UNIT - I SOLUTION OF EQUATIONS AND EIGENVALUE PROBLEMS :

Solution of equation - Fixed point iteration : $x = g(x)$ method - Newton's method - solution of linear system by Gaussian elimination and Gauss - Jordan method - iterative method - Gauss-Seidel method - Inverse of a matrix by Gauss Jordan method - Eigen value of a matrix by power method and by Jacobi method for symmetric matrix.

Fixed point iteration : $x = g(x)$ method : (or) Iteration method:

Let $f(x) = 0$ be the given equation whose roots are to be determined.

In this iteration method, first we write the given equation in the form $x = \phi(x)$.

Let $x = x_0$ be an initial approximation of the required root α, then the first approximation x_1 is given by $x_1 = \phi(x_0)$

The second, third, etc approximation are given by

$$x_2 = \phi(x_1)$$
$$x_3 = \phi(x_2)$$
$$x_4 = \phi(x_3)$$
$$\cdots\cdots$$
$$\cdots\cdots$$
$$x_n = \phi(x_{n-1})$$

Here x_n is the n^{th} iteration and the value of x_n gives the root of the given equation at the n^{th} iteration.

Condition for the convergence of the iteration method for solving $x = \phi(x)$ is $|\phi'(x)| < 1$ in the range.

Order of convergence for fixed point iteration is 1.

The convergence is linear.

Problems:

1. Find a real root of the equation $x^3 + x^2 - 1 = 0$ by iteration method.

Solution:

Let $f(x) = x^3 + x^2 - 1$

$f(0) = (0)^3 + (0)^2 - 1 = -1 = -ve$

$f(1) = (1)^3 + (1)^2 - 1 = 1 + 1 - 1 = 2 - 1 = 1 = +ve$

Hence a real root lies between 0 and 1.

Now, $x^3 + x^2 - 1 = 0$ can be written as

$$x^2(x+1) - 1 = 0$$

$$x^2(x+1) = 1$$

$$x^2 = \frac{1}{x+1}$$

$$x = \frac{1}{\sqrt{x+1}}$$

2

$$\phi(x) = \frac{1}{\sqrt{x+1}}$$

$$\phi(x) = \frac{1}{(x+1)^{1/2}}$$

$$\phi(x) = (x+1)^{-1/2}$$

$$\phi'(x) = -\frac{1}{2}(x+1)^{-1/2-1} = -\frac{1}{2}(x+1)^{-3/2}$$

$$|\phi'(x)| = \left|\frac{1}{2}(x+1)^{-3/2}\right|$$

$$= \left|\frac{1}{2}\frac{1}{(x+1)^{3/2}}\right|$$

$$\therefore |\phi'(x)| = \left|\frac{1}{2(x+1)^{3/2}}\right|$$

When $x = 0.5$, $|\phi'(0.5)| = \left|\frac{1}{2(0.5+1)^{3/2}}\right|$

$$= \left|\frac{1}{2(1.5)^{3/2}}\right| = \left|\frac{1}{2(1.8371)}\right|$$

$$= \left|\frac{1}{3.6742}\right| = 0.2722$$

$\therefore |\phi'(x)| < 1$ in $(0,1)$

Let the initial approximation be $x_0 = 0.5$

$$x_1 = \phi(x_0) = \frac{1}{\sqrt{x_0+1}} = \frac{1}{\sqrt{0.5+1}} = \frac{1}{1.2245} = 0.81649$$

$$x_2 = \phi(x_1) = \frac{1}{\sqrt{x_1+1}} = \frac{1}{\sqrt{0.81649+1}} = \frac{1}{1.34777} = 0.74196$$

$$x_3 = \phi(x_2) = \frac{1}{\sqrt{x_2+1}} = \frac{1}{\sqrt{0.74196+1}} = \frac{1}{1.31988} = 0.75767$$

$$x_4 = \phi(x_3) = \frac{1}{\sqrt{x_3+1}} = \frac{1}{\sqrt{0.75767+1}} = \frac{1}{1.32577} = 0.75428$$

$$x_5 = \phi(x_4) = \frac{1}{\sqrt{x_4+1}} = \frac{1}{\sqrt{0.75428+1}} = \frac{1}{1.32449} = 0.75501$$

$$x_6 = \phi(x_5) = \frac{1}{\sqrt{x_5+1}} = \frac{1}{\sqrt{0.75501+1}} = \frac{1}{1.32477} = 0.75485$$

$$x_7 = \phi(x_6) = \frac{1}{\sqrt{x_6+1}} = \frac{1}{\sqrt{0.75485+1}} = \frac{1}{1.32477} = 0.75485$$

$\therefore$ The value of x_6 and x_7 are equal.

$\therefore$ The real root of the given equation is 0.75485.

2. Solve by iteration method $2x - \log_{10} x = 7$.

Solution:

Given $2x - \log_{10} x = 7$

$$2x - \log_{10} x - 7 = 0$$

Let $f(x) = 2x - \log_{10} x - 7$

$f(1) = 2(1) - \log_{10}(1) - 7 = -5 = -ve$

$f(2) = 2(2) - \log_{10}(2) - 7 = -3.3010 = -ve$

$f(3) = 2(3) - \log_{10}(3) - 7 = -1.4771 = -ve$

$f(4) = 2(4) - \log_{10}(4) - 7 = 0.39794 = +ve$

Hence a real root lies between 3 and 4.

Now, $2x - \log_{10} x - 7 = 0$ can be written as

(3)

$$2x = \log_{10} x + 7$$

$$x = \frac{1}{2}\left[\log_{10} x + 7\right]$$

$$\phi(x) = \frac{1}{2}\left[\log_{10} x + 7\right]$$

$$\phi(x) = \frac{1}{2}\left[\log_e x \cdot \log_{10} e + 7\right]$$ $\left\{ \because \text{Change base rule } \log_{10} x = \log_e x \cdot \log_{10} e \right\}$

$$\phi'(x) = \frac{1}{2}\left[\frac{1}{x} \cdot \log_{10} e + 7\right]$$

$$\phi'(x) = \frac{1}{2}\left[\frac{1}{x}(0.434)\right]$$

$$|\phi'(x)| = \left|\frac{1}{2}\left[\frac{1}{x}(0.434)\right]\right|$$

When $x = 3.7$ lies in $(3,4)$

$$|\phi'(3.7)| = \left|\frac{1}{2}\left[\frac{1}{3.7}(0.434)\right]\right|$$

$$= |0.0586| < 1$$

$\therefore |\phi'(3.7)| < 1$

$|\phi'(x)| < 1$ lies in $(3,4)$

Let the initial approximation be $x_0 = 3.7$

$$x_1 = \phi(x_0) = \phi(3.7) = \frac{1}{2}\left[\log_{10}(3.7) + 7\right] = \frac{1}{2}[7.5682]$$

$$x_1 = 3.7841$$

$$x_2 = \phi(x_1) = \phi(3.7841) = \frac{1}{2}\left[\log_{10}(3.7841) + 7\right] = \frac{1}{2}[7.57796]$$

$$x_2 = 3.78898$$

$$x_3 = \phi(x_2) = \phi(3.78898) = \frac{1}{2}\left[\log_{10}(3.78898) + 7\right] =$$

$$\frac{1}{2}[7.57852] = 3.78926$$

$x_4 = \phi(x_3) = \phi(3.78926) = \frac{1}{2}\left[\log_{10}(3.78926) + 7\right]$

$= \frac{1}{2}\left[7.57855\right] = 3.78927$

$\therefore$ The Value of x_3 and x_4 are equal.

$\therefore$ The real root is 3.78927.

3. Find the real root of the equation $\cos x = 3x - 1$ using iteration method.

Solution:

Let $f(x) = \cos x - 3x + 1$

$f(0) = \cos(0) - 3(0) + 1 = 2 = +ve$

$f(1) = \cos(1) - 3(1) + 1 = -1.4597 = -ve$

Hence a real root lies between 0 and 1.

Now, $\cos x - 3x + 1 = 0$ can be written as

$\cos x + 1 = 3x$

$x = \frac{1}{3}\left[\cos x + 1\right]$

$\phi(x) = \frac{1}{3}\left[\cos x + 1\right]$

$\phi'(x) = \frac{1}{3}\left[-\sin x\right]$

$|\phi'(x)| = \left|\frac{1}{3}\sin x\right|$

When $x = 0.6$, $|\phi'(0.6)| = \left|\frac{1}{3}\sin(0.6)\right|$

$$= \left| \frac{1}{3}(0.0105) \right| = |0.1882| < 1.$$

$\therefore |\phi'(x)| < 1$ in $(0,1)$

Let the initial approximation be $x_0 = 0.6$

$$x_1 = \phi(x_0) = \frac{1}{3}\left[\cos x_0 + 1\right] = \frac{1}{3}\left[\cos(0.6)+1\right]$$

$$x_1 = \frac{1}{3}\left[1.8253\right] = 0.6084$$

$$x_2 = \phi(x_1) = \frac{1}{3}\left[\cos x_1 + 1\right] = \frac{1}{3}\left[\cos(0.6084)+1\right]$$

$$x_2 = \frac{1}{3}\left[1.8206\right] = 0.6069$$

$$x_3 = \phi(x_2) = \frac{1}{3}\left[\cos(0.6069)+1\right] = \frac{1}{3}\left[1.8214\right] = 0.6071$$

$$x_4 = \phi(x_3) = \frac{1}{3}\left[\cos(0.6071)+1\right] = \frac{1}{3}\left[1.8213\right] = 0.6071$$

$\therefore$ The value of x_3 and x_4 are equal.

Hence the real root is 0.6071.

Newton's method (or) Newton - Raphson method;

Formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Condition for convergence of Newton - Raphson method is

$$|f(x) \cdot f''(x)| < |f'(x)|^2$$

Rate of Convergence of Newton's method is 2

(or)

The order of Convergence of Newton's method is 2.

Problems:

1. Find the positive root of $x^4 - x = 10$ correct to Three decimal places using Newton's Raphson method.

Solution:

Let $f(x) = x^4 - x - 10$

$f'(x) = 4x^3 - 1$

$f(1) = (1)^4 - 1 - 10 = -10 = -ve$

$f(2) = (2)^4 - 2 - 10 = 16 - 2 - 10 = 4 = +ve$

Hence The root lies between 1 and 2.

$\therefore |f(1)| = 10$

$|f(2)| = 4$

$\therefore |f(1)| > |f(2)|$

$\therefore$ The root is nearer to 2.

Let we take $x_0 = 2$.

Newton's formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

put $n = 0$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$f(x_0) = f(2) = (2)^4 - 2 - 10 = 16 - 2 - 10 = 4$

$f'(x_0) = f'(2) = 4(2)^3 - 1 = 32 - 1 = 31$

$x_1 = 2 - \frac{4}{31}$

$x_1 = 2 - 0.1290 = 1.871.$

put $n = 1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.871 - \frac{f(1.871)}{f'(1.871)}$$

$f(1.871) = (1.871)^4 - 1.871 - 10 = 12.254 - 1.871 - 10$
$= 0.883.$

$f'(1.871) = 4(1.871)^3 - 1$
$= 26.199 - 1 = 25.199$

$$x_2 = 1.871 - \frac{0.383}{25.199} = 1.871 - 0.015 = 1.856$$

$x_2 = 1.856.$

put $n = 2$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 1.856 - \frac{f(1.856)}{f'(1.856)}$$

$f(1.856) = (1.856)^4 - 1.856 - 10 = 0.010$

$f'(1.856) = 4(1.856)^3 - 1 = 25.574$

$$x_3 = 1.856 - \frac{0.010}{25.574}$$

$$= 1.856.$$

$\therefore$ The value of x_2 and x_3 are equal.

Hence the real root is 1.856.

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2. Find a root of $x\log_{10}^{x} - 1.2 = 0$ by Newton-Raphson method correct to 3-decimal places.

Solution:

Let $f(x) = x\log_{10}^{x} - 1.2$

$f(x) = x[\log_e^x \cdot \log_{10}^e] - 1.2$ [Using change base rule $\log_{10}^x = \log_e^x \cdot \log_{10}^e$]

$f(x) = \log_{10}^e[x\log_e^x] - 1.2$

$f'(x) = \log_{10}^e[x\frac{1}{x} + \log_e^x]$

$f'(x) = \log_{10}^e[1 + \log_e^x]$

$f'(x) = \log_{10}^e + \log_{10}^e \log_e^x = \log_{10}^e + \log_{10}^x$

$f(1) = 1 \cdot \log_{10}^1 - 1.2 = -1.2 = -ve$

$f(2) = 2 \cdot \log_{10}^2 - 1.2 = -0.598 = -ve$

$f(3) = 3\log_{10}3 - 1.2 = 0.231 = +ve$

Hence the root lies between 2 and 3

$\therefore |f(2)| = 0.598.$

$\therefore |f(3)| = 0.231$

$|f(2)| > |f(3)|$

$\therefore$ The root is nearer to 3

Let $x_0 = 3$

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Newton's formula is,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

put $n=0$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 3 - \frac{f(3)}{f'(3)}$$

$f(3) = 3\log_{10}3 - 1.2 = 0.231$, $f'(3) = \log_{10}e + \log_{10}3 = 0.9111$

$$x_1 = 3 - \left(\frac{0.231}{0.9111}\right)$$

$$x_1 = 3 - 0.2535 = 2.747$$

put $n=1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.747 - \frac{f(2.747)}{f'(2.747)}$$

$f(x_1) = f(2.747) = 2.747\log_{10}(2.747) - 1.2 = 0.0055$

$f'(x_1) = \log_{10}e + \log_{10}(2.747) = 0.8729$

$$x_2 = 2.747 - \left(\frac{0.0055}{0.8729}\right) = 0.2741$$

put $n=2$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= (2.741) - \frac{f(2.741)}{f'(2.741)}$$

$f(2.741) = (2.741)\log_{10}(2.741) - 1.2 = 0.0003$

$f'(2.741) = \log_{10}e + \log_{10}(2.741) = 0.434 + 0.4379 = 0.8719$

$$x_3 = 2.741 - \left(\frac{0.0003}{0.8719}\right)$$

$$x_3 = 2.741$$

$\therefore$ The value of x_2 and x_3 are equal.

$\therefore$ The real root is 2.741.

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3. Find the iterative formula for finding the value of $\frac{1}{N}$ where N is a real x_0 using Newton's Raphson method. Hence evaluate $1/26$ correct to 4 decimal place.

Solution:

Let $x = 1/N$

$N = 1/x$

$1/x - N = 0$

Let $f(x) = 1/x - N$

$f(x) = x^{-1} - N$

$f'(x) = (-1)x^{-1-1} - 0 = -x^{-2} = -1/x^2$

Newton's formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{(1/x_n - N)}{(-1/x_n^2)}$$

$$= x_n + x_n^2(1/x_n - N)$$

$$= x_n + x_n - x_n^2 N$$

$$x_{n+1} = 2x_n - x_n^2 N = x_n(2 - x_n N)$$

$\therefore x_{n+1} = x_n(2 - x_n N)$ is a iterative formula.

To find $1/26$:

Now, $1/N = 1/26$ Take $N = 26$

Let $x_0 = 1/26$ put $n = 0$

$x_1 = x_0(2 - x_0 N) = \frac{1}{26}\left(2 - \frac{1}{26}26\right) = \frac{1}{26}(2-1)$

$= 1/26(1) = 0.0385.$

put $n = 1$

$x_2 = x_1(2 - x_1 N)$

$= (0.0385)[2 - 0.0385(26)]$

$x_2 = 0.0385.$

$\therefore$ The value of x_1 and x_2 are equal.

Hence the value of $1/26 = 0.0385.$

4. Derive the newton's algorithm for finding the p^{th} root of a number N and hence find the cube root of 17.

<u>Solution:</u> Given $x = \sqrt[p]{N}$

$x = (N)^{1/p}$

$x^p = N$

$x^p - N = 0$

$f(x) = x^p - N$

$f'(x) = p x^{p-1}$

By Newton's formula,

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$= x_n - \frac{(x_n^p - N)}{p x_n^{p-1}}$

$$= \frac{x_n p x_n^{p-1} - x_n^p + N}{p x_n^{p-1}} = \frac{x_n^{1+p-1} \cdot p - x_n^p + N}{p x_n^{p-1}}$$

$$x_{n+1} = \frac{x_n^p \cdot p - x_n^p + N}{p x_n^{p-1}} = \frac{x_n^p (p-1) + N}{p x_n^{p-1}}$$

To find $\sqrt[3]{17}$:

The value of $\sqrt[3]{17}$ is 2.5713

Hence the root lies between 2 and 3.

Let take $x_0 = 2$ and $p = 3$.

$\sqrt[3]{17} = \sqrt[p]{N}$

put $n = 0$

$$x_1 = \frac{(p-1)\, x_0^p + N}{p x_0^{p-1}} = \frac{(3-1)(2)^3 + (17)}{(3)(2)^{3-1}}$$

$$x_1 = \frac{2(8)+17}{3(2)^2} = 2.75.$$

put $n = 1$

$$x_2 = \frac{(p-1)\, x_1^p + N}{p x_1^{p-1}} = \frac{(3-1)\, (2.75)^3 + 17}{(2.75)^{3-1} \cdot 3}$$

$$x_2 = \frac{(2)(20.7969)+17}{3(2.75)^2} = 2.5826$$

put $n = 2$

$$x_3 = \frac{(p-1)\, (2.5826)^3 + 17}{3(2.5826)^{3-1}} = \frac{2(17.2255)+17}{20.0095}$$

$$x_3 = \frac{51.451}{20.0095} = 2.5713$$

put $n = 3$

$$x_4 = \frac{(3-1)\,(2.5713)^3 + 17}{3(2.5713)^{3-1}} = \frac{2(2.5713)^3 + 17}{3(2.5713)^2}$$

$x_4 = 2.5713$

$\therefore$ The value of x_3 and x_4 are equal.

Hence the real root is 2.5713.

Solution of linear system

Gaussian Elimination method: (or) Gauss-Elimination method:

Problems:

1. Apply the Gauss Elimination method to find the Solution of the following system.

$2x + 3y - z = 5$, $4x + 4y - 3z = 3$, $2x - 3y + 2z = 2$

Solution:

The given system of equations can be written in matrix form $AX = B$

$$\begin{pmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ 2 & -3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$$

$\therefore$ The augmented matrix is

$$(A/B) \sim \begin{pmatrix} 2 & 3 & -1 & 5 \\ 4 & 4 & -3 & 3 \\ 2 & -3 & 2 & 2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 3/2 & -1/2 & 5/2 \\ 4 & 4 & -3 & 4 \\ 2 & -3 & 2 & 2 \end{pmatrix} \quad R_1 \longrightarrow R_1/2$$

$$\sim \begin{pmatrix} 1 & 3/2 & -1/2 & 5/2 \\ 0 & 10 & -7 & -1 \\ 0 & -6 & 3 & -3 \end{pmatrix} \begin{matrix} R_2 \to R_2 - 2R_3 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 3/2 & -1/2 & 5/2 \\ 0 & 10 & -7 & -1 \\ 0 & 0 & -12 & -36 \end{pmatrix} R_3 \to 10R_3 + 6R_2$$

Use back Substitution to find the Solution to the System.

$$-12z = -36$$

$$z = \frac{-36}{-12}$$

$$z = 3$$

$$10y - 7z = -1$$

$$10y - 7(3) = -1$$

$$10y = -1 + 21$$

$$y = 2$$

$$x + 3/2\,y - 1/2\,z = 5/2$$

$$x + 3/2(2) - 1/2(3) = 5/2$$

$$x + 3 - 3/2 = 5/2$$

$$x = 5/2 + 3/2 - 3$$

$$\therefore x = 1$$

$$\therefore x = 1,\ y = 2,\ z = 3.$$

2. Solve the system of equations $x_1 - x_2 + x_3 = 1$, $-3x_1 + 2x_2 - 3x_3 = -6$, $2x_1 - 5x_2 + 4x_3 = 5$. by using Gauss Elimination method.

Solution:

The given system of equations can be written in matrix form $AX = B$

$$\begin{pmatrix} 1 & -1 & 1 \\ -3 & 2 & -3 \\ 2 & -5 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -6 \\ 5 \end{pmatrix}$$

$\therefore$ The augmented matrix is

$$(A/B) \sim \begin{pmatrix} 1 & -1 & 1 & 1 \\ -3 & 2 & -3 & -6 \\ 2 & -5 & 4 & 5 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & -1 & 0 & -3 \\ 0 & -3 & 2 & 3 \end{pmatrix} \quad \begin{matrix} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & 2 & 12 \end{pmatrix} \quad R_3 \to R_3 - 3R_2$$

Use back substitution to find the solution to the system

$$\therefore 2x_3 = 12$$
$$x_3 = 6$$

$$-x_2 + 0Z = -3$$
$$-x_2 = -3$$
$$x_2 = 3$$
$$x_1 - x_2 + x_3 = 1$$
$$6 - 3 + x_3 = 1 \qquad \therefore x_3 = -2$$

3. Solve the system of equations by Gauss-Elimination method $x_1+x_2+x_3+x_4=2$, $x_1+x_2+3x_3-2x_4=-6$, $2x_1+3x_2-x_3+2x_4=7$, $x_1+2x_2+x_3-x_4=-2$

<u>Solution:</u>

The given system of equation can be written in matrix form $AX=B$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & -2 \\ 2 & 3 & -1 & 2 \\ 1 & 2 & 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \\ 7 \\ -2 \end{pmatrix}$$

∴ The augmented matrix is

$$(A/B) \sim \begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 3 & -2 & -6 \\ 2 & 3 & -1 & 2 & 7 \\ 1 & 2 & 1 & -1 & -2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 2 & -3 & -8 \\ 0 & 1 & -3 & 0 & 3 \\ 0 & 1 & 0 & -2 & -4 \end{pmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 2 & -3 & -8 \\ 0 & 1 & -3 & 0 & 3 \\ 0 & 0 & 3 & -2 & -7 \end{pmatrix} \quad R_4 \to R_4 - R_3$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 0 & 3 \\ 0 & 0 & 2 & -3 & -8 \\ 0 & 0 & 0 & 5 & 10 \end{pmatrix} \quad R_2 \leftrightarrow R_3$$

Use back substitution to find the solution to the system

$\therefore 5x_4 = 10$

$x_4 = 2$

$$2x_3 - 3x_4 = -8$$

$$2(x_3) - 3(2) = -8$$

$$2x_3 = -8+6$$

$$x_3 = -1$$

$$\therefore x_2 - 3x_3 + 0x_4 = 3$$

$$x_2 - 3(-1) + 0 = 3$$

$$x_2 = 0$$

$$\therefore x_1 + x_2 + x_3 + x_4 = 2$$

$$x_1 + 0 - 1 + 2 = 2$$

$$x_1 + 1 = 2 - 2 + 1$$

$$x_1 = 1$$

$$\therefore x_1 = 0, \quad x_2 = 0, \quad x_3 = -1, \quad x_4 = 2.$$

GAUSS - JORDAN METHOD:

Problems:

1. Solve $x+3y+3z=16$, $x+4y+3z=18$, $x+3y+4z=19$, by Gauss-Jordan method.

Solution:

The given system of equation can be written in matrix form $AX=B$.

$$\begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 16 \\ 18 \\ 19 \end{pmatrix}$$

$\therefore$ The augmented matrix is

$$[A|B] \sim \begin{pmatrix} 1 & 3 & 3 & 16 \\ 1 & 4 & 3 & 18 \\ 1 & 3 & 4 & 19 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 3 & 3 & 16 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 3 & 10 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix} R_1 \to R_1 - 3R_2$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix} R_1 \to R_1 - 3A_1$$

$\therefore x = 1$

$y = 2$

$z = 3$

---

2. Using the Gauss-Jordan method solve the following equations. $10x + y + z = 12$, $x + 10y + z = 12$, $x + y + 10z = 12$.

Solution:

The given system of equation can be written in matrix form $Ax = B$

$$\begin{pmatrix} 10 & 1 & 1 \\ 1 & 10 & 1 \\ 1 & 1 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ 12 \\ 12 \end{pmatrix}$$

$\therefore$ The augmented matrix is

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$$(A/B) \sim \begin{pmatrix} 10 & 1 & 1 & 12 \\ 1 & 10 & 1 & 12 \\ 1 & 1 & 10 & 12 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 10 & 1 & 12 \\ 10 & 1 & 1 & 12 \\ 1 & 1 & 10 & 12 \end{pmatrix} \quad R_1 \longleftrightarrow R_2$$

$$\sim \begin{pmatrix} 1 & 10 & 1 & 12 \\ 0 & -99 & -9 & -108 \\ 0 & -9 & 9 & 0 \end{pmatrix} \quad \begin{matrix} R_2 \to R_2 - 10R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 10 & 1 & 12 \\ 0 & -99 & -9 & -108 \\ 0 & 0 & 108 & 108 \end{pmatrix} \quad R_3 \to 11R_3 - R_2$$

$$\sim \begin{pmatrix} 99 & 0 & -9 & 108 \\ 0 & -99 & -9 & -108 \\ 0 & 0 & 108 & 108 \end{pmatrix} \quad R_1 \to 99R_1 + 10R_2$$

$$\sim \begin{pmatrix} 10692 & 0 & 0 & 10692 \\ 0 & -99 & -9 & -108 \\ 0 & 0 & 108 & 108 \end{pmatrix} \quad R_1 \to 108R_1 - 9R_3$$

$$\sim \begin{pmatrix} 10692 & 0 & 0 & 10692 \\ 0 & -10692 & 0 & -10692 \\ 0 & 0 & 108 & 108 \end{pmatrix} \quad R_2 \to 108R_2 + 9R_3$$

$\therefore 10692\,x = 10692$

$x = 1$

$\therefore -10692\,y = -10692$

$y = 1$

$\therefore 108\,z = 108$

$z = 1$

$\therefore x = y = z = 1.$

---

3. Solve the following system by Gauss - Jordan method
$x_1 + x_2 + x_3 + 4x_4 = -6$, $x_1 + 7x_2 + x_3 + x_4 = 12$,
$x_1 + x_2 + 6x_3 + x_4 = -5$, $5x_1 + x_2 + x_3 + x_4 = 4$.

Solution:

The given system of equation can be written in matrix form $AX = B$.

$$\begin{pmatrix} 1 & 1 & 1 & 4 \\ 1 & 7 & 1 & 1 \\ 1 & 1 & 6 & 1 \\ 5 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -6 \\ 12 \\ -5 \\ 4 \end{pmatrix}$$

$\therefore$ The augmented matrix is

$$(A|B) \sim \begin{pmatrix} 1 & 1 & 1 & 4 & -6 \\ 1 & 7 & 1 & 1 & 12 \\ 1 & 1 & 6 & 1 & -5 \\ 5 & 1 & 1 & 1 & 4 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 4 & -6 \\ 0 & 6 & 0 & -3 & -18 \\ 0 & 0 & 5 & -3 & 1 \\ 0 & -4 & -4 & -19 & 34 \end{pmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 5R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 4 & -6 \\ 0 & 6 & 0 & -3 & 18 \\ 0 & 0 & 5 & -3 & 1 \\ 0 & 0 & -24 & -126 & 276 \end{pmatrix} \quad R_4 \to R_4 + 4R_1$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 4 & -6 \\ 0 & 6 & 0 & -3 & 18 \\ 0 & 0 & 5 & -3 & 1 \\ 0 & 0 & 0 & -702 & 1404 \end{pmatrix} \quad R_4 \to 5R_4 + 24R_3$$

$$\sim \begin{pmatrix} 6 & 0 & 6 & 27 & -54 \\ 0 & 6 & 0 & -3 & 18 \\ 0 & 0 & 5 & -3 & 1 \\ 0 & 0 & 0 & -702 & 1404 \end{pmatrix} \quad R_1 \to 6R_1 - R_2$$

$$\sim \begin{pmatrix} 30 & 0 & 0 & 153 & -276 \\ 0 & 6 & 0 & -3 & 18 \\ 0 & 0 & 5 & -3 & 1 \\ 0 & 0 & 0 & -702 & 1404 \end{pmatrix} \quad R_1 \to 5R_1 - 6R_3$$

$$\sim \begin{pmatrix} 30 & 0 & 0 & 153 & -276 \\ 0 & 6 & 0 & -3 & 18 \\ 0 & 0 & 3510 & 0 & -3510 \\ 0 & 0 & 0 & -702 & 1404 \end{pmatrix} \quad R_3 \to 702R_3 - 3R_4$$

$$\sim \begin{pmatrix} 21060 & 0 & 0 & 0 & 21060 \\ 0 & -6 & 0 & -3 & 18 \\ 0 & 0 & 3510 & 0 & -3510 \\ 0 & 0 & 0 & -702 & 1404 \end{pmatrix} \quad R_1 \to 702R_1 + 153R_4$$

$$\sim \left(\begin{array}{ccccc} 21060 & 0 & 0 & 0 & 21060 \\ 0 & 4212 & 0 & 0 & 8424 \\ 0 & 0 & 3510 & 0 & -3510 \\ 0 & 0 & 0 & -702 & 1404 \end{array}\right) \quad R_2 \rightarrow 702R_2 - 3R_4$$

$\therefore 21060\, x_1 = 21060$

$x_1 = 1$

$4212\, x_2 = 8424$

$x_2 = 2$

$3510\, x_3 = -3510$

$x_3 = -1$

$-702\, x_4 = 1404$

$x_4 = -2$

ITERATIVE METHOD

GAUSS - SEIDEL METHOD:

Formula:

$$x = \frac{1}{a_{11}}\left[c_1 - a_{12}y - a_{13}z\right]$$

$$y = \frac{1}{a_{22}}\left[c_2 - a_{21}x - a_{23}z\right]$$

$$z = \frac{1}{a_{33}}\left[c_3 - a_{31}x - a_{32}y\right]$$

Problems:

1. Solve the following equation by Gauss Seidel method $4x+2y+z=14$, $x+5y-z=10$, $x+y+8z=20$.

Solution:

First check the given Simultaneous equations is diagonally dominant.

$$|4| > |2| + |1|$$
$$|5| > |1| + |-1|$$
$$|8| > |1| + |1|$$

$\therefore$ The given system of equation satisfy the diagonally dominant.

Gauss - seidel Method formula,

$$x = \frac{1}{a_{11}}\left[c_1 - a_{12}y - a_{13}z\right]$$

$$y = \frac{1}{a_{22}}\left[c_2 - a_{21}x - a_{23}z\right]$$

$$z = \frac{1}{a_{33}}\left[c_3 - a_{31}x - a_{32}y\right]$$

First iteration:

putting $y=0$ and $z=0$

$$x = \frac{1}{4}\left[14 - 2(0) - 1(0)\right] = 3.5$$

putting $x = 3.5$ and $z=0$

$$y = \frac{1}{5}\left[10 - 1(3.5) - (-1)(0)\right] = 1.3$$

Putting $x = 3.5$ and $y = 1.3$

$$z = \frac{1}{8}\left[20 - 1(3.5) - 1(1.3)\right] = 1.9$$

Second iteration:

putting $y = 1.3$ and $z = 1.9$

$$x = \frac{1}{a_{11}}\left[c_1 - a_{12}y - a_{13}z\right] = \frac{1}{4}\left[14 - 2(1.3) - 1(1.9)\right] = 2.375$$

Putting $x = 2.375$ and $z = 1.9$

$$y = \frac{1}{5}\left[10 - 1(2.375) - (-1)(1.9)\right] = 1.905$$

putting $x = 2.375$ and $y = 1.905$

$$z = \frac{1}{8}\left[20 - 1(2.375) - 1.905\right] = 1.965.$$

Third iteration:

put $y = 1.905$ and $z = 1.965$.

$x = \frac{1}{4}[14 - 2(1.905) - 1(1.965)] = 2.056$

put $x = 2.056$ and $z = 1.965$.

$y = \frac{1}{5}[10 - 2.056 - (-1)(1.965)] = 1.982$

put $x = 2.056$ and $y = 1.982$

$z = \frac{1}{8}[20 - 2.056 - 1.982] = 1.995$

Fourth iteration:

put $y = 1.982$ and $z = 1.995$

$x = \frac{1}{4}[14 - 2(1.982) - 1(1.995)] = 2.010$

put $x = 2.010$ and $z = 1.995$.

$y = \frac{1}{5}[10 - 2.010 - (-1)(1.995)] = 1.997$

put $y = 1.997$ and $x = 2.010$

$z = \frac{1}{8}[20 - 2.010 - 1.997] = 1.999$

Fifth iteration:

put $y = 1.997$ and $z = 1.999$

$x = \frac{1}{4}[14 - 2(1.997) - 1(1.999)] = 2.002$

put $x = 2.002$ and $z = 1.999$

$y = \frac{1}{5}[10 - 2.002 + 1.999] = 1.999$

put $x = 2.002$ and $z = 1.999$

$z = \frac{1}{8}[20 - 2.002 - 1.999] = 2$

Sixth iteration:

put $y = 1.999$ and $z = 2$

$x = \frac{1}{4}[14 - 2(1.999) - 1(2)] = 2.001$

put $x = 2.001$ and $z = 2$

$y = \frac{1}{5}[10 - 2.001 - (-1)(2)] = 1.9998$

put $x = 2.001$ and $y = 1.9998$

$z = \frac{1}{8}[20 - 2.001 - 1.9998] = 2.$

After sixth iteration, we get

$x = 2.001, \quad y = 1.9998, \quad z = 1.9999$ (or) $2.$

---

2. Solve the following system of equation using Gauss-Seidel iteration method. $28x + 4y - z = 32$, $x + 3y + 10z = 24$, $2x + 17y + 4z = 35$.

Solution:

First check the given simultaneous equations is diagonally dominant.

$|28| > |4| + |1|$

$|3| > |1| + |10|$

$|4| < |2| + |17|$

$\therefore$ The above equations are not satisfy diagonally dominant.

$\therefore$ We interchange second and third equations

$28x + 4y - z = 32$

$2x + 17y + 4z = 35$

$x + 3y + 10z = 24.$

$|28| > |4| + |-1|$

$|17| > |2| + |4|$

$|10| > |1| + |3|.$

$\therefore$ The above equations are diagonally dominant.

Gauss - Seidel formula is,

$$x = \frac{1}{a_{11}}\left[c_1 - a_{12}y - a_{13}z\right]$$

$$y = \frac{1}{a_{22}}\left[c_2 - a_{21}x - a_{23}z\right]$$

$$z = \frac{1}{a_{33}}\left[c_3 - a_{31}x - a_{32}y\right]$$

First iteration:

put $y=0$ and $z=0$

$$x = \frac{1}{28}\left[32 - 4y + (1)z\right] = \frac{1}{28}\left[32 - 4(0) + 0\right] = 1.143$$

put $x = 1.143$ and $z=0$

$$y = \frac{1}{17}\left[35 - 2x - 4z\right] = \frac{1}{17}\left[35 - 2(1.143) - 4(0)\right] = 1.924$$

put $x = 1.143$ and $y = 1.924$

$$z = \frac{1}{10}\left[24 - x - 3y\right] = \frac{1}{10}\left[24 - (1.143) - 3(1.924)\right] = 1.709$$

Second iteration:

put $y = 1.924$, $z = 1.709$

$$x = \frac{1}{28}\left[32 - 4y + (1)z\right] = \frac{1}{28}\left[32 - 4(1.924) + 1.709\right] = 0.929$$

put $x = 0.929$, $z = 1.709$

$$y = \frac{1}{17}\left[35 - 2(0.929) - 4(1.709)\right] = 1.547$$

put $x = 0.929$, $y = 1.547$.

$$z = \frac{1}{10}\left[24 - 0.929 - 3(1.547)\right] = 1.843.$$

Third iteration:

put $y = 1.547$ and $z = 1.843$

$$x = \frac{1}{28}\left[32 - 4(1.547) + 1.843\right] = 0.988$$

put $x = 0.988$ and $z = 1.843$

$$y = \frac{1}{17}\left[35 - 2(0.988) - 4(1.843)\right] = 1.509$$

put $x = 0.988$ and $y = 1.509$

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$z = \frac{1}{10}[24 - 0.988 - 3(1.509)] = 1.849.$

Fourth iteration:

Put $y = 1.509$ and $z = 1.849$

$x = \frac{1}{28}[32 - 4(1.509) + 1.849] = 0.993$

put $x = 0.993$ and $z = 1.849$

$y = \frac{1}{17}[35 - 2(0.993) - 4(1.849)] = 1.506.$

put $x = 0.993$ and $y = 1.506.$

$z = \frac{1}{10}[24 - x - 3y] = \frac{1}{10}[24 - 0.993 - 3(1.506)]$

Fifth iteration:

put $y = 1.506$ and $z = 1.849$

$x = \frac{1}{28}[32 - 4(1.506) + 1.849] = 0.994$

put $x = 0.994$ and $z = 1.849$

$y = \frac{1}{17}[35 - 2(0.994) - 4(1.849)] = 1.507$

put $x = 0.994$ and $y = 1.507$

$z = \frac{1}{10}[24 - 0.994 - 3(1.507)] = 1.849.$

Sixth iteration:

put $y = 1.507$ and $z = 1.849$

$x = \frac{1}{28}[32 - 4(1.507) + 1.849] = 0.994$

put $x = 0.994$ and $z = 1.849.$

$y = \frac{1}{17}[35 - 2(0.994) - 4(1.849)] = 1.507.$

put $x = 0.994$ and $y = 1.507$

$z = \frac{1}{10}[24 - 0.994 - 3(1.507)] = 1.849.$

$\therefore x = 0.994, \quad y = 1.507$ and $z = 1.849$

INVERSE OF A MATRIX BY GAUSS JORDON METHOD:

Problems:

1. Find the inverse of the matrix $\begin{pmatrix} 5 & -2 \\ 3 & 4 \end{pmatrix}$ by Gauss-Jordon method.

Solution:

Let $AX = I$

where $A = \begin{pmatrix} 5 & -2 \\ 3 & 4 \end{pmatrix}$, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$

$\therefore$ The augmented matrix is

$$(A/I) \sim \begin{pmatrix} 5 & -2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 5 & -2 & 1 & 0 \\ 0 & 26 & -3 & 5 \end{pmatrix} \quad R_2 \rightarrow 5R_2 - 3R_1$$

$$\sim \begin{pmatrix} 130 & 0 & 20 & 10 \\ 0 & 26 & -3 & 5 \end{pmatrix} \quad R_1 \rightarrow 26R_1 + 2R_2$$

$$\sim \begin{pmatrix} 1 & 0 & 20/130 & 10/130 \\ 0 & 1 & -3/26 & 5/26 \end{pmatrix} \quad \begin{matrix} R_1 \rightarrow R_1/130 \\ R_2 \rightarrow R_2/26 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 2/13 & 1/13 \\ 0 & 1 & -3/26 & 5/26 \end{pmatrix}$$

Hence the inverse of the given matrix is

$$\begin{pmatrix} 2/13 & 1/13 \\ -3/26 & 5/26 \end{pmatrix} = 1/26 \begin{pmatrix} 4 & 2 \\ -3 & 5 \end{pmatrix}$$

2. Find the inverse of the matrix (10) $\begin{pmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{pmatrix}$ using Gauss - Jordan method.

Solution:

Let $AX = I$

where $A = \begin{pmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{pmatrix}$, $X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\therefore$ The augmented matrix is

$$(A/I) \sim \begin{pmatrix} 1 & 2 & 6 & 1 & 0 & 0 \\ 2 & 5 & 15 & 0 & 1 & 0 \\ 6 & 15 & 46 & 0 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 6 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 3 & 10 & -6 & 0 & 1 \end{pmatrix} \begin{matrix} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 6R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 6 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & -3 & 1 \end{pmatrix} R_3 \to R_3 - 3R_2$$

$$\sim \begin{pmatrix} 1 & 2 & 0 & 1 & 18 & -6 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & -3 & 1 \end{pmatrix} R_1 \to R_1 - 6R_3$$

2. Find the largest Eigenvalue and the corresponding Eigenvector $A = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ find also the least Eigenvalue and hence find the 3^{rd} Eigenvalue also. (or) Using power method find all the Eigenvalues.

Solution:

Let $A = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

Let $X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ be an arbitary Eigenvalue.

$$AX_1 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+0+0 \\ 1+0+0 \\ 0+0+0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1X_2$$

$$AX_2 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+6+0 \\ 1+2+0 \\ 0+0+0 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ 0 \end{pmatrix} = 7\begin{pmatrix} 1 \\ 0.4286 \\ 0 \end{pmatrix} = 7X_3$$

$$AX_3 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0.4286 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+2.5716+0 \\ 1+0.8572+0 \\ 0+0+0 \end{pmatrix} = \begin{pmatrix} 3.5716 \\ 1.8572 \\ 0 \end{pmatrix}$$

$$= 3.5716\begin{pmatrix} 1 \\ 0.52 \\ 0 \end{pmatrix} = 3.5716X_4$$

$$AX_4 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0.52 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+3.12+0 \\ 1+1.04+0 \\ 0+0+0 \end{pmatrix} = \begin{pmatrix} 4.12 \\ 2.04 \\ 0 \end{pmatrix} =$$

$$4.12\begin{pmatrix} 1 \\ 0.4951 \\ 0 \end{pmatrix} = 4.12X_5.$$

Problems:

1. Find the Numerically largest eigenvalue of $A = \begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}$ also find the corresponding Eigenvector.

Solution:

$$A = \begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}$$

Let $X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ be the arbitrary Eigenvector.

$$AX_1 = \begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 25 \\ 1 \\ 2 \end{pmatrix} = 25\begin{pmatrix} 1 \\ 0.04 \\ 0.08 \end{pmatrix}$$

$$AX_2 = \begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 0.04 \\ 0.08 \end{pmatrix} = \begin{pmatrix} 25.2 \\ 1.12 \\ 1.68 \end{pmatrix} = 25.2\begin{pmatrix} 1 \\ 0.044 \\ 0.0667 \end{pmatrix}$$

$$AX_3 = \begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 0.044 \\ 0.0667 \end{pmatrix} = \begin{pmatrix} 25.177 \\ 1.132 \\ 1.733 \end{pmatrix} = 25.177\begin{pmatrix} 1 \\ 0.045 \\ 0.0688 \end{pmatrix}$$

$$AX_4 = \begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 0.045 \\ 0.0688 \end{pmatrix} = \begin{pmatrix} 25.1826 \\ 1.135 \\ 1.7248 \end{pmatrix} = 25.1826\begin{pmatrix} 1 \\ 0.0451 \\ 0.0685 \end{pmatrix}$$

$$AX_5 = \begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 0.0451 \\ 0.0685 \end{pmatrix} = \begin{pmatrix} 25.1821 \\ 1.135 \\ 1.726 \end{pmatrix} = 25.1821\begin{pmatrix} 1 \\ 0.0451 \\ 0.0685 \end{pmatrix}$$

$\therefore$ The largest Eigenvalue is 25.18121

$\therefore$ The corresponding Eigenvector is $\begin{pmatrix} 1 \\ 0.0451 \\ 0.0685 \end{pmatrix}$

2. Find the largest Eigenvalue and the corresponding Eigenvector $A = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ find also the least Eigenvalue and hence find the 3^{rd} Eigenvalue also. (or) Using power method find all the Eigenvalues.

<u>Solution:</u>

Let $A = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

Let $X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ be an arbitrary Eigenvalue.

$$AX_1 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+0+0 \\ 1+0+0 \\ 0+0+0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1X_2$$

$$AX_2 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+6+0 \\ 1+2+0 \\ 0+0+0 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ 0 \end{pmatrix} = 7 \begin{pmatrix} 1 \\ 0.4286 \\ 0 \end{pmatrix} = 7X_3$$

$$AX_3 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0.4286 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+2.5716+0 \\ 1+0.8572+0 \\ 0+0+0 \end{pmatrix} = \begin{pmatrix} 3.5716 \\ 1.8572 \\ 0 \end{pmatrix}$$

$$= 3.5716 \begin{pmatrix} 1 \\ 0.52 \\ 0 \end{pmatrix} = 3.5716 X_4$$

$$AX_4 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0.52 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+3.12+0 \\ 1+1.04+0 \\ 0+0+0 \end{pmatrix} = \begin{pmatrix} 4.12 \\ 2.04 \\ 0 \end{pmatrix} =$$

$$4.12 \begin{pmatrix} 1 \\ 0.4951 \\ 0 \end{pmatrix} = 4.12 X_5$$

$$Ax_5 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0.4951 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+2.9706+0 \\ 1+0.9902+0 \\ 0+0+0 \end{pmatrix} = \begin{pmatrix} 3.9706 \\ 1.9902 \\ 0 \end{pmatrix}$$

$$= 3.9706 \begin{pmatrix} 1 \\ 0.5012 \\ 0 \end{pmatrix} = 3.9706\, x_6$$

$$Ax_6 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0.5012 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+3.0072+0 \\ 1+1.0024+0 \\ 0+0+0 \end{pmatrix} = \begin{pmatrix} 4.0072 \\ 2.0024 \\ 0 \end{pmatrix}$$

$$= 4.0072 \begin{pmatrix} 1 \\ 0.4997 \\ 0 \end{pmatrix} = 4.0072\, x_7$$

$$Ax_7 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0.4997 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+2.9982+0 \\ 1+0.9994+0 \\ 0+0+0 \end{pmatrix} = \begin{pmatrix} 3.9982 \\ 1.9994 \\ 0 \end{pmatrix}$$

$$= 3.9982 \begin{pmatrix} 1 \\ 0.500 \\ 0 \end{pmatrix} = 3.9982\, x_8.$$

$$Ax_8 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0.500 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+3+0 \\ 1+1+0 \\ 0+0+0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix}$$

$$= 4\, x_9$$

$\therefore$ The largest Eigen Value of $A = 4$ and their corresponding Eigenvector is $\begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix}$.

<u>To find the least Eigen Value of A:</u>

Let $B = A - \lambda_1 I$

$B = A - 4I$ $\quad$ [Since $\lambda_1 = 4$]

$$= \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} - 4 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$B = \begin{pmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Let $Y_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ be the aabitaly Eigen Valeee of B.

$$BY_1 = \begin{pmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3+0+0 \\ 1+0+0 \\ 0+0+0 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$$

$$= -3\begin{pmatrix} 1 \\ -0.3333 \\ 0 \end{pmatrix} = -3Y_2$$

$$BY_2 = \begin{pmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ -0.3333 \\ 0 \end{pmatrix} = \begin{pmatrix} -3-1.9998+0 \\ 1+0.6666+0 \\ 0+0+0 \end{pmatrix}$$

$$= \begin{pmatrix} -4.9998 \\ 1.6666 \\ 0 \end{pmatrix} = -4.9998\begin{pmatrix} 1 \\ -0.3333 \\ 0 \end{pmatrix} = -4.9998Y_3$$

$$BY_3 = \begin{pmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ -0.3333 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ 1.6666 \\ 0 \end{pmatrix} = -5\begin{pmatrix} 1 \\ -0.3333 \\ 0 \end{pmatrix}$$

$\therefore$ The largest Eigen Valeee of B is -5 and their Corresponding Eigenvector $\begin{pmatrix} 1 \\ -0.3333 \\ 0 \end{pmatrix}$

Smallest Eigenvalue of A = Largest Eigenvalue of A + Largest Eigenvalue of B

$= 4 + (-5)$

$= -1$

To find the third Eigen value of A:

Sum of the Eigenvalue = Sum of the diagonal elements

$$\lambda_1 + \lambda_2 + \lambda_3 = 1+2+3$$
$$4 - 1 + \lambda_3 = 1+2+3$$
$$3 + \lambda_3 = 6$$
$$\lambda_3 = 6 - 3 = 3$$

$\therefore$ All the three Eigen values are $4, 3, -1$.

EIGEN VALUE OF A MATRIX BY JACOBI METHOD FOR SYMMETRIC MATRIX

Problem:

1. Find all the Eigenvalues and Eigenvectors of the matrix.

$A = \begin{pmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{pmatrix}$ using Jacobi's method.

Solution:

Choose the largest off-diagonal element of the given matrix. Here the largest off-diagonal element is 2 which is in the position a_{13}.

$$A = \begin{pmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{pmatrix}$$

Here the largest off-diagonal element is in the position a_{13}. Therefore $\cos\theta, -\sin\theta, \sin\theta, \cos\theta$ are located in a_{11}, a_{13}, a_{31}, and a_{33} positions respectively and the remaining elements are similar to the unit matrix.

$$S_1 = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2a_{13}}{a_{11}-a_{33}}\right)$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{2\times 2}{1-1}\right)$$

$$= \frac{1}{2}\tan^{-1}(\infty) = \frac{1}{2}\cdot\frac{\pi}{2}$$

$$= \pi/4$$

$$S_1 = \begin{pmatrix} \cos(\pi/4) & 0 & -\sin(\pi/4) \\ 0 & 1 & 0 \\ \sin(\pi/4) & 0 & \cos(\pi/4) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$B_1 = S^{-1} A S_1$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}}+0+\frac{2}{\sqrt{2}} & 0+\sqrt{2}+0 & -\frac{1}{\sqrt{2}}+0+\frac{2}{\sqrt{2}} \\ \frac{\sqrt{2}}{\sqrt{2}}+0+\frac{\sqrt{2}}{\sqrt{2}} & 0+3+0 & -\frac{\sqrt{2}}{\sqrt{2}}+0+\frac{\sqrt{2}}{\sqrt{2}} \\ \frac{2}{\sqrt{2}}+0+\frac{1}{\sqrt{2}} & 0+\sqrt{2}+0 & -\frac{2}{\sqrt{2}}+0+\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt2} & \sqrt2 & \frac{1}{\sqrt2} \\ 2 & 3 & 0 \\ \frac{3}{\sqrt2} & \sqrt2 & -\frac{1}{\sqrt2} \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{3}{2}+0+\frac{3}{2} & \frac{\sqrt2}{\sqrt2}+0+\frac{\sqrt2}{\sqrt2} & 1+0-\frac{1}{2} \\ 0+2+0 & 0+3+0 & 0+0+0 \\ -\frac{3}{2}+0+\frac{3}{2} & \frac{-\sqrt2}{\sqrt2}+0+\frac{\sqrt2}{\sqrt2} & -\frac{1}{2}+0-\frac{1}{2} \end{pmatrix}$$

$$B = \begin{pmatrix} 6/2 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 1 & -2/2 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

Repeat The steps:

In B_1 The largest off diagonal element is 2 which is in The position a_{12}.

$$S_2 = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\cos\theta$, $-\sin\theta$, $\sin\theta$, $\cos\theta$ are located in The position a_{11}, a_{12}, a_{21}, a_{22}

where $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2a_{12}}{a_{11}-a_{22}}\right) = \frac{1}{2}\tan^{-1}\left(\frac{2\times2}{3-3}\right)$

$= \frac{1}{2}\tan^{-1}(\infty) = \frac{1}{2}\cdot\pi/2$

$\theta = \pi/4$

$$\therefore S_2 = \begin{pmatrix} \cos(\pi/4) & -\sin(\pi/4) & 0 \\ \sin(\pi/4) & \cos(\pi/4) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt2 & -\frac{1}{\sqrt2} & 0 \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B_2 = S_2^{-1} B_1 S_2$$

$$= \begin{pmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt2} & 0 \\ -\frac{1}{\sqrt2} & \frac{1}{\sqrt2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} & 0 \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt2} & 0 \\ -\frac{1}{\sqrt2} & \frac{1}{\sqrt2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt2} + \frac{2}{\sqrt2} + 0 & -\frac{3}{\sqrt2} + \frac{2}{\sqrt2} + 0 & 0+0+0 \\ \frac{2}{\sqrt2} + \frac{3}{\sqrt2} + 0 & -\frac{2}{\sqrt2} + \frac{3}{\sqrt2} + 0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0-1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt2} & 0 \\ -\frac{1}{\sqrt2} & \frac{1}{\sqrt2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{5}{\sqrt2} & -\frac{1}{\sqrt2} & 0 \\ 5/\sqrt2 & 1/\sqrt2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 5/2 + 5/2 + 0 & -\frac{1}{2} + \frac{1}{2} + 0 & 0+0+0 \\ -5/2 + 5/2 + 0 & 1/2 + 1/2 + 0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0-1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Here B_2 is a diagonal matrix.

$\therefore$ Eigen values are $5, 1, -1$

<u>To find the Eigen vectors:</u>

$$S_1 S_2 = \begin{pmatrix} \frac{1}{\sqrt2} & 0 & -\frac{1}{\sqrt2} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} & 0 \\ -\frac{1}{\sqrt2} & \frac{1}{\sqrt2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}+0+0 & -\frac{1}{2}+0+0 & 0+0-\frac{1}{\sqrt{2}} \\ 0-\frac{1}{\sqrt{2}}+0 & 0+\frac{1}{\sqrt{2}}+0 & 0+0+0 \\ \frac{1}{2}+0+0 & -\frac{1}{2}+0+0 & 0+0+\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/2 & -1/2 & 1/\sqrt{2} \end{pmatrix}$$

Hence the Eigen values are $5, 1, -1$ and the corresponding Eigenvectors are $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$, $\left(-\frac{1}{2}, \frac{1}{\sqrt{2}}, -1/2\right)$ and $\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$.

UNIT-I ASSIGNMENT QUESTIONS

1. Solve by iteration method $3x - \log_{10} x = 6$.
2. Solve $e^x - 3x = 0$ using fixed point iteration.
3. Find by Newton's Raphson method, the root of $\log_{10} x = 1.2$.
4. Find the iterative formula for finding the value of $\frac{1}{N}$ where N is a real x_0 using Newton's Raphson method. Hence evaluate $1/23$ correct to 4 decimal place.
5. Write down Newton's Raphson formula for finding $\sqrt{N}$ where N is a positive number and hence find $\sqrt{142}$
6. Solve the system of equations $3x+y-z=3$, $2x-8y+z=-5$ and $x-2y+9z=8$ using Gauss elimination method.

The system of ... $= 17$ and $5x - 2y + 7z = 20$.

8. Solve $10x_1 + x_2 + x_3 = 12$, $2x_1 + 10x_2 + x_3 = 13$,

$x_1 + x_2 + 5x_3 = 7$ by Gauss - Jordan method.

9. Solve the following equations using Gauss - Jordan method.

$2x_1 + 2x_2 - x_3 + x_4 = 4$, $4x_1 + 3x_2 - x_3 + 2x_4 = 6$,

$8x_1 + 5x_2 - 3x_3 + 4x_4 = 12$, $3x_1 + 3x_2 - 2x_3 + 2x_4 = 6$.

10. Solve the following system of equation using Gauss - Seidel iteration method. $8x - 3y + 2z = 20$, $6x + 3y + 12z = 35$, $4x + 11y - z = 33$

11. Solve the following system of equations using Gauss - Seidel method $27x + 6y - z = 85$, $6x + 15y + 2z = 72$, $x + y + 54z = 110$.

12. Find the inverse of the matrix $\begin{pmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{pmatrix}$ using Gauss-Jordan method.

13. Using power method, find all the Eigen values of $A = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}$

14. Find all the Eigenvalues and Eigenvectors of the matrix $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ using Jacobi's method.