PHYS281 Project Report

Simulating Physical Systems That Evolve With Time

Sam Blythe

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Abstract

Using computer programs to simulate physical systems that evolve with time is one of the most important breakthroughs in modern technology. A computer can do calculations at incredibly higher speeds than a human and they can process huge amounts of data. This breakthrough is what allows modern GPS systems to work, how the exact positions of bodies in our solar system are calculated or even how objects in video games are rendered. The aim of this project is to program a simulation in Python code that evolves a physical system over time. The system I have chosen is the solar system and how planets and the sun move under their mutual gravitational influence through time. I have produced plots of orbits of planets and done tests to show that my simulation does not violate any laws of conservation. There are also different algorithms to use when making calculations such as the Euler method or Euler-Cromer method. After comparison it can be shown that the Euler-Cromer method is the more accurate of the two.

1 Introduction

The motion of a solar system is determined by the gravitational interactions of all the bodies in the system. Gravity is an attractive force that works over very long ranges so all bodies in a solar system exert some force onto each other. This therefore means each body has a particular acceleration at any given moment, cuased by all other bodies. This is determined by equation 1

$$\bar{a}_i = \sum_{j \neq i}^{N} \frac{-Gm_j}{\mid \bar{r}_{ij} \mid^2} \hat{r}_{ij} \tag{1}$$

where N is the number of bodies, G the gravitational constant, m_j the mass of the other body, $\hat{r}_{ij}$ the unit vector in the direction of the other body, and $\bar{r}$ the vector between the two bodies. This equation sums all the accelerations caused by all other bodies on one body. The acceleration of the body in question can then be updated and used to calculate position and velocity. There are two ways to calculate position and velocity. The first is known as the Euler Method and is given by equations 2 & 3

$$\bar{x}_{n+1} \approx \bar{x}_n + \bar{v}_n \Delta t \tag{2}$$

$$\bar{v}_{n+1} \approx \bar{v}_n + \bar{a}_n \Delta t \tag{3}$$

Where new position $\bar{x}_{n+1}$ is calculated first using old position $\bar{x}_n$, velocity $\bar{v}_n$ and time step Δt and new velocity $\bar{v}_{n+1}$ is calculated second using old velocity $\bar{v}_n$, acceleration $\bar{a}_n$ and time step.

The second is known is the Euler-Cromer method and it is given by equations 3 & 4

$$\bar{x}_{n+1} \approx \bar{x}_n + \bar{v}_{n+1} \Delta t \tag{4}$$

Where new velocity is calculated first then new position is calculated using the new velocity. In this report I will analyse the the effectiveness of these two methods and my simulation. Firstly, I will compare my simulation result with the known analytical solution for a two body problem. I will then plot my orbits of planets to see if they match with expected results. Finally I will calculate the conserved quantities of the system such as Angular Momentum

$$\bar{L} = r \times \bar{p} \tag{5}$$

and Linear Momentum

$$\bar{p} = m \times \bar{v} \tag{6}$$

Where the values of $\bar{L}$ and $\bar{p}$ are summed for each body to find the total momentums of the system as a whole (which is the quantity that is being tested to see if it is conserved) I will then plot fractional change

$$\frac{\Delta L}{L} = \mid \frac{L(t) - L(0)}{L(0)} \mid \tag{7}$$

of these quantities against time and compare the euler abnd euler cromer methods to see which one conserves these quantities the best.

2 Results

2.1 Testing Simulation Against an Analytical Solution - The Two Body Problem

The first test to check if the simulation is working is to test it for only 2 bodies. This is because there is an analytical solution to this problem that the simulation results can be compared to. I used the bodies Earth and Satellite from "Coding Exercises: Final Project Part 2" [1] to run the simulation. The simulation ran for 12000 seconds with a delaT of 6s.

The results in Figure 1 show that the Analytical solution for the 2 body problem is very close to my numerical solution. Therefore my simulation is accurate. There is little difference between the Euler and Euler Cromer methods. This could be because there were too few iterations in this simulation to notice a difference.

```
The Earth and Satellite's locations after 12000 seconds are:
  Particle: Earth
    Mass: 5.972e+24,
    position: [4.82818964e-17 3.88875786e-18 0.00000000e+00]
    velocity: [8.01183291e-21 9.71266602e-22 0.00000000e+00]
    acceleration: [6.54637548e-25 1.61090392e-25 0.00000000e+00]
  Particle: Satellite
    Mass: 1.000e+02,
    position: [96612426.50526263 23786338.06955715        0.
    velocity: [-478.49630542 1943.54145462    0.        ]
    acceleration: [-0.03909738 -0.00962091  0.        ]
```

```
The Earth and Satellite's locations after 12000 seconds are:
  Particle: Earth
    Mass: 5.972e+24,
    position: [4.83309006e-17 3.89472752e-18 0.00000000e+00]
    velocity: [8.01206451e-21 9.71313882e-22 0.00000000e+00]
    acceleration: [6.54675475e-25 1.61102189e-25 0.00000000e+00]
  Particle: Satellite
    Mass: 1.000e+02,
    position: [96609499.78922324 23785981.53895841        0.
    velocity: [-478.51013694 1943.53863089    0.        ]
    acceleration: [-0.03909964 -0.00962162  0.        ]
```

```
The Earth and Satellite's locations after 12000 seconds are:
  Particle: Earth
    Mass: 5.972e+24,
    position: [4.82814259e-17 3.89458545e-18 0.00000000e+00]
    velocity: [8.01171550e-21 9.72233618e-22 0.00000000e+00]
    acceleration: [6.54618089e-25 1.61169404e-25 0.00000000e+00
  Particle: Satellite
    Mass: 1.000e+02,
    position: [96612426.50526263 23786338.06955715        0.
    velocity: [-478.49630542 1943.54145462    0.        ]
    acceleration: [-0.03909738 -0.00962091  0.        ]
```

Figure 1: The Expected Results (Top) from the analytical solution , Simulation results using Euler method (middle) and simulation results using the Euler-Cromer method(bottom) for a 2 body problem with deltaT =6s and total time = 12000s

2.2 Producing Stable Orbits for 3 or More Bodies

When more bodies are added to the simulation it is important to check that each body is in a stable orbit. A good test for this is to plot the X and Y coordinates over a long period of time to get a 2D trajectory of the bodies in the XY plane. For this simulation-bodies in the Solar System - the expected result is to see circular / slightly elliptical orbits of the planets and the sun in the centre effectively not moving. (It will wiggle slightly as it orbits the solar system barycentre but this movement is negligible for this simulation)

Figure 2 shows stable circular orbits of the planets in the correct positions. Therefore the simulation is working as expected. The trajectories produced

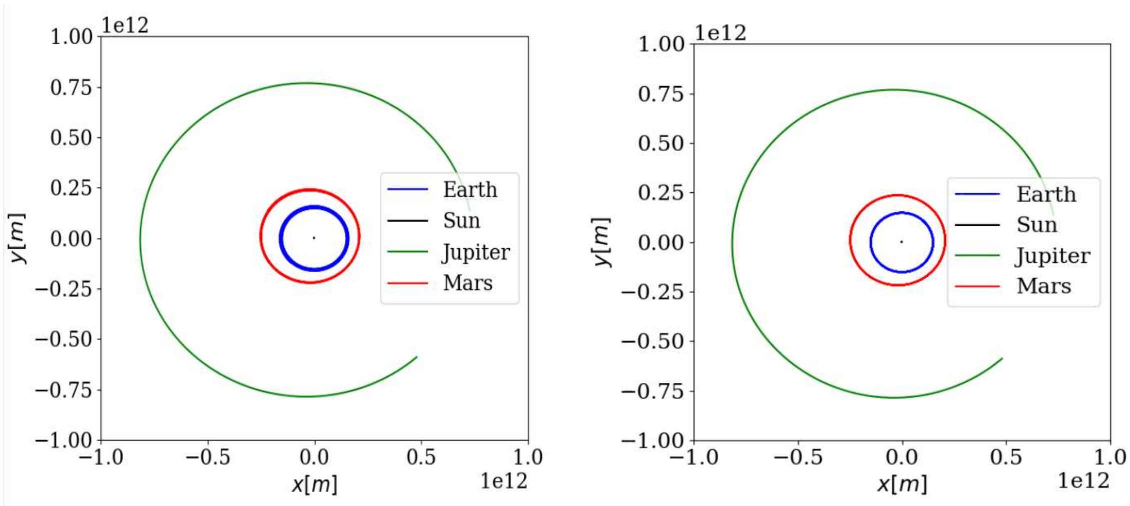

Figure 2: Trajectories in the xy plane of Sun (black), Earth (blue), Mars (red) and Jupiter (green), obtained integrating the equations of motion with the Euler method (left panel) and the Euler Cromer method (right panel). A Dt of 1 hour has been considered and the system has been advanced for ten years.

by the Euler Method and Euler-Cromer method in this test appear to be very similar so a further test was done to change the Dt to see how it may change the effectiveness of the algorithms.
Figure 3 is the simulation done with a larger dt of 1 day. This version has planets moving in spirals when calculated with the Euler Method which is not correct. However the Euler-Cromer method still works. This suggests the Euler-Cromer method is more versatile and less susceptible to breaking down the simulation and that the Euler Method is not as accurate for a larger Dt.

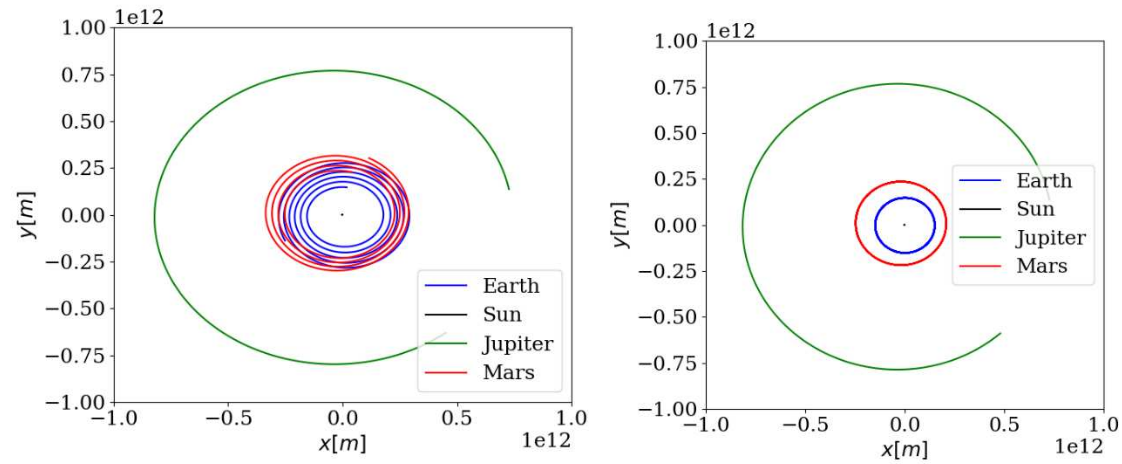

Figure 3: Trajectories in the xy plane of Sun (black), Earth (blue), Mars (red) and Jupiter (green), obtained integrating the equations of motion with the Euler method (left panel) and the Euler Cromer method (right panel). A Dt of 1 day has been considered and the system has been advanced for ten years.

2.3 Conserved Quantities

Another test to prove that the simulation is working as expected is to see if the laws of conservation of linear and angular momentum are being adhered to. If the total momentum of the system only fluctuates by a small enough amount it is reasonable to say that it is conserved by the simulation. Different algorithms will also conserve these quantities better than others.

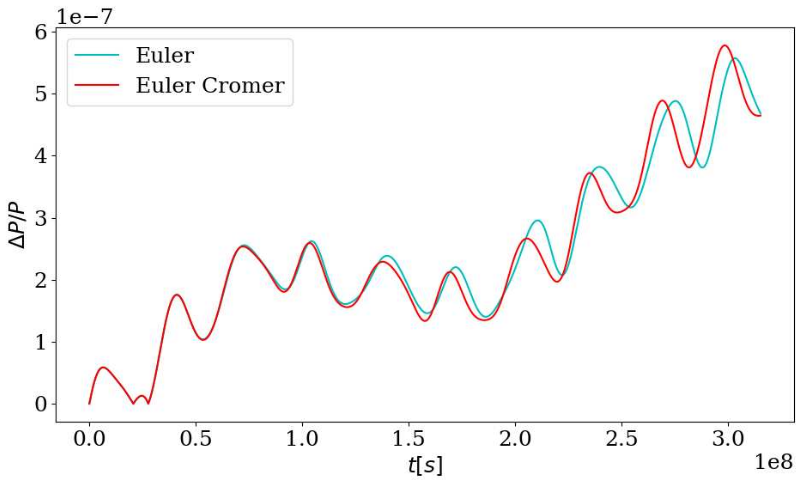

Figure 4: evolution of the fractional variation of the total linear momentum for the same system of Fig. 1. Blue lines are results obtained advancing the system with the Euler method, while red lines are results obtained advancing the system with the Euler-Cromer method.

In Figure 4 the fractional variation of linear momentum is shown to never go higher than 0.000006%. This means it is reasonable to suggest that the simulation is effective at conserving linear momentum. There is also little difference between the plots produced when advancing the system with Euler and Euler Cromer methods which means neither one is more effective at conserving linear momentum. However the results shown in figure 5 suggest that Euler Cromer actually is more effective at conserving Angular Momentum. Advancing the system with the Euler method gives you up to a 0.07% variation in Angular momentum whereas Euler-Cromer appears to be almost 0 (scale of the graph to large to show).

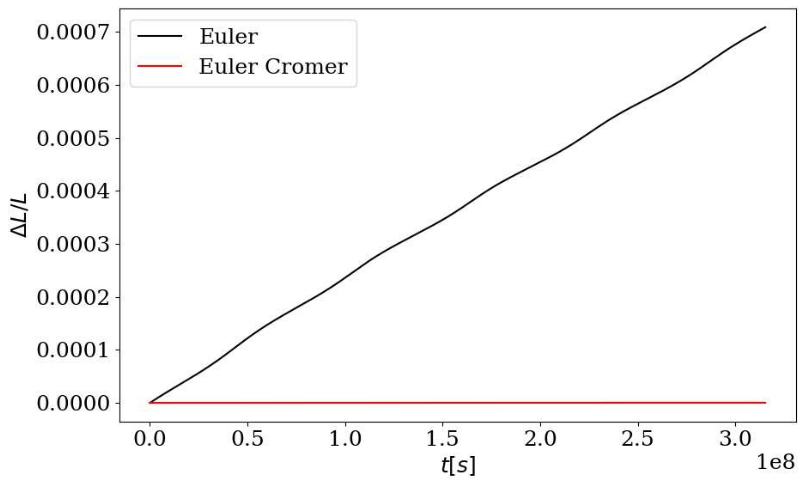

Figure 5: evolution of the fractional variation of the total angular momentum for the same system of Fig. 1. Black lines are results obtained advancing the system with the Euler method, while red lines are results obtained advancing the system with the Euler-Cromer method.

3 Summary

To conclude the simulation almost exactly matches the analytical solution for a 2 body problem. The plots of orbits can be made using either Euler or Euler Cromer method when delta t is small like 1 hour. However when delta t is larger ie. 1 day the simulation breaks down when using the Euler method. Therefore Euler-Cromer is better for a larger delta T. The simulation also conserves linear and angular momentum to with a reasonable degree of accuracy. Euler-Cromer has also been found to be far more efficient in conserving Angular Momentum than Euler method. My simulation is limited by the slight inaccuracy of my numerical methods. To improve I could get more methods to try such as Euler-Richardson or the Veret formula and expand my code to fit more planets. Also I could prove if the laws of conservation of energy are not violated by my simulation.

References

[1] https://modules.lancaster.ac.uk/course/view.php?id=37020section-8

[2] Giorgini, J. D., Yeomans, D. K., Chamberlin, A. B., Chodas, P. W., Jacobson, R. A., Keesey, M. S., Lieske, J. H., Ostro, S. J., Standish, E. M., Wimberly, R. N., "JPL's On-Line Solar System Data Service", Bulletin of the American Astronomical Society, Vol 28, p. 1099, 1997.

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Feedback comments

Abstract provides a good and understandable summary of the project. Abstract includes some qualitative description of results from the simulations. Quantitative information on the results would have enriched the abstract. A good description of the background theory behind the simulation and algorithms being applied.Two numerical approximation methods such as Euler and Euler-Cromer are correctly described.The introduction is lacking in references. Any errors are minor and do not effect the interpretation of the resultsResults include a qualitative description of outcomes of a suite of tests validating the simulation. Comparisons between different numerical approximations are described including a qualitative assessment of the differences. Comparing the results of the two-body problem with exercise 2 does not really count as a comparison with theory, but it is encouraging that results are similar. Plotting also the conserved quantities for different dt would have enriched this section. The conservation of energy is not reported. A good summary of the results is provided including some qualitative interpretation. Limitations of the simulation and areas for improvement are acknowledged, but lack detail. The conclusion should report some quantitative information. Figure 4 and 5 captions do not report the dt used.