Representation of Finite groups

- Some Solutions to Exercises*

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2 Character Theory

Exercise 2.1 One can realize $\chi + \chi'$ as the character of $\rho \oplus \rho'$. Pick $s \in G$. Suppose

$$(e_1, \ldots, e_m), \quad (e'_1, \ldots, e'_n)$$

are eigenbasis for ρ, ρ' with eigenvalues (λ_i, λ'_j are $\mathbb{C}$-valued functions of G)

$$(\lambda_1, \ldots, \lambda_m), \quad (\lambda'_1, \ldots, \lambda'_n)$$

then one can define a basis of $\rho \oplus \rho'$ by

$$(e''_1, \ldots, e''_{m+n}) := (e_1, \ldots, e_m, e'_1, \ldots, e'_n)$$

with corresponding eigenvalues

$$(\lambda''_1, \ldots, \lambda''_{m+n}) := (\lambda_1, \ldots, \lambda_m, \lambda'_1, \ldots, \lambda'_n)$$

Now we compute :

$$(\chi + \chi')^2_\sigma = \mathrm{Tr}(\rho \oplus \rho') = \sum_{i \le j} \lambda''_i \lambda''_j = \left(\sum_{i \le m < j} + \sum_{i \le j \le m} + \sum_{m < i \le j} \right) \lambda''_i \lambda''_j = \chi\chi' + \chi^2_\sigma + \chi'^2_\sigma$$

We omit the similar computation for alternating squares.

Exercise 2.2 Let $(e_i)_{i \in X}$ be a basis of X, suppose for each $s \in G$ that

$$\rho_s(e_i) = \sum_{j \in X} r_{ji}(s) e_j$$

then we have

$$\chi = \sum_i r_{ii}$$

Since ρ is permutation representation of the group action G on X, we get

$$r_{ji}(s) = \delta_{si,j}$$

From this observation, we get

$$\chi(s) = \sum_i r_{ii}(s) = \sum_i \delta_{si,i}$$

This is the number of elements in X fixed by s.

Exercise 2.3 The idea is to define $\rho'_s(x') = x' \circ \rho_s^{-1}$. In a more informal way :

$$\rho' = \circ \rho^{-1}$$

We have the following calculation :

$$< \rho'_s(x'), \rho_s x >=< x' \rho_s^{-1}, \rho_s x >=< x', x >$$

This shows existence of ρ'; uniqueness follows from nondegeneracy of $<, >$.
To compute the character χ' of ρ', take an eigenbasis e_i of ρ_s with values λ_i define $e'_i \in V'$ by

$$< e'_i, e_j >= \delta_{ij}$$

then we have

$$< \rho'_s(e'_i), e_j >=< e'_i, \rho_s^{-1} e_j >=< \lambda_i^* e'_i, e_j >$$

so by non-degeneracy of $<, >$, we see that $\rho'_s(e'_i) = \lambda_i^* e'_i$, hence

$$\chi' = \sum_i \lambda_i^* = (\sum_i \lambda_i)^* = \chi^*$$

Exercise 2.4 The fact that it is a representation is straightfoward. To show that $\rho, \rho'_1 \otimes \rho_2$ are isomorphic representations, where ρ'_1 the contragredient of ρ_1 in **Exercise 2.3** , it suffices to show that they have the same character. Fix $s \in G$, let $(d_1, \ldots, d_m), (e_1, \ldots, e_n)$ be eigenbasis for $\rho_{1,s}, \rho_{2,s}$ with eigenvalues $(\lambda_1, \ldots, \lambda_m), (\mu_1, \ldots, \mu_m)$.

- Define basis $(d'_1, \ldots, d'_m)$ for $\rho'_{1,s}$ as in **Exercise 2.3** ; it has eigenvalues $(\lambda_1^*, \ldots, \lambda_m^*)$. Now we get an eigenbasis $(d'_i \otimes e_j)$ for $\rho'_{1,s} \otimes \rho_{2,s}$ with eigenvalues $\lambda_i^* \mu_j$.
- Define basis $(f_{11}, f_{21}, \ldots, f_{mn})$ of ρ by

$$f_{ij}(d_k) = \delta_{ik} e_j$$

then we have

$$\rho_s(f_{ij})(d_k) = (\rho_{2,s} f_{ij} \rho_{1,s}^{-1})(d_k) = \lambda_k^* \mu_j \delta_{ik} e_j = \lambda_k^* \mu_j f_{ij}(d_k) = \begin{cases} 0, & \text{if } k \neq i \\ \lambda_i^* \mu_j f_{ij}(d_k), & \text{if } k = i \end{cases}$$

so we get $\rho_s(f_{ij})(d_k) = \lambda_i^* \mu_j f_{ij}(d_k)$, hence f_{ij} has eigenvalue $\lambda_i^* \mu_j$.

These two observations shows that the two representations have the same character.

Exercise 2.5 The number of times ρ contains 1 is given by

$$(\rho|1) = \frac{1}{g} \sum_{t \in G} \chi(t)$$

by using Theorem 4 and the fact that the character of 1 is the constant function with value 1.

Exercise 2.6

(a) By decomposing the representation by restriction to orbits, it suffices to show :

> If G acts transitively on X, then ρ decomposes as $1 \oplus \psi$, and that the corresponding this decomposition, we have a decomposition of χ into $1 + \psi$ such that $(\psi|1) = 0$.

This follows directly from the computation

$$(\chi|1) = \frac{1}{g} \sum_{t \in G} \chi(t) = \frac{1}{g} \sum_{t \in G} |X_t| = \frac{1}{g} \sum_{x \in X} |G_x| = \frac{1}{g} \sum_{x \in X} \frac{g}{|Gx|} = \frac{1}{g} \sum_{x \in X} \frac{g}{|X|} = 1$$

(the first equality is by **Exercise 2.5** , the second by **Exercise 2.2** , the third by a counting argument, the fourth by orbit-stabilizer, the fifth by the assumption that G acts transitively.)

(b) Identify this representation with the tensor product $\rho \otimes \rho$.

(c) Equivalence between (i)-(iii) is already established in the hint.
The hint also said that (iii) is equivalent to $(\psi^2|1) = 1$.
Since ψ is real-valued (since χ, 1 are), we get $(\psi^2|1) = (\psi|\psi)$.
These observations establishes the equivalence between (iii) and (iv).

Exercise 2.7 Suppose χ is one of such with dimension 1, let χ_i be the irreducible characters of G of dimension n_i, we have the following observation :

$$(\chi, \chi_i) = \frac{1}{g} \sum_{t \in G} \chi(t) \chi_i(t^{-1}) = \frac{n n_i}{g} n_i$$

Take $\chi_i = 1$, we see that n/g is a non-negative integer; this will suffice.

Exercise 2.8 In the canonical decomposition of V given by

$$V = \bigoplus_i V_i$$

choose a decomposition of V_i as

$$V_i = \bigoplus_j W_{i,j}$$

where each $W_{i,j}$ is isomorphic to W_i, then we also have canonical injections :

$$\alpha_{i,j} : \; W_i \xrightarrow{\sim} W_{i,j} \longrightarrow V_i \longrightarrow V$$

and projection maps

$$\rho_{i,j} : \; V \longrightarrow V_i \longrightarrow W_{i,j} \xrightarrow{\sim} W_i$$

Notice that these maps are all morphisms of representations.

(a) Assume $h \neq 0$. By assumption, h is a morphism of representations.
Take the subrepresentation $\mathrm{Ker}(h)$, we see that h is injective.
Compose h with the projection maps $\rho_{i,j}$, we see that as a vector space, we have

$$H_i = \bigoplus_j \mathrm{Span}(\alpha_{i,j})$$

From this, we see that H_i has the required dimension.

(b) Since G acts trivially on each $\mathrm{Span}(\alpha_{i,j})$ (the G-structure on H_i given in **Exercise 2.4**), the above decomposition is readily a direct sum decomposition. Restriction of F to each $\mathrm{Span}(\alpha_{i,j}) \otimes W_i$ induce an isomorphism (of representations) to $W_{i,j}$.

(c) Since G acts trivially on each $\mathrm{Span}(\alpha_{i,j})$ (as remarked in (b)), H_i is a trivial G-space, hence any direct sum decomposition H_i as direct sum of lines (as vector spaces), is readily a direct sum decomposition of trivial representations.

Exercise 2.9 Recall the maps $\alpha_{i,j}$ in **Exercise 2.8** . The image of the evaluation morphism is

$$\mathrm{Span}(\alpha_{i,j}(e_\alpha)) = V_{i,\alpha}$$

so we are done already.

Exercise 2.10 Since $x \in V_i$, we have

$$x = p_i(x) = (\sum_\alpha p_{\alpha\alpha})(x) = (\sum_\alpha p_{\alpha 1} p_{1\alpha})(x) = \sum_\alpha p_{\alpha 1}(x_1^\alpha) \in \sum_\alpha W(x_1^\alpha)$$

It is easy to see that $V(x)$ admits another description :

$$V(x) = \mathrm{Span}(\rho_t(x))_{t \in G}$$

By the direct description of $p_{1\alpha}$, we get

$$x_1^\alpha = p_{1\alpha}(x) = \frac{n}{g} \sum_{t \in G} r_{\alpha 1}(t^{-1}) \rho_t(x) \in \mathrm{Span}(\rho_t(x))_{t \in G} = V(x)$$

These two observations establishes the identity

$$V(x) = \sum_\alpha W(x_1^\alpha)$$

3 Subgroups, Products, Induced Representations

Exercise 3.1 Each ρ_t is an automorphism of representation, so are homotheties, hence any one dimensional subspace is a subrepresentation.

Exercise 3.2

(a) Since ρ_s is a morphism for $s \in \mathrm{C}$, we have by Schur's lemma that ρ_s is a homothety. Since the eigenvalues of ρ_s lies on the unit circle, we get $|\chi(s)| = n$ consequencely.

(b) By orthogonality of characters and (a) that

$$g = \sum_{t \in \mathrm{G}} |\chi(s)|^2 \geq \sum_{t \in \mathrm{C}} |\chi(s)|^2 = cn^2$$

(c) Recall that each element in G has finite order. Define $\xi = \exp(2\pi i/g)$. By (a), for each $t \in \mathrm{C}$, there exists an integer μ_s such that

$$\rho_s = \xi^{\mu_s}$$

Let $d = \gcd(\mu_s)_{s \in \mathrm{C}}$, then there exists integers a_s such that

$$d = \sum_{s \in \mathrm{C}} a_s \mu_s$$

Define an element t in C by

$$t = \prod_{s \in \mathrm{C}} s^{a_s}$$

then we get $\rho_t = \xi^d$. Since ρ is faithful, we see that t generates C.

Exercise 3.3 A character from an abelian group is just a homomorphism from G to $\mathbb{C}^*$. The group structure on $\mathbb{C}^*$ then endows $\hat{\mathrm{G}}$ with a structure of an abelian group. To check that the map into the double dual of G is an injection, notice that

$$\chi(x) = \chi(y) \text{ for all } \chi \in \hat{\mathrm{G}} \text{ iff } \chi(xy^{-1}) = 1 \text{ for all } \chi \in \hat{\mathrm{G}} \text{ iff } x = y$$

This will suffice. For reasons of cardinality, this map is a bijection.

Exercise 3.4 Use the hint and Example 1,3 in that subsection.

Exercise 3.5 We already know that W can be identified as the space of elements in V that vanishes off H. Notice that the explicit description of the action of G on V implies that $\rho_s \mathrm{W}$ is the space of elements in V that that vanishes off $\mathrm{H}s^{-1}$; from this observation, the conditions of the definition of an induced representation is easily checked.

Exercise 3.6 The idea is to calculate the characters of ρ and $\theta \otimes r_{\mathrm{K}}$. Let $(u, v) \in \mathrm{H} \times \mathrm{K}$, we get

$$\chi_\rho(uv) = \sum_{\substack{t \in \mathrm{K} \\ t^{-1}uvt \in \mathrm{H}}} \chi_\theta(t^{-1}uvt)$$

We pause for a bit to consider what does this summation does.

- Suppose $v = 1$, then we have for all $t \in \mathrm{K}$ that $t^{-1}uvt = u$, and that

$$\{t \in \mathrm{K} : t^{-1}uvt \in \mathrm{H}\} = \mathrm{K}$$

- Suppose $v \neq 1$, then we have for all $t \in \mathrm{K}$ that $t^{-1}uvt \notin \mathrm{H}$, so

$$\{t \in \mathrm{K} : t^{-1}uvt \in \mathrm{H}\} = \emptyset$$

From these two observations, we get

$$\chi_\rho(uv) = \sum_{\substack{t \in \mathrm{K} \\ t^{-1}uvt \in \mathrm{H}}} \chi_\theta(t^{-1}uvt) = \begin{cases} k\chi_\theta(u) & \text{if } v = 1 \\ 0 & \text{if } v \neq 1 \end{cases}$$

On the other hand, we know that

$$\chi_{\theta \otimes r_\mathrm{K}}(uv) = \chi_\theta(u)\chi_{r_\mathrm{K}}(v) = \begin{cases} k\chi_\theta(u) & \text{if } v = 1 \\ 0 & \text{if } v \neq 1 \end{cases}$$

so ρ is isomorphic to $\theta \otimes r_\mathrm{K}$.

5 Examples

Convention We use $\mathfrak{A}_n$ (resp. $\mathfrak{S}_n$, C_n, D_n) to denote the n-th alternating (resp. symmetric, cyclic, dihedral) groups.

Exercise 5.1 We have the following identity :

$$s^\epsilon r^\alpha s^\eta r^\beta r^{-\alpha} s^\epsilon = \begin{cases} s^\eta r^\beta & \text{if } (\eta, \epsilon) = (0,0) \\ s^\eta r^{-\beta} & \text{if } (\eta, \epsilon) = (0,1) \\ s^\eta r^{\beta - 2\alpha} & \text{if } (\eta, \epsilon) = (1,0) \\ s^\eta r^{-(\beta-2\alpha)} & \text{if } (\eta, \epsilon) = (1,1) \end{cases}$$

The rest follows easily from this calculation.

Exercise 5.2 Write $w = \exp(2\pi i/n)$. We have

$$\chi_h(s^\epsilon r^\alpha) = \delta_{\epsilon 0}(w^{hk} + w^{-hk})$$

So we get

$$\begin{aligned} (\chi_h \cdot \chi_{h'})(s^\epsilon r^\alpha) &= \left(\delta_{\epsilon 0}(w^{hk} + w^{-hk})\right)\left(\delta_{\epsilon 0}(w^{h'k} + w^{-h'k})\right) \\ &= \delta_{\epsilon 0}\left((w^{(h+h')k} + w^{-(h+h')k}) + (w^{(h-h')k} + w^{-(h-h')k})\right) \\ &= (\chi_{h+h'} + \chi_{h-h'})(s^\epsilon r^\alpha) \end{aligned}$$

this establishes $\chi_h \cdot \chi_{h'} = \chi_{h+h'} + \chi_{h-h'}$. When $h = h'$, we get

$$\chi_h \cdot \chi_h = \chi_{2h} + \chi_0 = \chi_{2h} + \psi_1 + \psi_2$$

where we have used the identity $\chi_0 = \psi_1 + \psi_2$ destablished in the text.
Let us show that $\psi_2 = \mathrm{Alt}^2 \chi_h$. It suffices to show that

$$\psi_2(s^\epsilon r^\alpha) = \frac{1}{2}\left(\chi_h(s^\epsilon r^\alpha)^2 + \chi_h((s^\epsilon r^\alpha)^2)\right), \quad \epsilon \in \{0,1\}, \alpha \in \{0, \dots, h-1\}$$

Notice that

$$(s^\epsilon r^\alpha)^2 = \begin{cases} 1 & \text{if } \epsilon = 1 \\ r^{2\alpha} & \text{if } \epsilon = 0 \end{cases}$$

we get

$$\frac{1}{2}\left(\chi_h(s^\epsilon r^\alpha)^2 + \chi_h((s^\epsilon r^\alpha)^2)\right) = \begin{cases} \frac{1}{2}(0-2) = -1 & \text{if } \epsilon = 0 \\ \frac{1}{2}\left((w^{hk} + w^{-hk})^2 - (w^{2hk} + w^{-2hk})\right) = 1 & \text{if } \epsilon = 1 \end{cases}$$

so we are done.

Exercise 5.4 Since H is normal, we get

$$(\mathrm{Ind}\,\theta)(u) = \begin{cases} 0 & \text{if } u \notin \mathrm{H} \\ \frac{1}{4}\sum_{v \in \mathrm{G}} \theta(v^{-1}uv) & \text{if } u \in \mathrm{H} \end{cases}$$

Decompose G as the disjoint union of the cosets H, tH, t^2H, we get

$$\begin{aligned}
(\operatorname{Ind}\theta)(x) &= \frac{1}{4}\sum_{v\in G}\theta(v^{-1}xv) \\
&= \frac{1}{4}\left(\sum_{v\in H}+\sum_{v\in tH}+\sum_{v\in t^2H}\right)\theta(v^{-1}xv) \\
&= \frac{1}{4}\left(\sum_{v\in H}\theta(v^{-1}xv)+\theta(v^{-1}yv)+\theta(v^{-1}zv)\right) \\
&= (\theta(x)+\theta(y)+\theta(z)) = -1
\end{aligned}$$

and that

$$(\operatorname{Ind}\theta)(1) = \deg(\operatorname{Ind}\theta) = \deg(\theta)[G : H] = 3\deg(\theta) = 3$$

hence, this character coincides with ψ.

6 The Group Algebra

Exercise 6.1 Let us show (i) implies (ii). Define the nontrivial proper submodule V as given in the hint, suppose it has a complementary summand W. Take some nontrivial element

$$x = \sum_{t \in G} a_t t \in W, \quad \sum_{t \in G} a_t \neq 0$$

then the element $\sum_{s \in G} sx$ is non-trivial and lies in the intersection of W and V.

Exercise 6.2 The formula is bilinear in each argument, so it suffices to treat the case where $u, v \in G$. By definition, we get

$$< u, v >= g\delta_{uv^{-1}}$$

so we only need to show

$$\delta_{uv^{-1}} = \frac{1}{g} \sum_i n_i \chi_i (uv^{-1})$$

which is clear by the character theory of the regular representation of G.

Exercise 6.3

(a) Since U is finite and contains G, the first assertion is clear. For the identity

$$\mathrm{Tr}_{W_i}(\rho_i(s^{-1})u_i)^* = \mathrm{Tr}_{W_i}(u_i' \rho_i(s))$$

recall that we may choose an eigenbasis for W_i with respect to ρ_i, so we may safely say that $\mathrm{Tr}_{W_i}(\rho_i(s^{-1})u_i)$ is the sum of eigenvalues. Observe also that

$$(\rho_i(s^{-1})u_i)(u_i'\rho_i(s)) = 1$$

so we deduce the identity by using an argument like

$$\mathrm{Tr}_{W_i}(\rho_i(s^{-1})u_i)^* = \sum \lambda_j^* = \sum \lambda_j^{-1} = \mathrm{Tr}_{W_i}(u_i'\rho_i(s))$$

The identity

$$\mathrm{Tr}_{W_i}(u_i'\rho_i(s)) = \mathrm{Tr}_{W_i}(\rho_i(s)u_i')$$

is clear by noticing that $u_i'\rho_i(s), \rho_i(s)u_i'$ are conjugates.

(b) By (a) and **Exercise 6.1**, we see that

$$\sum_{t \in G} |u(t)|^2 = \sum_{t \in G} u(t)u'(t^{-1}) = \frac{1}{g} < u, u' >= \frac{1}{g} \sum_{i=1}^h n_i \, \mathrm{Tr}_{W_i}(uu') = \frac{1}{g} \sum_{i=1}^h n_i^2 = 1$$

(c) Obvious.

(d) Take $U = \mathbb{Z}[G]$.

Exercise 6.4 By the computation

$$\omega_k(p_i) = \frac{1}{n_k}\sum_{t\in G}\frac{n_i}{g}\chi_i(t^{-1})\chi_k(t) = \delta_{ik}$$

we see that the image of the elements p_i under the isomorphism described in proposition 13 form a basis of $\mathbb{C}^h$, so p_i form a basis, and it also follows from the computations

$$\omega_k(p_ip_j) = \omega_k(p_i)\omega_k(p_j) = \delta_{ik}\delta_{jk} = \delta_{ij}\delta_{ik} = \begin{cases}\omega_k(p_i) & \text{if } i = j\\ 0 & \text{if } i \neq j\end{cases}$$

$$\omega_k(\sum_i p_i) = \sum_i \delta_{ik} = 1 = \omega_k(1)$$

that the other required identities are verified.

Exercise 6.5 Let ν be a homomorphism from the center of $\mathbb{C}[G]$ to $\mathbb{C}$. **Exercise 6.4** says

$$\begin{cases}\nu(p_i) \in \{0, 1\}\\ \sum_i \nu(p_i) = 1\end{cases}$$

It is easy to deduce from this that ν is just one of the ω_i.

Exercise 6.6 Firstly, since

$$\bigoplus_i \mathbb{Z}e_i \subseteq \text{Cent.}(\mathbb{C}[G])$$

we get $\bigoplus_i \mathbb{Z}e_i \subseteq \text{Cent.}(\mathbb{Z}[G])$. For the reverse, choose some

$$u = \sum_{t\in G} u(t)t \in \text{Cent.}(\mathbb{Z}[G])$$

Suppose $t, t' \in G$ lie in the same conjugacy class, say,

$$st = t's$$

for some $s \in G$, then by $us = su$, we get $u(t) = u(t')$. This observation shows $u \in \bigoplus_i \mathbb{Z}e_i$.

Exercise 6.7 By the hint, we are done by applying the triangle inequality.

Exercise 6.8 (For each nonnegative integer N, we define $\xi_N := \exp(2\pi i/N)$.)
Let $f(x)$ be the minimal polynomial of a over $\mathbb{Q}$, then $f(x) \in \mathbb{Z}[x]$ by integrality.
In order to show $|A| \leq 1$, it suffices to show the following :

Claim : The conjugates of a (roots of $f(x)$) over $\mathbb{Q}$ all have length ≤ 1.

Since λ_i are roots of unities, may take an integer $N \gg 0$ such that $\lambda_i \in \{\xi_N^0, \dots, \xi_N^{N-1}\}$.
We have the following diagram of field extensions :

$$\mathbb{Q}(\xi_N) \;\text{———}\; \mathbb{Q}(a) \;\text{———}\; \mathbb{Q}$$

Recall that the Galois group of the cyclotomic extension $\mathbb{Q}(\xi_N)/\mathbb{Q}$ is given by

$$\text{Gal}(\mathbb{Q}(\xi_N)/\mathbb{Q}) = \{\theta_d : \xi_N \mapsto \xi_N^d | (d, N) = 1\} \simeq (\mathbb{Z}/N\mathbb{Z})^\times$$

Define a polynomial

$$F(x) := \prod_{\sigma\in \mathrm{Gal}(\mathbb{Q}(\xi_N)/\mathbb{Q})} (x-\sigma(a)) = \prod_{d:(d,N)=1} (x-\theta_d(a))$$

then $F(x) \in \mathbb{Q}[x]$ and $f(x)|F(x)$ by the property of minimal polynomials.
By the definition of θ_d, we have the following calculation :

$$\theta_d(a) = \theta_d(\frac{1}{n}(\sum_{i=1}^{h} \lambda_i)) = \frac{1}{n}(\sum_{i=1}^{h} \lambda_i^d)$$

so $|\theta_d(a)| \leq 1$, so the roots of $F(x)$ (hence those of $f(x)$) all have length ≤ 1; this proves the claim.
By the definition of $f(x)$, we see that the constant term of $f(x)$ is $\pm A$, so we get $A \in \mathbb{Z}$.
Since $|A| \leq 1$, $A \in \{0, \pm 1\}$. If $A = 0$, $a = 0$; If $A = \pm 1$, $|a| = 1$ by our claim, and hence $\lambda_i = a$ for all i.

Exercise 6.9 Since the e_i lies in the center of the group algebra, it follows immediately that

$$\frac{c(s)}{n}\chi(s)$$

are all algebraic integers (for each $s \in G$). For the second assertion, we claim that

$$\frac{1}{n}\chi(s)$$

is an algebraic integer; this follows from the fact that $\chi(s)/n$ is a $\mathbb{Z}$-combination of the algebraic integers $\chi(s)$ and $c(s)\chi(s)/n$ by using the assumption that $(c(s), n) = 1$.
By **Exercise 6.8** , the eigenvalues of ρ_s are all the same given $\chi(s) \neq 0$, so ρ_s is a homothety.

Exercise 6.10 By the character theory of the regular representation of G, we get

$$1 + \sum_{\chi\neq 1} \chi(1)\chi(s) = 0$$

whenever $s \neq 1$. Notice that suppose every irreducible character of G satisfies

$$\text{Either } \chi(s) = 0 \text{ or } p|\chi(1)$$

then we see that

$$-1 = \sum_{\chi\neq 1} \chi(1)\chi(s) = \sum_{\substack{\chi\neq 1 \\ p|\chi(1)}} \chi(1)\chi(s)$$

This formula then exhibits $1/p$ as an algebraic integer in view of Proposition 15.
For the second assertion, notice that given ρ, χ satisfying the condition, we see that

$$\chi(s) \neq 0 \text{ and } (\chi(1), c(s)) = 1$$

so in view of **Exercise 6.9**, $\rho(s)$ is just a homothety.
For the last part, notice first that since χ isn't trivial, $N \neq G$. Next, since

$$sN \in \text{Cent.}(G/N) \text{ iff the commutator } [st] \text{ with any } t \in G \text{ lies in G iff } \rho([st]) = 1 \text{ for all } t \in G$$

we see that the last assertion is clearly true by the observation that $\rho(s)$ is a homothety.

7 Induced Representations; Mackey's Criterion

Exercise 7.1 It is easier to verify via the universal property of Ind.

(a) Factoring α through image, may assume α is a surjection and that α is a projection

$$\alpha : \ H \longrightarrow H/N$$

for some normal subgroup N of H. Recall that

$$\mathrm{Ind}_\alpha(W) = W \otimes_{\mathbb{C}[H]} \mathbb{C}[H/N]$$

Let us show first that this space is isomorphic to the one described in (b). Call this space V. Notice that we can define maps of abelian groups

$$V \underset{\nu}{\overset{\mu}{\rightleftarrows}} \mathrm{Ind}_\alpha(W)$$

given by

$$\mu(v) = v \otimes N, \quad \nu(w \otimes hN) = wh$$

It can be checked that the maps are well-defined, [1] inverse to each other, and are all G-maps. Now, let us construct natural bijections between

$$\mathrm{Hom}^H(W, \mathrm{Res}_\alpha(E)) \underset{\eta}{\overset{\xi}{\rightleftarrows}} \mathrm{Hom}^{H/N}(V, E)$$

That is : restriction and induction are adjoints. We verify via universal property. Suppose we are given an H-map

$$g : \ W \longrightarrow \mathrm{Res}_\alpha(E)$$

we want to show that there exists a unique G-map f

$$f : \ V \longrightarrow E$$

so that $\mathrm{Res}_\alpha(f) = g$. There is only one way to proceed : define f by

$$V \longrightarrow W \longrightarrow \mathrm{Res}_\alpha(E) = E$$

We also omit the verification that these bijections are natural. To deduce the formula

$$< \psi, \mathrm{Res}_\alpha(\varphi) >_H = < \mathrm{Ind}_\alpha \psi, \varphi >_{H/N}$$

notice that these quantities are just dimensions of the Hom-sets.

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[1] To show legitimacy, let us briefly recall a construction of tensor product of two algebras over a ring. Suppose given right-R algebra A and left-R algebra B, define the underlying set of the set $A \otimes_R B$ to be the same as that of the tensor product of R-modules $A \otimes_R B$, with algebra structure given by $(a \otimes b)(a' \otimes b') = (aa' \otimes bb')$. Hence, well-definedness of the map ν can be verified using this description.
On the other hand, we know that as an abelian group, $A \otimes_R B$ can be described as the quotient group $(A \times B)/R$, where R is generated over $\mathbb{Z}$ by the elements of the form $(a+a', b) - ((a, b) + (a', b))$, $(a, b+b') - ((a, b) + (a, b'))$, $(ar, b) - (a, rb)$. This gives us a structural map (only as a homomorphism of abelian groups !) of $A \otimes_R B$ given by

$$A \times_R B \longrightarrow A \otimes B, \quad (a, b) \longmapsto a \otimes b$$

This description allows one to check that μ is well-defined.

(b) We have seen the identification between V and $\mathrm{Ind}_\alpha(\mathrm{W})$ in (a).
For the formula, define a class function on H/N by

$$\mu(\sigma) = \frac{1}{n} \sum_{\substack{t\in \mathrm{H} \\ \alpha(t)=\sigma}} \psi(t)$$

then by applying the generalized Frobenius reciprocity, we get

$$< \mathrm{Ind}_\alpha \psi, \varphi >_{\mathrm{H/N}} = < \psi, \mathrm{Res}_\alpha \varphi >_{\mathrm{H}} = \frac{1}{h} \sum_{t\in \mathrm{H}} \psi(t)\varphi(\alpha(t)^{-1}) = \frac{1}{h} \sum_{\sigma\in \mathrm{H/N}} \sum_{\substack{t\in \mathrm{H} \\ \alpha(t)=\sigma}} \psi(t)\varphi(\sigma^{-1})$$

$$= \frac{1}{h/n} \sum_{\sigma \in \mathrm{H/N}} \left(\frac{1}{n} \sum_{\substack{t\in \mathrm{H} \\ \alpha(t)=\sigma}} \psi(t) \right) \varphi(\sigma^{-1}) = \frac{1}{h/n} \sum_{\sigma\in \mathrm{H/N}} \mu(\sigma)\varphi(\sigma^{-1}) = < \mu, \varphi >_{\mathrm{H/N}}$$

so we deduce that $\mathrm{Ind}_\alpha \psi = \mu$ by using the non-degeneracy of $<,>_{\mathrm{H/N}}$.

Exercise 7.2 By the formula of induced representations, we have

$$\mathrm{Ind}\, 1(s) = \sum_{\substack{t\in \mathrm{G} \\ t^{-1}st\in \mathrm{H}}} 1(t^{-1}st) = |\{t \in \mathrm{G} : t^{-1}st \in \mathrm{H}\}| = |\{t \in \mathrm{G} : st\mathrm{H} = t\mathrm{H}\}|$$

By description of a permutation representation, this quantity is just $\chi(s)$.
To show that $\psi := \chi - 1$ is a character, notice that

$$< \chi, 1 >_{\mathrm{G}} = < \mathrm{Ind}\, 1, 1 >_{\mathrm{G}} = < 1, \mathrm{Res}\, 1 >_{\mathrm{H}} = < 1, 1 >_{\mathrm{H}} = 1$$

so ψ is a character. We refer the remaining assertions to **Exercise 2.6**.

Exercise 7.3

(a) By the description of N, we see that

$$\mathrm{G} \setminus \mathrm{N} = \bigcup_{t\in \mathrm{G}} t\mathrm{H}t^{-1} = \bigcup_{r\in \mathrm{R}} r\mathrm{H}r^{-1}$$

where R a set of representatives in G for the cosets G/H. Applying the condition that H is Frobenius, we see that this set has cardinality $g - (g/h - 1)$, so we are done.

(b) It is easy to see that the property that an element lies in N or not is stable under conjugation. The condition that H is Frobenius and that $\bar{f}$ is a class function gives a unique extension of f to $\mathrm{G} \setminus \mathrm{N}$, and since $\mathrm{N} \cap \mathrm{H} = \emptyset$, extending by $f(1)$ to N is also legitimate.

(c) By the formula of the characters of induced class functions, we get

$$(\mathrm{Ind}\, f - f(1)\psi)(s) = (\mathrm{Ind}\, f - \mathrm{Ind}\, 1)(s) + f(1) = \frac{1}{h} \sum_{\substack{t\in \mathrm{G} \\ tst^{-1}\in \mathrm{H}}} \big(f(tst^{-1}) - f(1)\big) + f(1)$$

Suppose $s \in \mathrm{N}$, we get from the definition of N that

$$\{t \in \mathrm{G} : tst^{-1} \in \mathrm{H}\} = \emptyset$$

hence the result if $f(1)$. Now suppose $s \notin \mathrm{N}$ and that $tst^{-1} \in \mathrm{H}$, then

$$\{t \in \mathrm{G} : tst^{-1} \in \mathrm{H}\} = s\mathrm{H}$$

so we get

$$\frac{1}{h} \sum_{\substack{t \in \mathrm{G} \\ tst^{-1} \in \mathrm{H}}} f(tst^{-1}) = f(tst^{-1}), \quad \frac{1}{h} \sum_{\substack{t \in \mathrm{G} \\ tst^{-1} \in \mathrm{H}}} f(1) = f(1)$$

so the result is $f(tst^{-1})$. Therefore, $\bar{f} = \operatorname{Ind} f - f(1)\psi$.

(d) We can do the computation explicitly as follows : By definition, we have

$$< \bar{f_1}, \bar{f_2} >_{\mathrm{G}} = \frac{1}{g} \left(\sum_{s \in \mathrm{G} \setminus \mathrm{N}} + \sum_{s \in \mathrm{N}} \right) \bar{f_1}(s) \bar{f_2}(s^{-1})$$

For the first summation, by the explicit description of the set $\mathrm{G} \setminus \mathrm{N}$, we have

$$\begin{aligned} \sum_{s \in \mathrm{G} \setminus \mathrm{N}} \bar{f_1}(s) \bar{f_2}(s) &= \left((\sum_{r \in \mathrm{R}} \sum_{t \in \mathrm{H}}) - (\frac{g}{h} - 1)(\sum_{h=e}) \right) f_1(t) f_2(t^{-1}) \\ &= \frac{g}{h} \left(\sum_{t \in \mathrm{H}} f_1(t) f_2(t^{-1}) \right) - (\frac{g}{h} - 1) f_1(1) f_2(1) = g < f_1, f_2 >_{\mathrm{H}} -(\frac{g}{h} - 1) f_1(1) f_2(1) \end{aligned}$$

On the other hand, we have

$$\sum_{s \in \mathrm{N}} \bar{f_1}(s) \bar{f_2}(s^{-1}) = (\frac{g}{h} - 1) f_1(1) f_2(1)$$

Combining these together, we get $< \bar{f_1}, \bar{f_2} >_{\mathrm{G}} = < f_1, f_2 >_{\mathrm{H}}$.

(e) Under the assumption that f is an irreducible character, we have the following :

$$< \bar{f}, \bar{f} >_{\mathrm{G}} = < f, f >_{\mathrm{H}} = 1$$

$$\bar{f}(1) = (\operatorname{Ind} f)(1) - f(1)\psi(1) = f(1)(g/h) - f(1)(g/h - 1) = f(1)$$

On the other hand, suppose given an irreducible character g of G, we have by reciprocity that

$$< \bar{f}, g >_{\mathrm{G}} = < \operatorname{Ind} f - f(1)((\operatorname{Ind} 1) - 1), g >_{\mathrm{G}} = < f - f(1), \operatorname{Res} g >_{\mathrm{H}} + f(1) < \operatorname{Ind} f, 1 >_{\mathrm{G}}$$

hence $< \bar{f}, g >_{\mathrm{G}}$ are all integers; this shows f is $\mathbb{Z}$-combination of irreducible characters.
To show the last assertion, it suffices by **Exercise 6.7** to show that $\bar{f}(s) = \bar{f}(1)$ for $s \in \mathrm{N}$, but this is by the definition of $\bar{f}$.

(f) Let $\rho_1, \ldots, \rho_h$ be the irreducible characters of H with corresponding characters $\chi_1, \ldots, \chi_h$. By (e), each $\bar{\chi}_i$ corresponds to some representation of $\bar{\rho}_i$ of G. Define a subgroup of G

$$\mathrm{N}^+ := \bigcap_{i=1}^{h} \operatorname{Ker}(\bar{\rho}_i)$$

so N^+ is a normal subgroup. By (e), we already know that $\mathrm{N} \cup \{1\} \subseteq \mathrm{N}^+$.
The reverse inclusion is clear by recalling that the irreducible characters of H span the space of class functions on H along with the explicit description of $\mathrm{G} \setminus \mathrm{N}$ and $\bar{f}$.
Let us show $\mathrm{HN}^+ = \mathrm{G}$, but this is clear by the fact that $\mathrm{H} \cap \mathrm{N}^+ = \{1\}$ and that $|\mathrm{H}||\mathrm{N}^+| = |\mathrm{G}|$.

(g) Assume first that H is Frobenius. Suppose there exists some $(s,t) \in \mathrm{H} \times \mathrm{A}$ such that $st = ts$, we get $s = tst^{-1} \in \mathrm{H} \cap t\mathrm{H}t^{-1}$. Suppose $t \neq 1$, then $s = 1$.
Conversely, the action described in the question can be identified with another one : if we regard A as the set of left cosets G/H, then the conjugation action of H on A can be described as the action of H on G/H by multiplying on left, and the coset representatives of G/H can be chosen to be A. Now let us make the following assumption :

Suppose there exists $t \in \mathrm{A}$ and $s \in \mathrm{H}$ both not equal to 1 with $t^{-1}st \in \mathrm{H}$.

then we have $st\mathrm{H} = t\mathrm{H}$, so the action of H on G/H isn't free, a contratdiction.

Exercise 7.4 The idea is to use Mackey's irreducibility criterion.
Fix $\mathrm{M} \in \mathrm{G} \setminus \mathrm{H}$, form the group $\mathrm{H_M} = \mathrm{MHM}^{-1} \cap \mathrm{H}$, and let $\mathrm{X} \in \mathrm{H_M}$. Write M, X as

$$\mathrm{M} = \{\begin{bmatrix} p & q \\ r & s \end{bmatrix}, \quad \mathrm{X} = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

satisfying the relations

$$ps - qr = 1, r \neq 0, \quad ad = 1, c = 0$$

Using the condition that $\mathrm{X} \in \mathrm{MHM}^{-1}$, we have

$$\mathrm{M}^{-1}\mathrm{XM} = \begin{bmatrix} s & -q \\ -r & p \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} \in \mathrm{H}$$

Let us calculate the 1st column of the resulting matrix :

$$\mathrm{M}^{-1}\mathrm{XM} = \begin{bmatrix} sa - qc & sb - qd \\ -ra + pc & -rb + pd \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} psa + rsb - qrd & * \\ -pra - r^2b + prd & * \end{bmatrix} \in \mathrm{H}$$

where we have plugged in the condition $c = 0$. For brevity, let us write

$$\begin{bmatrix} psa + rsb - qrd \\ -pra - r^2b + prd \end{bmatrix} = \begin{bmatrix} \mathrm{A} \\ \mathrm{B} \end{bmatrix}$$

We have the following :

$$\begin{cases} \mathrm{B} & = 0 \\ \mathrm{A} + r^{-1}s\mathrm{B} & = (ps - qr)d \end{cases}$$

we get $\mathrm{A} = a^{-1}$ using the identities $ad = 1, ps - qr = 1$.
From this preliminary calculation, we find that

$$\chi_\omega^{\mathrm{M}} = \mathrm{Res_M}(\chi_\omega^{-1})$$

for each $\mathrm{M} \in \mathrm{G} \setminus \mathrm{H}$. Therefore, we have

$$< \mathrm{Res_M}(\chi_\omega), \chi_\omega^{\mathrm{M}} >_{\mathrm{H_M}} = < \mathrm{Res_M}(\chi_\omega), \mathrm{Res_M}(\chi_\omega^{-1}) >_{\mathrm{H}_m} = \frac{1}{|\mathrm{H}_m|} \sum_{\mathrm{X} \in \mathrm{H}_m} \chi_\omega^2(\mathrm{X})$$

Since $|\chi_\omega(\mathrm{X})| \leq 1$ for each $\mathrm{X} \in \mathrm{H}$, we see that $\mathrm{Res_M}(\chi_\omega)$ is isomorphic to χ_ω^{M} iff $\mathrm{Res_M}(\chi_\omega)^2 = 1$.
Letting M vary, we obtain from Mackey's criterion that χ_ω is irreducible iff $\chi_\omega^2 = 1$.

8 Example of Induced Representations

Exercise 8.1 The statement that
$$a = \sum_i \frac{h}{h_i}$$
follows from the orbit-stabilizer theorem applied to the action of H on A. As for
$$\sum_\rho (\deg \theta_{i,\rho})^2 = \frac{h^2}{h_i}$$
is by noticing that
$$\deg \theta_{i,\rho} = \deg \left(\mathrm{Ind}_{G_i}^{G}(\chi_i \otimes \bar{\rho})\right) = [G : G_i] \deg(\rho(1))$$
Since we also have
$$\sum_\rho (\deg \rho)^2 = h_i$$
the identity is proved. Finally, for the proof of (c) of Proposition 25, notice that
$$g = ha = h(\sum_i \frac{h}{h_i}) = \sum_i \frac{h^2}{h_i} = \sum_i \left(\sum_\rho \deg(\theta_{i,\rho})^2\right)$$
so by (a),(b) of Proposition 25, we are done.

Exercise 8.2 .

Exercise 8.3 The group D_n has a filtration
$$0 \quad \trianglelefteq \quad C_n \quad \trianglelefteq \quad D_n$$
which exhibits D_n as a supersolvable group. Let us consider nilpotency.
By the calculation in **Exercise 5.1** , we get
$$\mathrm{Cent}.(D_n) = \begin{cases} \{1\} & \text{if } n \text{ if odd.} \\ \{1, r^{n/2}\} & \text{if } n \text{ if even.} \end{cases}$$
so this shows for D_n to be nilpotent, it is necessary that n be a power of 2.
For sufficiency, let $n = 2^k$, then one can consider a filtration
$$\{1\} \quad = \quad C_1 \quad \trianglelefteq \quad C_2 \quad \trianglelefteq \quad \ldots \quad \trianglelefteq \quad C_{2^k} \quad \trianglelefteq \quad D_{2^k}$$
This filtration exhibits D_n as a nilpotent group.

Exercise 8.4 By **Exercise 8.5** , it suffices to show
$$\mathfrak{A}_4 \text{ isn't supersolvable, } \mathfrak{S}_4 \text{ is solvable.}$$
The fact that $\mathfrak{S}_4$ is solvable can be seen from the subnormal series
$$\{1\} \quad \trianglelefteq \quad H \quad \trianglelefteq \quad \mathfrak{A}_4 \quad \trianglelefteq \quad \mathfrak{S}_4$$
where H is the normal subgroup of $\mathfrak{A}_4$ consisting of 1 and product of disjoint 2-cycles.
Conversely, recall that the conjugacy classes of $\mathfrak{A}_4$ are
$$\{1\}, H \setminus \{1\}, tH, t^2H$$
It is easy to see that the only normal subgroups of $\mathfrak{A}_4$ are H, $\{1\}$, but H isn't cyclic.

Exercise 8.5 Suppose we are given a defining series of subgroups of G exhibiting the solvability (resp. supersolvability, nilpotency)

$$\{1\} = G_0 \trianglelefteq \dots G_{n-1} \trianglelefteq G_n = G$$

and another subgroup H of G, one get - by defining $H_i = G_i \cap H$ - a series of H

$$\{1\} = H_0 \subseteq \dots H_{n-1} \subseteq H_n = H$$

We claim that these series really exhibits solvability (resp. supersolvability, nilpotency) of H.
As a preliminary step, let G', H be subgroups of G and N a normal subgroup of G', observe that

- (Normality) $N \cap H$ is still a normal subgroup of $G' \cap H$.
- (Factor groups) We have
$$\frac{G' \cap H}{N \cap H} \xrightarrow{\sim} \frac{(G' \cap H)N}{N} \longrightarrow \frac{G'}{N}$$
- (Center) We have the relation $\mathrm{Cent.}(G' \cap H) \leq \mathrm{Cent.}(G')$.

These observations shows that a defining series for solvability (resp. supersolvability, nilpotency) is still a defining series for solvability (resp. supersolvability, nilpotency) under passage to subgroups.
For quotients, suppose given a normal subgroup N of G, we have a series

$$N = NG_0 \subseteq \dots NG_{n-1} \subseteq NG_n = G$$

As a preliminary step, let $H \leq G$, $N \trianglelefteq G$, $K \trianglelefteq H$, observe that

- (Normality) NK is still a normal subgroup of NH.
- (Factor groups) We have
$$\frac{NH}{NK} \xleftarrow{\sim} \frac{H}{NK \cap H} \longleftarrow \frac{H}{K}$$
- (Center) We have the relation $(N\,\mathrm{Cent.}(G'))/N \leq \mathrm{Cent.}(NG'/N)$.

one sees that after passing the above series to quotients

$$\{1\} = NG_0/N \subseteq \dots NG_{n-1}/N \subseteq NG_n/N = G/N$$

we get a defining series for solvability (resp. supersolvability, nilpotency) of G/N.

Exercise 8.6

(i) As indicated in the hint, if $c(s) \equiv 0 (\mathrm{mod}\, p)$ for all $s \neq 1$, we would have $q | (g-1)$, this shows existence of s. By **Exercise 6.10**, get a nontrivial irreducible character χ of G with

$$\chi(s) \neq 0, \quad \chi(1) \not\equiv 0 (\mathrm{mod}\, p)$$

so by taking kernels, one gets a nontrivial proper normal subgroup of G.

(ii) Do induction. By (i), G always have a nontrivial proper normal subgroup N. Take exact sequence

$$0 \longrightarrow N \longrightarrow G \longrightarrow G/N \longrightarrow 0$$

By inductive hypothesis and theorem 14, get solvability of G/N and N and hence G.
Base case is done in theorem 14.

(iii) Consider the groups $\mathfrak{A}_4, \mathfrak{S}_4$.

(iv) Since $\mathfrak{A}_5$ (resp. $\mathfrak{A}_6$) is a normal simple subgroup of $\mathfrak{S}_5$ (resp. $\mathfrak{S}_6$), $\mathfrak{S}_5$ (resp. $\mathfrak{S}_6$), is nonsolvable. For $\mathbf{GL}_2(\mathbb{F}_7)$, we note that $\mathbf{PSL}(2, 7)$ is simple, non-abelian (I don't know how to show this).

Exercise 8.7 As a preliminary step, notice that by surjectivity of the map

$$G \longrightarrow G/H$$

such a group P must be contained in the inverse image Q of P_H under the map.

(a) Take any Sylow p-subgroup P of Q. Schematically :

$$\begin{tikzcd}[column sep=small] & & H \arrow[dr, dash] & \\ G \arrow[r, dash] & Q \arrow[ur, dash] \arrow[dr, dash, "*"'] & & \{1\} \\ & & P \arrow[ur, dash] & \end{tikzcd} \qquad \begin{tikzcd} G/H \arrow[r, dash, "*"] & P_H \arrow[r] & \{1\} \end{tikzcd}$$

where the "$*$" means "the right one is a Sylow p-subgroup of the left one". Notice that :

- A p-subgroup is p-Sylow iff the index is prime to p.
- P is a Sylow p-subgroup of G : we already know P is a p-group. Now by

$$[G : P] = [G : Q][Q : P] = [G/H : P_H][Q : P]$$

  this gives $([G : P], p) = 1$.
- PH/H is a subgroup of $Q/H = P_H$.
- PH/H is a Sylow p-subgroup : we already know that it is a p-group. Now by

$$[G/H : PH/H] = [G : PH] = \frac{[G : P]}{[PH : P]}$$

  we get $([G/H : PH/H], p) = 1$.

Therefore, PH/H - the image of P under the projection map from G to G/H - is P_H.

(b) The case where H is a p-group is clear : Q already had the correct cardinality.
For the case where $H \leq \text{Cent.}(G)$, let us first reduce to the case where $(|H|, p) = 1$:
take a Sylow p-subgroup of H - denote it as K, then one has quotient maps :

$$G \longrightarrow G/K \longrightarrow G/H$$

By induction, we can find a unique group in G/K, and by (a), a unique group in G.
Now we may safely assume $(|H|, p) = 1$.
By second isomorphism theorem, if P is a solution, it must satisfy $P \cap H = 1$.
In this sense, if P, P' are solutions, we have $P_1H = P_2H = Q$ and $P_1 \cap H = P_2 \cap H = \{1\}$.
From this, we define a homomorphism φ from P_H to H as follows :

- Let $x \in Q$, it has unique expressions

$$x = x_1 s_1 = x_2 s_2, \quad x_1 \in P_1, \quad x_2 \in P_2, \quad s_1, s_2 \in H_2$$

- Define first a map Φ from Q to H by $x \mapsto x_1^{-1} x_2$. This is well-defined.

- This map is a homomorphism : If we take

$$x = x_1 s_1 = x_2 s_2, \quad y = y_1 t_1 = y_2 t_2$$

for suitable x_i, y_i, s_i, t_i, we get first that

$$xy = x_1 y_1 s_1 t_1 = x_2 y_2 s_2 t_2$$

which gives us

$$\Phi(x)\Phi(y) = (x_1^{-1}x_2)(y_1^{-1}y_2) = y_1^{-1}x_1^{-1}x_2 y_2 = (x_1 y_1)^{-1}(x_2 y_2) = \Phi(xy)$$

where we have used the observation that $\mathrm{H} \leq \mathrm{Cent.}(\mathrm{G})$.
- This homomorphism clearly factors through H; this finishes the construction of φ.
- Schematically, these homomorphisms are organized as follows :

$$\begin{tikzcd} \mathrm{Q} \arrow[rr, "\Phi"] \arrow[dr, two heads] & & \mathrm{H} \\ & \mathrm{P_H} \arrow[ur, dashed, "\varphi"'] & \end{tikzcd}$$

Now since $(|\mathrm{H}|, p) = 1$ and $\mathrm{P_H}$ is a p-group, $\varphi = 1$, which gives $\mathrm{P}_1 = \mathrm{P}_2^{-1} = \mathrm{P}_2$.

Exercise 8.8 One can do an induction on the length of the defining series for nilpotency. For a length 1 series, that is, for the case where G is abelian, we are done by conjugacy of Sylow-ps. For length $n+1$, apply inductive hypothesis on G_n, then apply the base case to G/G_n.

Exercise 8.9 Denote the subgroup as T. We have the following :

$$|\mathrm{G}| = (k^n - 1)\ldots(k^n - k^{n-1}), \quad |\mathrm{T}| = k^{1+\ldots+(n-1)}$$

it is easy to see that $(|\mathrm{G}|/|\mathrm{T}|, p) = 1$.

Exercise 8.10 Recall from Proposition 25 that each irreducible representation of HA has form

$$\theta_{i,\rho} = \mathrm{Ind}_{\mathrm{H}_i\mathrm{A}}^{\mathrm{HA}}\left(\mathrm{Res}_{\mathrm{A}}^{\mathrm{H}_i\mathrm{A}}\chi_i \otimes \mathrm{Res}_{\mathrm{H}_i}^{\mathrm{H}_i\mathrm{A}}\rho\right)$$

where the operations Res are in the sense of restriction along projection maps to components in the sense of **Exercise 7.1** . By Theorem 16 (and the fact that subgroups of supersolvables are supersovables), ρ can be written as

$$\rho = \mathrm{Ind}_{\mathrm{K}}^{\mathrm{H}_i}\varphi$$

for some subgroup K of H_i and for some degree 1 representation of K. Focus on the representation

$$\mathrm{Res}_{\mathrm{H}_i}^{\mathrm{H}_i\mathrm{A}}\rho = \mathrm{Res}_{\mathrm{H}_i}^{\mathrm{H}_i\mathrm{A}}\mathrm{Ind}_{\mathrm{K}}^{\mathrm{H}_i}\varphi$$

We have diagram of group homomorphisms and corresponding constructions :

$$\begin{tikzcd} & \mathrm{H}_i & \\ \mathrm{H}_i\mathrm{A} \arrow[ur, two heads] & & \mathrm{K} \arrow[ul, hook'] \\ & \mathrm{KA} \arrow[ul, hook'] \arrow[ur, two heads] & \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{C}\mathrm{H}_i - \mathrm{mod} \arrow[dl, "\mathrm{Res}"] & \\ \mathbb{C}\mathrm{H}_i\mathrm{A} - \mathrm{mod} & & \mathbb{C}\mathrm{K} - \mathrm{mod} \arrow[ul, "\mathrm{Ind}"'] \arrow[dl, "\mathrm{Res}"] \\ & \mathbb{C}\mathrm{KA} - \mathrm{mod} \arrow[ul, "\mathrm{Ind}"'] & \end{tikzcd}$$

It can be checked that the following identity holds : [2]

$$\mathrm{Res}^{\mathrm{H}_i\mathrm{A}}_{\mathrm{H}_i} \mathrm{Ind}^{\mathrm{H}_i}_{\mathrm{K}} \varphi \simeq \mathrm{Ind}^{\mathrm{H}_i\mathrm{A}}_{\mathrm{KA}} \mathrm{Res}^{\mathrm{KA}}_{\mathrm{K}} \varphi$$

so plug it back to the expression of $\theta_{i,\rho}$, we get

$$\begin{aligned}
\theta_{i,\rho} &= \mathrm{Ind}^{\mathrm{HA}}_{\mathrm{H}_i\mathrm{A}} \left(\mathrm{Res}^{\mathrm{H}_i\mathrm{A}}_{\mathrm{A}} \chi_i \otimes \mathrm{Res}^{\mathrm{H}_i\mathrm{A}}_{\mathrm{H}_i} \rho\right) \\
&\simeq \mathrm{Ind}^{\mathrm{HA}}_{\mathrm{H}_i\mathrm{A}} \left(\mathrm{Res}^{\mathrm{H}_i\mathrm{A}}_{\mathrm{A}} \chi_i \otimes \mathrm{Res}^{\mathrm{H}_i\mathrm{A}}_{\mathrm{H}_i} \mathrm{Ind}^{\mathrm{H}_i}_{\mathrm{K}} \varphi\right) \\
&\simeq \mathrm{Ind}^{\mathrm{HA}}_{\mathrm{H}_i\mathrm{A}} \left(\mathrm{Res}^{\mathrm{H}_i\mathrm{A}}_{\mathrm{A}} \chi_i \otimes \mathrm{Ind}^{\mathrm{H}_i\mathrm{A}}_{\mathrm{KA}} \mathrm{Res}^{\mathrm{KA}}_{\mathrm{K}} \varphi\right) \\
&\simeq \mathrm{Ind}^{\mathrm{HA}}_{\mathrm{H}_i\mathrm{A}} \mathrm{Ind}^{\mathrm{H}_i\mathrm{A}}_{\mathrm{KA}} \left(\mathrm{Res}^{\mathrm{KA}}_{\mathrm{H}_i\mathrm{A}} \mathrm{Res}^{\mathrm{H}_i\mathrm{A}}_{\mathrm{A}} \chi_i \otimes \mathrm{Res}^{\mathrm{KA}}_{\mathrm{K}} \varphi\right) \\
&\simeq \mathrm{Ind}^{\mathrm{HA}}_{\mathrm{KA}} \left(\mathrm{Res}^{\mathrm{KA}}_{\mathrm{H}_i\mathrm{A}} \mathrm{Res}^{\mathrm{H}_i\mathrm{A}}_{\mathrm{A}} \chi_i \otimes \mathrm{Res}^{\mathrm{KA}}_{\mathrm{K}} \varphi\right)
\end{aligned}$$

This exhibits $\theta_{i,\rho}$ as the induced representation of some 1-dimensional representation.

Exercise 8.11 The assertion that G is the semidirect product of the normal subgroup E by an order 3 subgroup H can be deduced from the fact that conjugation by i, j, k for elements in E only changes the signs of 2 elements in it : one is itself, the other one is 1. We may take

$$\mu := \frac{1}{2}(1 + i + j + k), \quad \mathrm{H} :=< -\mu >= \{1, -\mu, \mu - 1\}$$

To show G is solvable, it suffices to show that E is solvable, but this is clear by

$$\{1\} \quad \trianglelefteq \quad \{\pm 1\} \quad \trianglelefteq \quad \{\pm 1, \pm i\} \quad \trianglelefteq \quad \{\pm 1, \pm i, \pm j, \pm k\} \quad = \quad \mathrm{E}$$

The isomorphism

$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} \mathbf{M}_2(\mathbb{C})$$

is given by the assignment [3]

$$1 \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad i \longmapsto \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \qquad j \longmapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad k \longmapsto \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

which gives us a representation of G by using the composition

$$\rho : \ \mathrm{G} \longrightarrow \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} \mathbf{M}_2(\mathbb{C})$$

The fact that ρ is irreducible can be seen from $< \chi, \chi >_{\mathrm{G}} = 1$ (to calculate, sum over each coset). Let's show that G has no subgroup of index 2. If N is one of such, it must be normal. In the diagram

$$\begin{tikzcd}
 & \mathrm{N} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathrm{G} \arrow[dr, dash] & & \mathrm{N} \cap \mathrm{E} \arrow[dl, dash] \\
 & \mathrm{E} &
\end{tikzcd}$$

[2] I think that this is true, but am too lazy to check it.

[3] Found this in Wikipedia : https://tinyurl.com/y24hdxzr

each line indicates normality. We have by 2nd isomorphism theorem that

$$[\mathrm{G}:\mathrm{N}] = [\mathrm{E}:\mathrm{N}\cap\mathrm{E}]$$

The normal subgroups in E with index 2 are

$$\mathrm{L}_i := \{\pm 1, \pm i\}, \quad \mathrm{L}_j := \{\pm 1, \pm j\}, \quad \mathrm{L}_k := \{\pm 1, \pm k\}$$

If $\mathrm{E}\cap\mathrm{N} = \mathrm{L}_i$, we get $\mathrm{N} = \mathrm{HL}_i$, but this will contradict normality of N : we have

$$\mu i = k\mu, \quad (-\mu)i(-\mu)^{-1} = -\mu i(\mu - 1) = -k(\mu^2 - \mu) = k \notin \mathrm{N}$$

The proof is similar for other cases.

Exercise 8.12 Let X be the set of irreducible characters and X′ be those with degree bigger than the degree of χ, we get the following :

$$\sum_{\chi'\in\mathrm{X}} \chi'(1)^2 = g \equiv 0 (\operatorname{mod} p)$$
$$\sum_{\chi'\in\mathrm{X}'} \chi'(1)^2 \equiv \sum_{\chi'\in\mathrm{X}'} 0 \equiv 0 (\operatorname{mod} p)$$

By taking the difference of the sum, one gets the desired assertion.

9 Artin's Theorem

Exercise 9.1 For each $s \in G$, we always have

$$\operatorname{Re}(\chi(s)) \leq \operatorname{Re}(\chi(1))$$

by arguing on eigenvalues. This gives

$$\operatorname{Re}(\varphi(s^{-1})\chi(s)) \geq \operatorname{Re}(\varphi(s^{-1})\chi(1))$$

so we get the following :

$$\operatorname{Re}(<\varphi,\chi>_G) = \operatorname{Re}(\frac{1}{g}\left(\sum_{s\in G}\varphi(s^{-1})\chi(s)\right)) \geq \operatorname{Re}(\frac{1}{g}\left(\sum_{s\in G}\varphi(1)\chi(s)\right)) = \varphi(1)\operatorname{Re}(<\chi,1>_G) = 0$$

Exercise 9.2 Suppose that

$$\chi = \sum_i n_i\chi_i, \quad n_i \in \mathbb{Z}$$

then $<\chi,\chi>_G = 1$ shows that

$$\sum_i n_i^2 = 1$$

so either χ or $-\chi$ is irreducible. The condition $\chi(1) > 0$ ensures that χ is the one.

Exercise 9.3

(a) Fix $s \in G$. Recall that we can diagonalize ρ_s. Let $\lambda_1, \ldots, \lambda_n$ be its eigenvalues. We get

$$\chi_\sigma^k(s) = \sum_{1\leq i(1)\leq\ldots\leq i(k)\leq n}\lambda_{i(1)}\ldots\lambda_{i(k)}, \quad \chi_\lambda^k(s) = \sum_{1\leq i(1)<\ldots<i(k)\leq n}\lambda_{i(1)}\ldots\lambda_{i(k)}$$

On the other hand, we also have

$$\det(1-\rho_s T)^{-1} = \left(\prod_{k=1}^{n}(1-\lambda_k T)\right)^{-1} = \prod_{k=1}^{n}\left(\sum_{l=1}^{\infty}\lambda_k^l T^l\right), \quad \det(1+\rho_s T) = \prod_{k=1}^{n}(1+\lambda_k T)$$

so by comparing degree-wise, we get

$$\det(1-\rho_s T)^{-1} = \sum_{k=1}^{\infty}\chi_\sigma^k T^k = \sigma_T(\chi), \quad \det(1+\rho_s T) = \sum_{k=1}^{\infty}\chi_\lambda^k T^k = \lambda_T(\chi)$$

For the last part, notice first that

$$\Psi^k(\chi)(s) = \chi(s^k) = \sum_{i=1}^{n}\lambda_i^k$$

We first prove the identities

$$m\chi_\sigma^n = \sum_{i=1}^{k}\Psi^k(\chi)\chi_\sigma^{m-k}, \quad m\chi_\lambda^m = \sum_{i=1}^{k}(-1)^{k-1}\Psi^k(\chi)\chi_\lambda^{m-k}$$

Let $[n]$ be the (lienarly-ordered set) $\{1, \ldots, n\}$, define the following sets :

$$\mathrm{S}(m,n) = \{\text{increasing functions } i : [m] \to [n]\}$$
$$\mathrm{L}(m,n) = \{\text{strictly increasing functions } i : [m] \to [n]\}$$

Let $f \in \mathrm{S}(m,n)$ or $\mathrm{L}(m,n)$, we also denote λ_f as $\lambda_{f(1)} \ldots \lambda_{f(n)}$. We get

$$\chi_\sigma^m(s) = \sum_{f \in \mathrm{S}(m,n)} \lambda_f, \quad \chi_\lambda^m(s) = \sum_{f \in \mathrm{L}(m,n)} \lambda_f$$

so we get the following :

$$\sum_{i=1}^{k} \Psi^k(\chi)\chi_\sigma^{m-k} = \sum_{i=1}^{n}\sum_{k=1}^{m} \lambda_i^k \left(\sum_{g \in \mathrm{S}(m-k,n)} \lambda_g \right)$$

One deduce that for each $f \in \mathrm{S}(m,n)$, λ_f occurs in this triple sum m times, hence

$$= \sum_{f \in \mathrm{S}(m,n)} \lambda_f = \chi_\sigma^m(s)$$

Now we consider the summation

$$\sum_{i=1}^{k} (-1)^{k-1}\Psi^k(\chi)\chi_\lambda^{m-k} = \sum_{i=1}^{n}\sum_{k=1}^{m} (-1)^{k-1}\lambda_i^k \left(\sum_{g \in \mathrm{L}(m-k,n)} \lambda_g \right)$$

For each $f \in \mathrm{L}(m,n)$, the number of times λ_f occurs in this triple is also m.
For each $f \notin \mathrm{L}(m,n)$, it is also easy to deduce that there is no λ_f in the triple sum. Hence :

$$= \sum_{f \in \mathrm{L}(m,n)} \lambda_f = \chi_\lambda^m(s)$$

To prove the formulas

$$\sigma_{\mathrm{T}}(\chi)(s) = \exp\left\{ \sum_{k=1}^{\infty} \Psi^k(\chi)\frac{\mathrm{T}^k}{k} \right\}, \quad \lambda_{\mathrm{T}}(\chi)(s) = \exp\left\{ \sum_{k=1}^{\infty} (-1)^{k-1}\Psi^k(\chi)\frac{\mathrm{T}^k}{k} \right\}$$

It suffices to show the identity

$$\exp(f(\mathrm{T}))' = f'(\mathrm{T})\exp(f(\mathrm{T}))$$

where every operation defined are considered as formal operations in the power series ring; for if this were to be established, then one can prove the identities by first noticing that the degree 0, 1 terms of the power series below

$$\exp\left\{ \sum_{k=1}^{\infty} \Psi^k(\chi)\frac{\mathrm{T}^k}{k} \right\}, \quad \sum_{k=0}^{\infty} \chi_\sigma^k \mathrm{T}^k$$

coincides, and that the "chain rule" above gives a recursion formula for the coefficients of the higher degree terms of the first series, coinciding with the recursion formula for the second series we've just established; one has analogous statements for exterior powers.
To establish the general formula, let $f(\mathrm{T}) = \sum_{m=0}^{\infty} a_m \mathrm{T}^m$ be a formal power series, we have

$$\exp(f(\mathrm{T})) = \sum_{n=0}^{\infty} \frac{f(\mathrm{T})^n}{n!}$$

giving us

$$\exp(f(\mathrm{T}))' = \sum_{n=1}^{\infty} \left(\frac{f(\mathrm{T})^{n-1}}{(n-1)!} f'(\mathrm{T}) \right) = f'(\mathrm{T}) \left(\sum_{n=1}^{\infty} \frac{f(\mathrm{T})^{n-1}}{(n-1)!} \right) = f'(\mathrm{T})\exp(f(\mathrm{T}))$$

(b) By the recursion formula for χ_σ^n, one gets

$$\Psi^n(\chi) = n\chi_\sigma^n - \left(\sum_{k=1}^{n-1} \Psi^k(\chi)\chi_\sigma^{n-k}\right)$$

One concludes that $\mathrm{R}(\mathrm{G})$ is stable under each Ψ^n easily from an induction argument.

Exercise 9.4

(a) By the assumption that $(n, g) = 1$, one sees that the set map

$$\mathrm{G} \longrightarrow \mathrm{G}\ , \qquad x \longmapsto x^n$$

is a bijection, so we get

$$< \Psi^n(\chi), \Psi^n(\chi) >_{\mathrm{G}} = \frac{1}{g}\left(\sum_{s\in \mathrm{G}} \chi(s^n)\chi(s^{-n})\right) = \frac{1}{g}\left(\sum_{s\in \mathrm{G}} \chi(s)\chi(s^{-1})\right) = 1$$

By **Exercise 9.3** , $\Psi^n(\chi) \in \mathrm{R}(\mathrm{G})$, and by **Exercise 9.2** , $\Psi^n(\chi)$ is irreducible if $\Psi^n(\chi)(1) > 0$, but it is immediate that $\Psi^n(\chi)(1) = \chi(1) > 0$.

(b) The linear operator ψ_n (on $\mathbb{C}[\mathrm{G}]$) has an inverse :
Take m with $mn \equiv 1 (\operatorname{mod} g)$, then ψ_m, ψ_n are mutually inverses.
Therefore, we only need to show :

- $\psi_n|(\mathrm{Cent.}\,\mathbb{C}[\mathrm{G}])$ has image in $\mathrm{Cent.}\,\mathbb{C}[\mathrm{G}]$.
- $\psi_n(uv) = \psi_n(u)\psi_n(v)$ for each $u, v \in \mathrm{Cent.}\,\mathbb{C}[\mathrm{G}]$.

Recall that $\mathrm{Cent.}\,\mathbb{C}[\mathrm{G}]$ has the description (section 6.3)

$$\mathrm{Cent.}\,\mathbb{C}[\mathrm{G}] = \oplus_{i=1}^h \mathbb{C}e_i$$

then the first assertion is done by noticing the calculation :

$$\psi_n\left(\sum_{s\in \mathrm{G}} s^{-1}ts\right) = \sum_{s\in \mathrm{G}} \psi_n(s^{-1}ts) = \sum_{s\in \mathrm{G}} s^{-1}t^n s$$

and let t vary through G.
To prove the second assertion, recall that we have an isomorphism of algebras :

$$\omega : \ \mathrm{Cent.}(\mathbb{C}[\mathrm{G}]) \longrightarrow \mathbb{C}^h$$

(this is proposition 13 in the book) with each component function ω_i given by

$$\omega_i(u) = \frac{1}{n_i}\sum_{s\in \mathrm{G}} u(s)\chi_i(s)$$

We have the following calculation :

$$\omega_i(\psi_n(u)) = \frac{1}{n_i}\sum_{s\in \mathrm{G}} u(s)\chi_k(s^n) = \frac{1}{n_i}\sum_{s\in \mathrm{G}} u(s)\Psi^n(\chi_i)(s)$$

By (a), $\Psi^n(\chi_i)$ is an irreducible character, so we get the following :

There exists a bijection σ from $\{1, \dots, h\}$ to itself such that $\omega_i \circ \psi_n = \omega_{\sigma(i)}$ for each i.

From this, we obtain the calculation :

$$\omega_i(\psi_n(u)\psi_n(v)) = \omega_i(\psi_n(u))\omega_i(\psi_n(v)) = \omega_{\sigma(i)}(u)\omega_{\sigma(i)}(v) = \omega_{\sigma(i)}(uv) = \omega_i(\psi_n(uv))$$

Combining each i, we get that

$$\omega(\psi_n(u)\psi_n(v)) = \omega(\psi_n(uv))$$

By bijectivity of ω, we get $\psi_n(u)\psi_n(v) = \psi_n(uv)$.

Exercise 9.6

(a) For each irreducible character φ of G, we have the following :

$$<\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}\chi,\varphi>_{\mathrm{G}}=<\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}\mathrm{Ind}_{\mathrm{H}_i}^{\mathrm{H}}\chi',\varphi>_{\mathrm{G}}=<\mathrm{Ind}_{\mathrm{H}_i}^{\mathrm{G}}\chi',\varphi>_{\mathrm{G}}$$

This shows that $\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}\chi-\mathrm{Ind}_{\mathrm{H}_i}^{\mathrm{G}}\chi'$ is orthogonal to every class function of G, hence 0.

(b) Again, for each irreducible character φ of G, we have the following :

$$\begin{aligned}<\mathrm{Ind}_{{}^s\mathrm{H}}^{\mathrm{G}}{}^s\chi,\varphi>_{\mathrm{G}}&=\frac{1}{gh}\sum_{u\in\mathrm{G}}\sum_{\substack{v\in\mathrm{G}\\ v^{-1}uv\in{}^s\mathrm{H}}}\left({}^s\chi(v^{-1}uv)\varphi(u^{-1})\right)\\&=\frac{1}{gh}\sum_{u\in\mathrm{G}}\sum_{\substack{v\in\mathrm{G}\\ (vs)^{-1}u(vs)\in\mathrm{H}}}\left(\chi((vs)^{-1}u(vs))\varphi(u^{-1})\right)\\&=\frac{1}{gh}\sum_{u\in\mathrm{G}}\sum_{\substack{v\in\mathrm{G}\\ (vs)^{-1}u(vs)\in\mathrm{H}}}\left(\chi((vs)^{-1}u(vs))\varphi((vs)u^{-1}(vs)^{-1})\right)\\&=\frac{1}{gh}\sum_{u\in\mathrm{G}}\sum_{\substack{w\in\mathrm{G}\\ w^{-1}uw\in\mathrm{H}}}\left(\chi(w^{-1}uw)\varphi(w^{-1}uw)\right)\\&=\frac{1}{gh}\sum_{u\in\mathrm{G}}\sum_{\substack{w\in\mathrm{G}\\ w^{-1}uw\in\mathrm{H}}}\left(\chi(w^{-1}uw)\varphi(u)\right)\\&=<\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}\chi,\varphi>_{\mathrm{G}}\end{aligned}$$

This shows $\mathrm{Ind}_{{}^s\mathrm{H}}^{\mathrm{G}}{}^s\chi=\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}\chi$.

(c) Recall the exact sequences :

$$\bigoplus_{\mathrm{H}\in\mathrm{X}}\mathbb{Q}\otimes\mathrm{R}(\mathrm{H})\xrightarrow{\ \iota\ }\mathbb{Q}\otimes\mathrm{R}(\mathrm{G})\longrightarrow 0$$

$$0\longrightarrow\mathbb{Q}\otimes\mathrm{R}(\mathrm{G})\xrightarrow{\ \rho\ }\bigoplus_{\mathrm{H}\in\mathrm{X}}\mathbb{Q}\otimes\mathrm{R}(\mathrm{H})$$

given by induction, restriction. The space $\bigoplus_{\mathrm{H}\in\mathrm{X}}\mathbb{Q}\otimes\mathrm{R}(\mathrm{H})$ has an inner product $<,>_{\mathrm{X}}$ by

$$<(f_{\mathrm{H}}),(g_{\mathrm{H}})>_{\mathrm{X}}:=\sum_{\mathrm{H}\in\mathrm{X}}<f_{\mathrm{H}},g_{\mathrm{H}}>_{\mathrm{H}}$$

then we have a generalized reciprocity law :

$$<\iota((f_{\mathrm{H}})),g>_{\mathrm{G}}=\sum_{\mathrm{H}\in\mathrm{X}}<\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}f_{\mathrm{H}},g>_{\mathrm{G}}=\sum_{\mathrm{H}\in\mathrm{X}}<f_{\mathrm{H}},\mathrm{Res}_{\mathrm{H}}^{\mathrm{G}}g>_{\mathrm{H}}=<(f_{\mathrm{H}}),\rho(g)>_{\mathrm{X}}$$

By this reciprocity formula, we immediately get the following :

$$\mathrm{N}=\mathrm{Ker}(\iota)=\mathrm{Im}(\rho)^{\perp}$$

Let $\mathrm{N}'\subseteq\bigoplus_{\mathrm{H}\in\mathrm{X}}\mathbb{Q}\otimes\mathrm{R}(\mathrm{H})$ be the orthogonal complement of the $\mathbb{Q}$-vector space spanned by the $(f_{\mathrm{H}})\in\bigoplus_{\mathrm{H}\in\mathrm{X}}\mathbb{Q}\otimes\mathrm{R}(\mathrm{H})$ satisfying the conditions :

- For each $H, H' \in X$ with $H' \leq H$, we have

$$\mathrm{Ind}_{H'}^{H} f_{H'} = f_H$$

- For each $H \in X$ and $s \in G$, we have

$${}^{s}f_H = f_{{}^{s}H}$$

then one sees that N' is precisely the vector space what the question wants. Need $N = N'$.
The fact that $N' \subseteq N$ is by (a),(b) and our generalized reciporcity law.
For the converse ($N'^{\perp} \supseteq N^{\perp}$), take some $(f_H)_{H\in X} \in N'^{\perp}$.
For each $s \in G$, one can find a unique smallest subgroup $X(s) \in X$ of G containing s by

$$X(s) = \bigcap_{\substack{H\in X \\ x\in H}} H$$

Now define an $f \in \mathbb{Q} \otimes R(G)$ by $f(s) = f_{X(s)}(s)$; this definition is well-defined, and that

$$\rho(f) = (\mathrm{Res}_H^G f) = (f_H)$$

for each $H \in X$. This shows that

$$N^{\perp} = \mathrm{Im}(\rho) \supseteq N'^{\perp}$$

Exercise 9.7 Just apply **Exercise 9.6** .

Exercise 9.8 The fact that the class function

$$\lambda_A := \varphi(a) r_A - \theta_A$$

is orthogonal to the unit character is a straightfoward calculation :

$$< \lambda_A, 1 >_A = \frac{1}{a}\left(\sum_{s\in G} \varphi(a) r_A(s) - \theta_A(s)\right) = \frac{1}{a}\left(\varphi(a)a - \varphi(a)a\right) = 0$$

To show that λ_A is a character, it suffices by **Exercise 9.1** to verify that λ_A is real-valued and that $\lambda_A(s) \leq 0$ for each $s \neq 1$, which is clear from the definition.
Let X be the set of cyclic subgroups of G. We have for each $s \in G$ that

$$\sum_{A\in X} \mathrm{Ind}_A^G(\lambda_A)(s) = \sum_{A\in X} \left(\varphi(a)\,\mathrm{Ind}_A^G r_A(s) - \mathrm{Ind}_A^G \theta_A(s)\right) = g\delta_{1s} g - g = g(r_G(s) - 1)(s)$$

this gives

$$\sum_{A\in X} \mathrm{Ind}_A^G(\lambda_A) = g(r_G - 1)$$

10 Brauer's Theorem

Exercise 10.1 It is clear that $P \leq Z(x)$. Now since each p-group of $Z(x)$ is contained in a Sylow p-subgroup of $Z(x)$, we're done.

Exercise 10.2 For the first statement, notice that for $q = p^k$, we have

$$(x-1)^q = x^q - 1$$

Let x be a p-element, then $(x-1)$ is nilpotent by this identity; on the other hand, if $(x-1)$ is nilpotent, by raising the power to some $q = p^k$, we see that the order of x divides q, hence is a p-element.
On the other hand, suppose that x is a p' element of order d, then the polynomial

$$x^d - 1$$

has no repeated roots (becasue it is prime to its derivative dx^{d-1}). Now take a finite extension l of k so that $x^d - 1$ can be factored into product of linear polynomials in this field, then the minimal polynomial of x in this field has no repeated roots because it divides the minimal polynomial of x in k[4], which divides the polynomial $x^d - 1$ hence x is diagonalizable in this field.
For the converse, let x be a non-p' element. We have seen that any nontrivial p-element can't be semisimple. Now since a power x^d of x is a nontrivial p-element, x^d isn't semisimple, and hence x isn't semisimple.

Exercise 10.3 The idea is similar : let χ be an A-valued class function, we have

$$g\chi = (\sum_{C} \mathrm{Ind}_C^G \theta_C)\chi = \sum_{C} \mathrm{Ind}_C^G(\theta_C . \mathrm{Res}_C^G \chi)$$

where the summation is taken over the set of cyclic subgroups of G. Now we want

$$\chi_C := \theta_C . \mathrm{Res}_C^G \chi \in A \otimes R(G)$$

Since θ_C is a $c\mathbb{Z}$-valued class function, we have for each character ψ of G that

$$< \chi_C, \psi >_C = \frac{1}{c} \sum_{s \in G} \theta_C(s)\chi(s)\psi(s^{-1}) = \sum_{s \in G} (\theta_C/c)(s)\chi(s)\psi(s^{-1}) \in A$$

This shows $\chi_C \in A \otimes R(G)$.

Exercise 10.4 This can be seen as follows :

- Reduce to the case where $G =< x >$, that is, the case where G is cyclic, generated by x.

[4] In fact, they are the same. To show this, we recall the following fact from linear algebra :

Let l be a field extension of k, and let $v_1, \ldots, v_n$ be vectors in k^n, we have :
$v_1, \ldots, v_n$ are linearly independent over k iff linearly independent over l.

The fact can be proved as follows : choose basis e_i of l over k, and suppose $\sum_j \lambda_j v_j = 0$, $(\lambda_j \in l)$.
By using the basis e_i, we may write for some $a_{ij} \in k$ that $\lambda_j = \sum_i a_{ij} e_i$.
Plugging this expression into the above, we get $\sum_{i,j} a_{ij} e_i v_j = 0$.
Since $e_i v_j$ are linearly independent over k, we get $a_{ij} = 0$ for all i, j, and hence $\lambda_j = 0$ for all j.
We will use the fact to consider minimal polynomial of x in different fields. Suppose $x \in \mathbf{GL}_n(k)$, and let l be a field extension of k, then by taking the k-vector space spanned by the powers of x, we get our claim.

- Let $\chi \in A \otimes R(G)$ be an A-coefficient character, write it as

$$\chi = \sum_i a_i \chi_i, \quad a_i \in A, \chi_i : G \longrightarrow \mathbb{C}^*$$

- Since $k := A/\mathfrak{p}$ is a finite field of characterisitic p, we can find some $q = p^n$ such that

$$\text{For all } \alpha \in A, \alpha^q \equiv \alpha (\operatorname{mod} \mathfrak{p})$$

- On the other hand, we can even assume that $x^q = x_r^q$.
- This gives us the calculation :

$$\begin{aligned}
\chi(x) = \left(\sum_i a_i \chi_i\right)(x) &= \sum_i a_i \chi_i(x) \\
&\equiv \sum_i a_i \chi_i(x)^q = \sum_i a_i \chi_i(x^q) = \sum_i a_i \chi_i(x_r^q) = \sum_i a_i \chi_i(x_r)^q \\
&\equiv \sum_i a_i \chi_i(x_r) = \left(\sum_i a_i \chi_i\right)(x_r) = \chi(x_r) \pmod{\mathfrak{p}}
\end{aligned}$$

this gives us the desired result. I don't know any counterexamples though.

Exercise 10.5

(a) Let χ_j be the irreducible characters of G with $\chi = \chi_{j_0}$ for some j_0. Write

$$\chi = \sum_i c_i \operatorname{Ind}_{H_i}^G \psi_i$$

with each $c_i \in \mathbb{R}_{>0}$ and ψ_i a degree 1 character of H_i. Also write

$$\operatorname{Ind}_{H_i}^G \psi_i = \sum_j a_{ij} \chi_j \in R^+(G)$$

Plugging in this expression, we get

$$\chi = \chi_{i_0} = \sum_{i,j} a_{ij} c_j \chi_j$$

By orthogonality of characters, we get for each j that

$$\sum_i a_{ij} c_i = \delta_{j_0 j}$$

but since each $a_{ij} \geq 0, c_i > 0$, we get $a_{ij} = 0$ for all $j \neq j_0$. Therefore :

$$\operatorname{Ind}_{H_i}^G \psi_i = \sum_j a_{ij} \chi_j = a_{i j_0} \chi_{j_0} \in R^+ G$$

Therefore, we see that a positive integer multiple of χ_{j_0} is a monomial character.

(b) By **Exercise 2.6** , we see that χ is indeed irreducible ($\mathfrak{A}_5$ acts doubly transitively).
If $m\chi = \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}\psi$ for some subgroup H and deg 1 representation ψ of H and $m \in \mathbb{Z}_{>0}$, then

$$m = \frac{g}{h}\frac{\psi(1)}{\chi(1)} = \frac{15}{h}$$

this gives us $m = 1, 3, 5, 15$. We can exclude the possibility that $m = 1$: if $h = 15$, then Sylow's third theorem says that H has a normal Sylow-3 subgroup N, now take a 5-cycle $s \in \mathrm{H}$ and a 3-cycle $t \in \mathrm{N}$, it is easy to see that $sts^{-1} \notin \mathrm{H}$, hence a contradiction.
Now by Frobenius reciprocity, we have

$$< \mathrm{Res}_{\mathrm{H}}^{\mathrm{G}}\chi, \psi >_{\mathrm{H}} = < \chi, \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}\psi >_{\mathrm{G}} = < \chi, m\chi >_{\mathrm{G}} = m$$

Since $\mathrm{Res}_{\mathrm{H}}^{\mathrm{G}}\chi(1) = 4$, we see that $h \neq 1, 3$.
Now if $(m, h) = (3, 5)$, then H is generated by some 5-cycle s. We have :

$$< \mathrm{Res}_{\mathrm{H}}^{\mathrm{G}}\chi, \mathrm{Res}_{\mathrm{H}}^{\mathrm{G}}\chi >_{\mathrm{H}} = \frac{1}{5}\sum_{k=0}^{4}\chi(s^k)\chi(s^{-k}) = \frac{(4)^2 + (-1)^2 + (-1)^2 + (-1)^2 + (-1)^2}{5} = 4$$

since $\mathrm{Res}_{\mathrm{H}}^{\mathrm{G}}(\chi + 1)$ is the permutation representation.
Therefore, $\mathrm{Res}_{\mathrm{H}}^{\mathrm{G}}\chi$ can at most contain ψ at most 2 times, a contradiction (since $m = 3 > 2$).
By (a), χ can't be a linear combination of monomial characters with $\mathbb{R}_{\geq 0}$ coefficients.

Exercise 10.6 .

(a) By transitivity of induction, i.e.,

$$\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}\mathrm{Ind}_{\mathrm{E}}^{\mathrm{H}}(\alpha - 1) = \mathrm{Ind}_{\mathrm{E}}^{\mathrm{G}}(\alpha - 1)$$

we are done.

(b) I only know how to solve the case where G is elementary[5]
By (a), it suffices to show that $\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(1) \in \mathrm{R}'(\mathrm{G})$. Take $t \in \mathrm{G}$, we have by normality that

$$\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(1)(t) = \frac{1}{h}\sum_{\substack{s\in\mathrm{G} \\ sts^{-1}\in\mathrm{H}}} 1(sts^{-1}) = \begin{cases} g/h & \text{if } t \in \mathrm{H} \\ 0 & \text{if } t \notin \mathrm{H} \end{cases}$$

This shows that we have the following identity :

$$\mathrm{Res}_{\mathrm{G}}^{\mathrm{G/H}} r_{\mathrm{G/H}} = \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(1)$$

where $r_{\mathrm{G/H}}$ the character of the regular representation of G/H,
and that $\mathrm{Res}_{\mathrm{G}}^{\mathrm{G/H}}$ is in the generalized sense via a quotient map.
Now let χ_i be irreducible characters of G/H, we have

$$\mathrm{Res}_{\mathrm{G}}^{\mathrm{G/H}}(r_{\mathrm{G/H}}) = \mathrm{Res}_{\mathrm{G}}^{\mathrm{G/H}}(\sum_i \chi_i) = \sum_i \mathrm{Res}_{\mathrm{G}}^{\mathrm{G/H}}(\chi_i)$$

so we have decomposed $\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(1)$ into a sum of degree 1 characters. Since

$$\psi_i = 1 + (\psi_i - 1) = 1 + \mathrm{Ind}_{\mathrm{G}}^{\mathrm{G}}(\psi - 1)$$

(where $\psi_i := \mathrm{Res}_{\mathrm{G}}^{\mathrm{G/H}}(\chi_i)$), we are done.

[5]This case will suffice for the purpose of (c),(d) of **Exercise 10.6** .

(c) Let $H \in Y$. Take normalizer $N_G(H)$, then $N_G(H)$ is H or G.
If we can show $N_G(H) = G$, we're done. By nilpotency of G, choose a defining chain

$$0 = G_0 < G_1 < \dots < G_n = G$$

Notice first that $G_1 \leq \mathrm{Cent}.(G)$ is contained in $N_G(H)$ by definition. Now we do induction : Suppose for some $k > 0$ that $G_k \leq N_G(H)$, then since

$$G_{k+1}/G_k \leq \mathrm{Cent}.(G/G_k)$$

we get $G_{k+1} \leq N_G(H)$ by the following manipulation on symbols :

$$xN_G(H) = (xG_k)(N_G(H)G_k) = (N_G(H)G_k)(xG_k) = (N_G(H)G_k)(G_k x) = N_G(H)x$$

The fact that $[G : H]$ is prime is by maximality of H and solvability of G/H.
Since $N_G(H)$ contains all G_k, $N_G(H) = G$, so H is normal in G.
By Theorem 16, each character of G is monomial, so if the subgroup isn't G, we can take a maximal subgroup containing it, and apply transitivity of induction.
Let us show $R(G) = R'(G)$. Let $S(G)$ be the space of degree-1 characters of G. We have :

$$R'(G) \leq R(G) = S + \left(\sum_{H \in Y} \mathrm{Ind}_H^G R(H)\right) = S + \left(\sum_{H \in Y} \mathrm{Ind}_H^G R'(H)\right) \leq S + R'(G) \leq R'(G)$$

The first two equalities are from definition. The third is by inductive hypothesis.[6]
The fourth is by (b), since these H are normal with G/H of prime order (hence abelian).
The last equality is by the identity $\alpha = 1 + \mathrm{Ind}_G^G(\alpha - 1) \in R'(G)$ (where $\alpha \in S(G)$).
Therefore, the inductive step is prove. The base case is trivial.

(d) If we write

$$1 = \sum_{E \in X} \mathrm{Ind}_E^G(f_E)$$

we get

$$\varphi = \varphi(1) = \sum_{E \in X} \varphi \, \mathrm{Ind}_E^G(f_E) = \sum_{E \in X} \mathrm{Ind}_E^G(f_E . \mathrm{Res}_E^G(\varphi)) := \sum_{E \in X} \varphi_E$$

By the condition $\varphi(1) = 0$, we get

$$\varphi_E(1) = \frac{g}{e} f_E(1) \, \mathrm{Res}_E^G \, \varphi(1) = \frac{g}{e} f_E(1)\varphi(1) = 0$$

We show $f_E . \mathrm{Res}_E^G(\varphi) \in R_0'(E)$. By (c), we get $R'(E) = R(E)$, so $f_E . \mathrm{Res}_E^G(\varphi) \in R'(E)$. Since

$$R'(E) = R_0'(E) + \mathbb{Z}$$

and that an element in $R'(E)$ vanish on 1 iff it belongs to $R_0'(E)$, so we're done.
By (a), we see $\varphi \in R_0'(G)$.
For general $\varphi \in R(G)$, since $\varphi - \varphi(1)1 \in R_0'(G)$, we get $\varphi \in R'(G)$, hence $R(G) = R'(G)$.

[6] To make sense of this reasoning, we need to show that subgroups of elementary subgroups are elementary. Suppose we are given $G = C \times P$ with P a p-group and c with $(c, p) = 1$, and a subgroup $H \leq G$. Let

$$C \longleftarrow G \longrightarrow P$$

be canonical projection maps, and let H_1, H_2 be images of H under the projection maps, I claim that

$$H = H_1 \times H_2$$

It is clear that $H \leq H_1 \times H_2$. Since $(|C|, |P|) = 1$ take $u, v \in \mathbb{Z}$ with $u|P| + v|C| = 1$, then $(s, 1) = (s, t)^{u|P|}$.
This shows $H_1 \times \{1\} \leq H$. Similarly one deduces $H_2 \times \{1\} \leq H$, and hence $H_1 \times H_2 \leq H$.
Since subgroups of p-groups (resp. cyclic groups) are p-groups (resp. cyclic), H is elementary.

11 Application of Brauer's Theorem

Exercise 11.1 In view of Theorem 21' and the fact that the order of any subgroup of G must divide g, we may just assume G is cyclic. Pick m with $(m, p) = 1$, then $x \mapsto x^m$ is an automorphism of G, so we deduce that f is a rational multiple of the unit character.
The statement that if f has values in $\mathbb{Z}$ implies $f \in \mathrm{R(G)}$ is the special case $\mathrm{B} = \mathbb{Z}$ of Theorem 23.
If f is the characteristic function of the unit class, that is, if

$$f(s) = \delta_{1s}$$

then $f \in \mathbb{Q} \otimes \mathrm{R(G)}$. On the other hand, if we consider the function $\Psi^n f$ which has description

$$\Psi^n f(s) = \delta_{1s^n}$$

then we have $(g/(g,n))\Psi^n f \in \mathrm{R(G)}$.

Exercise 11.2 Let κ be the quotient of $\mathrm{A} \otimes \mathrm{R(G)}$ by $\mathrm{P}_{\mathrm{M},c}$, this gives an exact sequence

$$0 \longrightarrow \mathrm{P}_{\mathrm{M},c} \longrightarrow \mathrm{A} \otimes \mathrm{R(G)} \longrightarrow \kappa \longrightarrow 0$$

then we get another exact sequence

$$0 \longrightarrow \mathrm{M} \longrightarrow \mathrm{A} \longrightarrow \kappa$$

by intersecting with A. We are done if we can show that the composite map

$$\mathrm{A} \longrightarrow \mathrm{A} \otimes \mathrm{R(G)} \longrightarrow \kappa$$

is surjective, but this is clear : let $f \in \mathrm{A} \otimes \mathrm{R(G)}$, then take

$$g := f - f(c)1$$

Notice that $f(c)1$ is in the image of A in $\mathrm{A} \otimes \mathrm{R(G)}$ and that $g \in \mathrm{P}_{\mathrm{M},c}$.

Exercise 11.3 First, we still have injections :

$$\mathrm{B} \longrightarrow \mathrm{B} \otimes \mathrm{R(G)} \longrightarrow \mathrm{B}^{\mathrm{Cl(G)}}$$

The fact that R(G) is integral over $\mathbb{Z}$ implies that the corresponding maps

$$\mathrm{Spec}(\mathrm{B}^{\mathrm{Cl(G)}}) \longrightarrow \mathrm{Spec}(\mathrm{B} \otimes \mathrm{R(G)}) \longrightarrow \mathrm{Spec(B)}$$

are all surjections. Let $f : \mathrm{A} \longrightarrow \mathrm{B}$ be the structure map of B, we have commutative diagrams :

$$\begin{tikzcd} \mathrm{A} \arrow[r] \arrow[d, "f"] & \mathrm{A} \otimes \mathrm{R(G)} \arrow[r] \arrow[d, "f\otimes 1"] & \mathrm{A}^{\mathrm{Cl(G)}} \arrow[d, "f^{\mathrm{Cl(G)}}"] \\ \mathrm{B} \arrow[r] & \mathrm{B} \otimes \mathrm{R(G)} \arrow[r] & \mathrm{B}^{\mathrm{Cl(G)}} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Spec}(\mathrm{B}^{\mathrm{Cl(G)}}) \arrow[r] \arrow[d, "f^{\mathrm{Cl(G)}*}"] & \mathrm{Spec}(\mathrm{B} \otimes \mathrm{R(G)}) \arrow[r] \arrow[d, "(f\otimes 1)^*"] & \mathrm{Spec(B)} \arrow[d, "f^*"] \\ \mathrm{Spec}(\mathrm{A}^{\mathrm{Cl(G)}}) \arrow[r] & \mathrm{Spec}(\mathrm{A} \otimes \mathrm{R(G)}) \arrow[r] & \mathrm{Spec(A)} \end{tikzcd}$$

The description of $\mathrm{Spec}(\mathrm{B}^{\mathrm{Cl(G)}})$ is clear : it can be thought of as the set of $\mathrm{N}_{\mathfrak{p},c}$ defined by

$$\mathrm{N}_{\mathfrak{p},c} := \{f : \mathrm{Cl(G)} \longrightarrow \mathrm{B} : f(c) \in \mathfrak{p} \in \mathrm{Spec(B)}\}$$

Take the map from $\mathrm{Spec}(\mathrm{B}^{\mathrm{Cl(G)}})$ to $\mathrm{Spec}(\mathrm{A} \otimes \mathrm{R(G)})$, then the map is

$$\mathrm{N}_{\mathfrak{p},c} \longmapsto \mathrm{M}_{f^*\mathfrak{p},c} \longmapsto \mathrm{P}_{f^*\mathfrak{p},c}$$

so by the description of $\mathrm{Spec}(\mathrm{A} \otimes \mathrm{R(G)})$, we see by surjectivity that

$$\begin{aligned} \mathrm{Spec}(\mathrm{B} \otimes \mathrm{R(G)}) = \{&\mathrm{N}_{\mathfrak{p},c} \cap \mathrm{B} \otimes \mathrm{R(G)} : f^*(\mathfrak{p}) = 0, c : p - \text{regular}\} \\ &\cup \{\mathrm{N}_{\mathfrak{p},c} \cap \mathrm{B} \otimes \mathrm{R(G)} : f^*(\mathfrak{p}) \neq 0, c \in \mathrm{Cl(G)}\} \end{aligned}$$

I don't know if this counts everything only once, but it does counts everything.

Exercise 11.4 The group Γ (isomorphic to $(\mathbb{Z}/g\mathbb{Z})^*$) acts on A by

$$\sigma_t : \xi \mapsto \xi^t, \quad t \in (\mathbb{Z}/g\mathbb{Z})^*$$

hence we get an action σ on $A \otimes R(G)$ and an action σ^* on Spec.
This also gives an action on $A \otimes R(G)$ and hence on $\mathrm{Spec}(A \otimes R(G))$, given by

$$P_{M,c} \longmapsto P_{\sigma^*(M),c} = P_{\sigma^{-1}(M),c}$$

Since it is easy to see that $(A \otimes R(G))^\Gamma = R(G)$, we get

$$\mathrm{Spec}(R(G)) = \{P_{M,c} : M \in \mathrm{Spec}(A \otimes R(G))^\Gamma\}$$

where $\mathrm{Spec}(A \otimes R(G))^\Gamma$ is the set of prime ideals fixed by Γ.

Exercise 11.5 Since G is isomorphic to its double dual, we may try to show

$$A \otimes R(G) \text{ is isomorphic to } A[\hat{G}]$$

It is clearly true that $\mathbb{Z}[\hat{G}]$ is isomorphic to $R(G)$ by definition, so the hint is proved.
To determine $\mathrm{Spec}(A[G])$, use the identity above and apply proposition 30.

Exercise 11.6 Let χ be a character of G and $M \in \mathrm{Spec}(A)$ with

$$M \cap \mathbb{Z} = p\mathbb{Z}$$

then we have by **Exercise 10.4** that $A\chi \subseteq B$, hence $A \otimes R(G) \subset B$.
Now we compare the Specs. We want to show that the map (induced from inclusion)

$$\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A \otimes R(G))$$

is a bijection, but this is clear from the proof of Proposition 30 and 30'. (In short : by integrality, one still has a surjection from $\mathrm{Spec}(A^{\mathrm{Cl}(G)})$ to $\mathrm{Spec}(B)$; the condition on B guarantees stability among passage to regular class; the fact that $A \otimes R(G) \subset B$ guarantees that one can distinguish different regular classes.) I haven't thought of an example where $A \otimes R(G) \subsetneq B$.

Exercise 11.7 .

(a) By the formula of induced characters, we see that $H \cap c = \emptyset$ implies $I_H \subseteq P_{0,c}$.
On the contrary, if $H \cap c \neq \emptyset$, we get for each $\chi \in R(G)$ and $t \in c$ that

$$\mathrm{Ind}_H^G(\chi)(t) = \frac{1}{h}\left(\frac{g}{|c|}\sum_{s \in c \cap H} \chi(s)\right)$$

Since $H \cap c$ is a union of conjugacy classes of H, one can easily find some $f \in A \otimes R(H)$ such that $\mathrm{Ind}_H^G(f)(t) \neq 0$; this implies I_H isn't contained in $P_{0,c}$.

(b) Suppose first that H contains an elementary subgroup E associated to some element in c, then we get from Lemma 8 a class function ψ on E such that

$$\mathrm{Ind}_H^G \mathrm{Ind}_E^H \psi(s) \not\equiv 0 (\mathrm{mod}\, p), \quad s \in c$$

(Notice that $p\mathbb{Z} \subseteq M$.)
The converse can be seen as a consequence of Lemma 11 (since A/M has characteristic p).

(c) **I don't have an answer to this problem yet.** Below is my attempt :
Let us consider Theorem 18 first. Notice that the set V_p can be written as

$$A \otimes V_p = \sum_{H \in X(p)} I_H = \sum_{H \in X'(p)} I_H$$

where $X(p)$ the set of p-elementary subgroups of H, and $X'(p)$ those associated ones (the second quality uses the transitivity of induction and **Exercise 10.2**).
We've seen some properties of $\operatorname{Spec}(A)$. Let us consider $\operatorname{Spec}(A/pA)$.
so there exists N such that $|A/M|$ divides p^N for all $M \in \operatorname{Spec}(A/pA)$.
By (b), there exists for each p-regular class c and $M \in \operatorname{Spec}(A/pA)$ some $f_{M,c}$ with

$$f_{M,c} \in A \otimes V_p \setminus P_{M,c}$$

take the following product

$$G := \prod_{M} (f_{M,c}^q - 1)$$

where $q := p^N$, then $G(c)$ belongs to some nilradical.
Therefore, we have an element $F_c \in A \otimes V(p)$ with

$$F_c(c) \equiv 1 (\operatorname{mod} pA)$$

and that for each other p-regular class c' that

$$F_c(c') \equiv 0, 1 (\operatorname{mod} pA)$$

12 Rationality Questions

Exercise 12.2 Referring to the table in 5.7, we have homomorphisms :

$$\mathbb{Q}[G] \longrightarrow \mathbb{Q} \times \mathbb{Q}(\omega) \times \mathbf{M}_3(\mathbb{Q})$$

given by $s \mapsto (\chi_0(s), (\chi_1 + \chi_2)(s), \mu(s))$ where μ the character corresponding to ψ.

Exercise 12.3 The hommomorphisms from G into $\{\pm 1\}$ are given by

$$\chi_{\epsilon,\eta} : i \longmapsto (-1)^{\epsilon}, j \longmapsto (-1)^{\eta}$$

The assertion that the Schur index of the last component is equal to 2 follows from the fact that

$$\mathrm{Cent.}(\mathbb{H}_{\mathbb{Q}}) = \mathbb{Q}$$

The trace (as $\mathbb{Q}$-endomorphism) of this character is given by

$$\chi(\pm 1) = \pm 4, \chi(s) = 0 \quad (s \neq \pm 1)$$

so this shows the assertions regarding ψ.
Therefore, K[G] is quasisplit iff $K \otimes \mathbb{H}_{\mathbb{Q}}$ is isomorphic to $\mathbf{M}_2(K)$.
Now we show that this is true iff -1 is a sum of two square in K.
Suppose first that -1 is a sum of two squares in K, that is, if

$$-1 = \alpha^2 + \beta^2$$

define from $K \otimes \mathbb{H}_{\mathbb{Q}}$ to $\mathbf{M}_2(K)$ a map by

$$i \longmapsto \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \qquad j \longmapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad k \longmapsto \begin{bmatrix} -b & a \\ a & b \end{bmatrix}$$

and extend by $\mathbb{Q}$-linearity; this is a well-defined isomorphism.
I don't know how to prove the converse.

Exercise 12.4 We've seen that the irreducible representations of G (over C) has degrees $n_i m_i$.
On the other hand, these degrees divides the index a by Proposition 17, so each m_i divides a.
The last assertion can be seen by Proposition 35, we are done.

Exercise 12.5 Write $d = [L : K]$, we have by the proof of Lemma 12 that

$$d \cdot \bar{R}_K(G) \subseteq R_K(G)$$

By Proposition 35, each m_i must divide d.

Exercise 12.6 Let ξ_m be an m-th roots of unity and $L = K(\xi)$, then $[L : K]$ divides $\varphi(m)$.
By **Exercise 12.5** and the fact that L[G] quasisplits, m_i divides $[L : K]$ and hence $\varphi(m)$.

Exercise 12.7 First, notice that one can write

$$1 = \sum_{H \in X_K} \mathrm{Ind}_H^G(f_H), \quad f_H \in R_K(H)$$

Take $\varphi \in \bar{R}_K(G)$, one gets

$$\varphi = \sum_{H \in X_K} \mathrm{Ind}_H^G(f_H \cdot \mathrm{Res}_H^G \varphi)$$

Now since $\mathrm{Res}^G(\varphi) \in \bar{R}_K(H)$ by definition of $\bar{R}_K(G)$, we have

$$f_H \cdot \mathrm{Res}_H^G \varphi \in R_K(H) \cdot \bar{R}_K(H) = \bar{R}_K(H)$$

This shows that the map is surjective.

Exercise 12.8 Let $\mathrm{Cl}_K(G)$ be the set of Γ_K classes, we have inclusion maps (Lemma 16)

$$A \longrightarrow A \otimes R_K(G) \longrightarrow A^{\mathrm{Cl}_K(G)}$$

and surjective maps between spectrums (by integrality) :

$$\mathrm{Spec}(A^{\mathrm{Cl}_K(G)}) \longrightarrow \mathrm{Spec}(A \otimes R_K(G)) \longrightarrow \mathrm{Spec}(A)$$

One can also define analogous notions of $M_c^K, P_{M,c}^K$, with respect to Γ_K-conjugacy classes,
and classify $\mathrm{Spec}(A^{\mathrm{Cl}_K(G)})$ as sets of some Γ_K-class functions.
Analogous to the results about $\mathrm{Spec}(A \otimes R(G))$, we get the following (stated in a rough fashion) :

Generalized Proposition 30 : If

1. with each $c \in \mathrm{Cl}_K(G)$, we associate $P_{0,c}^K$
2. with each Γ_K-p-regular class c and nonzero $M \in \mathrm{Spec}(A)$ with residue characteristic p, we associate $P_{M,c}^K$

Then we obtain once and only once each element of $\mathrm{Spec}(A \otimes R_K(G))$.

The fact that everything is counted is by Lemma 16.
The fact that everything is counted at most once is by Lemma 18.

13 Rationality Questions : Examples

Exercise 13.1 .

(a) The representation of G in $\mathbb{C}$ is clear.
Take a primitive g-th root of unity α and a generator x of G, define a homomorphism

$$\theta : \mathbb{Q}[G] \longrightarrow \mathbb{C} , \quad x \longmapsto \alpha$$

Notice that $\mathrm{Im}(\theta) = \mathbb{Q}(g)$, and that $\mathrm{Im}(\theta^d) = \mathbb{Q}(g/d)$ for each $d|g$. Define

$$\chi_d := \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(d)/\mathbb{Q})} \sigma\theta^{g/d} = \sum_{t \in (\mathbb{Z}/d\mathbb{Z})^*} \theta^{tg/d} = \mathrm{Tr}_{\mathbb{Q}(d)/\mathbb{Q}}(\theta^{g/d}), \quad (d|g)$$

then each χ_d is $\mathbb{Q}$-valued and is really a character.
We have to show that they are all irreducible, and mutually orthogonal.
For irreducibility, notice that $\mathbb{Q}(d)$ is field, so has no proper ideals, hence a simple $\mathbb{Q}$-algebra.
Orthogonality also follows from simplicity.

(b) The ring $\mathbb{Q}[G]$ can be described as the polynomial ring

$$\mathbb{Q}[G] \simeq \mathbb{Q}[x]/(x^g - 1)$$

We have the decomposition of rings :

$$\mathbb{Q}[x]/(x^g - 1) \simeq \mathbb{Q}[x]/(\prod_{d|g} \Phi_d(x)) \simeq \prod_{d|g} \mathbb{Q}[x]/(\Phi_d(x)) \simeq \prod_{d|g} \mathbb{Q}(d)$$

(c) Notice that since $1 \in R_{\mathbb{Q}}(G_d)$, $\psi_d = 1_{G_d}^G \in R_{\mathbb{Q}}(G)$. Also, $\psi_d(1) = |G|/|G_d| = d$.
Notice that since

$$d = \sum_{d'|d} \varphi(d')$$

in order to show

$$\psi_d = \sum_{d'|d} \chi_{d'}$$

it suffices to show that for each $d'|d$ that

$$< \psi_d, \chi_{d'} >_G \neq 0$$

By Frobenius reciprocity, this is the same as showing

$$< 1, \mathrm{Res}_{G_d}^G \chi_{d'} >_{G_d} \neq 0$$

By our description in (a), we have

$$\frac{d}{g} < 1, \mathrm{Res}_{G_d}^G \chi_{d'} >_{G_d} = \frac{d}{g} \sum_{x \in G_d} \chi_{d'}(t)$$

Since $G_d \subset G_{d'}$, and $\chi_{d'}|G_{d'} = \varphi(d')1$ this value certainly isn't 0; this proves the identity.
Using the theory of arithmetic functions, the assertion involving μ is the calculation

$$\psi = \chi * I, \quad \chi = \psi * I^{-1} = \psi * \mu$$

hence the identity

$$\chi_d = \sum_{d'|d} \mu(\frac{d}{d'})\psi_{d'} = \sum_{d'|d} \mu(\frac{d}{d'}) 1_{G_{d'}}^G$$

Exercise 13.2 First, Theorem 26 (generalized Artin theorem) gives a surjection

$$\bigoplus_{H \in T} \mathbb{Q} \otimes R_{\mathbb{Q}}(H) \longrightarrow \mathbb{Q} \otimes R_{\mathbb{Q}}(G)$$

Next, **Exercise 13.1** shows that Theorem 30 is true when G is cyclic.
Now since induction is transitive, we are done.

Exercise 13.3 Notice that the permutation representation of G/H is just 1_H^G.
Let V be the space of ρ, and D the opposite ring of the $\mathbb{Q}[G]$-endomorphism algebra of V.
We have $[V : D] = n$. Notice the calculation :

$$< 1_H^G, \chi >_G = < 1, \mathrm{Res}_H^G \chi >_H = \frac{1}{h} \sum_{s \in H} \chi(s)$$

When $H = \{1\}$, this value is $\chi(1) = \dim_{\mathbb{Q}}(V) = n \dim_{\mathbb{Q}}(D)$. Also, we also have

$$\dim_{\mathbb{Q}} \left(\mathrm{Hom}_{\mathbb{Q}[G]}(n\chi, \chi)\right) = n \dim_{\mathbb{Q}} \left(\mathrm{Hom}_{\mathbb{Q}[G]}(\chi, \chi)\right) = n \dim_{\mathbb{Q}} \left(\mathrm{End}_{\mathbb{Q}[G]}(V)\right) = n \dim_{\mathbb{Q}}(D)$$

Therefore, the permutation representation contains ρ precisely n times.
Now suppose $H \neq \{1\}$, we want to show $< 1, \mathrm{Res}_H^G \chi >_H = 0$.
First, when G is cyclic, this is true by (a).
Next,

Exercise 13.4 .

Exercise 13.5 .

Exercise 13.6 .

(a)

(b)

(c)

Exercise 13.7 .

(a)

(b)

(c)

Exercise 13.8 .

Exercise 13.9 .

(a)

(b)

Exercise 13.10 .

Exercise 13.11 .

Exercise 13.12 .