survey of algebra hw 1

nxr8dq: Sankalpa Banjade

September 2023

Collaborators Michael Walker, azd9vv

1 Beginning

1.1.3 Deduce from (A0)–(A4) that additive inverses are unique. In other words, given x R, that there is a *unique* element x in R such that x + (x) = 0 = (x) + x.

Proof 1.1.3

First, let us assume that there exists an element in R, -x, such that $x + (-x) = 0$. So, by this, we must prove that for any additive inverse in R, namely in this case, we'll call x' of x, then $x' = -x$.

1. ($x + (-x) = 0$) and ($x + x' = 0$) through the definition of an additive inverse since by definition of an additive inverse, it is the inverse of a number a such that when added to said number a, its sum is 0 (Axiom A4).

2. We will use the zero property: ($-x + 0 = -x$), as any number a when added with 0 results in the number a (Axiom A3).

3. We will then substitute zero from the two separate equations we made earlier: ($-x + (x + x') = -x$), Just Substitution, no need for an axiom.

4. Then we utilize the associativity property such that ($(-x + x) + x' = -x$) (Axiom A2).

5. Finally, with the definition of zero, we have ($0 + x' = -x$), which boils down to ($x' = -x$), proving our statement that for every additive inverse of a number x, namely x', it must be unique by only being equal to -x (Axiom A3, definition of additive identity).

Proof 1.4.1:

Let R be a ring. Prove the following identity for (a, b, c, d in R):

$(a + b) + (c + d) = (a + c) + (b + d)$

We must utilize associativity and commutativity to change the respective grouping of the equation. So initially, we have ($(a + b) + (c + d)$). Utilizing associativity, we can then change the grouping to $(a + (b + (c + d)))$ (Associativity), and then we can change the location of b by moving it to the other side of the parentheses(commutative property of addition), so that it becomes $(a + ((c + d) + b))$ (Associativity). Then, we group a with the summation of c and d to become $((a + (c + d)) + b)$ (Associativity), which can be simplified down to (((a + c) + d) + b), and then once again regrouped to ((a + c) + (b + d)), hence proving our original statement only with associativity.

Proof 2:

Prove the identity: (-(a + b) = -a - b)

Since R isn't a commutative ring, we are limited in our options for the axioms to use. In this case, we begin with ((a + b) + (-(a + b)) = 0) (additive inverse of ((a + b))). Now, we drop the first set of parentheses and add the respective inverses of a and b, so (a + (-a) + b + (-(a + b)) = -a). Simplifying this, we get (0 + b + (-(a + b)) = -a). Further simplifying, we have (-b + b + (-(a + b)) = -a-b).

Now, using the definition of additive inverse, we get (0 + (-(-a + b)) = -a - b) (additive inverse property). Continuing, we get -(a + b) = -a - b (definition of additive inverse). Thus we prove our original statement

Proof 3:

Prove the identity: (-(-a) = a)

Definition of additive inverse for a number a in the set of reals:

(x + (-x) = 0)

Let (x = a) in one equation and let (x = -a) in the other equation. Our two equations become:

(a + (-a) = 0) and (-a + (-(-a)) = 0). We then set these two equations equal to each other as they both equal 0, so (a + (-a) = -a + (-(-a))).

We then add the additive inverse of -a to both sides, which in this case is (-(-a)). So, (a + (-a) + (-(-a)) = -a + (-(-a)) + (-a)), which simplifies down to (0 + a = 0 + (-(-a))) due to the definition of the additive inverse. This turns into (a = -(-a)) due to the 0 property of addition.

Proof 4. a(b) = (ab). (Warning: we have not assumed that R is commutative). We assume that a(-b) is the additive inverse of a(b) due to the fact that a(-b) + ab = a(-b+b) = a(0) = 0 from there we since we've already proved that additive inverses are unique the additive inverse of ab which is also -(ab) must be equal to a(-b) .

1.4.2 Let R be a commutative ring. Prove that (a+b)(a-b)=$a^2 - b^2$

We first start off with Axiom D2 utilising the fact that we can assume (a-b) to be a "number" and write it like (a+b)*c which when distributed out will equate to ac+bc = (a+b)(a-b), we then sub back in the value of c (a-b) into the equation which turns this into, a(a-b)+b(a-b)=(a+b)(a-b), once again utilising Axiom D2 of distribution we can simplify the right hand side to be a^2-ab+ba-b^2=(a+b)(a-b), due to the law of additive inverses we know that ab+-ba equates to 0 so we will simplify that. so now we have a^2-b^2=(a+b)(a-b) proving our original statement. In a non commutative ring such as M_2(R) we can apply axioms D1-D2, however since this is a non-commutative ring we must not change the direction of the multiplication. From the previous proof when applying this we have a^2+ba+-ab-b^2, however we can't continue after this point due to the fact that the additive inverse in this non-commutative ring of ba is not necessarily -ab since the order of multiplication matters. A specific example of commutativity failing would be for example take two matrices a and b. Let a = $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ and let b = $\begin{bmatrix} 2 & 2 \\ 2 & 8 \end{bmatrix}$ we then have that ab = $\begin{bmatrix} 6 & 18 \\ 4 & 10 \end{bmatrix}$ However ba = $\begin{bmatrix} 4 & 6 \\ 10 & 12 \end{bmatrix}$ With these two matrices, and by attempting to show the commutative property, we demonstrate that M_2R is a proper counterexample, proving that not all non commutative rings adhere to the original statement.

2.1.3 Begin with a truth table, assuming 0 is the definition of 0 as in the additive identity ie 0 + a = a

A	B	C	$A+B$	$B+A$	$B+C$	$(A+B)+C$	$A(B+C)$	0	$A+0$
T	T	T	F	F	F	T	T	F	T
T	T	F	F	F	T	F	F	F	T
T	F	F	T	T	F	T	T	F	T
T	F	T	T	T	T	F	F	F	T
F	F	F	F	F	F	F	F	F	F
F	F	T	F	F	T	T	T	F	F
F	T	T	T	T	F	F	F	F	F
F	T	F	T	T	T	T	T	F	F

Axiom A0). in the case $A, B \subseteq R$ their symmetric difference is like the "xor" operator in binary would be written as $A + B \subseteq R$. The definition of symmetric difference is $(A/B) \cup (B/A)$. If we wanted to show that A and B are elements of R then the sum of A and B would be an element of R. This would be written with out symmetric difference definition as $(A/B) \cup (B/A)$. $\subseteq R$. If A,B are elements of R then $(A/B)\cup$ (B/A)$\subseteq R$ would be true as well, since (A/B) or (B/A) has to be in A or B. If $(A/B), (B/A) \subseteq R$ then $(A/B)\cup$ (B/A)$\subseteq R$ must be in R as well due to the face each side of the union must also reside with the other elements of the union. Due to this $(A/B) \cup (B/A)$ = A + B $\subseteq R$, which satisfies Axiom 0.

Axiom A1. Since A + B and B + A have equivalent columns commutativity holds true.

Axiom A2. Since the columns for (A+B) + C and A +(B+C) are equivalent then associativity is true.

Axiom A3. We need to find a subset of "0" as a subset of R such that it satisfies the equation a + "0" = a for any a $\subseteq x$. With this we must assume that "0" must be the empty set which is 0 because in order for a + "0" = a then in the truth table 0 must be false entirely which is only true when "0" = 0. Also since commutativity is already verified than 0 + a = a is also proved, Also proving the existence of the additive identity of 0.

Axiom A4. A + (-A) = 0 is only true when A = |-A| and when based on the table it confirms the existence of additive identities.

A	$-A$	$A+(-A)=0$
T	T	F
F	F	F

Axiom D1. For A,B,C in R $\implies A * (B + C) = A * B + A * C$

A	B	C	$B+C$	$A*(B+C)$	$A*B$	$A*C$	$(A*B)+(B*C)$
T	T	T	F	F	T	T	F
T	T	F	T	T	T	F	T
T	F	F	F	F	F	F	F
T	F	T	T	T	F	T	T
F	F	F	F	F	F	F	F
F	F	T	T	F	F	F	F
F	T	T	F	F	F	F	F
F	T	F	T	F	F	F	F

Since the columns for A *(B+C) and (A*B) + (B*C) are the same, so Axiom D1 is proven true.

Axiom D2. A,B,C in R $\implies$ $(A+B)*C = (A*C)+(B*C)$

C	$A+B$	$A*C$	$B*C$	$(A+B)*C$	$(A*C)+(B*C)$
T	F	T	T	F	F
F	F	F	F	F	F
F	T	F	F	F	F
T	T	T	F	T	T
F	F	F	F	F	F
T	F	F	F	F	F
T	T	F	T	T	T
F	T	F	F	F	F

The columns for (A+B) * C and (A*C) + (B*C) are equivalent so Axiom D2 is proven true.

2.3.2 We proceed by contradiction such that R is not the zero ring. Which is implying that there exists an x that is fixed in r such that x does not equal 0. Utilizing the multiplicative identity we come up with these two statements.

x * 0 = 0 due to the zero property Now using that 0 = 1 in this specific ring we replace 0 with 1 in the above statement. x * 1 = 0. In order for both of these statements to be true x must be equal to 0 which goes against our assumption, which then proves that there can't exist any other value in R other than 0. This applies to all x as we assumed for any x in R.

2.3.4 We first assume that a is not a duplicate in this set ie that a does not equal 1 or 0, As we have already proven the case which it is equal to 1 or 0. Using similar thinking process from the last problem, the multiplication table would look alike in the sense that the first column and row in the table will all equal 0 since for all C in R any c multiplied with 0 yields 0. Now we have three different cases of a being multiplied by itself and what it can equal. a * a = 0, a * a = a and a * a = 1. In the first cause a * a cant equal 0, because a can't equal 0 so there is no way possible for a 8 a to equal 0. In our next case when a * a = a, this also can't be true because in this case a must equal 1 which would be going agains tour assumption. So that only leaves that a * a = 1, which will be our final conclusion.

$\cdot$	0	1	a
0	0	0	0
1	0	1	a
a	0	a	1

Table 1: Multiplication Table

Moving on from multiplication to addition we will then explore the additive operations in this table. The first row and column are the same due to the face that 0 is the additive identity. Now for the second column, since 1 + 1 can't equal 1 due to the face that it would go against the additive identity, so then a + a must equal 1 since that is the only case left. 3.1.1

$+$	0	1	a
0	0	1	a
1	1	a	0
a	a	0	1

Table 2: Addition Table

(2) if x < y it can be reasoned that y -x is in the set of reals that are positive ie y-x is a natural number. now using the property that any number minus itself is 0 (additive inverse) in this case z -z = 0, and then by additive identity we have y -x + 0 is in the set of positive reals, and subbing back in what equals 0, which is z-z, that means y -x + z -z is in the set of positive reals. Then utilizing the associative property of addition since this is an ordered ring we have y + z + -x - z is in the set of reals. By back tracing we have that this quantity y + z + -x - z must be greater than 0, and then by utilizing the additive inverse property to get the terms on the other side we have x + z < y + z. Proving our original statement

(4) Since x < y ywe have that y -x is in the set of negative reals. and since we know that z is also less than 0 than z must be in the set of the negative reals as well. In addition whenever we multiply two numbers in the set of the negative reals we will get a number in the set of the positive reals, thus we multiply the quantity (y-x) * z to get a number in the set of reals that are positive. Then utilizing Axiom D2, we have yz - xz is in the set of negative reals so that must mean that xz is greater than yz if the difference of yz and xz is in the set of negative reals which proves our statement. xz > yz.

3.1.7 Since we know that ab < ac, ac - ab lies in the set of the positive reals, we have Axiom D1 stating that a(c-b) must also lie in the set of the positive reals, by using another variable in this case d we will let d = c - b, and since we already know that a is in the set of the positive reals (Given) and that a * d must also lie in the set of the positive reals then d must also lie in the set of the positive reals as if d was in the set of negative reals then the result of a positive and a negative through multiplication would be an element in the set of the negative reals. So now that we know that d is in the set of positive reals we have c - b $>$ 0 utilizing additive inverse we can conclude that b $<$ c. Proving our original statement.