Cohere Research Talk: Massively Parallel Inference & Bayesian Evals

Sam Bowyer

10 November 2025

Outline

1. About Me
2. Alan: Massively Parallel Probabilistic Programming
3. Bayesian Evals: Uncertainty Quantification for LLM Evals

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- Currently working on discrete diffusion models (training an 'auxilliary' model with VI to suggest the order in which to decode tokens).
- Two projects I'll be talking about today: Alan (massively parallel probabilistic programming) & Bayesian Evals.

Alan: A Massively Parallel Probabilistic Programming Language

  

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- Dual goals:
  - Develop ‘massively parallel’ Bayesian inference algorithms: fast, accurate, and scalable; desinged for GPU acceleration.
  - Implement these algorithms in a probabilistic programming language in pytorch (`alan`), allowing users to specify general probabilistic models.

Regular Bayesian Inference

- **Bayesian inference**: Prior $P(z)$ and likelihood $P(x|z)$ for latent variables z and data x.

$$P(z|x) = \frac{P(x|z)P(z)}{\int_{\mathcal{Z}} P(x, z')dz'}$$

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  $$r_k(z) = \frac{P(x, z^k)}{Q(z^k)}.$$
  3. Approximate the normalising constant using the 'global' estimator:

$$\mathcal{P}_{\text{global}}(z) = \frac{1}{K}\sum_{k=1}^{K} r_k(z) \quad \text{such that} \quad \mathbb{E}_{z\sim Q}[\mathcal{P}_{\text{global}}(z)] = P(x).$$

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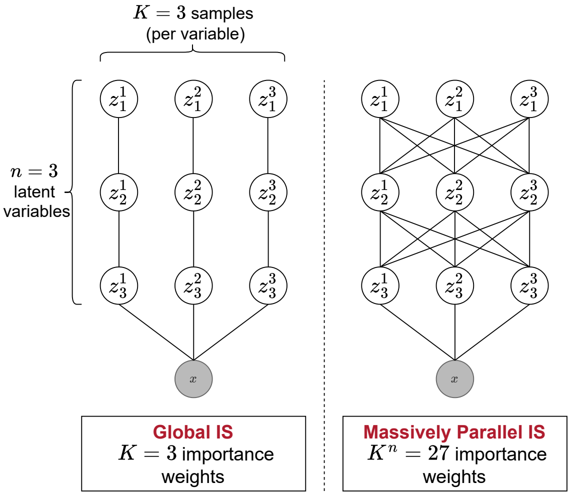
- Solution: **Massively Parallel Importance Sampling (MP-IS)**
  - Reason about all K^N possible joint samples at once.

Massively Parallel Importance Sampling (MP-IS)



- Suppose each latent sample $z^k = (z_1^k, \ldots, z_n^k) \sim Q(z)$ is comprised of n variables.

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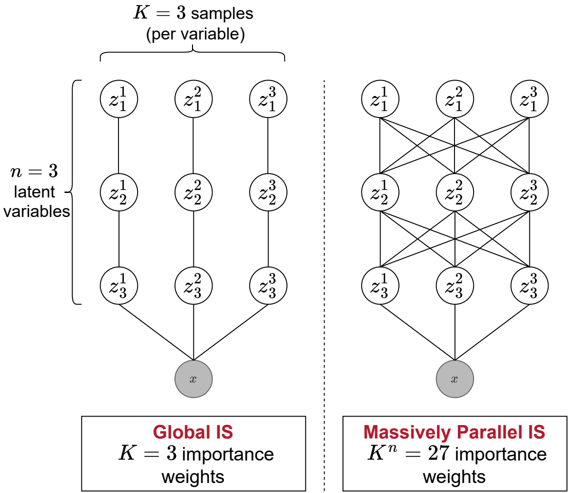

- Suppose each latent sample $z^k = (z_1^k, \ldots, z_n^k) \sim Q(z)$ is comprised of n variables.
- We can construct K^n different samples from the full joint space

$$(z_1^{k_1}, \ldots, z_n^{k_n}) \in \mathcal{Z}$$

where $\mathbf{k} = (k_1, \ldots, k_n) \in [K]^n$ is the indexing vector for each latent variable.

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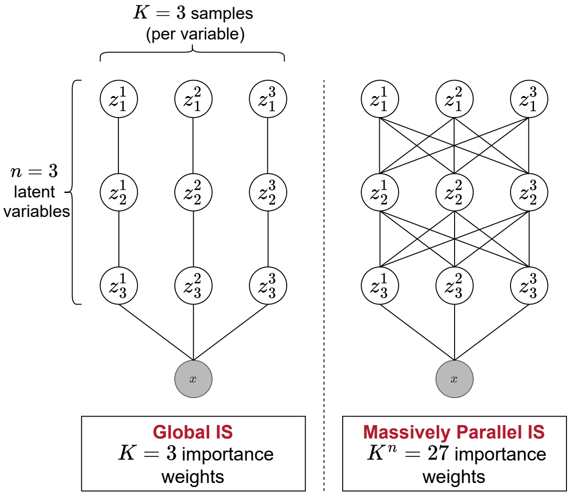

- Rather than using the global IS estimator

$$\mathcal{P}_{\text{global}}(z) = \frac{1}{K}\sum_{k=1}^{K}\frac{P(x, z^k)}{Q(z^k)}.$$

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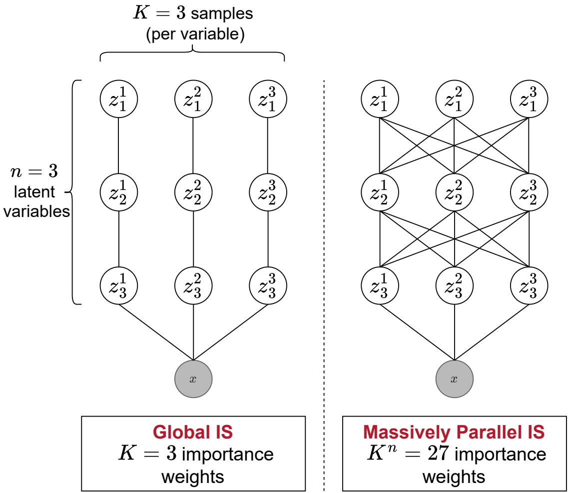

- Rather than using the global IS estimator

$$\mathcal{P}_{\text{global}}(z) = \frac{1}{K} \sum_{k=1}^{K} \frac{P(x, z^k)}{Q(z^k)}.$$

- ...we can use the MP-IS estimator

$$\mathcal{P}_{\text{MP}}(z) = \frac{1}{K^n} \sum_{\mathbf{k} \in [K]^n} \frac{P(x, z^{\mathbf{k}})}{Q_{\text{MP}}(z^{\mathbf{k}}, \mathbf{k})}.$$

(Which is still unbiased.)

MP-IS: Some Complications...

$$\mathcal{P}_{\text{MP}}(z) = \frac{1}{K^n} \sum_{\mathbf{k} \in [K]^n} \frac{P(x, z^{\mathbf{k}})}{Q_{\text{MP}}(z^{\mathbf{k}}, \mathbf{k})}.$$

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- We can use a hierarchical model:

$$Q_{\text{MP}}(z, \mathbf{k}) = \prod_{i=1}^{n} Q_{\text{MP}}(z_i^{k_i} | z_j \text{ for } j \in \text{qa}(i)),$$

where $\text{qa}(i)$ is the set of indices of parents of z_i in the proposal model.

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  where $\text{qa}(i)$ is the set of indices of parents of z_i in the proposal model.
- If variable z_i has a parent samples $z_j = (z_j^1, \ldots, z_j^K) \sim Q(z_j)$, then we can sample $z_i^{k_i}$ from $Q(z_i^{k_i} | z_j^{\pi(k_i)})$ for a (uniformly) random permutation π of $[K]$.

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- At first glance, this thing doesn't look all that nice to compute...
- But we can exploit the conditional independencies in the model to render it tractable.

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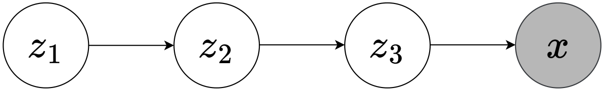

- E.g. with the model from before with $n = 3$, $P(x, z) = P(z_1)P(z_2|z_1)P(z_3|z_2)P(x|z_3)$, we can move the sums inside the product and get a bunch of tensor products:

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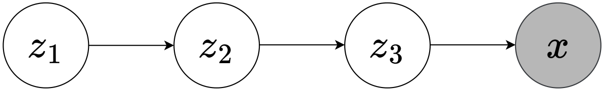

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$$\mathcal{P}_{\text{MP}}(z) = \frac{1}{K^3} \sum_{k_1 \in [K]} \sum_{k_2 \in [K]} \sum_{k_3 \in [K]} \frac{P(z_1^{k_1})P(z_2^{k_2}|z_1^{k_1})P(z_3^{k_3}|z_2^{k_2})P(x|z_3^{k_3})}{Q_{\text{MP}}(z^{\mathbf{k}}, \mathbf{k})}$$

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$$= \frac{1}{K^3} \sum_{k_1 \in [K]} \underbrace{\frac{P(z_1^{k_1})}{Q_{\text{MP}}(z_1^{k_1}, k_1)}}_{\text{Vector of size } K} \sum_{k_2 \in [K]} \underbrace{\frac{P(z_2^{k_2}|z_1^{k_1})}{Q_{\text{MP}}(z_2^{k_2}, \mathbf{k}_{1:2})}}_{\text{Matrix of size } K \times K} \sum_{k_3 \in [K]} \underbrace{\frac{P(z_3^{k_3}|z_2^{k_2})}{Q_{\text{MP}}(z_3^{k_3}, \mathbf{k}_{2:3})}}_{\text{Matrix of size } K \times K} \underbrace{\frac{P(x|z_3^{k_3})}{Q_{\text{MP}}(x|z_3^{k_3}, k_3)}}_{\text{Vector of size } K}$$

Can we hide these complications from the user?

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Can we hide these complications from the user?

- Yes! We do this with `alan`.
- User specifies the model with P and Q as pytorch modules, and we handle the massively parallel inference for them.

Alan: A Probabilistic Programming Language

```python
from alan import Normal, Bernoulli, Plate, BoundPlate, OptParam, Data, Problem
import torch as t

# Set up the model
d_z = 10

P = Plate(
    mu_z = Normal(t.zeros((d_z,)), t.ones((d_z,))),
    psi_z = Normal(t.zeros((d_z,)), t.ones((d_z,))),
    plate_1 = Plate(
        z = Normal("mu_z", lambda psi_z: psi_z.exp()),
        plate_2 = Plate(
            obs = Bernoulli(logits = lambda z, x: z @ x),
        )
    ),
)

Q = Plate(
    mu_z = Normal(OptParam(t.zeros((d_z,))), OptParam(t.zeros((d_z,)), transformation=t.exp)),
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P = BoundPlate(P, platesizes={'plate_1': num_users, 'plate_2': num_movies}, inputs = {'x': x})
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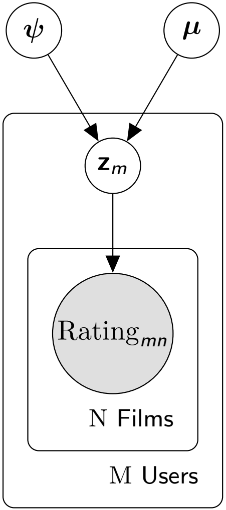
prob = Problem(P, Q)
```



MP-VI

- Using $\mathcal{P}_{\text{MP}}(z)$, we can do variational inference (VI) by maximising the ELBO:

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- Aitchison (2019) showed that MP-VI is a tighter bound than the global VI objective (IWAE):

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prob = Problem(P, Q)
opt = t.optim.Adam(prob.Q.parameters(), lr=lr)

# Train Q with VI
for i in range(num_iterations):
    opt.zero_grad()
    elbo = prob.sample(K=K).elbo_vi()
    elbo.backward()
    opt.step()
```

MP Algorithms

- We showed that we can obtain unbiased posterior moment estimates efficiently using autodiff (Bowyer et al. (2024)).

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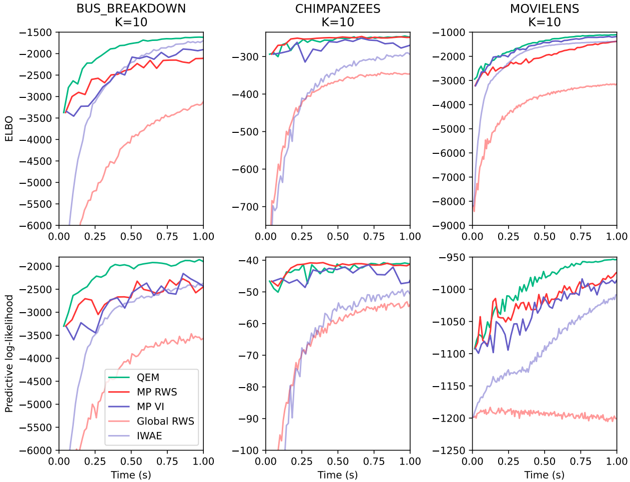
- Through a similar argument, we can set J to be vector- or tensor-valued based on K and the shape of z, allowing us to compute marginal likelihoods and importance samples too.

QEM: An Adaptive Importance Sampling Algorithm

- **QEM:** use these posterior moment estimates to iteratively update the approximate posterior Q_{MP} in an EM-like algorithm for adaptive imortance sampling (Heap et al. (2025)).



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- This project was a great (if, at times, complicated) way to learn about Bayesian inference and numerical and probabilistic programming.
- The results were pretty promising, but but there are some drawbacks to massively parallel methods:
  - The algorithms are complex to implement (hence wrapping them in a PPL).
  - Not all models have lots of conditional independencies to exploit.
  - Although it's slower and harder to tune, HMC is often hard to beat in terms of quality of inference.

Bayesian Evals: Uncertainty Quantification for LLM Evals

Work done with Laurence Aitchison and Desi R. Ivanova.

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- Two directions:
  - Improved UQ for evals with Bayesian methods.
  - Interpretability of evals with Bayesian hierarchical modelling and SAE-like approaches.
- The former direction led to an ICML spotlight position paper.
- The latter fell by the wayside, but is something I'd like to come back to at some point.

Motivation

Central Limit Theorem (CLT)

If $X_1, \ldots, X_N$ are IID r.v.s with mean $\mu \in \mathbb{R}$ and finite variance σ^2, then

$$\sqrt{N}(\hat{\mu} - \mu) \xrightarrow{d} \mathcal{N}(0, \sigma^2) \text{ as } N \to \infty,$$

where $\hat{\mu} = \frac{1}{N} \sum_{i=1}^{N} X_i$ is the sample mean.

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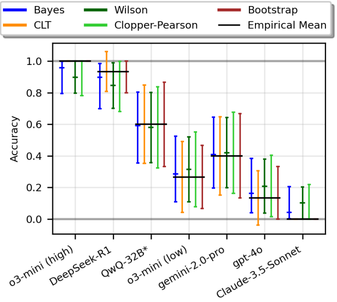
- In the case of binary data, the CLT-based confidence interval is:

$$\mathrm{CI}_{1-\alpha}(\theta) = \bar{X} \pm z_{\alpha/2} \sqrt{\frac{\bar{X}(1 - \bar{X})}{N}}$$

where $z_{\alpha/2}$ is the $100(1 - \alpha/2)$-th percentile of $\mathcal{N}(0, 1)$.

Real-World Failures of the CLT

- If N is too small, CLT-based error bars can collapse to zero-width or extend past $[0, 1]$.



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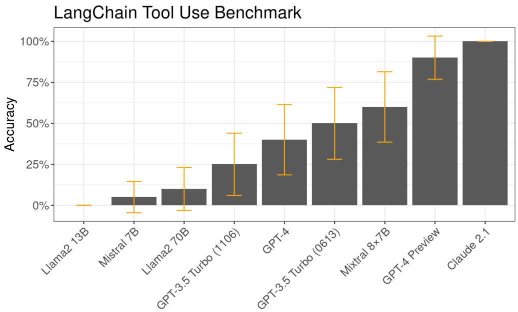
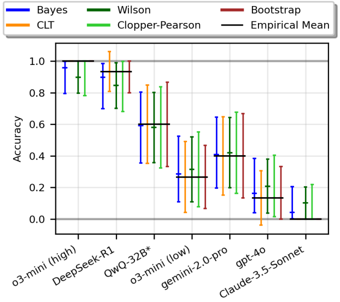


Langchain Typewriter Tool Use Benchmark (N=20)

Real-World Failures of the CLT

- If N is too small, CLT-based error bars can collapse to zero-width or extend past $[0, 1]$.
- Smaller, more intricate, and expensive LLM benchmarks are becoming increasingly common, so we need to find alternatives for the few-data regime.



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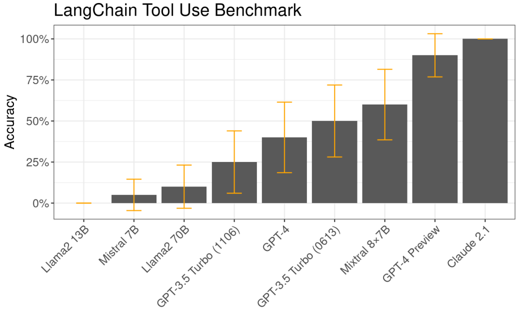

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Beta-Bernoulli Bayesian Credible Interval

```
posterior = scipy.stats.beta(1 + sum(y), 1 + N - sum(y))
bayes_ci  = posterior.interval(confidence=0.95)
```

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where $B(\alpha, a, b)$ is the α-th quantile of the Beta(a, b) distribution.

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Wilson & Clopper-Pearson Confidence Interval

```
result = scipy.stats.binomtest(k=sum(y), n=N)
wilson_ci = result.proportion_ci("wilson", 0.95)
clop_ci    = result.proportion_ci("exact",  0.95)
```

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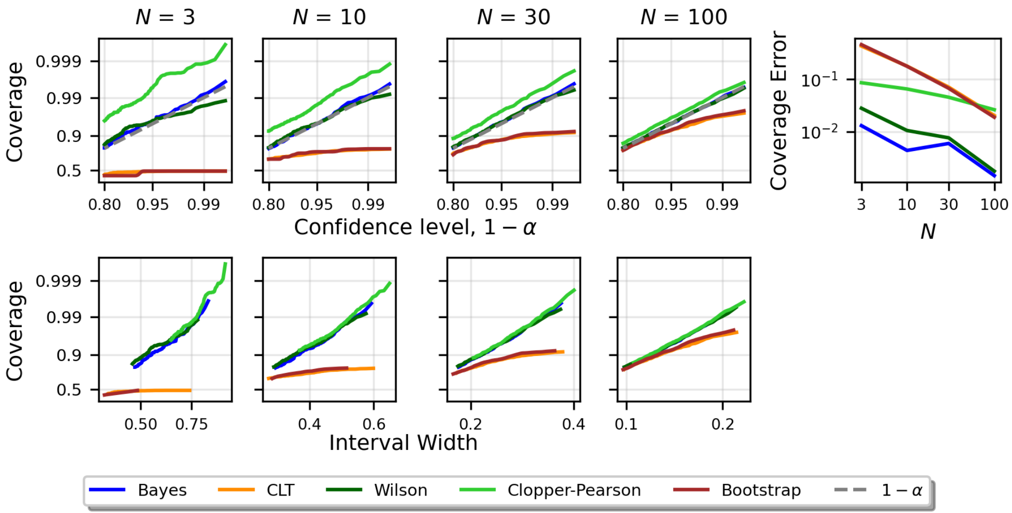
  - Coverage: What proportion of the time does a $1 - \alpha$ confidence-level interval actually contain the true parameter? (A frequentist metric, really.)

IID Questions Setting: Results



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$$\mathrm{SE}_{\mathrm{Clustered}} = \sqrt{\mathrm{SE}^2_{\mathrm{CLT}} + \frac{1}{N^2}\sum_{t=1}^{T}\sum_{i=1}^{N_t}\sum_{j\neq i}(y_{i,t} - \bar{y})(y_{j,t} - \bar{y})}$$

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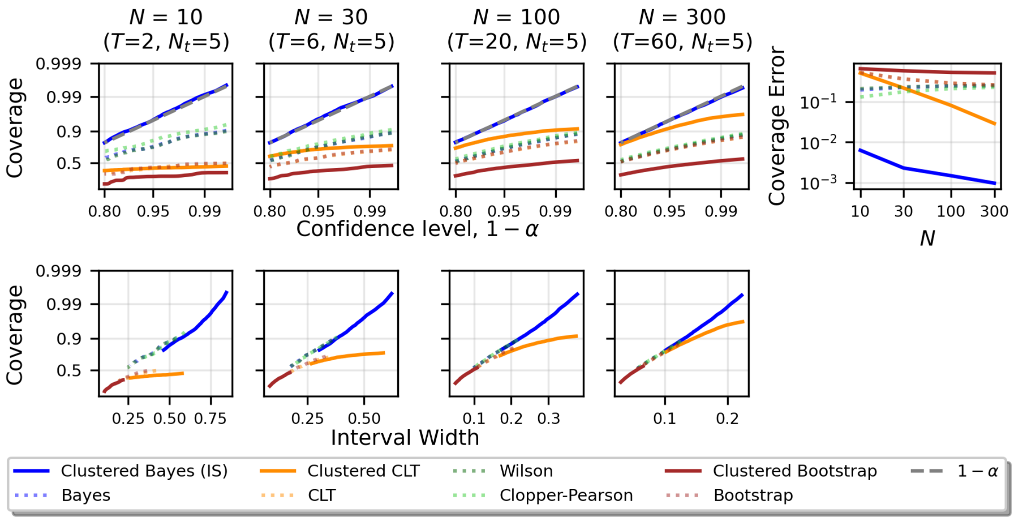

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References I

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