Colegio Canadiense

PRE-CALCULUS 12

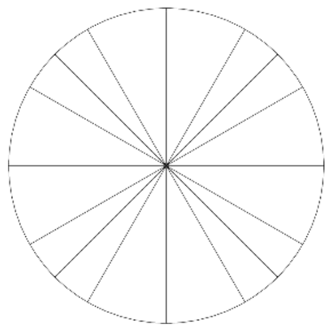
CHAPTER 5: TRIGONOMETRY II

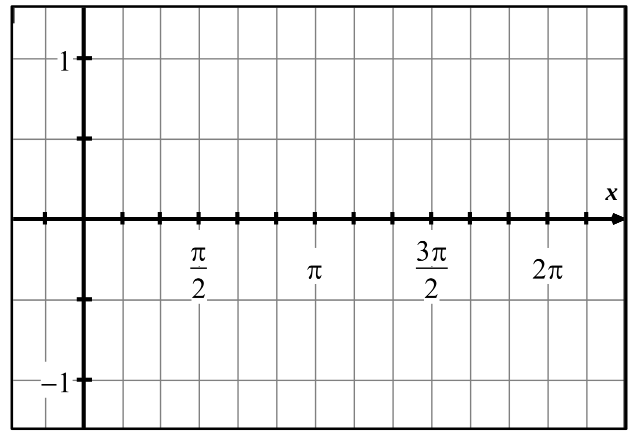
Mr. Best

LESSON 5.1: GRAPHING SINE AND COSINE FUNCTIONS

▶ ***Example 1.1:*** Sketch the graph of the function $y = \sin x$. Use the unit circle to fill in the table of values.

x	y
0°	
30°	
60°	
90°	
120°	
150°	
180°	
210°	
240°	
270°	
300°	
330°	
360°	





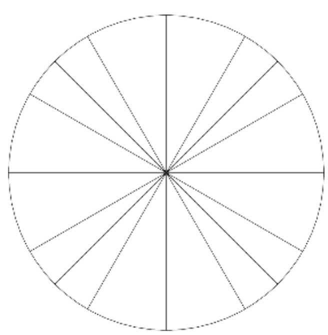
Using a graphing calculator, we can extend the graph of the function to the left of the origin. We observe that $y = \sin x$ is a ________________ function with an ____________________ domain. The above graph shows a portion of the graph on the restricted domain ____________.

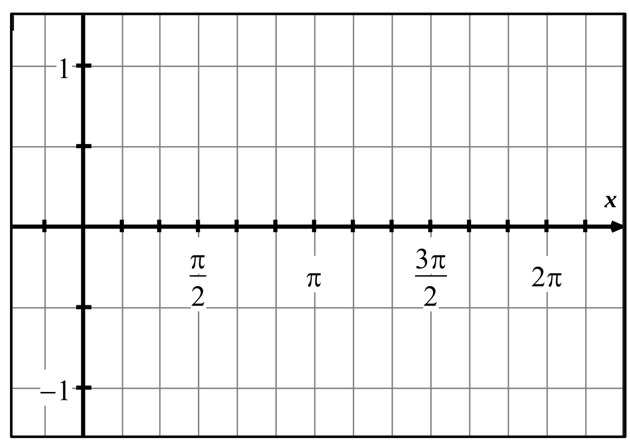
Fill in the following table for the function $y = \sin x$

Domain		***Range***	
Amplitude		***Period***	
x-intercepts		***y-intercept***	

▶ ***Example 1.2:*** Sketch the graph of the function $y = \cos x$. Use the unit circle to fill in the table of values.

x	y
0°	
30°	
60°	
90°	
120°	
150°	
180°	
210°	
240°	
270°	
300°	
330°	
360°	





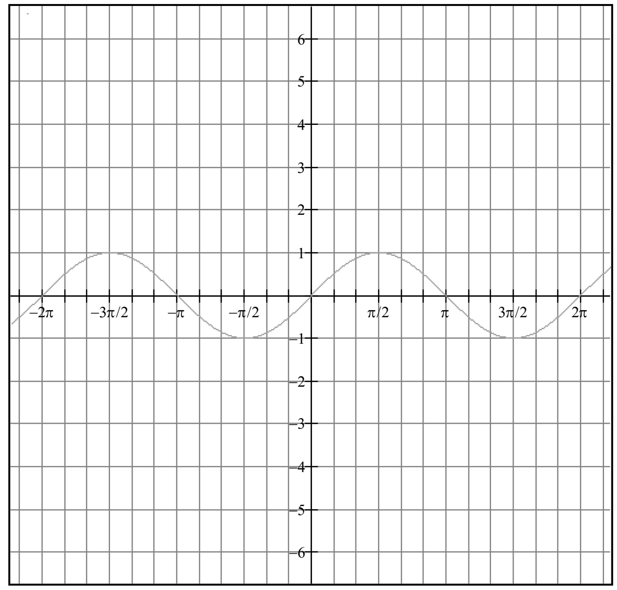
Using a graphing calculator, we can extend the graph of the function to the left of the origin. We observe that $y = \cos x$ is a ________________ function with an ____________________ domain. The above graph shows a portion of the graph on the restricted domain ____________.

Fill in the following table for the function $y = \cos x$

Domain		***Range***	
Amplitude		***Period***	
x-intercepts		***y-intercept***	

▶ ***Example 1.3:*** Sketch the graph of the function $y = \csc x$. Use the mapping $(a,b) \to (a, \quad)$

To graph $y = \csc x$, we simply graph the equivalent function $y = \frac{1}{\sin x}$ which is the graph of the reciprocal of the function $y = \sin x$.



Recall the steps for graphing reciprocal functions.

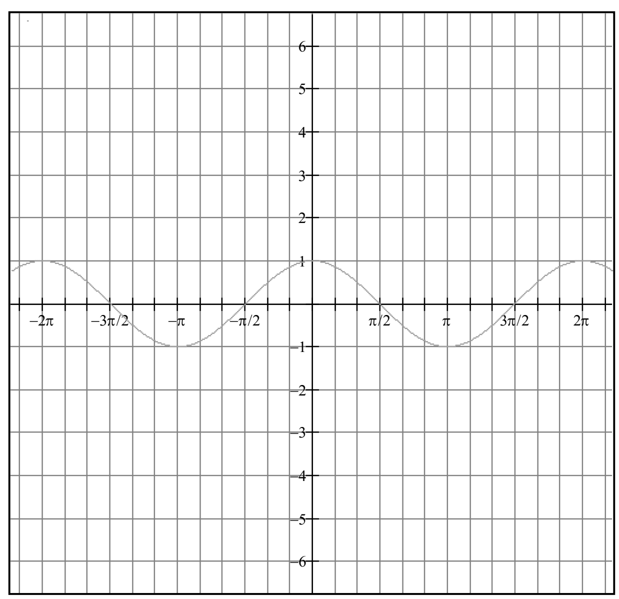
1. Draw vertical asmyptotes through the *x*-intercepts.
2. Plot invariant points whose *y*-coordinates are equal to either 1 or -1.
3. Plot other points by taking the reciprocal of the *y*-coordinate of points on the original graph.
4. Recognize that the graph of the reciprocal function will be positive and increasing (decreasing) whenever the original graph is positve and decreasing (increasing). Likewise, the graph of the reciprocal function will be negative and increasing (decreasing) whenever the original graph is negative and decreasing (increasing).

Fill in the following table for the function $y = \csc x$. NOTE: amplitude is not applicable to $y = \csc x$.

Domain		***Range***	
equations of asymptotes		***Period***	
x-intercept		***y-intercept***	

▶ ***Example 1.4:*** Sketch the graph of the function $y = \sec x$. Use the mapping $(a,b) \to (a, \quad)$

To graph $y = \sec x$, we simply graph the equivalent function $y = \frac{1}{\cos x}$ which is the graph of the reciprocal of the function $y = \cos x$.



Recall the steps for graphing reciprocal functions.

1. Draw vertical asmyptotes through the *x*-intercepts.
2. Plot invariant points whose *y*-coordinates are equal to either 1 or -1.
3. Plot other points by taking the reciprocal of the *y*-coordinate of points on the original graph.
4. Recognize that the graph of the reciprocal function will be positive and increasing (decreasing) whenever the original graph is positve and decreasing (increasing). Likewise, the graph of the reciprocal function will be negative and increasing (decreasing) whenever the original graph is negative and decreasing (increasing).

Fill in the following table for the function $y = \sec x$. NOTE amplitude is not applicable to $y = \sec x$.

Domain		***Range***	
equations of asymptotes		***Period***	
x-intercept		***y-intercept***	

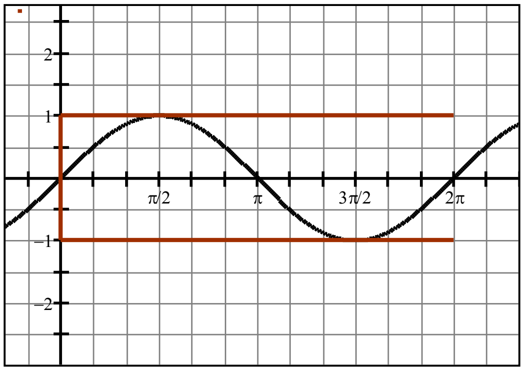
LESSON 5.2: TRANSFORMATION OF TRIGONOMETRIC FUNCTIONS

In this Lesson we will be graphing functions of the form $y = a\sin b(x-c)+d$ and $y = a\cos b(x-c)+d$.

Recall the effect of each parameter on the graph of a function

	Transformation	Description	Effect on Graph
a	Vertical stretch by a factor of a	$y=f(x) \Rightarrow y=af(x)$ If $0<\lvert a\rvert<1$, then ________________ by a factor of a If $\lvert a\rvert>1$, then ________________ by a factor of a $a<0$, then ________________ in the $x-$axis.	Amplitude $\lvert a\rvert$
b	Horizontal stretch by a factor of $\frac{1}{b}$	$y=f(x) \Rightarrow y=f(bx)$ If $0<\lvert b\rvert<1$, then ________________ by a factor of $\frac{1}{\lvert b\rvert}$ If $\lvert b\rvert>1$, then ________________ by a factor of $\frac{1}{\lvert b\rvert}$ $b<0$, then ________________ in the $y-$axis.	Period $2\pi\times\frac{1}{\lvert b\rvert}=\frac{2\pi}{\lvert b\rvert}$
c	Horizontal Translation	$y=f(x) \Rightarrow y=f(x-c)$ If $c>0$, then shift c units to the ________________ If $c<0$, then shift c units to the ________________	Phase Shift c
d	Vertical Translation	$y=f(x) \Rightarrow y=f(x)+d$ If $d>0$, then shift d units ________________ If $d<0$, then shift d units ________________	Vertical Displacement d

The graph of $y = \sin x$

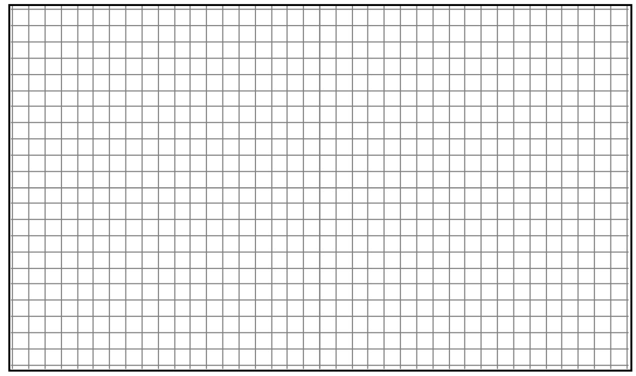


$a =$	$b =$	$c =$	$d =$
amplitude	***period***	***phase shift***	***vertical displacement***

Box Dimensions

Left c	***Right*** $c+P$	***Top*** $d+\lvert a\rvert$	***Bottom*** $d-\lvert a\rvert$

▶ ***Example 2.1*** Graph $y = 3\sin\left(2\left(x - \frac{\pi}{3}\right)\right) + 4$

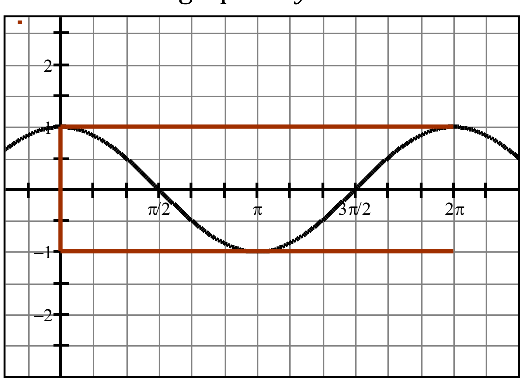


$a =$	$b =$	$c =$	$d =$
amplitude	***period***	***phase shift***	***vertical displacement***

Box Dimensions

Left c	***Right*** $c+P$	***Top*** $d+\lvert a\rvert$	***Bottom*** $d-\lvert a\rvert$

The graph of $y = \cos x$

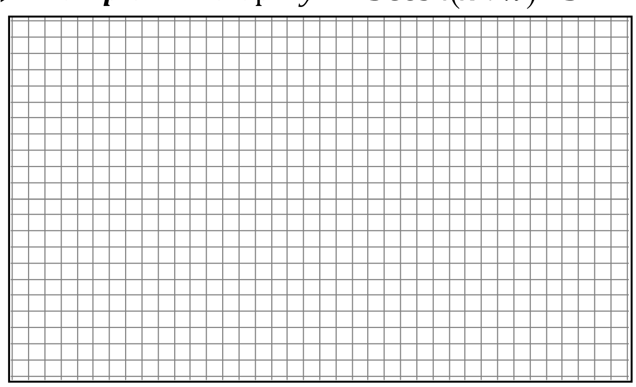


$a =$	$b =$	$c =$	$d =$
amplitude	***period***	***phase shift***	***vertical displacement***

Box Dimensions

Left c	***Right*** $c+P$	***Top*** $d+\lvert a\rvert$	***Bottom*** $d-\lvert a\rvert$

▶ ***Example 2.2*** Graph $y = -5\cos 4(x + \pi) - 3$



$a =$	$b =$	$c =$	$d =$
amplitude	***period***	***phase shift***	***vertical displacement***

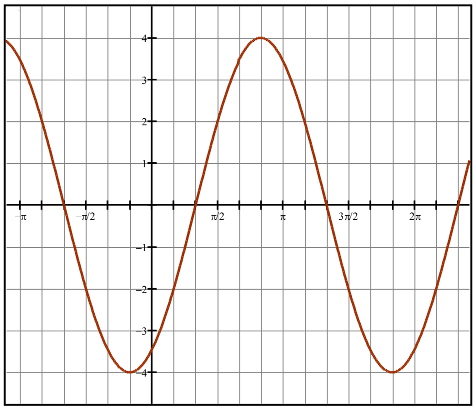
Box Dimensions

Left c	***Right*** $c+P$	***Top*** $d+\lvert a\rvert$	***Bottom*** $d-\lvert a\rvert$

Writing equations of Trigonometric functions

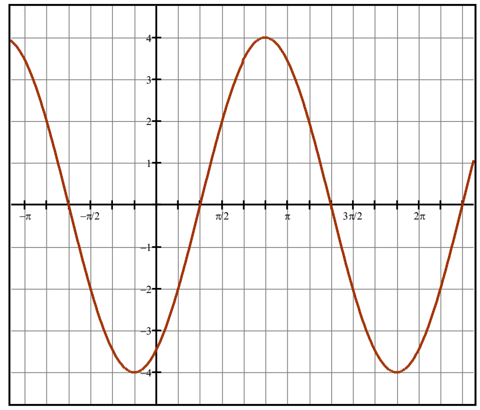
▶ ***Example 2.3:*** The four graphs on this page are identical. Use the following conditions to write four different equations for the same graph. Use the minimum possible phase shift for each equation.

(a) Write the equation of the graph in the form $y = a\sin(x - c)$ where $a > 0$.



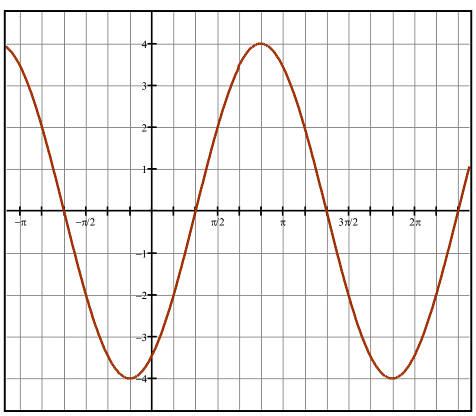
amplitude	***period***	***phase shift***	***vertical displacement***
$a =$	$b =$	$c =$	$d =$
Equation $(a > 0)$			

(b) Write the equation of the graph in the form $y = a\sin(x - c)$ where $a < 0$.



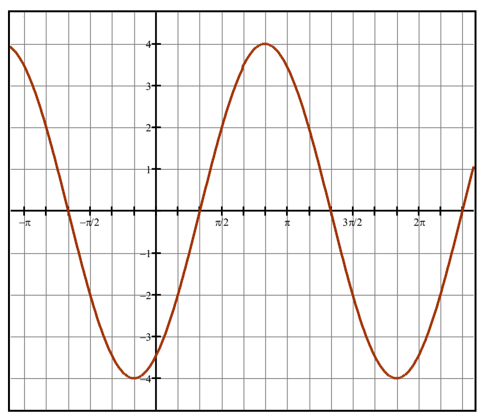
amplitude	***period***	***phase shift***	***vertical displacement***
$a =$	$b =$	$c =$	$d =$
Equation $(a < 0)$			

(c) Write the equation of the graph in the form $y = a\cos(x - c)$ where $a > 0$.



amplitude	***period***	***phase shift***	***vertical displacement***
$a =$	$b =$	$c =$	$d =$
Equation $(a > 0)$			

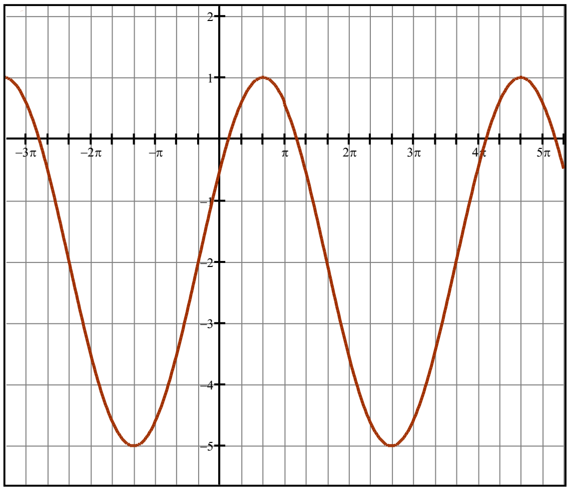
(d) Write the equation of the graph in the form $y = a\cos(x - c)$ where $a < 0$.



amplitude	***period***	***phase shift***	***vertical displacement***
$a =$	$b =$	$c =$	$d =$
Equation $(a < 0)$			

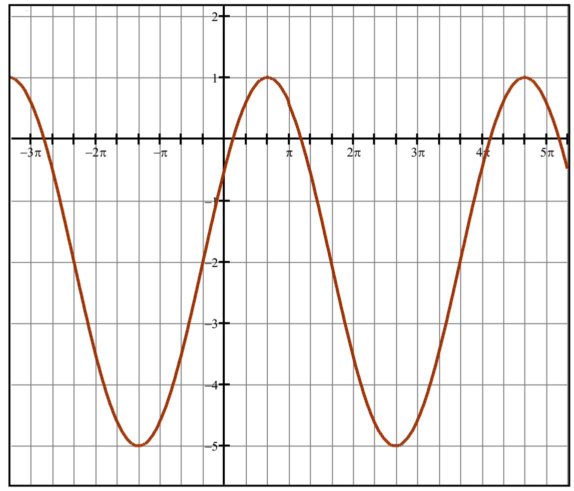
▶ ***Example 2.4:*** The four graphs on this page are identical. Use the following conditions to write four different equations for the same graph. Use the minimum possible phase shift for each equation.

(a) Write an equation for the graph using a ***sine*** function where $a > 0$



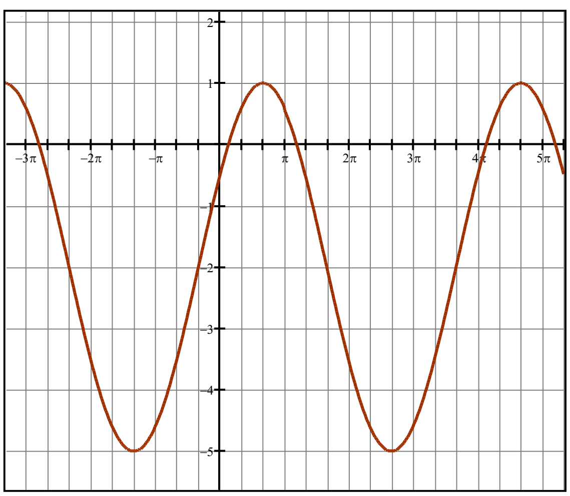
amplitude	***period***	***phase shift***	***vertical displacement***
$a =$	$b =$	$c =$	$d =$
Equation $(a > 0)$			

(b) Write an equation for the graph using a ***sine*** function where $a < 0$



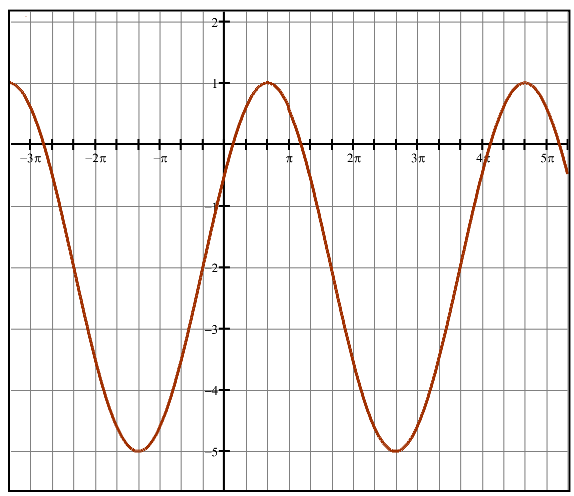
amplitude	***period***	***phase shift***	***vertical displacement***
$a =$	$b =$	$c =$	$d =$
Equation $(a < 0)$			

(c) Write an equation for the graph using a ***cosine*** function where $a > 0$.



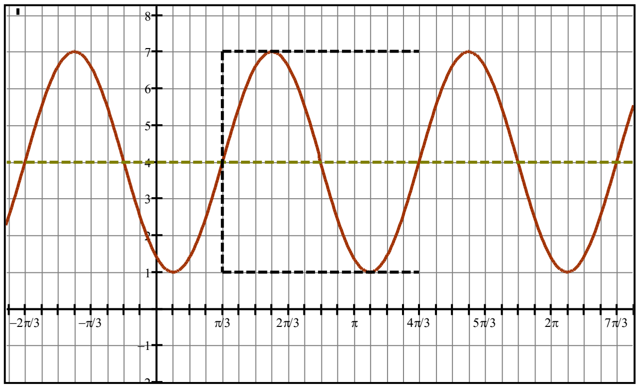
amplitude	***period***	***phase shift***	***vertical displacement***
$a =$	$b =$	$c =$	$d =$
Equation $(a > 0)$			

(d) Write an equation for the graph using a ***cosine*** function where $a < 0$

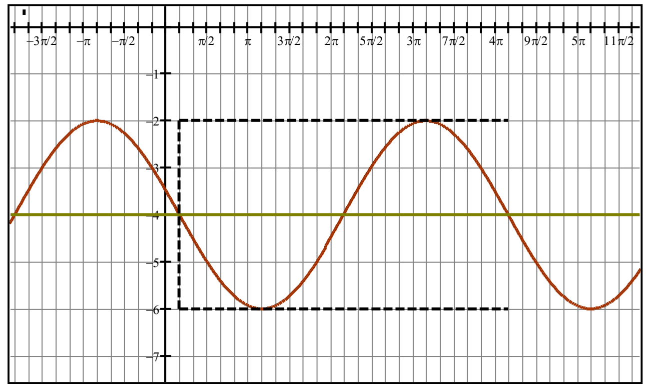


amplitude	***period***	***phase shift***	***vertical displacement***
$a =$	$b =$	$c =$	$d =$
Equation $(a < 0)$			

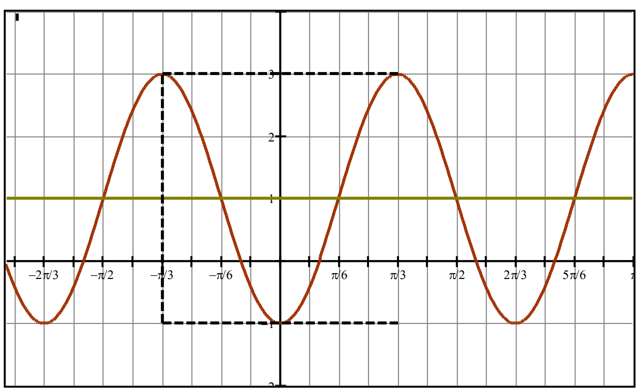
Practice Worksheet: Writing Trigonometric Equations from Graphs



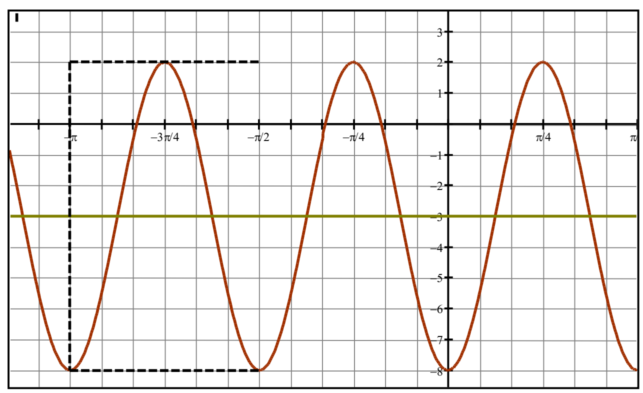
amplitude	***period***	***phase shift***	***vertical displacement***
sine: $a > 0$			



amplitude	***period***	***phase shift***	***vertical displacement***
sine: $a < 0$			



amplitude	***period***	***phase shift***	***vertical displacement***
cosine: $a > 0$			

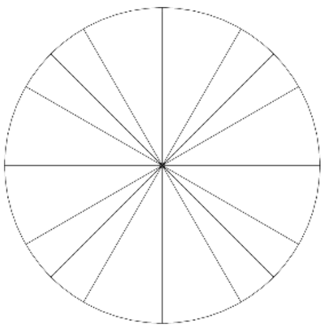


amplitude	***period***	***phase shift***	***vertical displacement***
cosine: $a < 0$			

LESSON 5.3: GRAPHING THE TANGENT FUNCTION

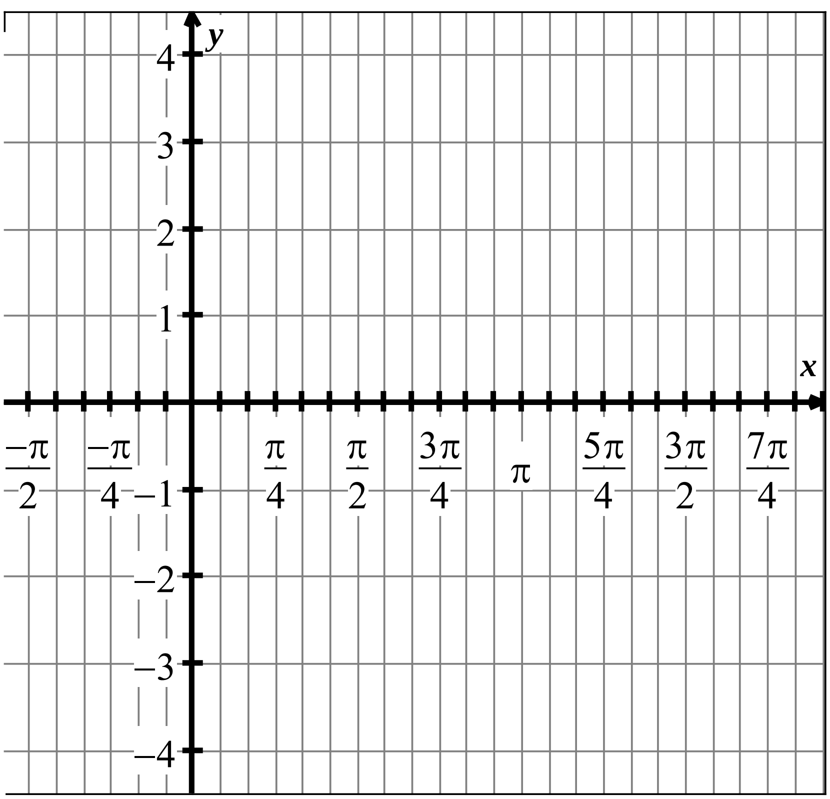
▶ ***Example 3.1:*** Sketch the graph of the function $y = \tan x$. Use a calculator to fill in the table of values.

There are a number of ways to show the connection between the tangent function and the unit circle.



x	y
0°	
15°	
30°	
45°	
60°	
75°	
90°	
105°	
120°	
135°	
150°	
165°	
180°	

Plot the points on the grid provided and using a graphing calculator to guide you, draw a smooth curve through the plotted points. Extend the graph of the function to the left of the origin.



We observe that $y = \tan x$ is a periodic function with vertical asymptotes every odd multiple of $\frac{\pi}{2}$.
NOTE: the concept of amplitude does not apply to $y = \tan x$.

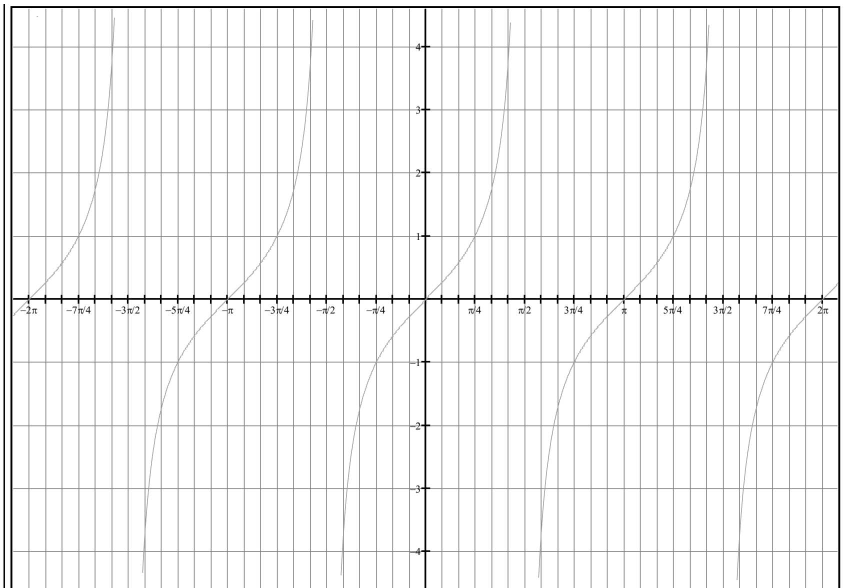
	$y = \tan x$
Domain	
Range	
Equations of asymptotes	
Period	
x-intercept(s)	
y-intercept	

▶ ***Example 3.2:*** Sketch the graph of the function $y = \cot x$.

To graph $y = \cot x$, we simply graph the equivalent function $y =$ ______ which is the graph of the reciprocal of the function $y = \tan x$.

Recall the steps for graphing reciprocal functions.

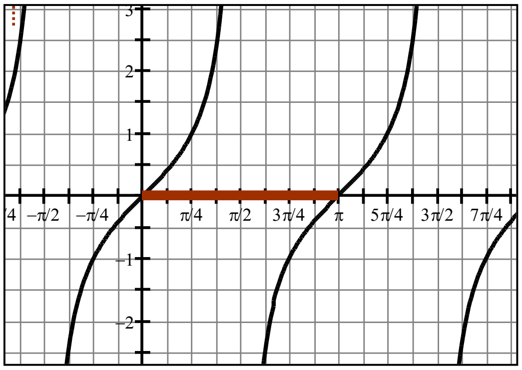
1. Draw vertical asymptotes through the *x*-intercepts.
2. Plot invariant points whose *y*-coordinates are equal to either 1 or -1.
3. Plot other points by taking the reciprocal of the *y*-coordinate of points on the original graph.
4. Recognize that the graph of the reciprocal function will be positive and increasing (decreasing) whenever the original graph is positve and decreasing (increasing). Likewise, the graph of the reciprocal function will be negative and increasing (decreasing) whenever the original graph is negative and decreasing (increasing).



In this Lesson we will be graphing functions of the form $y = a\tan b(x-c)+d$. The only difference here is that the tangent function has a different period. From Example 3.1 we can see that for $y = \tan x$, $P = \pi$.

	Transformation	Description	Effect on Graph
a	Vertical stretch by a factor of a	$y = f(x) \Rightarrow y = af(x)$ If $0 < \lvert a \rvert < 1$, then ______________ by a factor of a If $\lvert a \rvert > 1$, then ______________ by a factor of a $a < 0$, then ______________ in the $x-$axis.	Amplitude does not apply here.
b	Horizontal stretch by a factor of $\frac{1}{b}$	$y = f(x) \Rightarrow y = f(bx)$ If $0 < \lvert b \rvert < 1$, then ______________ by a factor of $\frac{1}{\lvert b \rvert}$ If $\lvert b \rvert > 1$, then ______________ by a factor of $\frac{1}{\lvert b \rvert}$ $b < 0$, then ______________ in the $y-$axis.	Period $\pi \times \frac{1}{\lvert b \rvert} = \frac{\pi}{\lvert b \rvert}$

The graph of $y = \tan x$



$a=$	$b=$	$c=$	$d=$
amplitude	***period***	***phase shift***	***vertical displacement***

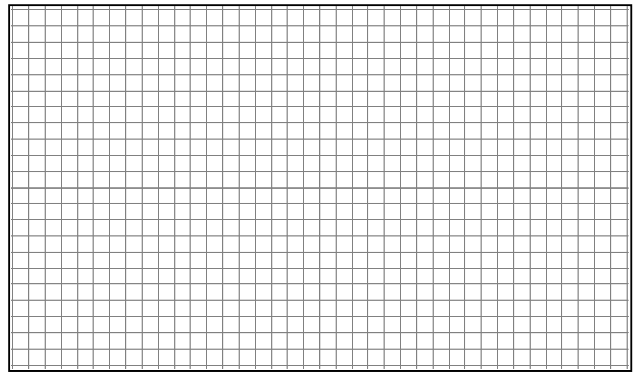
Baseline Dimensions

Left c	*Right* $c+P$	*Vertical Position* d

Note: critical points for $a > 0$. (if $a < 0$ then reverse up/down)

- ¼ cycle → up $|a|$ from the baseline
- ½ cycle → vertical asymptote
- ¾ cycle → down $|a|$ from the baseline
- points on baseline at left and right of baseline

▶ ***Example 3.3:*** Graph $y = 3\tan\left(2\left(x+\frac{\pi}{3}\right)\right) - 4$



$a=$	$b=$	$c=$	$d=$
amplitude	***period***	***phase shift***	***vertical displacement***

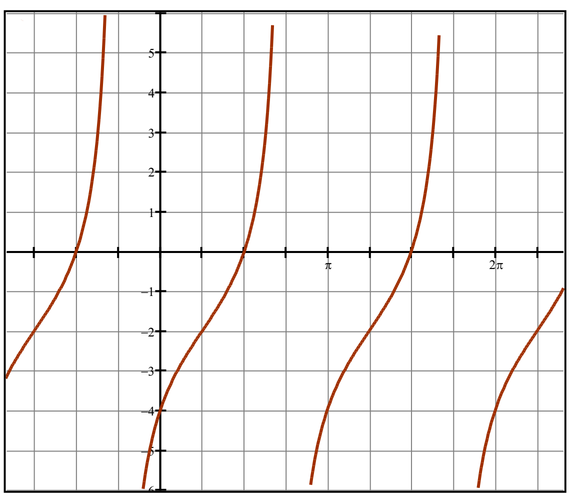
Baseline Dimensions

Left c	*Right* $c+P$	*Vertical Position* d

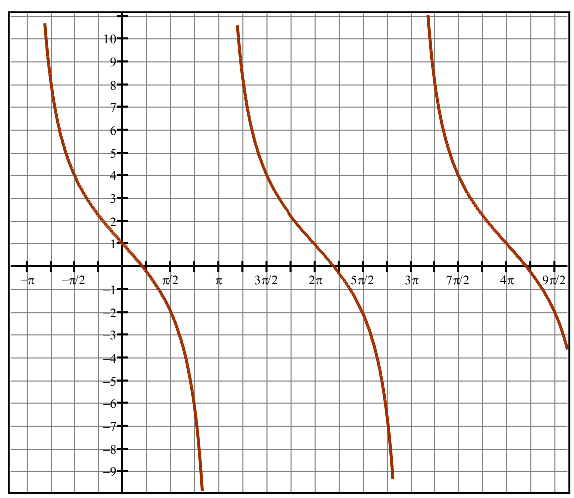
Note: critical points for $a > 0$. (if $a < 0$ then reverse up/down)

- ¼ cycle → up $|a|$ from the baseline
- ½ cycle → vertical asymptote
- ¾ cycle → down $|a|$ from the baseline
- points on baseline at left and right of baseline

▶ ***Example 3.4:*** Write an equation for the graph using a ***tangent*** function.



amplitude	***period***	***phase shift***	***vertical displacement***
$a=$	$b=$	$c=$	$d=$
Equation			



amplitude	***period***	***phase shift***	***vertical displacement***
$a=$	$b=$	$c=$	$d=$
Equation			

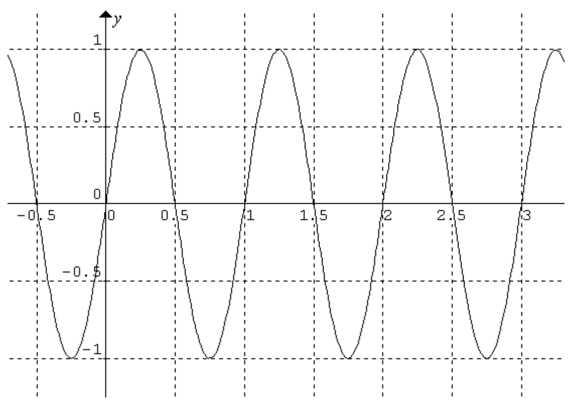
LESSON 5.4: MODELLING WITH SINUSOIDAL FUNCTIONS

In all the graphs that we have looked at so far, the periods have been ____________ numbers because they are multiples of π . In this lesson we will explore trigonometric functions whose periods are ______________ numbers.

FUNCTION	SIMPLIFIED FORM	HORIZONTAL COMPRESSION/ EXPANSION FACTOR	PERIOD
(a) $y = \sin \frac{2\pi}{2} x$			
(b) $y = \sin \frac{2\pi}{3} x$			
(c) $y = \sin \frac{2\pi}{0.5} x$			
(d) $y = \sin \frac{2\pi}{3.5} x$			

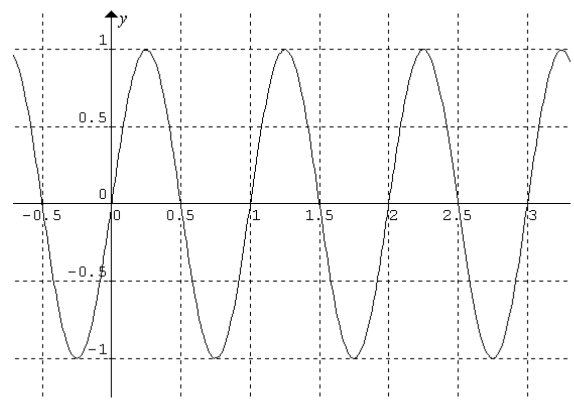
Graph the functions over 1 cycle on the grids below. The graph of $y = \sin 2\pi x$ is provided as a guide.

(a)



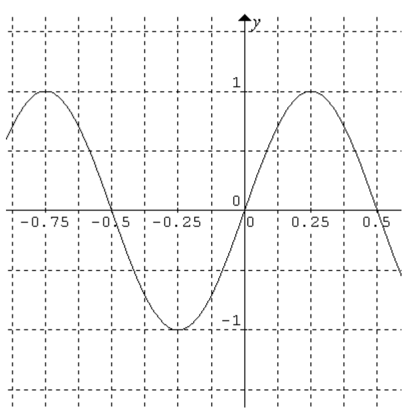
THE GRAPH OF :______________________

(b)



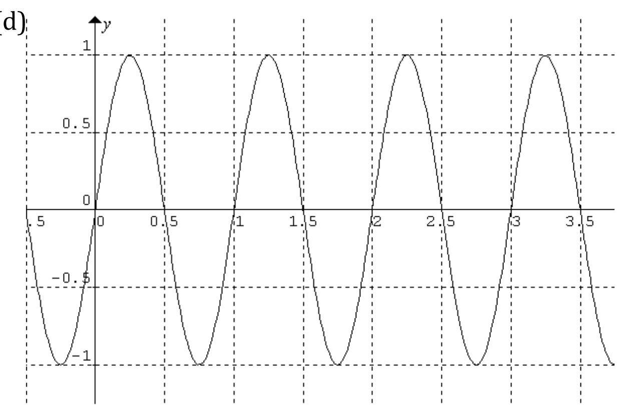
THE GRAPH OF :______________________

(c)



THE GRAPH OF :____________

(d)



THE GRAPH OF :______________________

Graphing Sinusoidal Functions

▶ ***Example 4.1:*** Graph the function $y = 3\cos 2\pi \frac{(x-2)}{4} + 6$ over 2 cycles.

$a =$	$b =$	$c =$	$d =$
amplitude	***period***	***phase shift***	***vertical displacement***

Left	***Right***	***Top***	***Bottom***
c	P	$d+\lvert a\rvert$	$d-\lvert a\rvert$

In general, sinusoidal functions have the form where p is the period

$$f(x) = a\cos\frac{2\pi}{P}(x-c)+d \quad \text{or} \quad f(x) = a\sin 2\pi\frac{(x-c)}{P}+d$$

▶ ***Example 4.2:*** Graph the function $y = 3.5\sin 2\pi\frac{(t-8.4)}{9.2} + 2.5$ over 2 cycles.

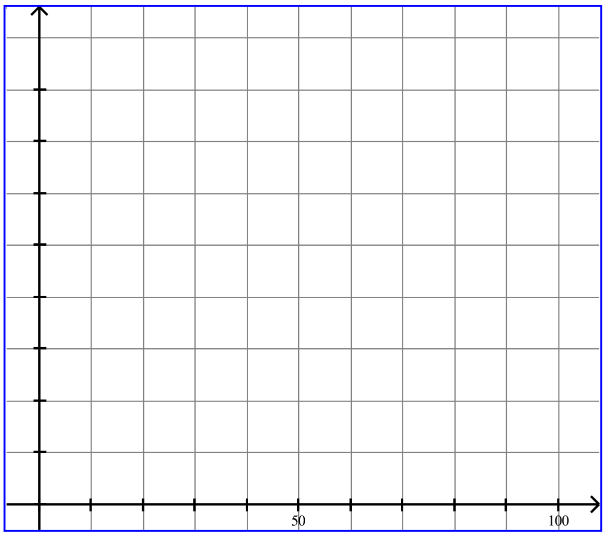
$a =$	$b =$	$c =$	$d =$
amplitude	***period***	***phase shift***	***vertical displacement***

Left	***Right***	***Top***	***Bottom***
c	P	$d+\lvert a\rvert$	$d-\lvert a\rvert$

Modelling Real Situations using Trigonometric Functions

▶ ***Example 4.3:*** A Ferris wheel with a radius of 30 m rotates once every 100s. At time $t = 0\,s$, Carl is at the lowest position on the Ferris wheel which is 5 m off the ground.

(a) Using the grid below, graph the height, *h,* of a passenger vs. the elapsed time, *t,* during at least one rotation of the Ferris wheel. Clearly show at least 5 points on your graph and indicate the scale on the vertical axis.



(b) Write the equation that gives the passenger's height, *h* metres, above the ground as a function of time, *t* seconds.

(c) Use the equation to find the times when Carl is at the lowest position.

(d) Use the equation to find the times when Carl is at the highest position.

(e) Use the equation to find the times when Carl is at a height of 50 m

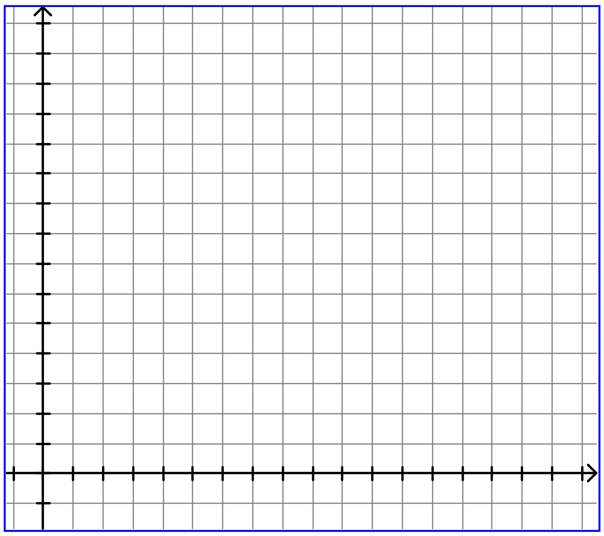
(f) Hence, find the time that Carl is above a height of 50 m during each complete cycle.

(g) The ride lasts for 10 minutes. What vertical distance does Carl travel during the ride?

(h) What total distance does Carl travel during the ride?

▶ ***Example 4.4:*** A deep sea harbour has ship berths that are 20 m at high tide and 4 m at low tide. The period of the tide is 12 hours. Let t represent the time, in hours, since the last high tide at 6:00 AM.

(a) Using the grid below, graph depth d of the ship berth vs. the elapsed time t. Clearly show at least 5 points on your graph and indicate the scale on the vertical axis.



(a) Write the equation that gives the depth d, in metres, as a function of time, t in hours..

(c) Use the equation to find the times when the depth is a minimum.

(d) Use the equation to find the times when the depth is a maximum.

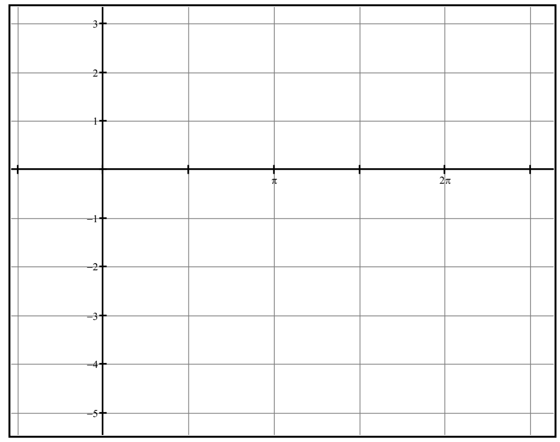
(e) Use the equation to find the times when the depth is 10 m.

(f) If tankers require a depth of at least 10 m to dock safely, how much time in a 24 hour interval starting at 00:00, is available for safe docking?

LESSON 5.5: SOLVING EQUATIONS BY GRAPHING

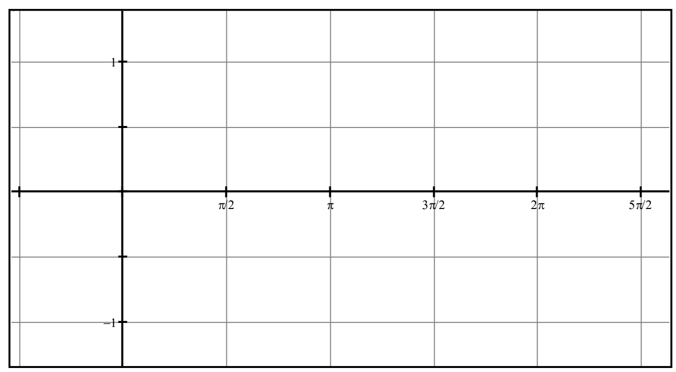
► ***Example 5.1:*** Solve the equation $1+\sin 2x=-1-3\sin 2x$ to 2 decimal places over the domain $0\leq x<2\pi$.

- What is the period of each function?
- How many solutions should we expect to find in the given domain?
- Solve graphically for the general solution.

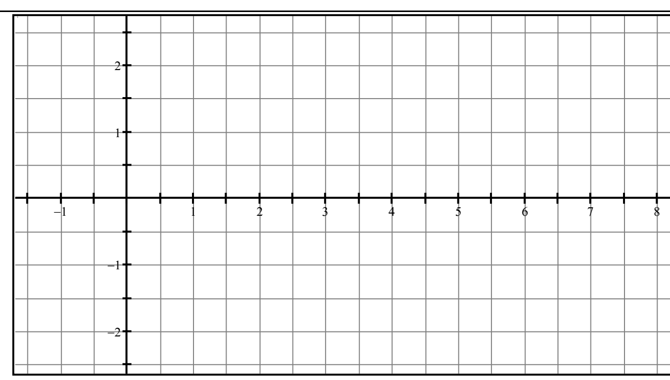


► ***Example 5.2:*** Solve the following equations and write the general solution for each equation.

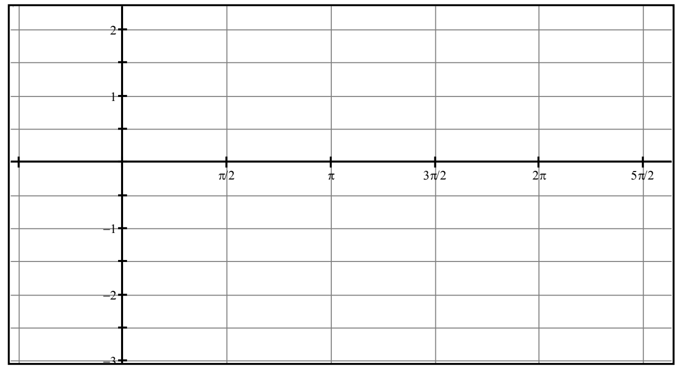
(a) $\cos(x+\pi)=\dfrac{1}{2}$



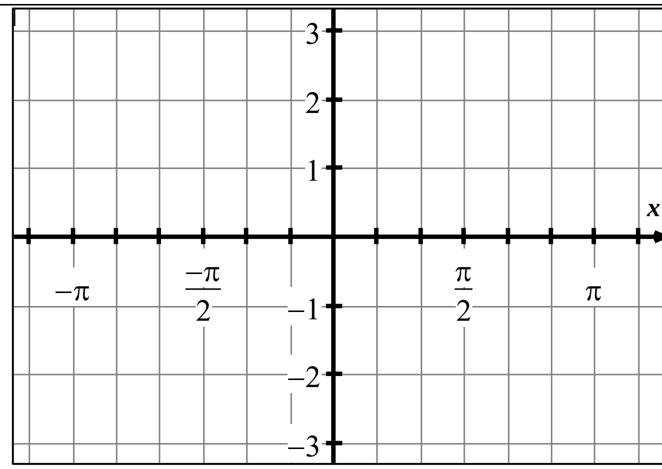
(b) $\cos\dfrac{\pi}{2}(x+1)=\cos\dfrac{\pi}{2}x$



(c) $2\sin^2 x+\sin x-2=0$



(d) $2\sin x = x$



(e) $\tan x+\sin x=2$