Position-dependent power spectrum of 21cm radiation background

The constrast in 21cm brightness temperature (difference between 21cm brightness T and CMB T) is

$$T_b(\vec{r}) = \tilde{T}_b \left[1 - X_i(\vec{r})\right](1+\delta)\left[1 - \frac{1}{Ha}\frac{\partial v_r}{\partial r}\right]\left(\frac{T_s - T_{CMB}}{T_s}\right)$$

where $\tilde{T}_b \simeq 23.88\,\text{mK}\left(\frac{\Omega_b h^2}{0.02}\right)\left(\frac{0.15}{\Omega_m h^2}\frac{1+z}{10}\right)^{1/2}$, $X_i(\vec{r}) \equiv 1 - X_{HI}(\vec{r})$

and δ is the hydrogen mass density contrast. Ignoring redshift space distortions and assuming $T_s \gg T_{CMB}$, this becomes

$$T_b(\vec{r}) = \tilde{T}_b\, X_{HI}(\vec{r})(1+\delta)$$

Let us further define $\delta_X \equiv \frac{X_{HI}(\vec{r}) - \bar{X}_{HI}}{\bar{X}_{HI}}$ such that

$$T_b(\vec{r}) = \tilde{T}_b\, \bar{X}_{HI}(1+\delta_X)(1+\delta)$$

From this, the power spectrum of $\Delta T_b = T_b(\vec{r}) - \bar{T}_b$ is (to leading order)

$$P_{\Delta T}(\vec{k}) = \tilde{T}_b^2\, \bar{X}_{HI}^2\left[P_{\delta\delta} + P_{\delta_X\delta_X} + 2P_{\delta\delta_X}\right]$$

Or, using $P_{XX} = \bar{X}_{HI}^2 P_{\delta_X\delta_X}$ and $P_{X\delta} = \bar{X}_{HI} P_{\delta_X\delta}$,

$$P_{\Delta T}(\vec{k}) = \tilde{T}_b^2\left(\bar{X}_{HI}^2 P_{\delta\delta} + P_{XX} + \bar{X}_{HI} P_{X\delta}\right)$$

(This is close to what Eiichiro wrote in his email)

Now let us make this a function of the long-wavelength mode δ_l (suppress k-dependence for brevity)

$$P_{\Delta T}(\delta_l) = \tilde{T}_b^2 \left(\bar{x}_{HI}^2(\delta_l)\, P_{\delta\delta}(\delta_l) + P_{xx}(\delta_l) + 2\bar{x}_{HI}(\delta_l)\, P_{x\delta}(\delta_l) \right)$$

We want to compute $dP_{\Delta T}/d\delta_l \Big|_{\delta_l=0}$

I think Eiichiro's suggestion is to first explore the regime where dPx,x/ddelta and dPx,delta/ddelta are negligible. This is like assuming that the neutral fraction in each sub volume is uniform (i.e. ignoring the bubbles). In this case, the response just becomes

$$\left.\frac{dP_{\Delta T}}{d\delta_l}\right|_{\delta_l=0} = \tilde{T}_b^2 \left[2x_{HI}(0) \left.\frac{d\bar{x}_{HI}}{d\delta_l}\right|_{\delta_l=0} P_{\delta\delta}(0) + \bar{x}_{HI}^2(0) \left.\frac{dP_{\delta\delta}}{d\delta_l}\right|_{\delta_l=0} \right]$$

It is quite easy to compute dxHI/ddelta from the excursion-set model of reionization. In that model, the volume-weighted average ionized fraction is simply proportional to the number of ionizing photons produced within the volume, which itself is proportional to the fraction of mass that has collapsed to form stars. This, in turn, is assumed to be proportional to the fraction of mass within halos. Let Mmin be the minimum mass threshold for a halo to form stars. Then the ionized fraction within some large sphere with mean density contrast delta_l can be written

$$\bar{x}_i(\delta_l) = \xi \,\mathrm{erfc}\left[\frac{\delta_c - \delta_l}{\sqrt{2(S_{min} - S_l)}} \right]$$

Here, S_min = sigma^2(Mmin) is the variance of linear density fluctuations within spheres that are the size of the Lagrangian spheres of halos with mass Mmin, S_l is the same but for the large-scale sphere, delta_c is the critical threshold for collapse in the spherical top-hat collapse model, and the pre-factor is some efficiency parameter for star-formation and ionizing photon output of the sources.

Since $\bar{x}_{HI} = 1 - \bar{x}_i$, we can easily compute

$$\boxed{\left.\frac{d\bar{x}_{HI}}{d\delta_l}\right|_{\delta_l=0} = -\sqrt{\frac{2}{\pi}} \frac{\exp\left[\frac{-\delta_c^2}{2(S_{min}-S_l)}\right]}{\sqrt{S_{min}-S_l}}}$$

Note that, in evaluating this expression, one must be careful about whether delta_l is linearly extrapolated to present-day.