Rational Inattention

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1 Introduction

Rational inattention models attempt to formalize what economist Herbert A. Simon called 'bounded rationality.' Simon argued that our decision making capabilities are limited by a finite capacity to process information and that "a wealth of information creates a poverty of attention, and a need to allocate that attention efficiently" (1971). Using the mathematics of information theory, rational inattention models allow us to explain the emergent properties of bounded rationality across a wide range of empirical phenomena. In the words of the rational inattention's pioneer, Christopher Sims, the theory's approach is "to construct optimizing-agent models that are consistent with people not using freely available information" so as to "explain why some freely available information is not used, or imperfectly used" (2010).

2 Mutual Information Costs

In this project inspired by Foley 2012 and Maćkowiak et al. 2020, we explore a simple discrete-choice model where the agent decides to pick an action from a basket of multiple choices $a_1, a_2, ...a_k$. The payoffs of these actions are $u[a_1], u[a_2], ...u[a_k]$. The behavior of the agent corresponds to a mixed-strategy probability distribution $\{f_1, f_2, ...f_k\}$ which determines the relative frequency at which the agent picks each action. The prior on the agent's behavior before receiving any information (e.g. the price signals) is uniform over all actions, i.e. $S = \{\frac{1}{k}, \frac{1}{k}, ...\frac{1}{k}\}$. The posterior $S|\mathbf{v} = \{p_1, p_2, ...p_k\}$ on the agent's behavior after receiving the price signals $\mathbf{v}$ must solve the optimization problem:

$$\max \sum_{i=1}^{k} u[a_i] \cdot p_i - T \cdot I(S; \mathbf{v})$$

subject to the constraints:

$$\sum_{i=1}^{k} p_i = 1 \text{ and } \forall i \in [1, k],\ p_i \geq 0$$

where $I(S; \mathbf{v})$ is the mutual information between the agent's behavior and the price signals, and T is the unit cost of information or 'behavior temperature.'

To solve the problem first note:

$$\begin{aligned} I(S; \mathbf{v}) &= H(S) - H(S|\mathbf{v}) \\ &= \log(k) - \sum_{i=1}^{k} p_i \cdot \log\left(\frac{1}{p_i}\right) \end{aligned}$$

Thus we can rewrite our objective function as:

$$\sum_{i=1}^{k} u[a_i] \cdot p_i - T \cdot \left(\log(k) - \sum_{i=1}^{k} p_i \cdot \log\left(\frac{1}{p_i}\right)\right)$$

Since $-T \cdot \log(k)$ is constant this is equivalent to considering:

$$\sum_{i=1}^{k} u[a_i] \cdot p_i + T \cdot \sum_{i=1}^{k} p_i \cdot \log\left(\frac{1}{p_i}\right)$$

Since our de facto objective function is the sum of a linear function and a concave function and our constraints form a hyperplane, we know the first-order conditions of the Lagrangian:

$$\mathcal{L}[p_i, \mu] = \sum_{i=1}^{k} u[a_i] \cdot p_i + T \cdot \sum_{i=1}^{k} p_i \cdot \log\left(\frac{1}{p_i}\right) - \mu \cdot \left(\sum_{i=1}^{k} p_i - 1\right)$$

are necessary and sufficient to characterize the solution. Thus we just need to solve the system:

$$\frac{\partial \mathcal{L}}{\partial p_i} = u[a_i] - T \cdot \log(p_i) - T - \mu = 0$$

Rearranging yields:

$$\log(p_i) = \frac{u[a_i] - T - \mu}{T} = \frac{u[a_i]}{T} - \left(1 + \frac{\mu}{T}\right)$$

and hence:

$$p_i = \frac{e^{u[a_i]/T}}{e^{1+\mu/T}}$$

Since:

$$\sum_{i=1}^{k} p_i = \frac{\sum_{i=1}^{k} e^{u[a_i]/T}}{e^{1+\mu/T}} = 1$$

we must have:

$$e^{1+\mu/T} = \sum_{i=1}^{k} e^{u[a_i]/T}$$

and:

$$p_i = \frac{e^{u[a_i]/T}}{\sum_{i=1}^{k} e^{u[a_i]/T}}$$

Thus the agent's optimal posterior strategy is the Boltzmann distribution!

3 Entropy Constrained Behavior

The reason the Boltzmann distribution pops up here is a little subtle. Let's look at an equivalent (although less generalizable) rational inattention model where we solve:

$$\max \sum_{i=1}^{k} u[a_i] \cdot p_i$$

subject to the constraints:

$$\sum_{i=1}^{k} p_i \cdot \log\left(\frac{1}{p_i}\right) = H$$

$$\sum_{i=1}^{k} p_i = 1 \text{ and } \forall i \in [1,k],\ p_i \geq 0$$

, i.e. the agent maximizes their expected utility subject to the constraint that their strategy must have a threshold entropy. This is a sort of dual to the derivation of the Boltzmann distribution where entropy is maximized subject to a mean energy constraint. The equivalence of this alternative formulation with our original problem follows since the objective function is linear, the constraint set is convex, and the Lagrangian of the new problem:

$$\mathcal{L}[p_i, \mu, T] = \sum_{i=1}^{k} u[a_i] \cdot p_i + T \cdot \left(-H + \sum_{i=1}^{k} p_i \cdot \log\left(\frac{1}{p_i}\right)\right) - \mu \cdot \left(\sum_{i=1}^{k} p_i - 1\right)$$

is virtually identical to the Lagrangian of the previous problem (up to a handful of constants which disappear when we take partials).

4 Binary Choice

Consider the binary-choice problem where a buyer decides whether to purchase a good priced at p which they value at p^*. If they purchase the good the payoff is p^* whereas if they don't purchase the good they get to keep their p dollars. Thus the frequency with which they will buy the good is given by:

$$f_B(p) = \frac{e^{p^*/T}}{e^{p/T} + e^{p^*/T}} = \frac{1}{1 + e^{(p-p^*)/T}}$$

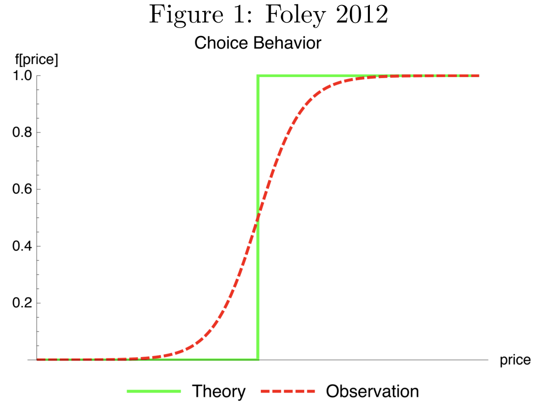
Similarly for a seller:

$$f_S(p) = \frac{e^{p/T}}{e^{p/T} + e^{p^*/T}} = \frac{1}{1 + e^{(p^*-p)/T}}$$

In both cases, the agent's optimal response follows a logistic curve and in the case of the buyer it is the Fermi-Dirac distribution! This is a promising feature of rational inattention models since "logistic quantal response behavior occurs reliably in a wide range of choice situations, and there is no reason not to believe that the experimental data reported as supporting loss aversion is highly replicable" (Foley).

The binary-choice problem also sheds light on the effects of imposing a unit cost of T per bit of information. In models where the agent has unbounded information processing capacity, T is effectively 0 and logistic response approaches the green curve below. However when $T > 0$, logistic response starts to increasingly resemble what we see in the experimental data. We also see that the higher the behavior temperature (i.e. unit cost per bit) of an agent, the more uncertain they ultimately are in their valuation.

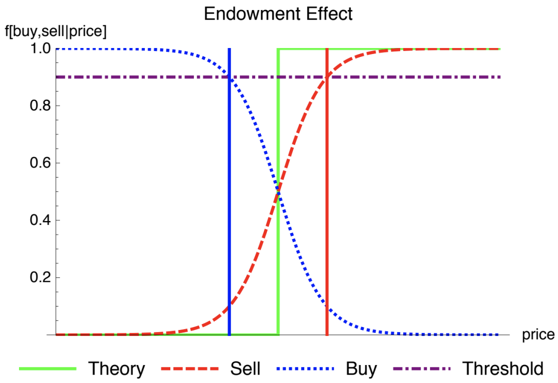
Figure 1: Foley 2012



5 Emergent Properties

One particularly interesting feature of logistic response is that it provides a cute explanation of the endowment effect. The famous studies of behavioral economists Kahneman and Tversky have shown the value attributed to a good by an agent appears to be higher when they are a seller than when they are a buyer. However, logistic response reveals that the agent can actually maintain consistent valuation across contexts by simply varying the assumption that they are either very willing to buy the good or very willing to sell the good. The figure below demonstrates how logistic response causes an agent's selling price to be higher than the buying price when we place a horizontal line at $f(p) = 90\%$ for an agent with $T > 0$:

Figure 2: Foley 2012

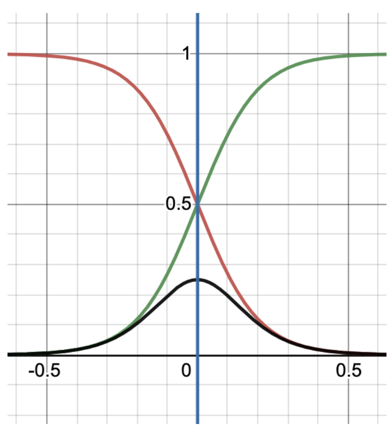


Another cool emergent property of rational inattention models is noise trading. Consider the case where a buyer and seller with identical valuations p^* of a good and behavior temperatures T_B and T_S are offered to transact at price p. Both buyer and seller will accept with frequency:

$$f_S(p) \cdot f_B(p) = \frac{1}{1 + e^{(p^* - p)/T_S}} \cdot \frac{1}{1 + e^{(p - p^*)/T_B}}$$

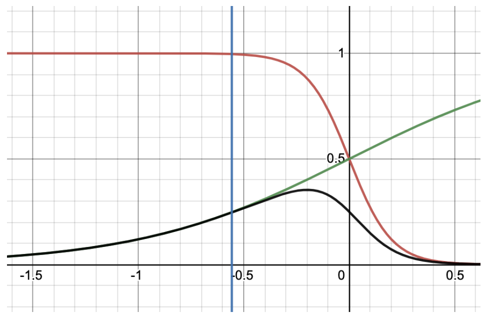
Thus, if all prices are offered with equal relative frequency, when $T_B = T_S$ the distribution of transaction prices will look like:

Figure 3: Noise trading when $T_B = T_S$



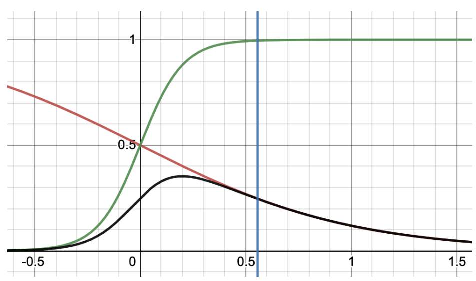
when $T_B < T_S$ the distribution of transaction prices will look like:

Figure 4: Noise trading when $T_B < T_S$



and when $T_B > T_S$ the distribution of transaction prices will look like:

Figure 5: Noise trading when $T_B > T_S$



, where the x-axis corresponds to $p - p^*$, the blue line corresponds to the mean transaction price, and the red and green curves correspond to the buyer's response and seller's response respectively.

So long as $T_B, T_S > 0$, the model assigns non-zero probabilities to transactions where the agents take losses. Such transactions would never occur if the agents had infinite information processing capacity. Additionally, in scenarios where the two agents have different behavior temperatures, we find the lower temperature agent extracts a surplus from the high temperature agent! Not only does this make for a nice thermodynamic analogy, but it reflects real phenomena in markets where agents who are less inexperienced/have more spread in their valuation can be exploited by agents who are more experienced/have less spread in their valuation.

6 Conclusion

Rational inattention models provide a simple yet powerful way to examine the mechanisms and consequences of bounded rationality in the market. In this project, we demonstrated that the mutual information cost and entropy-constrained models applied to the discrete-choice problem are equivalent, and that the Bolztmann distribution:

$$p_i = \frac{e^{u[a_i]}}{\sum_{i=1}^{k} e^{u[a_i]}}$$

dictates the optimal response in both scenarios.

In particular, we found the optimal solution to the buyer's binary-choice problem was the Fermi-Dirac distribution:

$$f_S(p) = \frac{1}{1 + e^{(p^* - p)/T}}$$

This then allowed us to examine the emergent properties of logistic quantal response more broadly and provide possible explanations for phenomena such as the endowment effect and noise trading.

7 Works Cited

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