Prediction Markets: A Mechanism for Information Aggregation

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1 Introduction

In *The Use of Knowledge in Society*, Hayek argues that the central issue facing contemporary economics does not simply consist in figuring out how to optimally allocate a 'given' set of resources. Such problems are relatively tractable from a mathematical point of view, and the marginalists more or less demonstrated the bliss point to be where the "marginal rates of substitution between any two commodities" (Hayek 1945) are the same for all consumers. While Hayek acknowledges the significance of this insight, he also believes answering the question of effective allocation is functionally useless without concrete knowledge of the scarcity which constrains economic welfare in the first place. The truly important function of economics then involves determining what resources actually are 'given' to society. The difficulty in making such a determination stems from the distributed and contradictory nature of relevant economic data amongst millions of agents. Thus, for Hayek, the question of how to best aggregate this decentralized patchwork of information lies at the heart of 'the dismal science.'

Hayek proceeds to identify three primary modes of information aggregation: 1) Central planning, 2) Competition, and 3) Monopoly. Central planning involves "direction of the whole economic system according to one unified plan," (ibid) competition consists of "decentralized planning by many persons," (ibid) and monopoly aims at the "delegation of planning to organized industries" (ibid). Importantly, for Hayek, all economic activity consists of planning. But who ought to be doing the planning? The answer, says Hayek, depends on the kind of knowledge necessary to carry out the planning in question. Far too often, people assume that "scientific knowledge...is the sum of all knowledge" (ibid). This erroneously suggests that the direction of all economic activity ought to be left to a small group of technical experts versed in operations research who make decisions via a central committee. To the contrary, scientific knowledge of 'general rules' cannot account for "knowledge of the particular circumstances of time and place" (ibid). This latter form of knowledge belongs to virtually every economic agent, and in particular, the agents whose occupation demands rapid adaptation to local changes in their environment. However, agents cannot rely exclusively on local signals to make optimal decisions and require "further information...of the larger economic system" (ibid). Crucially, since agents do not require knowledge of why a particular resource is more or less difficult to acquire, and only to what extent a particular resource is more or less difficult to acquire, it is sufficient to summarize all the data which comprise the totality

of macroeconomic trends into prices which the agent can use to gauge in what manner they must alter their activities. In other words, Hayek paints a picture of the market as, first and foremost, a mechanism to compress information into prices.

But if one were to design a market tabula rasa to fit the aim of information compression, exactly what form should such a price system take? Prediction markets are one example as to what such an idealized information aggregation mechanism might look like. Inspired directly by Hayek's argument in *The Use of Knowledge in Society*, prediction markets attempt to solve the age old problem of aggregating opinions by using the mathematics of information theory to incentivize participants to reveal their beliefs to one another. Namely, the opinions which prediction markets aggregate are opinions on the odds of a specific event occurring (e.g. a candidate winning an election or a startup delivering on its goals). We might reasonably wonder "why use a prediction market instead just taking a straight forward average of everyone's beliefs?" The answer to this question is rather deep and begins with a classic result– Arrow's impossibility theorem.

2 Arrow's Impossibility Theorem

We follow the proof of Arrow's impossibility theorem laid out by Yu 2012 in "A one-shot proof of Arrow's impossibility theorem."

2.1 3 Reasonable Demands

Let's start with some definitions. Given n voters $\{x_1, x_2..., x_n\}$ and m candidates $\{y_1, y_2..., y_m\}$, we define $\mathcal{O}$ to be the set of all $m!$ *order preferences* over our candidates.
A *preference profile* is an element $\theta \in \mathcal{O}^n$, where we let θ_i be the order preference of the ith voter. In particular, we write $\theta_i(a) \succ \theta_i(b)$ if voter x_i prefers candidate y_a to candidate y_b.
Finally, a *social aggregation function* is any function $f : \mathcal{O}^n \to \mathcal{O}$ which maps a given preference profile to a unique order preference (you can think of this as our 'voting system').
Now, we are ready to describe the 3 demands we want our social aggregation function f to satisfy. I will define each demand formally and then use plain English to make each one seem reasonable.

First, f satisfies *unanimity* iff $\forall \theta \in \mathcal{O}^n$:

$$(\forall i \in [n], \; \theta_i(a) \succ \theta_i(b)) \implies f(\theta)(a) \succ f(\theta)(b)$$

This means if each of our voters prefers y_a to y_b, then the aggregate preference must also favor y_a to y_b.

Second, f satisfies *binary independence* iff $\forall \theta, \tau \in \mathcal{O}^n$:

$$(\forall i \in [n], \, \theta_i(a) \succ \theta_i(b) \leftrightarrow \tau_i(a) \succ \tau_j(b)) \implies (f(\theta)(a) \succ f(\theta)(b) \leftrightarrow f(\tau)(a) \succ f(\tau)(b))$$

In other words, the aggregate preference between y_a and y_b is uniquely determined by each voter's preference between y_a and y_b.

Third, f satisfies *non-dictatorship* iff:

$$\neg(\exists i \in [n] \text{ s.t. } \forall \theta \in \mathcal{O}, \, f(\theta) = \theta_i)$$

That is to say, there does not exist a voter whose order preference is *always* the aggregate preference.

Hopefully, these demands seem like sensible things to ask from a voting system. If they don't, maybe try convincing yourself that a simple majority rule procedure between two candidates *does* satisfy them.
Unfortunately, when there are more than 2 candidates, turns out we're asking for too much...

2.2   Proof of the Theorem

STAC there exists a social aggregation function f satisfying unanimity, binary independence, and non-dictatorship in an election with more than 2 candidates...

2.2.1   Defining Pivots

Given two candidates y_a, y_b, we define the *pivot* from a to b as follows.
Consider a preference profile $\theta^0 \in \mathcal{O}^n$ where $\forall i \in [n]$:

$$\theta_i^0(a) \succ \theta_i^0(b)$$

By unanimity, we must have:

$$f(\theta^0)(a) \succ f(\theta^0)(b)$$

Now pick $\theta^k \in \mathcal{O}^n$ so that $\forall i \in [n] \backslash [k]$:

$$\theta_i^k(a) \succ \theta_i^k(b)$$

but $\forall i \in [k]$:

$$\theta_i^k(a) \prec \theta_i^k(b)$$

In particular, the last preference profile θ^n will have $\forall i \in [n]$:

$$\theta_i^n(a) \prec \theta_i^n(b)$$

Thus by unanimity, we must have:

$$f(\theta^n)(a) \prec f(\theta^n)(b)$$

Moreover, we are guarenteed the existence of an integer $b|a \in [n]$ satisfying:

$$b|a = \min\{k \mid f(\theta^k)(a) \prec f(\theta^k)(b)\}$$

We call the candidate $x_{b|a}$ our "pivot" from a to b. Note $b|a$ is *unique* despite our particular choice of $\theta^0, \theta^1..., \theta^n$ since f satisfies binary independence. Additionally, if $\theta \in \mathcal{O}^n$ is a preference profile so that $\forall i \in [1, b|a)$:

$$\theta_i(a) \succ \theta_i(b)$$

and $\forall i \in (b|a, n]$:

$$\theta_i(a) \prec \theta_i(b)$$

then by construction:

$$f(\theta)(b) \succ f(\theta)(a) \iff \theta_{b|a}(b) \succ \theta_{b|a}(a)$$

2.2.2 The Power of Pivots

Let's introduce a third candidate y_c and suppose we have a preference profile $\tau \in \mathcal{O}^n$ so that $\forall i \in [1, b|a]$:

$$\tau_i(a) \succ \tau_i(b) \succ \tau_i(c)$$

, and $\forall i \in (b|a, n]$:

$$\tau_i(a) \prec \tau_i(c) \prec \tau_i(b)$$

Since $b|a$ is the pivot from a to b, we know that:

$$f(\tau)(a) \succ f(\tau)(b)$$

Also by unanimity:

$$f(\tau)(b) \succ f(\tau)(c)$$

Thus:

$$f(\tau)(a) \succ f(\tau)(b) \succ f(\tau)(c)$$

Now, if voter $x_{b|a}$ switches their order preference so:

$$\tau_{b|a}(b) \succ \tau_{b|a}(a) \succ \tau_{b|a}(c)$$

and the rest of the voters arbitrarily shuffle their preferences between y_b and y_c, then binary independence implies we still have:

$$f(\tau)(a) \succ f(\tau)(c)$$

Moreover, since $b|a$ is the pivot from a to b:

$$f(\tau)(b) \succ f(\tau)(a)$$

Thus:

$$f(\tau)(b) \succ f(\tau)(a) \succ f(\tau)(c)$$

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Finally, if all the voters arbitrarily change their ranking of y_a, then binary independence implies we still have:

$$f(\tau)(b) \succ f(\tau)(c)$$

Thus, so long as voter $x_{b|a}$ prefers candidate y_b to y_c, this dictates $f(\tau)(b) \succ f(\tau)(c)$, *regardless of everyone else's preferences.*

In other words the pivot from a to b gets to force all candidates (except for y_a) below y_b in the aggregate preference if they so chose. Interesting, n'est ce pas?

2.2.3   All Pivots are the Same

Consider a preference profile $\theta \in \mathcal{O}^n$ such that $\forall i \in [1, b|a)$:

$$\theta_i(b) \succ \theta_i(c)$$

and $\forall i \in (b|a, n]$:

$$\theta_i(c) \succ \theta_i(b)$$

As we saw at the end of the last section, since $\theta_{b|a}(b) \succ \theta_{b|a}(c)$ we must have:

$$f(\theta)(b) \succ f(\theta)(c)$$

Thus $b|c \leq b|a$.
Now consider a preference profile $\theta \in \mathcal{O}^n$ such that $\forall i \in [1, b|a)$:

$$\theta_i(c) \succ \theta_i(b)$$

and $\forall i \in [b|a, n]$:

$$\theta_i(b) \succ \theta_i(c)$$

As we saw at the end of the last section, since $\theta_{b|a}(b) \succ \theta_{b|a}(c)$ we must have:

$$f(\theta)(b) \succ f(\theta)(c)$$

Thus $c|b \geq b|a$. Putting this together with the previous inequality:

$$b|c \leq b|a \leq c|b$$

By symmetry we have:

$$c|b \leq c|a \leq b|c$$

Thus:

$$b|c = c|b = a|b = b|a = c|a = a|c$$

Since all pivots are the same, there exists a voter who gets to force each candidate below any other candidate in the aggregate preference. Thus this voter is a dictor, contradicting our assumption that f is non-dictatorial!

3   Probabilistic Arrow's Theorem

Now that we've seen Arrow's theorem proper, we are ready to examine a slightly tweaked version which answers why we can't just take a straight forward average of people's beliefs about an event!

Our proof will follow/combine the results in Mcconway 1981 and Lehrer & Wagner 1983.

3.1   3 Reasonable Demands

Instead of aggregating a group of individuals' preference orders over of set of candidates, we are now going to aggregate their *probability distributions* over a measurable space $(\Omega, \mathcal{F})$.
We require Ω contain more than 2 measurable outcomes (this is analagous to having more than 2 candidates). Also, we will denote the set of all possible measures over $(\Omega, \mathcal{F})$ by $\mathcal{M}$.

In particular, the probability distributions of our n individuals will be characterized by a *opinion pool* $\{P_1, P_2, ...P_n\} \in \mathcal{M}^n$.
A *consensus function* will be a function $C : \mathcal{M}^n \to \mathcal{M}$ which maps a given opinion pool to a unique probability distribution over $(\Omega, \mathcal{F})$.
Once again, we have three seemingly reasonable demands:

First, C satisfies the *independence preservation property* (IPP) iff :

$$(\forall i \in [n], \ P_i(A \cap B) = P_i(A) \cdot P_i(B))$$
$$\implies C(P_1, P_2, ...P_n)(A \cap B) = C(P_1, P_2, ...P_n)(A) \cdot C(P_1, P_2, ...P_n)(B)$$

IPP parallels our previous notion of *unanimity* in that if each individual thinks events A and B are independent, then the aggregate distribution also treats A and B as independent.

Second, C satisfies the *strong setwise function property* (SSFP) iff $\exists G : [0,1]^n \to [0,1]$ such that for any given opinion pool $\{P_1, P_2, ..., P_n\} \in \mathcal{M}^n$:

$$\forall A \in \mathcal{F}, \ C(P_1, P_2, ...P_n)(A) = G(P_1(A), P_2(A), ...P_n(A))$$

SSFP is analogous to *binary independence* since it implies the aggregate probability of a given event is uniquely determined by the probabilities each individual assigns to that event.

Third, C satisfies *non-dictatorship* iff:

$$\neg(\exists i \in [n] \text{ s.t. } \forall \{P_1, P_2, ..., P_n\} \in \mathcal{M}^n, \ \forall A \in \mathcal{F}, \ C(P_1, P_2, ...P_n)(A) = P_i(A))$$

3.2 Proof of the Theorem

As before, we will proceed by showing IPP + SSFP necessarily violate non-dictatorship. However, we need to first establish a crucial and somewhat surprising lemma...

3.2.1 SSFP $\iff$ Consensus is Linear

We claim C satisfies SSFP iff $\exists w_1, w_2, ... w_n \in [0, 1]$ such that $\sum_{i=1}^{n} w_i = 1$ and for any given opinion pool $\{P_1, P_2, ... P_n\} \in \mathcal{M}^n$, $\forall A \in \mathcal{F}$:

$$C(P_1, P_2, ... P_n)(A) = G(P_1(A), P_2(A), ... P_n(A)) = \sum_{i=1}^{n} w_i \cdot P_i(A)$$

Proof:
It trivially follows that linear consensus $\implies$ SSFP so we will only show the other direction.

Consider three disjoint non-empty subsets $A, B, C \in \mathcal{F}$ which partition Ω (we are guarenteed these exist by the assumption Ω has more than 2 measurable outcomes).

Let $(a_1, b_1), (a_2, b_2), ... (a_n, b_n) \in \mathbb{R}_{\geq 0}^2$ so that $\forall i \in [n]$, $a_i + b_i \leq 1$. We construct an opinion pool $\{P_1', P_2', ... P_n'\} \in \bar{\mathcal{M}}^n$ so that:

$$P_i'(A) = a_i$$
$$P_i'(B) = b_i$$
$$P_i'(C) = 1 - (a_i + b_i)$$

By SSFP:

$$
\begin{aligned}
G(a_1 + b_1, a_2 + b_2, ... a_n + b_n) &= G(P_1'(A) + P_1'(B), P_2'(A) + P_2'(B), ... P_n'(A) + P_n'(B)) \\
&= G(P_1'(A \cup B), P_2'(A \cup B), ... P_n'(A \cup B)) \\
&= C(P_1', P_2', ... P_n')(A \cup B) \\
&= C(P_1', P_2', ... P_n')(A) + C(P_1', P_2', ... P_n')(B) \\
&= G(P_1'(A), P_2'(A), ... P_n'(A)) + G(P_1'(B), P_2'(B), ... P_n'(B)) \\
&= G(a_1, a_2, ... a_n) + G(b_1, b_2, ... b_n)
\end{aligned}
$$

In particular, this implies $\forall x = \{x_1, x_2, ... x_n\} \in [0, 1]^n$:

$$G(x_1, x_2, ... x_n) = G(x_1, 0, ... 0) + G(0, x_2, ... x_n) = \sum_{i=1}^{n} G(0, ... 0, x_i, 0, ... 0)$$

More concisely, if we write $G_i(a) = G(\underbrace{0, ... 0}_{i-1 \ 0's}, a, 0, ... 0)$ then:

$$G(x) = \sum_{i=1}^{n} G_i(x_i)$$

Now if $a, b \in [0, 1]$ so that $a + b \in [0, 1]$, then from before:

$$G_i(a + b) = G(0, ...0, a + b, 0, ...0)$$
$$= G(0, ...0, a, 0, ...0) + G(0, ...0, b, 0, ...0)$$
$$= G_i(a) + G_i(b)$$

, i.e. each G_i satisfies the Cauchy functional equation on the unit interval. Note $\forall n \in \mathbb{N}^+$:

$$G_i(x) = G_i\left(\frac{n}{n} \cdot x\right) = n \cdot G_i\left(\frac{x}{n}\right)$$

, i.e.

$$\frac{1}{n} \cdot G_i(x) = G_i\left(\frac{x}{n}\right)$$

Now if $m \in \mathbb{N}_{\geq 0}$ so that $\frac{m}{n} \in \mathbb{Q}_{\geq 0} \cap [0, 1]$ then:

$$G_i\left(\frac{m}{n} \cdot x\right) = m \cdot G_i\left(\frac{x}{n}\right) = \frac{m}{n} \cdot G_i(x)$$

Finally $\forall r \in [0, 1]$ can pick a rational $q_r \in [0, 1]$ arbitrarily close so that $d_r = r - q_r \in [0, \frac{1}{N}]$ for any given $N \in \mathbb{N}$. Note:

$$G_i(r) - G_i(1) \cdot r = G_i(r - q_r + q_r) - G_i(1) \cdot r$$
$$= G_i(r - q_r) + G_i(q_r) - G_i(1) \cdot r$$
$$= G_i(r - q_r) + (q_r - r) \cdot G_i(1)$$
$$= G_i(d_r) - d_r \cdot G_i(1)$$

We know $\text{Im}(G_i) \subseteq [0, 1]$ as its values must be probabilities. Thus $G_i(d_r) \leq \frac{1}{N}$ (otherwise $G_i(N \cdot d_r) = N \cdot G_i(d_r) > 1$) and $G_i(1) \leq 1$. Hence:

$$|G_i(r) - G_i(1) \cdot r| = |G_i(d_r) - d_r \cdot G_i(1)| \leq |G_i(d_r)| + d_r \cdot |G_i(1)| \leq \frac{2}{N}$$

Taking the limit as $N \to \infty$ yields:

$$G_i(r) = G_i(1) \cdot r$$

Thus $\forall i \in [n]$, G_i is of the form:

$$G_i(x) = w_i \cdot x$$

where $w_i \in [0, 1]$.

Putting things together, this implies for any given opinion pool $\{P_1, P_2, ...P_n\} \in \mathcal{M}^n$, $\forall A \in \mathcal{A}$:

$$C(P_1, P_2, ...P_n)(A) = G(P_1(A), P_2(A), ...P_n(A))$$
$$= \sum_{i=1}^{n} G_i(P_i(A))$$
$$= \sum_{i=1}^{n} w_i \cdot P_i(A)$$

In particular, when $A = \Omega$ we have:

$$1 = C(P_1, P_2, ... P_n)(\Omega) = \sum_{i=1}^{n} w_i \cdot P_i(\Omega) = \sum_{i=1}^{n} w_i$$

as desired. Phew! □

3.2.2   IPP + SSFP $\implies$ Dictatorship

As before, consider three disjoint non-empty subsets $A, B, C \in \mathcal{F}$ which partition Ω. By the above, SSFP implies $\exists w_1, w_2 ... w_n \in [0, 1]$ summing to 1 so that for any given opinion pool $\{P_1, P_2, ... P_n\} \in \mathcal{M}^n$:

$$C(P_1, P_2, ... P_n)(A) = \sum_{i=1}^{n} w_i P_i(A)$$

$$C(P_1, P_2, ... P_n)(B) = \sum_{i=1}^{n} w_i P_i(B)$$

$$C(P_1, P_2, ... P_n)(C) = \sum_{i=1}^{n} w_i P_i(C)$$

Since not all w_i can be zero we know $\exists k \in [n]$ so that $w_k > 0$. We construct an opinion pool $\{P_1', P_2', ... P_n'\}$ so that:

$$(P_k'(A), \ P_k'(B), P_k'(C)) = \left(0, \frac{1}{2}, \frac{1}{2}\right)$$

and $\forall i \in [n] \backslash \{k\}$:

$$(P_i'(A), \ P_i'(B), P_i'(C)) = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$$

Note:

$$C(P_1', P_2', ... P_n')(A) = \frac{1 - w_k}{2}$$

$$C(P_1', P_2', ... P_n')(B) = \frac{1}{2}$$

$$C(P_1', P_2', ... P_n')(C) = \frac{w_k}{2}$$

Consider the events $S = A \cup B$ and $T = B \cup C$. By the above:

$$C(P_1', P_2', ... P_n')(S) = \frac{1 - w_k}{2} + \frac{1}{2} = 1 - \frac{w_k}{2}$$

$$C(P_1', P_2', ... P_n')(T) = \frac{1}{2} + \frac{w_k}{2}$$

Since $P'_k(S) = P'_k(A) + P'_k(B) = \frac{1}{2}$ and $P'_k(T) = P'_k(B) + P'_k(C) = 1$:

$$P'_k(S \cap T) = P'_k(B) = \frac{1}{2} = \frac{1}{2} \cdot 1 = P'_k(S) \cdot P'_k(T)$$

Additionally, since $\forall i \in [n]\setminus\{k\}$, $P'_i(S) = P'_i(A) + P'_i(B) = 1$ and $P'_i(T) = P'_i(B) + P'_i(C) = \frac{1}{2}$:

$$P'_i(S \cap T) = P'_i(B) = \frac{1}{2} = 1 \cdot \frac{1}{2} = P'_i(S) \cdot P'_i(T)$$

Thus by IPP, we must have:

$$C(P'_1, P'_2, ...P'_n)(B) = C(P'_1, P'_2, ...P'_n)(S \cap T) = C(P'_1, P'_2, ...P'_n)(S) \cdot C(P'_1, P'_2, ...P'_n)(T)$$

Hence:

$$\frac{1}{2} = \left(1 - \frac{w_k}{2}\right) \cdot \left(\frac{1}{2} + \frac{w_k}{2}\right)$$

The only solutions to the above quadratic are $w_k = 0$ and $w_k = 1$. However, $w_k > 0$ by assumption, implying $w_k = 1$ and all other $w_i = 0$. This means for any given opinion pool $\{P_1, P_2, ...P_n\} \in \mathcal{M}^n$, $\forall A \in \mathcal{F}$:

$$C(P_1, P_2, ...P_n)(A) = \sum_{i=1}^{n} w_i \cdot P_i(A) = P_k(A)$$

In other words, we have found our dictator! Thus, given more than 2 measurable outcomes, *no* consensus function can satisfy IPP, SSFP, and non-dictatorship simultaneously.

Given the above impossibility result, how are we then supposed aggregate beliefs surrounding the odds of an event? Note that in our "probabilistic Arrow's theorem", the agents' beliefs are *static*. Thus a clever way to side step the problem of aggregating differing beliefs might be to instead let the agents interact with one another, causing their beliefs to become *dynamic*, and hope they all agree with each other at the end. Indeed, a generalization of Aumann's agreement theorem (which we will now explore) states that this will exactly be the case when two given agents are interacting...

4 Aumann's Agreement Theorem

Aumann's agreement theorem was first articulated by Robert Aumann in a paper titled "Agreeing to Disagree" which asserts that "people with the same priors cannot agree to disagree" (Aumann 1976).

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, and let Q^a, Q^b be partitions of Ω such that $Q^a \vee Q^b$ (the coarsest common refinement) consists of non-null events. $(\Omega, \mathcal{F}, \mu)$ is our space of states of the world.

μ is a prior common to agent a and agent b.
$Q^a = \{Q_1^a, Q_2^a, ... Q_K^a\}$ is the information partition of agent a.
$Q^b = \{Q_1^b, Q_2^b, ... Q_L^b\}$ is the information partition of agent b.

If $\omega \in \Omega$ is the true state of the world then agent i is informed of $Q^i(\omega)$, which is the element of Q^i that contains ω.
For an event $E \in \mathcal{F}$, if we have $E \subseteq Q^i(\omega)$, then we say agent i knows E.
Additionally, if E includes that member of the meet $Q^a \wedge Q^b$ (the finest common coarsening) which contains ω then we call E common knowledge.
More informally, an event E is common knowledge if a knows E, b knows E, a knows that b knows E, b knows that a knows E etc.

Let A be an event and let $q^i(\omega) = \frac{P(A \cap Q^i(\omega))}{P(Q^i(\omega))}$ be agent i's posterior.
If it is common knowledge at ω that $q^i(\omega) = q_i$ then let $P(\omega)$ be that member of the meet which contains ω. Note we can write:

$$P(\omega) = \cup_j \ Q_j^a$$

where the $Q_j^a \in Q^a$ partition $P(\omega)$. Hence:

$$\mu(A \mid P(\omega)) = \frac{\mu(\cup_j \ A \cap Q_j^a)}{\mu(\cup_j \ Q_j^a)} = \frac{\sum_j \mu(Q_j^a) \cdot \mu(A \mid Q_j^a)}{\sum_j \mu(Q_j^a)} = \frac{\sum_j \mu(Q_j^a) \cdot q_a}{\sum_j \mu(Q_j^a)} = q_a$$

by symmetry it also follows that:

$$\mu(A \mid P(\omega)) = q_b$$

thus:

$$q_a = q_b$$

Hence two agents cannot agree (have common knowledge of posteriors) to disagree (which are unequal).

4.1   Bringing Things to Life

We will now examine a generalization of Aumann's theorem by Geanakoplos and Polemarchakis 1982. Continuing with the notation above agents a and b will interact dynamically as follows:
Step 1- Agent a announces their posterior:

$$q_1^a = \frac{\mu(A \cap Q^a(\omega))}{\mu(Q^a(\omega))}$$

, meaning a could be in partitions:

$$a_1 = \{k \mid \frac{\mu(A \cap Q_k^a)}{\mu(Q_k^a)} = q_1^a\}$$

Agent b will then adjust their posterior and announce it as:

$$q_1^b = \frac{\mu(A \cap (Q^b(\omega) \cap (\cup_{k \in a_1} Q_k^a)))}{\mu(Q^b(\omega) \cap (\cup_{k \in a_1} Q_k^a))}$$

meaning b could be in partitions:

$$b_1 = \{l \mid \frac{\mu(A \cap (Q_l^b \cap (\cup_{k \in a_1} Q_k^a)))}{\mu(Q_l^b \cap (\cup_{k \in a_1} Q_k^a))} = q_1^b\}$$

Step t- Agent a will then adjust their posterior and announce it as:

$$q_t^a = \frac{\mu(A \cap (Q^a(\omega) \cap (\cup_{l \in b_{t-1}} Q_l^b)))}{\mu(Q^a(\omega) \cap (\cup_{l \in b_{t-1}} Q_l^b))}$$

, meaning a could be in partitions:

$$a_t = \{k \in a_{t-1} \mid \frac{\mu(A \cap (Q_k^a \cap (\cup_{l \in b_{t-1}} Q_l^b)))}{\mu(Q_k^a \cap (\cup_{l \in b_{t-1}} Q_l^b))} = q_t^a\}$$

Agent b will then adjust their posterior and announce it as:

$$q_t^b = \frac{\mu(A \cap (Q^b(\omega) \cap (\cup_{k \in a_t} Q_k^a)))}{\mu(Q^b(\omega) \cap (\cup_{k \in a_t} Q_k^a))}$$

meaning b could be in partitions:

$$b_t = \{l \in b_{t-1} \mid \frac{\mu(A \cap (Q_l^b \cap (\cup_{k \in a_t} Q_k^a)))}{\mu(Q_l^b \cap (\cup_{k \in a_t} Q_k^a))} = q_t^b\}$$

Importantly note that q_t^a is a function of b_{t-1}, a_t is a function of a_{t-1}, b_{t-1} and q_t^a, and q_t^b is a function of a_t. Thus if $a_t = a_{t-1}$ and $b_t = b_{t-1}$ then $q_{t+1}^a = q_t^a$, $a_{t+1} = a_t$ and $q_{t+1}^b = q_t^b$. Since b_t is a function of b_{t-1}, a_t, and q_t^b, it also follows that $b_{t+1} = b_t$. In other words, if $a_t = a_{t-1}$ and $b_t = b_{t-1}$ then our agents have reached a communication equilibrium.

Moreover, since $\forall i \in \mathbb{N}$, $(a_i \supseteq a_{i+1}) \wedge (b_i \supseteq b_{i+1})$ and $\max |a_1| = K$ and $\max |b_1| = L$, we know $\exists j \in [K + L]$ s.t.:

$$(a_j = a_{j+1}) \wedge (b_j = b_{j+1})$$

Thus we are guaranteed convergence.

To illustrate the point consider the following example where $(\Omega, \mathcal{F}, \mu) = ([8], 2^{[8]}, \#)$, $\omega = 1$ and:

$$Q^a = \{\{1, 2, 3, 4, 6\}, \{5, 7, 8\}\}$$
$$Q^b = \{\{1, 3, 5, 7\}, \{2, 4, 6, 8\}\}$$

Additionally, suppose the event being communicated about is $A = \{3, 4\}$.

Initially, a will report a posterior of $\frac{2}{5}$. Then, b will realize a must be in the partition $\{1,2,3,4,5,6\}$ and the only possible true states of the world are $\{1,3\}$. Thus b will report a posterior of $\frac{1}{2}$. Next, a will realize b must have been in the partition $\{1,3,5,7\}$ since otherwise b would have reported a posterior of $\frac{1}{3}$. Subsequently, a will also report a posterior of $\frac{1}{2}$ and a communication equilibrium will be reached since each agent knows the partition of the other.

Just for fun, what would have occurred if b started? Well the initial posterior reported would be $\frac{1}{4}$. Since to a, this probability is consistent with both of b's partitions, a will still think $\{1,2,3,4,6\}$ are the possible true states of the world. Thus a will report a posterior of $\frac{2}{5}$ and the process proceeds as outlined above.

Odd that our agents posteriors converged to the same thing no?
Suppose at some step at some step we have $a_{t+1} = a_t$ and $b_{t+1} = b_t$ (as we are guaranteed). Since $\forall k \in a_{t+1} = a_t$:

$$\mu(A \cap (Q_k^a \cap (\cup_{l \in b_t} Q_l^b))) = q_{t+1}^a \cdot \mu(Q_k^a \cap (\cup_{l \in b_t} Q_l^b))$$

and $\forall l \in b_{t+1} = b_t$:

$$\mu(A \cap (Q_l^b \cap (\cup_{k \in a_t} Q_k^a))) = q_{t+1}^b \cdot \mu(Q_l^b \cap (\cup_{k \in a_t} Q_k^a))$$

we have:

$$\sum_{k \in a_t} \mu(A \cap (Q_k^a \cap (\cup_{l \in b_t} Q_l^b))) = q_{t+1}^a \cdot \sum_{k \in a_t} \mu(Q_k^a \cap (\cup_{l \in b_t} Q_l^b))$$

and:

$$\sum_{l \in b_t} \mu(A \cap (Q_l^b \cap (\cup_{k \in a_t} Q_k^a))) = q_{t+1}^b \cdot \sum_{l \in b_t} \mu(Q_l^b \cap (\cup_{k \in a_t} Q_k^a))$$

Hence:

$$\frac{\mu(A \cap (\cup_{k \in a_t} Q_k^a \cap (\cup_{l \in b_t} Q_l^b)))}{\mu(\cup_{k \in a_t} Q_k^a \cap (\cup_{l \in b_t} Q_l^b))} = \frac{\mu(A \cap (\cup_{k \in a_t} \cup_{l \in b_t} Q_k^a \cap Q_l^b))}{\mu(\cup_{k \in a_t} \cup_{l \in b_t} Q_k^a \cap Q_l^b)} = q_{t+1}^a$$

and:

$$\frac{\mu(A \cap (\cup_{l \in b_t} Q_l^b \cap (\cup_{k \in a_t} Q_k^a)))}{\mu(\cup_{l \in b_t} Q_l^b \cap (\cup_{k \in a_t} Q_k^a))} = \frac{\mu(A \cap (\cup_{l \in b_t} \cup_{k \in a_t} Q_l^b \cap Q_k^a))}{\mu(\cup_{l \in b_t} \cup_{k \in a_t} Q_l^b \cap Q_k^a)} = q_{t+1}^b$$

, i.e. $q_{t+1}^a = q_{t+1}^b$

4.2 Common Knowledge Will Equilibriate

Suppose the posteriors of agents a and b are common knowledge in world ω.

Since agent b knows a's initial posteriors, when a announces their posteriors,

b already knows this posterior from their current partition. Thus b will just report their initial posteriors.

Since a knows b knows a's posteriors, every element of a's partition already indicated the announcement would not eliminate anything from b's initial partition.

Additionally, since a knows b's initial posteriors, a's current partition already contains all relevant information. Thus a will just report their initial posteriors.

Since b knows a knows b knows a's posteriors and b knows a knows b's posteriors, every element of b's partition already indicated the announcement would not eliminate anything from a's initial partition.

Additionally, since b knows a's initial posteriors, b's current partition already contains all relevant information. Thus b will just report their initial posteriors.

Through symmetry, we can continue this process identically ad infinitum. Thus, if the agents' posteriors are common knowledge at ω, then agents a and b can't disagree forever!

5 Prediction Markets

Now that we are heuristically justified in adopting a dynamic approach to aggregation, we are finally ready to see the technology which drives this dynamic– prediction markets! Note that all we need is a mechanism to incentivize traders to report true beliefs and we are good to go since the traders will accordingly change their minds upon listening to one another and submit new reports...

5.1 Scoring Rules

Our description of prediction market's follows that laid out by Hanson 2002. To begin with, the common approach used in belief elicitation for a single individual is to use a *scoring rule*. Let our subject's probability distribution over a disjoint partition of states $\{1, 2, ...N\}$ be represented by the vector $\vec{p}$. If their report is represented by $\vec{r}$ then a scoring rule s awards the subject $s_i(\vec{r})$ dollars in case of event i. A *proper* scoring rule satisfies:

$$\vec{p} = \operatorname{argmax} \mathbb{E}[s(\vec{r})] = \operatorname{argmax} \sum_{i=1}^{N} \vec{p}_i \cdot s_i(\vec{r})$$

and:

$$\mathbb{E}[s(\vec{p})] = \sum p_i \cdot s_i(\vec{p}) \geq 0$$

for any $\vec{p}$. These constraints together imply that a subject has incentive to report their true probability distribution when paid out according to the scoring rule s.

The most natural/common proper scoring rule is the logarithmic scoring rule of the form:
$$a_i + b \cdot \ln(r_i)$$

To see why this scoring rule is proper note a_i and b can be set so that the agent's expected reward for telling the truth is non-negative and:

$$\text{argmax} \sum_{i=1}^{N} p_i \cdot s_i(\vec{r}) = \text{argmax} \sum_{i=1}^{N} p_i \cdot (a_i + b\ln(r_i))$$

$$= \text{argmax} \sum_{i=1}^{N} p_i \cdot a_i + \sum_{i=1}^{N} p_i \cdot b\ln(r_i)$$

$$= \text{argmax}\, b \sum_{i=1}^{N} p_i \cdot \ln(r_i)$$

$$= \text{argmin}\, -b \sum_{i=1}^{N} p_i \cdot \ln(r_i)$$

$$= \text{argmin}\, b \sum_{i=1}^{N} p_i \cdot \ln(p_i) - b \sum_{i=1}^{N} p_i \cdot \ln(r_i)$$

$$= \text{argmin}\, b \sum_{i=1}^{N} p_i \cdot \ln\left(\frac{p_i}{r_i}\right)$$

$$= \text{argmin}\, b \cdot KL(\vec{p} \,||\, \vec{r})$$

$$= \vec{p}$$

5.2   Market Scoring Rules

How do we go about eliciting the beliefs of *multiple traders*? Hypothetically, we could simply use a separate scoring rule for each individual. However, this is expensive. Alternatively, in a market scoring rule, the market makes some initial default report and whenever a future trader submits a new report, they always agree to pay the award of the report last submitted. Thus, given that the nth report was r^n the person who submits the $n + 1$th report expects to make:
$$s_i(r^{n+1}) - s_i(r^n)$$

in case of event i. This person also has incentive to report their true beliefs since $s_i(r^n)$ is given and thus their decision ought to simply maximize $\mathbb{E}[s(r^{n+1})]$. The 0th trader is the market, so if there are N reports submitted, the market only need pay out:

$$\sum_{i=1}^{N} s_i(r^i) - s_i(r^{i-1}) = s_i(r^N) - s_i(r^0)$$

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in case of event i.

If we use a *logarithmic* market scoring rule (LMSR) each person expects to make:

$$b \cdot \ln\left(\frac{r_i^{n+1}}{r_i^n}\right)$$

in case of event i and:

$$\sum_{i=1}^{k} b \cdot r_i^{n+1} \ln\left(\frac{r_i^{n+1}}{r_i^n}\right) = b \cdot KL(r^{n+1} \mid\mid r^n)$$

overall. That is to say, each trader in a prediction market using a LMSR expects to profit proportional to the surprisal between their report and the report before them.

Moreover, if the market starts with an initial report that is uniform over all k outcomes, then worst case scenario it will need to pay out:

$$\max s_i(r^f) - s_i(r^0) = \max b \cdot \left(\ln(r_i^f) - \ln\left(\frac{1}{N}\right)\right) = \max b \cdot \ln\left(\frac{r_i^f}{1/N}\right) = b \cdot \ln(N)$$

which can be thought of as the subsidy necessary to run the market.

5.3   Reneging

The logarithmic market scoring rule provides us with a tool which creates incentives for agents to tell us the truth during each step of the process in Aumann's agreement theorem. However, this incentive is only myopic. Consider two agents interacting via a LMSR using the example where $\omega = 1$,:

$$Q^a = \{\{1,2,3,4,6\},\{5,7,8\}\}$$
$$Q^b = \{\{1,3,5,7\},\{2,4,6,8\}\}$$

, and $A = \{3,4\}$. Suppose the market's default estimate is $\mu(A) = \frac{1}{2}$ and $\mu(\neg A) = \frac{1}{2}$

If agent a were to start then $q_1^a = \frac{2}{5}$, $q_1^b = \frac{1}{2}$, and $q_2^a = \frac{1}{2}$. In this case, after adjusting their initial prior, agent a would expect to make an amount proportional to:

$$\frac{1}{2} \cdot \left(\ln\left(\frac{2/5}{1/2}\right) + \ln\left(\frac{3/5}{1/2}\right)\right) \approx -0.0204$$

on their first report (i.e. lose money)!

If agent b were to start then $q_1^b = \frac{1}{4}$, $q_1^a = \frac{2}{5}$, and $q_1^b = \frac{1}{2}$. Hence, agent b will eventually expect to lose money on their first report as well since:

$$\frac{1}{2} \cdot \left(\ln\left(\frac{1/4}{1/2}\right) + \ln\left(\frac{3/4}{1/2}\right)\right) \approx -0.1438$$

This is no good! Both agents might refuse to participate if they are extremely risk averse and think there is even a slight chance they lose money a report.

Alternatively, consider the scenario where $\omega = 1$,:

$$Q^a = \{\{1, 2, 3, 4, 6\}, \{5, 7, 8, 9\}\}$$
$$Q^b = \{\{1, 2, 5, 7\}, \{3, 4, 6, 8, 9\}\}$$

, $A = \{3, 6, 9\}$, and the market's default estimate is set to $\mu(A) = \frac{1}{2}$ as before.

If agent b chooses to start and plays truthfully then they will announce a posterior of 0, causing agent a to also revise their posterior to 0. Thus in this case, agent a would expect to make nothing, while agent b would expect to make an amount proportional to:

$$1 \cdot \ln\left(\frac{1}{1/2}\right) = \ln(2)$$

However, agent b could make more by lying at agent a's expense! For instance, if agent b reports a posterior of $\frac{3}{5}$ and agent a assumes agent b is playing truthfully, then a will think b is in the partition $\{3, 4, 6, 8, 9\}$ and report a posterior of $\frac{2}{3}$. Next, agent b could reveal their lie and report a posterior of 0 causing agent a to also revise their posterior to 0. Overall then, agent b would expect to net an amount proportional to:

$$1 \cdot \ln\left(\frac{2/5}{1/2}\right) + 1 \cdot \ln\left(\frac{1}{1/3}\right) = \ln\left(\frac{12}{5}\right)$$

While agent a would expect to net:

$$\ln\left(\frac{1/3}{2/5}\right) = \ln\left(\frac{5}{6}\right)$$

Luckily, we can remedy the myopic nature of the LMSR's incentives if we slightly modify it in the manner outlined below.

When an agent submits a report, they no longer agree to pay the reward of only the prediction before them. Rather, they agree to pay the reward of *some* prediction submitted before them. Firstly, this maintains the incentive to submit predictions since an agent expects to make money if the prediction they pay is different from theirs and lose nothing if it is identical. Secondly, this allows agents who change their beliefs to safely *renege* (or withdraw) prior reports. Thus, we can quell agent's fears about retroactively expecting to lose money by announcing their predictions and also protect agents from being lied to!

6 Rational Inattention and Prediction Markets

If we are designing a prediction market with the intent of agents incorporating each others beliefs into their reports (as they do in Aumann's model of interaction), then we might wonder how to incentivize such behavior. Alternatively, if

we have a group of experts who can be negatively influenced by outside infor-
mation, we might wonder how to prevent the acquisition of extra information.
If a scoring rule disincentivizes the acquisition of outside information we call it
robust (Tsakas 2020). Alternatively, if a scoring rule incentivizes information
acquisition we call it *frail*. Luckily, there is a straight forward way to control
whether a LMSR is robust or frail, but before visiting this we will need to briefly
discuss *rational inattention*.

Rational inattention models attempt to formalize what economist Herbert A.
Simon called 'bounded rationality.' Simon argued that our decision making ca-
pabilities are limited by a finite capacity to process information and that "a
wealth of information creates a poverty of attention, and a need to allocate that
attention efficiently" (1971). By simply assigning a cost to information process-
ing, rational inattention models are able to explain the emergent properties of
bounded rationality across a wide range of empirical phenomena. In the words
of the rational inattention's pioneer, Christopher Sims, the theory's approach
is "to construct optimizing-agent models that are consistent with people not
using freely available information" so as to "explain why some freely available
information is not used, or imperfectly used" (2010).

6.1 Mutual Information Costs

Consider an event S which takes on some value $s \in \{s_1, s_2, ...s_N\}$. Suppose an
agent's beliefs about S assigns probability $\mathbb{P}(S = s_n)$ to each outcome. Addi-
tionally suppose the agent can run an experiment X which takes on some value
$x \in \{x_1, x_2, ...x_M\}$ to gain information about S and revise their probabilities to
$\mathbb{P}(S = s_i \mid X = x)$. By running the experiment X, the agent expects to gain:

$$
\begin{aligned}
H(S) - H(S \mid X) &= \sum_n P(S = s_n) \cdot \ln\left(\frac{1}{P(S = s_n)}\right) \\
&\quad - \mathbb{E}\left[\sum_n P(S = s_n \mid X = x) \cdot \ln\left(\frac{1}{P(S = s_n \mid X = x)}\right)\right] \\
&= I(S; X)
\end{aligned}
$$

bits of information about S. In other words, the amount of information X
reveals about S is simply the mutual information between S and X. Rational
inattention models assign the cost of an agent running such an experiment to
be:

$$c \cdot I(S; X)$$

,i.e. the cost of running of an experiment is \$$c$ per bit of information that the
agent pays attention to.

6.2 Mutual Information Benefits

Continuing the above, suppose our agent is being paid out according to an LMSR for reporting their beliefs and the most recent report made assigns $\mathbb{P}'(S = s_n)$ to each outcome. Then if our agent reports their beliefs without running the experiment X, they will expect to make:

$$b \cdot KL(\mathbb{P}_S, \mathbb{P}'_S) = \sum_n b \cdot \mathbb{P}(S = s_n) \ln\left(\frac{\mathbb{P}(S = s_n)}{\mathbb{P}'(S = s_n)}\right)$$

$$= \sum_n b \cdot \mathbb{P}(S = s_n) \ln\left(\frac{1}{\mathbb{P}'(S = s_n)}\right) - \sum_n b \cdot \mathbb{P}(S = s_n) \ln\left(\frac{1}{\mathbb{P}(S = s_n)}\right)$$

$$= \sum_n b \cdot \mathbb{P}(S = s_n) \ln\left(\frac{1}{\mathbb{P}'(S = s_n)}\right) - H(S)$$

However, if our agent runs the experiment and discovers $X = x$ and reports their new probabilities, they will expect to make:

$$b \cdot KL(\mathbb{P}_{S \mid X=x}, \mathbb{P}'_S) = \sum_n b \cdot \mathbb{P}(S = s_n \mid X = x) \ln\left(\frac{\mathbb{P}(S = s_n \mid X = x)}{\mathbb{P}'(S = s_n)}\right)$$

Thus, by running the experiment, on average they will expect to make:

$$\mathbb{E}\left[b \cdot KL(\mathbb{P}_{S \mid X}, \mathbb{P}'_S)\right] = \sum_m \mathbb{P}(X = x_m) \sum_n b \cdot \mathbb{P}(S = s_n \mid X = x_m) \ln\left(\frac{\mathbb{P}(S = s_n \mid X = x_m)}{\mathbb{P}'(S = s_n)}\right)$$

$$= \sum_m \sum_n b \cdot \mathbb{P}(S = s_n, X = x_m) \ln\left(\frac{\mathbb{P}(S = s_n \mid X = x_m)}{\mathbb{P}'(S = s_n)}\right)$$

$$= \sum_m \sum_n b \cdot \mathbb{P}(S = s_n, X = x_m) \ln\left(\frac{1}{\mathbb{P}'(S = s_n)}\right)$$

$$- \sum_m \sum_n b \cdot \mathbb{P}(S = s_n, X = x_m) \ln\left(\frac{1}{\mathbb{P}(S = s_n \mid X = x_m)}\right)$$

$$= \sum_n \sum_m b \cdot \mathbb{P}(S = s_n, X = x_m) \ln\left(\frac{1}{\mathbb{P}'(S = s_n)}\right) - b \cdot H(S \mid X)$$

$$= \sum_n b \cdot \mathbb{P}(S = s_n) \ln\left(\frac{1}{\mathbb{P}'(S = s_n)}\right) - b \cdot H(S \mid X)$$

Hence on average, running the experiment will increase the amount the agent expects to make by:

$$\mathbb{E}\left[b \cdot KL(\mathbb{P}_{S \mid X}, \mathbb{P}'_S)\right] - b \cdot KL(\mathbb{P}_S, \mathbb{P}'_S) = b \cdot H(S) - b \cdot H(S \mid X) = b \cdot I(S; X)$$

, i.e. they will benefit \$$b$ per bit of information gained from the experiment.

6.3   Cost Benefit Analysis

When an agent assesses whether or not to run an experiment, they will weigh whether the expected benefits of running the experiment outweigh the expected costs. In other words, they will evaluate whether:

$$b \cdot I(S; X) - c \cdot I(S; X) = (b - c) \cdot I(S; X) > 0$$

Since $I(S; X) \geq 0$ (as mutual information is always non-negative), this boils down to evaluating whether $b > c$. Thus an individual agent will choose to acquire outside information iff the market's award of \$$b$ per bit of information is greater than their personal cost of \$$c$ per bit of information.

Thus if we are running a prediction market with N participants whose information costs are $c_1, c_2, ...c_N$, a *robust* LMSR will set:

$$b < \min_i c_i$$

so that all agents are disinectivized from acquiring additional information.

Similarly, a *frail* LMSR will set:

$$b > \max_i c_i$$

so that all agents are incentivized to acquire additional information.

Naturally, one might wonder how to find the information costs associated with each agent. As suggested in Foley 2012, we can deduce the c_i by running lotteries to find the slope of the agents' logistic quantal response curves. If we assume rationally inattentive behavior, these slopes will directly correspond to the each agent's respective information cost.

7   Thermodynamics and Prediction Markets

Intuitively, it isn't all too surprising that there is a deep connection between prediction markets and thermodynamics given the logarithmic market scoring rule's relationship to information theory. However, to see the full extent of the connection we will need discuss an alternative way of running prediction markets...

7.1   Automated Market Makers

Automated market makers (AMM's) are an elegant technology that exploit the duality between prices and probabilities to run prediction markets (Chen and Vaughan 2010).

Given a disjoint partition of outcomes $\{1, 2, ..., N\}$ for some event, an AMM

buys and sells securities corresponding to each outcome i of the form "pays \$1 if i occurs." Let $\vec{q}$ be the vector whose ith entry represents the number of shares of security i held by the traders. Since the point of an AMM is to easily interpret prices as probabilities, we want security i's price to increase if q_i increases (the traders purchase some amount of security i) and decrease if q_i decreases (the traders sell some amount of security i). However, this means that the price of a security must change for each infinitesimal amount of its shares that are bought or sold. Thus, prediction markets must use a 'cost function' C to dictate how much an traders will have to pay for a specific transaction given the market's current state. For instance, if the market's participants currently hold q_i of each security collectively and a trader wishes to now transact r_i shares of each security, then this action would cost them:

$$C(\vec{q} + \vec{r}) - C(\vec{q})$$

Note if we parameterize the path from $\vec{q}$ to $\vec{q} + \vec{r}$ by some function $a(t)$, then the price of security i at each point along the transaction is given by:

$$p_i(a(t)) = \frac{\partial C}{\partial a_i}$$

There are many different cost functions which can be used to run a prediction market, however the cost function:

$$C(\vec{q}) = b \cdot \ln \sum_{i=1}^{N} e^{q_i/b}$$

is particularly nice since its prices are of the form:

$$p_i(\vec{q}) = \frac{e^{q_i/b}}{\sum_{i=1}^{N} e^{q_i/b}}$$

, i.e. the Boltzmann distribution!

The Boltzmann AMM is equivalent to a LMSR since if the market's current probabilties are given by the above vector of prices and a trader wishes to change them to:

$$p_i(\vec{q'}) = \frac{e^{q_i'/b}}{\sum_{i=1}^{N} e^{q_i'/b}}$$

then in case of event i they will make:

$$
\begin{aligned}
(q_i' - q_i) - (C(\vec{q'}) - C(\vec{q})) &= b \cdot \left(\frac{q_i'}{b} - \ln \sum_{i=1}^{N} e^{q_i'/b} \right) - b \cdot \left(\frac{q_i}{b} - \ln \sum_{i=1}^{N} e^{q_i/b} \right) \\
&= b \cdot \ln \left(\frac{e^{q_i'/b}}{\sum_{i=1}^{N} e^{q_i'/b}} \right) - b \cdot \ln \left(\frac{e^{q_i/b}}{\sum_{i=1}^{N} e^{q_i/b}} \right) \\
&= b \cdot \ln \left(\frac{p_i(\vec{q'})}{p_i(\vec{q})} \right)
\end{aligned}
$$

That is to say, their expected payoff is:

$$b \sum_{i=1}^{N} p_i(\vec{q'}) \cdot \ln\left(\frac{p_i(\vec{q'})}{p_i(\vec{q})}\right) = b \cdot KL(p(\vec{q'}) \parallel p(\vec{q}))$$

7.2 Jarzynski's Equality and The Second Law

Now that we have established the Boltzmann AMM is equivalent to a LMSR, it follows the maximum amount of money the market must pay out to its participants is:

$$b \cdot \ln(N)$$

which can be thought of as the subsidy the market uses for its own funding pool. Additionally since:

$$C(\vec{0}) = b \cdot \ln\left(\sum_{i=1}^{N} e^{0/b}\right) = b \cdot \ln(N)$$

and $C(\vec{q}) - C(\vec{0})$ represents how much money agents must give the market for it to issue $\vec{q}$ of each security, the cost function of a Boltzmann AMM indicates how much money agents have given the market in the course of making bets in addition to the market's own funding pool. Therefore, the amount of money the market has given out at any point in time is:

$$F_{\text{AMM}}(\vec{q}) = -(C(\vec{q}) - b \cdot \ln(N)) = -b \cdot \ln \frac{\sum_{i=1}^{N} e^{q_i/b}}{N} = -b \cdot \ln\langle e^{q_i/b}\rangle$$

This exactly parallels Jarzynski's equality and suggests we should start drawing some analogies! Note that if we interpret the amount of security i held by the traders as the work performed by the AMM in state i, then F_{AMM} (or the amount of money the market has given out) is identical to the equilibrium free energy in a thermodynamic process. This makes sense since the more money the market gives out, the more its capacity to perform work (have securities bought from it).

This also allows us to immediately conclude that the money given out by the market at any given point in time is bounded above by the average number of shares sold to the market for each security i since by Jensen's:

$$\langle e^{q_i/b}\rangle \geq e^{\frac{\sum_{i=1}^{N} q_i/b}{N}}$$

$$\rightarrow \ln\langle e^{q_i/b}\rangle \geq \frac{\sum_{i=1}^{N} q_i/b}{N}$$

$$\rightarrow -b \cdot \ln\langle e^{q_i/b}\rangle \leq -\frac{\sum_{i=1}^{N} q_i}{N}$$

$$\rightarrow F_{\text{AMM}}(\vec{q}) \leq -\frac{\sum_{i=1}^{N} q_i}{N}$$

, giving as a version of the second law of thermodynamics!

7.3 The Second Law for any Transaction

The above version of the second law only applies for a transaction in which there is intially $\vec{0}$ of each security issued.

In general, if the amount of each security sold changes from $\vec{q}$ to $\vec{q}'$ then the average work performed on the market (or the expected number of paying shares the market bought) is:

$$\overline{W} = \sum_{i=1}^{N} -(q_i' - q_i) \cdot p_i(\vec{q}) = \sum_{i=1}^{N} W_i \cdot p_i(\vec{q})$$

and the change in free energy of our market (or the amount of money it gives out in the transaction) is:

$$\Delta F_{\text{AMM}} = \left(-b \cdot \ln\langle e^{q_i/b} \rangle + (C(\vec{q}) - C(\vec{q}')) \right) - \left(-b \cdot \ln\langle e^{q_i/b} \rangle \right)$$

$$= b \cdot \ln \left(\frac{\sum_{i=1}^{N} e^{q_i/b}}{\sum_{i=1}^{N} e^{q_i'/b}} \right)$$

$$= -b \cdot \ln \left(\sum_{i=1}^{N} e^{(q_i'-q_i)/b} \cdot \frac{e^{q_i/b}}{\sum_{j=1}^{N} e^{q_j/b}} \right)$$

$$= -b \cdot \ln \left(\sum_{i=1}^{N} e^{-W_i/b} \cdot p_i(\vec{q}) \right)$$

We wish to show:

$$\Delta F_{\text{AMM}} \leq \overline{W}$$

i.e., the amount of money given out by the market in a transaction is bounded above by the expected number of paying shares the market bought.

Note by Jensen's:

$$\sum_{i=1}^{N} e^{-W_i/b} \cdot p_i(\vec{q}) \geq e^{\sum_{i=1}^{N} -\frac{W_i}{b} \cdot p_i(\vec{q})}$$

$$\rightarrow \ln \left(\sum_{i=1}^{N} e^{-W_i/b} \cdot p_i(\vec{q}) \right) \geq \sum_{i=1}^{N} -\frac{W_i}{b} \cdot p_i(\vec{q})$$

$$\rightarrow -b \cdot \ln \left(\sum_{i=1}^{N} e^{-W_i/b} \cdot p_i(\vec{q}) \right) \leq \sum_{i=1}^{N} W_i \cdot p_i(\vec{q})$$

as desired.

Hence we have recovered the second law of thermodynamics for any arbitrary transaction that takes $\vec{q}$ to $\vec{q}'$!

7.4 Thermodynamic Operations

We can also interpret the quantities of each security i issued at any given point in time as determining the energy landscape of our system.

If a trader's beliefs $\text{Pr}_{\text{trader}}$ disagrees with the current probability distribution given by the AMM's prices Pr_{AMM} then according to their coarse graining the free energy is:

$$F_{\text{trader}} = -b \cdot \ln\langle e^{q_i/b}\rangle + b \cdot KL(\text{Pr}_{\text{trader}} \parallel \text{Pr}_{\text{AMM}})$$

, i.e. they believe the market has given out $b \cdot KL(\text{Pr}_{\text{trader}} \parallel \text{Pr}_{\text{AMM}})$ more dollars than the market believes. This strengthens our analogy since this expression is identical to that of the non-equilibrium free energy in a thermodynamic system. If the trader carries out the appropriate transaction to adjust the market's prices to $\text{Pr}_{\text{trader}}$ (by changing the amount of each security sold from $\vec{q}$ to $\vec{q}'$), then the AMM will believe the change in free energy is:

$$\Delta F_{\text{AMM}} = \left(-b \cdot \ln\langle e^{q_i/b}\rangle + (C(\vec{q}) - C(\vec{q}'))\right) - \left(-b \cdot \ln\langle e^{q_i/b}\rangle\right)$$

$$= b \cdot \ln\left(\frac{\sum_{i=1}^{N} e^{q_i/b}}{\sum_{i=1}^{N} e^{q_i'/b}}\right)$$

and the trader will believe the change in free energy is:

$$\Delta F_{\text{trader}} = \left(-b \cdot \ln\langle e^{q_i/b}\rangle + (C(\vec{q}) - C(\vec{q}'))\right) - \left(-b \cdot \ln\langle e^{q_i/b}\rangle + b \cdot KL(\text{Pr}_{\text{trader}} \parallel \text{Pr}_{\text{AMM}})\right)$$

$$= b \cdot \ln\left(\frac{\sum_{i=1}^{N} e^{q_i/b}}{\sum_{i=1}^{N} e^{q_i'/b}}\right) - b \cdot KL(\text{Pr}_{\text{trader}} \parallel \text{Pr}_{\text{AMM}})$$

This is nice because the first quantity tracks with the change in free energy for a system that starts and ends in equilibrium and the second quantity tracks with the change in free energy for a system that starts out of and ends in equilibrium.

Moreover, we can interpret the discrepancy between the free energies as due to the 'arbitrage oppurtunity' arising from the system being out of equilibrium no longer existing for the trader and never having existed for the market.

Alternatively, suppose our trader waits to adjust the market's prices to their beliefs and another participant comes along in the mean time and adjusts the market's prices to $\text{Pr}_{\text{participant}}$ (by changing the amount of each security sold from $\vec{q}$ to $\vec{q}'$). Then the trader will now believe the change in free energy is:

$$\Delta F_{\text{trader}} = \left(-b \cdot \ln\langle e^{q_i/b}\rangle + (C(\vec{q}) - C(\vec{q}')) + b \cdot KL(\text{Pr}_{\text{trader}} \parallel \text{Pr}_{\text{participant}})\right)$$

$$- \left(-b \cdot \ln\langle e^{q_i/b}\rangle + b \cdot KL(\text{Pr}_{\text{trader}} \parallel \text{Pr}_{\text{AMM}})\right)$$

$$= b \cdot \ln\left(\frac{\sum_{i=1}^{N} e^{q_i/b}}{\sum_{i=1}^{N} e^{q_i'/b}}\right) + (b \cdot KL(\text{Pr}_{\text{trader}} \parallel \text{Pr}_{\text{participant}}) - b \cdot KL(\text{Pr}_{\text{trader}} \parallel \text{Pr}_{\text{AMM}}))$$

which tracks with the change in free energy for a system that starts and ends out of equilibrium.

Thus, we can summarize the basic process underlying prediction markets as below:

$\rightarrow$ [Initial market/landscape has certain amount of free energy corresponding to $\vec{q}$]

$\rightarrow$ [Trader carries out transaction/performs work to profit from extra non-equilibrium free energy]

$\rightarrow$ [Adjustment shifts market/landscape so that free energy corresponds to new $\vec{q'}$]

$\rightarrow$ [Rinse and repeat]

8   Conclusion

Prediction markets are a powerful tool for aggregating opinions which in a sense bypass an Arrow-esque impossibility result. The justification for their use is largely heuristic, but empirical evidence demonstrates that they perform quite well in making forecasts. There is potentially room for improvement in these forecasts which is yet to be explored as described in the section on reneging. The best way to gauge whether allowing reneging will actually improve market performance will be through controlled experiments which I plan on carrying out during the coming year. Additionally, prediction markets have deep connections to information theory and non-equilibrium statistical mechanics, many of which were illustrated above. In particular, viewing the money given out by the market as the free energy of our system turns out to be an extremely fruitful analogy ripe for many other future developments.

9   Works Cited

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