

Cohere Research Talk: Massively Parallel Inference & Bayesian Evals

Sam Bowyer

10 November 2025

Outline

- 1 About Me
- 2 Alan: Massively Parallel Probabilistic Programming
- 3 Bayesian Evals: Uncertainty Quantification for LLM Evals

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- Currently working on discrete diffusion models (training an ‘auxilliary’ model with VI to suggest the order in which to decode tokens).
- Two projects I'll be talking about today: Alan (massively parallel probabilistic programming) & Bayesian Evals.

Alan: A Massively Parallel Probabilistic Programming Language



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 - Develop ‘massively parallel’ Bayesian inference algorithms: fast, accurate, and scalable; designed for GPU acceleration.

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- Dual goals:

- Develop ‘massively parallel’ Bayesian inference algorithms: fast, accurate, and scalable; designed for GPU acceleration.
- Implement these algorithms in a probabilistic programming language in pytorch (`alan`), allowing users to specify general probabilistic models.

Regular Bayesian Inference

- **Bayesian inference:** Prior $P(z)$ and likelihood $P(x|z)$ for latent variables z and data x .

$$P(z|x) = \frac{P(x|z)P(z)}{\int_{\mathcal{Z}} P(x, z') dz'}$$

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- ① Sample K latent variables from a proposal distribution Q (usually IID):

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- ③ Approximate the normalising constant using the ‘global’ estimator:

$$\mathcal{P}_{\text{global}}(z) = \frac{1}{K} \sum_{k=1}^K r_k(z) \quad \text{such that} \quad \mathbb{E}_{z \sim Q}[\mathcal{P}_{\text{global}}(z)] = P(x).$$

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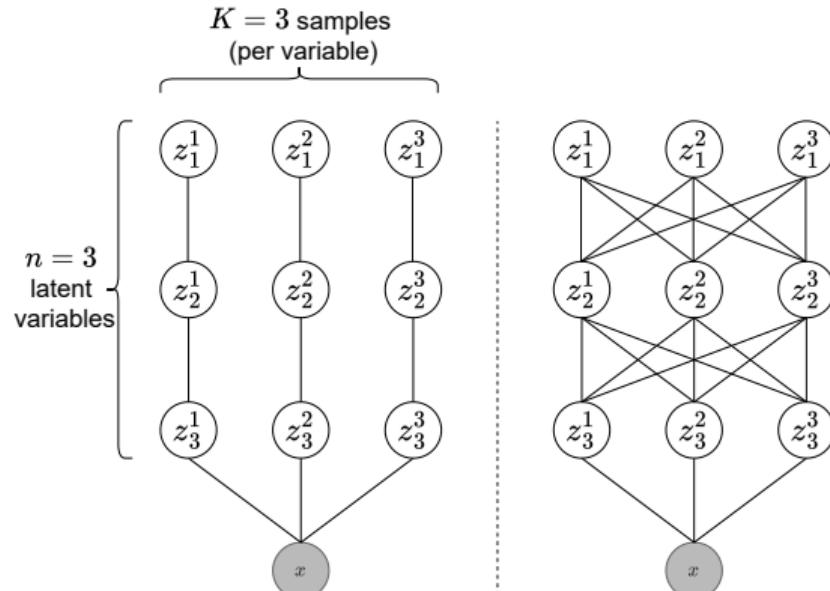
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- Solution: **Massively Parallel Importance Sampling (MP-IS)**
 - Reason about all K^n possible joint samples at once.

Massively Parallel Importance Sampling (MP-IS)

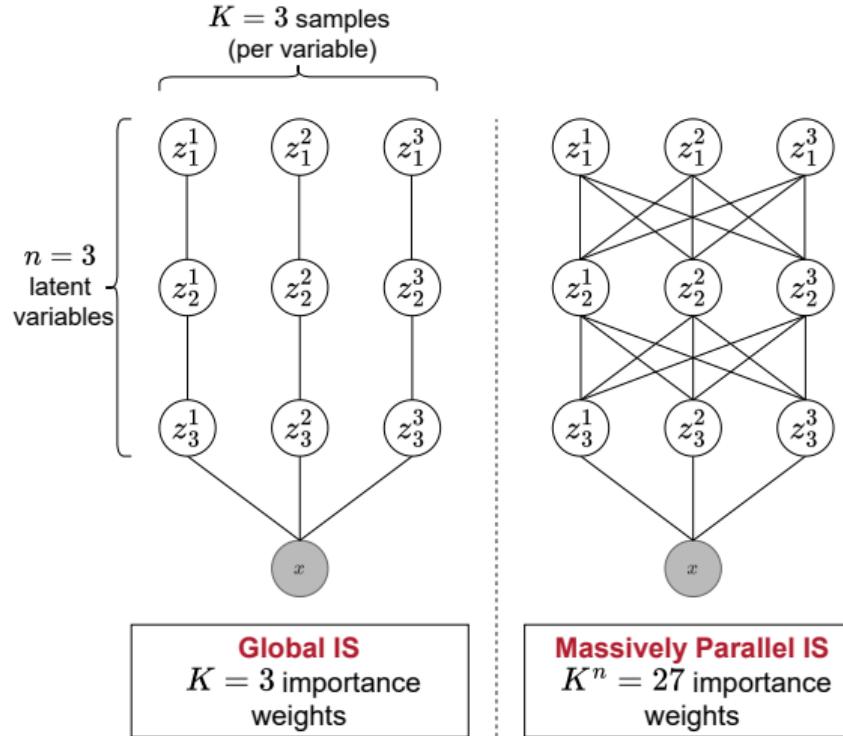


Global IS
 $K = 3$ importance weights

Massively Parallel IS
 $K^n = 27$ importance weights

- Suppose each latent sample $z^k = (z_1^k, \dots, z_n^k) \sim Q(z)$ is comprised of n variables.

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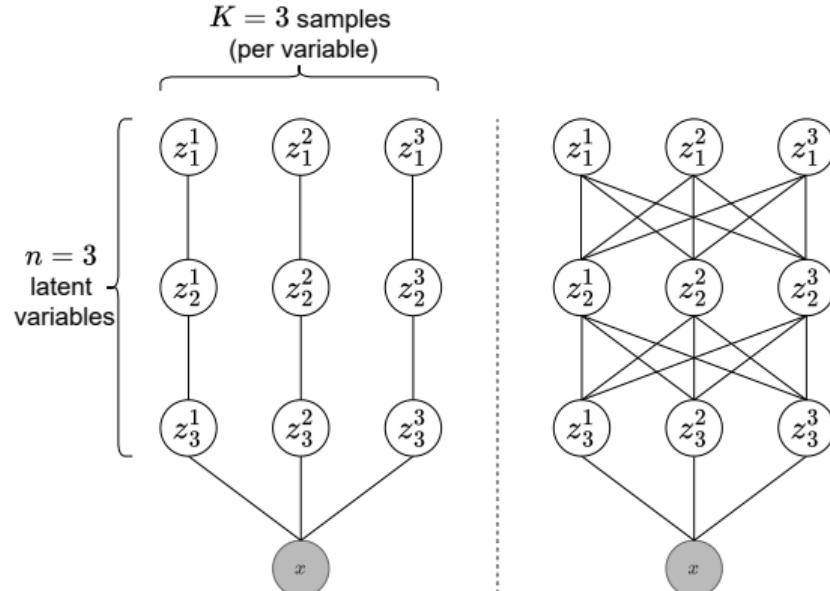


- Suppose each latent sample $z^k = (z_1^k, \dots, z_n^k) \sim Q(z)$ is comprised of n variables.
- We can construct K^n different samples from the full joint space

$$(z_1^{k_1}, \dots, z_n^{k_n}) \in \mathcal{Z}$$

where $\mathbf{k} = (k_1, \dots, k_n) \in [K]^n$ is the indexing vector for each latent variable.

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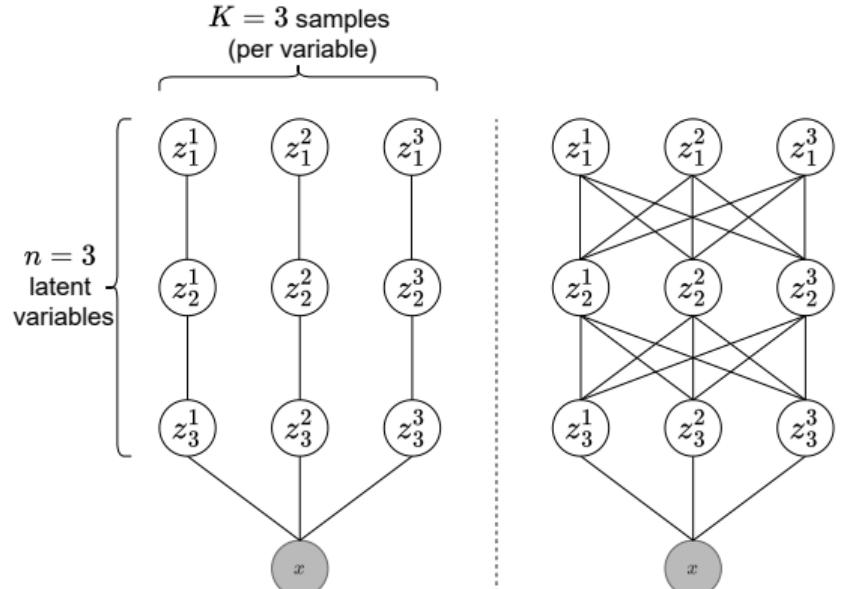
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- Rather than using the global IS estimator

$$\mathcal{P}_{\text{global}}(z) = \frac{1}{K} \sum_{k=1}^K \frac{P(x, z^k)}{Q(z^k)}.$$

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$$\mathcal{P}_{\text{global}}(z) = \frac{1}{K} \sum_{k=1}^K \frac{P(x, z^k)}{Q(z^k)}.$$

- ...we can use the MP-IS estimator

$$\mathcal{P}_{\text{MP}}(z) = \frac{1}{K^n} \sum_{\mathbf{k} \in [K]^n} \frac{P(x, z^{\mathbf{k}})}{Q_{\text{MP}}(z^{\mathbf{k}}, \mathbf{k})}.$$

(Which is still unbiased.)

MP-IS: Some Complications...

$$\mathcal{P}_{\text{MP}}(z) = \frac{1}{K^n} \sum_{\mathbf{k} \in [K]^n} \frac{P(x, z^\mathbf{k})}{Q_{\text{MP}}(z^\mathbf{k}, \mathbf{k})} = \frac{1}{K^n} \sum_{\mathbf{k} \in [K]^n} r_\mathbf{k}(z).$$

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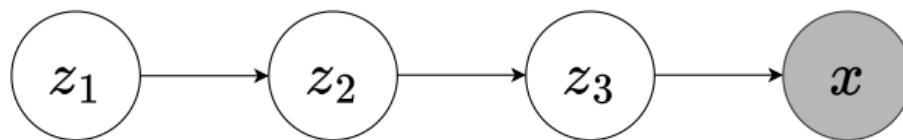
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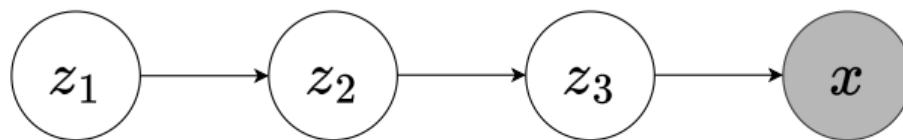
- We have to be careful about how we define Q_{MP} over the space of all K^n joint samples.
- Also, at first glance, this thing doesn't look all that nice to compute...
- But we can exploit the conditional independencies in the model to render it tractable.

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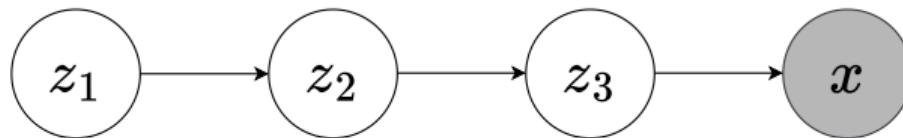
- E.g. with the model from before with $n = 3$, $P(x, z) = P(z_1)P(z_2|z_1)P(z_3|z_2)P(x|z_3)$, we can move the sums inside the product and get a bunch of tensor products:

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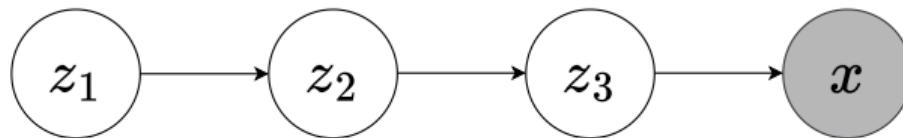
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$$\mathcal{P}_{\text{MP}}(z) = \frac{1}{K^3} \sum_{k_1 \in [K]} \sum_{k_2 \in [K]} \sum_{k_3 \in [K]} \frac{P(z_1^{k_1})P(z_2^{k_2}|z_1^{k_1})P(z_3^{k_3}|z_2^{k_2})P(x|z_3^{k_3})}{Q(z_1^{k_1})Q(z_2^{k_2})Q(z_3^{k_3})}$$

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$$= \frac{1}{K^3} \underbrace{\sum_{k_1 \in [K]} \frac{P(z_1^{k_1})}{Q(z_1^{k_1})}}_{\text{Vector of size } K} \underbrace{\sum_{k_2 \in [K]} \frac{P(z_2^{k_2}|z_1^{k_1})}{Q(z_2^{k_2})}}_{\text{Matrix of size } K \times K} \underbrace{\sum_{k_3 \in [K]} \frac{P(z_3^{k_3}|z_2^{k_2})}{Q(z_3^{k_3})}}_{\text{Matrix of size } K \times K} \underbrace{P(x|z_3^{k_3})}_{\text{Vector of size } K}$$

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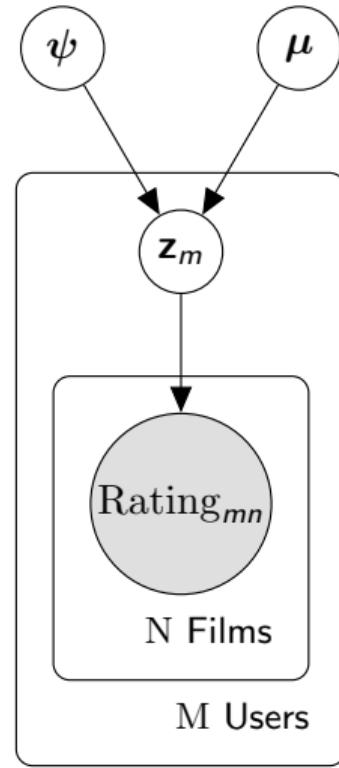
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Can we hide these complications from the user?

- Yes! We do this with `alan`.
- User specifies the model with `P` and `Q` as pytorch modules, and we handle the massively parallel inference for them.

Alan: A Probabilistic Programming Language

```
1 from alan import Normal, Bernoulli, Plate, BoundPlate, OptParam, Data, Problem
2 import torch as t
3
4 # Set up the model
5 d_z = 10
6
7 P = Plate(
8     mu_z = Normal(t.zeros((d_z,)), t.ones((d_z,))),
9     psi_z = Normal(t.zeros((d_z,)), t.ones((d_z,))),
10    plate_1 = Plate(
11        z = Normal("mu_z", lambda psi_z: psi_z.exp()),
12        plate_2 = Plate(
13            obs = Bernoulli(logits = lambda z, x: z @ x),
14        )
15    ),
16)
17
18 Q = Plate(
19     mu_z = Normal(OptParam(t.zeros((d_z,))), OptParam(t.zeros((d_z,)), transformation=t.exp)),
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25         )
26    ),
27)
28
29 P = BoundPlate(P, platesizes={'plate_1': num_users, 'plate_2': num_movies}, inputs = {'x': x})
30 Q = BoundPlate(Q, platesizes={'plate_1': num_users, 'plate_2': num_movies}, inputs = {'x': x})
31
32 prob = Problem(P, Q)
```



- Using $\mathcal{P}_{\text{MP}}(z)$, we can do variational inference (VI) by maximising the ELBO:

$$\log P(x) \geq \mathcal{L}_{\text{MP}}(\theta) = \mathbb{E}_{z \sim Q_{\text{MP}}(\theta)}[\log \mathcal{P}_{\text{MP}}(z)]$$

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- Aitchison (2019) showed that MP-VI is a tighter bound than the global VI objective (IWAE):

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32 prob = Problem(P, Q)
33 opt = t.optim.Adam(prob.Q.parameters(), lr=lr)
34
35 # Train Q with VI
36 for i in range(num_iterations):
37     opt.zero_grad()
38     elbo = prob.sample(K=K).elbo_vi()
39     elbo.backward()
40     opt.step()
```

MP Algorithms

- We can obtain unbiased posterior moment estimates via autodiff (Bowyer et al. (2024)).

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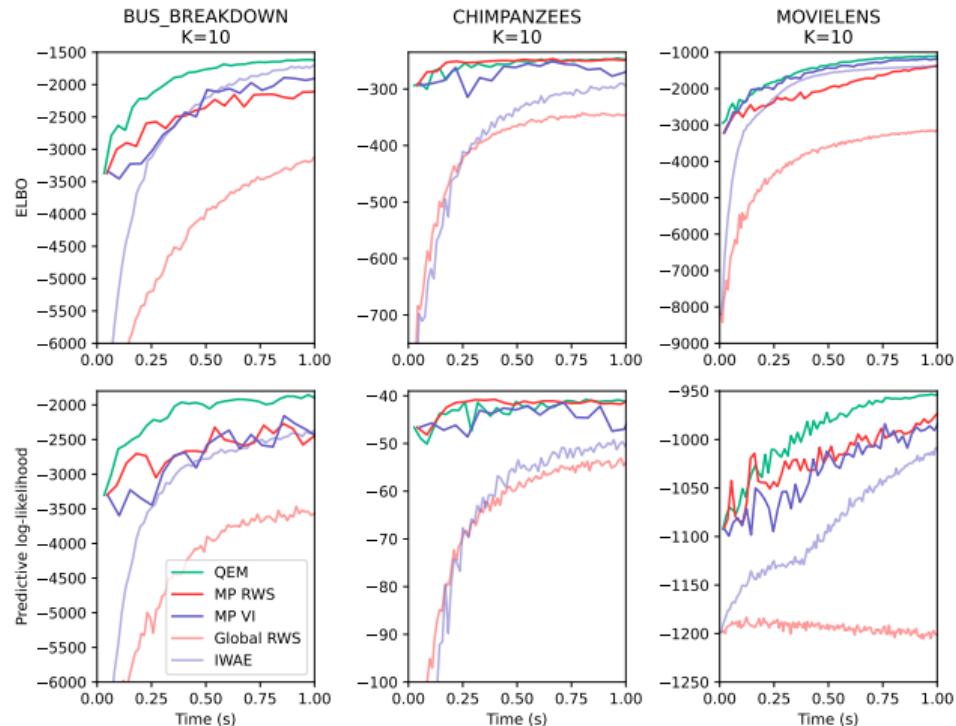
- By similar arguments: $J \in \mathbb{R}^K$ gives us marginal importance weights; $J \in \mathbb{R}^{K^{1+|\text{pa}(i)|}}$ gives us importance samples for z_i , given its parents $\text{pa}(i)$.

QEM: An Adaptive Importance Sampling Algorithm

QEM (Heap et al. (2025))

- ① Start with an initial approximate posterior Q_0 .
- ② Compute posterior moment estimates $m_{\text{MP}}(z)$ using MP-IS.
- ③ Update the approximate posterior Q_{t+1} using the moment estimates.

Can be seen as an EM-like algorithm for adaptive importance sampling.



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- This project was a great (if, at times, complicated) way to learn about Bayesian inference and numerical and probabilistic programming.
- The results were pretty promising, but there are some drawbacks to massively parallel methods:
 - The algorithms are complex to implement (hence wrapping them in a PPL).
 - Not all models have lots of conditional independencies to exploit.
 - Although it's slower and harder to tune, HMC is often hard to beat in terms of quality of inference.

Bayesian Evals: Uncertainty Quantification for LLM Evals



Work done with Laurence Aitchison and Desi R. Ivanova.

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- The former direction led to an ICML spotlight position paper: '*Position: Don't Use the CLT in LLM Evals With Fewer Than a Few Hundred Datapoints*'.

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- The former direction led to an ICML spotlight position paper: '*Position: Don't Use the CLT in LLM Evals With Fewer Than a Few Hundred Datapoints*'.
- The latter fell by the wayside, but is something I'd like to come back to at some point.

Motivation

Central Limit Theorem (CLT)

If X_1, \dots, X_N are IID r.v.s with mean $\mu \in \mathbb{R}$ and finite variance σ^2 , then

$$\sqrt{N}(\hat{\mu} - \mu) \xrightarrow{d} \mathcal{N}(0, \sigma^2) \text{ as } N \rightarrow \infty,$$

where $\hat{\mu} = \frac{1}{N} \sum_{i=1}^N X_i$ is the sample mean.

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- The CLT-based confidence interval is:

$$\text{CI}_{1-\alpha}(\mu) = \hat{\mu} \pm z_{\alpha/2} \text{SE}(\hat{\mu})$$

where $z_{\alpha/2}$ is the $100(1 - \alpha/2)$ -th percentile of $\mathcal{N}(0, 1)$ and $\text{SE}(\hat{\mu}) = \sqrt{\frac{\hat{\sigma}^2}{N}}$ is the standard error of the sample mean.

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where $\hat{\mu} = \frac{1}{N} \sum_{i=1}^N X_i$ is the sample mean.

- The CLT-based confidence interval is:

$$\text{CI}_{1-\alpha}(\mu) = \hat{\mu} \pm z_{\alpha/2} \text{SE}(\hat{\mu})$$

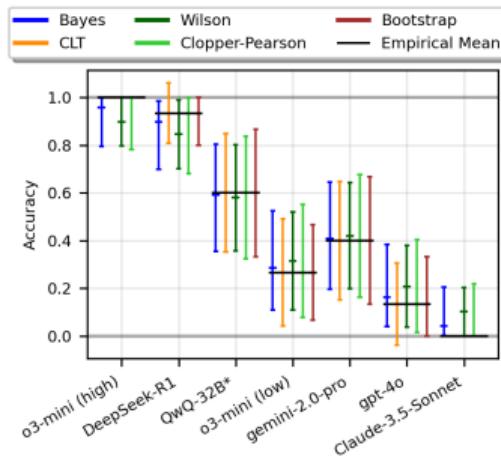
where $z_{\alpha/2}$ is the $100(1 - \alpha/2)$ -th percentile of $\mathcal{N}(0, 1)$ and $\text{SE}(\hat{\mu}) = \sqrt{\frac{\hat{\sigma}^2}{N}}$ is the standard error of the sample mean.

- In the case of binary data $X_i \in \{0, 1\}$, this becomes:

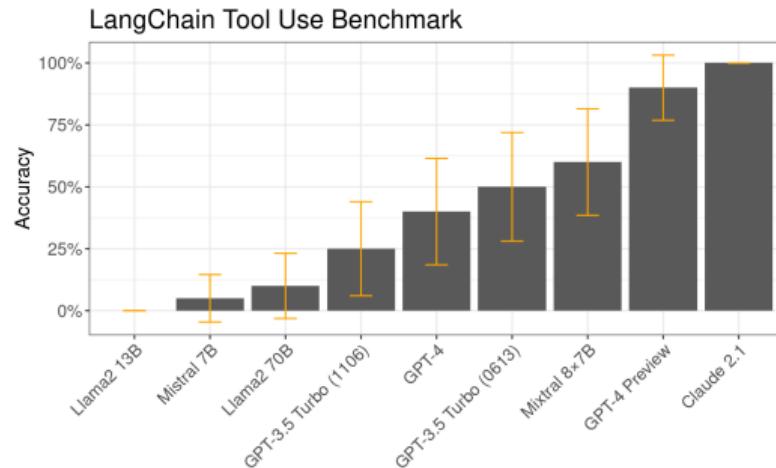
$$\text{CI}_{1-\alpha}(\theta) = \bar{X} \pm z_{\alpha/2} \sqrt{\bar{X}(1 - \bar{X})/N}.$$

Real-World Failures of the CLT

- If N is too small, CLT-based error bars can collapse to zero-width or extend past $[0, 1]$.



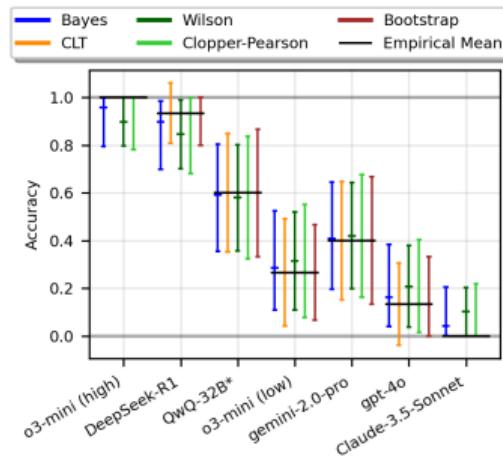
Math Arena's AIME II 2025 Benchmark
($N=15$). Various 95% interval types shown.



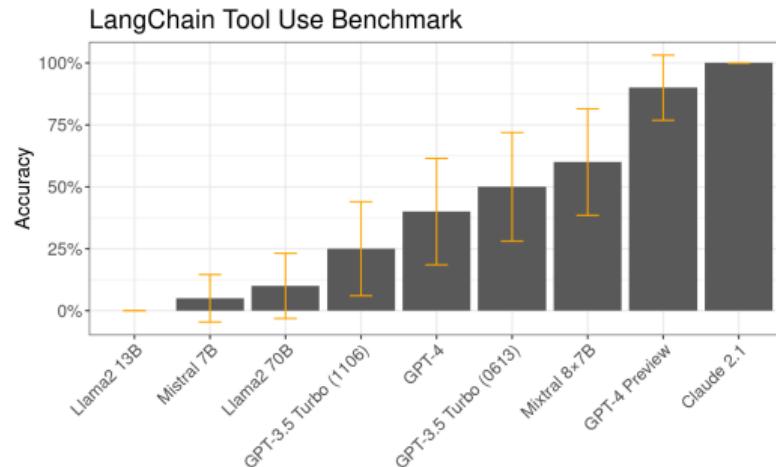
Langchain Typewriter Tool Use Benchmark
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Real-World Failures of the CLT

- If N is too small, CLT-based error bars can collapse to zero-width or extend past $[0, 1]$.
- Smaller, more intricate, and expensive LLM benchmarks are becoming increasingly common, so we need to find alternatives for the few-data regime.



Math Arena's AIME II 2025 Benchmark
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Bayesian Alternative: Beta-Binomial Model

- Treat the data as IID Bernoulli with a **uniform prior** on the parameter θ .

$$\theta \sim \text{Beta}(1, 1) = \text{Uniform}[0, 1]$$

$$y_i \sim \text{Bernoulli}(\theta) \text{ for } i = 1, \dots, N$$

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Beta-Bernoulli Bayesian Credible Interval

```
1 posterior = scipy.stats.beta(1 + sum(y), 1 + N - sum(y))
2 bayes_ci  = posterior.interval(confidence=0.95)
```

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Wilson & Clopper-Pearson Confidence Interval

```
1 result = scipy.stats.binomtest(k=sum(y), n=N)
2 wilson_ci = result.proportion_ci("wilson", 0.95)
3 clop_ci   = result.proportion_ci("exact", 0.95)
```

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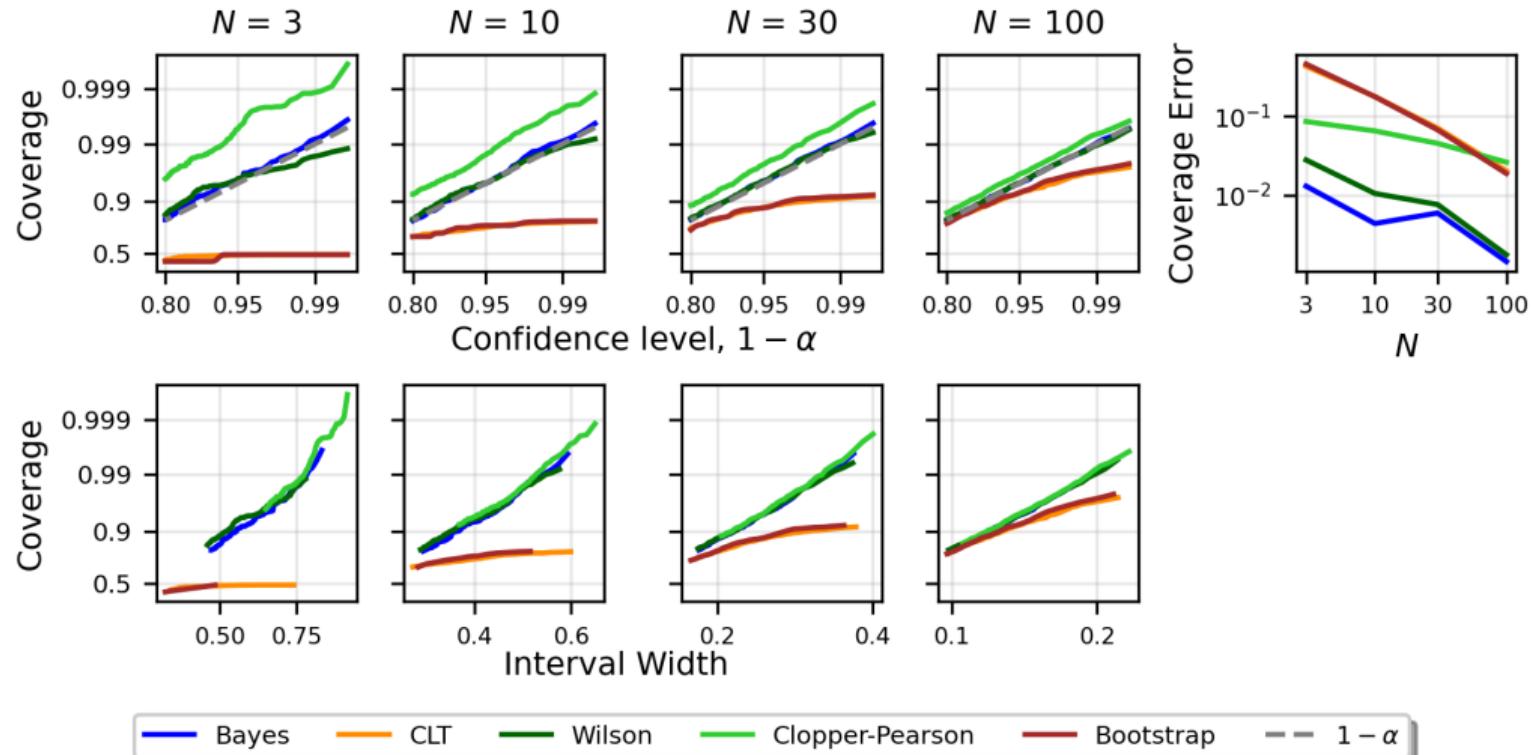
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 - Ideally, coverage = $1 - \alpha$.

IID Questions Setting: Results



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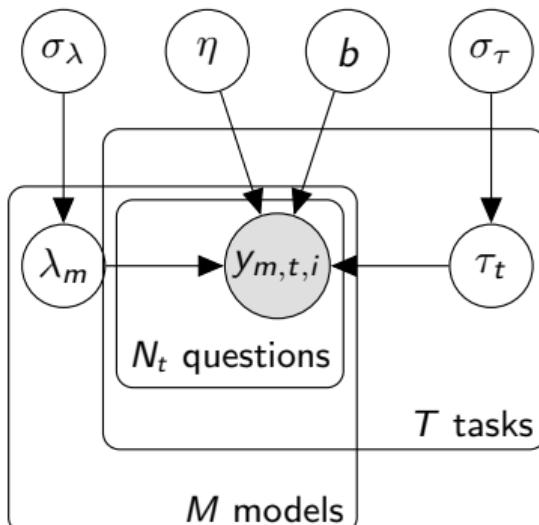
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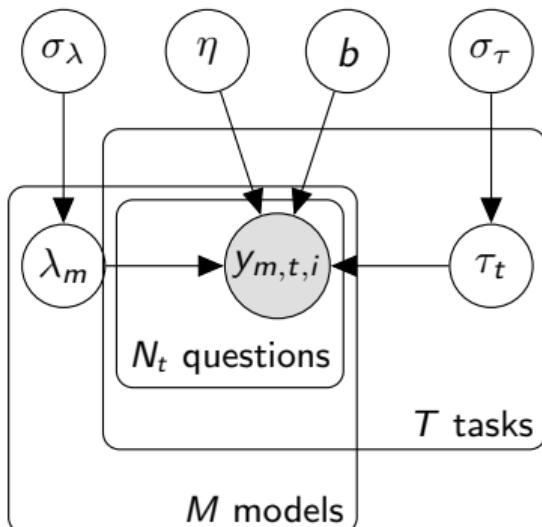
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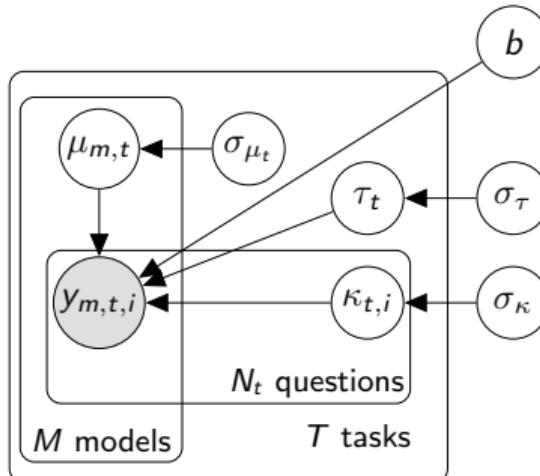
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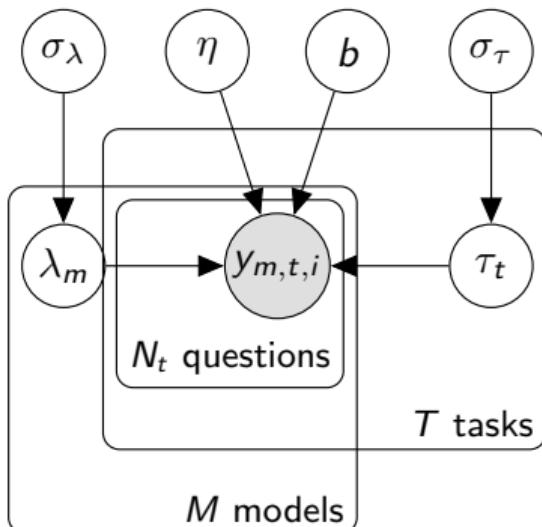
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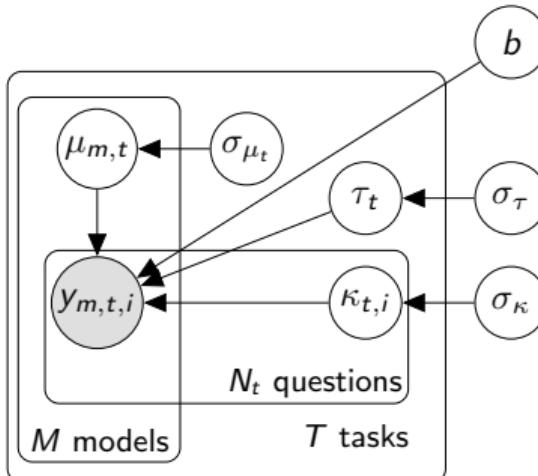
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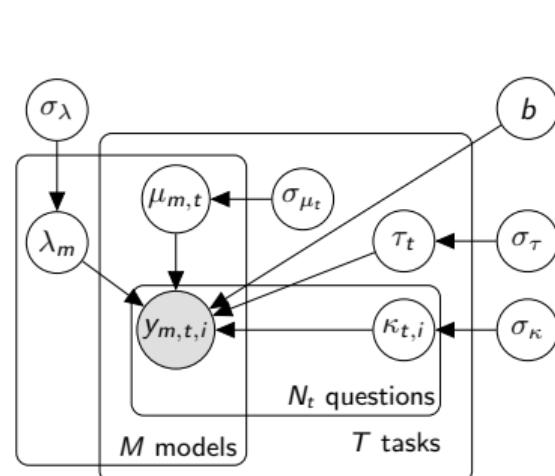
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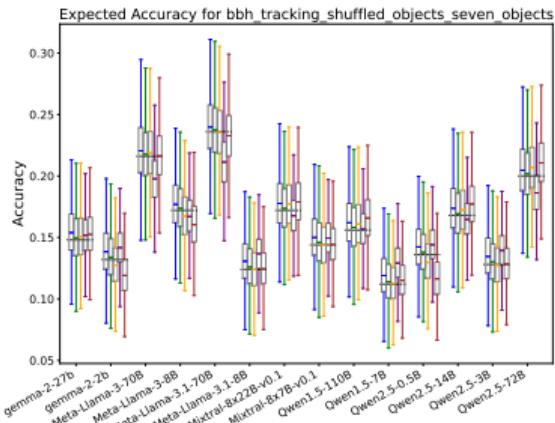
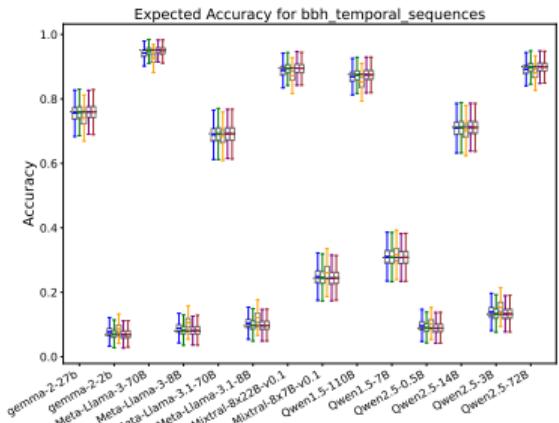
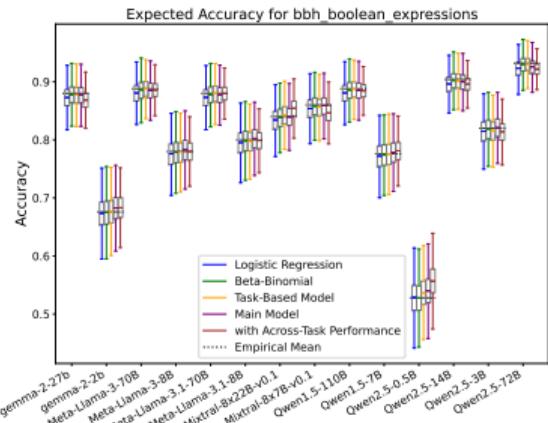
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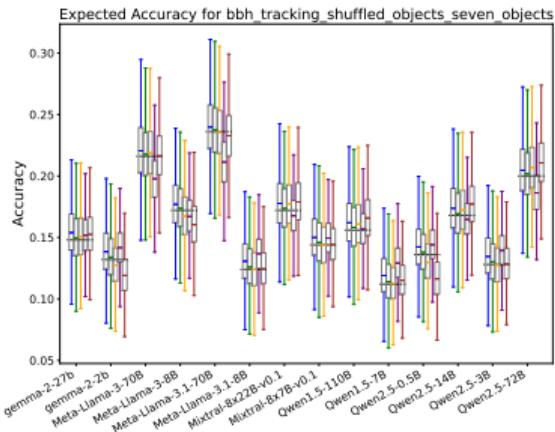
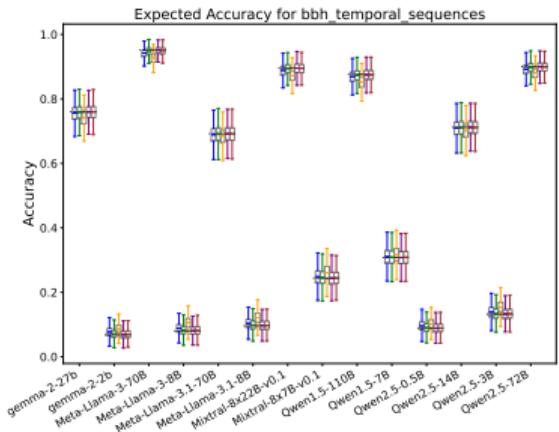
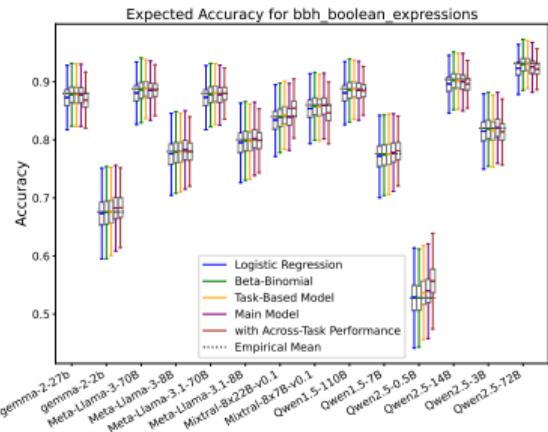
+ Across-task performance latent variable.

Graphical models of the proposed hierarchical models. **Way too complicated!**

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So just use Beta-Binomial!

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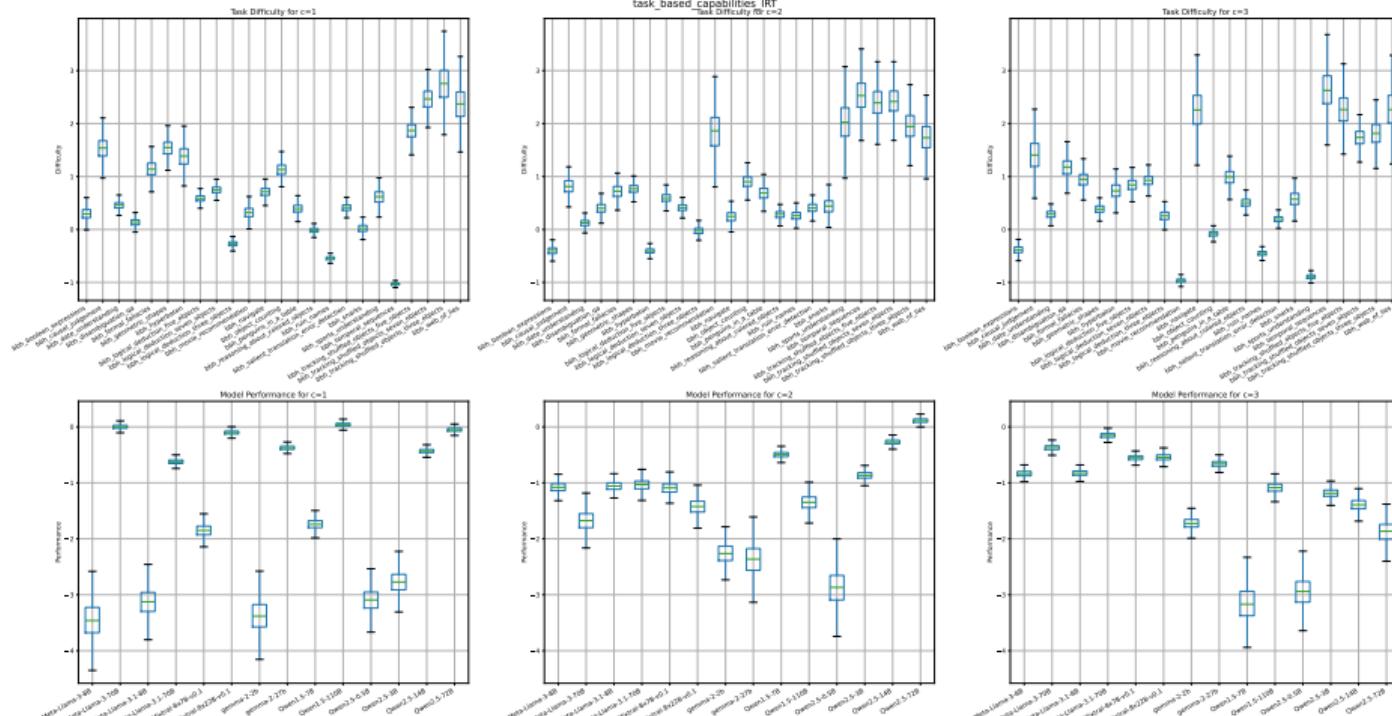
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- Perhaps these techniques could be used more successfully for predicting eval performance based on something like pretraining metrics?

Thank You

- Any questions?

References I

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$$\text{SE}_{\text{Clustered}} = \sqrt{\text{SE}_{\text{CLT}}^2 + \frac{1}{N^2} \sum_{t=1}^T \sum_{i=1}^{N_t} \sum_{j \neq i} (y_{i,t} - \bar{y})(y_{j,t} - \bar{y})}$$

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- θ_t is the difficulty of the questions in task t (we ensure that $\mathbb{E}[\theta_t | \theta] = \theta$),

$$\theta_t \sim \text{Beta}(d\theta, d(1 - \theta)).$$

- $\text{Var}(\theta_t | \theta) = \frac{\theta(1-\theta)}{d^2+1}.$

Clustered Questions Setting: A Bayesian Approach

- ‘Dispersion’ parameter d controls the range of difficulty across tasks,

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- Can we integrate out the per-task difficulty parameters θ_t ?
- Yes! The number of successes per task is **Beta-Binomial** distributed:

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- Get an **importance-weighted posterior** for θ : draw prior samples $\{(\theta^{(k)}, d^{(k)})\}_{k=1}^K$, then compute weights

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- Compute quantile-based Bayesian credible intervals for θ .

Clustered Questions Setting: Bayesian Implementation

Snippet 4: Bayesian analysis for clustered evals

```
1 # S_t, N_t: np.arrays of length T with total
2 # successes & questions per task
3 import numpy as np
4 from scipy.stats import betabinom
5
6 # set number of samples, K
7 K = 10_000
8
9 # get K samples from the prior (with extra dimension for broadcasting over tasks)
10 thetas = np.random.beta(1,1, size=(K,1))
11 ds = np.random.gamma(1,1, size=(K,1))
12
13 # obtain weights via the likelihood (sum the per-task log-probs)
14 log_weights = betabinom(N_t, (ds*thetas), (ds*(1-thetas))).logpmf(S_t).sum(-1)
15
16 # normalise the weights
17 weights = np.exp(log_weights - log_weights.max())
18 weights /= weights.sum()
19
20 # obtain samples from the posterior
21 posterior = thetas[np.random.choice(K, size=K, replace=True, p=weights)]
22
23 # Bayesian credible interval
24 bayes_ci = np.percentile(posterior, [2.5, 97.5])
```

Clustered Questions Setting: Results

