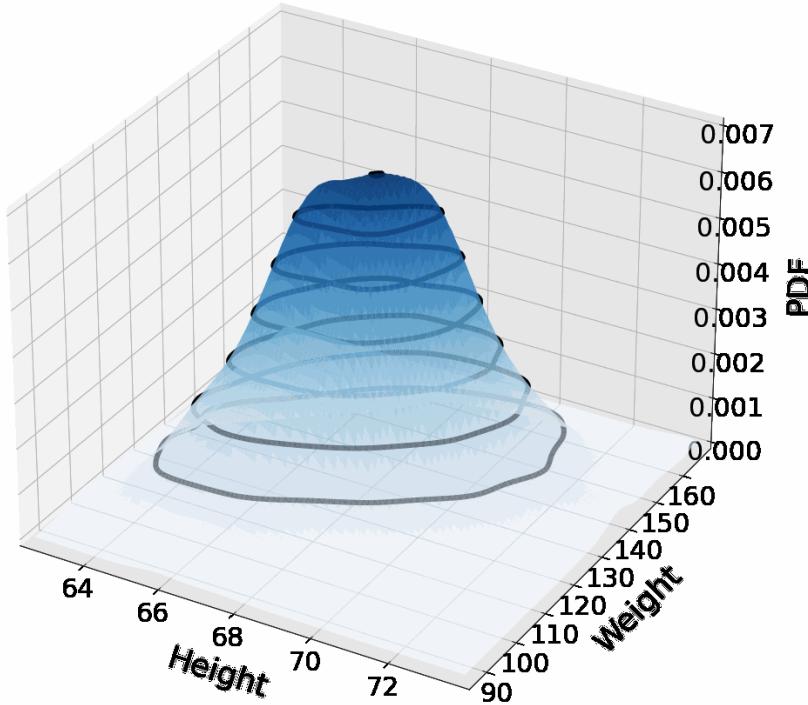
$$H \sim \mathcal{N}(\mu_H, \sigma_H)$$

W : Weight of an adult in pounds

$$W \sim \mathcal{N}(\mu_W, \sigma_W)$$



Multivariate Gaussian Distribution: an Example



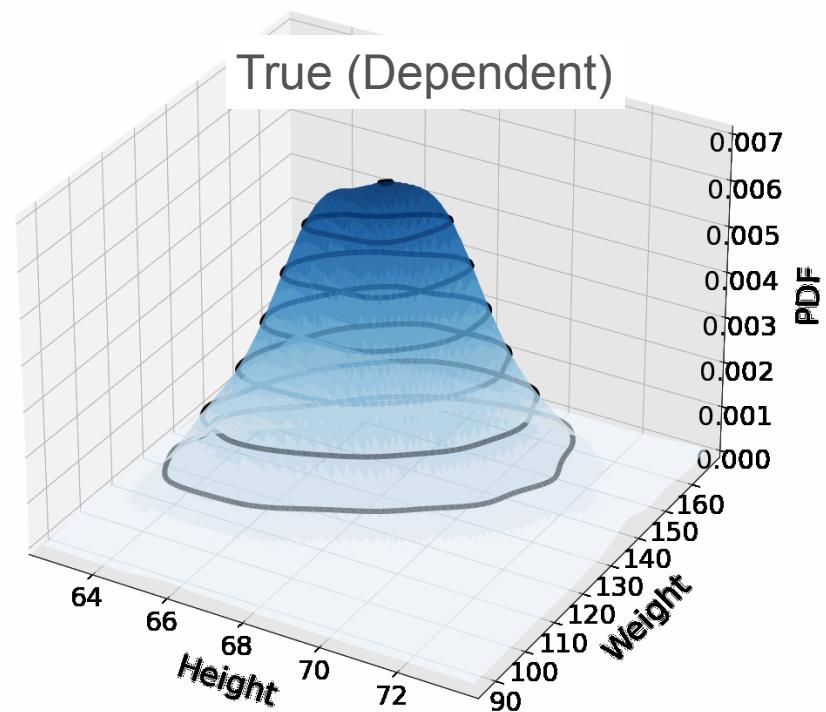
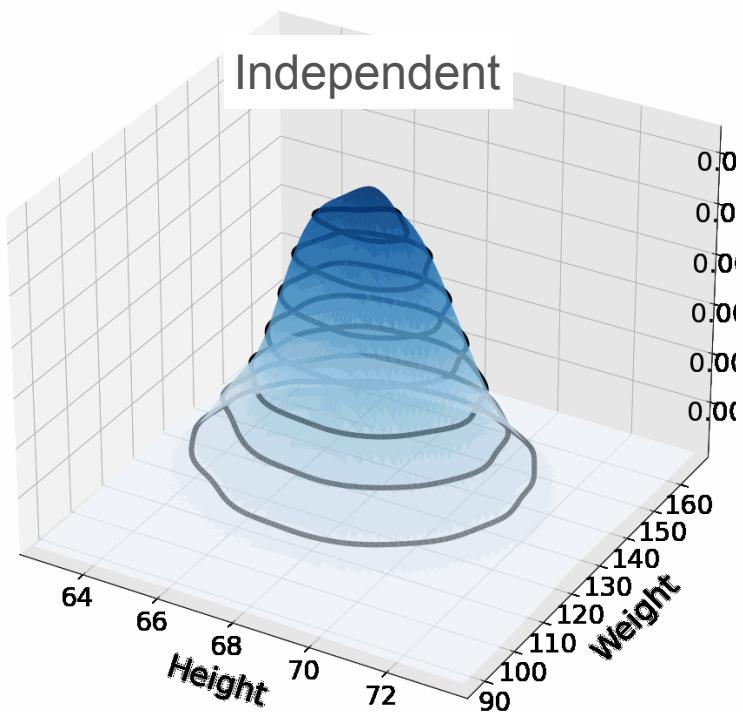
If W, H were independent

$$f_{HW}(h, w) = f_H(h)f_W(w)$$

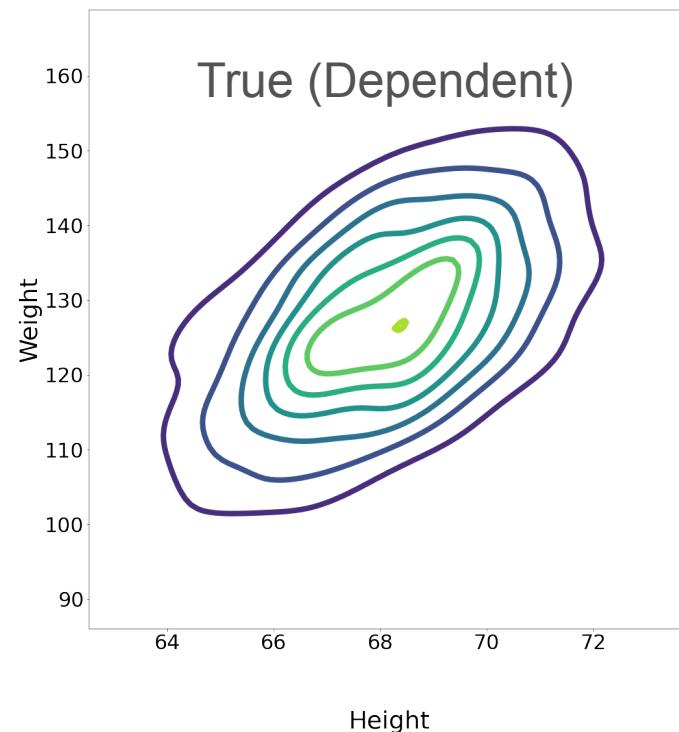
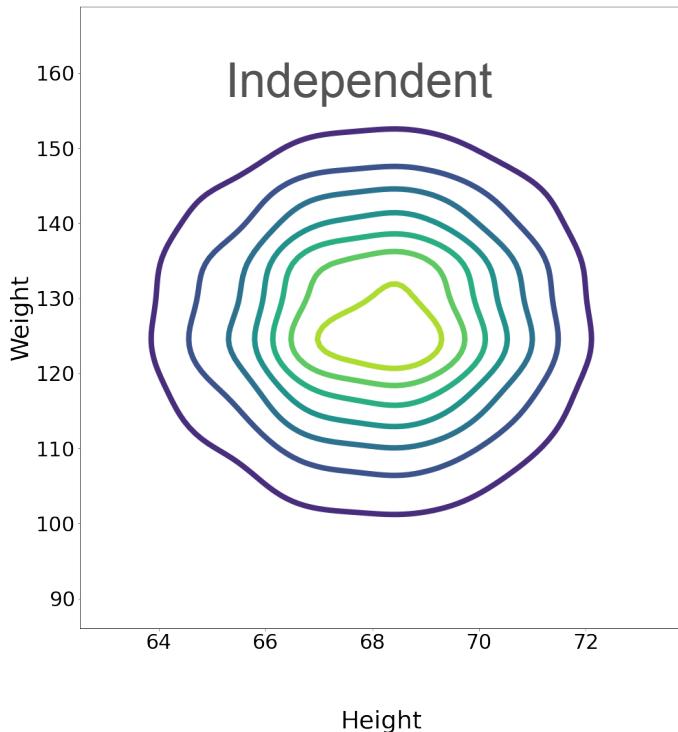
$$= \frac{1}{\sqrt{2\pi}\sigma_H} e^{-\frac{1}{2}\frac{(h-\mu_H)^2}{\sigma_H^2}} \cdot \frac{1}{\sqrt{2\pi}\sigma_W} e^{-\frac{1}{2}\frac{(w-\mu_W)^2}{\sigma_W^2}}$$

$$= \frac{1}{2\pi\sigma_H\sigma_W} e^{-\frac{1}{2}\left(\frac{(h-\mu_H)^2}{\sigma_H^2} + \frac{(w-\mu_W)^2}{\sigma_W^2}\right)}$$

Multivariate Gaussian Distribution: an Example



Multivariate Gaussian Distribution: an Example



Multivariate Gaussian Distribution: an Example

If W, H were independent

$$f_{HW}(h, w) = f_H(h)f_W(w)$$
$$= \frac{1}{2\pi\sigma_H\sigma_W} e^{-\frac{1}{2}\left(\frac{(h-\mu_H)^2}{\sigma_H^2} + \frac{(w-\mu_W)^2}{\sigma_W^2}\right)}$$

$$\left\| \begin{bmatrix} \frac{h-\mu_H}{\sigma_H} \\ \frac{w-\mu_W}{\sigma_W} \end{bmatrix} \right\|^2 = \begin{bmatrix} h-\mu_H & w-\mu_W \\ \sigma_H & \sigma_W \end{bmatrix} \begin{bmatrix} h-\mu_H \\ w-\mu_W \end{bmatrix}$$

$\left[\begin{array}{c} h - \mu_h \\ w - \mu_w \end{array} \right] = \left[\begin{array}{c} h \\ w \end{array} \right] - \left[\begin{array}{c} \mu_h \\ \mu_w \end{array} \right]$

Multiply by diagonal matrix

Multivariate Gaussian Distribution: an Example

If W, H were independent

$$f_{HW}(h, w) = f_H(h)f_W(w)$$
$$= \frac{1}{2\pi\sigma_H\sigma_W} e^{-\frac{1}{2}\left(\frac{(h-\mu_H)^2}{\sigma_H^2} + \frac{(w-\mu_W)^2}{\sigma_W^2}\right)}$$

det (Σ)^{1/2}

$$\left\| \begin{bmatrix} \frac{h-\mu_H}{\sigma_H} \\ \frac{w-\mu_W}{\sigma_W} \end{bmatrix} \right\|_2^2 = \begin{bmatrix} h-\mu_H & w-\mu_W \\ \sigma_H & \sigma_W \end{bmatrix} \begin{bmatrix} \frac{1}{\sigma_H^2} & 0 \\ 0 & \frac{1}{\sigma_W^2} \end{bmatrix} \begin{bmatrix} h-\mu_H \\ w-\mu_W \\ \sigma_W \end{bmatrix}$$
$$= ([h \ w] - [\mu_H \ \mu_W]) \begin{bmatrix} \frac{1}{\sigma_H^2} & 0 \\ 0 & \frac{1}{\sigma_W^2} \end{bmatrix} \left([h \ w] - [\mu_H \ \mu_W] \right)$$

Covariance matrix!
(Σ)

$$= \left([h \ w] - [\mu_H \ \mu_W] \right)^T \begin{bmatrix} \sigma_H^2 & 0 \\ 0 & \sigma_W^2 \end{bmatrix}^{-1} \left([h \ w] - [\mu_H \ \mu_W] \right)$$

μ

Multivariate Gaussian Distribution: an Example

If W, H were independent

$$f_{HW}(h, w) = f_H(h)f_W(w)$$

$$= \frac{1}{2\pi\sigma_H\sigma_W} e^{-\frac{1}{2}\left(\frac{(h-\mu_H)^2}{\sigma_H^2} + \frac{(w-\mu_W)^2}{\sigma_W^2}\right)}$$

$$= \frac{1}{2\pi\sigma_H\sigma_W} \exp\left(-\frac{1}{2} \left(\begin{bmatrix} h \\ w \end{bmatrix} - \begin{bmatrix} \mu_H \\ \mu_W \end{bmatrix} \right)^T \begin{bmatrix} \sigma_H^2 & 0 \\ 0 & \sigma_W^2 \end{bmatrix}^{-1} \left(\begin{bmatrix} h \\ w \end{bmatrix} - \begin{bmatrix} \mu_H \\ \mu_W \end{bmatrix} \right)\right)$$

Multivariate Gaussian Distribution: an Example

If W, H were independent

$$f_{HW}(h, w) = f_H(h)f_W(w)$$

$$= \frac{1}{2\pi\sigma_H\sigma_W} e^{-\frac{1}{2}\left(\frac{(h-\mu_H)^2}{\sigma_H^2} + \frac{(w-\mu_W)^2}{\sigma_W^2}\right)}$$

$$= \frac{1}{2\pi \det \Sigma^{1/2}} \exp \left(-\frac{1}{2} \left(\begin{bmatrix} h \\ w \end{bmatrix} - \boldsymbol{\mu} \right)^T \boldsymbol{\Sigma^{-1}} \left(\begin{bmatrix} h \\ w \end{bmatrix} - \boldsymbol{\mu} \right) \right)$$

Multivariate Gaussian Distribution: an Example

Dependent case:

$$f_{HW}(h, w) = \cancel{f_H(h)f_W(w)}$$
$$= \frac{1}{2\pi\sigma_H\sigma_W} e^{-\frac{1}{2}\left(\frac{(h-\mu_H)^2}{\sigma_H^2} + \frac{(w-\mu_W)^2}{\sigma_W^2}\right)}$$

$$\Sigma = \begin{bmatrix} \sigma_H^2 & Cov(H, W) \\ Cov(H, W) & \sigma_W^2 \end{bmatrix}$$

$$= \frac{1}{2\pi \det \Sigma^{1/2}} \exp \left(-\frac{1}{2} \left(\begin{bmatrix} h \\ w \end{bmatrix} - \boldsymbol{\mu} \right)^T \Sigma^{-1} \left(\begin{bmatrix} h \\ w \end{bmatrix} - \boldsymbol{\mu} \right) \right)$$

Multivariate Gaussian Distribution: General Definition

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

$x = [x_1 \ x_2 \ \dots \ x_n]^T$

$\mu = [\mu_1, \mu_2, \dots, \mu_n]^T$

Mean vector

$f_X(x_1, x_2, \dots, x_n)$

random variables

$X = [X_1 \ X_2 \ \dots \ X_n]$

• For univariate, we work with scalar values and variances

• For multivariate, we work with vectors and the covariance matrix

covariance matrix / spread of the bell

$|\Sigma|$ determinant of the covariance matrix



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Probability Distributions with Multiple Variables

Conclusion