

REPRESENTATION OF FINITE GROUPS

- Some Solutions to Exercises

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2 CHARACTER THEORY

EXERCISE 2.1 One can realize $\chi + \chi'$ as the character of $\rho \oplus \rho'$. Pick $s \in G$. Suppose

$$(e_1, \dots, e_m), \quad (e'_1, \dots, e'_n)$$

are eigenbasis for ρ, ρ' with eigenvalues (λ_i, λ'_j) are \mathbb{C} -valued functions of G

$$(\lambda_1, \dots, \lambda_m), \quad (\lambda'_1, \dots, \lambda'_n)$$

then one can define a basis of $\rho \oplus \rho'$ by

$$(e''_1, \dots, e''_{m+n}) := (e_1, \dots, e_m, e'_1, \dots, e'_n)$$

with corresponding eigenvalues

$$(\lambda''_1, \dots, \lambda''_{m+n}) := (\lambda_1, \dots, \lambda_m, \lambda'_1, \dots, \lambda'_n)$$

Now we compute :

$$(\chi + \chi')^2_\sigma = \text{Tr}(\rho \oplus \rho') = \sum_{i \leq j} \lambda''_i \lambda''_j = \left(\sum_{i \leq m < j} + \sum_{i \leq j \leq m} + \sum_{m < i \leq j} \right) \lambda''_i \lambda''_j = \chi\chi' + \chi^2_\sigma + \chi'^2_\sigma$$

We omit the similar computation for alternating squares.

EXERCISE 2.2 Let $(e_i)_{i \in X}$ be a basis of X , suppose for each $s \in G$ that

$$\rho_s(e_i) = \sum_{j \in X} r_{ji}(s) e_j$$

then we have

$$\chi = \sum_i r_{ii}$$

Since ρ is permutation representation of the group action G on X , we get

$$r_{ji}(s) = \delta_{si,j}$$

From this observation, we get

$$\chi(s) = \sum_i r_{ii}(s) = \sum_i \delta_{si,i}$$

This is the number of elements in X fixed by s .

EXERCISE 2.3 The idea is to define $\rho'_s(x') = x' \circ \rho_s^{-1}$. In a more informal way :

$$\rho' = \circ \rho^{-1}$$

We have the following calculation :

$$\langle \rho'_s(x'), \rho_s x \rangle = \langle x' \rho_s^{-1}, \rho_s x \rangle = \langle x', x \rangle$$

This shows existence of ρ' ; uniqueness follows from nondegeneracy of \langle, \rangle .

To compute the character χ' of ρ' , take an eigenbasis e_i of ρ_s with values λ_i define $e'_i \in V'$ by

$$\langle e'_i, e_j \rangle = \delta_{ij}$$

then we have

$$\langle \rho'_s(e'_i), e_j \rangle = \langle e'_i, \rho_s^{-1} e_j \rangle = \langle \lambda_i^* e'_i, e_j \rangle$$

so by non-degeneracy of \langle, \rangle , we see that $\rho'_s(e'_i) = \lambda_i^* e'_i$, hence

$$\chi' = \sum_i \lambda_i^* = \left(\sum_i \lambda_i \right)^* = \chi^*$$

EXERCISE 2.4 The fact that it is a representation is straightforward. To show that $\rho, \rho'_1 \otimes \rho_2$ are isomorphic representations, where ρ'_1 the contragredient of ρ_1 in **EXERCISE 2.3**, it suffices to show that they have the same character. Fix $s \in G$, let $(d_1, \dots, d_m), (e_1, \dots, e_n)$ be eigenbasis for $\rho_{1,s}, \rho_{2,s}$ with eigenvalues $(\lambda_1, \dots, \lambda_m), (\mu_1, \dots, \mu_n)$.

- Define basis (d'_1, \dots, d'_m) for $\rho'_{1,s}$ as in **EXERCISE 2.3**; it has eigenvalues $(\lambda_1^*, \dots, \lambda_m^*)$.
Now we get an eigenbasis $(d'_i \otimes e_j)$ for $\rho'_{1,s} \otimes \rho_{2,s}$ with eigenvalues $\lambda_i^* \mu_j$.
- Define basis $(f_{11}, f_{21}, \dots, f_{mn})$ of ρ by

$$f_{ij}(d_k) = \delta_{ik} e_j$$

then we have

$$\rho_s(f_{ij})(d_k) = (\rho_{2,s} f_{ij} \rho_{1,s}^{-1})(d_k) = \lambda_k^* \mu_j \delta_{ik} e_j = \lambda_k^* \mu_j f_{ij}(d_k) = \begin{cases} 0, & \text{if } k \neq i \\ \lambda_i^* \mu_j f_{ij}(d_k), & \text{if } k = i \end{cases}$$

so we get $\rho_s(f_{ij})(d_k) = \lambda_i^* \mu_j f_{ij}(d_k)$, hence f_{ij} has eigenvalue $\lambda_i^* \mu_j$.

These two observations shows that the two representations have the same character.

EXERCISE 2.5 The number of times ρ contains 1 is given by

$$(\rho|1) = \frac{1}{g} \sum_{t \in G} \chi(t)$$

by using Theorem 4 and the fact that the character of 1 is the constant function with value 1.

EXERCISE 2.6

- (a) By decomposing the representation by restriction to orbits, it suffices to show :

If G acts transitively on X , then ρ decomposes as $1 \oplus \psi$, and that the corresponding this decomposition, we have a decomposition of χ into $1 + \psi$ such that $(\psi|1) = 0$.

This follows directly from the computation

$$(\chi|1) = \frac{1}{g} \sum_{t \in G} \chi(t) = \frac{1}{g} \sum_{t \in G} |X_t| = \frac{1}{g} \sum_{x \in X} |G_x| = \frac{1}{g} \sum_{x \in X} \frac{g}{|G_x|} = \frac{1}{g} \sum_{x \in X} \frac{g}{|X|} = 1$$

(the first equality is by **EXERCISE 2.5**, the second by **EXERCISE 2.2**, the third by a counting argument, the fourth by orbit-stabilizer, the fifth by the assumption that G acts transitively.)

- (b) Identify this representation with the tensor product $\rho \otimes \rho$.

- (c) Equivalence between (i)-(iii) is already established in the hint.

The hint also said that (iii) is equivalent to $(\psi^2|1) = 1$.

Since ψ is real-valued (since $\chi, 1$ are), we get $(\psi^2|1) = (\psi|\psi)$.

These observations establishes the equivalence between (iii) and (iv).

EXERCISE 2.7 Suppose χ is one of such with dimension 1, let χ_i be the irreducible characters of G of dimension n_i , we have the following observation :

$$(\chi, \chi_i) = \frac{1}{g} \sum_{t \in G} \chi(t) \chi_i(t^{-1}) = \frac{nn_i}{g}$$

Take $\chi_i = 1$, we see that n/g is a non-negative integer; this will suffice.

EXERCISE 2.8 In the canonical decomposition of V given by

$$V = \bigoplus_i V_i$$

choose a decomposition of V_i as

$$V_i = \bigoplus_j W_{i,j}$$

where each $W_{i,j}$ is isomorphic to W_i , then we also have canonical injections :

$$\alpha_{i,j} : W_i \xrightarrow{\sim} W_{i,j} \longrightarrow V_i \longrightarrow V$$

and projection maps

$$\rho_{i,j} : V \longrightarrow V_i \longrightarrow W_{i,j} \xrightarrow{\sim} W_i$$

Notice that these maps are all morphisms of representations.

(a) Assume $h \neq 0$. By assumption, h is a morphism of representations.

Take the subrepresentation $\text{Ker}(h)$, we see that h is injective.

Compose h with the projection maps $\rho_{i,j}$, we see that as a vector space, we have

$$H_i = \bigoplus_j \text{Span}(\alpha_{i,j})$$

From this, we see that H_i has the required dimension.

(b) Since G acts trivially on each $\text{Span}(\alpha_{i,j})$ (the G -structure on H_i given in **EXERCISE 2.4**), the above decomposition is readily a direct sum decomposition. Restriction of F to each $\text{Span}(\alpha_{i,j}) \otimes W_i$ induce an isomorphism (of representations) to $W_{i,j}$.

(c) Since G acts trivially on each $\text{Span}(\alpha_{i,j})$ (as remarked in (b)), H_i is a trivial G -space, hence any direct sum decomposition H_i as direct sum of lines (as vector spaces), is readily a direct sum decomposition of trivial representations.

EXERCISE 2.9 Recall the maps $\alpha_{i,j}$ in **EXERCISE 2.8**. The image of the evaluation morphism is

$$\text{Span}(\alpha_{i,j}(e_\alpha)) = V_{i,\alpha}$$

so we are done already.

EXERCISE 2.10 Since $x \in V_i$, we have

$$x = p_i(x) = \left(\sum_\alpha p_{\alpha\alpha} \right)(x) = \left(\sum_\alpha p_{\alpha 1} p_{1\alpha} \right)(x) = \sum_\alpha p_{\alpha 1}(x_1^\alpha) \in \sum_\alpha W(x_1^\alpha)$$

It is easy to see that $V(x)$ admits another description :

$$V(x) = \text{Span}(\rho_t(x))_{t \in G}$$

By the direct description of $p_{1\alpha}$, we get

$$x_1^\alpha = p_{1\alpha}(x) = \frac{n}{g} \sum_{t \in G} r_{\alpha 1}(t^{-1}) \rho_t(x) \in \text{Span}(\rho_t(x))_{t \in G} = V(x)$$

These two observations establishes the identity

$$V(x) = \sum_\alpha W(x_1^\alpha)$$

3 SUBGROUPS, PRODUCTS, INDUCED REPRESENTATIONS

EXERCISE 3.1 Each ρ_t is an automorphism of representation, so are homotheties, hence any one dimensional subspace is a subrepresentation.

EXERCISE 3.2

- (a) Since ρ_s is a morphism for $s \in C$, we have by Schur's lemma that ρ_s is a homothety. Since the eigenvalues of ρ_s lies on the unit circle, we get $|\chi(s)| = n$ consequencely.
- (b) By orthogonality of characters and (a) that

$$g = \sum_{t \in G} |\chi(s)|^2 \geq \sum_{t \in C} |\chi(s)|^2 = cn^2$$

- (c) Recall that each element in G has finite order. Define $\xi = \exp(2\pi i/g)$. By (a), for each $t \in C$, there exists an integer μ_s such that

$$\rho_s = \xi^{\mu_s}$$

Let $d = \gcd(\mu_s)_{s \in C}$, then there exists integers a_s such that

$$d = \sum_{s \in C} a_s \mu_s$$

Define an element t in C by

$$t = \prod_{s \in C} s^{a_s}$$

then we get $\rho_t = \xi^d$. Since ρ is faithful, we see that t generates C .

EXERCISE 3.3 A character from an abelian group is just a homomorphism from G to \mathbb{C}^* . The group structure on \mathbb{C}^* then endows \hat{G} with a structure of an abelian group. To check that the map into the double dual of G is an injection, notice that

$$\chi(x) = \chi(y) \text{ for all } \chi \in \hat{G} \text{ iff } \chi(xy^{-1}) = 1 \text{ for all } \chi \in \hat{G} \text{ iff } x = y$$

This will suffice. For reasons of cardinality, this map is a bijection.

EXERCISE 3.4 Use the hint and Example 1,3 in that subsection.

EXERCISE 3.5 We already know that W can be identified as the space of elements in V that vanishes off H . Notice that the explicit description of the action of G on V implies that $\rho_s W$ is the space of elements in V that that vanishes off Hs^{-1} ; from this observation, the conditions of the definition of an induced representation is easily checked.

EXERCISE 3.6 The idea is to calculate the characters of ρ and $\theta \otimes r_K$. Let $(u, v) \in H \times K$, we get

$$\chi_\rho(uv) = \sum_{\substack{t \in K \\ t^{-1}uvt \in H}} \chi_\theta(t^{-1}uvt)$$

We pause for a bit to consider what does this summation does.

- Suppose $v = 1$, then we have for all $t \in K$ that $t^{-1}uvt = u$, and that

$$\{t \in K : t^{-1}uvt \in H\} = K$$

- Suppose $v \neq 1$, then we have for all $t \in K$ that $t^{-1}uvt \notin H$, so

$$\{t \in K : t^{-1}uvt \in H\} = \emptyset$$

From these two observations, we get

$$\chi_\rho(uv) = \sum_{\substack{t \in K \\ t^{-1}uvt \in H}} \chi_\theta(t^{-1}uvt) = \begin{cases} k\chi_\theta(u) & \text{if } v = 1 \\ 0 & \text{if } v \neq 1 \end{cases}$$

On the other hand, we know that

$$\chi_{\theta \otimes r_K}(uv) = \chi_\theta(u)\chi_{r_K}(v) = \begin{cases} k\chi_\theta(u) & \text{if } v = 1 \\ 0 & \text{if } v \neq 1 \end{cases}$$

so ρ is isomorphic to $\theta \otimes r_K$.

5 EXAMPLES

CONVENTION We use \mathfrak{A}_n (resp. \mathfrak{S}_n, C_n, D_n) to denote the n -th alternating (resp. symmetric, cyclic, dihedral) groups.

EXERCISE 5.1 We have the following identity :

$$s^\epsilon r^\alpha s^\eta r^\beta r^{-\alpha} s^\epsilon = \begin{cases} s^\eta r^\beta & \text{if } (\eta, \epsilon) = (0, 0) \\ s^\eta r^{-\beta} & \text{if } (\eta, \epsilon) = (0, 1) \\ s^\eta r^{\beta-2\alpha} & \text{if } (\eta, \epsilon) = (1, 0) \\ s^\eta r^{-(\beta-2\alpha)} & \text{if } (\eta, \epsilon) = (1, 1) \end{cases}$$

The rest follows easily from this calculation.

EXERCISE 5.2 Write $w = \exp(2\pi i/n)$. We have

$$\chi_h(s^\epsilon r^\alpha) = \delta_{\epsilon 0}(w^{hk} + w^{-hk})$$

So we get

$$\begin{aligned} (\chi_h \cdot \chi_{h'})(s^\epsilon r^\alpha) &= \left(\delta_{\epsilon 0}(w^{hk} + w^{-hk}) \right) \left(\delta_{\epsilon 0}(w^{h'k} + w^{-h'k}) \right) \\ &= \delta_{\epsilon 0} \left((w^{(h+h')k} + w^{-(h+h')k}) + (w^{(h-h')k} + w^{-(h-h')k}) \right) \\ &= (\chi_{h+h'} + \chi_{h-h'})(s^\epsilon r^\alpha) \end{aligned}$$

this establishes $\chi_h \cdot \chi_{h'} = \chi_{h+h'} + \chi_{h-h'}$. When $h = h'$, we get

$$\chi_h \cdot \chi_h = \chi_{2h} + \chi_0 = \chi_{2h} + \psi_1 + \psi_2$$

where we have used the identity $\chi_0 = \psi_1 + \psi_2$ destablished in the text.

Let us show that $\psi_2 = \text{Alt}^2 \chi_h$. It suffices to show that

$$\psi_2(s^\epsilon r^\alpha) = \frac{1}{2} (\chi_h(s^\epsilon r^\alpha)^2 + \chi_h((s^\epsilon r^\alpha)^2)), \quad \epsilon \in \{0, 1\}, \alpha \in \{0, \dots, h-1\}$$

Notice that

$$(s^\epsilon r^\alpha)^2 = \begin{cases} 1 & \text{if } \epsilon = 1 \\ r^{2\alpha} & \text{if } \epsilon = 0 \end{cases}$$

we get

$$\frac{1}{2} (\chi_h(s^\epsilon r^\alpha)^2 + \chi_h((s^\epsilon r^\alpha)^2)) = \begin{cases} \frac{1}{2}(0-2) = -1 & \text{if } \epsilon = 0 \\ \frac{1}{2} ((w^{hk} + w^{-hk})^2 - (w^{2hk} + w^{-2hk})) = 1 & \text{if } \epsilon = 1 \end{cases}$$

so we are done.

EXERCISE 5.4 Since H is normal, we get

$$(\text{Ind } \theta)(u) = \begin{cases} 0 & \text{if } u \notin H \\ \frac{1}{4} \sum_{v \in G} \theta(v^{-1}uv) & \text{if } u \in H \end{cases}$$

Decompose G as the disjoint union of the cosets H, tH, t^2H , we get

$$\begin{aligned} (\text{Ind } \theta)(x) &= \frac{1}{4} \sum_{v \in G} \theta(v^{-1}xv) \\ &= \frac{1}{4} \left(\sum_{v \in H} + \sum_{v \in tH} + \sum_{v \in t^2H} \right) \theta(v^{-1}xv) \\ &= \frac{1}{4} \left(\sum_{v \in H} \theta(v^{-1}xv) + \theta(v^{-1}yv) + \theta(v^{-1}zv) \right) \\ &= (\theta(x) + \theta(y) + \theta(z)) = -1 \end{aligned}$$

and that

$$(\text{Ind } \theta)(1) = \deg(\text{Ind } \theta) = \deg(\theta)[G : H] = 3 \deg(\theta) = 3$$

hence, this character coincides with ψ .

6 THE GROUP ALGEBRA

EXERCISE 6.1 Let us show (i) implies (ii). Define the nontrivial proper submodule V as given in the hint, suppose it has a complementary summand W . Take some nontrivial element

$$x = \sum_{t \in G} a_t t \in W, \quad \sum_{t \in G} a_t \neq 0$$

then the element $\sum_{s \in G} sx$ is non-trivial and lies in the intersection of W and V .

EXERCISE 6.2 The formula is bilinear in each argument, so it suffices to treat the case where $u, v \in G$. By definition, we get

$$\langle u, v \rangle = g \delta_{uv^{-1}}$$

so we only need to show

$$\delta_{uv^{-1}} = \frac{1}{g} \sum_i n_i \chi_i(uv^{-1})$$

which is clear by the character theory of the regular representation of G .

EXERCISE 6.3

(a) Since U is finite and contains G , the first assertion is clear. For the identity

$$\text{Tr}_{W_i}(\rho_i(s^{-1})u_i)^* = \text{Tr}_{W_i}(u'_i \rho_i(s))$$

recall that we may choose an eigenbasis for W_i with respect to ρ_i , so we may safely say that $\text{Tr}_{W_i}(\rho_i(s^{-1})u_i)$ is the sum of eigenvalues. Observe also that

$$(\rho_i(s^{-1})u_i)(u'_i \rho_i(s)) = 1$$

so we deduce the identity by using an argument like

$$\text{Tr}_{W_i}(\rho_i(s^{-1})u_i)^* = \sum \lambda_j^* = \sum \lambda_j^{-1} = \text{Tr}_{W_i}(u'_i \rho_i(s))$$

The identity

$$\text{Tr}_{W_i}(u'_i \rho_i(s)) = \text{Tr}_{W_i}(\rho_i(s)u'_i)$$

is clear by noticing that $u'_i \rho_i(s), \rho_i(s)u'_i$ are conjugates.

(b) By (a) and **EXERCISE 6.1**, we see that

$$\sum_{t \in G} |u(t)|^2 = \sum_{t \in G} u(t)u'(t^{-1}) = \frac{1}{g} \langle u, u' \rangle = \frac{1}{g} \sum_{i=1}^h n_i \text{Tr}_{W_i}(uu') = \frac{1}{g} \sum_{i=1}^h n_i^2 = 1$$

(c) Obvious.

(d) Take $U = \mathbb{Z}[G]$.

EXERCISE 6.4 By the computation

$$\omega_k(p_i) = \frac{1}{n_k} \sum_{t \in G} \frac{n_i}{g} \chi_i(t^{-1}) \chi_k(t) = \delta_{ik}$$

we see that the image of the elements p_i under the isomorphism described in proposition 13 form a basis of \mathbb{C}^h , so p_i form a basis, and it also follows from the computations

$$\omega_k(p_i p_j) = \omega_k(p_i) \omega_k(p_j) = \delta_{ik} \delta_{jk} = \delta_{ij} \delta_{ik} = \begin{cases} \omega_k(p_i) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\omega_k\left(\sum_i p_i\right) = \sum_i \delta_{ik} = 1 = \omega_k(1)$$

that the other required identities are verified.

EXERCISE 6.5 Let v be a homomorphism from the center of $\mathbb{C}[G]$ to \mathbb{C} . **EXERCISE 6.4** says

$$\begin{cases} v(p_i) \in \{0, 1\} \\ \sum_i v(p_i) = 1 \end{cases}$$

It is easy to deduce from this that v is just one of the ω_i .

EXERCISE 6.6 Firstly, since

$$\bigoplus_i \mathbb{Z}e_i \subseteq \text{Cent.}(\mathbb{C}[G])$$

we get $\bigoplus_i \mathbb{Z}e_i \subseteq \text{Cent.}(\mathbb{Z}[G])$. For the reverse, choose some

$$u = \sum_{t \in G} u(t)t \in \text{Cent.}(\mathbb{Z}[G])$$

Suppose $t, t' \in G$ lie in the same conjugacy class, say,

$$st = t's$$

for some $s \in G$, then by $us = su$, we get $u(t) = u(t')$. This observation shows $u \in \bigoplus_i \mathbb{Z}e_i$.

EXERCISE 6.7 By the hint, we are done by applying the triangle inequality.

EXERCISE 6.8 (For each nonnegative integer N , we define $\xi_N := \exp(2\pi i/N)$.)

Let $f(x)$ be the minimal polynomial of a over \mathbb{Q} , then $f(x) \in \mathbb{Z}[x]$ by integrality.

In order to show $|A| \leq 1$, it suffices to show the following :

Claim : The conjugates of a (roots of $f(x)$) over \mathbb{Q} all have length ≤ 1 .

Since λ_i are roots of unities, may take an integer $N \gg 0$ such that $\lambda_i \in \{\xi_N^0, \dots, \xi_N^{N-1}\}$.

We have the following diagram of field extensions :

$$\mathbb{Q}(\xi_N) \text{ ——— } \mathbb{Q}(a) \text{ ——— } \mathbb{Q}$$

Recall that the Galois group of the cyclotomic extension $\mathbb{Q}(\xi_N)/\mathbb{Q}$ is given by

$$\text{Gal}(\mathbb{Q}(\xi_N)/\mathbb{Q}) = \{\theta_d : \xi_N \mapsto \xi_N^d \mid (d, N) = 1\} \simeq (\mathbb{Z}/N\mathbb{Z})^\times$$

Define a polynomial

$$F(x) := \prod_{\sigma \in \text{Gal}(\mathbb{Q}(\xi_N)/\mathbb{Q})} (x - \sigma(a)) = \prod_{d: (d, N)=1} (x - \theta_d(a))$$

then $F(x) \in \mathbb{Q}[x]$ and $f(x) \mid F(x)$ by the property of minimal polynomials.

By the definition of θ_d , we have the following calculation :

$$\theta_d(a) = \theta_d\left(\frac{1}{n} \left(\sum_{i=1}^h \lambda_i\right)\right) = \frac{1}{n} \left(\sum_{i=1}^h \lambda_i^d\right)$$

so $|\theta_d(a)| \leq 1$, so the roots of $F(x)$ (hence those of $f(x)$) all have length ≤ 1 ; this proves the claim.

By the definition of $f(x)$, we see that the constant term of $f(x)$ is $\pm A$, so we get $A \in \mathbb{Z}$.

Since $|A| \leq 1$, $A \in \{0, \pm 1\}$. If $A = 0$, $a = 0$; If $A = \pm 1$, $|a| = 1$ by our claim, and hence $\lambda_i = a$ for all i .

EXERCISE 6.9 Since the e_i lies in the center of the group algebra, it follows immediately that

$$\frac{c(s)}{n} \chi(s)$$

are all algebraic integers (for each $s \in G$). For the second assertion, we claim that

$$\frac{1}{n} \chi(s)$$

is an algebraic integer; this follows from the fact that $\chi(s)/n$ is a \mathbb{Z} -combination of the algebraic integers $\chi(s)$ and $c(s)\chi(s)/n$ by using the assumption that $(c(s), n) = 1$.

By **EXERCISE 6.8**, the eigenvalues of ρ_s are all the same given $\chi(s) \neq 0$, so ρ_s is a homothety.

EXERCISE 6.10 By the character theory of the regular representation of G , we get

$$1 + \sum_{\chi \neq 1} \chi(1)\chi(s) = 0$$

whenever $s \neq 1$. Notice that suppose every irreducible character of G satisfies

$$\text{Either } \chi(s) = 0 \text{ or } p \mid \chi(1)$$

then we see that

$$-1 = \sum_{\chi \neq 1} \chi(1)\chi(s) = \sum_{\substack{\chi \neq 1 \\ p \mid \chi(1)}} \chi(1)\chi(s)$$

This formula then exhibits $1/p$ as an algebraic integer in view of Proposition 15.

For the second assertion, notice that given ρ, χ satisfying the condition, we see that

$$\chi(s) \neq 0 \text{ and } (\chi(1), c(s)) = 1$$

so in view of **EXERCISE 6.9**, $\rho(s)$ is just a homothety.

For the last part, notice first that since χ isn't trivial, $N \neq G$. Next, since

$$sN \in \text{Cent.}(G/N) \text{ iff the commutator } [st] \text{ with any } t \in G \text{ lies in } G \text{ iff } \rho([st]) = 1 \text{ for all } t \in G$$

we see that the last assertion is clearly true by the observation that $\rho(s)$ is a homothety.

7 INDUCED REPRESENTATIONS; MACKEY'S CRITERION

EXERCISE 7.1 It is easier to verify via the universal property of Ind .

(a) Factoring α through image, may assume α is a surjection and that α is a projection

$$\alpha : H \longrightarrow H/N$$

for some normal subgroup N of H . Recall that

$$\text{Ind}_\alpha(W) = W \otimes_{\mathbb{C}[H]} \mathbb{C}[H/N]$$

Let us show first that this space is isomorphic to the one described in (b). Call this space V . Notice that we can define maps of abelian groups

$$V \xrightleftharpoons[v]{\mu} \text{Ind}_\alpha(W)$$

given by

$$\mu(v) = v \otimes N, \quad v(w \otimes hN) = wh$$

It can be checked that the maps are well-defined,¹ inverse to each other, and are all G -maps. Now, let us construct natural bijections between

$$\text{Hom}^H(W, \text{Res}_\alpha(E)) \xrightleftharpoons[\eta]{\xi} \text{Hom}^{H/N}(V, E)$$

¹To show legitimacy, let us briefly recall a construction of tensor product of two algebras over a ring.

Suppose given right- R algebra A and left- R algebra B , define the underlying set of the set $A \otimes_R B$ to be the same as that of the tensor product of R -modules $A \otimes_R B$, with algebra structure given by $(a \otimes b)(a' \otimes b') = (aa' \otimes bb')$.

Hence, well-definedness of the map v can be verified using this description.

On the other hand, we know that as an abelian group, $A \otimes_R B$ can be described as the quotient group $(A \times B)/R$, where R is generated over \mathbb{Z} by the elements of the form $(a+a', b) - ((a, b) + (a', b))$, $(a, b+b') - ((a, b) + (a, b'))$, $(ar, b) - (a, rb)$. This gives us a structural map (only as a homomorphism of abelian groups!) of $A \otimes_R B$ given by

$$A \times_R B \longrightarrow A \otimes B, \quad (a, b) \longmapsto a \otimes b$$

This description allows one to check that μ is well-defined.

That is : restriction and induction are adjoints. We verify via universal property.
Suppose we are given an H-map

$$g : W \longrightarrow \text{Res}_\alpha(E)$$

we want to show that there exists a unique G-map f

$$f : V \longrightarrow E$$

so that $\text{Res}_\alpha(f) = g$. There is only one way to proceed : define f by

$$V \longrightarrow W \longrightarrow \text{Res}_\alpha(E) = E$$

We also omit the verification that these bijections are natural. To deduce the formula

$$\langle \psi, \text{Res}_\alpha(\varphi) \rangle_H = \langle \text{Ind}_\alpha \psi, \varphi \rangle_{H/N}$$

notice that these quantities are just dimensions of the Hom-sets.

- (b) We have seen the identification between V and $\text{Ind}_\alpha(W)$ in (a).
For the formula, define a class function on H/N by

$$\mu(\sigma) = \frac{1}{n} \sum_{\substack{t \in H \\ \alpha(t) = \sigma}} \psi(t)$$

then by applying the generalized Frobenius reciprocity, we get

$$\begin{aligned} \langle \text{Ind}_\alpha \psi, \varphi \rangle_{H/N} &= \langle \psi, \text{Res}_\alpha \varphi \rangle_H = \frac{1}{h} \sum_{t \in H} \psi(t) \varphi(\alpha(t)^{-1}) = \frac{1}{h} \sum_{\sigma \in H/N} \sum_{\substack{t \in H \\ \alpha(t) = \sigma}} \psi(t) \varphi(\sigma^{-1}) \\ &= \frac{1}{h/n} \sum_{\sigma \in H/N} \left(\frac{1}{n} \sum_{\substack{t \in H \\ \alpha(t) = \sigma}} \psi(t) \right) \varphi(\sigma^{-1}) = \frac{1}{h/n} \sum_{\sigma \in H/N} \mu(\sigma) \varphi(\sigma^{-1}) = \langle \mu, \varphi \rangle_{H/N} \end{aligned}$$

so we deduce that $\text{Ind}_\alpha \psi = \mu$ by using the non-degeneracy of $\langle, \rangle_{H/N}$.

EXERCISE 7.2 By the formula of induced representations, we have

$$\text{Ind } 1(s) = \sum_{\substack{t \in G \\ t^{-1}st \in H}} 1(t^{-1}st) = |\{t \in G : t^{-1}st \in H\}| = |\{t \in G : stH = tH\}|$$

By description of a permutation representation, this quantity is just $\chi(s)$.

To show that $\psi := \chi - 1$ is a character, notice that

$$\langle \chi, 1 \rangle_G = \langle \text{Ind } 1, 1 \rangle_G = \langle 1, \text{Res } 1 \rangle_H = \langle 1, 1 \rangle_H = 1$$

so ψ is a character. We refer the remaining assertions to **EXERCISE 2.6**.

EXERCISE 7.3

(a) By the description of N , we see that

$$G \setminus N = \bigcup_{t \in G} tHt^{-1} = \bigcup_{r \in R} rHr^{-1}$$

where R a set of representatives in G for the cosets G/H . Applying the condition that H is Frobenius, we see that this set has cardinality $g - (g/h - 1)$, so we are done.

(b) It is easy to see that the property that an element lies in N or not is stable under conjugation. The condition that H is Frobenius and that \bar{f} is a class function gives a unique extension of f to $G \setminus N$, and since $N \cap H = \emptyset$, extending by $f(1)$ to N is also legitimate.

(c) By the formula of the characters of induced class functions, we get

$$(\text{Ind } f - f(1)\psi)(s) = (\text{Ind } f - \text{Ind } 1)(s) + f(1) = \frac{1}{h} \sum_{\substack{t \in G \\ tst^{-1} \in H}} (f(tst^{-1}) - f(1)) + f(1)$$

Suppose $s \in N$, we get from the definition of N that

$$\{t \in G : tst^{-1} \in H\} = \emptyset$$

hence the result if $f(1)$. Now suppose $s \notin N$ and that $tst^{-1} \in H$, then

$$\{t \in G : tst^{-1} \in H\} = sH$$

so we get

$$\frac{1}{h} \sum_{\substack{t \in G \\ tst^{-1} \in H}} f(tst^{-1}) = f(tst^{-1}), \quad \frac{1}{h} \sum_{\substack{t \in G \\ tst^{-1} \in H}} f(1) = f(1)$$

so the result is $f(tst^{-1})$. Therefore, $\bar{f} = \text{Ind } f - f(1)\psi$.

(d) We can do the computation explicitly as follows : By definition, we have

$$\langle \bar{f}_1, \bar{f}_2 \rangle_G = \frac{1}{g} \left(\sum_{s \in G \setminus N} + \sum_{s \in N} \right) \bar{f}_1(s) \bar{f}_2(s^{-1})$$

For the first summation, by the explicit description of the set $G \setminus N$, we have

$$\begin{aligned} \sum_{s \in G \setminus N} \bar{f}_1(s) \bar{f}_2(s) &= \left(\left(\sum_{r \in R} \sum_{t \in H} \right) - \left(\frac{g}{h} - 1 \right) \left(\sum_{h=e} \right) \right) f_1(t) f_2(t^{-1}) \\ &= \frac{g}{h} \left(\sum_{t \in H} f_1(t) f_2(t^{-1}) \right) - \left(\frac{g}{h} - 1 \right) f_1(1) f_2(1) = g \langle f_1, f_2 \rangle_H - \left(\frac{g}{h} - 1 \right) f_1(1) f_2(1) \end{aligned}$$

On the other hand, we have

$$\sum_{s \in N} \bar{f}_1(s) \bar{f}_2(s^{-1}) = \left(\frac{g}{h} - 1 \right) f_1(1) f_2(1)$$

Combining these together, we get $\langle \bar{f}_1, \bar{f}_2 \rangle_G = \langle f_1, f_2 \rangle_H$.

(e) Under the assumption that f is an irreducible character, we have the following :

$$\langle \bar{f}, \bar{f} \rangle_G = \langle f, f \rangle_H = 1$$

$$\bar{f}(1) = (\text{Ind } f)(1) - f(1)\psi(1) = f(1)(g/h) - f(1)(g/h - 1) = f(1)$$

On the other hand, suppose given an irreducible character g of G , we have by reciprocity that

$$\langle \bar{f}, g \rangle_G = \langle \text{Ind } f - f(1)((\text{Ind } 1) - 1), g \rangle_G = \langle f - f(1), \text{Res } g \rangle_H + f(1) \langle \text{Ind } f, 1 \rangle_G$$

hence $\langle \bar{f}, g \rangle_G$ are all integers; this shows f is \mathbb{Z} -combination of irreducible characters.

To show the last assertion, it suffices by **EXERCISE 6.7** to show that $\bar{f}(s) = \bar{f}(1)$ for $s \in N$, but this is by the definition of \bar{f} .

(f) Let ρ_1, \dots, ρ_h be the irreducible characters of H with corresponding characters χ_1, \dots, χ_h . By (e), each $\bar{\chi}_i$ corresponds to some representation of $\bar{\rho}_i$ of G . Define a subgroup of G

$$N^+ := \bigcap_{i=1}^h \text{Ker}(\bar{\rho}_i)$$

so N^+ is a normal subgroup. By (e), we already know that $N \cup \{1\} \subseteq N^+$.

The reverse inclusion is clear by recalling that the irreducible characters of H span the space of class functions on H along with the explicit description of $G \setminus N$ and \bar{f} .

Let us show $HN^+ = G$, but this is clear by the fact that $H \cap N^+ = \{1\}$ and that $|H||N^+| = |G|$.

(g) Assume first that H is Frobenius. Suppose there exists some $(s, t) \in H \times A$ such that $st = ts$, we get $s = tst^{-1} \in H \cap tHt^{-1}$. Suppose $t \neq 1$, then $s = 1$.

Conversely, the action described in the question can be identified with another one : if we regard A as the set of left cosets G/H , then the conjugation action of H on A can be described as the action of H on G/H by multiplying on left, and the coset representatives of G/H can be chosen to be A . Now let us make the following assumption :

Suppose there exists $t \in A$ and $s \in H$ both not equal to 1 with $t^{-1}st \in H$.

then we have $stH = tH$, so the action of H on G/H isn't free, a contradiction.

EXERCISE 7.4 The idea is to use Mackey's irreducibility criterion.

Fix $M \in G \setminus H$, form the group $H_M = MHM^{-1} \cap H$, and let $X \in H_M$. Write M, X as

$$M = \begin{bmatrix} p & q \\ r & s \end{bmatrix}, \quad X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

satisfying the relations

$$ps - qr = 1, r \neq 0, \quad ad = 1, c = 0$$

Using the condition that $X \in MHM^{-1}$, we have

$$M^{-1}XM = \begin{bmatrix} s & -q \\ -r & p \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} \in H$$

Let us calculate the 1st column of the resulting matrix :

$$M^{-1}XM = \begin{bmatrix} sa - qc & sb - qd \\ -ra + pc & -rb + pd \end{bmatrix} \begin{bmatrix} p \\ r \end{bmatrix} = \begin{bmatrix} psa + rsb - qrd & * \\ -pra - r^2b + prd & * \end{bmatrix} \in H$$

where we have plugged in the condition $c = 0$. For brevity, let us write

$$\begin{bmatrix} psa + rsb - qrd \\ -pra - r^2b + prd \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix}$$

We have the following :

$$\begin{cases} B &= 0 \\ A + r^{-1}sB &= (ps - qr)d \end{cases}$$

we get $A = a^{-1}$ using the identities $ad = 1, ps - qr = 1$.

From this preliminary calculation, we find that

$$\chi_\omega^M = \text{Res}_M(\chi_\omega^{-1})$$

for each $M \in G \setminus H$. Therefore, we have

$$\langle \text{Res}_M(\chi_\omega), \chi_\omega^M \rangle_{H_M} = \langle \text{Res}_M(\chi_\omega), \text{Res}_M(\chi_\omega^{-1}) \rangle_{H_M} = \frac{1}{|H_m|} \sum_{X \in H_m} \chi_\omega^2(X)$$

Since $|\chi_\omega(X)| \leq 1$ for each $X \in H$, we see that $\text{Res}_M(\chi_\omega)$ is isomorphic to χ_ω^M iff $\text{Res}_M(\chi_\omega)^2 = 1$. Letting M vary, we obtain from Mackey's criterion that χ_ω is irreducible iff $\chi_\omega^2 = 1$.

8 EXAMPLE OF INDUCED REPRESENTATIONS

EXERCISE 8.1 The statement that

$$a = \sum_i \frac{h}{h_i}$$

follows from the orbit-stabilizer theorem applied to the action of H on A . As for

$$\sum_\rho (\deg \theta_{i,\rho})^2 = \frac{h^2}{h_i}$$

is by noticing that

$$\deg \theta_{i,\rho} = \deg \left(\text{Ind}_{G_i}^G (\chi_i \otimes \bar{\rho}) \right) = [G : G_i] \deg (\rho(1))$$

Since we also have

$$\sum_\rho (\deg \rho)^2 = h_i$$

the identity is proved. Finally, for the proof of (c) of Proposition 25, notice that

$$g = ha = h \left(\sum_i \frac{h}{h_i} \right) = \sum_i \frac{h^2}{h_i} = \sum_i \left(\sum_\rho \deg(\theta_{i,\rho})^2 \right)$$

so by (a),(b) of Proposition 25, we are done.

EXERCISE 8.2 .

EXERCISE 8.3 The group D_n has a filtration

$$0 \leq C_n \leq D_n$$

which exhibits D_n as a supersolvable group. Let us consider nilpotency. By the calculation in **EXERCISE 5.1**, we get

$$\text{Cent.}(D_n) = \begin{cases} \{1\} & \text{if } n \text{ is odd.} \\ \{1, r^{n/2}\} & \text{if } n \text{ is even.} \end{cases}$$

so this shows for D_n to be nilpotent, it is necessary that n be a power of 2. For sufficiency, let $n = 2^k$, then one can consider a filtration

$$\{1\} = C_1 \leq C_2 \leq \dots \leq C_{2^k} \leq D_{2^k}$$

This filtration exhibits D_n as a nilpotent group.

EXERCISE 8.4 By **EXERCISE 8.5**, it suffices to show

$$\mathfrak{A}_4 \text{ isn't supersolvable, } \mathfrak{S}_4 \text{ is solvable.}$$

The fact that \mathfrak{S}_4 is solvable can be seen from the subnormal series

$$\{1\} \leq H \leq \mathfrak{A}_4 \leq \mathfrak{S}_4$$

where H is the normal subgroup of \mathfrak{A}_4 consisting of 1 and product of disjoint 2-cycles. Conversely, recall that the conjugacy classes of \mathfrak{A}_4 are

$$\{1\}, H \setminus \{1\}, tH, t^2H$$

It is easy to see that the only normal subgroups of \mathfrak{A}_4 are $H, \{1\}$, but H isn't cyclic.

EXERCISE 8.5 Suppose we are given a defining series of subgroups of G exhibiting the solvability (resp. supersolvability, nilpotency)

$$\{1\} = G_0 \leq \dots G_{n-1} \leq G_n = G$$

and another subgroup H of G , one get - by defining $H_i = G_i \cap H$ - a series of H

$$\{1\} = H_0 \subseteq \dots H_{n-1} \subseteq H_n = H$$

We claim that these series really exhibits solvability (resp. supersolvability, nilpotency) of H . As a preliminary step, let G', H be subgroups of G and N a normal subgroup of G' , observe that

- (Normality) $N \cap H$ is still a normal subgroup of $G' \cap H$.
- (Factor groups) We have

$$\frac{G' \cap H}{N \cap H} \xrightarrow{\sim} \frac{(G' \cap H)N}{N} \longrightarrow \frac{G'}{N}$$

- (Center) We have the relation $\text{Cent.}(G' \cap H) \leq \text{Cent.}(G')$.

These observations shows that a defining series for solvability (resp. supersolvability, nilpotency) is still a defining series for solvability (resp. supersolvability, nilpotency) under passage to subgroups. For quotients, suppose given a normal subgroup N of G , we have a series

$$N = NG_0 \subseteq \dots NG_{n-1} \subseteq NG_n = G$$

As a preliminary step, let $H \leq G$, $N \trianglelefteq G$, $K \trianglelefteq H$, observe that

- (Normality) NK is still a normal subgroup of NH .
- (Factor groups) We have

$$\frac{NH}{NK} \xleftarrow{\sim} \frac{H}{NK \cap H} \xleftarrow{\sim} \frac{H}{K}$$

- (Center) We have the relation $(N \text{Cent.}(G'))/N \leq \text{Cent.}(NG'/N)$.

one sees that after passing the above series to quotients

$$\{1\} = NG_0/N \subseteq \dots NG_{n-1}/N \subseteq NG_n/N = G/N$$

we get a defining series for solvability (resp. supersolvability, nilpotency) of G/N .

EXERCISE 8.6

- (i) As indicated in the hint, if $c(s) \equiv 0 \pmod{p}$ for all $s \neq 1$, we would have $q|(g-1)$, this shows existence of s . By **EXERCISE 6.10**, get a nontrivial irreducible character χ of G with

$$\chi(s) \neq 0, \quad \chi(1) \not\equiv 0 \pmod{p}$$

so by taking kernels, one gets a nontrivial proper normal subgroup of G .

- (ii) Do induction. By (i), G always have a nontrivial proper normal subgroup N . Take exact sequence

$$0 \longrightarrow N \longrightarrow G \longrightarrow G/N \longrightarrow 0$$

By inductive hypothesis and theorem 14, get solvability of G/N and N and hence G .
Base case is done in theorem 14.

- (iii) Consider the groups $\mathfrak{A}_4, \mathfrak{S}_4$.

- (iv) Since \mathfrak{A}_5 (resp. \mathfrak{A}_6) is a normal simple subgroup of \mathfrak{S}_5 (resp. \mathfrak{S}_6), \mathfrak{S}_5 (resp. \mathfrak{S}_6), is nonsolvable. For $\text{GL}_2(\mathbb{F}_7)$, we note that $\text{PSL}(2, 7)$ is simple, non-abelian (I don't know how to show this).

EXERCISE 8.7 As a preliminary step, notice that by surjectivity of the map

$$G \longrightarrow G/H$$

such a group P must be contained in the inverse image Q of P_H under the map.

- (a) Take any Sylow p -subgroup P of Q . Schematically :

$$\begin{array}{ccccc} & & H & & \\ & \swarrow & & \searrow & \\ G & \longrightarrow & Q & & \{1\} \\ & \searrow & & \swarrow & \\ & & P & & \end{array} \quad G/H \xrightarrow{*} P_H \longrightarrow \{1\}$$

where the "*" means "the right one is a Sylow p -subgroup of the left one". Notice that :

- A p -subgroup is p -Sylow iff the index is prime to p .
- P is a Sylow p -subgroup of G : we already know P is a p -group. Now by

$$[G : P] = [G : Q][Q : P] = [G/H : P_H][Q : P]$$

this gives $([G : P], p) = 1$.

- PH/H is a subgroup of $Q/H = P_H$.
- PH/H is a Sylow p -subgroup : we already know that it is a p -group. Now by

$$[G/H : PH/H] = [G : PH] = \frac{[G : P]}{[PH : P]}$$

we get $([G/H : PH/H], p) = 1$.

Therefore, PH/H - the image of P under the projection map from G to G/H - is P_H .

- (b) The case where H is a p -group is clear : Q already had the correct cardinality.
For the case where $H \leq \text{Cent.}(G)$, let us first reduce to the case where $(|H|, p) = 1$:
take a Sylow p -subgroup of H - denote it as K , then one has quotient maps :

$$G \longrightarrow G/K \longrightarrow G/H$$

By induction, we can find a unique group in G/K , and by (a), a unique group in G .

Now we may safely assume $(|H|, p) = 1$.

By second isomorphism theorem, if P is a solution, it must satisfy $P \cap H = 1$.

In this sense, if P, P' are solutions, we have $P_1H = P_2H = Q$ and $P_1 \cap H = P_2 \cap H = \{1\}$.

From this, we define a homomorphism ϕ from P_H to H as follows :

- Let $x \in Q$, it has unique expressions

$$x = x_1s_1 = x_2s_2, \quad x_1 \in P_1, \quad x_2 \in P_2, \quad s_1, s_2 \in H_2$$

- Define first a map Φ from Q to H by $x \mapsto x_1^{-1}x_2$. This is well-defined.
- This map is a homomorphism : If we take

$$x = x_1s_1 = x_2s_2, \quad y = y_1t_1 = y_2t_2$$

for suitable x_i, y_i, s_i, t_i , we get first that

$$xy = x_1y_1s_1t_1 = x_2y_2s_2t_2$$

which gives us

$$\Phi(x)\Phi(y) = (x_1^{-1}x_2)(y_1^{-1}y_2) = y_1^{-1}x_1^{-1}x_2y_2 = (x_1y_1)^{-1}(x_2y_2) = \Phi(xy)$$

where we have used the observation that $H \leq \text{Cent.}(G)$.

- This homomorphism clearly factors through H ; this finishes the construction of ϕ .
- Schematically, these homomorphisms are organized as follows :

$$\begin{array}{ccc} Q & \xrightarrow{\Phi} & H \\ & \searrow & \nearrow \phi \\ & P_H & \end{array}$$

Now since $(|H|, p) = 1$ and P_H is a p -group, $\phi = 1$, which gives $P_1 = P_2^{-1} = P_2$.

EXERCISE 8.8 One can do an induction on the length of the defining series for nilpotency. For a length 1 series, that is, for the case where G is abelian, we are done by conjugacy of Sylow- p s. For length $n + 1$, apply inductive hypothesis on G_n , then apply the base case to G/G_n .

EXERCISE 8.9 Denote the subgroup as T . We have the following :

$$|G| = (k^n - 1) \dots (k^n - k^{n-1}), \quad |T| = k^{1+\dots+(n-1)}$$

it is easy to see that $(|G|/|T|, p) = 1$.

EXERCISE 8.10 Recall from Proposition 25 that each irreducible representation of HA has form

$$\theta_{i,\rho} = \text{Ind}_{H_i A}^{\text{HA}} \left(\text{Res}_A^{H_i A} \chi_i \otimes \text{Res}_{H_i}^{H_i A} \rho \right)$$

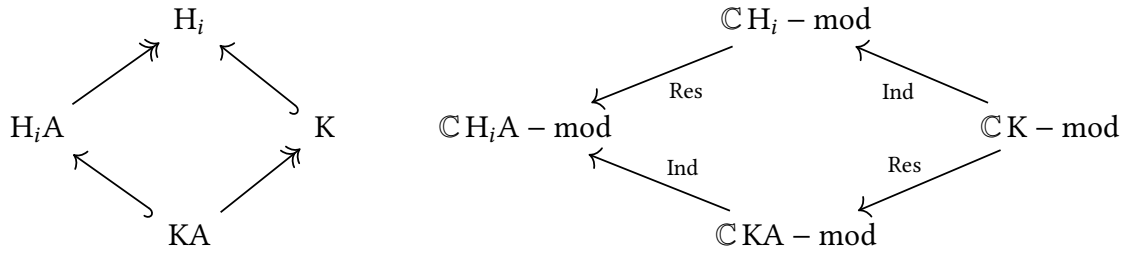
where the operations Res are in the sense of restriction along projection maps to components in the sense of **EXERCISE 7.1** . By Theorem 16 (and the fact that subgroups of supersolvables are supersolvables), ρ can be written as

$$\rho = \text{Ind}_K^{H_i} \varphi$$

for some subgroup K of H_i and for some degree 1 representation of K . Focus on the representation

$$\text{Res}_{H_i}^{H_i A} \rho = \text{Res}_{H_i}^{H_i A} \text{Ind}_K^{H_i} \varphi$$

We have diagram of group homomorphisms and corresponding constructions :



It can be checked that the following identity holds : ²

$$\text{Res}_{H_i}^{H_i A} \text{Ind}_K^{H_i} \varphi \simeq \text{Ind}_{K A}^{H_i A} \text{Res}_K^{K A} \varphi$$

so plug it back to the expression of $\theta_{i,\rho}$, we get

$$\begin{aligned} \theta_{i,\rho} &= \text{Ind}_{H_i A}^{\text{HA}} \left(\text{Res}_A^{H_i A} \chi_i \otimes \text{Res}_{H_i}^{H_i A} \rho \right) \\ &\simeq \text{Ind}_{H_i A}^{\text{HA}} \left(\text{Res}_A^{H_i A} \chi_i \otimes \text{Res}_{H_i}^{H_i A} \text{Ind}_K^{H_i} \varphi \right) \\ &\simeq \text{Ind}_{H_i A}^{\text{HA}} \left(\text{Res}_A^{H_i A} \chi_i \otimes \text{Ind}_{K A}^{H_i A} \text{Res}_K^{K A} \varphi \right) \\ &\simeq \text{Ind}_{H_i A}^{\text{HA}} \text{Ind}_{K A}^{H_i A} \left(\text{Res}_{H_i A}^{K A} \text{Res}_A^{H_i A} \chi_i \otimes \text{Res}_K^{K A} \varphi \right) \\ &\simeq \text{Ind}_{K A}^{\text{HA}} \left(\text{Res}_{H_i A}^{K A} \text{Res}_A^{H_i A} \chi_i \otimes \text{Res}_K^{K A} \varphi \right) \end{aligned}$$

This exhibits $\theta_{i,\rho}$ as the induced representation of some 1-dimensional representation.

²I think that this is true, but am too lazy to check it.

EXERCISE 8.11 The assertion that G is the semidirect product of the normal subgroup E by an order 3 subgroup H can be deduced from the fact that conjugation by i, j, k for elements in E only changes the signs of 2 elements in it : one is itself, the other one is 1. We may take

$$\mu := \frac{1}{2}(1 + i + j + k), \quad H := \langle -\mu \rangle = \{1, -\mu, \mu - 1\}$$

To show G is solvable, it suffices to show that E is solvable, but this is clear by

$$\{1\} \trianglelefteq \{\pm 1\} \trianglelefteq \{\pm 1, \pm i\} \trianglelefteq \{\pm 1, \pm i, \pm j, \pm k\} = E$$

The isomorphism

$$H \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} M_2(\mathbb{C})$$

is given by the assignment ³

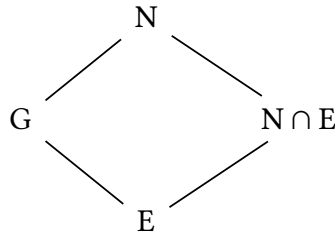
$$1 \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad i \longmapsto \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \quad j \longmapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad k \longmapsto \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

which gives us a representation of G by using the composition

$$\rho : G \longrightarrow H \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} M_2(\mathbb{C})$$

The fact that ρ is irreducible can be seen from $\langle \chi, \chi \rangle_G = 1$ (to calculate, sum over each coset).

Let's show that G has no subgroup of index 2. If N is one of such, it must be normal. In the diagram



each line indicates normality. We have by 2nd isomorphism theorem that

$$[G : N] = [E : N \cap E]$$

The normal subgroups in E with index 2 are

$$L_i := \{\pm 1, \pm i\}, \quad L_j := \{\pm 1, \pm j\}, \quad L_k := \{\pm 1, \pm k\}$$

If $E \cap N = L_i$, we get $N = HL_i$, but this will contradict normality of N : we have

$$\mu i = k\mu, \quad (-\mu)i(-\mu)^{-1} = -\mu i(\mu - 1) = -k(\mu^2 - \mu) = k \notin N$$

The proof is similar for other cases.

EXERCISE 8.12 Let X be the set of irreducible characters and X' be those with degree bigger than the degree of χ , we get the following :

$$\sum_{\chi' \in X} \chi'(1)^2 = g \equiv 0 \pmod{p}$$

$$\sum_{\chi' \in X'} \chi'(1)^2 \equiv \sum_{\chi' \in X'} 0 \equiv 0 \pmod{p}$$

By taking the difference of the sum, one gets the desired assertion.

³Found this in Wikipedia : <https://tinyurl.com/y24hdxzr>

9 ARTIN'S THEOREM

EXERCISE 9.1 For each $s \in G$, we always have

$$\operatorname{Re}(\chi(s)) \leq \operatorname{Re}(\chi(1))$$

by arguing on eigenvalues. This gives

$$\operatorname{Re}(\varphi(s^{-1})\chi(s)) \geq \operatorname{Re}(\varphi(s^{-1})\chi(1))$$

so we get the following :

$$\operatorname{Re}(\langle \varphi, \chi \rangle_G) = \operatorname{Re}\left(\frac{1}{g} \left(\sum_{s \in G} \varphi(s^{-1})\chi(s) \right)\right) \geq \operatorname{Re}\left(\frac{1}{g} \left(\sum_{s \in G} \varphi(1)\chi(s) \right)\right) = \varphi(1) \operatorname{Re}(\langle \chi, 1 \rangle_G) = 0$$

EXERCISE 9.2 Suppose that

$$\chi = \sum_i n_i \chi_i, \quad n_i \in \mathbb{Z}$$

then $\langle \chi, \chi \rangle_G = 1$ shows that

$$\sum_i n_i^2 = 1$$

so either χ or $-\chi$ is irreducible. The condition $\chi(1) > 0$ ensures that χ is the one.

EXERCISE 9.3

(a) Fix $s \in G$. Recall that we can diagonalize ρ_s . Let $\lambda_1, \dots, \lambda_n$ be its eigenvalues. We get

$$\chi_\sigma^k(s) = \sum_{1 \leq i(1) \leq \dots \leq i(k) \leq n} \lambda_{i(1)} \dots \lambda_{i(k)}, \quad \chi_\lambda^k(s) = \sum_{1 \leq i(1) < \dots < i(k) \leq n} \lambda_{i(1)} \dots \lambda_{i(k)}$$

On the other hand, we also have

$$\det(1 - \rho_s T)^{-1} = \left(\prod_{k=1}^n (1 - \lambda_k T) \right)^{-1} = \prod_{k=1}^n \left(\sum_{l=1}^{\infty} \lambda_k^l T^l \right), \quad \det(1 + \rho_s T) = \prod_{k=1}^n (1 + \lambda_k T)$$

so by comparing degree-wise, we get

$$\det(1 - \rho_s T)^{-1} = \sum_{k=1}^{\infty} \chi_\sigma^k T^k = \sigma_T(\chi), \quad \det(1 + \rho_s T) = \sum_{k=1}^{\infty} \chi_\lambda^k T^k = \lambda_T(\chi)$$

For the last part, notice first that

$$\Psi^k(\chi)(s) = \chi(s^k) = \sum_{i=1}^n \lambda_i^k$$

We first prove the identities

$$m\chi_\sigma^n = \sum_{i=1}^k \Psi^k(\chi) \chi_\sigma^{m-k}, \quad m\chi_\lambda^m = \sum_{i=1}^k (-1)^{k-1} \Psi^k(\chi) \chi_\lambda^{m-k}$$

Let $[n]$ be the (linearly-ordered set) $\{1, \dots, n\}$, define the following sets :

$$S(m, n) = \{\text{increasing functions } i : [m] \rightarrow [n]\}$$

$$L(m, n) = \{\text{strictly increasing functions } i : [m] \rightarrow [n]\}$$

Let $f \in S(m, n)$ or $L(m, n)$, we also denote λ_f as $\lambda_{f(1)} \dots \lambda_{f(n)}$. We get

$$\chi_\sigma^m(s) = \sum_{f \in S(m, n)} \lambda_f, \quad \chi_\lambda^m(s) = \sum_{f \in L(m, n)} \lambda_f$$

so we get the following :

$$\sum_{i=1}^k \Psi^k(\chi) \chi_\sigma^{m-k} = \sum_{i=1}^n \sum_{k=1}^m \lambda_i^k \left(\sum_{g \in S(m-k, n)} \lambda_g \right)$$

One deduce that for each $f \in S(m, n)$, λ_f occurs in this triple sum m times, hence

$$= \sum_{f \in S(m, n)} \lambda_f = \chi_\sigma^m(s)$$

Now we consider the summation

$$\sum_{i=1}^k (-1)^{k-1} \Psi^k(\chi) \chi_\lambda^{m-k} = \sum_{i=1}^n \sum_{k=1}^m (-1)^{k-1} \lambda_i^k \left(\sum_{g \in L(m-k, n)} \lambda_g \right)$$

For each $f \in L(m, n)$, the number of times λ_f occurs in this triple is also m .

For each $f \notin L(m, n)$, it is also easy to deduce that there is no λ_f in the triple sum. Hence :

$$= \sum_{f \in L(m, n)} \lambda_f = \chi_\lambda^m(s)$$

To prove the formulas

$$\sigma_T(\chi)(s) = \exp \left\{ \sum_{k=1}^{\infty} \Psi^k(\chi) \frac{T^k}{k} \right\}, \quad \lambda_T(\chi)(s) = \exp \left\{ \sum_{k=1}^{\infty} (-1)^{k-1} \Psi^k(\chi) \frac{T^k}{k} \right\}$$

It suffices to show the identity

$$\exp(f(T))' = f'(T) \exp(f(T))$$

where every operation defined are considered as formal operations in the power series ring; for if this were to be established, then one can prove the identities by first noticing that the degree 0, 1 terms of the power series below

$$\exp \left\{ \sum_{k=1}^{\infty} \Psi^k(\chi) \frac{T^k}{k} \right\}, \quad \sum_{k=0}^{\infty} \chi_\sigma^k T^k$$

coincides, and that the "chain rule" above gives a recursion formula for the coefficients of the higher degree terms of the first series, coinciding with the recursion formula for the second series we've just established; one has analogous statements for exterior powers.

To establish the general formula, let $f(T) = \sum_{m=0}^{\infty} a_m T^m$ be a formal power series, we have

$$\exp(f(T)) = \sum_{n=0}^{\infty} \frac{f(T)^n}{n!}$$

giving us

$$\exp(f(T))' = \sum_{n=1}^{\infty} \left(\frac{f(T)^{n-1}}{(n-1)!} f'(T) \right) = f'(T) \left(\sum_{n=1}^{\infty} \frac{f(T)^{n-1}}{(n-1)!} \right) = f'(T) \exp(f(T))$$

(b) By the recursion formula for χ_σ^n , one gets

$$\Psi^n(\chi) = n\chi_\sigma^n - \left(\sum_{k=1}^{n-1} \Psi^k(\chi) \chi_\sigma^{n-k} \right)$$

One concludes that $R(G)$ is stable under each Ψ^n easily from an induction argument.

EXERCISE 9.4

(a) By the assumption that $(n, g) = 1$, one sees that the set map

$$G \longrightarrow G, \quad x \longmapsto x^n$$

is a bijection, so we get

$$\langle \Psi^n(\chi), \Psi^n(\chi) \rangle_G = \frac{1}{g} \left(\sum_{s \in G} \chi(s^n) \chi(s^{-n}) \right) = \frac{1}{g} \left(\sum_{s \in G} \chi(s) \chi(s^{-1}) \right) = 1$$

By **EXERCISE 9.3**, $\Psi^n(\chi) \in R(G)$, and by **EXERCISE 9.2**, $\Psi^n(\chi)$ is irreducible if $\Psi^n(\chi)(1) > 0$, but it is immediate that $\Psi^n(\chi)(1) = \chi(1) > 0$.

(b) The linear operator ψ_n (on $\mathbb{C}[G]$) has an inverse :

Take m with $mn \equiv 1 \pmod{g}$, then ψ_m, ψ_n are mutually inverses.

Therefore, we only need to show :

- $\psi_n|(\text{Cent. } \mathbb{C}[G])$ has image in $\text{Cent. } \mathbb{C}[G]$.
- $\psi_n(uv) = \psi_n(u)\psi_n(v)$ for each $u, v \in \text{Cent. } \mathbb{C}[G]$.

Recall that $\text{Cent. } \mathbb{C}[G]$ has the description (section 6.3)

$$\text{Cent. } \mathbb{C}[G] = \oplus_{i=1}^h \mathbb{C}e_i$$

then the first assertion is done by noticing the calculation :

$$\psi_n \left(\sum_{s \in G} s^{-1}ts \right) = \sum_{s \in G} \psi_n(s^{-1}ts) = \sum_{s \in G} s^{-1}t^n s$$

and let t vary through G .

To prove the second assertion, recall that we have an isomorphism of algebras :

$$\omega : \text{Cent.}(\mathbb{C}[G]) \longrightarrow \mathbb{C}^h$$

(this is proposition 13 in the book) with each component function ω_i given by

$$\omega_i(u) = \frac{1}{n_i} \sum_{s \in G} u(s) \chi_i(s)$$

We have the following calculation :

$$\omega_i(\psi_n(u)) = \frac{1}{n_i} \sum_{s \in G} u(s) \chi_i(s^n) = \frac{1}{n_i} \sum_{s \in G} u(s) \Psi^n(\chi_i)(s)$$

By (a), $\Psi^n(\chi_i)$ is an irreducible character, so we get the following :

There exists a bijection σ from $\{1, \dots, h\}$ to itself such that $\omega_i \circ \psi_n = \omega_{\sigma(i)}$ for each i .

From this, we obtain the calculation :

$$\omega_i(\psi_n(u)\psi_n(v)) = \omega_i(\psi_n(u))\omega_i(\psi_n(v)) = \omega_{\sigma(i)}(u)\omega_{\sigma(i)}(v) = \omega_{\sigma(i)}(uv) = \omega_i(\psi_n(uv))$$

Combining each i , we get that

$$\omega(\psi_n(u)\psi_n(v)) = \omega(\psi_n(uv))$$

By bijectivity of ω , we get $\psi_n(u)\psi_n(v) = \psi_n(uv)$.

EXERCISE 9.5 .

EXERCISE 9.6

(a)

(b)

(c)

EXERCISE 9.7 .

EXERCISE 9.8 .

10 BRAUER'S THEOREM