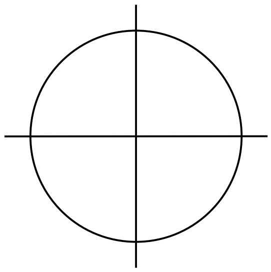
PRE-CALCULUS 12

CHAPTER 4: TRIGONOMETRY I

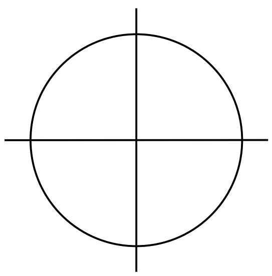
LESSON 4.1: ANGLES IN STANDARD POSITION

Rotation Angles

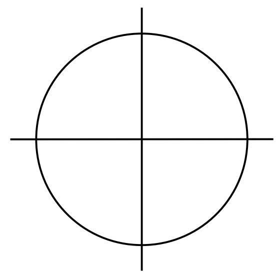


Let $P(x, y)$ represent a point which moves around a circle with radius r and centre $O(0,0)$. P starts at the point $A(r,0)$ on the $x-$axis. For any position of P, an angle θ is defined, which represents the amount of rotation about the origin. When the vertex of the angle is $(0,0)$, the ________________ arm is OA, the __________________ arm is OP, and we say the angle θ is in ___________________ position.

If $\theta > 0$, the rotation is ________________.



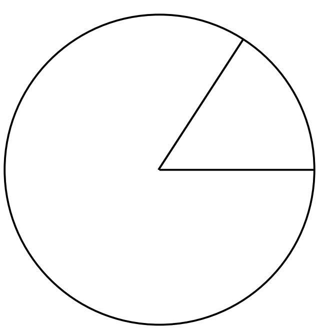
If $\theta < 0$, the rotation is __________________.

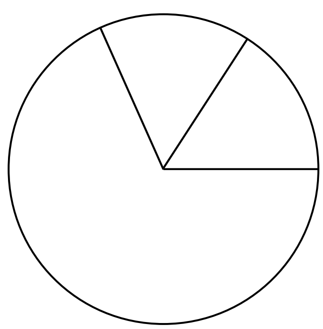


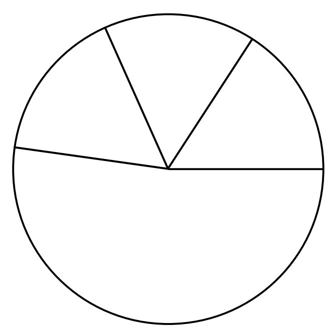
In order to simplify some of the calculations involved in Trigonometry and Calculus, mathematicians use an alternative angular measure called _________________ measure.

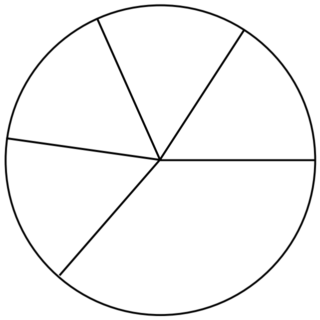
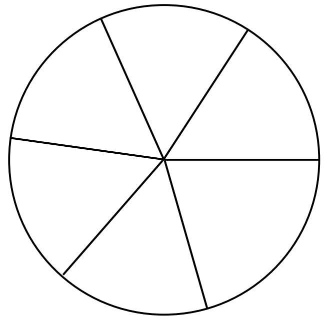
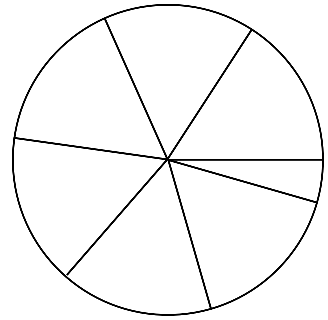
Definition: The radian measure of an angle is the ratio of the length of an arc of a circle to the radius of the circle.

$\begin{array}{l} \textit{measure of an} \\ \textit{angle in radians} \end{array} = \dfrac{\textit{length of arc subtending the angle}}{\textit{length of radius}}$	$\theta = \dfrac{\quad}{\quad}$



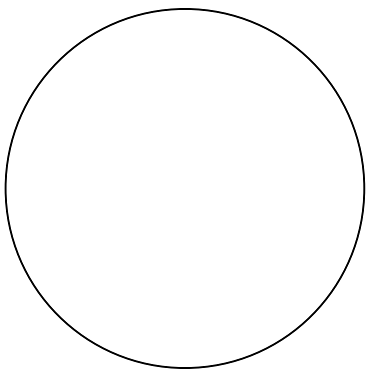




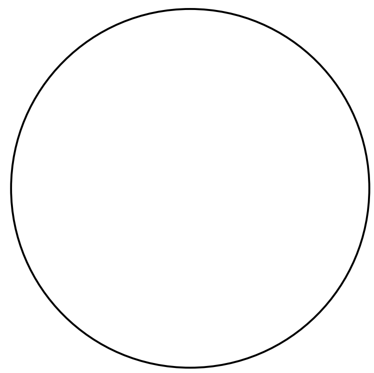




Converting between Degrees and Radians

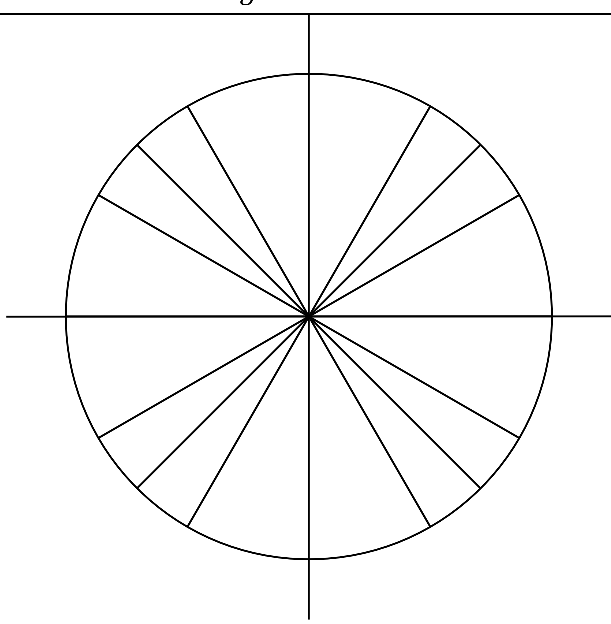
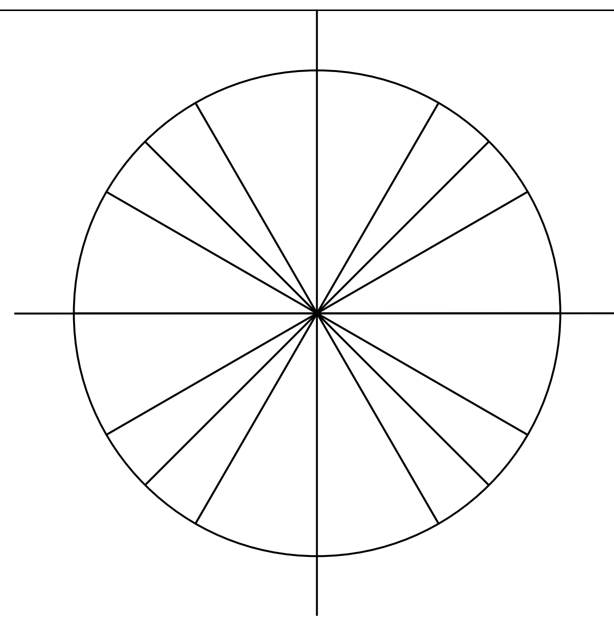
▶ ***Example 1.1:***
Determine the radian measure of $360°$



▶ ***Example 1.2:***
Determine the radian measure of $180°$



Common Conversions

Degree Measure	*Radian Measure*
	

Other Conversions: Degrees to Radians

▶ ***Example 1.3:*** Convert from degrees to radians (give your answer as an exact value in terms of π).

(a) $15°$

(b) $480°$

▶ ***Example 1.4:*** Convert from degrees to radians (round to two decimal places).

(a) $70°$

(b) $213°$

Other Conversions: Radians to Degrees

▶ ***Example 1.5:*** Convert from radians to degrees.

(a) $\frac{\pi}{4}$

(b) $\frac{-7\pi}{3}$

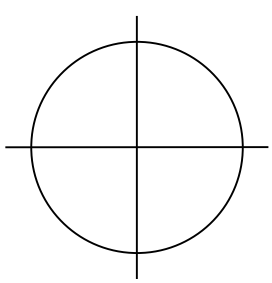
▶ ***Example 1.6:*** Convert from radians to degrees (to the nearest hundredth).

(a) $1.57\ radians$

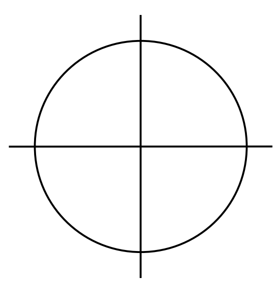
(b) 3.2

When the terminal arm is rotated 360°, the terminal arm of the new angle will be in the same position as the terminal arm of the original angle. We say the two angles are ____________________.

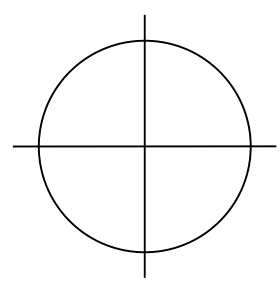
Given an angle of 60°



Add 360°



Add another 360°



Coterminal Angles in Degrees

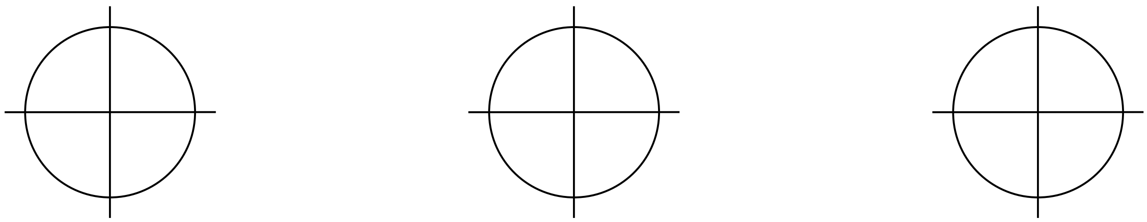
1. Two or more angles in standard position are coterminal if the terminal arm is in the same position for each angle.
2. If θ represents an angle in standard position, then any angle coterminal with θ is represented by the expression $\theta + n(360°)$, where n is an integer.

▶ ***Example 1.7:*** Given $\theta = 150°$

(a) Draw an angle in standard position

(b) Find two other angles which are coterminal with θ and illustrate them on the diagram.

(c) Write an expression to represent any angle coterminal with θ



Coterminal Angles in Radian Measure

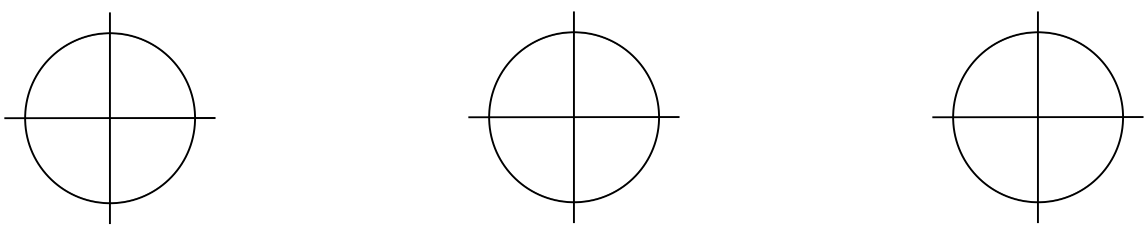
If θ represents any angle in radian measure, then any other angle coterminal with θ is represented by the expression $\theta + n(2\pi)$ or $\theta + 2n\pi$, where n is an integer.

▶ ***Example 1.8:*** Given $\theta = \dfrac{3\pi}{4}$

(a) Draw an angle in standard position

(b) Find two other angles which are coterminal with θ (one positive, one negative) and illustrate them on the diagram.

(c) Write an expression to represent any angle coterminal with *θ.*

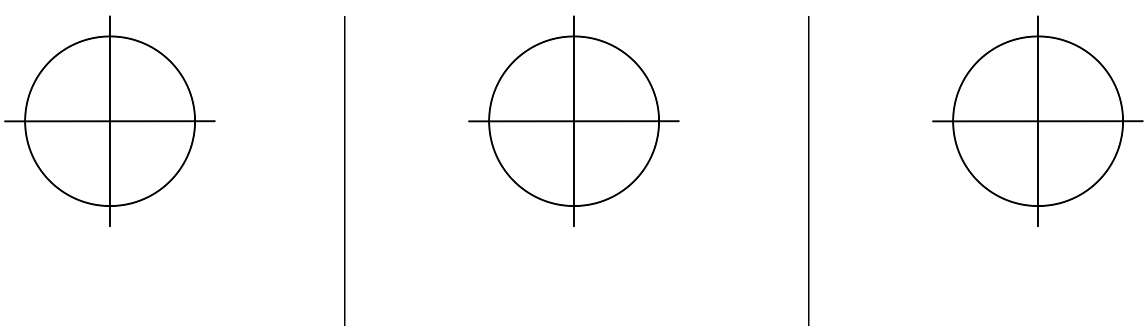


Reference Angles in Degrees

For any angle θ in standard position, its reference angle is the acute angle between its terminal arm and the $x-$ axis.

▶ ***Example 1.9:*** Find the reference angle for each angle

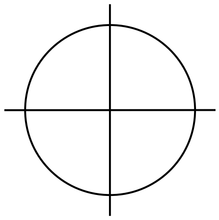
$\theta = 150°$ $\theta = -280°$ $\theta = 472°$



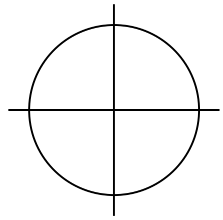
Reference Angles in Radian Measure

▶ ***Example 1.10***: Find the reference angle for each angle

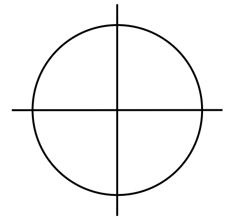
(a) $\theta = \dfrac{5\pi}{6}$



(b) $\theta = -\dfrac{5\pi}{4}$

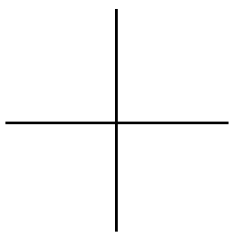


(c) $\theta = \dfrac{11\pi}{3}$

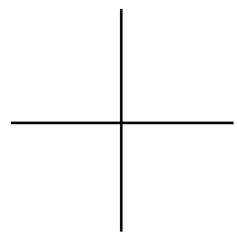


▶ ***Example 1.11***: Find all angles θ, where $0 \le \theta \le 2\pi$ that have the same reference angle as the following:

(a) $\theta_R = \dfrac{5\pi}{12}$

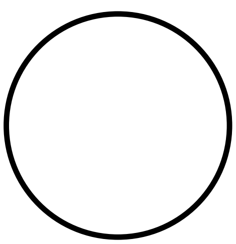


(b) $\theta_R = 0.376$

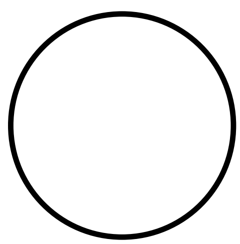


Solving Problems involving Arc Length

▶ ***Example 1.12:*** A pendulum 30 cm long swings through an arc of 45 cm. Through what angle does the pendulum swing? (Answer in degrees and in radians to the nearest tenth.)

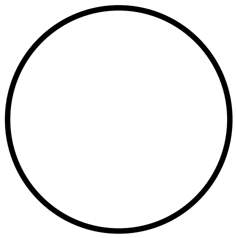


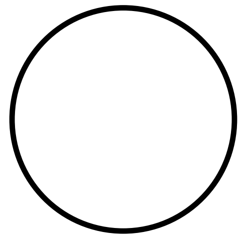
▶ ***Example 1.13:*** A circle of radius 5 cm contains a central angle of 2.4 radians. Calculate the length of the arc subtended by the central angle, to the nearest tenth of a centimeter.



▶ ***Example 1.14:*** Calculate the arc length (the nearest tenth of a metre) of a sector of a circle with diameter 9.2 m if the sector angle is $150°$.

▶ ***Example 1.15:*** Find the radius of a circle in which the central angle of 1.5 radians is subtended by an arc of length 4.5 cm.

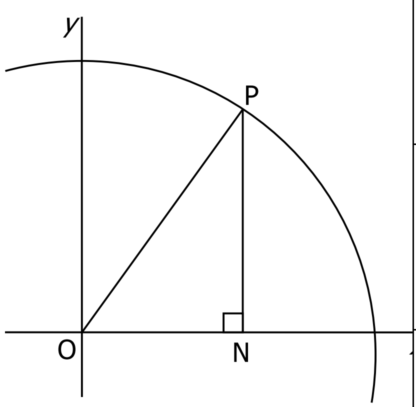




LESSON 4.2: EXACT VALUES OF TRIGONOMETRIC FUNCTIONS

Trigonometric Ratios of Angles in Standard Position

Let $P(x, y)$ represent any point in the first quadrant on a circle with radius r. Then P is on the terminal arm of an angle θ in standard position. Draw a right triangle ΔPON by dropping a perpendicular from P to the $x-$axis.



$r =$ ____________________

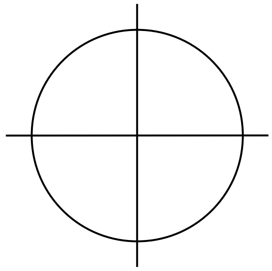
Primary Ratios	Reciprocal Ratios
SINE RATIO $\sin\theta = \dfrac{opposite}{hypotenuse} = \_\_\_\_$	**COSECANT RATIO** $\csc\theta = \dfrac{1}{\sin\theta} = \_\_\_\_$
COSINE RATIO $\cos\theta = \dfrac{adjacent}{hypotenuse} = \_\_\_\_$	**SECANT RATIO** $\sec\theta = \dfrac{1}{\cos\theta} = \_\_\_\_$
TANGENT RATIO $\tan\theta = \dfrac{opposite}{adjacent} = \_\_\_\_$	**COTANGENT RATIO** $\cot\theta = \dfrac{1}{\tan\theta} = \_\_\_\_$

Determining the Sign of Trigonometric Ratios

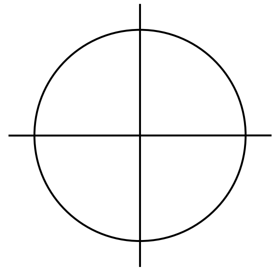
(a) In each quadrant, draw the rotation angle θ in standard position
(b) Complete the chart to determine the sign of each ratio.

$\sin\theta =$		
$\cos\theta =$		
$\tan\theta =$		

Quadrant 2

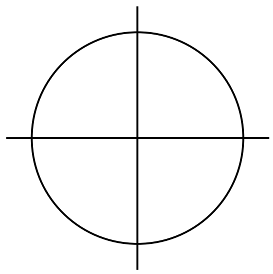


Quadrant 1

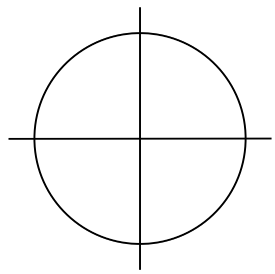


$\sin\theta =$		
$\cos\theta =$		
$\tan\theta =$		

$\sin\theta =$		
$\cos\theta =$		
$\tan\theta =$		



Quadrant 3

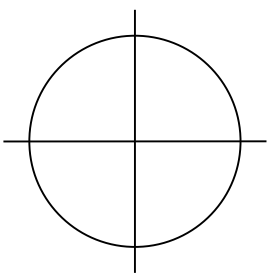


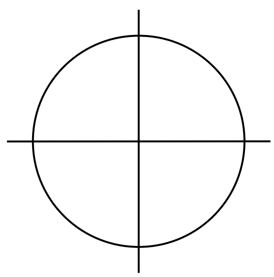
Quadrant 4

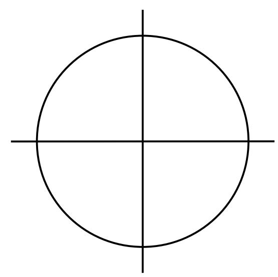
$\sin\theta =$		
$\cos\theta =$		
$\tan\theta =$		

- The signs of the primary and reciprocal ratios can be remembered by using the following quadrant diagrams

$\sin\theta$, $\csc\theta$ depend on _____ $\cos\theta$, $\sec\theta$ depend on _____ $\tan\theta$, $\cot\theta$ depend on __________





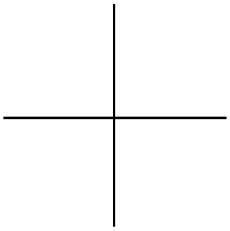
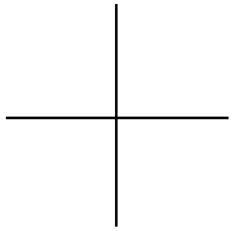
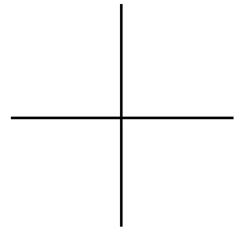


- Another mnemonic device is to form an acronym

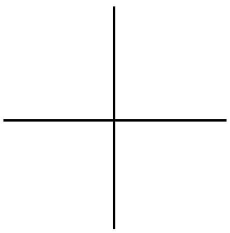
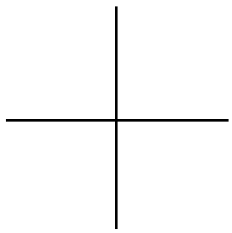
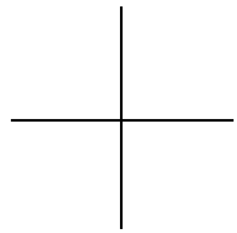
Quadrant I:	(**A**ll ratios are positive)	▶	A=
Quadrant II:	(**S**ine ratios are positive)	▶	S=
Quadrant III:	(**T**angent ratios are positive)	▶	T=
Quadrant IV:	(**C**osine ratios are positive)	▶	C=

▶ ***Example 2.1:*** Rewrite as the same trigonometric function of a positive acute angle.

(a) $\sin 140°$ (b) $\tan 323°$ (c) $\cos 258°$

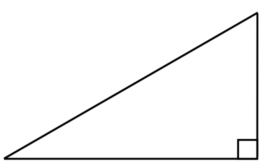
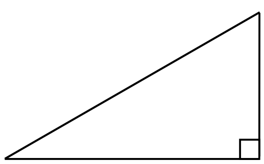
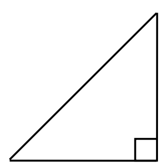
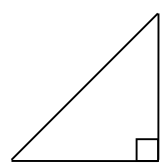
  

(d) $\sin\left(-\frac{2\pi}{3}\right)$ (e) $\sec\left(\frac{7\pi}{18}\right)$ (f) $\cot(-2.49)$

The trigonometric functions $\sin\theta$ and $\cos\theta$ (and the other functions that are formed from these), have values that are sometimes irrational (non-repeating and non-terminating decimals). For some of these angles it is possible to find the exact value of the trigonometric functions using the following triangles.

Special Triangles

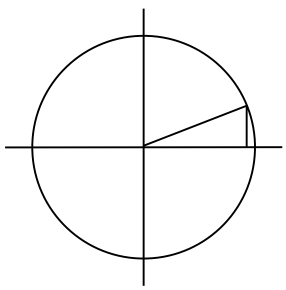
   

▶ ***Example 2.2:*** Fill in the following table

x in degrees	$30°$	$45°$	$60°$
x in radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\sin x$			
$\cos x$			
$\tan x$			

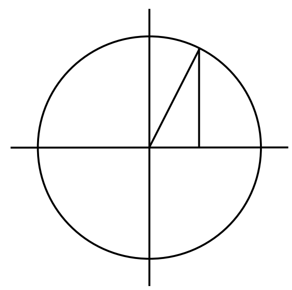
Finding exact Trigonometric Values for $0°$ and $90°$

Given any point (x, y) on a circle with radius r, then when $\theta = 0°$ we observe that $x = r$ and $y = 0$



$\sin\theta = \dfrac{y}{r}$	$\sin 0° = \text{——} =$
$\cos\theta = \dfrac{x}{r}$	$\cos 0° = \text{——} =$
$\tan\theta = \dfrac{y}{x}$	$\tan 0° = \text{——} =$

Given any point (x, y) on a circle with radius r, then when $\theta = 90°$ we observe that $x = 0$ and $y = r$



$\sin\theta = \dfrac{y}{r}$	$\sin 90° = \text{——} =$
$\cos\theta = \dfrac{x}{r}$	$\cos 90° = \text{——} =$
$\tan\theta = \dfrac{y}{x}$	$\tan 90° = \text{——} =$

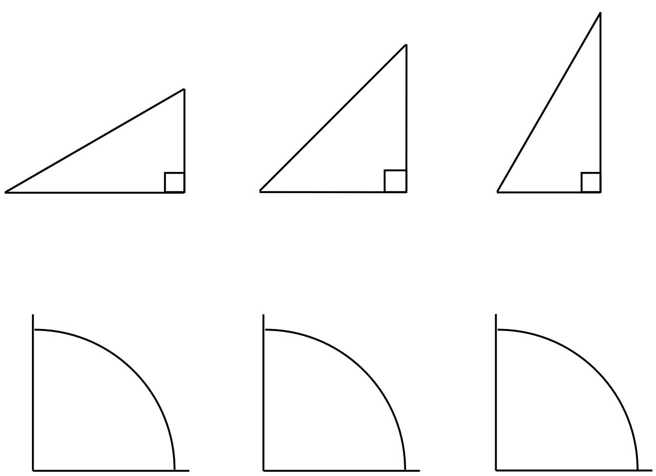
Recognizing patterns in the table

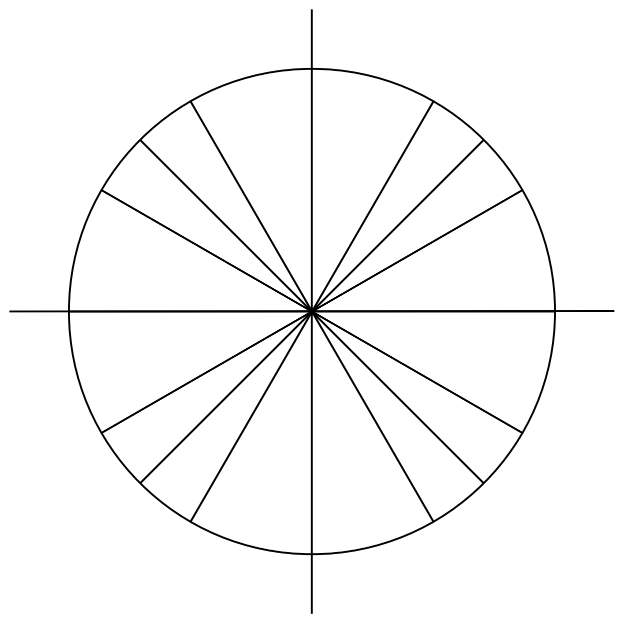
θ in degrees	$0°$	$30°$	$45°$	$60°$	$90°$
θ in radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$					
$\cos\theta$					
$\tan\theta$					

Using the Unit Circle

The unit circle is the circle centred at the origin $(0,0)$ with radius equal to 1. If we construct the special triangles so that the hypotenuse of each is equal to 1 unit, then we can arrange the triangles around the unit circle to establish the exact values of all the multiples of $30°$, $45°$, $60°$, and $90°$.

$30°$ $45°$ $60°$



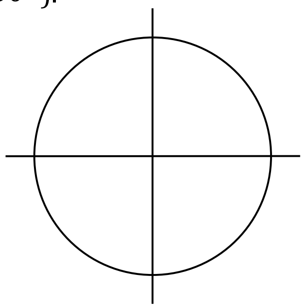


Given the unit circle with $r=1$, we observe that the trigonometric ratios simplify as follows

$\sin\theta = \frac{y}{r} =$	$\csc\theta = \frac{r}{y} =$
$\cos\theta = \frac{x}{r} =$	$\sec\theta = \frac{r}{x} =$
$\tan\theta = \frac{y}{x} =$	$\cot\theta = \frac{x}{y} =$

This means that every point (x, y) on the unit circle can be written in the form $(\cos\theta, \sin\theta)$. In other words, to find the cosine ratio of an angle in standard position whose terminal arm passes through the point (x, y) on the unit circle, all we need to do is write down the $x-$coordinate of the point.
The $y-$coordinate represents the sine ratio.

The unit circle is also useful in finding the trigonometric ratios for the quadrant angels (multiples of $90°$).



▶ ***Example 2.3*** Determine the exact value of the following:

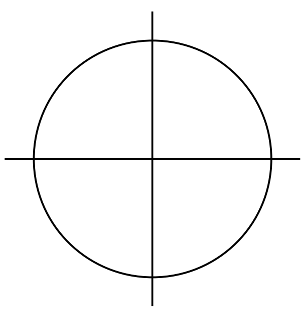
$\sin\theta$				
$\cos\theta$				
$\tan\theta$				
$\csc\theta$				
$\sec\theta$				
$\cot\theta$				

For other angles (the multiples of $30°$, $45°$, $60°$), we use the special triangles.

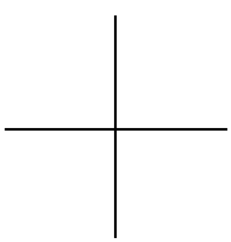
▶ ***Example 2.4:*** Determine the exact value of the following.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
$240°$						
$-\frac{7\pi}{2}$						

▸ ***Example 2.5:*** Find the $y-$coordinate of the point on the unit circle in quadrant II where $x=-\frac{2}{3}$. Find the values of the primary and reciprocal trigonometric ratios for the angle in standard position whose terminal are passes through this point.



▸ ***Example 2.6:*** If $\tan\theta=-\frac{3}{2\sqrt{5}}$ and $\sec\theta$ is positive, then find the value of $\sin\theta$.



TRIGONOMETRY I PRACTICE: SPECIAL ANGLES

Determine the exact value of each

$\sin\left(\frac{\pi}{6}\right)=$	$\cos\left(\frac{\pi}{6}\right)=$	$\tan\left(\frac{\pi}{6}\right)=$
$\sin\left(\frac{\pi}{2}\right)=$	$\cos\left(\frac{\pi}{2}\right)=$	$\tan\left(\frac{\pi}{2}\right)=$
$\sin\left(\frac{\pi}{3}\right)=$	$\cos\left(\frac{\pi}{3}\right)=$	$\tan\left(\frac{\pi}{3}\right)=$
$\sin\left(\frac{5\pi}{6}\right)=$	$\cos\left(\frac{5\pi}{6}\right)=$	$\tan\left(\frac{5\pi}{6}\right)=$
$\sin^2\left(\frac{\pi}{4}\right)=$	$\cos^2\left(\frac{\pi}{4}\right)=$	$\tan^2\left(\frac{\pi}{4}\right)=$
$\sin\left(-120^\circ\right)=$	$\cos\left(-120^\circ\right)=$	$\tan\left(-120^\circ\right)=$
$\sin(\pi)=$	$\cos(\pi)=$	$\tan(\pi)=$
$\csc\left(\frac{7\pi}{4}\right)=$	$\sec\left(\frac{7\pi}{4}\right)=$	$\cot\left(\frac{7\pi}{4}\right)=$
$\csc^2\left(\frac{7\pi}{6}\right)=$	$\sec^2\left(\frac{7\pi}{6}\right)=$	$\cot^2\left(\frac{7\pi}{6}\right)=$
$\sin 2\left(\frac{4\pi}{3}\right)=$	$2\cos\left(\frac{8\pi}{3}\right)=$	$\tan^2\left(\frac{8\pi}{3}\right)=$
$\sin^2\left(\frac{11\pi}{4}\right)+\cos^2\left(\frac{11\pi}{4}\right)=$	$\cos^2\left(\frac{11\pi}{4}\right)-\sin^2\left(\frac{11\pi}{4}\right)=$	$\tan^2\left(\frac{11\pi}{4}\right)-\sec^2\left(\frac{11\pi}{4}\right)=$

LESSON 4.3: SOLVING TRIGONOMETRIC EQUATIONS

▶ ***Example 3.1:*** Determine the positive rotation angle θ where $0 \le \theta \le 2\pi$

Reference Angle	Quadrant	Sketch	Rotation Angle
$\frac{\pi}{3}$	1		
$\frac{\pi}{4}$	2		
$\frac{\pi}{18}$	3		
$\frac{5\pi}{12}$	4		
$\frac{\pi}{2}$	*between* 3 *and* 4		

Solving Trigonometric Equations in Degrees

▶ ***Example 3.2:*** Solve for θ, *where* $0° \le \theta < 360°$

(a) $\sin\theta = -0.8090$

(b) $\cos\theta = -0.8090$

▶ ***Example 3.3:*** Solve for θ, *where* $-180° \le \theta < 180°$

(a) $\cot\theta = 0.5$

(b) $\csc\theta = -2.86$

Solving Trigonometric Equations in Radian Measure

▶ ***Example 3.4:*** Solve for θ, *where* $0 \le \theta < 2\pi$

(a) $\sin\theta = \frac{1}{2}$

(b) $\sec\theta = -\sqrt{2}$

▶ ***Example 3.5:*** Solve for θ, *where* $-\pi \le \theta \le \pi$

(a) $\sin\theta = 0.425$

(b) $2\cot\theta + 3 = 0$

▶ ***Example 3.6:*** Solve for the exact value of θ

(a) $\cot\theta = -\sqrt{3}$, $0° \le \theta < 360°$

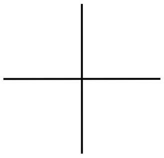
(b) $2\sec\theta - 7 = \sec\theta - 5$, $-\pi \le \theta \le \pi$

(c) $2\tan^2\theta - 6 = 0$ where $0 \le \theta < 2\pi$

(d) $\csc^3\theta + 8 = 0$ where $-2\pi \le \theta \le 2\pi$

▶ ***Example 3.7:*** Solve the equation $\cos^2 x - \cos x = 0$ for $-\pi \le x \le \pi$.

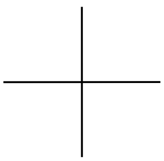
Solution:



General Solution:

▶ ***Example 3.8:*** Solve the equation $2\sin^2 x - 5\sin x - 3 = 0$ for $0 \le x < 2\pi$.

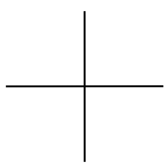
Solution:



General Solution:

▶ ***Example 3.9:*** Solve the equation $\sin 3x = \dfrac{1}{\sqrt{2}}$

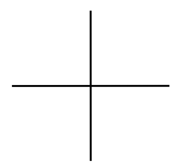
Solution: where $0 \le x < 2\pi$



General Solution:

▶ ***Example 3.10:*** Solve the equation $\sec 2x = 2$

Solution: where $0 \le x < 2\pi$



General Solution: