CHAPTER 06

TIME AND FREQUENCY CHARACTERIZATION OF SIGNALS AND SYSTEMS

CONTENTS

INTRODUCTION

The frequency-domain representation of an LTI system offers advantages over time-domain analysis, simplifying differential equations to algebraic operations and providing clear visualization of frequency-selective filtering.
However, system design often requires balancing time and frequency domain considerations. Oscillatory behavior in filters' impulse responses may necessitate compromising frequency selectivity for improved time-domain behavior.

THE MAGNITUDE-PHASE REPRESENTATION OF THE FOURIER TRANSFORM

The magnitude-phase representation of the continuous-time Fourier transform X(jw) is

$$X(j\omega) = |X(j\omega)|e^{j\sphericalangle X(j\omega)}.$$

Similarly the magnitude-phase representation of the discrete-time Fourier transform is

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{j\sphericalangle X(e^{j\omega})}.$$

From the Fourier transform synthesis equation, we can interpret that:

- X(jw) is providing with the decomposition of the signal x(t) into a sum of complex exponentials at different frequencies.
- $|X(jw)|^2$ may be interpreted as the energy-density spectrum of x(t).
- $|X(jw)|^2 dw/2\pi$ can be thought of as the amount of energy in the signal x(t) that lies in the infinitesimal frequency band between w and w + dw .

- The phase angle of X(jw), on the other hand , does not affect the amplitudes of the individual components, but instead provides us with information concerning the relative phases of these exponential.
- The phase relationships captured by X(jw) have a significant effect on the nature of the signal x(t) and thus typically contain a substantial amount of information about the signal.

As another illustration of the effect of phase, consider the signal

$$x(t) = 1 + \frac{1}{2}\cos(2\pi t + \phi_1) + \cos(4\pi t + \phi_2) + \frac{2}{3}\cos(6\pi t + \phi_3).$$

Now the effect of the phase is demonstrated on the next page .

Now what if we reverse the sentence and by reversing the sentence we mean to replace t by (-t).

$$\mathfrak{F}\{x(-t)\} = X(-j\omega) = |X(j\omega)|e^{-j\sphericalangle X(j\omega)}.$$

That is, the spectrum of a sentence played in reverse has the same magnitude function as the spectrum of the original sentence and differs only in phase.

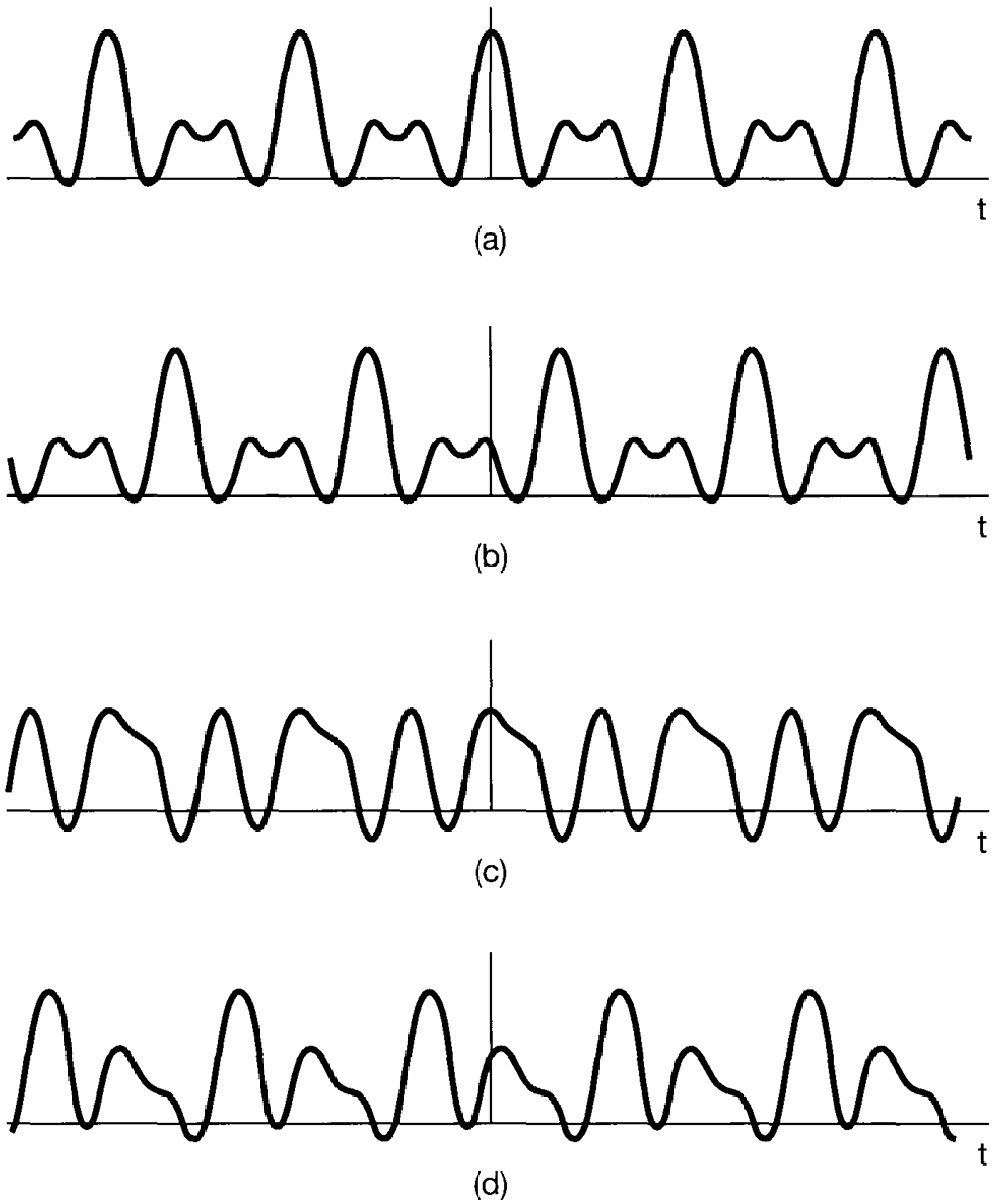


Figure 6.1 The signal $x(t)$ given in eq. (6.3) for several different choices of the phase angles ϕ_1, ϕ_2, and ϕ_3: (a) $\phi_1 = \phi_2 = \phi_3 = 0$; (b) $\phi_1 = 4$ rad, $\phi_2 = 8$ rad, $\phi_3 = 12$ rad; (c) $\phi_1 = 6$ rad, $\phi_2 = -2.7$ rad, $\phi_3 = 0.93$ rad; (d) $\phi_1 = 1.2$ rad, $\phi_2 = 4.1$ rad, $\phi_3 = -7.02$ rad

The Magnitude-Phase Representation of the Frequency Response of LTI Systems

1. Linear and Nonlinear Phase
2. Group Delay
3. Log-Magnitude and Bode Plots

From the convolution property for continuous-time Fourier transforms, the transform Y(jw) of the output of an LTI system is related to the transform X(jw) of the input to the system by the equation.

$$Y(j\omega) = H(j\omega)X(j\omega),$$

where H(jw) is the frequency response of the system-i.e., the Fourier transform of the system's impulse response. Similarly, in discrete time, the Fourier transforms of the input X(ei(<)) and ouput Y(ei(<)) of an LTI system with frequency response H(eiw) are related by

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}).$$

Thus, the effect that an LTI system has on the input is to change the complex amplitude of each of the frequency components of the signal. By looking at this effect in terms of the magnitude-phase representation, we can understand the nature of the effect in more detail. Specifically, in continuous time

$$|Y(j\omega)| = |H(j\omega)||X(j\omega)|$$

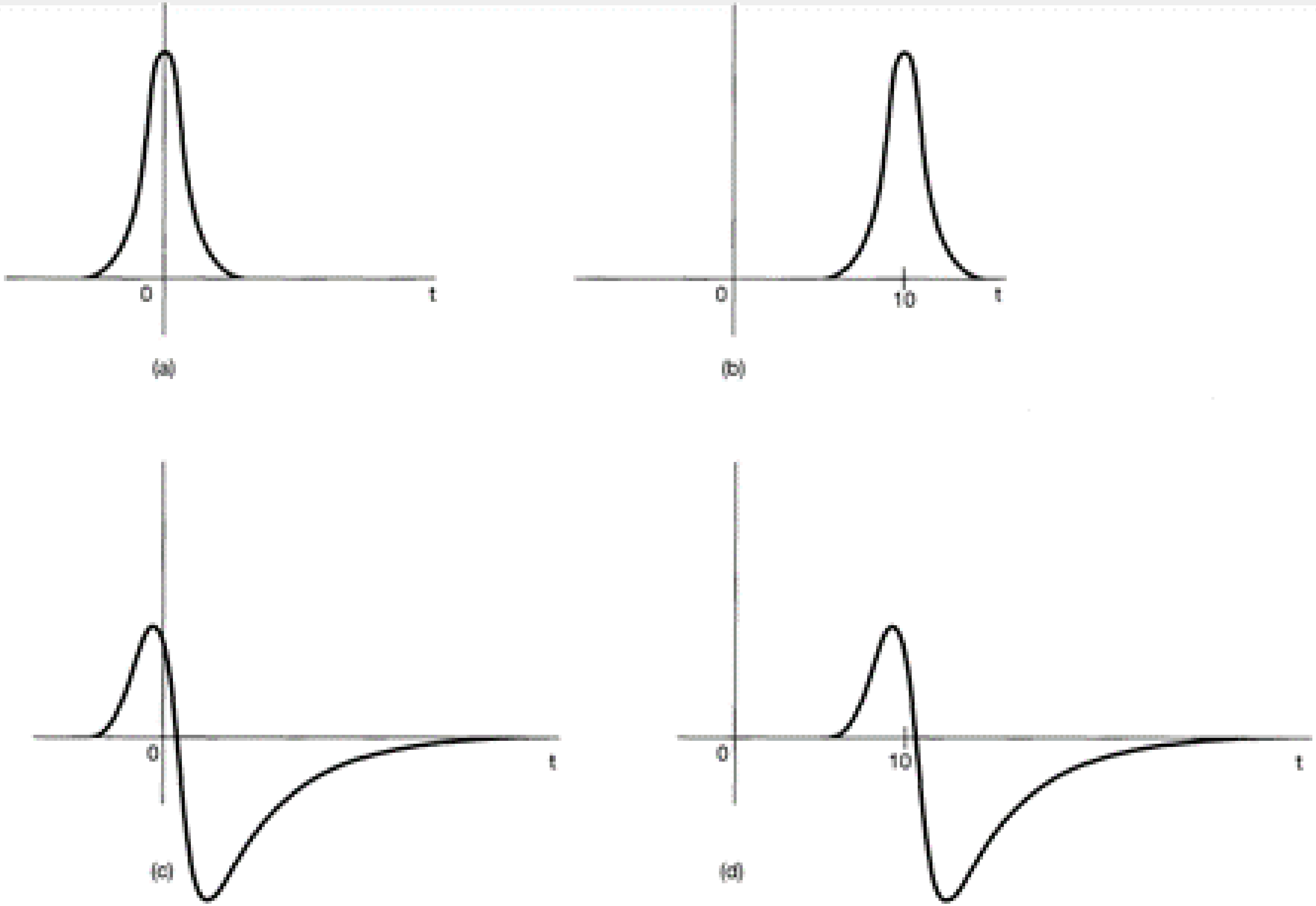
$$\sphericalangle Y(j\omega) = \sphericalangle H(j\omega) + \sphericalangle X(j\omega),$$

Linear and Nonlinear Phase

When the phase shift at the frequency w is a linear function of w, there is a particularly straightforward interpretation of the effect in the time domain. Consider the continuoustime LTI system with frequency response.

$$H(j\omega) = e^{-j\omega t_0},$$

$$|H(j\omega)| = 1, \quad \sphericalangle H(j\omega) = -\omega t_0.$$

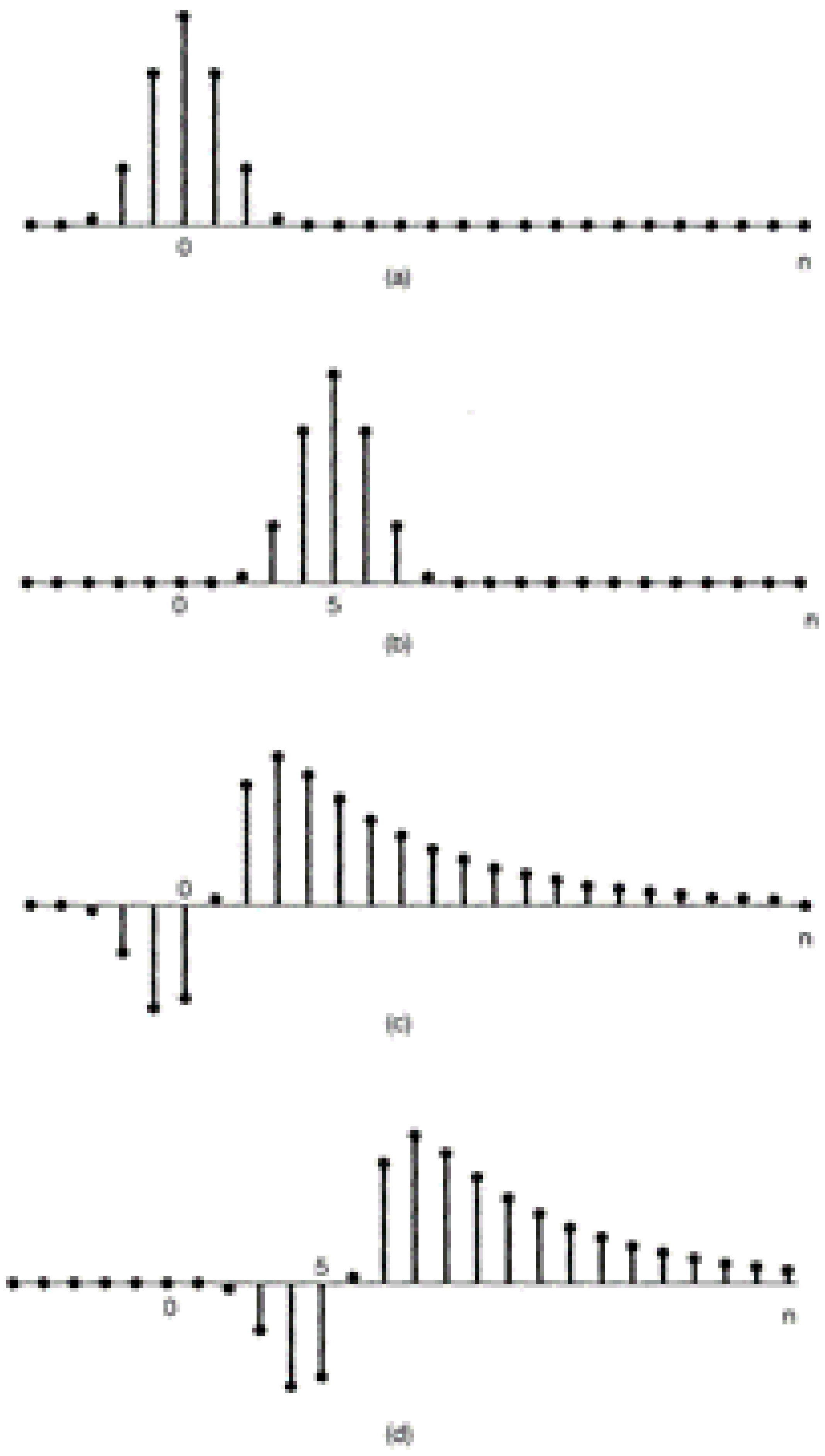

0
t
(a)
0
10
t
(b)
0
t
(c)
0
10
t
(d)

Group Delay

The concept of delay can be very naturally and simply extended to include nonlinear phase characteristics. Suppose that we wish to examine the effects of the phase of a continuous-time LTI system on a narrowband input-i.e., an input x(t) whose Fourier Transform is zero or negligibly small outside a small band of frequencies centered at w = w0 . By taking the band to be very small, we can accurately approximate the phase of this system in the band with the linear approximation.

$$\sphericalangle H(j\omega) \simeq -\phi - \omega\alpha,$$

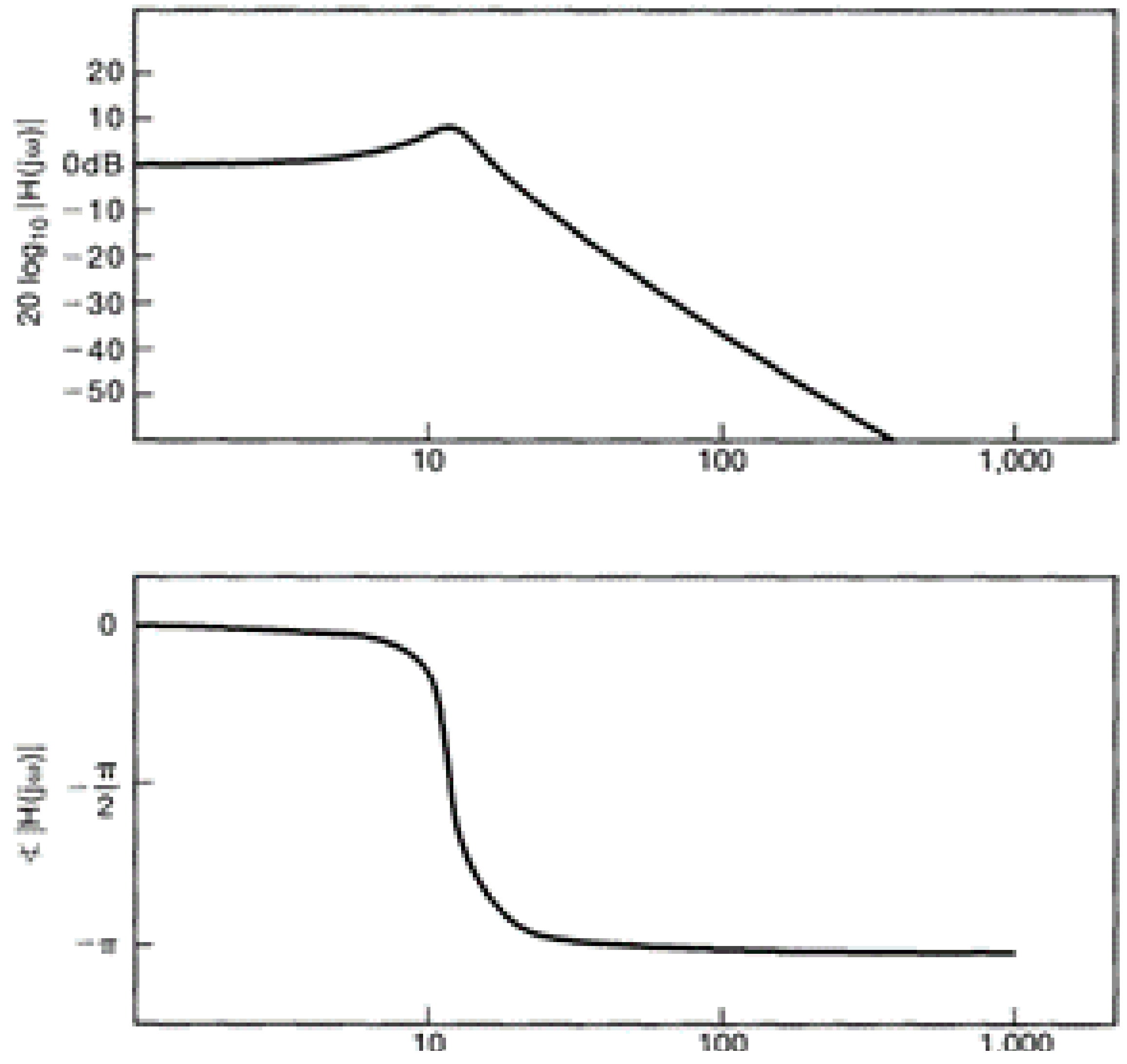
$$Y(j\omega) \simeq X(j\omega)|H(j\omega)|e^{-j\phi}e^{-j\omega\alpha}.$$

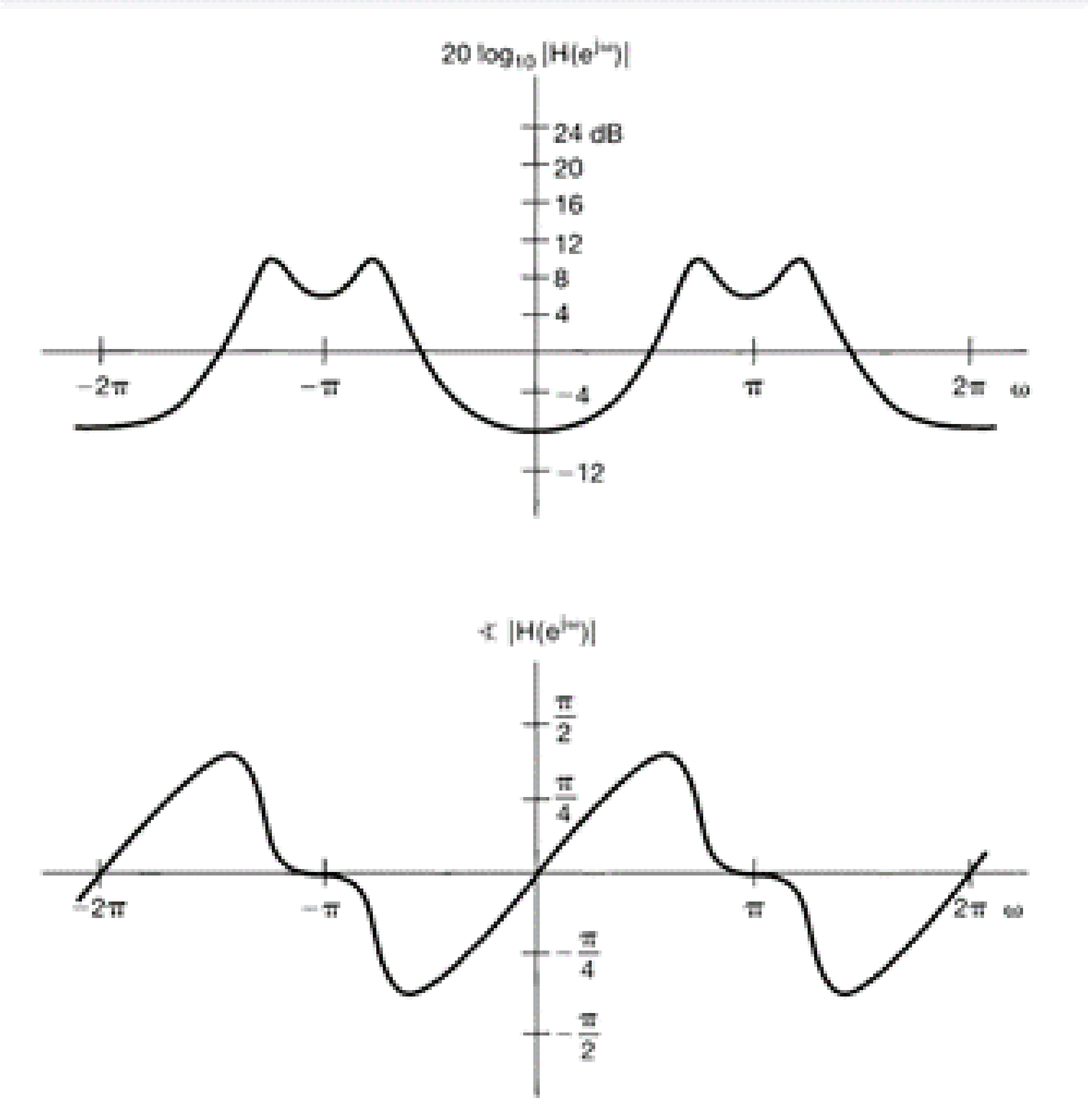


Log-Magnitude and Bode Plots

Note that the phase relationship is additive, while the magnitude relationship involves the product of |H(jw)| and |X(jw)|. Thus, if the magnitudes of the Fourier transform are displayed on a logarithmic amplitude scale takes the form of an additive relationship.

$$\log|Y(j\omega)| = \log|H(j\omega)| + \log|X(j\omega)|,$$





TIME-DOMAIN PROPERTIES OF IDEAL FREQUENCY-SELECTIVE FILTERS

A continuous-time ideal lowpass filter has a frequency response of the form

$$H(j\omega) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & |\omega| > \omega_c \end{cases}.$$

Similarly,a discrete-time ideal lowpass filter has a frequency response and is periodic with period w.

$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \omega_c < |\omega| \leq \pi \end{cases}$$

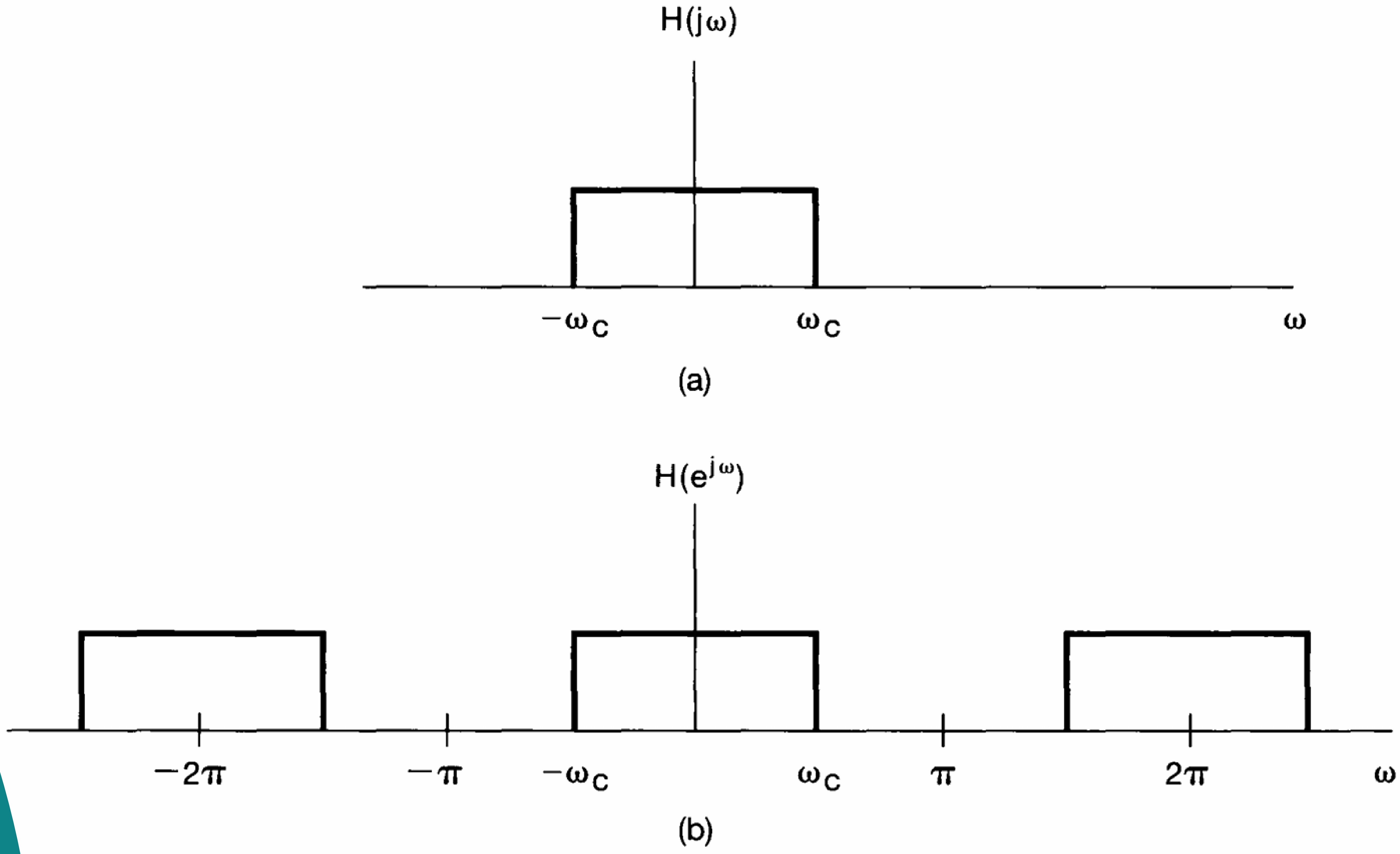


Figure 6.10 (a) The frequency response of a continuous-time ideal lowpass filter; (b) the frequency response of a discrete-time ideal lowpass filter.

- Ideal lowpass filters have perfect frequency selectivity.
- That is, they pass without attenuation all frequencies at or lower than the cutoff frequency and completely stop all frequencies in the stopband (i.e., higher than cutoff frequency).
- Moreover, these filters have zero phase characteristics, so they introduce no phase distortion.

Nonlinear phase characteristics can lead to significant changes in the time-domain characteristics of a signal even when the magnitude of its spectrum is not changed by the system.
On the other hand, an ideal filter with linear phase over the passband, as illustrated introduces only a simple time shift relative to the response of the ideal lowpass filter with zero phase characteristic.

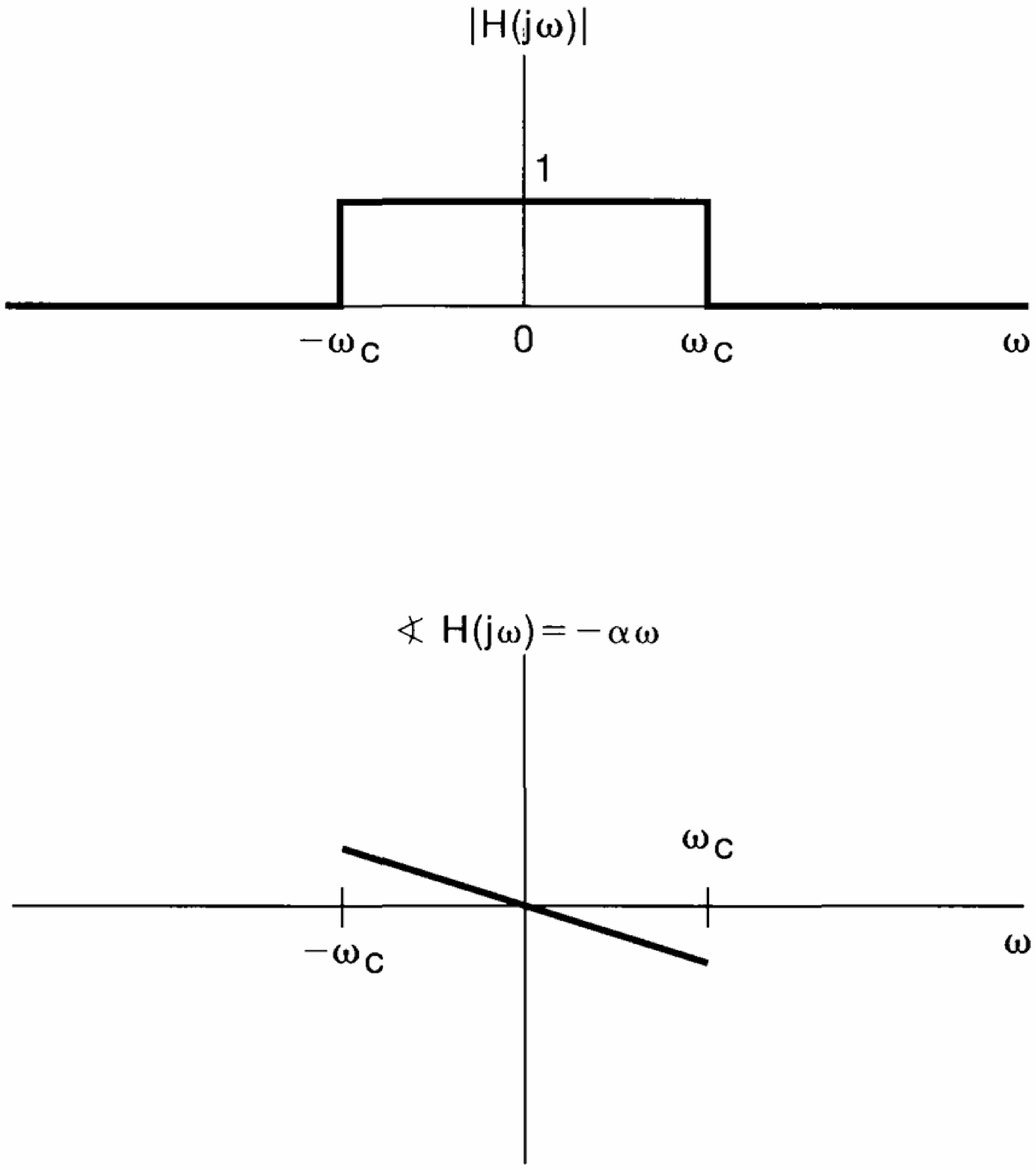


Figure 6.11 Continuous-time ideal lowpass filter with linear phase characteristic.

- If ideal frequency responses is augmented with a linear phase characteristic, the impulse response is simply delayed by an amount equal to the negative of the slope of this phase function for the continuous-time impulse response.
- Note that in both continuous and discrete time, the width of the filter passband is proportional to w, while the width of the main lobe of the impulse is proportional to 1/w. As the bandwidth of the filter increases, the impulse response becomes narrower.

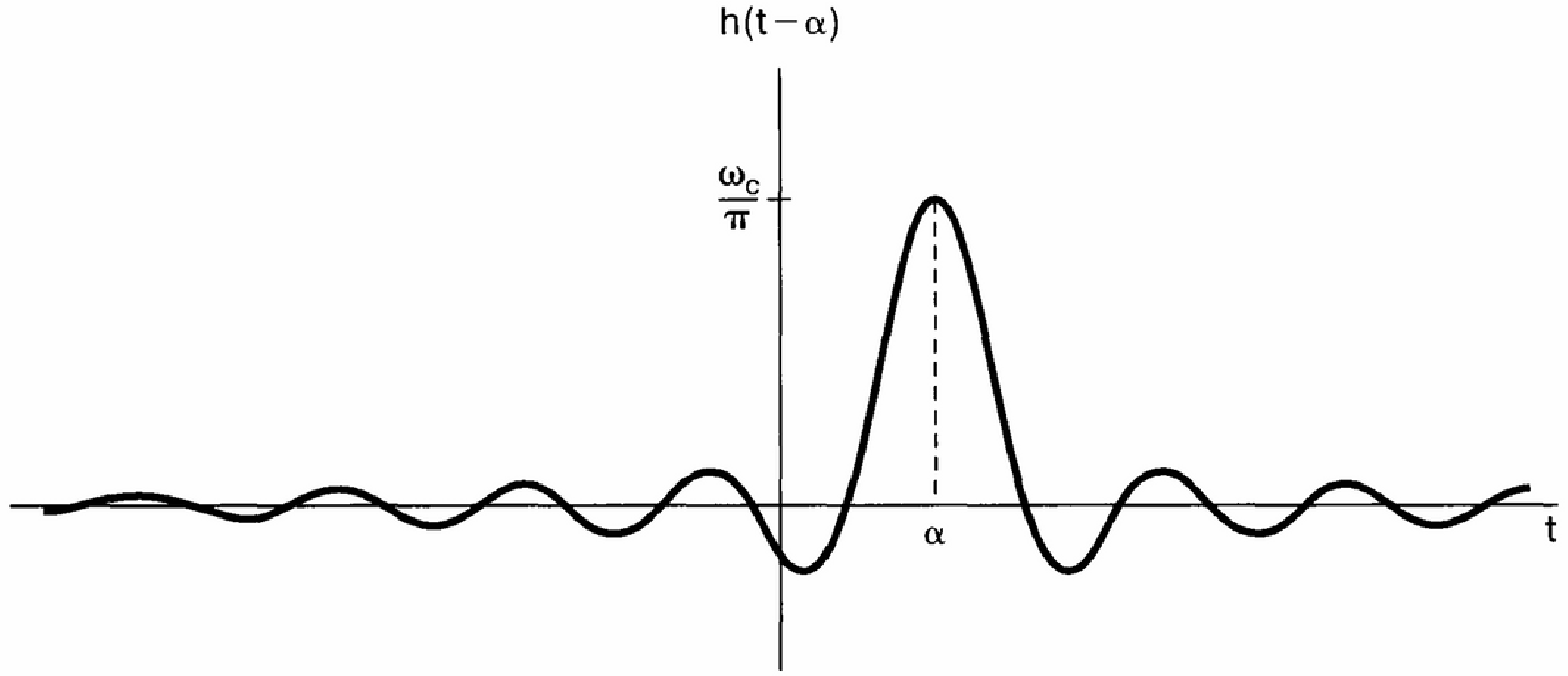


Figure 6.13 Impulse response of an ideal lowpass filter with magnitude and phase shown in Figure 6.11.

- The step responses s(t) and s[n] of the ideal lowpass filters in continuous time and discrete time are displayed on next page.
- In particular, for these filters, the step responses overshoot their long-term final values and exhibit oscillatory behavior, frequently referred to as ringing.
- The step response is the running integral or sum of the impulse response.

$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau,$$

$$s[n] = \sum_{m=-\infty}^{n} h[m].$$

The step responses undergo their most significant change in value over a time interval which is known as rise time of the step response, a rough measure of the response time of the filter, is also inversely related to the bandwidth of the filter.

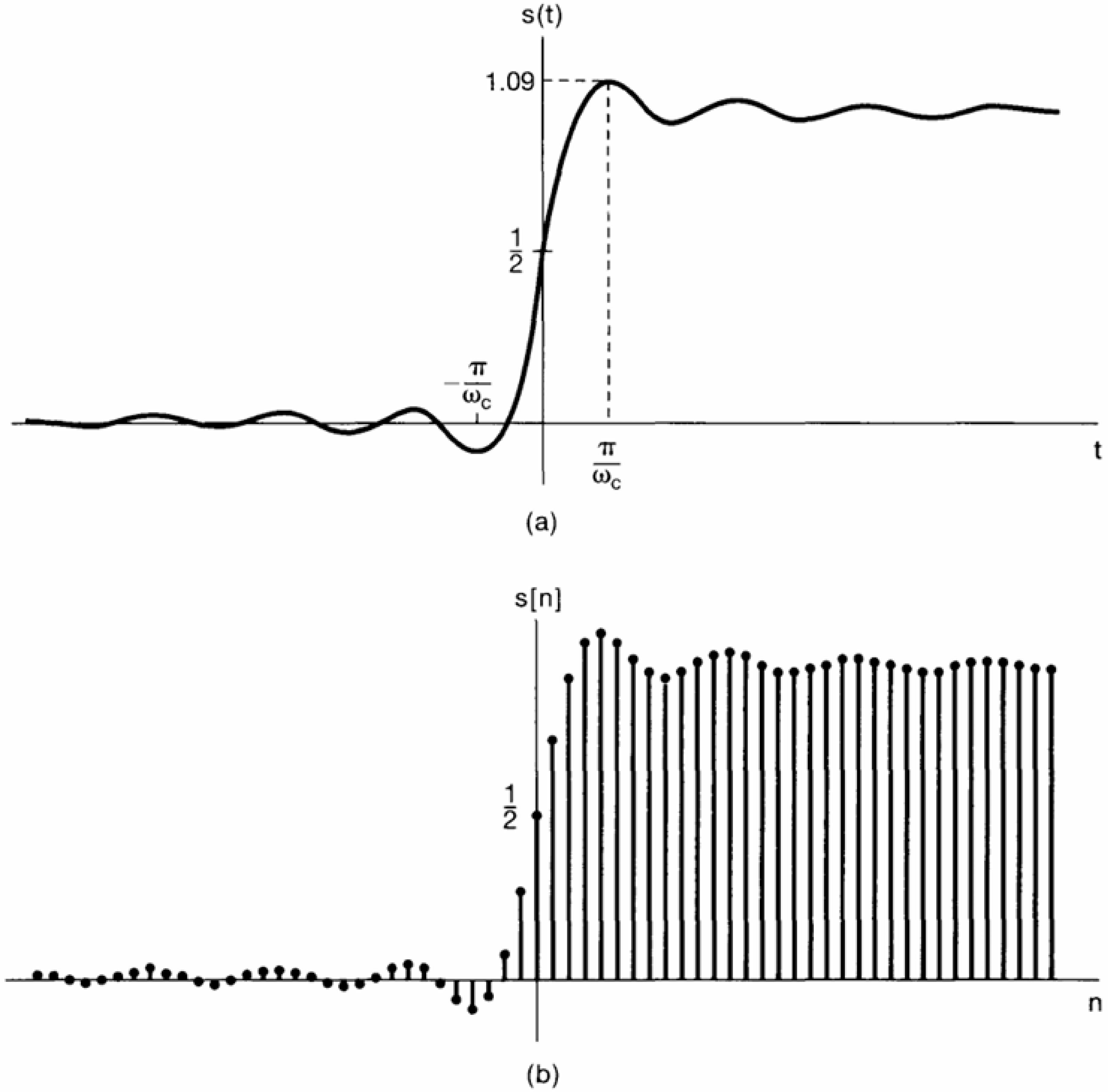
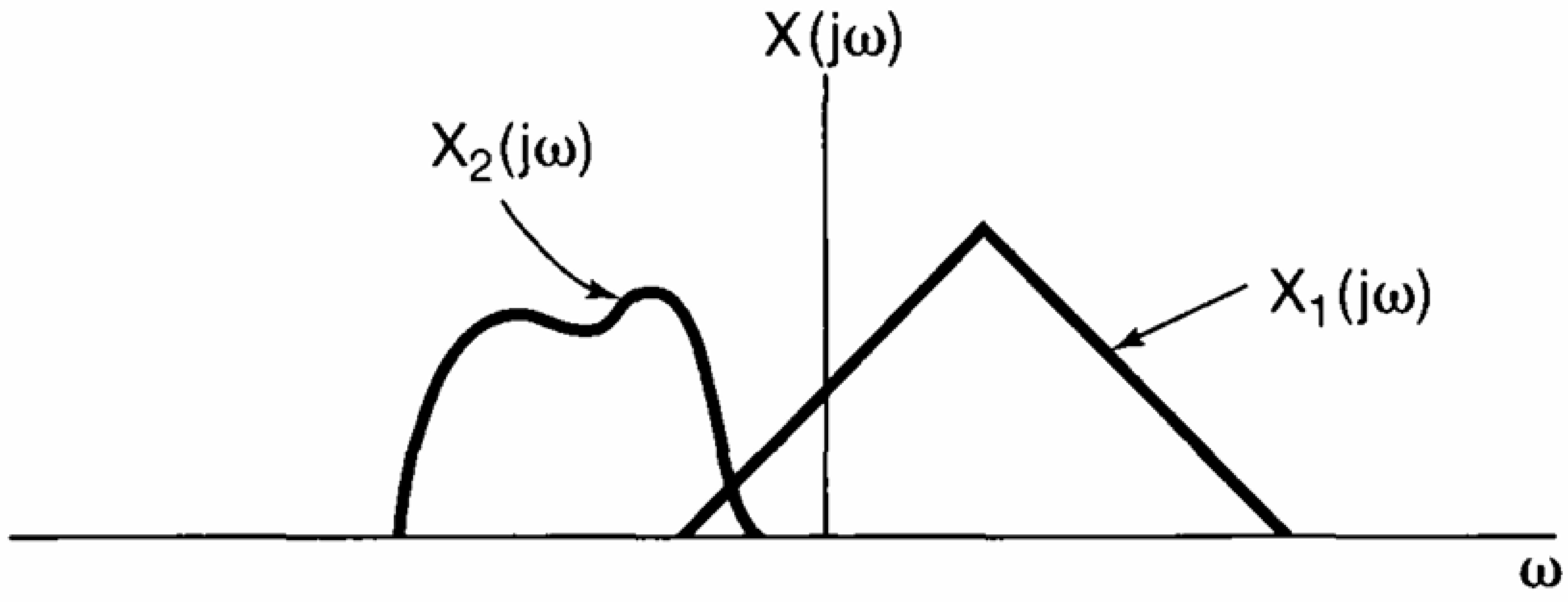


Figure 6.14 (a) Step response of a continuous-time ideal lowpass filter; (b) step response of a discrete-time ideal lowpass filter.

Time-Domain and Frequency-Domain Aspects of Non-ideal Filters

The characteristics of ideal filters are not always desirable in practice. For example, in many filtering contexts, the signals to be separated do not always lie in totally disjoint frequency bands. A typical situation might be that depicted in where the spectra of two signals overlap slightly. In such a case, we may wish to trade off the fidelity with which the filter preserves one of these signals say, x1(t) against the level to which frequency components of the second signal x2(t) are attenuated. A filter with a gradual transition from passband to stopband is generally preferable when filtering thesuperposition of signals with overlapping spectra.



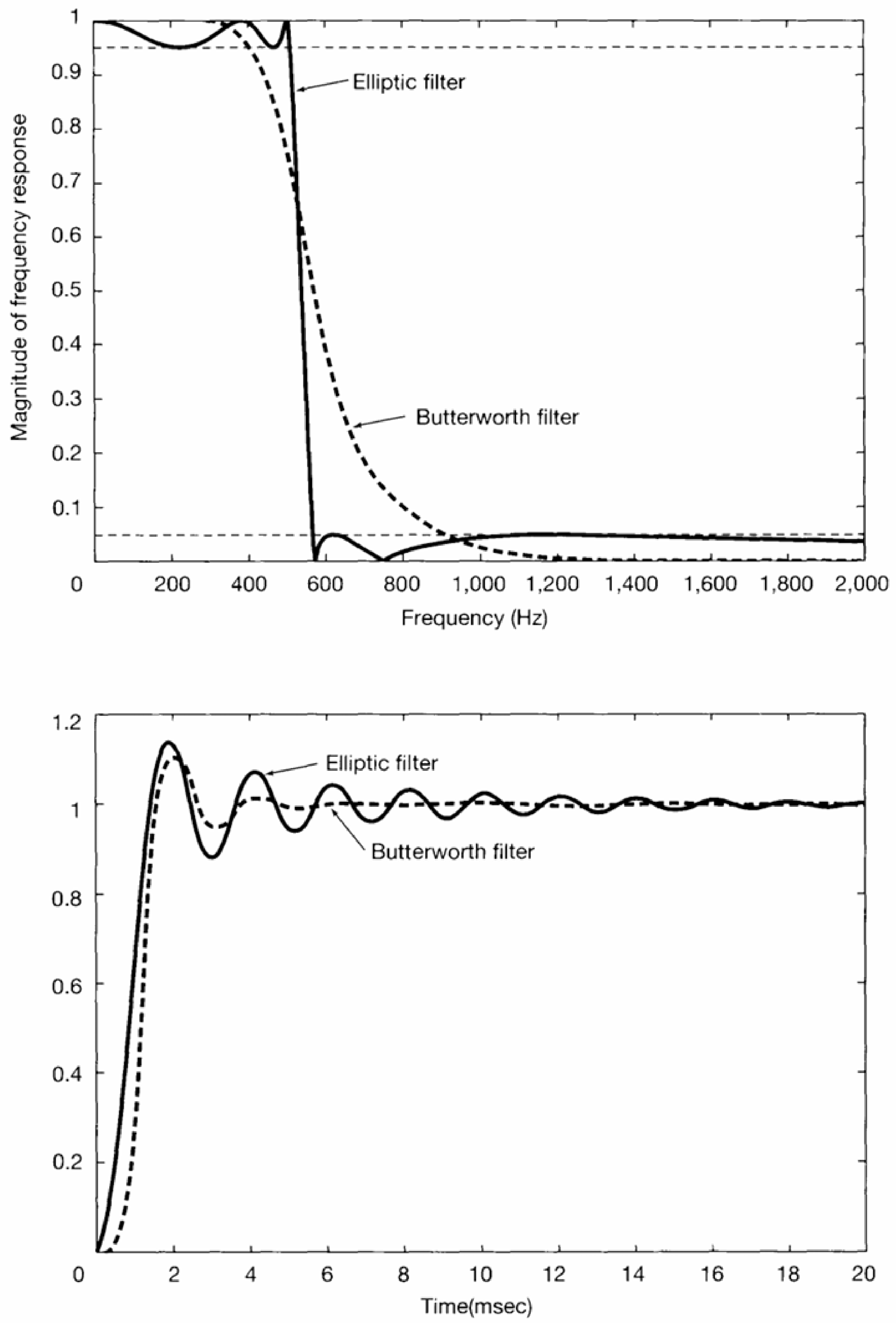
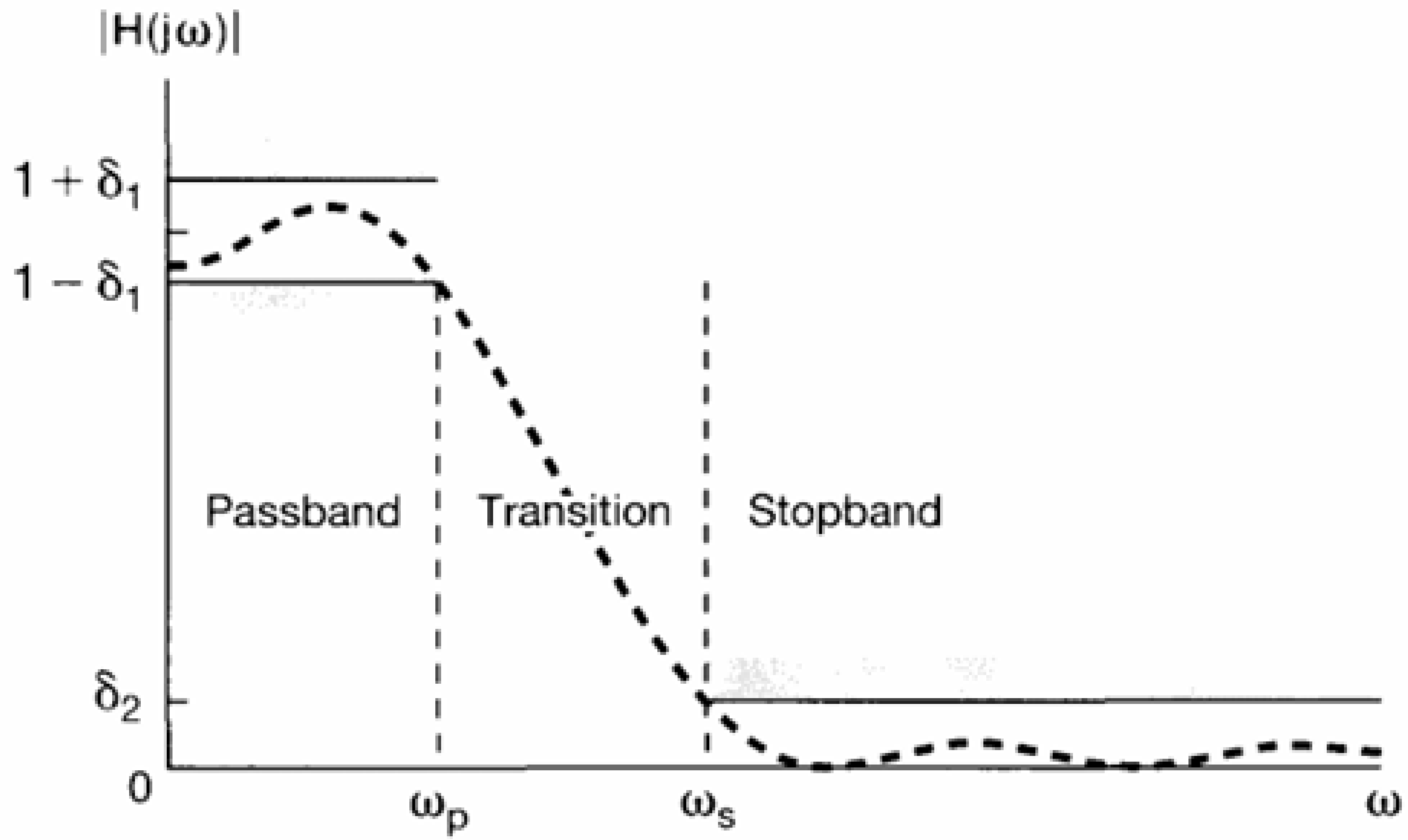


Figure 6.18 Example of a fifth-order Butterworth filter and a fifth-order elliptic filter designed to have the same passband and stopband ripple and the same cutoff frequency: (a) magnitudes of the frequency responses plotted versus frequency measured in Hertz; (b) step responses.



FIRST-ORDER AND SECOND-ORDER CONTINUOUS-TIME SYSTEMS

- Linear constant-coefficient differential equations model LTI systems, widely used in physical system modeling.
- High-order systems often simplify to combinations of first and second-order systems in cascade or parallel arrangements.
- Understanding first and second-order system properties is key for analyzing and designing higher-order systems.

First-Order Continuous-Time Systems

The differential equation for a first-order system is often expressed in the form

$$\tau \frac{dy(t)}{dt} + y(t) = x(t),$$

where the coefficient τ is a positive number whose significance will be made shortly. The corresponding frequency response for the first-order system is

$$H(j\omega) = \frac{1}{j\omega\tau + 1},$$

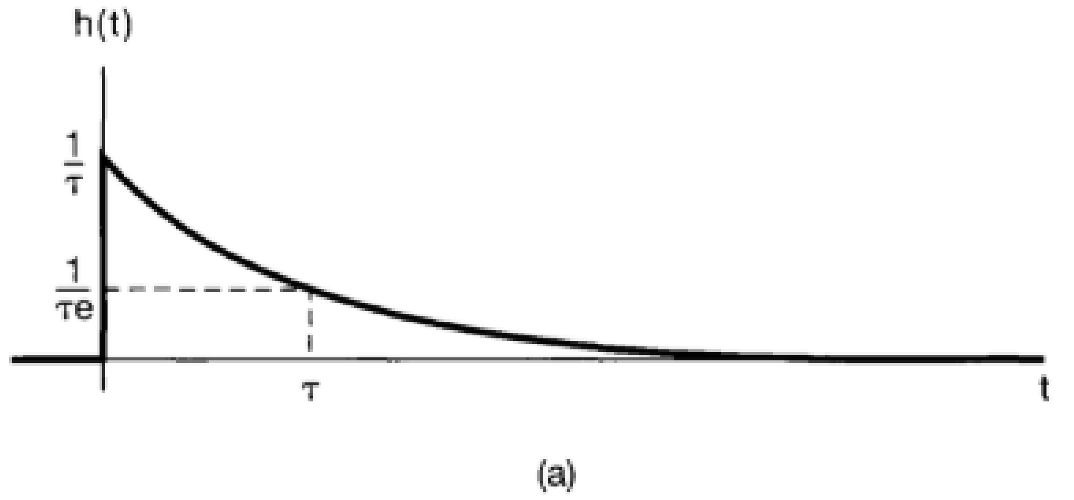
and the impulse response is

$$h(t) = \frac{1}{\tau} e^{-t/\tau} u(t),$$

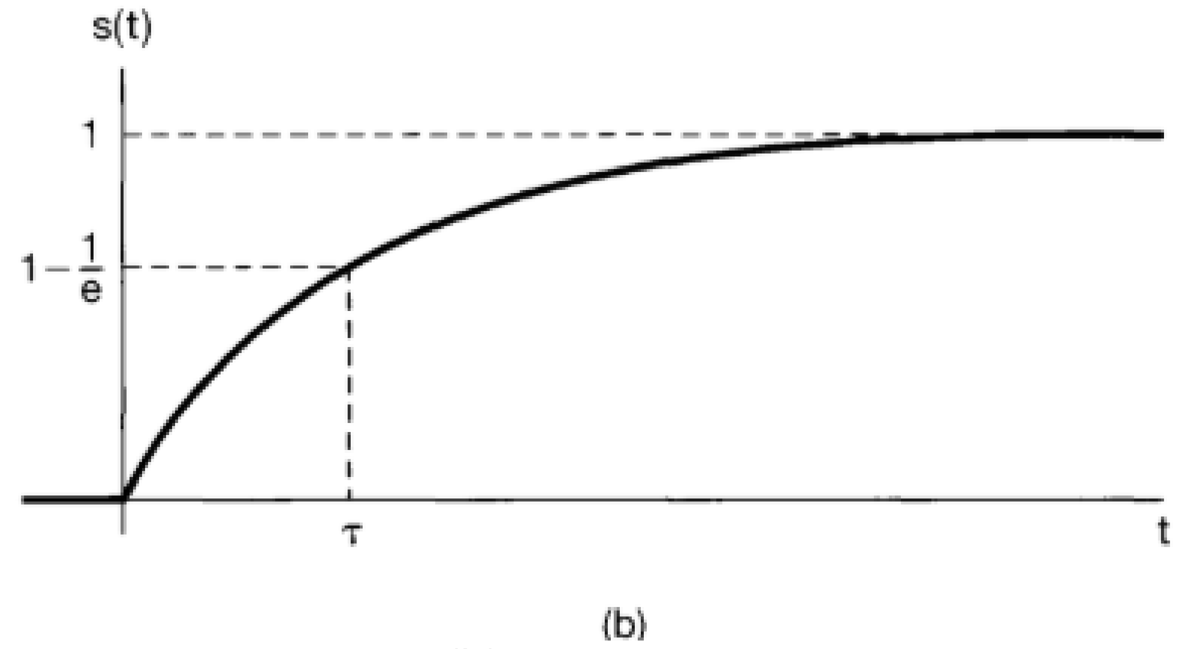
which is sketched in Figure 6.19(a). The step response of the system is

$$s(t) = h(t) * u(t) = [1 - e^{-t/\tau}]u(t).$$

Continuous-time first- order system:

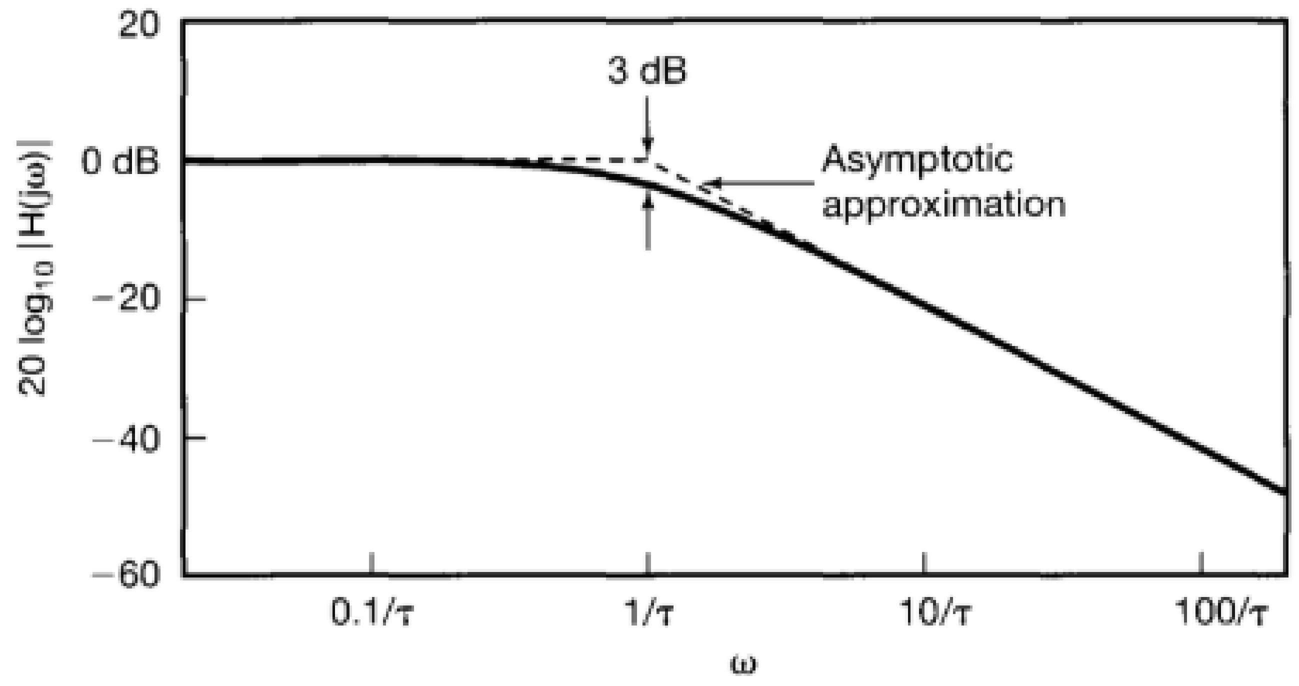


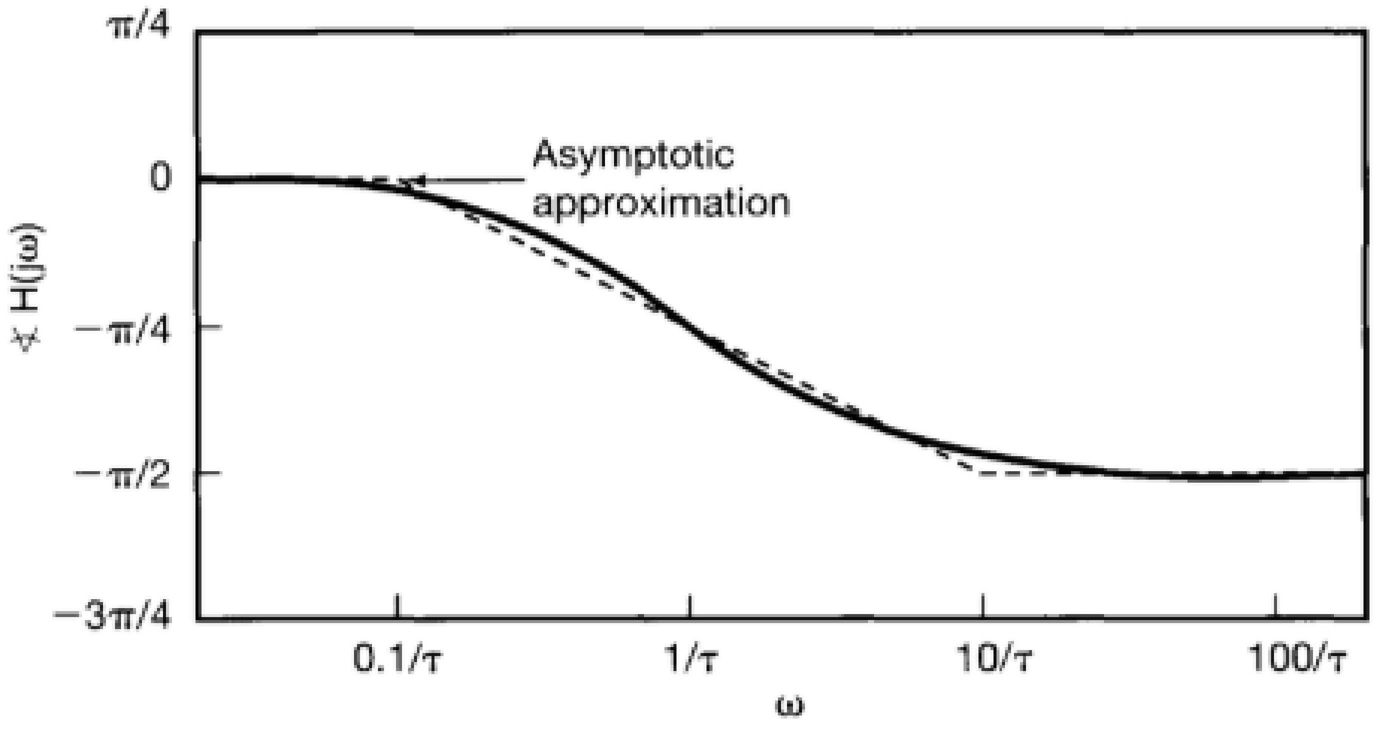
(a) impulse response



(b) step response

Bode plot for a w continuous-time first-order system.





Second-Order Continuous-Time Systems

The linear constant-coefficient differential equation for a second-order system is

$$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n\frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t).$$

$$m\frac{d^2y(t)}{dt^2} = x(t) - ky(t) - b\frac{dy(t)}{dt},$$

or

$$\frac{d^2y(t)}{dt^2} + \left(\frac{b}{m}\right)\frac{dy(t)}{dt} + \left(\frac{k}{m}\right)y(t) = \frac{1}{m}x(t).$$

Comparing this to eq. (6.31), we see that if we identify

$$\omega_n = \sqrt{\frac{k}{m}}$$

and

$$\zeta = \frac{b}{2\sqrt{km}},$$

The frequency response for the second-order system of eq. (6.31) is

$$H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}.$$

$$H(j\omega) = \frac{\omega_n^2}{(j\omega - c_1)(j\omega - c_2)},$$

$$c_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1},$$
$$c_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}.$$

$$H(j\omega) = \frac{M}{j\omega - c_1} - \frac{M}{j\omega - c_2},$$

$$M = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}.$$

$$h(t) = M[e^{c_1 t} - e^{c_2 t}]u(t).$$

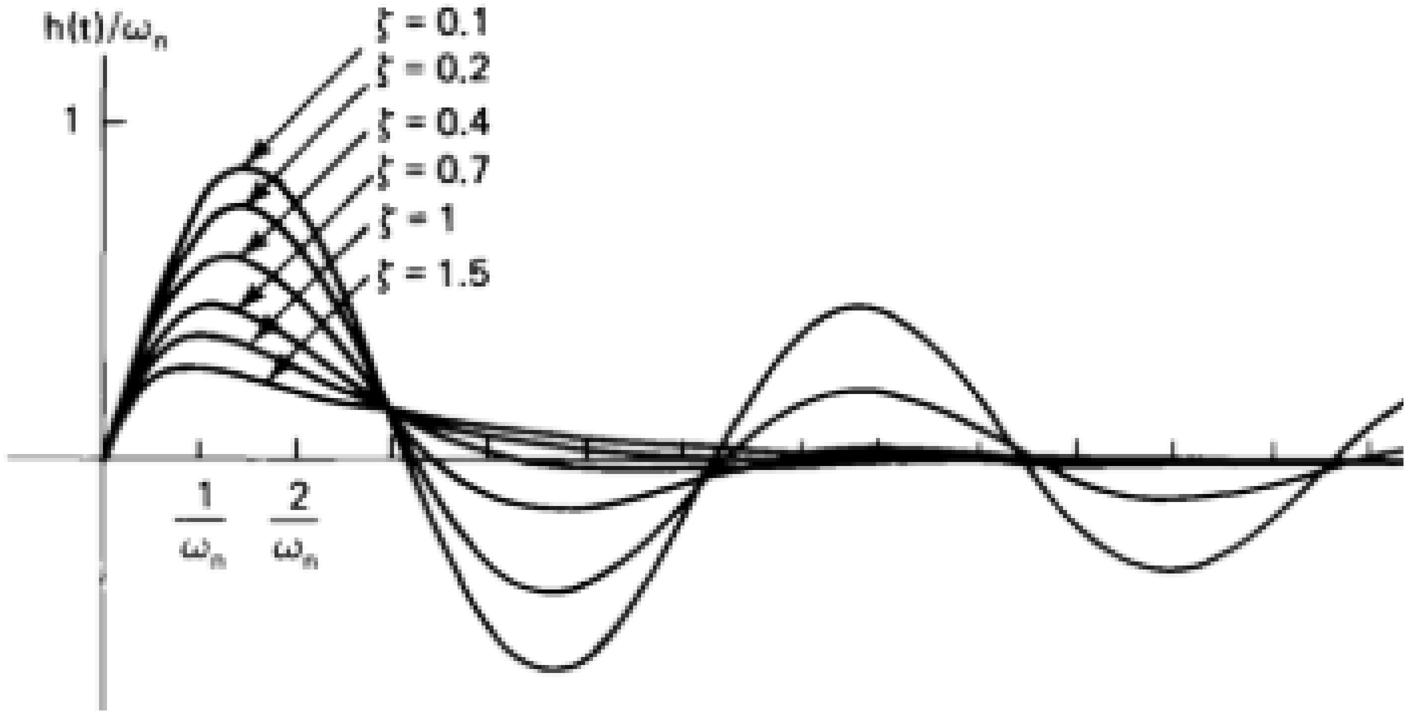
If $\zeta = 1$, then $c_1 = c_2 = -\omega_n$, and

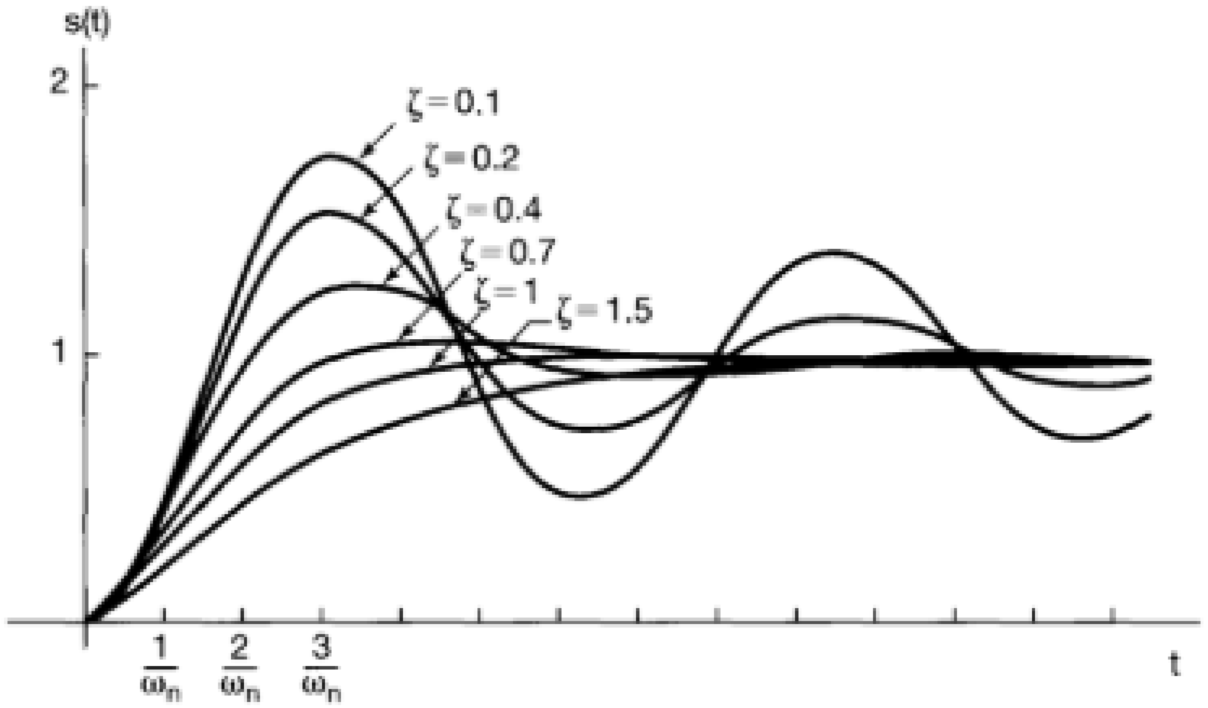
$$H(j\omega) = \frac{\omega_n^2}{(j\omega + \omega_n)^2}.$$

$$h(t) = \omega_n^2 t e^{-\omega_n t} u(t).$$

Note from eqs. (6.37) and (6.39), that $h(t)/\omega_n$ is a function of $\omega_n t$. eq. (6.33) can be rewritten as

$$H(j\omega) = \frac{1}{(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1},$$





Bode Plots for Rational Frequency Responses

$$H(j\omega) = 1 + j\omega\tau$$

and

$$H(j\omega) = 1 + 2\zeta\left(\frac{j\omega}{\omega_n}\right) + \left(\frac{j\omega}{\omega_n}\right)^2.$$

The Bode plots for eqs. (6.49) and (6.50) follow directly from Figures from the fact that

$$20\log_{10}|H(j\omega)| = -20\log_{10}\left|\frac{1}{H(j\omega)}\right|$$

.

$$H(j\omega) = K.$$

Since $K = |K|e^{j\cdot 0}$ if $K > 0$ and $K = |K|e^{j\pi}$ if $K < 0$, we see that

$$20\log_{10}|H(j\omega)| = 20\log_{10}|K|$$

$$\sphericalangle H(j\omega) = \begin{cases} 0, & \text{if } K > 0 \\ \pi, & \text{if } K < 0 \end{cases}$$

FIRST-ORDER AND SECOND-ORDER DISCRETE-TIME SYSTEMS

Any system with a frequency response that is a ratio of polynomials in (e-jw) i.e., any discrete-time LTI system described by a linear constant-coefficient difference equation-can be written as a product or sum of first- and second-order systems, implying that these basic systems are of considerable value in both implementing and analyzing more complex systems

Consider the first-order causal LTI system described by the difference equation

$$y[n] - ay[n-1] = x[n],$$

with $|a| < 1$. From Example 5.18, the frequency response of this system is

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}},$$

and its impulse response is

$$h[n] = a^n u[n],$$

Also, the step response of the system is:

$$s[n] = h[n] * u[n] = \frac{1 - a^{n+1}}{1 - a} u[n],$$

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Consider next the second-order causal LTI system described by

$$y[n] - 2r\cos\theta\, y[n-1] + r^2 y[n-2] = x[n],$$

with $0 < r < 1$ and $0 \leq \theta \leq \pi$. The frequency response for this system is

$$H(e^{j\omega}) = \frac{1}{1 - 2r\cos\theta\, e^{-j\omega} + r^2 e^{-j2\omega}}.$$

The denominator of $H(e^{j\omega})$ can be factored to obtain

$$H(e^{j\omega}) = \frac{1}{[1-(re^{j\theta})e^{-j\omega}][1-(re^{-j\theta})e^{-j\omega}]}.$$

For $\theta \neq 0$ or π, the two factors in the denominator of $H(e^{j\omega})$ are different, and a partial-fraction expansion yields

$$H(e^{j\omega}) = \frac{A}{1-(re^{j\theta})e^{-j\omega}} + \frac{B}{1-(re^{-j\theta})e^{-j\omega}},$$

where

$$A = \frac{e^{j\theta}}{2j\sin\theta}, \quad B = \frac{e^{-j\theta}}{2j\sin\theta}.$$

In this case, the impulse response of the system is

$$h[n] = [A(re^{j\theta})^n + B(re^{-j\theta})^n]u[n]$$
$$= r^n \frac{\sin[(n+1)\theta]}{\sin\theta} u[n].$$

For $\theta = 0$ or π, the two factors in the denominator of eq. (6.58) are the same. When $\theta = 0$,

$$H(e^{j\omega}) = \frac{1}{(1 - re^{-j\omega})^2}$$

and

$$h[n] = (n+1)r^n u[n].$$

When $\theta = \pi$,

$$H(e^{j\omega}) = \frac{1}{(1 + re^{-j\omega})^2}$$

and

$$h[n] = (n+1)(-r)^n u[n].$$

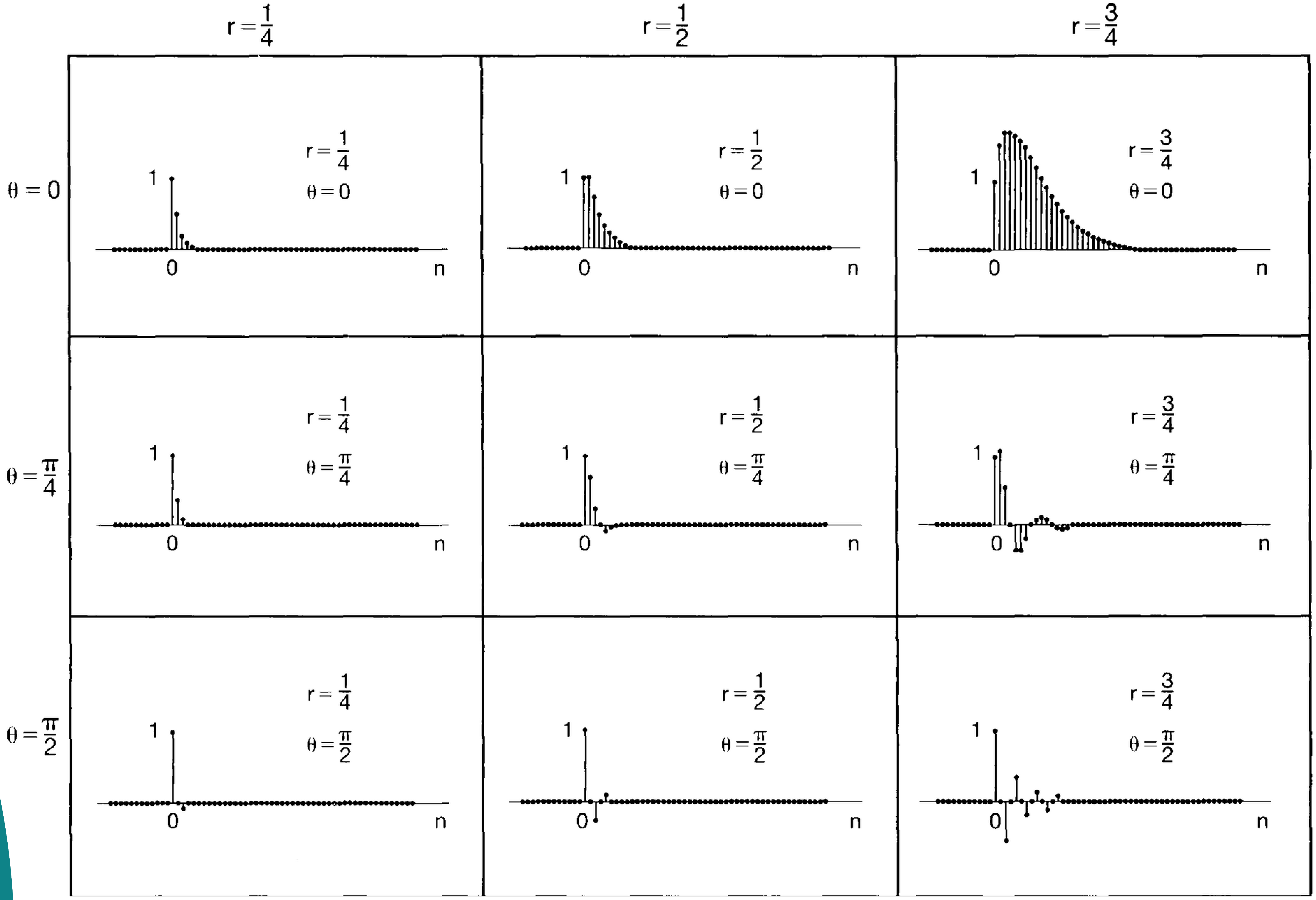
We see that the rate of decay of h[n] is controlled by r-i.e., the closer r is to 1, the slower is the decay in h[n]. Similarly, the value of theta determines the frequency of oscillation. For example, with theta = 0 there is no oscillation in h[n], while for theta = pi the oscillations are rapid. For theta = (

$$s[n] = h[n] * u[n] = \left[A\left(\frac{1-(re^{j\theta})^{n+1}}{1-re^{j\theta}}\right) + B\left(\frac{1-(re^{-j\theta})^{n+1}}{1-re^{-j\theta}}\right)\right]u[n].$$

$$s[n] = \left[\frac{1}{(r-1)^2} - \frac{r}{(r-1)^2}r^n + \frac{r}{r-1}(n+1)r^n\right]u[n],$$

while for $\theta = \pi$,

$$s[n] = \left[\frac{1}{(r+1)^2} + \frac{r}{(r+1)^2}(-r)^n + \frac{r}{r+1}(n+1)(-r)^n\right]u[n].$$



r = 1/4
r = 1/2
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θ = π/4
θ = π/2
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θ = 0
r = 3/4
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θ = π/4
r = 1/2
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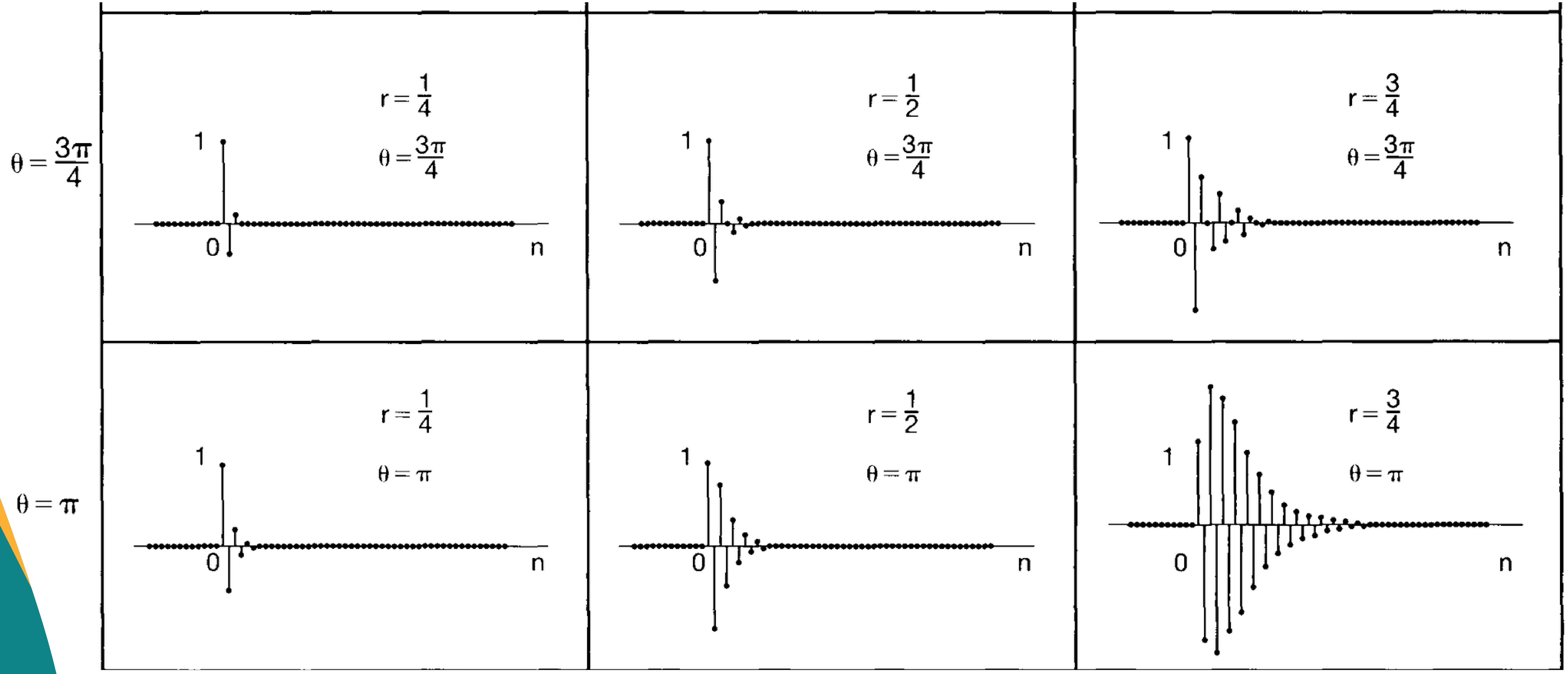

$\theta = \frac{3\pi}{4}$
$\theta = \pi$
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$\theta = \frac{3\pi}{4}$
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$r = \frac{3}{4}$
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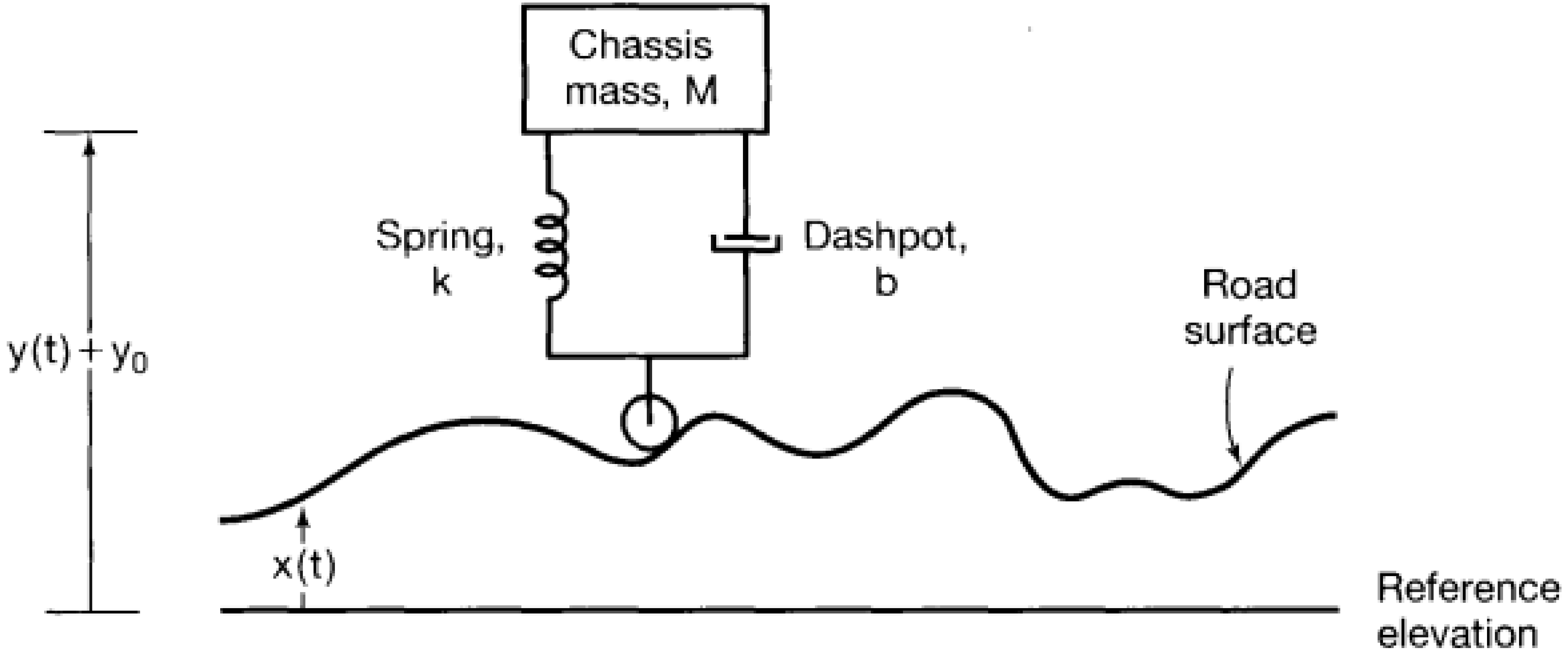
EXAMPLES OF TIME- AND FREQUENCY-DOMAIN ANALYSIS OF SYSTEMS

Analysis of an Automobile Suspension System

- The automobile suspension system is generally intended to filter out rapid variations in the ride caused by the road surface (i.e., the system acts as a lowpass filter).

- The basic purpose of the suspension system is to provide a smooth ride, and there is no sharp, natural division between the frequencies to be passed and those to be rejected.



Diagrammatic representation of an automotive suspension system. Here, Yo represents the distance between the chassis and the road surface when the automobile is at rest, y(t) + Yo the position of the chassis above the reference elevation, and x(t) the elevation of the road above the reference elevation.

- In the previous diagrammatic representation , y0 represents the distance between the chassis and the road surface when the automobile is at rest, y(t) + y0 the position of the chassis above the reference elevation, and x(t) the elevation of the road above the reference elevation.

- The differential equation governing the motion of the chassis is :-

$$\boxed{M\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + ky(t) = kx(t) + b\frac{dx(t)}{dt}}$$

where M is the mass of the chassis and k and b are the spring and shock absorber constants, respectively

The frequency response of the system is :-

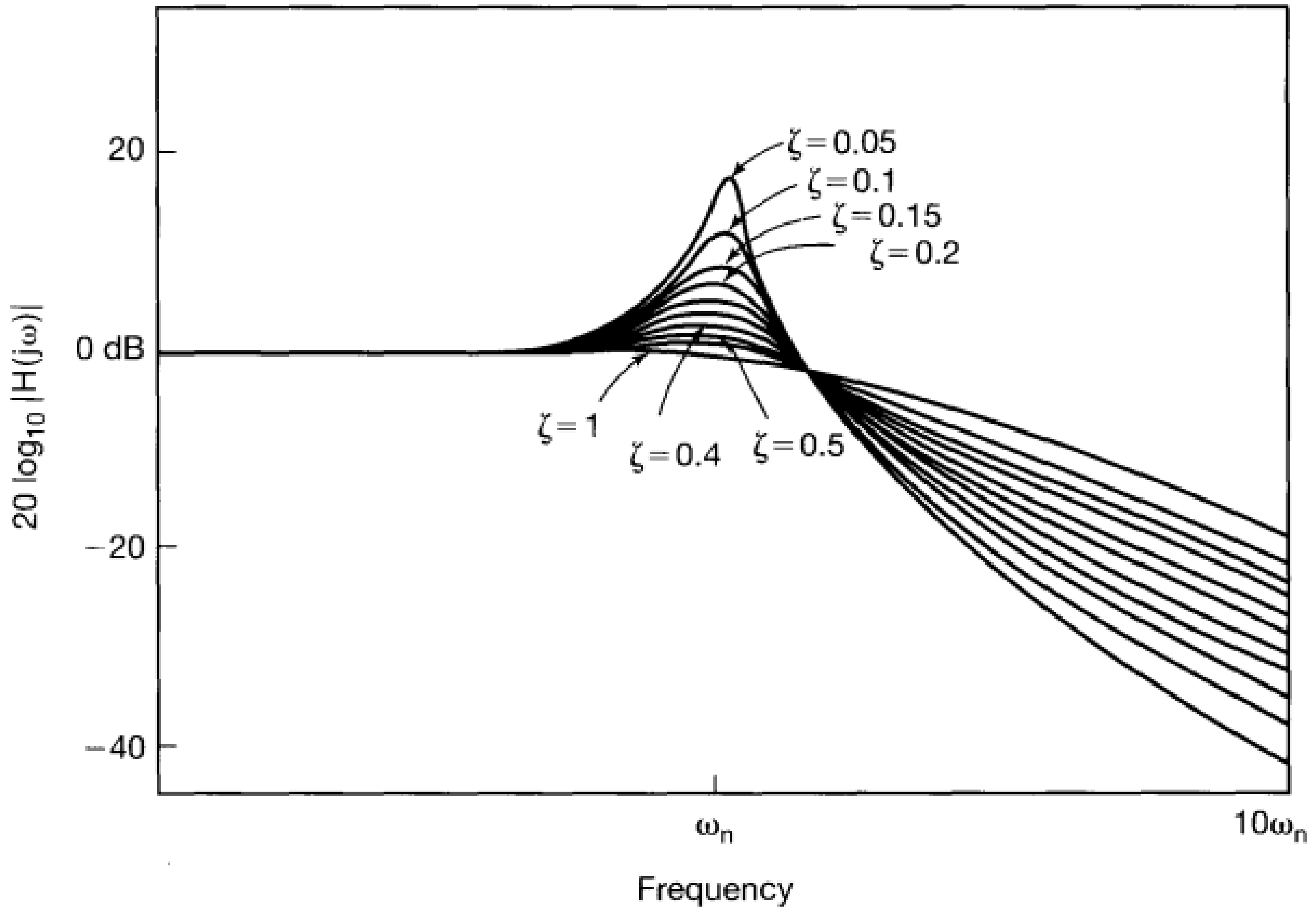
$$H(j\omega) = \frac{k + b\,j\omega}{(j\omega)^2 M + b(j\omega) + k},$$

or

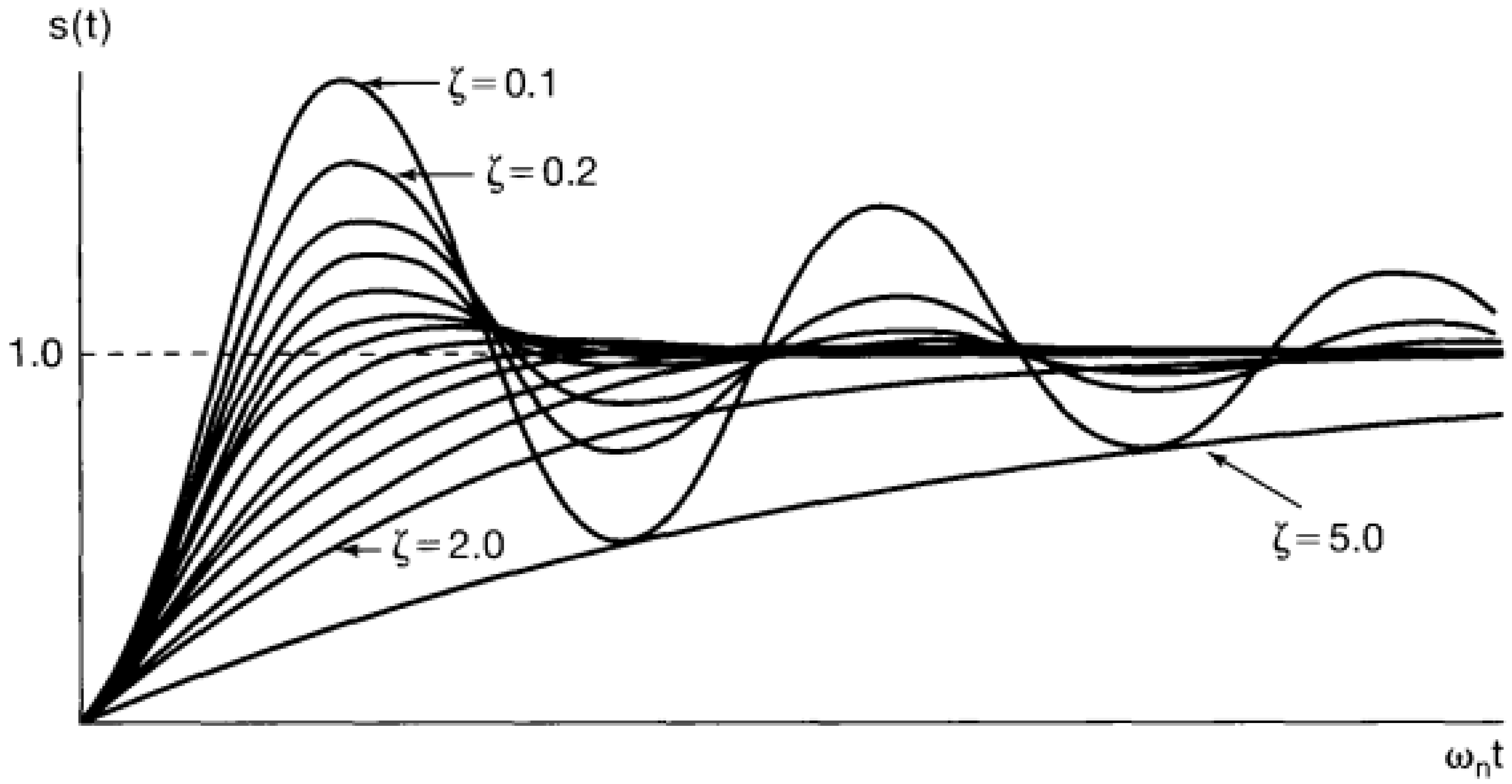
$$H(j\omega) = \frac{\omega_n^2 + 2\zeta\omega_n(j\omega)}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2},$$

where $\omega_n = \sqrt{\frac{k}{M}}$ and $2\zeta\omega_n = \frac{b}{M}.$

The parameter ω_n is referred to as the undamped natural frequency and ζ as the damping ratio. A Bode plot of the log magnitude of the frequency response in equation can be constructed by using first-order and second-order Bode plots.



Bode plot for the magnitude of the frequency response of the automobile suspension system for several values of the damping ratio.



Step response of the automotive suspension system for various values of the damping ratio (ζ = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.2, 1.5, 2.0, 5.0).