Survey HW Week of thanksgiving break

nxr8dq

November 2023

1 Introduction

1.1 22.5.1, 22.5.4, 22.5.7, 23.4.3, 23.4.6, 23.4.9, 23.4.10.

22.5.1 Question 1
(1 3 7 2)(4 6)(5)
Order of 4 2 1 respectively, so $\text{lcm}(4, 2, 1)$ is 4. Therefore, the order of the cyclic group is 4.

Question 2 For g in S_6 the cycle decomposiiton would be (1,2,3,4,5,6) so the order of the group would be 6, as you would need to apply g 6 times in order to get to the starting point.

22.5.4 $S_4 = \left\{ \begin{array}{l} (1), \\ (1,2),(1,3),(1,4),(2,3),(2,4),(3,4), \\ (1,2,3),(1,3,2),(1,2,4),(1,4,2),(1,3,4),(1,4,3),(2,3,4),(2,4,3), \\ (1,2,3,4),(1,2,4,3),(1,3,2,4),(1,3,4,2),(1,4,2,3),(1,4,3,2), \\ (1,2)(3,4),(1,3)(2,4),(1,4)(2,3) \end{array} \right\}$

22.5.7 The possible orders of the product of two transpositions $f = (i\ j)$ and $g = (k\ l)$ in S_n are as follows:

1. If f and g are disjoint, then the order of fg is 2.
2. If f and g have one element in common, then the order of fg is 3.
3. If f and g are the same transposition, then the order of fg is 1.

Therefore, the possible orders of the product fg are 1, 2, or 3.

23.4.3 Proof: Let $C = \langle g \rangle$ be a cyclic group generated by g. By definition, every element $c \in C$ can be expressed as g^n for some integer n. Given a group homomorphism $\phi : C \to G$, it preserves the group operation, which implies that $\phi(g^n) = \phi(g)^n$ for all integers n.

Now, consider the image of C under ϕ, denoted by $\phi(C)$. For every element $\phi(c) \in \phi(C)$, there exists an integer n such that $c = g^n$, and consequently, $\phi(c) = \phi(g^n) = \phi(g)^n$.

Hence, every element in $\phi(C)$ can be written as $\phi(g)^n$, which means $\phi(C)$ is generated by $\phi(g)$. Therefore, $\phi(C)$ is cyclic with $\phi(g)$ as its generator.

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23.4.6 Let $n \in Z^+$, $m \in Z$, and define $\phi_m : Z/(n) \to Z/(n)$ by $\phi_m([x]) = m[x]$.

1. To prove that ϕ_m is a homomorphism, we show that for all $x, y \in Z$,

$$\phi_m([x] + [y]) = \phi_m([x]) + \phi_m([y]).$$

2. For any homomorphism $\psi : Z/(n) \to Z/(n)$, ψ is determined by its action on the generator [1], hence $\psi = \phi_m$ for some m.

3. The order of $\mathrm{Ker}(\phi_m)$ is the number of solutions to $mx \equiv 0 \pmod{n}$, which is $\gcd(m, n)$.

4. The order of $\phi_m(Z/(n))$ is $n/\gcd(m, n)$, since it is the index of the kernel in $Z/(n)$.

5. ϕ_m is an isomorphism if and only if $\gcd(m, n) = 1$, as this ensures that ϕ_m is injective (trivial kernel) and surjective.

23.4.9

Let G and H be finite groups, and suppose that the orders of G and H are coprime. We aim to prove that the trivial homomorphism is the unique homomorphism from G to H.

Proof: Let $\phi : G \to H$ be a homomorphism. For any element $g \in G$, the order of g, denoted $|g|$, is the smallest positive integer m such that $g^m = e_G$, with e_G being the identity of G. Since ϕ is a homomorphism, we have $\phi(g^m) = \phi(g)^m = e_H$, where e_H is the identity of H.

However, the orders of G and H are coprime, so there are no shared nontrivial divisors between them. This implies that the only element in H that has an order dividing the order of g is e_H itself. Consequently, for $\phi(g)$ to have an order that divides the order of g, we must have $\phi(g) = e_H$.

Therefore, ϕ must send every element of G to e_H, which means that ϕ is the trivial homomorphism. Since any nontrivial homomorphism would contradict the coprimality of the orders of G and H, the trivial homomorphism is indeed the unique homomorphism from G to H.

() **23.4.10** Determine $\mathrm{Hom}(S_3, Z/(3))$.

Solution:

We consider homomorphisms $\phi : S_3 \to Z/(3)$. Since $Z/(3)$ is abelian, any such homomorphism factors through the abelianization of S_3, which is isomorphic to $Z/(2)$ because S_3 has a subgroup of index 2, namely the alternating group A_3.

The groups $Z/(2)$ and $Z/(3)$ have no nontrivial common quotients, implying that the only homomorphism from S_3 to $Z/(3)$ is the trivial one. Therefore, $\mathrm{Hom}(S_3, Z/(3))$ consists solely of the trivial homomorphism.

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