Real and Complex Analysis Solutions

- Solutions to the Real Analysis Part

Contents

This document is an attempt on writing down or collect other's solutions to as many exercises in the book *Real and Complex Analysis* by Walter Rudin as possible. The current plan is to finish the first 9 chapters. Suggestions and error corrections are highly-welcomed, so please feel free to contact me through taricpinkfriend@gmail.com.

This document is not entirely complete yet - namely, the exercises in chapter 7-9 still remains untouched - also, it may still undergo other revisions and corrections. Updates to this document can be found at the following GitHub page : github.com.

For those that plans to study the book by themselves, we also note here that there already exists some solutions to these exercises out there on the Internet, but personally, I found most help from answers on the Mathematics StackExchange site when I really don't have any idea on how to do some of the exercises in this book, so the readers are also encouraged to check out these other resources to accompany you along the way.

1 Abstract Integration

Exercise 1 (*Infinite σ-algebras are Uncountable*) The short answer is no. Suppose given a countably infinite σ-algebra $\mathfrak{M}$ on X, then it easily verified that for every subset Y of X, there exists a smallest measurable set $Y^\#$ containing it. We have the following observations :

(i) Given $y \in Y \subseteq X$, we have $\{y\}^\# \subseteq Y^\#$, and that each $Y^\#$ is a union of the $\{y\}^\#$'s.

(ii) Given $y, z \in X$, we have either $\{y\}^\# = \{z\}^\#$ or $\{y\}^\# \cap \{z\}^\# = \emptyset$.

We omit the proof of (i). For (ii), notice if $y \in \{z\}^\#$ then $\{y\}^\# \subseteq \{z\}^\#$ by (i); if not, we get $\{z\}^\# = \{z\}^\# \setminus \{y\}^\#$. By (ii), in order to arrive at a contradiction, it suffices to show that the cardinality of the collection of sets of the form $\{y\}^\#$ is infinite, but suppose this collection were to be finite, then we have by (i) that the σ-algebra $\mathfrak{m}$ is finite.

Exercise 2 An analogous version : given real measurable functions $u_i : X \to \mathbb{R}$ with i ranging from the integers between 1 and n, and given a continuous function $\Phi : \mathbb{R}^n \to Y$, then the composite

$$X \xrightarrow{u} \mathbb{R}^n \xrightarrow{\Phi} Y$$

is measurable, where we write $u := (u_1, \ldots, u_n)$. It suffices to show measurability of u. Since $\mathbb{R}$ is second countable with basis some open intervals, $\mathbb{R}^n$ is also second countable with basis some open n-cubes, but the preimage of these n-cubes under u may be described as a finite intersection of preimages of the component functions u_i of open intervals, so we are done.

Exercise 3 The identity

$$f^{-1}(x, +\infty] = \bigcup_{\substack{r > x \\ r \in \mathbb{Q}}} f^{-1}[r, +\infty]$$

shows that f is measurable (see Theorem 1.12 (c)).

Exercise 4 Assertion (a) follows from the calculation :

$$\limsup_{n\to\infty} (-a_n) = \inf_n \sup_{k\geq n} (-a_n) = \inf_n -\left(\inf_{k\geq n} a_n\right) = -\sup_n \inf_{k\geq n} a_n = -\left(\liminf_{n\to\infty} a_n\right)$$

For (b), we remark that in order for the summation of a pair of numbers (a, b) in the extended real number system to be well-defined, we have to impose the condition that $(a, b) \neq (-\infty, \infty), (\infty, -\infty)$. Associated to $\{a_n\}, \{b_n\}$, we define the sequences $\{c_n\}, \{A_n\}, \{B_n\}, \{C_n\}$ in $[-\infty, \infty]$ by the rule

$$c_n = a_n + b_n, \quad A_n = \sup_{k\geq n} a_k, \quad B_n = \sup_{k\geq n} b_k, \quad C_n = \sup_{k\geq n} c_k$$

then $\{A_n\}, \{B_n\}, \{C_n\}$ are all decreasing, and that the limits of these sequences are the limsups of $\{a_n\}, \{b_n\}, \{c_n\}$. WLOG, let us assume that $-\infty \leq \lim A_n \leq \lim B_n \leq +\infty$.

- Case $-\infty = \lim A_n < \lim B_n = \infty$: this is excluded from the question.
- Case $-\infty < \lim A_n \leq \lim B_n = \infty$: then $\lim C_n \leq \lim A_n + \lim B_n$ is trivially true.
- Case $-\infty = \lim A_n \leq \lim B_n < \infty$: then $a_n = -\infty$ for almost all n. By our previous remark, $b_n \neq -\infty$ for almost all n, so $c_n = -\infty$ for almost all n, so $\lim C_n = -\infty = \lim A_n + \lim B_n$.
- Case $-\infty < \lim A_n \leq \lim B_n < \infty$: then for almost all n, the sum $A_n + B_n$ is well-defined, say that for $n \geq n_0$ the sum is defined. Take $n \geq n_0$, we have $A_n + B_n \geq C_n$. Take limits on both sides, and use the fact that $\{A_n\}, \{B_n\}$ are decreasing, we get $\lim A_n + \lim B_n = \lim(A_n + B_n) \geq \lim C_n$.

By taking $a_n = (-1)^n = -b_n$, we see that strict inequality might occur in (b). For (c), use

$$\sup_{k\geq n} a_n \leq \sup_{k\geq n} b_n$$

and take limits on both sides.

Exercise 5 For (a), note that the set Y where $g - f$ is defined is measurable, and $f = g$ on Y^c. The sets $\{x : f(x) < g(x)\}, \{x : f(x) = g(x)\}$ are $(g-f)^{-1}(0,\infty), (g-f)^{-1}(0) \cup Y^c$, hence (a). For (b), for any triple of given natural numbers i, j, n, we may define a set

$$E_{i,j,n} = \{x : f_j(x) - n^{-1} < f_i(x) < f_j(x) + n^{-1}\}$$

The set of points x such that $\{f_i(x)\}$ converges to a finite limit is the following set :

$$E := \bigcap_n \bigcup_k \bigcap_{i,j \geq k} E_{i,j,n}$$

since if $x \in E$, then for any n, there exists k such that whenever $i, j \geq k$ we have $f_j(x) - n^{-1} < f_i(x) < f_j(x) + n^{-1}$, which implies $f_i(x) \neq \pm\infty$ for almost all i and that the sequence is Cauchy.

Exercise 6 Since X is uncountable, $\mu : \mathfrak{m} \to \{0, 1\}$ is well-defined. The fibers of μ partitions $\mathfrak{m} = \mu^{-1}(0) \cup \mu^{-1}(1)$; let us write $\mathfrak{m}_i = \mu^{-1}(i)$. We have the following :

(i) The set X trivailly lies in $\mathfrak{m}_1$.

(ii) The operation $E \mapsto E^c$ defines mutually inverse bijections between $\mathfrak{m}_0$ and $\mathfrak{m}_1$.

(iii) The sets $\mathfrak{m}_0, \mathfrak{m}_1$ are closed under countable unions and intersections.
(In particular, since $\emptyset \notin \mathfrak{m}_1$, for any $Y_0, Y_1 \in \mathfrak{m}_1$, we have $Y_0 \cap Y_1 \neq \emptyset$)

(iv) Given $Y_0 \in \mathfrak{m}_0, Y_1 \in \mathfrak{m}_1$ we have $Y_0 \cap Y_1 \in \mathfrak{m}_0, Y_0 \cup Y_1 \in \mathfrak{m}_1$.

Hence $\mathfrak{m}$ is a σ-algebra by (i)-(iv). That μ is a measure follows from (iii)-(iv).
Now we consider what are the measurable functions and the corresponding integral.
Let f be measurable, then the preimage of a ray in $[-\infty, \infty]$ lies in $\mathfrak{m}$. Take

$$x^* := \sup\{x : f^{-1}[-\infty, x) \in \mathfrak{m}_0\}, \quad x_* := \inf\{x : f^{-1}(x, +\infty] \in \mathfrak{m}_0\}$$

Now we claim that $x^* = x_*$. First we show $x^* \geq x_*$. Take $\delta > 0$, we have by (ii) that

$$f^{-1}[-\infty, x^* + \delta) \in \mathfrak{m}_1, \quad f^{-1}(x^* + \delta, \infty] \subset f^{-1}[-\infty, x^* + \delta)^c \in \mathfrak{m}_0$$

which shows $x^* + \delta \geq x_*$, so we get $x^* \geq x_*$. Similarly, $x_* \leq x^*$. Call this common value x_f, then $f(x) = x_f$ except at a countable subset - a measure zero subset - of X. On the contrary, a function of this form is clearly measurable. Therefore, a function f is measurable with respect to μ if and only if f is constant a.e.. The corresponding integral is then given by $\int_X f d\mu = x_f$.

Exercise 7 Using the dominated convergence theorem along with the fact that $f_1 \in L^1(\mu)$, we get the assertion. An example exhibiting the importance of the condition $f_1 \in L^1(\mu)$ can be constructed by taking $X = \mathbb{R}$ and $f_n = \chi_{(n,+\infty)}$.

Exercise 8 (*Strict Inequality in Fatou's Lemma*) In short, this example shows strict inequality might occur in the statement of Fatou's lemma. The idea is that the pointwise liminf of f_n is the constant function with value 0 while the integrals of f_n jumps between $\mu(X \setminus E)$ and $\mu(E)$.

Exercise 9 Since $c < \infty$, the set where $f(x) = \infty$ has measure 0, so we assume from the outset that f is a function $X \to [0, \infty)$. On the other hand, since $c > 0$, we see that $\mu(f^{-1}(0, \infty)) \neq 0$. Define $F_n : X \to [0, \infty]$ by $F_n = n \log[1 + (f/n)^\alpha]$. By hint, we distinguish two cases :

- Case $\alpha \geq 1$: We show F_n is dominated by αf. Let us show that

$$\alpha f - F_n = \alpha f - n \log[1 + (f/n)^\alpha]$$

is non-negative. Take $g_n = \alpha x - n \log[1 + (x/n)^\alpha]$, then

$$g(0) = 0, \quad g'(x) = \alpha \left(1 - \frac{nx^{\alpha-1}}{n^\alpha + x^\alpha}\right) \geq 0 \text{ for } x \geq 0$$

Since $f \in L^1(\mu)$, we get from dominated convergence theorem in this case that

$$\lim_{n\to\infty} \int_X F_n d\mu = \int_X \lim_{n\to\infty} F_n d\mu = \int_X \lim_{n\to\infty} \left(n^{1-\alpha} n^\alpha \log[1 + f^\alpha/n^\alpha]\right) d\mu = \begin{cases} \infty & \text{if } \alpha > 1 \\ c & \text{if } \alpha = 1 \end{cases}$$

- Case $0 < \alpha < 1$: since $F_n(x) \to \infty$ for $x \in f^{-1}(0, \infty)$, we get by Fatou's lemma that

$$\liminf_{n\to\infty} \int_X F_n d\mu \geq \int_X \left(\liminf_{n\to\infty} F_n\right) d\mu \geq \int_{f^{-1}(0,\infty)} \left(\liminf_{n\to\infty} F_n\right) d\mu = \infty$$

Exercise 10 Given $\varepsilon > 0$, take N so that $\|f - f_n\|_\infty < \varepsilon$ for all $n \geq N$, then

$$\left|\int_X (f_n - f) d\mu\right| \leq \varepsilon \mu(X)$$

Now the conditions (boundedness of f_n, measurability of f_n, f as a uniform limit of f_n) given in the question shows that each f_n and f are in $L^1(\mu)$, so

$$\int_X (f_n - f) d\mu = \int_X f_n d\mu - \int_X f d\mu$$

(see Theorem 1.32). Combining the first estimate with the second equality, we get the assertion. To see the importance of the hypothesis $\mu(X) < \infty$, take $X = \mathbb{R}$ and $f_n = n^{-1}$.

Exercise 11 Write $F_n = \bigcap_{k \geq n} E_k^c$, then $x \in A^c$ iff $x \in F_n$ for some n, hence

$$A^c = \bigcup_n F_n = \bigcup_n \bigcap_{k\geq n} E_k^c, \quad A = \bigcap_n F_n^c = \bigcap_n \bigcup_{k\geq n} E_k$$

To prove Theorem 1.41 based on this fact, note first that since $\mu(F_1^c) < \infty$ by assumption, we get $\mu(A) = \lim_{n\to\infty} \mu(F_n^c)$, while also that $\mu(F_n^c) \leq \sum_{k\geq n} \mu(E_k) \to 0$ by the same assumption.

Exercise 12 Define a measure λ on the same underlying measure space X by the law $d\lambda = |f| d\mu$ (see Theorem 1.29), then λ is a finite measure by assumption. Suppose there exists $\varepsilon > 0$ such that for any $n > 0$ we have a set E_n with the properties

$$\mu(E_n) < 2^{-n}, \quad \lambda(E_n) = \int_{E_n} |f| d\mu \geq \varepsilon$$

and define sets F_n, A as in **Exercise 11** , then we have $\sum_n \mu(E_n) < \infty$, so

$$\mu(A) = 0, \quad \lambda(A) = \int_A |f| d\mu = 0$$

but by the condition that λ is a finite measure, we also get

$$\lambda(A) = \lim_{n\to\infty} \lambda(F_n^c) \geq \varepsilon$$

which is a contradiction. Hence for any $\varepsilon > 0$, there is a $\delta > 0$ so that $\mu(E) < \delta$ implies $\lambda(E) < \varepsilon$.

Exercise 13 Recall that Proposition 1.24(c) is the statement $c\int_E f\,d\mu = \int_E cf\,d\mu$. Suppose $c = \infty$, let $Y = f^{-1}(0,+\infty] \cap E$, then cf is 0 on $E \setminus Y$ and is ∞ inside Y.
Now $c\int_E f\,d\mu = 0$ iff $\int_E f\,d\mu = 0$ iff $\mu(Y) = 0$ iff $\int_E cf\,d\mu = 0$.

2 Positive Borel Measures

Exercise 1 Consider first the assertion (b). Let $\mathbb{R}_d$ be the topological space with the same underlying set as $\mathbb{R}$ but with topology defined by letting the open rays (x,∞) be a subbasis. Addition of functions $f_1, f_2 : X \to \mathbb{R}_d$ can be written as a composition

$$f_1 + f_2 : \ X \xrightarrow{(f_1,f_2)} \mathbb{R}_d \times \mathbb{R}_d \xrightarrow{\ +\ } \mathbb{R}_d$$

so it suffices to determine the continuity of the addition map $+ : \mathbb{R}_d \times \mathbb{R}_d \to \mathbb{R}_d$ which is a routine exercise. Therefore we see that (b) holds in the generality where f_1, f_2 needn't have to be nonnegative, and that the domain X needn't has to be $\mathbb{R}$. Similarly, one can show (a) in this way.
Let us consider (d). Define $F_n = f_1 + \ldots + f_n$. Assuming $\{f_n\}$ are non-negative functions, we get

$$\left(\sum_{n=1}^{\infty} f_n\right)^{-1}(x,\infty] = \left(\sup_n F_n\right)^{-1}(x,\infty] = \bigcup_n F_n^{-1}(x,\infty]$$

for any $x \in [-\infty,+\infty]$; this shows (d) holds at least in the generality where X needn't be $\mathbb{R}$ but each f_n a non-negative function since each F_n is lower-semicontinuous by (a).
A counterexample for (c) can be constructed by $X = \mathbb{R}$, $f_n = \chi_{[(n+1)^{-1},n^{-1}]}$.
A counterexample for (d) can be constructed by $X = \mathbb{R}$, $f_1 = \chi_{(-1,2)}$, $f_n = -\chi_{[n^{-1},(n-1)^{-1}]}$ for $n > 1$.

Exercise 2 Given $x \in \mathbb{R}$, $\delta > 0$, write $U_{x,\delta} = (x-\delta, x+\delta)$. Notice $U_{y,\delta} \subseteq U_{x,2\delta}$ for $y \in U_{x,\delta}$, $\delta > 0$, and that $\varphi(x,\delta) = \operatorname{diam} f(U_{x,\delta})$, $\varphi(x) = \inf_{\delta>0} \varphi(x,\delta)$. Take $z \in \mathbb{R}$, want to show $\varphi^{-1}(-\infty, z)$ open. Suppose $\varphi(x) < z$, then there is some $\delta > 0$ with $\varphi(x, 2\delta) < z$. For $y \in U_{x,\delta}$, we have

$$\varphi(y) \le \varphi(y,\delta) = \operatorname{diam} f(U_{y,\delta}) \le \operatorname{diam} f(U_{x,2\delta}) = \varphi(x,2\delta) < z$$

which gives $x \in U_{x,\delta} \subseteq \varphi^{-1}(-\infty, z)$; up to this point, we have shown upper semicontinuity of φ.
Now we show $\varphi^{-1}(0)$ is the set of points where f is continuos. For a given $x \in \mathbb{R}$, $\varphi(x) = 0$ iff for all $\varepsilon > 0$ there exists $\delta > 0$ so that $\varphi(x,\delta) = \operatorname{diam} f(U_{x,\delta}) < \varepsilon$. Now by triangle inequality, we have

$$\sup_{y \in U_{x,\delta}} |f(x) - f(y)| \le \operatorname{diam} f(U_{x,\delta}) \le 2 \sup_{y \in U_{x,\delta}} |f(x) - f(y)|$$

so we are done. The above considerations can be generalized to metric spaces.

Exercise 3 (*Metric Spaces are T6*) Let $x, y \in X$, $e \in E$, we have the estimation

$$\rho_E(x) \le \rho(x,e) \le \rho(x,y) + \rho(y,e)$$

which gives $\rho_E(x) - \rho_E(y) \le \rho(x,y)$ hence by symmetry that $|\rho_E(x) - \rho_E(y)| \le \rho(x,y)$; so ρ_E is uniformly continuous. For the function $f = \rho_A/(\rho_A + \rho_B)$, it is bounded between 0 and 1, is 0 on A and 1 on B. If given $K \subset V$ with K compact V open, take $(A,B) = (V^c, K)$, then $K \prec f \prec V$. To see why f is supported in V, notice that $\rho_A(x) = 0$ iff $x \in A$ by closedness of A.

Exercise 4 For (a), we have by outer regularity of E_1, E_2 and the condition that E_1, E_2 can be separated by open sets, we can restrict attention to disjoint open neighborhoods of E_1, E_2. Relating $\mu(E_1) + \mu(E_2)$ and $\mu(E_1 \cup E_2)$ is now a routine calculation that we will omit here.
As for (b), using inner regularity and the fact that $\mathfrak{m}_F$ is closed under set difference, we can find compact sets K_n with $K_1, \ldots, K_n$ disjoint, all contained in V, such that $\mu(V \setminus (K_1 \cup \ldots \cup K_n)) < n^{-1}$; now let $N = E \setminus K$ where K the union of the K_n's, then we are done.

Exercise 5 (*Cantor Set*) The standard construction of E exhibits E as a countable intersection of descending Borel subsets $E_1 \supset E_2 \supset \ldots$ with $m(E_n) = (2/3)^n$, so $m(E) = 0$.

Exercise 6 (*A Totally Disconnected Compact subset of* $\mathbb{R}$ *with Positive Measure*) We will provide two examples of this. The first will be used in the answer of **Exercise 7** , the latter resembles the method in **Exercise 8** .

Example 1 : The Smith-Volterra-Cantor Set The construction of this example is similar to that of the Cantor set, but with the exception that the length of each deleted intervals is different from that of the Cantor set so that the measure of this set becomes positive. The construction goes as follows. Define a decreasing sequence $K_1 \supset K_2 \supset \ldots$ of closed subsets of $[0, 1]$ by specifying $K_1 = [0, 1]$, and with K_{n+1} inductively defined by removing away for each connected component of K_n an open interval of length 2^{-2n} centered at the center of that connected component, and define K to be the intersection of the K_n's. We have the following :

$$m(K_1) = 1, \quad m(K_{n+1}) = m(K_n) - 2^{n-1} \cdot 2^{-2n}, \quad \sup\{m([x, y]) : x \le y, [x, y] \subset K_n\} \le 2^{n-1}$$

These calculations shows that $m(K) = 2^{-1}$ and that K is disconnected.

Example 2 : The "Take an Enumeration of the Rationals" Trick We first remark that a subset of $\mathbb{R}$ containing no rationals is clearly totally disconnected. Take a bounded interval $[0, 1]$, we might consider deleting all the rational numbers in it, but the result might not be a closed set, so we make the following modification. Let r_n be an enumeration of the rationals, define $U_n = (r_n - 2^{-n-2}, r_n + 2^{-n-2})$, then $U = \bigcup U_n$ is an open set with measure $\le 2^{-1}$, so we are done by taking $K = [0, 1] \setminus U$.

Exercise 7 (*Measure of a Dense Open set of* $[0, 1]$) Replace the number 2 in Example 1 of **Exercise 6** with $(\sqrt{1 + 4(1 - \varepsilon)^{-1}} + 1)/2$.

Exercise 8 (*A Borel Set with Positive non-full Measure on each Interval*) Take an enumeration of the rationals $\{r_n\}$. Define subsets as follows :

- Let V_1 be an open interval of length 1 centered at r_1,
  and inductively define V_{n+1} as the open interval of length $3^{-1}m(V_n)$ centered at r_{n+1}.
- Let $W_n = V_n \setminus \bigcup_{k=1}^{\infty} V_{n+k}$.

then each W_n has positive measure, are mutually disjoint, that any non-empty open interval in $\mathbb{R}$ contains some W_n. Take Borel subsets E_n of W_n with $0 < \mu(E_n) < \mu(W_n)$, define $E = \bigcup E_n$.

Exercise 9 First, let $\{r_n\}$ be the following sequence of rational numbers in $[0, 1]$:

$$r_1 = 1/2, \quad r_2 = 1/3, \quad r_3 = 2/3, \quad r_4 = 1/4, \quad r_5 = 2/4, \quad r_6 = 3/4, \quad \ldots$$

and define integers $\{s_n\}$ associated to r_n :

$$s_1 = 2, \quad s_2 = 3, \quad s_3 = 3, \quad s_4 = 4, \quad s_5 = 4, \quad s_6 = 4, \quad \ldots$$

Write $a_{n,1} = 0, a_{n,2} = r_n - s_n^{-1}, a_{n,3} = r_n, a_{n,4} = r_n + s_n^{-1}, a_{n,5} = 1$. Define $f_n : [0, 1] \longrightarrow [0, 1]$ by

$$f_n(a_{n,i}) = \begin{cases} 1 & \text{if } i = 3 \\ 0 & \text{otherwise} \end{cases}, \text{ with } f_n \text{ being linear on each } [a_{n,i}, a_{n,i+1}]$$

then $\int_0^1 f_n(x)dx = s_n^{-1}$, $\{f_n(x)\}$ does not converge for any $x \in (0,1)$; to see this, notice that for a positive integer $N > 0$ and any $x \in [(2N)^{-1}, 1-(2N)^{-1}]$, we have $\sup_{i:s_i=N} f_i(x) \geq 2^{-1}$. Although f_n is pointwise convergent on $0, 1$, we may take $\varepsilon > 1$ and replace $f_n(x)$ by $f_n(\varepsilon^{-1}(x-2^{-1})+2^{-1})$.

Exercise 10 This follows almost directly from Lebesgue's monotone convergence theorem. If we don't want to use the theorem directly, here is a proof using only the elementary properties of measures : let $\varepsilon > 0$, define nested borel subsets $A_1 \supseteq A_2 \supseteq \dots$ of $I := [0,1]$ by the rule

$$A_n = \{x \in I : f_m(x) \geq \varepsilon \text{ for some } m \geq n\}$$

then $\bigcap A_n = \emptyset$, hence $\lim m(A_n) = m(A) = 0$. Take n with $m(A_n) \leq \varepsilon$, then for each $m \geq n$, we get :

$$\int_I f_m(x)dx = \int_{A_n} f_m(x)dx + \int_{I \setminus A_n} f_m(x)dx \leq \varepsilon + \varepsilon = 2\varepsilon$$

hence $\lim \int_I f_n(x)dx = 0$.

Exercise 11 Following the hint, let $\{K_\alpha\}$ be collection measure 1 compact sets, and let K be their intersection. Let us show $\mu(K) = 1$. By outer regularity, only need to show for each open $V \supset K$ that $\mu(V) = 1$, but since $\bigcap(K_\alpha \setminus V) = \emptyset$ and that X is compact, some finite intersection of $(K_\alpha \setminus V)$ is already empty, say that V contains $K_1 \cap \dots \cap K_n$, then $\mu(V^c) \leq \sum_{i=1}^n \mu(K_i^c) = 0$, hence $\mu(V) = 1$. The statement regarding proper compact subsets of K is straightforward from the construction of K.

Exercise 12 Let K be a compact subset of $\mathbb{R}$, then take a countable subset $L := \{x_n\}_{n=1}^\infty$ such that $\bar{L} = K$. Let δ_n be unit mass of x_n, take $\mu = \sum_n 2^{-n}\delta_n$, then the support of μ is exactly K. (short verification : a borel set has full measure iff it contains L, and if an open set contains L, it contains K, so proper open subsets hence proper compact subsets of K does not have full measure)

Exercise 13 Suppose $K \subset \mathbb{R}$ is a support of some function, then K cannot contain any isolated points. It is easy to see from this fact that K is the closure of its interior, while open sets in $\mathbb{R}$ are disjoint unions of open intervals (for example, by Lindelöf's lemma), so one can reduce to the case where K is closed interval, and the rest is straightforward.

Exercise 14 By the Vitali-Carathéodory theorem, if $f \in L^1(m)$, there is some Borel measurable function g with $g \leq f$ and $g = f$ a.e. $[m]$. Our issue is that we don't have integratability, so we construct on small Borel subsets and paste. Take countably many compact sets K_n with union $\mathbb{R}^k$, define $E_n = f^{-1}[-n,n]$. Take Borel sets $B_{m,n} \subset K_m \cap E_n$ with $\mu(K_m \cap E_n \setminus B_{m,n}) = 0$, then $\chi_{B_{m,n}} f$ is integratable and can be approximated by some Borel measurable function $g_{m,n}$. Now define g on $\bigcup_{m,n} B_{m,n}$ as the sup of the $g_{m,n}$ and $-\infty$ outside it. We omit the construction of h.

Exercise 15 Define $f_n(x) = \chi_{[0,n]}(1-xn^{-1})^n e^{x/2}$, then we are considering $\lim \int_0^\infty f_n(x)dx$. Now $f_n(x)$ is dominated by $e^{-x/2}$, so by dominated convergence theorem, the integral is $\int_0^\infty e^{-x/2}dx = 2$. The other one can be done in a similar way.

Exercise 16 (*Any Proper Subspace of $\mathbb{R}^k$ has Measure 0*) Let D_n be closed ball with radius n and center 0 in $\mathbb{R}^k$, let $Y_n = D_n \cap Y$, want $\mu(Y_n) = 0$. Take $v \in Y^\perp \setminus 0$, since $\{u + rv_n : u \in Y_n, r \in [0,1]\}$ is bounded, contains a disjoint union of countably infinitely many translations of Y_n, we're done.

Exercise 17 (*An Example of a Measure Given by Riesz's Theorem that is not Inner Regular*)
Let us use a more topological viewpoint of the construction of the space X. Firstly, the set maps

$$d, d' : \mathbb{R}\times\mathbb{R} \longrightarrow [0,\infty) \ , \quad d(y_1,y_2) = |y_1 - y_2|, \quad d'(x_1,x_2) = \delta_{x_1,x_2}$$

defines different metrics on $\mathbb{R}$; let Y_2 be the metric space $(\mathbb{R}, d)$, Y_1 be the metric space $(\mathbb{R}, d')$, then Y_1 has discrete topology, and X is $Y_1 \times Y_2$ with product topology, with a metric given by the one in the question. Let $\pi : X = Y_1 \times Y_2 \to Y_1$ be the projection. Take $f \in C_c(X)$, write $K = \operatorname{supp} f$, then $K \subseteq \pi^{-1}\pi(K) = \pi(K) \times Y_2$ with $\pi(K)$ compact hence finite. A compact subset K of E has cardinality $\leq \#\pi(K) < \infty$, so $\Lambda(\chi_K) = 0$. Since E is outer regular, only need $\mu(V) = \infty$ for any open set V containing E. Given map $\nu : \mathbb{R} \to (0, +\infty)$, define open set

$$U_\nu = \bigcup_{x\in\mathbb{R}} \{x\} \times (-\nu(x), \nu(x))$$

Sets of the form $\{x\} \times (-\delta, \delta)$ form an open neighborhood basis of $(x, 0)$, so V contains some U_ν, so it suffices to show $\mu(U_\nu) = \infty$ for any ν. Now $\nu^{-1}[n^{-1}, (n-1)^{-1})$ for $n \in \mathbb{N}$ (here $0^{-1} = \infty$) is a countable cover of $\mathbb{R}$, so some of them is uncountable, so may take some $n \in \mathbb{N}$ and countably infintely many x_k with $\nu(x_k) \geq n^{-1}$, giving $\mu(U_\nu) = \infty$.

Exercise 18 (*The condition on σ-compactness in the statement of Theorem 2.18 is Necessary*)

(1) Basically, the topology on X is the order topology - the topology generated by the subbasis P_α, S_α of downward, upward open rays in X.

(2) X is Hausdorff : Given $\alpha < \beta$, take $P_{\alpha+1}, S_\alpha$.

(3) X is compact : In general, given a totally ordered set with order topology, if it has a last element β, the upward open rays is then a nbd basis of β. Given an open cover $\{U_\lambda\}_{\lambda\in\Lambda}$ on X, we may define, recursively, (closed) subsets X_n and elements α_n : (here we write 0 as inf X)

$$\alpha_1 = \omega_1,\ X_n = {S_{\alpha_n}}^c,\ \alpha_{n+1} = 0 \text{ if } \alpha_n = 0, \text{ otherwise } \alpha_{n+1} \text{ as an element } < \alpha_n$$
$$\text{with } S_{\alpha_{n+1}} \cap X_n \subseteq U_\lambda \text{ for some } \lambda \in \Lambda.$$

Since X is well-ordered, $\alpha_n = 0$ for some n, so X has a finite subcover.

(4) Note : Basically, we have shown that a well-ordered set with a last element is compact in its order topology. In particular, ${P_\alpha}^c$ is compact for any $\alpha < \omega_1$.

(5) $\{\omega_1\}^c$ is not σ-compact : Notice that σ-compact sets are Lindelöf. Take the cover $\{P_\alpha\}_{\alpha\in X}$.

(6) For $f \in C(X)$, there is some $\alpha < \omega_1$ so that f is constant on S_α : Notice that $f(\omega_1)$ has a countable basis, so may find $\alpha < \omega_1$ with S_α inside the intersection of the inverse images.

(7) The collection of uncountable compact subsets is closed under countable intersection :
Let K_n be a such a collection, with intersection K. Notice that an uncountable compact subset of X must contain ω_1, so $\omega_1 \in K$. By cutting away some P_α for some K_n, only need $K \neq \{\omega_1\}$. Let us mention the following two facts. Firstly, any countable subset in $\{\omega_1\}^c$ has a supremum in $\{\omega_1\}^c$ by well-orderedness. Next, for any element $\alpha \in \{\omega_1\}^c$ and any $n \in \mathbb{N}$, there is some $\beta \in K_n$ with $\alpha \leq \beta < \omega_1$ by uncountability of ω_1. From this, we construct recursively a countable set of the form $\{\alpha_{m,n}\}_{m,n\in\mathbb{N}}$ such that :

$$\{\alpha_{m,n}\}_n \subseteq K_m \text{ for each } m \in \mathbb{N}, \text{ with } \alpha_{m,n} \text{ satisfying the following relation :}$$

$$
\begin{aligned}
0 \le \alpha_{1,1} \le \alpha_{2,1} \le \alpha_{3,1} \le \ldots &\le \sup_m \alpha_{m,1} \\
\le \alpha_{1,2} \le \alpha_{2,2} \le \alpha_{3,2} \le \ldots &\le \sup_m \alpha_{m,2} \\
\le \alpha_{1,3} \le \alpha_{2,3} \le \alpha_{3,3} \le \ldots &\le \sup_m \alpha_{m,3} \ldots < \omega_1
\end{aligned}
$$

Take $\sup_{m,n}\{\alpha_{m,n}\} = L < \omega_1$, then $\sup_n\{\alpha_{m,n}\} = L \in K_m$ for each m, and we're done.

(8) $\mathfrak{M}$ is a σ-algebra : (7) gives closedness under countable union; for the rest check by definition.

(9) $\mathfrak{M}$ contains the Borel sets : $S_\alpha, P_\alpha \in \mathfrak{m}$ by (4), and we are done by (0), (8).

(10) λ is measure : Follows from (7).

(11) λ is not regular : Since $\lambda(\omega_1) = 0$ but $\lambda(S_\alpha) = 1$ for any $\alpha < \omega_1$.

(12) For any $f \in C(X)$, $f(\omega_1) = \int_X f d\lambda$: Since f is constant on some full measure nbd of ω_1.

(13) Note : $\lambda(P_\alpha) = 0$, $\lambda(S_\alpha) = 1$ for any $\alpha < \omega_1$. This can be seen from (4).

(14) Describe the regular μ associated to the linear functional $\Lambda : f \mapsto \int_X f d\lambda$ by Riesz's theorem : Take $\alpha < \omega_1$. By (12), $\mu(S_\alpha) = \sup\{f(\omega_1) : f \prec S_\alpha\}$. By Urysohn's lemma, $\mu(S_\alpha) = 1$, so $\mu(\{\omega_1\}) = 1$ by regularity of μ. Similarly, since $\lambda(P_\alpha) = 0$, we get $\mu(P_\alpha) = 0$ by the definition of Λ. Therefore, $\mu(X) = 1$ and $\mu(\{\omega_1\}^c) = 0$.

Exercise 19 Here is a very rough attempt. The theorem can be reformulated as follows.

Riesz's Theorem - Compact Version Given compact Hausdorff X, a positive linear functional Λ on $C(X)$, there exists a σ-algebra $\mathfrak{M}$ and and a unique representing positive, finite measure μ on $\mathfrak{M}$, such that $\mathfrak{M}$ is complete, contains all Borel subsets, and that μ is regular.

A new outline of a proof can be given roughly as follows :

- Definition of μ : define on open sets, then on $P(X)$, in a similar manner as

$$\mu(V) = \sup\{\Lambda f : f \prec V\}, \quad \mu(E) = \inf\{\mu(V) : E \subseteq V\}$$

From these definitions, compact subsets then also have the following analogous property :

$$\mu(K) = \inf\{\Lambda f : K \prec f\}$$

(as in step II of the original proof) Therefore μ is a positive and finite function.

- μ has countable additivity : can be done as in step I,IV of the original proof.
- Definition of $\mathfrak{M}$: collection of the inner regular (hence regular) subsets; it already contains all the compact (i.e. closed) sets, and sets with measure 0. In fact, it also contains all open sets as in step III of the original proof.
- Small lemma : given $E \in \mathfrak{M}$ and $\varepsilon > 0$, there is $K \subseteq E \subseteq V$ with $\mu(V \setminus K) < \varepsilon$.
This is by regularity of E and countable additivity as in Step V of the original proof.
- $\mathfrak{M}$ is σ-algebra that is complete and contains all Borel subsets : Only need closedness under countable unions and complements. Countable union follows from countable additivity of μ as in step IV of the original proof, complements follows from the small lemma above as in step IV of the original proof. Also, $X \in \mathfrak{M}$.

- μ represents Λ : as in step X of the original proof.
- Uniqueness of μ : suffices to verify on compact sets by regularity.

In short, the fact that $\mathfrak{M}_F$ is just $\mathfrak{M}$ in this case made our life a lot more easier.

Exercise 20 (*The Required Conditions of DCT is not Necessarily Necessary*) .
Given a positive integer n, let $\delta_n = 2^{-n-1}$, take finitely many $f_{n1}, \dots, f_{nk_n} : [0,1] \to [0,\infty)$ with:

$$\int_0^1 f_{ni}(x)dx \leq \delta_n; \quad \sup_{1\leq i\leq k_n} f_{ni}(x) = \begin{cases} \delta_n^{-1} & \text{for } x \in [2\delta_n, 3\delta_n] \\ 0 & \text{for } x \notin [\delta_n, 4\delta_n] \end{cases}; \quad f_{ni} \in C_c([0,1])$$

Assemble these f_{ij} into a sequence using the lexicographic order, then we're done.

Exercise 21 For $r \in \mathbb{R}$, let $U_r = (-\infty, r)$. By considering $\{f^{-1}(U_n)\}_{n\in\mathbb{N}}$, $f(X)$ bounded above. Let $r = \sup f(X) < \infty$, by considering $\{f^{-1}(U_{r-n^{-1}})\}_{n\in\mathbb{N}}$, $r \in f(X)$.

Exercise 22 Given $p \in X$, write $g_n(x;p) = f(p) + nd(x,p)$. Also, let $Y = f^{-1}[0,\infty)$.
For (i), take $\varepsilon > 0$, $p \in Y$ with $g_n(y;p) \leq g_n(y) + \varepsilon$, then the following calculation will suffice :

$$g_n(x) - g_n(y) \leq g_n(x;p) - g_n(y;p) + \varepsilon \leq nd(x,y) + \varepsilon$$

For (ii), it follows from $0 \leq g_1(x;p) \leq g_2(x;p) \leq \dots$ and $g_1(x;x) = g_2(x;x) = \dots = f(x)$.
For (iii), assume first $x \in Y$. Take $\varepsilon > 0$, pick δ so that $f(x) - f(p) > \varepsilon$ whenever $d(x,p) < \delta$, then

$$f(x) - g_n(x;p) = f(x) - f(p) - nd(x,p) \leq \sup\{\varepsilon, f(x) - n\delta\} \to \varepsilon \text{ as } n \to \infty$$

so $g_n(x) \to f(x)$ as $n \to \infty$. If $x \notin Y$, for $L > 0$, take $R > 0$ with $f(p) > L$ when $d(x,p) < R$, then

$$g_n(x;p) = f(p) + nd(x,p) \geq \inf\{L, nR\} \to L \text{ as } n \to \infty$$

so $g_n(x) \to \infty = f(x)$ as $n \to \infty$ in this case.

Exercise 23 The answers are no. Let V be the unit open ball centered at 0. For upper semicontinuity, take unit mass at origin μ_0. For lower semicontinuity, take a countable dense subset $X = \{x_n\}_{n\in\mathbb{N}}$ of V and take $\sum_n 2^{-n}\mu_{x_n}$.

Exercise 24 It suffices to treat the case where f is positive by considering f^+, f^-. Since $f \in L^1(m)$, we may, by rescaling and shifting, reduce the question to that of finding for each $\varepsilon > 0$ a step function g on $[0,1]$ with $\int_0^1 |f-g|dx < \varepsilon$. By Vitali-Carathéodory, may assume f is upper semicontinuous. Using Lebesgue number lemma, may find $M \in \mathbb{N}$ so that whenever $|y-x| < M^{-1}$, $f(y) - f(x) < \varepsilon$. Partition $[0,1)$ into M equilength half-open subintervals $I_1, \dots, I_M$ and define g on each I_k to be the constant function with value the supremum (exists !) of f on that interval.

Exercise 25 By the following observations, we have $c = \log 2$:

$$\log(1+e^t) < c + t \text{ iff } \log(1+e^{-t}) < c; \quad \sup\{\log(1+e^{-t}) : t \in (0,\infty)\} = \log 2.$$

Write $X = f^{-1}[0,+\infty) \subseteq [0,1]$. In short, we have by dominated convergence theorem that

$$\int_0^1 n^{-1}\log(1+e^{nf(x)})dx = \left(\int_{X^c} + \int_X\right) n^{-1}\log(1+e^{nf(x)})dx \to \int_X f(x)dx = \int_0^1 f^+(x)dx$$

where $X^c = [0,1] \setminus X$. We explain this as follows :

- Write $g_n(x) := n^{-1}\log\left(1+e^{nf(x)}\right)$. Since log, exp are continuous, $g_n(x)$ is measurable. Also :

$$\lim_{n\to\infty} g_n(x) = \begin{cases} f(x) & \text{if } x \in \mathrm{X} \\ 0 & \text{if } x \in \mathrm{X}^c \end{cases}$$

- Now $g_n(x)$ is dominated by $\log 2 + f(x) \in \mathrm{L}^1$ on X (by (i)), and dominated by $\log 2$ on X^c.

3 L^p SPACES

CONVENTION FOR EXERCISE 1-3 For clarity, given $[x,y] \subseteq (a,b)$, we say φ is *vex* on (a,b) if :

$$\varphi((1-\lambda)x+\lambda y) \leq (1-\lambda)\varphi(x)+\lambda\varphi(y) \text{ for each } \lambda \in (0,1)$$

so in particular, φ is convex on (a,b) if it is vex on all closed subintervals in it. In exercises 1-3 of this chapter, we will assume $[0,1] \subset (a,b)$ and only check vexity on $[0,1]$. In this case, we will state that we "assume the convention".

EXERCISE 1 Assume the "convention".

- Given a collection $\{f_\lambda\}_{\lambda\in\Lambda}$ of convex functions on $(0,1)$, if $\sup_\lambda f_\lambda < \infty$, then it is convex : this can be seen by taking sup over the following inequality

$$f_\lambda(x) \leq x f_\lambda(0) + (1-x) f_\lambda(1), \text{ for } \lambda \in \Lambda, x \in (0,1)$$

- Given a collection $\{f_n\}_{n\in\mathbb{N}}$, then :
  - $\lim_n f_n$ is convex provided it exists : similar as above.
  - $\limsup_n f_n$ is convex : combine the preceding results.
  - $\liminf_n f_n$ might not be : take $f_n(x) = x$ for even n and $f_n(x) = 1-x$ for odd n.

EXERCISE 2 Assume the "convention", and let $\mathrm{A} = \varphi(0)$, $\mathrm{B} = \varphi(1)$. We have

$$\psi\varphi(x) \leq \psi(x\mathrm{A}+(1-x)\mathrm{B}) \leq x\psi(\mathrm{A})+(1-x)\psi(\mathrm{B}) = x\psi\varphi(0)+(1-x)\psi\varphi(1)$$

Suppose $\varphi > 0$ and $\log\varphi$ convex, we have $\log\varphi(x) \leq x\log\mathrm{A}+(1-x)\log\mathrm{B}$, hence

$$\varphi(x) = \exp\log\varphi(x) \leq \exp(x\log\mathrm{A}+(1-x)\log\mathrm{B}) = \mathrm{A}^x\mathrm{B}^{1-x} \leq x\mathrm{A}+(1-x)\mathrm{B}$$

To see why this is not why vice versa, notice that $\log x$ is not vex on $[0,1]$.

EXERCISE 3 Assume the "convention", take $\mathrm{A} = \varphi(0)$, $\mathrm{B} = \varphi(1)$, $\mathrm{L}(x) : x \mapsto x\mathrm{A}+(1-x)\mathrm{B}$. Write $\mathrm{T}_n = 2^{-n}\mathbb{Z} \cap [0,1]$, $\mathrm{T} = \bigcup_n \mathrm{T}_n$, T is dense in $[0,1]$, and $\mathrm{L}-\varphi \geq 0$ on T by induction.

EXERCISE 4 Given a measurable subset $\mathrm{Y} \subseteq \mathrm{X}$, define

$$\varphi_{\mathrm{Y}} : (0,\infty) \to [0,\infty], \quad p \mapsto \int_{\mathrm{Y}} |f|^p d\mu = \int_{\mathrm{X}} \chi_{\mathrm{Y}}|f|^p d\mu$$

Let $\mathrm{A} = |f|^{-1}[0,1]$, then $\chi_{\mathrm{A}}|f|^p, \chi_{\mathrm{A}^c}|f|^p$ pointwise monotone, so $\varphi_{\mathrm{A}}, \varphi_{\mathrm{A}^c}$ monotone, so

$$\varphi(p) = \varphi_{\mathrm{A}}(p)+\varphi_{\mathrm{A}^c}(p) \leq \varphi_{\mathrm{A}}(r)+\varphi_{\mathrm{A}^c}(s) \leq \varphi(r)+\varphi(s) < \infty$$

which gives (a). Also, (a) can be shown by Hölder's inequality : take $p = ur + (1-u)s$, then

$$\varphi(p) = \varphi(ur + (1-u)s) \leq \varphi(r)^u \varphi(s)^{1-u} < \infty$$

which also give first part of (b). Notice this also gives (d) : write $\mathrm{L} = \max(\|f\|_r, \|f\|_s)$, then

$$\|f\|_p^p = \varphi(p) \leq \varphi(r)^u \varphi(s)^{1-u} = \|f\|_r^{ur} \|f\|_s^{(1-u)s} \leq \mathrm{L}^{ur} \mathrm{L}^{(1-u)s} = \mathrm{L}^p$$

Now we show continuity of φ. Given $[p, p+\varepsilon] \subseteq \mathrm{E}$, by DCT applied to $\chi_{\mathrm{A}}|f|^p, \chi_{\mathrm{A}^c}|f|^p$ and monotonicity of $\varphi_{\mathrm{A}}, \varphi_{\mathrm{A}^c}$, we have

$$\varphi(p) = \varphi_{\mathrm{A}}(p) + \varphi_{\mathrm{A}^c}(p) = \lim_{q \to p^+} (\varphi_{\mathrm{A}}(q) + \varphi_{\mathrm{A}^c}(q)) = \lim_{q \to p^+} \varphi(q)$$

so φ is upper semicontinuous on E. Lower semicontinuity is alike.
Now we consider (c). Take $\mathrm{X} = (0,1), (2,\infty)$, $f(x) = x^{-1}, (x \log^2 x)^{-1}, e^{-x}$, we see

$$\text{E can be any of the } (0,1), (1,\infty), (0,1], [1,\infty), (0,\infty).$$

so E can be any closed or open rays in $(0,\infty)$ by considering a rescaling. Now since each connected subset of $(0,\infty)$ is an intersection of two rays, E can be any of the connected subset of $(0,\infty)$.
Now we only need (e). Write $\mathrm{X}_n = |f|^{-1}(n,\infty)$. If $\|f\|_\infty = \infty$, each X_n has positive measure. Also,

$$\|f\|_p \geq \mu(\mathrm{X}_n)^{p^{-1}} n$$

so $\lim_p \|f\|_p = \infty$. If $0 < \|f\|_\infty = \mathrm{L} \neq \infty$, may let $\mathrm{L} = 1$. For $\varepsilon \in (0,1)$, $\mu(\mathrm{X}_\varepsilon) < \infty$ (by $\|f\|_p < \infty$), so

$$\|f\|_p \geq \mu(\mathrm{X}_\varepsilon)^{p^{-1}} \varepsilon$$

so as we take $p \to \infty$, and $\varepsilon \to 1^-$, we have $\lim_p \|f\|_p \geq 1$; also we have for $p > r$ that

$$\|f\|_p = \int_{\mathrm{X}} \left(|f|^{p-r}|f|^r d\mu\right)^{p^{-1}} \leq \left(\|f\|_r^r\right)^{p^{-1}}$$

so as $p \to \infty$, we get $\lim_p \|f\|_p = 1$.

Exercise 5 Firstly, (a) follows from Hölder's inequality, by

$$\|f\|_r^r = \|f^r 1\|_1 \leq \|f^r\|_t \|1\|_{t'} = \|f\|_s^r$$

where $rt = s$. For (b), we see that if $s < \infty$, then equality holds iff f^s is constant a.e. on X (see the remark of Theorem 3.5). For $s = \infty$, it holds iff $f^r = \|f^r\|_\infty$ a.e. (see the proof of Theorem 3.8).
We omit (c). For (d), may assume $|f|$ real and ≥ 0 (so $\log|f| = \log f$) and $r = 1$ (so $f \in \mathrm{L}^1$).
Take log on both sides, what we want is the following :

$$\lim_{p \to 0^+} \log \|f\|_p = \lim_{p \to 0^+} p^{-1} \log \int_{\mathrm{X}} f^p d\mu = \int_{\mathrm{X}} \log f d\mu$$

Assume first RHS$\neq \infty$ (i.e., $\log f \in \mathrm{L}^1$). For LHS, limit exists : it is bounded (by 0) and decreasing (by (a)). Since $t \mapsto \log t$ is concave, LHS $\geq$ RHS by Jensen's inequality. Define for each $p \in (0,1)$:

$$g(p) = p^{-1} \log \int_{\mathrm{X}} f^p d\mu - \int_{\mathrm{X}} \log f d\mu$$

then it suffices to show $\lim_{p \to 0^+} g(p) \leq 0$. Notice by $\log(1+x) \leq x$, we have

$$g(p) \leq \int_{\mathrm{X}} \left(\frac{f^p - 1}{p} - \log f\right) d\mu$$

We show that the integrand is dominated by some L^1 function, then a standard argument using DCT will give us the desired result. For $a \in (0, \infty)$ we have

$$\frac{a^p - 1}{p} = \begin{cases} \int_1^a t^{p-1}dt \le \int_1^a 1dt = a - 1, & \text{if } a \in (0,1) \\ \int_a^1 t^{p-1}dt \le \int_1^a t^{-1}dt = \log(a), & \text{if } a \in [1,\infty) \end{cases}$$

and that our present assumptions gives $\log f, f \in L^1(\mu)$, so we are done.
Suppose now $\log f \notin L^1$. Take $X_n = f^{-1}[n^{-1}, \infty)$, $f_n = \chi_{X_n} f + \chi_{X_n^c}$, MCT says $\exp\left(\int_X \log f_n d\mu\right) \to 0$. Notice each f_n satisfies $f_n, \log f_n \in L^1$. For $p \in (0,1)$, $n \in \mathbb{N}$, we have

$$\|f\|_p^p = \int_X f^p d\mu = \int_{X_n} f^p d\mu + \int_{X_n^c} f^p d\mu \le \|f_n\|_p^p + \mu(X_n^c) n^{-p}$$

Let $\varepsilon > 0$. Take some $n \in \mathbb{N}$,$p^* \in (0,1)$ such that the following are satisfied :

$$\exp\left(\int_X \log f_n d\mu\right) < 4^{-1}\varepsilon, \quad \mu(X_n^c) < 2^{-1}, \quad n^{-1} < \varepsilon$$

$$\|f_n\|_p - \exp\left(\int_X \log f_n d\mu\right) < 4^{-1}\varepsilon, \text{ for } p < p^*$$

we have for each $0 < p < p^* < 1$ that $\|f\|_p < \varepsilon$. Therefore, $\|f\|_p \to 0$ as $p \to 0^+$; this finishes (d).

Exercise 6 Notice that by **Exercise 5** , we may use the identity

$$\Phi\left(\exp\left(\int_0^1 \log f dm\right)\right) = \int_0^1 (\Phi \circ f)\, dm$$

If given $0 = x_1 \le \ldots \le x_n = 1$, positive numbers $a_1, \ldots, a_n$, take $f = \sum a_i \chi_{[x_{i-1}, x_i]}$, we get

$$\Phi\left(\prod_i a_i^{m_i}\right) = \sum_i (m_i \Phi(a_i)), \quad \text{where } m_i := m([x_{i-1}, x_i]) = x_i - x_{i-1}$$

so to show $\Phi(x^c) = c\Phi(x) + (1-c)\Phi(1)$, it suffices to take $n = 2, x_1 = c, (a_1, a_2) = (x, 1)$. Now we classify all Φ. Define $\phi(x) := \Phi(x) - \Phi(1)$, we have $\phi(1) = 0, \phi(x^c) = c\phi(x)$, for $x \in (0,\infty), c \in [0,1]$, hence for all $c \in [0, \infty)$ by $c^{-1}\phi(x^c) = \phi(x)$ when $c > 1$. By this fact, we see $\Phi(x) = \Phi(1) + \phi(x) = A + B\log x$ for $x \in (0, \infty)$, where $A = \Phi(1), B = \phi(e) = \Phi(e) - A$. As for $\Phi(0)$, take a bounded, measurable f with $f \ge 0$, $\log f \notin L^1$ (for example, take $f : x \mapsto \exp(-[x^{-1}])$), we have

$$\Phi(0) = A + B\int_0^1 \log f(x)dx = \begin{cases} -\infty, & \text{if } B > 0 \\ \infty, & \text{if } B < 0 \\ A, & \text{if } B = 0 \end{cases}$$

In short, $\Phi(0)$ is uniquely determined by its value outside 0.
We omit the verification that every Φ defined by a pair (A, B) has the required property.

Exercise 7 For finite measures, $r < s$ gives $L^r \supset L^s$. For counting measures, $r < s$ gives $L^r \subset L^s$. For an example where $r \neq s$ implies $L^r \not\subseteq L^s$, look at (c) of **Exercise 4** .

Exercise 8 Consider the first statement. Given $\delta \in (0, \infty)$, define function h_δ as a continuous function that is linear on the intervals $(0, \delta]$, $[\delta, \infty)$ with $h_\delta(0) = 1, h_\delta(\delta) = 0, \lim_t h_\delta(t) = 0$.
Since $g \to \infty$ as $x \to 0$, there is some δ_1 with $0 \le h_{\delta_1} \le g$. One can construct inductively a sequence $\{\delta_n\}_{n\in\mathbb{N}}$ such that partial sums of h_{δ_n} are $\le g$ with $\delta_n \to 0^+$ as $n \to \infty$. Let h be the infinite sum.
For the second statement, consider $g = \log(x+1)$, then $g, g/x$ tends to $\infty, 0$ as $x \to \infty$, but any function convex on $(0, \infty)$ cannot have this property.

Exercise 9 May assume f real and ≥ 0. Given Φ, assume $\Phi(p) \geq 1$ for $p \geq \mathrm{M}$ for some M. We will construct f as an infinite sum of the form $\sum_n n\chi_{\mathrm{E}_n}$ with the E_n's being mutually disjoint measurable subsets. For each n, $n^p/\Phi(p)^p$ is bounded for $p \geq \mathrm{M}$ by some $\mathrm{A}_n > 0$, so by taking E_n with $0 < \mathrm{A}_n m(\mathrm{E}_n) \leq 2^{-n}$ with E_n mutually disjoint, we see for $p \geq \mathrm{M}$ that the following holds :

$$\|f\|_p^p = \sum_n n^p m(\mathrm{E}_n) \leq \Phi(p)^p, \quad \|f\|_\infty = \infty$$

so $\|f\|_p$ can tend to ∞ as slow as possible.

Exercise 10 Notice f is the L^p-limit of the f_n's, so by Theorem 3.12, there is a convergent subsequence converging pointwisely to f a.e., so $f = g$ a.e..

Exercise 11 By Hölder's inequality, $\|f\|_2^2\|g\|_2^2 \geq \|fg\|_1 \geq 1$. Replace f, g by their square roots.

Exercise 12 Since $\varphi : x \mapsto (1+x^2)^{1/2}$ is convex (since $\varphi''(x) = (1+x^2)^{-3/2} > 0$), we get

$$\varphi\left(\int_\Omega h d\mu\right) \leq \int_\Omega (\varphi \circ h) d\mu \leq \int_\Omega (1+h) d\mu$$

which is the desired inequality. Write $\int_\Omega (\varphi \circ h) d\mu$ as M, then $\mathrm{M} = \mathrm{A}+1$ iff $h = 0$ a.e., and $\mathrm{M} = \varphi(\mathrm{A})$ iff $h = \mathrm{A}$ a.e. (since φ is injective on $[0, \infty)$). Now specialize to the case where Ω, μ is $[0,1]$ with Lebesgue measure, $h \geq 0$ a continuous function that is also the derivative of some f, then f is increasing, and A is $f(1) - f(0)$. Let us specialize further to the case where $f(0) = 0, f(1) = 1$ so $\mathrm{A} = 1$. The integral of $\varphi \circ h$ is length of graph of f, 2 is the length of the path $(0,0) \to (1,0) \to (1,1)$, $\sqrt{2} = \varphi(1)$ is the length of the path $(0,0) \to (1,1)$. Hence, length of graph of f is bounded by $\sqrt{2}$, 2.

Exercise 13 Write $f \leftrightarrow g$ if there exists α, β that are not both 0 so that $\alpha f = \beta g$ a.e..
Assume first $p \in (1, \infty)$, it was stated in the book that in this case the equality holds in Hölder's inequality iff $|f|^p \leftrightarrow |g|^q$, and an examination of formula (9),(9') of the proof of Theorem 3.5, 3.9 says that the equality holds in Minkowski's inequality iff $|f|^p \leftrightarrow |f+g|^p$ and $|g|^q \leftrightarrow |f+g|^q$ and that $|f+g| = |f| + |g|$ a.e., which is equivalent to $f \leftrightarrow g$.
For the remaining cases, the proof of Theorem 3.8 says the equality holds in Hölder's inequality iff $|fg| = \|f\|_\infty |g|$ a.e. for $p = \infty$ and $|fg| = |f|\|g\|_\infty$ a.e. for $p = 1$, while the proof of Theorem 3.9 says equality in Minkowski's inequality holds iff $|f+g| = |f| + |g|$ a.e. for $p = 1, \infty$.

Comment on Exercise 14 It seems more natural to use Fubini's theorem to deal with this exercise, which wasn't used in the following solution. A solution using Fubini's theorem will appear again later in **Exercise 14** of Chapter 8 of the book. In the following solution, the answers to (a)-(c) are scattered around the different items, but I feel like it might also be useful to understand (a)-(c) as properties about the functional associated to Hardy's inequality.

Exercise 14 For convenience, we fix a few notations : let $\mathrm{X} = (0, \infty)$, $\Phi : \mathrm{L}^p(\mathrm{X}) \to \mathrm{L}^p(\mathrm{X})$ the function $f \mapsto \mathrm{F}$ although it is not clear yet if Φ does have range in $\mathrm{L}^p(\mathrm{X})$, $q = p/(p-1)$. Notice :

(i) Φ is $\mathbb{C}$-linear, and $0 \leq \Phi(f)$ given that $0 \leq f$.

(ii) $\|\Phi(f)\|_p \leq \|\Phi(g)\|_p$ for $0 \leq f \leq g$ in $\mathrm{L}^p(\mathrm{X})$: this is by (i).
$\|\Phi(f)\|_p \leq \|\Phi(|f|)\|_p$ for $f \in \mathrm{L}^p(\mathrm{X})$: this is by $|\Phi(f)| \leq \Phi(|f|)$.

(iii) $\Phi(f)$ is bounded by $\|f\|_p x^{q^{-1}-1}$: this is by Hölder's inequality

$$|\Phi(f)(x)| \leq \Phi(|f|)(x) \leq x^{-1}\int_0^x |f|dt \leq x^{-1}\left(\int_0^x |f|^p dt\right)^{1/p}\left(\int_0^x 1^q dt\right)^{1/q} \leq x^{q^{-1}-1}\|f\|_p$$

(iv) Φ has image in $\mathrm{C}(\mathrm{X})$: take $x, y \in \mathrm{X}$, write $\Phi(f) = \mathrm{F}$, then

$$|\mathrm{F}(x) - \mathrm{F}(y)| = \left|x^{-1}\int_0^x f dt - y^{-1}\int_0^y f dt\right| \leq |x^{-1} - y^{-1}|\int_0^x |f|dt + y^{-1}\left|\int_y^x |f|dt\right|$$

so as $x \to y$ we have $\mathrm{F}(x) \to \mathrm{F}(y)$.

(v) $\Phi(f_n) \to \Phi(f)$ pointwise if $f_n \to f$ pointwise with $0 \leq f_1 \leq f_2 \leq \dots$: by MCT.
$\Phi(f_n) \to \Phi(f)$ pointwise if $f_n \to f$ pointwise with each $|f_n| \leq g$ where $g \in \mathrm{L}^1(\mathrm{X})$: by DCT.
$\Phi(f_n) \to \Phi(f)$ pointwise if $f_n \to f$ in norm : this is by (iii).

(vi) $\|\Phi\| \leq q$: We first show this is true on $\mathrm{C}_c(\mathrm{X})$ then on $\mathrm{L}^p(\mathrm{X})$. Here we write again $\mathrm{F} = \Phi(f)$.

- For $f \in \mathrm{C}_c(\mathrm{X})$ with $f \geq 0$, integration by parts and the identity $x\mathrm{F}' = f - \mathrm{F}$ gives

$$\int_0^\infty \mathrm{F}^p dx = -p\int_0^\infty x\mathrm{F}'\mathrm{F}^{p-1}dx + \left(x\mathrm{F}^p\ |_0^\infty\right) = -p\int_0^\infty (f - \mathrm{F})\mathrm{F}^{p-1}dx$$

Here $\mathrm{F} = \Phi(f)$, and differentiataion justified by (iv). By Hölder's inequality, we have

$$\|\mathrm{F}\|_p^p = q\|f\mathrm{F}^{p-1}\|_1 \leq q\|f\|_p\|\mathrm{F}^{p-1}\|_q = q\|f\|_p\|\mathrm{F}\|_p^{p-1}, \text{ hence } \|\mathrm{F}\|_p \leq q\|f\|_p$$

- For $f \in \mathrm{C}_c(\mathrm{X})$, we have $\|\Phi(f)\|_p \leq \|\Phi(|f|)\|_p \leq q\|f\|_p$.
- For $f \in \mathrm{L}^p(\mathrm{X})$, take f_n converging to f in norm, then by (iv) and Fatou's lemma, we get

$$\|\Phi(f)\|_p \leq \liminf \|\Phi(f_n)\|_p \leq q \liminf \|f_n\|_p = q\|f\|_p$$

Notice this finishes the proof of (a), and shows that Φ has image in $\mathrm{L}^p(\mathrm{X})$.

(vii) $\|\Phi(f)\|_p = q\|f\|_p$ only when $f = 0$ a.e. : Take such an f and write $\mathrm{F} = \Phi(f)$.

- Suppose $f \in \mathrm{C}_c(\mathrm{X})$ with $f \geq 0$, an examination of the proof in (vi) shows $f^p \leftrightarrow \mathrm{F}^{(p-1)q}$ by **Exercise 13** (where $\leftrightarrow$ is defined in that exercise); that is, $f \leftrightarrow \mathrm{F}$.
May assume $f = \alpha\mathrm{F}$ a.e. for some α. We get from the definition of F that

$$x\mathrm{F}(x) = \int_0^x f(t)dt = \int_0^x \alpha\mathrm{F}(t)dt, \quad x\mathrm{F}' = (\alpha - 1)\mathrm{F}$$

where differentiaion is justified by (iv). Recall the identity in (vii) (true for $f \in \mathrm{C}_c(\mathrm{X})$) :

$$\int_0^\infty \mathrm{F}^p dx = -p\int_0^\infty x\mathrm{F}'\mathrm{F}^{p-1}dx = -(\alpha - 1)p\int_0^\infty \mathrm{F}^p dx$$

then α is determined, so we arrive at the identity $\mathrm{F} + px\mathrm{F}' = 0$, so F takes the form $\mathrm{C}x^{-1/p}$. Since $f \in \mathrm{L}^p(\mathrm{X})$, we must have $f = 0$ a.e.; this completes the case.

- Suppose $f \in \mathrm{C}_c(\mathrm{X})$, then by (ii) and the case just considered applied to $|f|$, $f = 0$ a.e..

- Suppose $f \in L^p(X)$. First, we may assume $f \geq 0$ by considering $|f|$.
  Recall that in our proof of the case $f \in C_c(X), f \geq 0$, the result is from the identity
  $$\int_0^\infty F^p dx = q \int_0^\infty F^{p-1} f dx$$
  deduced by elementary methods. In fact this holds if we only assume $f \in L^p(X)$ and that $f \geq 0$, and if this were established, we are done. We make the following reductions : (Here we abbreviate "pointwise limit of a pointwise increasing sequence" as "$*$")
  - Notice that if f is a $*$ of functions f_n with each f_n satisfying some property (P), then we may assume f_n has property (P); the idea is to use MCT along with (v),(i).
    In the following, (P) is either simple, compactly supported and simple, or $\in C_c(X)$.
  - Firstly, f is a $*$ of simple functions, so may assume f is simple.
  - Next, by σ-compactness of X, f is a $*$ of compactly supported simple functions, so may assume f has compact support.

  Therefore, may assume $0 \leq f \leq M\chi_{(0,L)}$ where $M, L \geq 0$. Lusin's theorem allows us to write f as an a.e. pointwise limit of functions in $C_c(X)$ each dominated by $M\chi_{(0,L)}$. Now notice that $\Phi(f_n)^p, f_n\Phi(f_n)^{p-1}$ are all dominated by some L^1 function by the following :
  $$\Phi(f_n)^p \leq M^p\chi_{(0,L)} + M^pL^p\chi_{(0,L)^c} \cdot x^{-p}, \quad f_n\Phi(f_n)^{p-1} \leq M^p\chi_{(0,L)}$$
  Finally, we have
  $$\begin{aligned}\int_0^\infty \Phi(f)^p dx = &\int_0^\infty \lim_n \Phi(f_n)^p dx = \lim_n \int_0^\infty \Phi(f_n)^p dx \\ =&q \lim_n \int_0^\infty \Phi(f_n)^{p-1} f_n dx = q \int_0^\infty \lim_n \Phi(f_n)^{p-1} f_n dx = q \int_0^\infty \Phi(f)^{p-1} f dx\end{aligned}$$
  the 1st and 5th equality by (v), the 2nd and 4th by DCT, the 3rd by the case $f \in C_c(X)$.

This completes the proof of (b).

(viii) $\|\Phi\| = q$: The hint suggests considering $\chi_{[1,A]}x^{-1/p}$, call this f_A. We have
$$\|f_A\|_p^p = \int_1^A x^{-1} dx = \ln(A)$$
On the other hand, if we write $F_A = \Phi(f_A)$, we get
$$F_A(x) = x^{-1} \int_1^x x^{-1/p} dx = q(x^{-1/p} - x^{-1}), \text{ for } x \in [1, A)$$
so we have
$$\|F_A\|_p^p \geq \int_1^A F_A^p dx = q^p \left(\int_1^A (x^{-1/p} - x^{-1})^p dx \right)$$
so we get from L'Hôspital's rule that
$$\frac{\|F_A\|_p^p}{\|f_A\|_p^p} \geq \frac{q^p \int_1^A (x^{-1/p} - x^{-1})^p dx}{\ln(A)} \to q^p, \text{ as } A \to \infty$$
hence the assertion, which is (c).

So up to this point, (a)-(c) have been established, and only (d) remains. One may test examples like $f(x) = \min(1, x^{-n})$. In general, if $f \in L^1(X)$ and $f > 0$, then by rescaling may assume $\int_0^1 f dx = 1$, hence $\Phi(f)(x) \geq x^{-1}$ for $x > 1$, so $\Phi(f) \notin L^1(X)$.

Exercise 15 Given these positive a_n, define $a(x) = \sum_n a_n \chi_{(n-1,n]}$, then the p-norm of a is the sum of the p-th powers. The corresponding $\mathrm{A}(x)$ defined in the sense of **Exercise 14** is given by

$$\mathrm{A}(x) = x^{-1}\int_0^x a(t)dt = x^{-1}\left((\sum_{n=1}^{[x]} a_n) + (x-[x])a_{n+1} \right)$$

Suppose first that a_n is decreasing, then $\mathrm{A}(x)$ is a decreasing function, so in this case we have

$$\sum_{\mathrm{N}=1}^{\infty}\left(\frac{1}{\mathrm{N}}\sum_{n=1}^{\mathrm{N}} a_n\right)^p = \int_0^\infty \mathrm{A}([x]+1)^p dx \leq \int_0^\infty \mathrm{A}(x)^p dx = \|\mathrm{A}\|_p^p \leq \left(\frac{p}{p-1}\right)^p \|a\|_p^p = \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p$$

On the other hand, suppose a_n not decreasing and $\|a\|_p < \infty$, then one can rearrange a_n into b_n such that b_n is decreasing (for in this case, over each subset S of $\mathbb{N}$, a_n obtains its supremum at some $n \in \mathrm{S}$ since $\|a\|_p \neq \infty$), but this time we have $\mathrm{A} \geq \mathrm{B}$ with $\|a\|_p = \|b\|_p$ (absolute convergence).

Exercise 16 Take $\varepsilon > 0$. Let us do the general case. Let $\mathrm{I}(n,k) = \mathbb{Q} \cap (n,\infty)$; this is a countable dense subset of (n,∞). Define $\mathrm{S}(n,k) = \bigcap_{i,j\in \mathrm{I}(n,k)}\{x : |f_i(x) - f_j(x)| < k^{-1}\}$; the result can be related to that of taking intersection over all $i, j > n$ by (ii). Now condition (i) gives $\bigcup_n \mathrm{S}(n,k) = \mathrm{X}$ for each k, so may find n_k with $\mu(\mathrm{X} \setminus \mathrm{S}(n_k,k)) < 2^{-k}\varepsilon$. Take $\mathrm{E} = \bigcap_k \mathrm{S}(n_k,k)$, then $\mu(\mathrm{X}\setminus\mathrm{E}) \leq \sum_k \mu(\mathrm{X}\setminus \mathrm{S}(n_k,k)) < \varepsilon$. For a σ-finite counterexample, consider $f_n = \chi_{[n-1,n]}$ on $\mathbb{R}^1$.

Exercise 17 For (a), by $|\alpha - \beta| \leq |\alpha| + |\beta|$ may let $\alpha, -\beta \in (0,\infty) \subset \mathbb{R}$. Homogenize, only need :

For each $p \in (0,\infty), t \in [1,\infty)$, we have $(1+t)^p \leq \gamma_p(1+t^p)$.

For $p \leq 1$, this is by concavity. For $p > 1$, write $p = n + q$ where $n = \lceil p \rceil - 1 < p$. "By induction" :

$$(1+t)^p = (1+t)^n(1+t)^q \leq 2^{n-1}(1+t^n)(1+t)^q$$

so we only need $f(t) := 2^q(1+t^p) - (1+t^n)(1+t)^q \geq 0$ for $t \geq 1$, which is by

$$f(1) = 0, \quad f'(t) = p2^q t^{p-1} - nt^{n-1}(1+t)^q - q(1+t^n)(1+t)^{q-1} \geq 0 \text{ for } t \geq 1$$

For (b), let us first consider part (i). The set Y where $f_n \to f$ pointwise is measurable with $\mu(\mathrm{X}\setminus\mathrm{Y}) = 0$. Let λ be the measure defined by $d\lambda = |f|^p d\mu$, then $\lambda(\mathrm{X}) < \infty$. Apply Egoroff's theorem to Y with respect to λ, we see that there exists some $\mathrm{E} \subset \mathrm{Y}$ with

$$\lambda(\mathrm{Y}\setminus\mathrm{E}) = \int_{\mathrm{Y}\setminus\mathrm{E}} |f|^p d\mu < \varepsilon, \quad f_n \to f \text{ pointwisely on E}$$

Now we want to modify E so that $\mu(\mathrm{E}) < \infty$. This can be forced by the following fact :

Let $(\mathrm{X}, \mathfrak{m}, \mu)$ be a measurable space, $g : \mathrm{X} \to [0,\infty)$ a measurable function. Suppose $\int_\mathrm{X} g d\mu < \infty$, then for any given $\varepsilon > 0$, there exists a subset $\mathrm{Y} \subset \mathrm{X}$ with $\mu(\mathrm{Y}) < \infty$ and $\int_\mathrm{X} g d\mu \leq \int_\mathrm{Y} g d\mu + \varepsilon$.

The idea is to note that $\mathrm{X}_n := g^{-1}(n^{-1},\infty)$ are of finite measure, and that $\chi_{\mathrm{X}_n} f \to f$ pointwise and monotonely on X, and conclude by MCT. In our situation, may take $(g,\mathrm{X}) = (|f|^p, \mathrm{E})$, we obtain some subset $\mathrm{B} \subset \mathrm{E}$ with $\mu(\mathrm{B}) < \infty, \lambda(\mathrm{E}\setminus\mathrm{B}) < \varepsilon$, so taking $\mathrm{A} = \mathrm{X}\setminus\mathrm{B}$, we immediately have :

$$\int_\mathrm{A} |f|^p d\mu = \lambda(\mathrm{A}) = \lambda(\mathrm{X}\setminus\mathrm{Y}) + \lambda(\mathrm{Y}\setminus\mathrm{E}) + \lambda(\mathrm{E}\setminus\mathrm{B}) < 0 + \varepsilon + \varepsilon = 2\varepsilon$$

$$\mu(\mathrm{B}) < \infty, \quad f_n \to f \text{ pointwisely on B}$$

So far we have shown the decomposition $X = A \cup B$ given in the hint of the question exists. This being done, let us show $\lim\|f - f_n\|_p = 0$. Notice that by (a), we have the estimate :

$$\int_X |f - f_n|^p d\mu \le \int_B |f - f_n|^p d\mu + \gamma_p \int_A |f|^p d\mu + \gamma_p \int_A |f_n|^p d\mu$$

The first term tends to 0 by uniform convergence and $\mu(B) < \infty$; the second term is bounded by $\gamma_p\varepsilon$. Now we estimate the integral $\int_A |f_n|^p d\mu$. Notice that we have by Fatou's lemma that

$$\limsup \int_A |f_n|^p d\mu = \limsup \left(\|f_n\|_p^p - \int_B |f_n|^p d\mu\right) \le \|f\|_p - \int_B |f|^p d\mu = \int_A |f|^p d\mu \le \varepsilon$$

Therefore, we have $\limsup \int_X |f - f_n|^p d\mu \le 2\gamma_p\varepsilon$ for any $\varepsilon > 0$, which completes (i).
For (ii), we take $h_n := \gamma_p(|f|^p + |f_n|^p) - |f - f_n|^p$. Since $\liminf h_n = 2\gamma_p|f|^p$ a.e., we have

$$\begin{aligned} 2\gamma_p\|f\|_p^p &\le \liminf \int_X h_n d\mu = \liminf \left(\gamma_p(\|f\|_p^p + \|f_n\|_p^p) - \|f - f_n\|_p^p\right) \\ &= 2\gamma_p\|f\|_p^p - \limsup\|f - f_n\|_p^p \end{aligned}$$

by Fatou's lemma, which shows $\lim\|f - f_n\|_p = 0$, and this concludes part (ii).
For (c), take $X = (0, 1)$, $f_n = n^{(p+1)/p}\chi_{(0,n^{-1})}$, then $f_n \to 0$ a.e. pointwise with $\|f_n\|_p = n \to \infty \ne 0$.

Suggested Edit on Exercise 18 Another way to state convergence in measure is :

For each $\delta, \varepsilon > 0$ there exists N so that $\mu\{x : |f_n(x) - f(x)| > \varepsilon\} < \delta$ for all $n \ge N$.

Exercise 18 For (a), take ε, δ, use Egoroff theorem to cut away a set of measure $< \varepsilon/2$, and use uniform convergence on the remaining set to conclude. For (b), suppose f_n doesn't converge in measure to f, then may take some ε, δ violating the condition of convergence in measure, so $\|f_n - f\|_p \ge \varepsilon^p\delta$ for infinitely many n, hence f_n does not converge in L^p-norm. For (c), take n_k with

$$A_k := \mu\{x : |f_{n_k}(x) - f(x)| > k^{-1}\} < 2^{-k}, \quad 0 < n_1 < n_2 < \dots$$

For $\{f_{n_k}\}_k$, the set of pointwise convergence contains $\bigcup_l \bigcap_{k\ge l} A_k^c$, with measure 0 complement.
Converses of both (a),(b) are false. Let $X = [0, 1]$, for $n > k > 0$ define $f_{nk} = n\chi_{[i/n,(i+1)/n]}$, consider f_{nk} in lexicographic order, then $f_{nk} \to 0$ in measure but not in norm and not a.e. pointwise.
Suppose $\mu(X) = \infty$. By our proof above, (b),(c) remains true except (a) by taking $X = \mathbb{R}$, $f_n = \chi_{[n-1,n]}$.

Comment on the definition of "averages" in Exercise 19 In this exercise, the author defined averages as in the form given in Theorem 1.40. In Theorem 1.40, the extra condition $\mu(X) < \infty$ is assumed. In the case $\mu(X) = \infty$, the averages of f is only defined over measurable sets with positive, finite measures. Therefore, in particular, we will define the set of averages A_f as the set

$$A_f = \{\frac{1}{\mu(E)} \int_E f d\mu : E \in \mathfrak{m}, 0 < \mu(E) < \infty\}$$

instead of the one without the condition $\mu(E) < \infty$ in that exercise. With essentially the same proof, Theorem 1.40 extends to the case where X doesn't have finite measure and $f \in L^\infty(\mu)$ or $L^1(\mu)$.
Notice that related to the convexity of A_f, we also refer - for the interested readers - to a theorem by Lyapunov (or Liapunoff) which gives some sufficient conditions on when a "vector measure" has convex image. This material can be found in Theorem 5.5 of Rudin's book *Functional Analysis.*

Exercise 19 Given $x \in \mathbb{C}$ and $r > 0$, we write $B_r(x)$ as the open set consisting of points with distance $< r$ from x. The following holds by the definition of $|\cdot\|_\infty$:

$$R_f \subseteq \overline{B_{\|f\|_\infty}(0)}$$

where the first inclusion is by Theorem 1.40, so to show compactness of R_f, only need R_f^c open :
If $\mu(f^{-1}B_{2\varepsilon}(x)) = 0$ then for any $y \in B_\varepsilon(x)$, we have $B_\varepsilon(y) \subset B_{2\varepsilon}(x)$, so $\mu(f^{-1}B_\varepsilon(y)) = 0$.
Let us consider statements involving A_f. Theorem 1.40 gives

$$\overline{A_f}^c \subseteq R_f^c$$

It is easy to see that A_f needn't be closed, for example, take $X = (0, 1)$ with Lebesgue measure and $f = \log 1 + x$, then its set of averages is not closed. A trivial example where A_f is convex for every $f \in L^\infty(\mu)$ can be taken by taking X to be a singleton; non-examples can be taken by considering unit masses. The considerations above are not altered if we consider $L^1(\mu)$ instead of $L^\infty(\mu)$.

Exercise 20 Try simple functions.

Exercise 21 Let $(Y, d), (Y', d')$be two completions. The inclusion $X \to Y'$ can be extended to a well-defined isometry $i : Y \to Y'$ by the fact that X is dense in Y and that Y′ is complete. Similarly one gets $j : Y' \to Y$. Now iji agrees with i when restricted to X, so $iji = i$, $ji = 1_Y$; similarly for ij.

Exercise 22 Let $\{x_k\}$ be Cauchy with $\{x_{n_k}\} \to L$, think of $|x_n - L| \leq |x_n - x_{n_k}| + |x_{n_k} - L|$.

Exercise 23 May assume f is real, ≥ 0, with $\|f\|_\infty = 1$, then $\alpha_{n+1}/\alpha_n \leq 1$. Recall by (e) of **Exercise 4** that $\lim_{p\to\infty}\|f\|_p = 1$. We can relate α_n, α_{n+1} by Hölder's inequality :

$$\alpha_n = \|f^n\|_1 \leq \|f^n\|_{(n+1)/n}\|1\|_{n+1} = \alpha_{n+1}^{n/(n+1)}\mu(X)^{1/(n+1)}$$

hence $\alpha_{n+1}/\alpha_n \geq \mu(X)^{-1/(n+1)}\|f\|_{n+1} \to 1$, so $\alpha_{n+1}/\alpha_n \to 1$.

Exercise 24 Notice for $p \in (0, \infty)$, the function $x \mapsto x^p$ is convex for $p \geq 1$, concave for $p \leq 1$. Now for $x, y \in (0, \infty)$, we get by concavity, intermediate value theorem that for $x, y \in (0, \infty)$:

$$|x^p - y^p| \leq |x - y|^p, \text{ for } p \leq 1; \quad |x^p - y^p| \leq p|x - y|(|x|^{p-1} + |y|^{p-1}), \text{ for } p \geq 1$$

Now the first inequality clearly gives (a). For (b), we have by Hölder's inequality that

$$\int \left||f|^p - |g|^p\right| d\mu \leq p \int |f - g| \left||f|^{p-1} + |g|^{p-1}\right| d\mu \leq p\|f - g\|_p \||f|^{p-1} + |g|^{p-1}\|_q$$

where q the conjugate exponent of p. By Minkowski's inequality, we have

$$\||f|^{p-1} + |g|^{p-1}\|_q \leq \|f^{p-1}\|_q + \|g^{p-1}\|_q = \|f\|_p^{p-1} + \|g\|_p^{p-1} \leq 2R^{p-1}$$

thus giving the estimation in (b).

Exercise 25 Let λ be the measure defined by $d\lambda = fd\mu$, so $\lambda(X) = 1$. Define $g : x \mapsto f(x)^{-1}$.
For the first inequality, notice that since $\varphi : t \mapsto -t \log t$ is concave, we have

$$\int_E (\log f)d\mu = \int_E (\varphi \circ g)d\lambda \leq \varphi(\int_E gd\lambda) = \mu(E) \log \frac{1}{\mu(E)}$$

For the second inequality, notice that since $\varphi : t \mapsto t^{1-p}$ is concave, we have

$$\int_E f^p d\mu = \int_E (\varphi \circ g)d\lambda \leq \varphi(\int_E gd\lambda) = \mu(E)^{1-p}$$

(Note that for the inequalities to make sense, must assume $0 < \mu(E) < \infty$.)

Exercise 26 Notice $x \mapsto x \log x, \log x$ are convex on $(0, \infty)$. Now use Jensen's inequality twice.

4 Elementary Hilbert Space Theory

Exercise 1 Clearly $M \subseteq (M^\perp)^\perp$. By closedness of $(M^\perp)^\perp$, $\overline{M} \subseteq (M^\perp)^\perp$. Conversely, if $x \notin \overline{M}$, decompose x with respect to the orthogonal projection $H \to \overline{M}$ to see $x \notin (\overline{M}^\perp)^\perp \supseteq (M^\perp)^\perp$.

Exercise 2 The first part is just a review of Gram-Schmidt's process in linear algebra; to check, note that $x_{n+1} - v_{n+1} \neq 0$ is the orthogonal projection of x_{n+1} to $X_n = U_n$, so $u_1, \ldots, u_{n+1}$ are mutually orthogonal, with $\|u_{n+1}\| = 1$ and $X_{n+1} = U_{n+1}$ (here X_n is $\mathbb{C}$-span of $x_1, \ldots, x_n$; similarly for U_n).
For the second part, if we were given such a dense subset $\{x_n\}_n$, may choose linearly independent subset $\{x_{n_k}\}_k$ with the same span and perform Gram-Schmidt.

Exercise 3 For any $p \in [1, \infty)$, since $C(T)$ is dense in $L^p(T)$ and that the trigonometric polynomials is a countable orthonormal system of $C(T)$, we conclude by **Exercise 4**. As for $p = \infty$, there is an uncountable subset X of $L^\infty(T)$ such that any two distinct points in X has $L^\infty(T)$-distance ≥ 1; to see this, take a sequence (a_n) of strictly increasing numbers between $(0, 1)$ with $a_1 = 0, \lim a_n = 1$, and for any $f : \mathbb{N} \to \mathbb{N}$, define $x_f : [0, 1) \to [0, \infty)$ by $x_f = \sum f(n)\chi_{[a_n, a_{n+1})}$, then take $X := \{x_f\}$.

Exercise 4 Note that $\mathbb{C}$ is separable, so existence of a countable maximal orthonormal basis implies separability. Conversely, given a countable dense subset $\{x_n\}$, use **Exercise 2**.

Exercise 5 Let $L =< \cdot, y >$, then $M = (\mathbb{C}y)^\perp$, $M^\perp = ((\mathbb{C}y)^\perp)^\perp = \mathbb{C}y$ by closedness of $\mathbb{C}y$.

Comment on Exercise 6 The set S is a subspace equipped with ℓ^2 metric instead of the product topology. We will also sometimes think of points in it as functions on $\mathbb{N}$; in view of completeness of H, this makes sense when the function is in ℓ^2. Also in the part where the question wants us to show that H is not locally compact, it should be noted that it is assumed H isn't finite dimensional.

Exercise 6 Write $U = \{u_n\}$. Two points in U either have distance 0 or $\sqrt{2}$, so U is closed in X and discrete in its subspace topology (hence non-compact). Also, U is bounded by 1.
Let us consider the statements involving S. Let S' be the space with the same underlying set but equipped with the product topology; the topology on S' is finer than S.
If $\sum \delta_n^2 < \infty$, then any open set U of S is a union of open sets in S', so $S = S'$, hence compact by Tychonoff's theorem. If one wants to avoid Tychonoff's theorem, here is another method. We show sequential compactness by a diagonal argument, which would suffice since S is a metric space. Take a sequence $x_n : \mathbb{N} \to \mathbb{C}$ in S; for convenience of our later use, call this $x_{0,n}$. Choose subsequence $x_{1,n}$ of $x_{0,n}$ with $x_{1,n}(1)$ converging to some $x(1)$ with $|x_{1,n}(1) - x(1)|^2 \leq (1n)^{-1}$ for each n, and repeat this with $0, 1$ replaced by $m, m+1$ for each m. Then $x_{n,n} \to x$ as $n \to \infty$ by :

$$\|x_{n,n} - x\|^2 = \sum_m \left|x_{n,n}(m) - x(m)\right|^2 \leq n^{-2}\left(\sum_{m \leq n} m^{-2}\right) + \left(\sum_{m \geq n} \delta_m^2\right) \to 0$$

where $(x_{n,n}(m) - x(m))^2 \leq (mn)^{-1}$ for $m \leq n$ is by the subsequence construction and assumption.
Now suppose $\sum \delta_n^2 = \infty$, then S contains an infinite subset X such that each pair of distinct points in X have distance ≥ 1 with each other. Cover S with balls of diameters < 1 shows non-compactness.
Now we prove that H is not locally compact. By taking closure, it suffices to treat the case where the orthonormal set $U := \{u_n\}$ is maximal. Notice for $\varepsilon > 0$, two points in the closed set εU either have distance 0 or $\sqrt{2}\varepsilon$, so no compact set contains εU, while εU forms a neighborhood basis of 0.

Exercise 7 A way is to force $c_k^2 \sum_{n\in \mathrm{E}_k} a_n = k^{-2}$ and use $\sum_k k^{-1} = \infty, \sum_k k^{-2} < \infty$.

Exercise 8 Note that H_i are isomorphic to some $\ell^2(\mathrm{A}_i)$ where A_i an index set of some maximal orthonormal basis of H_i, so we reduce to the case of ℓ^2 spaces. The rest is omitted.

Exercise 9 For any measurable subset A of $[-\pi, \pi]$, we have $\chi_\mathrm{A} \in \mathrm{L}^2([-\pi, \pi])$, so for the Fourier coefficients $(c_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, $c_n \to 0$ as $|n| \to \infty$. Details are left to the readers.

Exercise 10 Define $f : \mathrm{E} \to [0, \infty)$ by $f(x) = \lim_k \sin^2(n_k x)$. Define subsets

$$\mathrm{E}^+ = (f^2)^{-1}[1/2, +\infty),\ \mathrm{E}^- = (f^2)^{-1}[0, 1/2]; \quad \mathrm{F}^+ = f^{-1}(1/\sqrt{2}),\ \mathrm{F}^- = f^{-1}(-1/\sqrt{2}).$$

so $(f^2)^{-1}(1/2) = \mathrm{F}^+ \cup \mathrm{F}^- = \mathrm{E}^+ \cap \mathrm{E}^- \subset \mathrm{E}^+ \cup \mathrm{E}^- = \mathrm{E}$. By **Exercise 9** and DCT, we get

$$0 \leq \int_{\mathrm{E}^+} (2f^2(x) - 1)dx = \lim_k \int_{\mathrm{E}^+} (2\sin^2(n_k x) - 1)dx = -\lim_k \int_{\mathrm{E}^+} \cos(2n_k x)dx = 0$$

so $\mu(\mathrm{E}^+ \setminus \mathrm{F}^+ \cup \mathrm{F}^-) = 0$, similarly for E^-, but again by **Exercise 9** and DCT, $\mu(\mathrm{F}^+) = \mu(\mathrm{F}^-) = 0$.

Exercise 11 It suffices to consider $\ell^2(\mathbb{Z})$. Take $x_n : \mathbb{Z} \to \mathbb{C}$ by $x_n(m) = n^{-1}\delta_{nm}$.

Exercise 12 Recall that c_k is defined by

$$c_k = \left(\int_{-\pi}^{\pi} \left(\frac{1+\cos(t)}{2}\right)^k dt\right)^{-1}$$

Note that by Euler's identity, if we take $x = \exp(it/2)$, then we have

$$c_k^{-1}\sqrt{k} = 4^{-k}\sqrt{k}\int_{-\pi}^{\pi} (x^{-1} + x)^{2k} dt = 4^{-k}\sqrt{k}\binom{2k}{k} \to \frac{1}{\sqrt{\pi}} \in (0, \infty)$$

by the fact that the integral of x^{2n} is 0 if n is an integer that is not 0 along with Stirling's formula.

Exercise 13 It suffices to show the case $f(t) = \exp(2\pi ikt)$, since functions in $\mathrm{C}(\mathrm{T})$ can be approximated uniformly by trigonometric polynomials. This is true for $k = 0$. It also suffices to show the case $k = 1$ by taking integral multiples of α, but then we are done by

$$0 \leq \lim_\mathrm{N} \left|\frac{1}{\mathrm{N}}\sum_{n=1}^{\mathrm{N}} f(n\alpha)\right| = \lim_{\mathrm{N}\to\infty} \left|\frac{1}{\mathrm{N}}\sum_{n=1}^{\mathrm{N}} \exp(2\pi i\alpha n)\right| \leq 2|1 - \exp(2\pi i\alpha)|^{-1} \lim_\mathrm{N} \mathrm{N}^{-1} = 0$$

Exercise 14 Given a, b, c, we have

$$\int_{-1}^{1} (x^3 - cx^2 - bx - a)^2 dx = \frac{2}{7} + \frac{2}{15}\left((5b^2 - 6b) + (15a^2 + 10ac + 3c^2)\right)$$

so the minimum value is $\frac{2}{7} - \frac{2}{15}\frac{9}{5} = \frac{8}{175}$, where the term $\frac{9}{5}$ comes from completing squares in b.
For the other part, let $\mathrm{H} = \mathrm{L}^2[-1, 1]$, M the span of $1, x, x^2, x^3$, then M is closed in H. Let $\mathrm{P} : \mathrm{H} \to \mathrm{M}$ be the orthogonal projection and Q be that of the complement. The question is then

Under conditions $(g, 1) = (g, x) = (g, x^2) = 0, (g, g) \leq 1, g \in \mathrm{M}$, maximize (g, x^3).

Suppose $g = \alpha(x^3 - cx^2 - bx - a)$ with $\alpha \neq 0$, then $a = c = 0$ by $(g, 1) = (g, x^2) = 0$, and that $b = \frac{3}{5}$ by $(g, x) = 0$, so the conditions gives a 1-dimensional space. Note that $(g, g) = \frac{4}{175}\alpha^2$, $(g, x^3) = \frac{4}{175}\alpha$, so the constraint $(g, g) \leq 1$ determines the maximal value of (g, x^3).

Exercise 15 Note that $\int_0^\infty x^n e^{-x} = n!$. Therefore we have

$$\int_0^\infty (x^3 - cx^2 - bx - a)^2 e^{-x} dx = a^2 + 2b^2 + 24c^2 + 2ab + 4ac + 12bc - 12a - 48b - 240c + 720$$

By calculus methods (such as WolframAlpha), the minimum value is 36 at $(a, b, c) = (6, -18, 9)$. For the maximum problem, note that we are dealing with $x^k e^{-x/2}$ for $k = 0, 1, 2, 3$.
As a responsible author, these tedious calculations are left to the potentially interested readers.

Exercise 16 Let P be the orthogonal projection $\mathrm{H} \to \mathrm{M}$, Q that of complement, then

$$\min\{\|x_0 - x\|^2 : x \in \mathrm{M}\} = \min\{\|\mathrm{P}x_0 - \mathrm{P}x\|^2 + \|\mathrm{Q}x_0\|^2 : x \in \mathrm{M}\} = \|\mathrm{Q}x_0\|^2$$

here we use the fact that $\|x\|^2 = \|\mathrm{P}x\|^2 + \|\mathrm{Q}x\|^2$, while

$$\max\{(x_0, y)^2 : y \in \mathrm{M}^\perp\} = \max\{(\mathrm{Q}x_0, y)^2 : y \in \mathrm{M}^\perp\} \leq \|\mathrm{Q}x_0\|^2$$

but maximum is achieved when y is parallel to $\mathrm{Q}x_0$, so the inequality is an equality.

Comment on Exercise 17 In this exercise, H needs to be infinite dimensional for this to be true.

Exercise 17 Under the assumption that H is infinite dimensional, it is isomorphic to some $\ell^2(\mathrm{A})$, so contains a subspace isomorphic to $\ell^2(\mathbb{Z})$ or $\mathrm{L}^2([0,1])$. Take $[0,1] \to \mathrm{L}^2([0,1])$ by $t \to \chi_{[0,t)}$.

Exercise 18 Write $\mathrm{U} = \{u_s\}$ so $\mathrm{X} = \mathbb{C}\mathrm{U}$. To see the limit exists, note first that the formula is bilinear, so only need to check when $f, g \in \mathrm{U}$. Take $(f, g) = (u_s, u_t)$, then $(f, g) = (u_{s-t}, 1)$, so may assume $t = 0$. If $s = 0$, clearly $(1, 1) = 1$. If $s \neq 0$, we have for each $\mathrm{A} > 0$ that

$$\left|\frac{1}{2\mathrm{A}} \int_{-\mathrm{A}}^{\mathrm{A}} u_s dx\right| = \frac{1}{2\mathrm{CA}} \left|\int_{-\mathrm{CA}}^{\mathrm{CA}} u_\pi dx\right| \leq \frac{1}{2\mathrm{CA}} \int_{[\mathrm{CA}] \leq |x| \leq \mathrm{CA}} 1 dx \leq \frac{1}{\mathrm{CA}} \to 0$$

where $\mathrm{C} = \pi^{-1}s$, so $(u_s, 1) = 0$; this shows that the the inner product is well-defined and furnishes X as a unitary space. To see that its completion H is nonseparable, note that X itself does not have a countable basis. To see that H is also unitary, note that there exists a unique solution to the diagram

$$\begin{tikzcd} \mathrm{X} \times \mathrm{X} \arrow[rr, "{(-,-)}"] \arrow[dr, hook] & & \mathbb{C} \\ & \mathrm{H} \times \mathrm{H} \arrow[ur, dashed, "{(-,-)}"'] & \end{tikzcd}$$

by using the fact that $\mathrm{X} \times \mathrm{X}$ is dense in $\mathrm{H} \times \mathrm{H}$ and completeness of $\mathbb{C}$. We omit the routine verification that the resulting function on $\mathrm{H} \times \mathrm{H}$ furnishes H as a unitary space. The fact that U is an maximal orthonormal subset is just by the fact that $\mathbb{C}\mathrm{U} = \mathrm{X}$ is dense in H.

Exercise 19 All of these identities are routine calculations. For the first, use

$$\sum_{n=0}^{\mathrm{N}-1} (\omega^k)^n = \begin{cases} \mathrm{N}, & \text{if } \omega^k = 1 \\ (\omega^{k\mathrm{N}} - 1)(\omega^k - 1)^{-1}, & \text{otherwise} \end{cases}$$

For the second and third, start from $\|u + v\|^2 = (u + v, u + v)^2$ and see where it goes.

5 Examples of Banach Space Techniques

Notation Here we use k to denote either $\mathbb{R}$ or $\mathbb{C}$.

Exercise 1 Graph the equation $Ax^p + By^p = 2$. Due to bad drawing skills and lack of artistic talent of the author (with commutative diagrams being an exception), we omit the pictures here.

Exercise 2 By triangle inequality.

Exercise 3 For $p = 1, \infty$, take $L^p(\mu) = L^p([-1, 1])$, $E = [-\frac{2}{3}, \frac{1}{3}]$, $f = \chi_E$, $g = \chi_{1-E}$.
For other p, we get by **Exercise 13** of chapter 3 that $\|h\|_p = 1$ iff there exists $\alpha, \beta \in \mathbb{R}^1$ that are not both zero with $\alpha f = \beta g$ a.e., so the condition $\|f\|_p = \|g\|_p = 1$ gives $f = g$ a.e., or, "=" in $L^p(\mu)$.

Exercise 4 May define $\Lambda : C \to k$ so that $M = \Lambda^{-1}(1)$, then Λ is k-linear, continuous, and M is closed and convex. Under the norm function $\|\cdot\|_\infty : C \to [0, \infty)$, M has image in $(2, \infty)$. In fact, the infimum of the image of M is 2. A way to show this is this : fix first a small $\delta > 0$, let $x_0 = 0, x_2 = 1/2, x_3 = 1$, by intermediate value theorem, there is $x_1 \in (x_0, x_2)$ so that the function f defined to be linear on each $[x_i, x_{i+1}]$ with $f(x_0) = f(x_1) = 2 + \delta, f(x_2) = f(x_3) = 0$ lies in M.

Exercise 5 The verification that M is closed and convex can be shown as in **Exercise 4** . Under the norm function $\|\cdot\|_1 : L^1([0, 1]) \to [0, \infty)$, M has image in $[1, \infty)$, and for any $x \in (0, 1)$, take f to be the function linear on $[0, x]$, $[x, 1]$ with $f(0) = f(1) = 0, f(x) = 2$, then $f \in M$ with $\|f\|_1 = 1$.

Exercise 6 For existence, use the diagram below to construct F :

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, hook] & k \\
\overline{M} \arrow[ur, dashed, "\overline{f}"'] & H \arrow[l, two heads] \arrow[u, dashed, "F"']
\end{tikzcd}$$

We know $\|\overline{f}\| = \|f\|$ since the inner product $H \times H \to k$ is continuous, while $\|F\| = \|\overline{f}\|$ is easy.
For uniqueness, note that $\|F\| = \|f\|$ forces F to be 0 when restricted to $\overline{M}^\perp$, while M is dense in $\overline{M}$.

Exercise 7 Take μ to be the counting measure on a set of two points $X = \{a, b\}$, and let M be the subspace of functions that is 0 on a, then the inclusion $i : M \to L^1(\mu)$ has a continuous retraction r. Consider the following commutative diagram :

$$\begin{tikzcd}
M \arrow[rr, "\Lambda i"] \arrow[dr, shift left, "i"] & & \mathbb{C} \\
& L^1(\mu) \arrow[ul, shift left, "r"] \arrow[ur, "\Lambda"'] &
\end{tikzcd}$$

where Λ the functional defined by integration over X, then subject to Λi, both $\Lambda, \Lambda ir$ are norm preserving extensions of Λi to the space $L^1(\mu)$ (of norm 1).

Exercise 8 Let S be set of vectors in X of norm 1. For (a), suppose given a Cauchy sequence $f_n : X \to k$ in X^*, define f so that its restriction to S is the pointwise limit of f_n (exists by completeness of k) and extend over X, then f is the limit of f_n under the norm in X^*. To see this, suppose

For all $\varepsilon > 0$, there exists $N > 0$ so that $\sup_{x \in S}\{|f_n(x) - f_m(x)|\} < \varepsilon$ for all $n, m > N$.

then for all $n > \mathrm{N}$ and $x \in \mathrm{S}$, we have $|f_n(x) - f(x)| = \lim_m |f_n(x) - f_m(x)| \leq \varepsilon$, hence $\|f - f_n\| \leq \varepsilon$. On the other hand, we also get $\|f\| \leq \|f - f_n\| + \|f_n\| < \infty$, which shows f is bounded; linearity of f is straightfoward. X^* is clearly a normed, $\mathbb{C}$-vector space, so X^* is a Banach space on its own.
For (b), write $x^* : \mathrm{X}^{**} \to \mathrm{X}$ as the functional $f \mapsto f(x)$. The norm of x^* is then

$$\frac{\|x^*\|}{\|x\|} = \sup_{f \in \mathrm{X}^*} \frac{\|x^*(f)\|}{\|f\|\|x\|} = \sup_{f \in \mathrm{X}^*} \frac{f(\|x\|^{-1}x)}{\|f\|}$$

By the Banach-Hahn extension theorem, RHS ≥ 1, and by the definition of norm on X^*, RHS ≤ 1.
For (c), consider the contrapositive statement and use $\|f(x)\| \leq \|f\|\|x\|$.

Comment on Exercise 9 Recall the strict inclusions $\ell^1 \subset c_0 \subset \ell^\infty$.

Exercise 9 For (a), given $\Lambda : c_0 \to \mathbb{C}$ defined by some $\eta \in \ell^1$ and for $\xi \in c_0$, we have

$$|\Lambda\xi| = \left|\sum \eta_i \xi_i\right| \leq \sum |\eta_i \xi_i| \leq \|\eta\|_1 \|\xi\|_\infty$$

which shows $\|\Lambda\| \leq \|\eta\|_1$. To see $\|\Lambda\| = \|\eta\|_1$, take ξ with $\xi_i \eta_i = |\eta_i|$ with $\|\xi\|_\infty = 1$. On the other hand, any $\Lambda \in c_0^*$ is determined by its values on $\delta^i \in c_0$ defined by $\delta^i_j = \delta_{ij}$; let c_f be the subspace spanned by these δ^i, then we have determined $\Lambda|c_f$, but c_f is dense in c_0, so we are done with (a).
For (b), take $(\ell^1, c_0, \|\cdot\|_1, \|\cdot\|_\infty) \to (\ell^\infty, \ell^1, \|\cdot\|_\infty, \|\cdot\|_1)$, with $\|\Lambda\| = \|\eta\|_\infty$ by testing different $\xi = \delta^i$.
For first part of (c), modify (a) by $c_0 \to \ell^\infty$. For the second part, note first that c_f is not dense in ℓ^∞.
Take $\nu \in \ell^\infty \setminus c_0$ with $\lim \nu_i = 1$, let M span of ν and c_0 in ℓ^∞, define $\Lambda : \mathrm{M} \to \mathbb{C}$ by $\eta \mapsto \lim \eta_i$, and extend to ℓ^∞ (Hahn-Banach), the result is something $\neq 0$ in $(\ell^\infty)^*$ but 0 on c_0, hence $(\ell^\infty)^* \neq c_0^*$.
For (d), note that c_f is countably generated and dense in ℓ^1, c_0, while ℓ^∞ contains $\{0,1\}^{\prod \mathbb{Z}}$.

Exercise 10 Use **Exercise 9** (a).

Exercise 11 Let $\mathrm{X} = \mathrm{Lip}\,\alpha$. The following are all norm functions on X :

$$\mathrm{A} : f \mapsto |f(a)|, \quad \mathrm{I} : f \mapsto \|f\|_\infty, \quad \mathrm{M} : f \mapsto \mathrm{M}_f, \quad \mathrm{P} : \mathrm{M} + \mathrm{A}, \quad \mathrm{Q} = \mathrm{M} + \mathrm{I}$$

Let $\{f_n\}$ be a P or Q-Cauchy sequence, then it is M-Cauchy since $\mathrm{M} \leq \mathrm{P}, \mathrm{Q}$.
In fact, it is also I-Cauchy: for Q, this is by $\mathrm{I} \leq \mathrm{Q}$; for P, this is by

$$|f_{m,n}(x)| \leq |f_{m,n}(x) - f_{m,n}(a)| + |f_{m,n}(a)| \leq \mathrm{M}(f_{n,m})|x - a|^\alpha + |f_{m,n}(a)| \leq \max\{1, |b - a|^\alpha\}\mathrm{P}(f_{n,m})$$

where $f_{n,m} = f_n - f_m$. Therefore $\{f_n\}$ has an I-limit $f \in \mathrm{C}([a,b])$. We show the following assertions :

$$f \text{ is the M-limit of } \{f_n\} \text{ and } f \in \mathrm{X}.$$

Let us write $\mathrm{M}(f; s, t) = |s - t|^{-\alpha}|f(s) - f(t)|$, so $\mathrm{M}(f) = \sup_{s \neq t} \mathrm{M}(f; s, t)$.

- To show f is the M-limit of $\{f_n\}$, we have for $s \neq t$ that :

  $$\begin{gathered} \mathrm{M}(f - f_n; s, t) = \lim_m \mathrm{M}(f_m - f_n; s, t) \leq \varepsilon \text{ for all } n \geq \mathrm{N} \\ \text{if } \mathrm{I}(f_n - f_m) \leq \varepsilon \text{ for all } m, n \geq \mathrm{N} \end{gathered}$$

  so $\mathrm{M}(f - f_n) \to 0$, hence $\mathrm{M}(f_n) \to \mathrm{M}(f)$ since M is a norm function.

- To show $f \in \mathrm{X}$, or, $\mathrm{M}(f) < \infty$, we use

  $$\begin{aligned} \mathrm{M}(f; s, t) &\leq \mathrm{M}(f_n; s, t) + \mathrm{M}(f - f_n; s, t) \leq \mathrm{M}(f_n) + \sup_{m \to \infty} \mathrm{M}(f_n - f_m; s, t) \\ &\leq (\sup \mathrm{M}(f_n)) + 1 \text{ if } \mathrm{M}(f_n - f_m) \leq 1 \text{ for all } m \geq n \end{aligned}$$

  here we used the fact that $\{f_n\}$ is M-Cauchy. Since $\sup \mathrm{M}(f_n)$ is bounded, we are done.

Exercise 12 Note that such a function is determined by its values on the three vertices by the theory of finite dimensional linear algebra, so take a suitable counting measure on the vertices.

Exercise 13 Given $r > 0$, let $B(r) := \{z \in \mathbb{C} : |z| < r\}$. For (a), given $n, M > 0$, define

$$E_{n,m} = f_n^{-1}(\overline{B}(M)^c)$$

then the condition that f_n converges pointwise on all X gives

$$\bigcap_M \bigcup_n E_{n,m} = \emptyset, \quad \bigcup_M \bigcap_n E_{n,m}^c = X$$

Here the intersections, unions are countable ones. Since X is complete, by Baire's category theorem, some $\bigcap_n E_{n,m}^c$ is not no-where dense; this implies (a). For (b), define as in hint the closed sets A_n, and note that $\bigcap A_n = X$ by the pointwise convergence condition, and apply Baire's category theorem.

Exercise 14 For the first part, we follow the hint. Take $f \in C$. Given $m > 0$, $\varepsilon > 0$, let $\{x_{m,i}\}_{i=0}^m$ be points in $[0,1]$ that partitions $[0,1]$ into m equilength intervals, define f_m to be the function that is linear on each $[x_{m,i}, x_{m,i+1}]$ with $f_m(x_{m,i}) = (-1)^i\varepsilon + f(x_{m,i})$, then we have :

$$\left|\frac{f_m(x_{m,i}) - f_m(x_{m,i+1})}{x_{m,i} - x_{m,i+1}}\right| = m|2\varepsilon + f(x_{m,i}) - f(x_{m,i+1})|$$

For large enough m, uniform continuity of f on $[0,1]$ gives $|f(x_{m,i}) - f(x_{m,i+1})| \leq \varepsilon$, so $f_m \notin X_{[m\varepsilon]}$, and also in this case that $\|f - f\|_\infty \leq 2\varepsilon$; to see this, take $t \in [0,1]$, $x^- = x_{m,i}$, $x^+ = x_{m,i+1}$, $x = tx^- + (1-t)x^+$, then we have by linearity of f_m on $[x^-, x^+]$ that

$$|f(x) - f_m(x)| = |t(f(x) - f_m(x^-)) + (1-t)(f(x) - f_m(x^+))|$$

and use the following estimate to bound the first term :

$$|f(x) - f_m(x^-)| \leq |f(x) - f(x^-)| + |f(x^-) - f_m(x^-)| \leq \varepsilon + \varepsilon$$

similarly for the term $f(x) - f_m(x^+)$. Now for this fixed m, we know when $\|f_m - g\|_\infty \leq \varepsilon/8$, then $g \notin X_{[m\varepsilon/2]}$; the details are left to the readers. The facts proved above shows that each open set in C has an open subset contained in X_n^c, so X_n is nowhere dense, and then since the set of nowhere differentiable functions contains $\bigcap X_n^c \supset \bigcap (X_n^c)^o$, we're done by Baire's category theorem.

Comment on Exercise 15 Let us fix some notations. As in **Exercise 9** , we use $c_0 \subset c \subset \ell^\infty$ to denote the set of complex-valued functions on $\mathbb{Z}$ that tends 0 / has a limit / uniformly bounded. Note that under the norm $\|\cdot\|_\infty$, these spaces are all Banach spaces. From this viewpoint, it is useful to think of A as a linear map $c \longrightarrow c$, and that each row A_i of A defines a linear map $c \longrightarrow \mathbb{C}$.

Exercise 15 Suppose A defines a transformation $c \longrightarrow c$ that preserves limits, then (a) is true by testing Kronecker's delta functions, (c) is true by testing constant functions. We shall show (b).
Let us write $|A_i|$ as the sum $\sum_j |a_{ij}|$, and $\|A_i\|$ as the operator norm $c \longrightarrow \mathbb{C}$ of A_i. Firstly, we have :

$$|A_i| = \sum_j |a_{ij}| < \infty \text{ for all } i$$

For if on the contrary that $|A_i| = \infty$, take $0 = n_0 < n_1 < \ldots$ with $\sum_{j=n_k}^{n_{k+1}-1} |a_{ij}| > 1$, find $s \in c_0$ with $\sum_{j=0}^{n_k} s_j a_{ij} > \sum_{j=1}^{k} j^{-1}$ for each k, then A isn't well-defined. For (b), by Banach-Steinhauss, it suffices to show $\sup_i \|A_i s\|_\infty < \infty$ for all $s \in c$, and $|A_i| \le \|A_i\|$. For the first, use convergence of the sequence $As = (A_i s)_i$. For the other, by taking s^k with $A_i s^k = \sum_{j=0}^k |a_{ij}|$ for some k, we have

$$\|A_i\| = \sup_{s\in c} \frac{\|A_i s\|_\infty}{\|s\|_\infty} \ge \sup_{k\in\mathbb{N}} \frac{\|A_i s^k\|_\infty}{\|s^k\|_\infty} = \sup_{k\in\mathbb{N}} \sum_{j=0}^k |a_{ij}| = |A_i|$$

For the converse, condition (c) shows that only need to consider over c_0. Take $s \in c_0$, we get

$$|A_i s| = |\sum_j a_{ij} s_j| \le \sum_j |a_{ij}||s_j| \to 0$$

by conditions (a), (b). (Sketch : $\sum_j |a_{ij}||s_j| \le (\sum_{j\le N} + \sum_{j>N})|a_{ij}||s_j| \le N\|s\|_\infty \sum_{j=0}^N |a_{ij}| + M\varepsilon$, where M a bound for all $|A_i|$, with N an integer that makes ε a bound for remainder terms of $\sum_j |s_j|$).
For examples where A take some sequences outside c to c, try $s = (s_j) = ((-1)^{j+1} j)$.

Exercise 16 Follow the hint. Take $Z = X \times Y$ with norm function defined componentwisely, then Z is a Banach space, and the condition says that the subspace $G \subset Z$ of graph of $\Lambda : X \to Y$ is closed. In the commutative diagram below, the undecorated arrows are bounded linear maps :

$$\begin{tikzcd}
G \arrow[r, hook, "i"] \arrow[dr, "pi"] & X \times Y \arrow[r, "q"] \arrow[d, "p"] & Y \\
& X \arrow[ul, dashed, bend left, "r"] \arrow[ur, dashed, "\Lambda"'] &
\end{tikzcd}$$

Since $pi : G \to X$ is a bounded linear isomorphism between Banach spaces, its inverse is also bounded (Theorem 5.10). Now since $\Lambda = qir$, hence bounded and hence continuous.

Exercise 17 The identity $\|M_f\| \le \|f\|_\infty$ is straightfoward. When the measure space is a union of measurable subsets with finite measures, we always have $\|M_f\| = \|f\|_\infty$. For the "onto" question, note first that we must have $\mu(f^{-1}(0)) = 0$, so it makes sense to talk about the "$1/f$", and we are reduced to find conditions on f so that $g/f \in L^2$ if $g \in L^2$. At least when $1/f \in L^\infty$ this is true.

Exercise 18 Since Y complete, we show $\{\Lambda_n y\}_n$ is Cauchy for any $y \in X \setminus E$. Take $\varepsilon > 0$, $x \in E$ with $|x - y| < \varepsilon$, find N exhibiting Cauchiness of $\{\Lambda_n x\}$ with respect to ε, and use

$$|\Lambda_m y - \Lambda_n y| \le |\Lambda_m y - \Lambda_m x| + |\Lambda_n y - \Lambda_n x| + |\Lambda_m x - \Lambda_n x| \le (2M + 1)\varepsilon$$

Exercise 19 In the text, if we define $\Lambda_n : f \mapsto s_n(f; x)$ for some $x \in T$, we have the following :

$$\frac{\|D_n\|_1}{\log(n)} \ge \frac{4}{\pi^2} \frac{\sum_{k=1}^n \frac{1}{k}}{\log(n)} \to \frac{4}{\pi^2}, \quad \|\Lambda_n\| = \|D_n\|_1$$

If we take $\Lambda'_n = \Lambda_n / \lambda_n$ with $\lim \lambda_n / \log(n) = 0$, then $\|\Lambda'_n\|$ is unbounded.
Now let us show $\lim \|s_n\|_\infty / \log(n) = 0$. First we find an upper bound for $\|D_n\|_1$. Note that :

$$\|D_n\|_1 = \frac{1}{\pi} \int_0^\pi \left| \frac{\sin((n+1/2)t)}{\sin(t/2)} \right| dt = \frac{1}{\pi} (\int_0^{\pi/n} + \int_{\pi/n}^\pi) \left| \frac{\sin((n+1/2)t)}{\sin(t/2)} \right| dt$$

Since $\sin(nx) < n \sin(x)$ for $x \in (0, \pi/n)$ and $\sin(x) < x/\pi$ for $x \in (0, \pi/2)$, we have

$$\le \frac{1}{\pi} \left((\int_0^{\pi/n} \frac{(2n+1)\sin(t/2)}{\sin(t/2)}) + (\int_{\pi/n}^\pi \frac{1}{\sin(t/2)} dt) \right) \le 3 + 2 \int_{\pi/n}^\pi t^{-1} dt \le 3 + 2 \log n$$

Therefore, the operator $f \mapsto s_n(f; x)/\log(n)$ is uniformly (with respect to x) bounded.

Exercise 20 For (a), we remark first that $\mathbb{Q}$ is not a countable intersection of open sets by Baire's category theorem, so suppose there exists such a sequence of functions, we would have

$$\mathbb{Q} = \bigcap_m \bigcup_n f_n^{-1}(m, \infty)$$

which is a contradiction. For (b), let $\{q_m\}$ be an enumeration of $\mathbb{Q}$, and let $\mathrm{Q}_m = \{q_1, \dots, q_m\}$.
First we define piecewise linear functions $g_{m,n}$ that are 0 on Q_m with properties like $\sup_m g_{m,n} = \chi_{\mathrm{Q}_n^c}$.
For $n = 1$, let $q_{m,1}^{\pm} = q_1 \pm m^{-1}$, let $g_{m,1} = 1$ on $(q_{m,1}^-, q_{m,1}^+)^c$, linear on $[q_{m,1}^-, q_1]$, $[q_1, q_{m,1}^+]$ with $g(q_1) = 0$.
For $n > 1$, construct over each $q \in \mathrm{Q}_n$ as in the case $n = 1$ then take infimum.
(In short, $g_{m,n}(x)$ is defined by the formula $g_{m,n}(x) = \min(m|x - q_1|, \dots, m|x - q_n|, 1)$.)
Now define $\{f_{m,n}\}$ by $f_{m,n} = \sum_{k \le n} g_{m,k}$, then $\sup_m f_{m,n} = \sum_{k \le n} \chi_{\mathrm{Q}_k^c} = n\chi_{\mathrm{Q}_n^c} + \sum_{k<n}(k-1)\chi_{\{q_k\}}$, so

$$\mathbb{Q}^c = \bigcap_k \mathrm{Q}_k^c = \bigcap_k \bigcup_{m,n} f_{m,n}^{-1}(k, \infty)$$

This says that $\{f_{m,n}(x)\}_{m,n}$ is unbounded iff $x \in \mathbb{Q}^c$, so we are done after a re-indexing.
For (c), consider first the analogous statement (a') of (a). We will define $\{f_n\}$ supported in $[0, 1]$ with $f_n(0) = f_n(1)$ that tends to ∞ pointwisely only on $\mathbb{Q} \cap [0, 1]$, then the general case will follow by extending periodically. Note first that every element in $[0, 1]$ can be written as a sum

$$x = \sum_{n=1}^{\infty} \frac{a_n}{(n+1)!}, \quad a_n = 0, \dots, n$$

Note that for rationals, its representation can be chosen to be a finite sum, while it is impossible to do so for irrationals. For each $n > 0$, let $\{x_{n,k}\}_{k=0}^{n!}$ be partition of $[0, 1]$ into $n!$ equilength intervals, and define $x_{n,k}^{\pm} = x_{n,k} - 1/(n+1)!$, let f_n be linear on each subinterval defined by $\{x_{n,k}, x_{n,k}^{\pm}\}$, with

$$f_n(x_{n,k}) = n, \quad f_n(x_{n,k}^{\pm}) = 0$$

then we see that $f_n(x) \to \infty$ if $x \in \mathbb{Q}$. In fact, $\{f_n(x)\}$ has infinitely many 0s in it if $x \in \mathbb{Q}^c$: Write x as a sum as above with coefficients a_n. First, $f_n(x) = 0$ iff $|\{n!x\} - 1/2| \le 1/2 - 1/(n+1)$, while

$$\{n!x\} = \{n! \sum_{k=1}^{\infty} \frac{a_k}{(k+1)!}\} = \{\sum_{k=n}^{\infty} \frac{n!}{(k+1)!} a_k\}$$

Since $x \in \mathbb{Q}^c$, both $\{a_k\}$ and $\{a_k - k\}$ are not eventually zero, and also for each n that

$$0 < \sum_{k=n+1}^{\infty} \frac{n!}{(k+1)!} a_k < \frac{1}{n+1}$$

so we are done. Now only (b') remains. Consider our example $\{f_{m,n}\}$ in (b) and take diagonal $f_{n,n}$, using the fact that $\lim_{n\to\infty} \sum_{k=1}^{m} \min(m|x - q_1|, \dots, m|x - q_k|, 1) = \infty$ if $x \in \mathbb{Q}^c$.

Comments on Exercise 20 There are other methods to (a'). One of them uses the fact that there is a homeomorphism on $\mathbb{R}^1$ that induces a bijection between $\mathbb{Q}$ and the 2-adic rationals, thus reduce the problem to a simpler one. The existence of this homeomorphism is guaranteed by Minkowski's question mark function, or from a theorem of Sierpinski that countable metrizable spaces without any isolated points are isomorphic to each other.

Exercise 21 For the first question, note that such a set E has the property iff under the map

$$\sigma : \mathbb{R}^1 \times \mathbb{R}^1, \quad (x, y) \mapsto x - y$$

that $\sigma(\mathrm{E} \times \mathrm{E}) = \mathbb{R}^1$. Note that for Cantor's set C, $\sigma(\mathrm{C} \times \mathrm{C}) = [-1, 1]$, so we may take $\mathrm{E} = \mathbb{Z} + \mathrm{C}$.
For the other, a dense, measure zero G_δ would suffice (by the category theorem). Let $\{q_k\} = \mathbb{Q}$, set

$$\mathrm{U}_n = \bigcup (q_k - n^{-1}2^{-k}, q_k + n^{-1}2^{-k}), \quad \mathrm{E} = \bigcap \mathrm{U}_n$$

Exercise 22 As in hint, we assume $x = 0, f(0) = 0$. Want $s_n(f;0) \to 0$. We have :

$$s_n(f;0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(t)}{\sin(t/2)} \sin((n+1/2)t)dt$$

Since $f(t)/\sin(t/2) \in L^1(T)$ by $f \in \mathrm{Lip}\,\alpha$, we are done by the Riemann-Lebesgue lemma.

6 Complex Measures

Comment on Exercise 1 We assume the fact that $|\mu|$ is bounded.

Exercise 1 Use $|\mu|(E) - 2\varepsilon \le (\sum_{i\in I}|\mu(E_i)|) - \varepsilon \le \sum_{i\in J}|\mu(E_i)|$ with $|J| < \infty$ and $\subseteq I$.

Exercise 2 Let $(X, \mathfrak{m}, \mu)$ be the measure space $(0,1)$ with the Lebesgue measure. Suppose first λ has a decomposition (λ_a, λ_s). Since μ can only concentrate on sets with μ-measure 0 complement, λ_s can only concentrate on a μ-measure 0 subset A. Use a nonempty, μ-measure 0 subset of A^c. Suppose for some $h \in L^1(\mu)$ that $d\mu = hd\lambda$, integrate over each $\{x\} \subseteq X$ and conclude $h = 0$.

Comment on Exercise 3 That M(X) is a $\mathbb{C}$-vector space is clear from the fact that regularity is not altered under the formation of sum, scalar multiplication of complex measures. That the function $\mu \mapsto |\mu|(X)$ is a norm function follows from boundedness and some routine verifications of the identities in Definition 5.2 that we shall omit; it is also quite analogous to how one show that $x^* : f \mapsto f(x)$ is a norm function on X^* - the space of bounded functionals on a Banach space X - as in Chapter 5 **Exercise 8** . We will focus on showing completeness of M(X) in the solution below.

Exercise 3 Given a Cauchy sequence $\{\mu_n\}$ in M(X), want to find a limit in M(X). Let $\mathfrak{m}$ be the σ-algebra on X. For any $E \in \mathfrak{m}$, $\{\mu_n(E)\}$ is Cauchy. Define $\mu : \mathfrak{m} \to \mathbb{C}$ by $\mu(E) = \lim \mu_n(E)$.
We aim to show that $\mu \in M(X)$ and is a limit of $\{\mu_n\}$ in M(X).

- μ is a complex measure : notice first that we have the following :
  - μ has finite additivity : since it is a "pointwise limit" (of points in $\mathfrak{m}$) of $\{\mu_n\}$.
  - $\mu(E_k) \to \mu(E)$ if given $E_1 \subseteq E_2 \subseteq \dots$ in $\mathfrak{m}$ with $E = \bigcup E_k$: for each n, consider

    $$|\mu(E)-\mu(E_k)| \le |\mu(E)-\mu_n(E)|+|\mu(E_k)-\mu_n(E_k)|+|\mu_n(E_k^c)| \le |\mu_n(E_k^c)|+2\lim_{m\to\infty}\|\mu_m-\mu_n\|$$

    Using Cauchiness of $\{\mu_n\}$, take some n with $\lim_m\|\mu_m - \mu_n\| \le \varepsilon$, then take some k_0 with $|\mu_n(E_k^c)| \le \varepsilon$ for $k \ge k_0$, then we have $|\mu(X) - \mu(E_k)| \le 3\varepsilon$ for $k \ge k_0$.

  These two facts being established, we get countable additivity of μ.
- $|\mu|(X) < \infty$: since μ is a complex measure.
- $\|\mu - \mu_n\| \to 0$: by **Exercise 1** , we have $\|\mu - \mu_n\| \le \lim_m\|\mu_m - \mu_n\|$.
- μ is regular : for inner regularity, take open V, look at

  $$0 \le ||\mu|(V) - |\mu|(K)| \le ||\mu_n|(V) - |\mu_n|(K)| + 2\|\mu - \mu_n\|$$

  Choose n then choose compact K to bound the difference. Similarly for outer regularity.
- $\mu \in M(X)$ and is the limit of μ_n in M(X) : by the previous items.

Exercise 4 For $p = \infty$, take $f = 1$ to see $g \in L^q(\mu)$. For other p, we show that the linear functional $\Lambda : L^p(\mu) \longrightarrow L^q(\mu)$ defined by $f \mapsto \|gf\|_1$ is bounded, then the duality theorem (Theorem 6.16) along with the uniqueness part shows that $g \in L^q(\mu)$. The idea is to use the Banach-Steinhaus theorem. Write $X = \bigcup E_k$, with $E_1 \subseteq E_2 \subseteq \ldots$ each with $\mu(E_k) < \infty$, and $G_n = |g|^{-1}[0, n)$, define $g_n = \chi_{E_n \cap G_n} g$, then $|g_n| \longrightarrow |g|$ pointwisely from below, and if we define $\Lambda_n : f \mapsto \|g_n f\|_1$, then each Λ_n is a bounded linear functional (since $g_n \in L^1(\mu)$), and that MCT gives $\|\Lambda\| = \sup_n \|\Lambda_n\|$, so the assumptions along with Banach-Steinhaus theorem shows that $\|\Lambda\| < \infty$.

Exercise 5 Notice $\dim_{\mathbb{C}}(L^1(\mu)) = 1$, $\dim_{\mathbb{C}}(L^\infty(\mu)) = 2$.

Exercise 6 The association $L^q(\mu) \longrightarrow L^p(\mu)^*$ defined by $g \mapsto g^* : f \mapsto \int_X fg d\mu$ is well-defined.
It is injective : if $g^* = 0$ and $g \in L^q(\mu)$, set $E_n = |g|^{-1}(n^{-1}, \infty)$, then $\chi_{E_n} g = 0$, so $g = 0$ a.e..
It is surjective : Let $\Lambda \in L^p(\mu)^*$. Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be the set of σ-finite subsets of X. For each $E \in \mathcal{F}$, let μ_E be the measure μ restricted to E, then each μ_E is represented by some $g_E \in L^q(\mu_E)$; note that the association $\mathcal{F} \longrightarrow L^q(\mu)$ given by $E \mapsto g_E \in L^q(\mu_E) \subseteq L^q(\mu)$ is "compatible" with restriction maps induced from inclusions of σ-finite spaces. Consider the following map :

$$\xi : \mathcal{F} \longrightarrow L^q(\mu) \xrightarrow{\|\cdot\|_q} [0, \infty)$$

It turns out that $\xi(\mathcal{F})$ is actually bounded; to see this, notice that this is by

$$\|g_E\|_q^q = \int_X |g_E|^q d\mu \le \|g_E^{q-1}\|_p \|\Lambda\| = \|g_E\|_q^{q/p} \|\Lambda\|, \qquad \text{hence } \xi(E) = \|g_E\|_q \le \|\Lambda\|.$$

Let $M = \sup \xi(\mathcal{F})$, take $E_n \in \mathcal{F}$ so that $\xi(E_n) \longrightarrow M$. Since $\mathcal{F}$ is closed under countable union, MCT even gives some $E^+ \in \mathcal{F}$ with $\xi(E^+) = M$; this shows $g_E = \chi_E g_{E^+}$ a.e. for all $E \in \mathcal{F}$. Therefore, g_{E^+} actually represents Λ, by using the fact that $|f|^{-1}(0, \infty) \in \mathcal{F}$ for any $f \in L^p(\mu)$.

Exercise 7 For positive and bounded μ, use $\overline{\hat{\mu}(n)} = \hat{\mu}(-n)$. For general μ, we get by hint that $\widehat{|\mu|}(n) \longrightarrow 0$ as $|n| \longrightarrow \infty$. Therefore, by our previous case applied to $|\mu|$, it suffices to show that :

$$\text{"}\widehat{|\mu|}(n) \longrightarrow 0 \text{ as } |n| \longrightarrow \infty\text{" implies "}\widehat{\mu}(n) \longrightarrow 0 \text{ as } n \longrightarrow -\infty\text{".}$$

Notice again that by using a polar decomposition $|\mu| = hd\mu$, we have

$$\widehat{\mu}(-n) = \int_0^{2\pi} e^{int} d\mu(t) = \int_0^{2\pi} e^{int} h d|\mu|(t)$$

Now we conclude using a similar method as in the hint : Let P be a trigonometric polynomial, then

$$|\widehat{\mu}(-n)| = \left|\int_0^{2\pi} e^{int} h d|\mu|(t)\right| \le \|h - P\|_\infty \cdot |\mu|([0, 2\pi)) + \left|\int_0^{2\pi} P e^{int} d|\mu|(t)\right|$$

and note that for a fixed P, the last term $\longrightarrow 0$ as $|n| \longrightarrow \infty$.

Comment on Exercise 8 Here we directly assume $k \neq 0$.

Exercise 8 First $\widehat{\mu}$ is constant. If $\widehat{\mu} = 0$, approximate χ_E to see $\mu(E) = \int_E \chi_E d\mu = 0$. If $\widehat{\mu} \neq 0$, then

$$0 = \widehat{\mu}(n + k) - \widehat{\mu}(n) = \int_0^{2\pi} e^{-int}(e^{-ikt} - 1) d\mu(t) = \int_0^{2\pi} e^{-int} d\nu(t) = \widehat{\nu}(n)$$

Here ν is the (complex) measure defined by $d\nu = (e^{-ikt} - 1)d\mu$, so $\nu = 0$, so μ looks like $\sum_{l=0}^{k-1} c_l \delta_{2\pi l/k}$.

Exercise 9 No, and in fact, μ can even be taken to be m. Let us define g_n. Given n, define $\delta_n = 2^{-n}$, and given $k = 0, \ldots, \delta_n^{-1} - 1$, define $I_{n,k} = [k\delta_n, k\delta_n + \delta_n^2)$. Let g_n be a continuous function such that

$$\int_{I_{n,k}} g_n dm = \delta_n, \quad g_n = 0 \text{ outside } \bigcup_{k=0}^{\delta_n^{-1}-1} I_{n,k}$$

then g_n satisfies (i),(ii); to see (i), note that the set of points where $g_n \to 0$ is contained in $\bigcap_{n=1}^{\infty} \bigcup_{k=0}^{\delta_n^{-1}-1} I_{n,k}$, which has measure 0. For (iii), we have for each n that

$$\left| \sum_{k=0}^{\delta_n^{-1}-1} f(k\delta_n) - \int_I f g_n dm \right| \le \delta_n \sum_{k=0}^{\delta_n^{-1}-1} (\sup_{t \in I_{n,k}} f(t) - \inf_{t \in I_{n,k}} f(t))$$

To see this, use the inequality $|f(k\delta_n) - f| \le (\sup_{t\in I_{n,k}} f(t) - \inf_{t \in I_{n,k}} f(t))$ over each $I_{n,k}$ and integrate. Using uniform continuity, $\int_I f g_n dm \to \sum_{k=0}^{\delta_n^{-1}-1} f(k\delta_n)$, hence tends to $\int_I f dm$.

Comment on Exercise 10 Sometimes uniform integratability is defined in another form where it is required that $\int_E |f| d\mu < \varepsilon$ (instead of only $|\int_E f d\mu|$) whenever $\mu(E) < \delta$; these two definitions are actually equivalent, and we leave this fact as an additional exercise to the readers.
On the other hand, the reader may try doing items (c),(d) first before considering (b).

Exercise 10 For (a), may assume $|\Phi| = 1$, then use Theorem 6.11 with λ given by $d\lambda = f d\mu$.
For (b), we first show $f \in L^1(\mu)$. By (ii), may find some $\delta > 0$ such that $\mu(E) < \delta$ implies $\int_E |f_n| d\mu < 1$. Note that by Fatou's lemma, $\chi_E f \in L^1(\mu)$ if $\mu(E) < \delta$. By cutting away finitely many of these subsets, may assume any non-empty measurable subsets of X either has measure $\ge \delta$ or measure 0 by (i).
By (i), f restricted to the set E of $x \in X$ with $\mu(f^{-1}(f(x))) > 0$ has finite image.
By (iv) applied to each such fiber of f, we finally get $f \in L^1(\mu)$.
To see $\lim \int_X |f_n - f| d\mu = 0$, take $\varepsilon > 0$, use Egoroff's theorem to find E so that f_n converges uniformly on it, with the property that $\int_{E^c} |f_n| d\mu < \varepsilon$ for any n by using (ii), then Fatou's lemma gives

$$\liminf_n \int_X |f - f_n| d\mu \le 2\varepsilon + \liminf_n \int_E |f - f_n| d\mu = 2\varepsilon$$

where the last equality is by condition (i) and uniform convergence on E.
For (c), our counterexample $f_n = \chi_{[n-1,n]}$ in **Exercise 16** for Egoroff's theorem works.
For (d), a counterexample can be constructed by taking $(X, \mathfrak{m}, \mu)$ by $X = \{x\}$, $\mathfrak{m} = \{\emptyset, X\}$, $\mu(x) = 1$, with $f_n = n$. To see that (iv) is redundant when μ is the Lebesgue measure on a bounded interval, use the fact that given any δ, the interval can be covered by finitely many measurable sets E_i with $\mu(E_i) < \delta$, and use (ii) along with Fatou's lemma to show that $f \in L^1(\mu)$.
For (e), first suppose given $f_n \to f$ pointwise, f_n dominated by $g \in L^1(\mu)$, and $\mu(X) < \infty$. (ii) holds, since $\Phi := \{g\}$ is uniformly integrable with $|f_n| \le g$ (by (a)). The verification of (i),(iii),(iv) is omitted.
For (f), a way is to define $g_n = 2^{2n} \chi_{2^{-n}, 2^{-n-1}}$, $f_n(t) = g_n(t) - g_n(t - 1/2)$.
For (g), that ρ is a metric is straightfoward. That $(\mathfrak{m}, \rho)$ is a complete metric space can be seen as follows. Take a Cauchy sequence $A_n \in \mathfrak{m}$, define $A = \bigcup_m \bigcap_{n \ge m} A_n \in \mathfrak{m}$. One has

$$\rho(A, A_m) = \mu(A \setminus A_m) + \mu(A_m \setminus A) \le 2\mu(C_m \setminus B_m)$$

by the fact $B_m \subseteq A_m \subseteq C_m, B_m \subseteq A \subseteq C_m$; here $B_m = \bigcap_{n \ge m} A_n$, $C_m = \bigcup_{n \ge m} A_n$.
If A_n has the property $\rho(A_n, A_{n+1}) \le 2^{-n}$, one sees that $\rho(A, A_m) \to 0$ by using

$$\mu(B_m \setminus C_m) = \mu(B_m \setminus A_m) + \mu(A_m \setminus C_m) \le \sum_{n \ge m} \rho(A_n, A_m) \le 2^{-m+1}$$

If A_n does not have this property, consider a subsequence of A_n with this property.
The fact that $E \mapsto \int_E f_n d\mu$ is continuous follows from uniform integratability.
The rest of the hint is straightfoward, so are omitted here.

Exercise 11 If we can show $\{f_n\}$ is uniformly integrable, then we are done by **Exercise 10** (b), with condition (iv) ensured by boundedness of $\{\|f_n\|_1\}_n$ (which is by the condition $\mu(X) < \infty$) along with Fatou's lemma. To see uniform integratability, note that Hölder's inequality gives :

$$\int_E |f_n| d\mu \leq \mu(E)^{1/q} C^{1/p}$$

Exercise 12 The fact that g is not measurable can be done by noting that $g^{-1}[0, 1/2] \notin \mathfrak{m}$.
The fact that $f \mapsto \int f g d\mu$ is a bounded linear functional is by $|\int g f d\mu| \leq \int |f| d\mu$.
To see why this shows $(L^1(\mu))^* \neq L^\infty(\mu)$, say g is represented by some $h \in L^\infty(\mu)$, try $f = \chi_{\{x\}}$.

Exercise 13 Let M be subspace of $L^\infty(\mu)$ such that each $[f] \in M$ has a representative g such that $\lim_{t \to 1/2^+} g(t), \lim_{t \to 1/2^-} g(t)$ exists; note that $C(I) \subseteq M$. Define $\Lambda : M \to \mathbb{R}^1$ by

$$\Lambda[f] = \inf\{\lim_{t \to 1/2^+} \mathrm{Re}(g(t)) - \lim_{t \to 1/2^-} \mathrm{Re}(g(t)) : g \text{ has these limits and } g \in [f]\}$$

then $\Lambda \neq 0$, but is 0 on M. Extend Λ by using the Hahn-Banach theorem.

7 Differentiation

Exercise 1 .

Exercise 2 .

Exercise 3 .

Exercise 4 .

Exercise 5 .

Exercise 6 .

Exercise 7 .

Exercise 8 .

Exercise 9 .

Exercise 10 .

Exercise 11 .

Exercise 12 .

Exercise 13 .

Exercise 14 .

Exercise 15 .

Exercise 16 .

Exercise 17 .

Exercise 18 .

Exercise 19 .

Exercise 20 .

Exercise 21 .

Exercise 22 .

Exercise 23 .

8 Integration of Product Spaces

Exercise 1 .

Exercise 2 .

Exercise 3 .

Exercise 4 .

Exercise 5 .

Exercise 6 .

Exercise 7 .

Exercise 8 .

Exercise 9 .

Exercise 10 .

Exercise 11 .

Exercise 12 .

Exercise 13 .

Exercise 14 .

Exercise 15 .

Exercise 16 .

9 Fourier Transforms

Exercise 1 .

Exercise 2 .

Exercise 3 .

Exercise 4 .

Exercise 5 .

Exercise 6 .

Exercise 7 .

Exercise 8 .

Exercise 9 .

Exercise 10 .

Exercise 11 .

Exercise 12 .

Exercise 13 .

Exercise 14 .

Exercise 15 .

Exercise 16 .

Exercise 17 .

Exercise 18 .

Exercise 19 .