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PROBABILITY AMALGAMATION AND THE INDEPENDENCE ISSUE: A REPLY TO LADDAGA

1. INTRODUCTION

Suppose that a group of n individuals wishes to assign consensual probabilities to a sequence of propositions $s_1, \ldots, s_k$ which are pairwise contradictory ($i \neq j \Rightarrow (s_i \wedge s_j$ is logically false)) and exhaustive ($s_1 \vee \ldots \vee s_k$ is logically true). Suppose that after thorough discussion their opinions as to the most appropriate values of these probabilities are registered in an $n \times k$ matrix $P = (p_{ij})$, where p_{ij} denotes the probability assigned by individual i to proposition s_j. The terms of the problem dictate that the entries of each row of P be nonnegative and sum to one. Initial consensus on the values of the probabilities in question is reflected in a matrix with identical rows.

If consensus fails to obtain initially, there arises the problem of how to amalgamate the opinions in P into group estimates of the probabilities in question. One obvious possibility is to take a weighted arithmetic mean of the entries in each of the columns of P. Implementation of this procedure requires the selection of a sequence of weights $w_1, \ldots, w_n$, nonnegative and summing to one, following which the group assigns proposition s_j, for $j = 1, \ldots, k$, the probability $p_j = w_1 p_{1j} + \cdots + w_n p_{nj}$. Lehrer (1975, 1976) has proposed a method for determining such weights by a process of iterated mutual evaluation among members of the group.[1] The weights so selected are consensual and reflect the group's collective judgment about the expertise of each of its members as an assessor of the probabilities in question.

Laddaga (1977) has criticized Lehrer's model, taking issue not so much with the iterative method for choosing weights, but rather with the basic proposal to amalgamate probabilities by any sort of arithmetic averaging. At the core of Laddaga's complaint is the observation that individuals may assign probabilities in such a way that some pair of propositions turns out on each of their assignments to be independent, while for the group probabilities produced by weighted arithmetic averaging this pair of propositions turns out not to be independent. Laddaga thinks (1977, p. 475) that "prior theoretical concerns usually

Synthese **55** (1983) 339–346. 0039-7857/83/0553-0339 $00.80

determine which events are considered independent", and thus views the aforementioned possibility as a damning defect of arithmetic averaging. Our response to his claim is twofold. First we show, based on recent results of Wagner, that if the group probability assigned to a proposition depends only on the probabilities assigned by individuals to that proposition, then requiring a method of amalgamation always to respect individual attributions of independence allows only "dictatorial" amalgamation, unless one is willing to violate another condition on amalgamation which Laddaga and we both think desirable. Second, we argue that for a large class of probability assessment problems, there is neither a prior theoretical determination of independence nor even much interest in posterior observations that certain propositions are independent. We conclude, *contra* Laddaga, that the failure of an amalgamation method to respect individual attributions of independence is nothing to get excited about.

2. THE AXIOMATICS OF PROBABILITY AMALGAMATION

Suppose, as above, that n individuals are assessing probabilities over a sequence of propositions $s_1, \ldots, s_k$ which are pairwise contradictory and exhaustive. Denote by $\mathcal{P}(n, k)$ the set of all $n \times k$ matrices with nonnegative entries and rows summing to one, and by $\mathcal{P}(k)$ the set of all vectors $(p_1, \ldots, p_k)$ with nonnegative entries summing to one. Members of $\mathcal{P}(n, k)$ correspond to possible "profiles" of individual probability assignments and members of $\mathcal{P}(k)$ to possible group probability assignments. Allowing for the widest possible initial range of amalgamation methods, we make the following definition:

DEFINITION. *A probability amalgamation method* (PAM) *is a function* $F: \mathcal{P}(n, k) \to \mathcal{P}(k)$.

If $P = (p_{ij}) \in \mathcal{P}(n, k)$, we denote $F(P)$ by $(p_1, \ldots, p_k)$, where p_j denotes the group probability assigned to proposition s_j, $j = 1, \ldots, k$. Each such vector $F(P) = (p_1, \ldots, p_k)$ gives rise to a probability measure π defined on arbitrary disjunctions of the "atomic" propositions $s_1, \ldots, s_k$ in the following obvious way: If D is a subset of the index set $\{1, \ldots, k\}$ and $u = \bigvee s_j$, taken over all $j \in D$, one sets $\pi(u) = \sum p_j$, taken over all $j \in D$.[2] Similarly, the probabilities $p_{i1}, \ldots, p_{ik}$ assigned by individual i to propositions $s_1, \ldots, s_k$ give rise to a probability measure π_i by the rule $\pi_i(u) = \sum p_{ij}$, taken over all $j \in D$.

A measure π, derived as above from a PAM F and a matrix P, satisfies all of the probability axioms and is, in this minimal sense, coherent. However, since at this point there are no restrictions on F, the measure π might bear little or no relation to the opinions registered in P. A PAM, as we have defined it, might even ignore a total consensus on the probabilities of the propositions $s_1, \ldots, s_k$. Assuming that we wish the group probability measure π to reflect, in some sense, the opinions in P, there arises the question of what restrictions to place on F in order to achieve this goal.

A restriction which comes immediately to mind is that F ought to respect a consensus on the probabilities assigned to any atomic proposition s_j. Hence, if the entries of the jth column of P are identically equal to some α, then it should be the case that $p_j = \alpha$. Let us adopt as an axiom the following weak version of this restriction:

Z (*Zero Unanimity*): *For all* $P \in \mathcal{P}(n, k)$, *if the* jth *column of* P *consists entirely of zeros, then* $p_j = 0$.

It follows from Z for any proposition u that if $\pi_1(u) = \cdots = \pi_n(u) = 0$, then $\pi(u) = 0$. Recall that a pair of propositions u and t are *mutually exclusive relative to a probability measure* ρ if $\rho(u \wedge t) = 0$.[3] Clearly, a PAM satisfying Z respects individual attributions of mutual exclusivity, in the sense of the following axiom:

RME (*Respect for Individual Attributions of Mutual Exclusivity*): *For any propositions* u *and* t, *if* $\pi_1(u \wedge t) = \cdots = \pi_n(u \wedge t) = 0$, *then* $\pi(u \wedge t) = 0$.

Setting $u = t = s_j$, it is clear that RME implies Z, and so these two axioms are in fact equivalent. Laddaga (p. 474) has endorsed RME as a desirable restriction on probability amalgamation and on this we are in agreement.

Laddaga (pp. 474–475) also believes that probability amalgamation should respect individual attributions of independence. Recall that a pair of propositions u and t are independent *relative to a probability measure* ρ *if* $\rho(u \wedge t) = \rho(u)\rho(t)$. Laddaga thus endorses the following axiomatic restriction on amalgamation:

RI (*Respect for Individual Attributions of Independence*): *For any propositions* u *and* t, *if* $\pi_i(u \wedge t) = \pi_i(u)\pi_i(t)$ *for all* $i = 1, \ldots, n$, *then* $\pi(u \wedge t) = \pi(u)\pi(t)$.

Rather than discussing the reasonableness of RI *per se* at this point, let us identify the PAMS which satisfy both RME and RI. We shall carry out our analysis under the assumption that the group probability assigned to a proposition depends only on the probabilities assigned by individuals to that proposition. We capture this assumption by means of the following axiom of invariance:

IA (*Irrelevance of Alternatives*): *Let* P *and* $P' \in \mathcal{P}(n, k)$ *and denote* $F(P)$ *by* $(p_1, \ldots, p_k)$ *and* $F(P')$ *by* $(p'_p, \ldots, p'_k)$. *For any* $j = 1, \ldots, k$, *if the* jth *column of* P *is identical to the* jth *column of* P', *then* $p_j = p'_j$.

An equivalent way of stating IA is the following: A PAM F satisfies IA if and only if for each atomic proposition s_j there is a function f_j: $[0, 1]^n \to [0, 1]$ such that for each $P = (p_{ij}) \in \mathcal{P}(n, k)$, $F(P) = (p_1, \ldots, p_k)$ where $p_j = f_j(p_{1j}, \ldots, p_{nj})$. Since the group probability assigned to s_j is a function purely of the probabilities assigned by individuals to s_j, the probabilities which they assigned to propositions other than s_j are irrelevant to the determination of p_j. Standing alone, IA allows for the possibility that the aforementioned functions f_j vary with j. However, if there are at least three atomic propositions ($k \geq 3$) and RME (equivalently, Z) is postulated along with IA, then the functions f_j are identically equal to some weighted arithmetic mean:

THEOREM 1. *Let* F: $\mathcal{P}(n, k) \to \mathcal{P}(k)$, *where* $k \geq 3$. *Then* F *satisfies* RME *and* IA *if and only if there exists a sequence of weights* $w_1, \ldots, w_n$, *nonnegative and summing to one, such that for all* $P = (p_{ij}) \in \mathcal{P}(n, k)$, $F(P) = (p_1, \ldots, p_k)$, *where* $p_j = w_1 p_{1j} + \ldots + w_n p_{nj}$ *for each* $j = 1, \ldots, k$.

Several different proofs of Theorem 1 have been discovered (Lehrer and Wagner 1981; Aczél and Wagner 1981; and Aczél, Kannappan, Ng, and Wagner 1982). It follows from this theorem that if RI is postulated in addition to RME and IA, then there is a single individual whose probability assignments are always adopted as the group probability assignments:

THEOREM 2. *Let* F: $\mathcal{P}(n, k) \to \mathcal{P}(k)$, *where* $k \geq 3$. *Then* F *satisfies* RME, RI, *and* IA *if and only if there exists an individual* d *such that for all* $P = (p_{ij}) \in \mathcal{P}(n, k)$, $F(P) = (p_{d1}, \ldots, p_{dn})$.

The proof of Theorem 2 appears in the concluding technical section of this article.[5] This theorem can be recast as an impossibility result if one precludes dictatorial amalgamation by an additional axiom. In either form, its implications for probability amalgamation are oviously substantial.

One might simply conclude from the above that there is no reasonable way to amalgamate individual probability assignments, in effect giving up on the possibility of combining the expertise of a group of individuals. Alternatively, one might take issue with the reasonableness of one or more of the axioms RME, RI, and IA. This is the route we take, and RI is the axiom which we reject.[6] This leaves us with RME and IA, which as indicated in Theorem 1, are nicely (and exclusively) satisfied by the weighted arithmetic means originally proposed by Lehrer as amalgamation functions. Our reason for rejecting RI is that in countless cases independence is simply not of much interest. Suppose, for example,that individuals are assigning probabilities of winning to a set of racehorses. An individual assigns probabilities p_a, p_b, and p_c to horses a, b, and c in such a way that $(p_a + p_b)(p_b + p_c) = p_b$. We point out to him that this entails the independence of the propositions u: a or b wins and t: b or c wins. Is he likely to have the slightest interest in this observation? Could he possibly, à la Laddaga, have formulated a "prior theoretical" commitment to the unwieldy assertion that it is as probable that a or b is the winner, given that b or c is the winner, as it is that a or b is the winner, *tout court*? Suppose that everyone in the group happens to assign probabilities in such a way that u and t turn out to be independent. Are they likely to have the slightest interest in guaranteeing that group probabilities are assigned in such a way that these propositions turn out to be independent?

As with racehorses, so it goes, as in Lehrer's original example, with competing scientific hypotheses and, indeed, with any probability assessment situation in which the initial acts of assessment are directed at the probabilities of a set of pairwise contradictory, exhaustive propositions. In such situations the independence of certain compounds of these propositions is largely fortuitous.[7] Why then should an amalgamation method respect individual attributions of independence, failing consensus about the probability values on which such attributions are based?

3. TECHNICAL APPENDIX

Proof of Theorem 2. By Theorem 1 there is a sequence of weights $w_1, \dots, w_n$, nonnegative and summing to one, such that for all $P =$

$(p_{ij}) \in \mathcal{P}(n, k)$, $F(P) = (p_1, \ldots, p_k)$, where $p_j = w_1 p_{1j} + \cdots + w_n p_{nj}$, $j = 1, \ldots, k$. Clearly, at least one of these weights, call it w_d, is positive. We show in fact that $w_d = 1$ and hence that all remaining weights are zero, thus establishing the desired result.

Consider the matrix $P = (p_{ij})$, defined as follows: The first three entries of row d are 0, 1/2, and 1/2, and the remaining entries in that row, if any, are 0. The first three entries of each of the remaining rows are 1/2, 1/2, and 0, and the remaining entries in those rows, if any, are zero. Let $u = s_1 \vee s_2$ and $t = s_2 \vee s_3$. Then $\pi_i(u \wedge t) = 1/2 = \pi_i(u)\pi_i(t)$ for $i = 1, \ldots, n$. Hence by RI we must have $\pi(u \wedge t) = \pi(u)\pi(t)$. Since $\pi(u) = \pi(s_1) + \pi(s_2) = 1/2(1 - w_d) + 1/2 = 1 - 1/2\, w_d$, $\pi(t) = \pi(s_2) + \pi(s_3) = 1/2 + 1/2\, w_d$, and $\pi(u \wedge t) = \pi(s_2) = 1/2$, we have

$$(1 - 1/2\, w_d)(1/2 + 1/2\, w_d) = 1/2.$$

This quadratic equation in w_d has as its two solutions $w_d = 1$ and $w_d = 0$. Since by assumption $w_d > 0$, it follows that $w_d = 1$, as desired.

Remark. The foregoing proof exploits what might be viewed by some as an unusual case of independence. But it is easy to show that if $k = 3$ this sort of independence, or something much like it, is the only kind which can arise. (If s_1, s_2, and s_3 are assigned probabilities p, q, and $1 - p - q$, and $u = s_1 \vee s_2$ and $t = s_2 \vee s_3$ are independent, than $(p + q)(1 - p) = q$, and hence $p(1 - p - q) = 0$. Thus $p = 0$ or $p + q = 1$, as in the case which we exploited.)

If $k \geq 4$, we may construct a proof exploiting an ordinary case of independence which avoids any assignment of the extreme probabilities 0 and 1. For example, following the pattern of the above proof, we may consider the matrix P, where row d consists of the entries 1/9, 2/9, 4/9, $2/9(k-3), \ldots, 2/9(k-3)$, and all remaining rows consist of the entries 4/9, 2/9, 1/9, $2/9(k-3), \ldots, 2/9(k-3)$. For $u = s_1 \vee s_2$ and $t = s_2 \vee s_3$, we have $\pi_i(u \wedge t) = 2/9 = \pi_i(u) \cdot \pi_i(t)$, $i = 1, \ldots, n$. Hence by RI, $\pi(u \wedge t) = \pi(u) \cdot \pi(t)$, from which it follows that

$$(2/3 - 1/3\, w_d)(1/3 + 1/3\, w_d) = 2/9,$$

and thus that $w_d = 1$.

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NOTES

[1] Earlier proposals along similar lines appear in French (1956), Harary (1959), and De Groot (1974). See also Stone (1961). Refinements of Lehrer's elementary model appear in Wagner (1978, 1980, 1981), and in Lehrer and Wagner (1981).
[2] In particular $\pi(s_j) = p_j$ and $\pi(u) = 0$ if u is logically false, i.e., an "empty" disjunction.
[3] In particular, if $u \wedge t$ is logically false, then u and t are mutually exclusive relative to any probability measure. However, if u and t are mutually exclusive relative to some ρ, $u \wedge t$ need not be logically false.
[4] Aczél, Kannappan, and Ng have given a complete description, for the case $k = 2$, of those PAMs satisfying IA (with identical f_j) and RMU. Suffice it to say here that when $k = 2$ a wide variety of nonlinear amalgamation methods satisfy these axioms. Details appear in Lehrer and Wagner (1981), and Aczél, Kannappan, Ng, and Wagner (1982).
[5] A weaker version of Theorem 2, based on the assumption that all probabilities, including conditional probabilities, are amalgamated by means of a single function f, appears in Dalkey (1972, 1975). Dalkey's proof does not apply when $k = 2$, although he does not point this out explicitly. For an interesting discussion of Dalkey's results see McConway (1981).
[6] It would naturally be of theoretical interest to describe the PAMs satisfying just RME and RI. This is likely to be a very difficult task, however, since, without IA, the group probability assigned to each s_j might depend on *every* entry of P.
[7] Some authors of elementary probability texts emphasize this point by using the term "stochastic independence", hoping thereby to preclude automatic identification of the identity $\pi(u \wedge t) = \pi(u)\pi(t)$ with some higher level assertion about u and t. Of course there are situations, such as those involving repeated trials, where a prior attribution of independence is significant. In such cases arithmetic averaging may involve certain puzzling anomalies. See Dalkey (1972, 1975). In some such cases it is not clear what the appropriate method of amalgamation is. In others, where, for example, a class of "natural conjugate distributions" is being amalgamated, superior alternatives to arithmetic averaging have been developed. See Winkler (1968) for a penetrating analysis of the latter problem.

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