Freyd-Mitchell Embedding Theorem

- Notes on Abelian Categories

Contents

Any suggestions and error corrections are highly welcomed. E-mail :

taricpinkfriend@gmail.com

1 Foreword

~ In this document, we want to prove the *Freyd-Mitchell Embedding Theorem* :

Every small abelian category can be imbedded into some ${}_{\mathrm{R}}\mathsf{Mod}$.

Which is a theorem named after Peter Freyd and Barry Mitchell. Our treatment in this note closely follows from that given in Freyd's book on *Abelian Categories*.[1]
Some results in this note are given in the form of exercises or hints. They are mostly not difficult, but in case you are stuck, the answers are mostly in the Freyd's book.
Sometimes we will also assume some diagrammatic lemmas (e.g. five lemma, nine lemma. . .) without proving it, but to avoid endlessly proving many technical and hard-to-remember results, this might be an unfortunate necessary evil (and that we are absolutely sorry for it).[2]

Notations on Categories Here we fix some standard notations :

1. Categories are often written in the calligraphic font, such as $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \dots$.
2. The category of left R-modules is denoted as ${}_{\mathrm{R}}\mathsf{Mod}$, so $\mathsf{Ab} = {}_{\mathbb{Z}}\mathsf{Mod}$

2 The Weak Embedding Theorem

Summary In this section, we prove in **Theorem 2.1** a special case of the general embedding theorem (**Theorem 2.2**). In fact, **Theorem 2.2** can be deduced from **Theorem 2.1** (later).

Theorem 2.1 (*Weak F-M Embedding Theorem*) .
Cocomplete abelian categories admitting a projective generator are fully abelian. $*$

Recall The definition of some of the terms above are as follows :

1. $\mathcal{A}$ is *cocomplete* : Means that $\mathcal{A}$ admits coequalizers, small coproducts.
   We also say that $\mathcal{A}$ is *bicomplete* if it is complete and cocomplete.
2. $\mathcal{A}$ admits a *generator* : Means there exists an object in $\mathrm{G} \in \mathcal{A}$ such that the functor

$$\mathrm{Hom}_{\mathcal{A}}(\mathrm{G}, -) : \ \mathcal{A} \longrightarrow \mathsf{Ab}$$

   is a faithful embedding.
3. $\mathcal{A}$ is *fully abelian* : Means for every small, exact, full subcategory $\mathcal{A}'$ of $\mathcal{A}$, there is an exact faithful embedding of $\mathcal{A}'$ into some ${}_{\mathrm{R}}\mathsf{Mod}$.
4. $\mathcal{A}'$ is an *exact subcategory* of $\mathcal{A}$: Means the inclusion functor $\mathcal{A}' \longrightarrow \mathcal{A}$ is exact. $*$

[1] A copied version of this book can be found in https://www.maths.ed.ac.uk/~v1ranick/papers/freydab.pdf. The note given in https://arxiv.org/pdf/1901.08591.pdf by Arnold Tan Junhan is also nice.
[2] A possible convenient way to prove diagrammatic results in abelian categories (without assuming any embedding theorems) is described in example 0.5, 0.6 of Ravi Vakil's notes given in https://math.stanford.edu/~vakil/0708-216/216ss.pdf (on spectral sequences though).

Proof. Take a small, exact, full subcategory $\mathscr{A}'$ of $\mathscr{A}$ with an embedding functor

$$\iota : \mathscr{A}' \longrightarrow \mathscr{A}$$

We claim that we can assume that $\mathscr{A}$ admits a projective generator G such that

$$\text{For each } X \in \mathscr{A}', \text{ there is an epi } \quad G \longrightarrow X \longrightarrow 0 \ .$$

By **Exercise 2.1** , this can be done by considering the object

$$G^+ := \coprod_{X \in \mathscr{A}'} \left(\coprod_{\mathrm{Hom}(G,X)} G \right)$$

Define F to be the functor $\mathrm{Hom}(G, -)$. By **Exercise 2.2** , we have a factorization of functors

$$\begin{tikzcd} \mathscr{A} \arrow[rr, dashed] \arrow[dr, "F"'] & & \mathsf{Mod}_R \arrow[dl, "\lambda"] \\ & \mathsf{Ab} & \end{tikzcd}$$

where λ is the forgetful functor. Therefore from now on, we may just treat F as a functor into Mod_R. Since G is a projective generator, F is an exact, faithful embedding. It suffices to show

$$F\iota \text{ is full.}$$

This is **Exercise 2.3** . ■

Exercise 2.1 (*Replacing a Generator by a Fatter one*) Show that :

(i) G^+ is projective. (Hint : Are coproducts of projectives projective ?)

(ii) G^+ is a generator. (Hint : Are coproducts of generators generators ?)

(iii) Define for each $X \in \mathscr{A}'$ a morphism

$$\textstyle\coprod_{\mathrm{Hom}(G,X)} G \longrightarrow X \longrightarrow 0$$

Show that this morphism is an epi. Now define for each $X \in \mathscr{A}'$ an epi

$$G^+ \longrightarrow X \longrightarrow 0$$

*

Exercise 2.2 (*Enriched Structure on Hom-sets*) Recall that the Hom-sets in an additive category (in particular, abelian categories are additive) has a unique structure of an abelian group.

(i) Show that the following endomorphism set has a ring structure :

$$R := \mathrm{Hom}(G, G)$$

(Hint : What is the composition, addition law? What is the identity, zero element?)

(ii) Show that for each $X \in \mathscr{A}$, the following set has a right R-module structure :

$$\mathrm{Hom}(G, X)$$

(Hint : How should R act on the set ?)

(iii) Show that Mod_R is equivalent to ${}_{R^{op}}\mathsf{Mod}$, where R^{op} the opposite ring of R. *

Exercise 2.3 This exercise shows $F\iota$ is full (where $F := \mathrm{Hom}(G, -)$).
Take $A, B \in \mathcal{A}'$. We want to show that

$$F : \ \mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(FA, FB)$$

is surjective. Take an right R-map φ

$$\varphi : \ FA \longrightarrow FB$$

Recall we have canonical epis :

$$\rho_A : \ G \longrightarrow A \longrightarrow 0 \ , \quad \rho_B : \ G \longrightarrow B \longrightarrow 0$$

Extend ρ_A to an exact sequence

$$0 \longrightarrow K \xrightarrow{\iota_A} G \xrightarrow{\rho_A} A \longrightarrow 0$$

Applying the exact functor F, we get a diagram with exact rows (in Mod_R) :

$$\begin{tikzcd} 0 \arrow[r] & FK \arrow[r, "\iota_A\circ"] & R \arrow[r, "\rho_A\circ"] \arrow[d, dashed, "\psi"] & FA \arrow[r] \arrow[d, "\varphi"] & 0 \\ & & R \arrow[r, "\rho_B\circ"] & FB \arrow[r] & 0 \end{tikzcd}$$

(i) Show that there exists an right R-map ψ making the square in the above diagram commute. (Hint : R is a free (and hence projective) R-module)

(ii) Identify ψ as left multiplication by some element in R. (Hint : $\psi(r) = \psi(1)r$)
In other words, write $\psi(1) = s$, then $\psi = s\circ = F(s)$.

(iii) Consider diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\iota_A"] & G \arrow[r, "\rho_A"] \arrow[d, "s"] & A \arrow[r] \arrow[d, dashed, "\mu"] & 0 \\ & & G \arrow[r, "\rho_B"] & B \arrow[r] & 0 \end{tikzcd}$$

Show that there exists a μ making the square in the diagram commute.
(Hint 1 : Try showing $\rho_B s \iota_A = 0$) (Hint 2 : Apply F to the diagram) (Hint 3 : F is faithful)

(iv) Show that $F\mu = \varphi$. (Hint : $F(\rho_A) = \rho_A\circ$ is an epi) $*$

Remark We have said in the beginning of this section that Mitchell's Theorem is a special case of the general embedding theorem, but haven't give the general statement of the theorem. Having defined enough notions, we may now state the theorem. The proof is given later. $*$

Theorem 2.2 (*General F-M Embedding Theorem*) Every abelian category is fully abelian. $*$

Corollary 2.3 (*Ordinary F-M Embedding Theorem*) .
Every small abelian category can be imbedded into some ${}_R\mathsf{Mod}$. $*$

3 Yoneda Embedding of Small Abelian Categories

Summary In this section, we utilize the Yoneda embedding to embed categories.

Throughout this section, $\mathcal{A}$ is assumed to be a small abelian category.

Notation $[\mathcal{A}, \mathsf{Ab}]$ means the category of additive functors between $\mathcal{A}, \mathsf{Ab}$. $*$

Exercise 3.1 (*Yoneda Embedding*) Let $\mathcal{A}$ be a small abelian category.
For each $X \in \mathcal{A}$, we can define a functor

$$H(X) := \mathrm{Hom}(X,)$$

Each $H(X)$ has a factorization

$$\begin{tikzcd} \mathcal{A} \arrow[rr, dashed] \arrow[dr, "H(X)"'] & & \mathsf{Ab} \arrow[dl] \\ & \mathsf{Set} & \end{tikzcd}$$

with each $H(X)$ being additive. Therefore, we may regard H as a fully faithful embedding

$$H : \mathcal{A}^{\mathrm{op}} \longrightarrow [\mathcal{A}, \mathsf{Ab}]$$

(i) Show that the category $[\mathcal{A}, \mathsf{Ab}]$ is abelian, bicomplete.
(Hint : Consider pointwise, and notice that Ab is abelian and bicomplete.)

(ii) Show that the functor H is left exact. (Hint : Consider pointwise) $*$

Proposition 3.1 (*Proposition concerning the Existence of a Projective Generator*) .
The category $[\mathcal{A}, \mathsf{Ab}]$ admits a projective generator. $*$

Proof. Define a functor (an object in $[\mathcal{A}, \mathsf{Ab}]$)

$$G := \coprod_{X \in \mathcal{A}} H(X)$$

We have natural isomorphisms (the last isomorphism is by Yoneda lemma) :

$$\mathrm{Hom}(G, F) = \mathrm{Hom}(\coprod_{X \in \mathcal{A}} H(X), F) \cong \prod_{X \in \mathcal{A}} \mathrm{Hom}(H(X), F) \cong \prod_{X \in \mathcal{A}} F(X)$$

This shows that $\mathrm{Hom}(G, -)$ is an exact, faithful embedding, so G is a projective generator. ∎

The following proposition related to $[\mathcal{A}, \mathsf{Ab}]$ is also important later, but since many notions aren't introduced yet, the proof is postponed to section 5.

Proposition 3.2 (*Proposition concerning the Existence of Injective Envelopes*) .
The category $[\mathcal{A}, \mathsf{Ab}]$ is a Grothendieck category.[3] $*$

[3] We remark that the definition of a Grothendieck Category given in Freyd's book is different from what people might find from the ordinary or standard one.

4 Subobjects and Extensions in Abelian Categories

In this section, $\mathcal{A}$ is always an abelian category.

Summary In this section, we recall briefly the notion of subobjects and facts related to it, and introduce the notion of an extension, since these notions has some similarity between them.

Definition 4.1 (*Subobjects*) Let $X \in \mathcal{A}$. Let us consider exact sequences of the form

$$0 \longrightarrow X' \longrightarrow X$$

Suppose we are given a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & X' \arrow[r] \arrow[d] & X \arrow[d, equal] \\ 0 \arrow[r] & X'' \arrow[r] & X \end{tikzcd}$$

then we say in this case that $X' \subseteq X''$. We say that $X' \sim X''$ if $X' \subseteq X'' \subseteq X'$.
Take the family of equivalence classes :

$$\mathscr{S}(X) : \{X \in \mathcal{A} : \ 0 \longrightarrow X' \longrightarrow X \ \}/\sim$$

then each equivalence class in $\mathscr{S}(X)$ is called an *subobject* of X. $*$

Exercise 4.1 (*Equivalence and Isomorphisms*) Show that $X' \sim X''$ iff X' if isomorphic to X''.
(Hint : Use monomorphicity.) $*$

Remark By **Exercise 4.1** , we get a partial order on $\mathscr{S}(X)$ induced by "$\subseteq$"; furthermore, when we talk about a subobject, we may just mention a representing object instead of always talking about "the equivalence class of the object". $*$

Definition 4.2 (*Intersection,Union*) Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be subobjects of X, define

$$\bigcap_{\lambda\in\Lambda} X_\lambda, \quad \bigcup_{\lambda\in\Lambda} X_\lambda$$

called the *intersection*, *union* in X, to be the glb, lub of $\{X_\lambda\}_{\lambda\in\Lambda}$ in the poset $\mathscr{S}(X)$ (if exists). $*$

Exercise 4.2 (*Existence of Small/Finite Intersection/Union*) .

1. Let $\mathcal{A}$ be abelian, then finite intersection/union exists. (Hint : Take finite (co-)limits)
2. Let $\mathcal{A}$ be a bicomplete, abelian category, then small intersection/union exists.
   (Hint : Take small (co-)limits exists.) $*$

Definition 4.3 (*Extension*) Let $X \in \mathcal{A}$.
An *extension* E of X is a monic $0 \longrightarrow X \longrightarrow E$. If we extend it to an exact sequence :

$$0 \longrightarrow X \longrightarrow E \longrightarrow Y \longrightarrow 0$$

Then the extension is called *trivial* if the sequence splits.
The extension is called *essential* if X intersects each nontrivial subobjects of E nontrivially.
The extension is called *proper* if $Y \neq 0$. The extension is called *injective* if E is injective. $*$

5 Well-Powered Categories and Grothendieck Categories

In this section, $\mathcal{A}$ is always an abelian category.

Summary We introduce the notion of well-powered categories and Grothendieck categories, which are additional axioms on abelian categories related to operations on subobjects.

Definition 5.1 (*Well-Poweredness*) Let $\mathcal{A}$ be a category. We say that $\mathcal{A}$ is *well-powered* if

$$\text{For each } X \in \mathcal{A}, \mathscr{S}(X) \text{ is a set.}$$

(The definition of $\mathscr{S}(X)$ is given in **Definition 4.1**) $*$

The following proposition gives a criterion on well-poweredness.

Proposition 5.2 A cocomplete abelian category admitting a generator is well-powered. $*$

Proof. Let G be a generator of an abelian category $\mathcal{A}$. Fix $X \in \mathcal{A}$. By **Exercise 5.1** , may assume

$$(\star) \quad \text{If } X' \subsetneq X, \text{ then } \mathrm{Hom}(G, X') \subsetneq \mathrm{Hom}(G, X).$$

We claim that

$$|\mathscr{S}(X)| \leq |\mathscr{P}(\mathrm{Hom}(G, X))|$$

where $\mathscr{P}(\mathrm{Hom}(G, X))$ the power set of $\mathrm{Hom}(G, X)$. Take subobjects X', X'' of X. If they are comparable, we are done. If not, take their intersection, then we are done by **Exercise 5.2**. ∎

Exercise 5.1 (*Generators in an abelian category are Strong*) Show that the condition $(\star)$ can be assumed (Hint : Extend a proper inclusion into an exact sequence, and apply the left exact, faithful functor $\mathrm{Hom}(G, -)$.)

Exercise 5.2 Convince yourself that $\mathrm{Hom}(G, X' \cap X'') = \mathrm{Hom}(G, X') \cap \mathrm{Hom}(G, X'')$. $*$

Definition 5.3 (*Grothendieck Categories*) [4]Let $\mathcal{A}$ be bicomplete, well-powered.
We say that $\mathcal{A}$ is *Grothendieck* if for each $X \in \mathcal{A}$ and chain of subobjects $\{X_i\}_{i \in I} \subseteq \mathscr{S}(X)$, we have for each $X' \in \mathscr{S}(X)$ that

$$X' \cap \left(\bigcup_{i \in I} X_i \right) \simeq \bigcup_{i \in I} (X' \cap X_i)$$ $*$

Example Informally, "set-like" abelian categories are often Grothendieck, for example, ${}_R\mathsf{Mod}$. $*$

Recall the statement of **Proposition 3.2** . We can now give a proof here.

Proof of Proposition 3.2. By **Proposition 3.1** and **Proposition 5.2** , $[\mathcal{A}, \mathsf{Ab}]$ is well-powered; By (i) of **Exercise 3.1** , $[\mathcal{A}, \mathsf{Ab}]$ is bicomplete. Since Ab is Grothendieck (by the **Example** above), we see that $[\mathcal{A}, \mathsf{Ab}]$ is Grothendieck by considering pointwisely. ∎

[4]We warn again that the definition of a Grothendieck Category given in Freyd's book is non-standard.

6 Injective Envelopes

In this section, $\mathscr{A}$ is always assumed to be Grothendieck.
(so it is also assumed that $\mathscr{A}$ is bicomplete, well-powered, abelian)

Summary In this section, we define the notion of an injective envelope in terms of essential extensions. We also want to show in this section that in a Grothendieck category admitting a generator, every object has an injective envelope.

Proposition 6.1 (*Injective Objects in Grothendieck Categories*) .
Let $\mathrm{I} \in \mathscr{A}$. Then I is injective iff I has no proper essential extension. ∗

Proof. By **Exercise 6.1** and , it suffices to show the statement :

> Let $\mathrm{I} \in \mathscr{A}$. If I has no proper essential extension, then every extension of I is trivial.

Take some I with that property and take an extension $0 \longrightarrow \mathrm{I} \longrightarrow \mathrm{E}$.
Define a set $\mathcal{F} \subseteq \mathscr{S}(\mathrm{E})$ by

$$\mathcal{F} := \{\mathrm{X} \in \mathscr{S}(\mathrm{E}) : \mathrm{X} \cap \mathrm{I} = 0\}$$

If $\mathcal{F}$ is empty, we are done, so we may suppose $\mathcal{F}$ isn't. Since $\mathscr{A}$ is Grothendieck, we can apply Zorn's lemma to get a maximal element in $\mathcal{F}$, call it X^+ (**Exercise 6.2**). Take exact sequence

$$0 \longrightarrow \mathrm{X}^+ \longrightarrow \mathrm{E} \longrightarrow \mathrm{Y}^- \longrightarrow 0$$

Since $\mathrm{X}^+ \cap \mathrm{I} = 0$, we have a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{I} \arrow[r] \arrow[d, equal] & \mathrm{E} \arrow[d] \\ 0 \arrow[r] & \mathrm{I} \arrow[r] & \mathrm{Y}^- \end{tikzcd}$$

By **Exercise 6.3** , $0 \longrightarrow \mathrm{I} \longrightarrow \mathrm{Y}^-$ is an essential extension, hence an isomorphism.
Therefore the extension $0 \longrightarrow \mathrm{I} \longrightarrow \mathrm{E}$ is trivial. ■

Exercise 6.1 Show that I is injective iff every extension of I is trivial. ∗

Exercise 6.2 This exercise checks the condition for Zorn's lemma. Take a chain

$$\{\mathrm{X}_j\}_{j \in \mathrm{J}} \subseteq \mathcal{F}$$

Take the union of the X_j, show that I intersects this object trivially. (Hint : Grothendieckness) ∗

Exercise 6.3 Show that $0 \longrightarrow \mathrm{I} \longrightarrow \mathrm{Y}^-$ is essential. (Possible hint : Suppose $\mathrm{Y}^{--} \cap \mathrm{I} = 0$ for some subobject $\mathrm{Y}^{--} \neq 0$ of Y, construct a commutative diagram with exact rows :

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{X}^{++} \arrow[r] \arrow[d, dashed] & \mathrm{E} \arrow[r] \arrow[d] & \mathrm{Z} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathrm{Y}^{--} \arrow[r] & \mathrm{Y}^- \arrow[r] & \mathrm{Z} \arrow[r] & 0 \end{tikzcd}$$

Show that $\mathrm{X}^{++} \cap_{\mathrm{E}} \mathrm{I} \neq 0 = \mathrm{X}^{++} \cap_{\mathrm{Y}^-} \mathrm{I}$ and that $\mathrm{X}^{++} \cap_{\mathrm{E}} \mathrm{I}$ is actually a subobject of $\mathrm{X}^{++} \cap_{\mathrm{Y}^-} \mathrm{I}$.) ∗

Definition 6.2 (*Injective Envelopes*) Let $A \in \mathcal{A}$, an extension

$$0 \longrightarrow A \longrightarrow E$$

is called an *injective envelope* of A if the extension is injective and essential (**Definition 4.3**). $*$

Before we proceed to the main theorem, let us recall the following standard fact.

Recall ${}_{R}\mathsf{Mod}$ have enough injectives (for any ring R). $*$

We also give some useful facts related to faithful functors for the upcoming proof.

Exercise 6.4 (*Faithful Functors and Exact Sequences*) .
Let $\mathcal{A}, \mathcal{B}$ be abelian categories and $F \in [\mathcal{A}, \mathcal{B}]$, with F being faithful. Show the following :

(i) Given a morphism $X \xrightarrow{f} Y$, suppose the sequence

$$0 \longrightarrow F(X) \xrightarrow{F(f)} F(Y)$$

is exact, then $0 \longrightarrow X \xrightarrow{f} Y$ is exact.

(ii) Formulate the dual of (i).

(iii) Use (i),(ii) to show that F is *conservative* : that is, f is isom iff $F(f)$ is isom.
(More generally : faithful functors out of a *balanced* category are conservative) $*$

Theorem 6.3 (*Grothendieck Categories Admitting a Generator has Injective Envelopes*) .
Suppose $\mathcal{A}$ has a generator, then every object in it has an injective envelope. $*$

Proof of Theorem 6.3. Let G be a generator. Recall the result in **Exercise 5.1** .
As in the proof of **Theorem 2.1** , let $R = \mathrm{Hom}(G, G)$, we have a (left exact) functor

$$\mathcal{A} \xrightarrow{F} \mathsf{Mod}_R \ , \quad F := \mathrm{Hom}(G, -)$$

We claim that F sends essential extensions to essential extensions; that is,

$$\text{Suppose } \ 0 \longrightarrow X \xrightarrow{i} E \ \text{ is an essential extension,}$$
$$\text{then } \ 0 \longrightarrow F(X) \xrightarrow{i\circ} F(E) \ \text{ is an essential extension.}$$

Take $M \subseteq F(E) = \mathrm{Hom}(G, E)$ such that M is non-trivial. Want $F(X) \cap M \neq 0$.
Take non-trivial $f \in M$, form commtutative diagram with pullback square and exact rows :

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r, dashed, "j"] \arrow[d, dashed, "g"] & G \arrow[d, "f"] \\ 0 \arrow[r] & X \arrow[r, "i"] & E \end{tikzcd}$$

and take an exact sequence

$$0 \longrightarrow K \longrightarrow U \xrightarrow{fj} E$$

Since $0 \longrightarrow X \xrightarrow{i} E$ is essential, we have $U \neq 0$, and that K is a proper subobject of U. Apply F, we get exact sequence :

$$0 \longrightarrow F(K) \longrightarrow F(U) \xrightarrow{fj\circ} F(E)$$

Since $U \neq 0$, $F(U) \neq 0$. Since $fj \neq 0$, $F(fj) \neq 0$. Since K is a proper subobject, F(K) is a proper. Therefore, there exists $h \in F(U) = \mathrm{Hom}(G, U)$ such that $F(fj)(h) \neq 0$.
(By definition, h is a morphism $G \longrightarrow U$) Notice that

$$F(fj)(h) = fjh = igh = F(i)(gh) \in F(X)$$
$$F(fj)(h) = f(jh) \in MR \subseteq M$$

We get $F(fj)(h) \in F(X) \cap M$ with $F(fj)(h) \neq 0$; this proves the claim.
Since ${}_R\mathsf{Mod}$ has enough injectives, there exists an injective object Q, and an exact sequence

$$0 \longrightarrow F(X) \longrightarrow Q$$

For each essential extension $0 \longrightarrow X \xrightarrow{j} E$ of X, we have a commutative diagram

$$\begin{tikzcd} & Q & \\ 0 \arrow[r] & F(X) \arrow[u, "j"] \arrow[r, "F(i)"] & F(E) \arrow[ul, "f"'] \end{tikzcd}$$

Since j is monic, we get $\mathrm{Ker}(f) \cap F(X) = 0$, so $\mathrm{Ker}(f) = 0$ by essentialness.
Therefore, we get the following :

Each equivalence class of essential extension of X corresponds to a unique subobject of Q.

Now we are ready to prove the theorem, using a transfinite recursion argument.
Take an ordinal α bigger than the cardinality of $\mathscr{S}(Q)$ and give it a well-ordering.
We define a family of essential extensions $\{E^\gamma\}_{\gamma \leq \alpha}$ of X as follows :

1. Initial step : Suppose γ is the unique minimal. Define E^0 to be X.
2. On succesors : Suppose $\gamma = \beta + 1$. Define E^γ as a proper essential extension if E^β isn't injective (existence follows from **Proposition 6.1**); otherwise, define $E^\gamma = E^\beta$.
3. On limit ordinals : Define E^γ as the colimit of $\{E^\beta\}_{\beta<\gamma}$.

By **Exercise 6.5** , **Exercise 6.6** , all the E^γ are essential. If none of them are injective, we get an injection from a set of cardinality α to $\mathscr{S}(Q)$, contradicting our assumption. ■

Exercise 6.5 (*Essential Extension of an Essential Extension is Essential*) Suppose

$$0 \longrightarrow X \longrightarrow E\ , \qquad 0 \longrightarrow E \longrightarrow E'$$

are essential extensions, show that $0 \longrightarrow X \longrightarrow E'$ is an essential extension. ∗

Exercise 6.6 (*Union of an Ascending Chain of Essential Extensions is Essential*) Suppose

$$\{E_i\}_{i \in I} \subseteq \mathscr{S}(E)$$

is an ascending chain of subobjects of some object E, and that $A \in \bigcap_i \mathscr{S}(E_i)$ such that

$$0 \longrightarrow A \longrightarrow E_i$$

are all essential, show that $0 \longrightarrow A \longrightarrow \bigcup_{i \in I} E_i$ is essential. (Hint : Grothendieckness) ∗

7 Idea of Proof of the General Embedding Theorem

In this section, we briefly recap on some results we have proved before, and describe the idea of the proof of the general embedding theorem (stated in **Theorem 2.2**), which asserts that

Every abelian category is fully abelian.

Review Suppose $\mathcal{A}$ is an abelian category. Let us review what we've done so far :

- **Weak Embedding Theorem** (**Theorem 2.1**)
  Cocomplete categories admitting a projective generator are fully abelian.
- **Facts about the Category of Additive Functors** $[\mathcal{A}, \mathsf{Ab}]$
  (**Exercise 3.1** , **Proposition 3.1** , **Proposition 5.2** , **Proposition 3.2** , **Theorem 6.3**)
  - The category $[\mathcal{A}, \mathsf{Ab}]$ is bicomplete, well-powered, admits a projective generator, and also Grothendieck. Most importantly, it admits injective envelopes.
  - The Yoneda embedding functor $\mathrm{H} : \mathcal{A}^{\mathrm{op}} \longrightarrow [\mathcal{A}, \mathsf{Ab}]$ defined by
    $$\mathrm{H}(\mathrm{A}) = \mathrm{Hom}(\mathrm{A}, -)$$
    is fully faithful, left exact. *

If we can embed the small category $\mathcal{A}$ exactly, fully faithfully into a fully abelian category, then the general embedding theorem is proved. Having known the weak embedding theorem and have more understanding on the functor category $[\mathcal{A}, \mathsf{Ab}]$, our first try might be the Yoneda embedding functor H. Unfortunately, the functor category $[\mathcal{A}, \mathsf{Ab}]$ still has some defect :

- The functor H^{op} is fully faithful, right exact, but we need exactness.
- The category $[\mathcal{A}, \mathsf{Ab}]^{\mathrm{op}}$ is bicomplete and admits an injective cogenerator,
  but we need a projective generator.

Definition 7.1 (*Full Subcategories of Mono Functors, Left Exact Functors*) .
We use the symbols
$$[\mathcal{A}, \mathsf{Ab}]_{\mathrm{L}}, \quad [\mathcal{A}, \mathsf{Ab}]_{\mathrm{M}}$$
to denote the subcategories of $[\mathcal{A}, \mathsf{Ab}]$ consisting of left exact, mono functors. *

Recall A functor is called *mono* if it preserves exactness of exact sequences of the form
$$0 \longrightarrow \mathrm{X} \longrightarrow \mathrm{Y}$$
and a functor is called *left exact* if it preserves exactness of exact sequences of the form
$$0 \longrightarrow \mathrm{X} \longrightarrow \mathrm{Y} \longrightarrow \mathrm{Z}$$
From the definition, we see that $[\mathcal{A}, \mathsf{Ab}]_{\mathrm{L}}$ is a full subcategory of $[\mathcal{A}, \mathsf{Ab}]_{\mathrm{M}}$.

Using the notation in above, notice that the functor H has a factorization

$$\begin{tikzcd} \mathcal{A}^{\mathrm{op}} \arrow[rr, dashed] \arrow[dr, "\mathrm{H}"'] & & {[\mathcal{A}, \mathsf{Ab}]_{\mathrm{L}}} \arrow[dl] \\ & {[\mathcal{A}, \mathsf{Ab}]} & \end{tikzcd}$$

The category $[\mathcal{A}, \mathsf{Ab}]_{\mathrm{L}}$ (along with $[\mathcal{A}, \mathsf{Ab}]_{\mathrm{M}}$) would be the key player in the upcoming sections. *

8 Category of Mono Functors as Mono Subcategories

Summary In this section, we prove that the subcategory $[\mathcal{A}, \mathsf{Ab}]_{\mathrm{M}}$ of $[\mathcal{A}, \mathsf{Ab}]$ is closed under products, subobjects, essential extensions.

Proposition 8.1 The subcategory $[\mathcal{A}, \mathsf{Ab}]_{\mathrm{M}}$ is closed under

(A) products (B) subobjects (C) essential extensions

Proof of (C). Suppose F is mono, let $0 \longrightarrow \mathrm{F} \longrightarrow \mathrm{E}$ be an essential extension.
Suppose E isn't mono, then there exists a monic

$$0 \longrightarrow \mathrm{X}' \xrightarrow{i} \mathrm{X}$$

such that $\mathrm{E}(\mathrm{X}') \xrightarrow{\mathrm{E}(i)} \mathrm{E}(\mathrm{X})$ isn't monic. Take nontrivial $x \in \mathrm{E}(\mathrm{X}')$ with $\mathrm{E}(i)(x) = 0$.
We construct a "subfunctor E' of E generated by x" as follows :

- On objects, define $\mathrm{E}'(\mathrm{Y}) = \mathrm{E}(\mathrm{Hom}(\mathrm{X}', \mathrm{Y}))(x)$.
- On morphisms, define by restriction.

then E' obviously lies in $[\mathcal{A}, \mathsf{Ab}]$, and is a nontrivial subobject of E.
By essentialness of E, we get $\mathrm{E}' \cap \mathrm{F} \neq 0$, so there exists some $\mathrm{Y} \in \mathcal{A}$ with $\mathrm{E}'(\mathrm{Y}) \cap \mathrm{F}(\mathrm{Y}) \neq 0$.
Let us take some non-trivial $y \in \mathrm{E}'(\mathrm{Y}) \cap \mathrm{F}(\mathrm{Y})$.
By the definition of E', there exists $f \in \mathrm{Hom}(\mathrm{X}', \mathrm{Y})$ such that $\mathrm{E}(f)(x) = y$.
Take pushout diagram with exact rows :

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{X}' \arrow[r, "i"] \arrow[d, "f"] & \mathrm{X} \arrow[d, dashed, "g"] \\
0 \arrow[r] & \mathrm{Y} \arrow[r, dashed, "j"] & \mathrm{V}
\end{tikzcd}$$

Applying the functor E, we get a diagram

$$\begin{tikzcd}
x & \in & \mathrm{E}(\mathrm{X}') \arrow[r, "\mathrm{E}(i)"] \arrow[d, "\mathrm{E}(f)"] & \mathrm{E}(\mathrm{X}) \arrow[d, dashed, "\mathrm{E}(g)"] \\
y & \in & \mathrm{E}(\mathrm{Y}) \arrow[r, dashed, "\mathrm{E}(j)"] & \mathrm{E}(\mathrm{V})
\end{tikzcd}$$

Notice first that by our assumption on x, we get

$$\mathrm{E}(j)(y) = \mathrm{E}(j)\mathrm{E}(f)(x) = \mathrm{E}(g)\mathrm{E}(i)(x) = 0$$

Recall that F is mono. By the following commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \mathrm{F}(\mathrm{Y}) \arrow[r, "\mathrm{F}(j)"] \arrow[d] & \mathrm{F}(\mathrm{V}) \arrow[d] \\
 & \mathrm{E}(\mathrm{Y}) \arrow[r, "\mathrm{E}(j)"] & \mathrm{E}(\mathrm{V})
\end{tikzcd}$$

and the fact that $y \in \mathrm{E}'(\mathrm{Y}) \cap \mathrm{F}(\mathrm{Y}) \subseteq \mathrm{E}(\mathrm{Y}) \cap \mathrm{F}(\mathrm{Y})$, we get

$$\mathrm{E}(j)(y) = \mathrm{F}(j)(y) \neq 0$$

This is a contradiction. ■

Exercise 8.1 Prove (A), (B). (Hint for (A) : Consider pointwisely. For (B) : Naturality) *

9 Reflection to Mono Subcategories

Summary In this section, we extract and abstract the properties of the subcategory $[\mathcal{A}, \mathsf{Ab}]_{\mathrm{M}}$ given in **Proposition 8.1**, then we introduce the notion of a reflection, and investigate the relation between reflections and the existence of a left adjoint of the inclusion functor.

In this section, $\mathcal{A}$ is always an Grothendieck category admitting injective envelopes.

Definition 9.1 (*Mono Subcategories*) [5]Let $\mathcal{A}'$ be a full subcategory of $\mathcal{A}$. If $\mathcal{A}'$ is closed under products, subobjects, essential extensions, then $\mathcal{A}'$ is called a *mono subcategory* of $\mathcal{A}$. Its objects are called *mono objects.* ∗

Definition 9.2 (*Reflection*) .
Given $X \in \mathcal{A}$, a *reflection* of X is an object $T(X) \in \mathcal{A}$ along with a map $X \longrightarrow T(X)$ such that

$$\text{For each morphism } X \xrightarrow{f} Y \text{ with } Y \in \mathcal{A}',$$
$$\text{there exists a unique morphism } T(X) \xrightarrow{T(f)} Y \text{ completing the diagram}$$

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "f"'] & & T(X) \arrow[dl, dashed, "T(f)"] \\ & Y & \end{tikzcd}$$
∗

Remark By definition, reflections are unique up to isomorphisms. ∗

Exercise 9.1 (*Adjoint functors via Reflection*) Show that given a full subcategory $\mathcal{B}'$ of a category $\mathcal{B}$, the existence of a left adjoint of the inclusion functor $\mathcal{B}' \longrightarrow \mathcal{B}$ is equivalent to the statement that every object in $\mathcal{B}$ admits a reflection in $\mathcal{B}'$. ∗

Proposition 9.3 (*Mono Subcategories are Reflexive*) .
Every object X in $\mathcal{A}$ admits a reflection T(X) in $\mathcal{A}'$. ∗

Proof. First, we claim that

$$\text{For every } X \in \mathcal{A},\ \mathcal{A}' \cap \mathcal{Q}(X) \text{ has a unique maximal element } X^{+}.$$

The idea is simple : consider the family of quotient objects of X

$$X \longrightarrow Y \longrightarrow 0$$

get a morphism (notice that $\mathcal{Q}(X)$ is small by **Exercise 9.2**)

$$X \longrightarrow \prod_{Y \in \mathcal{A}' \cap \mathcal{Q}(X)} Y$$

then take $X^{+} = \mathrm{Im}(\ X \longrightarrow \prod_{Y \in \mathcal{A}' \cap \mathcal{Q}(X)} Y\ )$.
Next we claim that every object in X admits a reflection : to do this, first consider the case where $Y \in \mathcal{A}' \cap \mathcal{Q}(X)$; for the general case, just consider $\mathrm{Im}(f)$. ■

[5]This term is also non-standard but useful for our use so we keep it.

Exercise 9.2 (*Well-poweredness in terms of Quotient Objects*) .
Let C be an abelian category. We may define for each object $X \in C$ the collections

$$\mathscr{S}(X), \quad \mathscr{Q}(X)$$

of subobjects, quotient objects. Show that :

$$\mathscr{S}(X) \text{ is small for all X iff } \mathscr{Q}(X) \text{ is small for all X. (Hint : Five Lemma)}$$

Definition 9.4 (*Reflection*) From now on, we assign for each $X \in \mathcal{A}$ a reflection

$$T(X) := \mathrm{Im}(\ X \longrightarrow \textstyle\prod_{Y \in \mathcal{A}' \cap \mathscr{Q}(X)} Y\ ) \in \mathcal{A}'$$

(By the proof of **Proposition 9.3** , it is well-defined; by **Exercise 9.1**, it extends to a functor.) ∗

Definition 9.5 (*Pure Subobjects*) Let $X \in \mathcal{A}'$. Let X' be a subobject of X. Take exact sequence

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

then X' is said to be a *pure subobject* of X if $X'' \in \mathcal{A}'$. ∗

Definition 9.6 (*Absolutely Pure Objects*) .
Let $X' \in \mathcal{A}'$. We say that X' is *absolutely pure* if for every exact sequence

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

with $X \in \mathcal{A}'$, we have $X'' \in \mathcal{A}'$.
(In other words : whenever X' appears as a subobject of a mono object, it is pure) ∗

Exercise 9.3 (*Injective Mono Objects*) Show if $X' \in \mathcal{A}'$ is injective, it is absolutely pure. ∗

Proposition 9.7 (*Pure Subobjects of Absolutely Pure Objects*) .
Suppose X' is a pure subobject of an absolutely pure object $X \in \mathcal{A}'$, then X' is absolutely pure. ∗

Proof. Suppose given an exact sequence (with $Y \in \mathcal{A}'$)

$$0 \longrightarrow X' \longrightarrow Y \longrightarrow Y'' \longrightarrow 0$$

Want $Y'' \in \mathcal{A}'$. Take a pushout diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

By general properties of pushout, we get commutative diagram with exact rows and columns :

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & X' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & Y'' \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & 0 & \\ & 0 & 0 & & \end{tikzcd}$$

Focus on the middle row, we have by **Exercise 9.4** that $Z \in \mathcal{A}'$.
Focus on the middle column, we have, by absolute purity of X, that $Y'' \in \mathcal{A}'$. ■

Exercise 9.4 Suppose given an exact sequence

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

with $X', X'' \in \mathcal{A}'$, show that $X \in \mathcal{A}'$. (Hint : Take an essential extension $0 \longrightarrow X' \longrightarrow E$. Extend it over X. Show that X embeds into $E \oplus X''$ and that $E \oplus X'' \in \mathcal{A}'$.) *

10 Torsion Objects

Summary In this section, we introduce the notion of a torsion object. This section isn't used before section 12, so it is recommended that this section be skipped in a first reading.

Definition 10.1 (*Torsion objects*) Let $X \in \mathcal{A}$, then we say that

$$X \text{ is torsion iff } \mathrm{Hom}(X, Y) = 0 \text{ for all } Y \in \mathcal{A}'.$$

Exercise 10.1 (*Torsion Object Criterion*) Show that X is torsion iff $T(X) = 0$. *

Proposition 10.2 (*Maximal Torsion Subobject*) Let $X \in \mathcal{A}$, consider the exact sequence

$$X \longrightarrow T(X) \longrightarrow 0$$

Take the kernel of this map, that is, an exact sequence

$$0 \longrightarrow K \longrightarrow X \longrightarrow T(X) \longrightarrow 0$$

then K is the unique maximal torsion subobject of X. *

Proof. Let us show that K is torsion. Take a morphism

$$K \xrightarrow{f} Y$$

where $Y \in \mathcal{A}'$, and let $0 \longrightarrow Y \longrightarrow E$ be an injective envelope of Y.
Construct a diagram with exact rows :

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "f"] \arrow[d] & X \arrow[r] \arrow[d, dashed] & T(X) \arrow[r] \arrow[dl, dashed] & 0 \\
0 \arrow[r] & Y \arrow[r] & E & &
\end{tikzcd}$$

(here we have used the fact that E is injective and is an essential extension) So we are done.
Now we show K is unique maximal. Take a torsion subobject of X

$$0 \longrightarrow X' \longrightarrow X$$

and construct a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X' \arrow[r] \arrow[d] & X \arrow[d] \\
& T(X') \arrow[r, dashed] & T(X)
\end{tikzcd}$$

then by **Exercise 10.1** , we are done. *

11 Left Exact Functors as Absolutely Pure Objects

Summary In this section, we will show that the full subcategory of $[\mathcal{A}, \mathsf{Ab}]$ consisting of absolutely pure objects with respect to the mono subcategory $[\mathcal{A}, \mathsf{Ab}]_{\mathrm{M}}$ is actually $[\mathcal{A}, \mathsf{Ab}]_{\mathrm{L}}$; hope this justifies the reason of introducing the notion of pureness, absolutely pureness in the previous section.

We've seen that subfunctors of mono functors are mono. What about left exact functors ?

Lemma 11.1 (*Subfunctor of Left Exact Functor*) Given exact sequence of functors

$$0 \longrightarrow \mathrm{F}' \longrightarrow \mathrm{F} \longrightarrow \mathrm{F}'' \longrightarrow 0$$

with F being left exact, then F′ is left exact iff F′ is a pure subobject of F (or : F″ is mono). $*$

Proof. Suppose given an exact sequence (in $\mathcal{A}$) :

$$0 \longrightarrow \mathrm{X}' \longrightarrow \mathrm{X} \longrightarrow \mathrm{X}''$$

we get a commutative diagram with exact columns and exact middle row :

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \mathrm{F}'(\mathrm{X}') \arrow[r] \arrow[d] & \mathrm{F}'(\mathrm{X}) \arrow[r] \arrow[d] & \mathrm{F}'(\mathrm{X}'') \arrow[d] \\
0 \arrow[r] & \mathrm{F}(\mathrm{X}') \arrow[r] \arrow[d] & \mathrm{F}(\mathrm{X}) \arrow[r] \arrow[d] & \mathrm{F}(\mathrm{X}'') \\
0 \arrow[r] & \mathrm{F}''(\mathrm{X}') \arrow[r] \arrow[d] & \mathrm{F}''(\mathrm{X}) \arrow[d] & \\
 & 0 & 0 &
\end{tikzcd}$$

A diagrammatic result shows that the top row is exact iff the bottom is.
This translates to F′ is left exact iff F″ is mono. ■

Theorem 11.2 (*Left Exact Functors are precisely the Absolutely Pure Objects*) .
A mono functor F′ is left exact iff it is absolutely pure. $*$

Proof. Take an injective envelope $0 \longrightarrow \mathrm{F} \longrightarrow \mathrm{E}$.
By **Exercise 11.1** and **Exercise 9.3** , E is left exact and absolutely pure.
By **Lemma 11.1** and **Proposition 9.7** , we are done. ■

Exercise 11.1 (*Injective objects in the Functor Category*) Let $\mathrm{E} \in [\mathcal{A}, \mathsf{Ab}]$ be an injective.

1. Show that E is right exact. (Hint : Given an exact sequence

$$\mathrm{A}' \longrightarrow \mathrm{A} \longrightarrow \mathrm{A}'' \longrightarrow 0$$

   apply the functors $\mathrm{H}, \mathrm{Hom}(-, \mathrm{E})$ and use Yoneda's lemma to conclude.)

2. Suppose further that E is mono, conclude that E is exact (and hence left exact). $*$

12 Reflection to Subcategory of Absolutely Pure Objects

Summary We have seen in section 11 that left exact functors can be characterized as absolutely pure objects. In this section, we go back to an abstract setting as in section 9 and 10.

Our setting is this : let $\mathscr{A}$ be a Grothendieck category admitting injective envelopes, $\mathscr{A}'$ be full subcategory of mono objects, and $\mathscr{A}''$ be that of absolutely pure objects.

We've seen reflections from $\mathscr{A}$ to $\mathscr{A}'$ in section 9. Now we consider reflections from $\mathscr{A}'$ to $\mathscr{A}''$.

(Definition and facts about torsion objects given in section 10 will be used from now on.)

Recall (Compare **Definition 9.1**)
Given $X \in \mathscr{A}'$, a *reflection* of X is an object $S(X) \in \mathscr{A}''$ along with a map $X \longrightarrow S(X)$ such that

For each morphism $X \xrightarrow{f} Y$ with $Y \in \mathscr{A}''$,
there exists a unique morphism $S(X) \xrightarrow{S(f)} Y$ completing the diagram

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "f"'] & & S(X) \arrow[dl, dashed, "S(f)"] \\ & Y & \end{tikzcd}$$ ∗

Proposition 12.1 (*Recognition Theorem*) Given an exact sequence in $\mathscr{A}$

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

with $X' \in \mathscr{A}'$, $X \in \mathscr{A}''$, X'' torsion, then $X' \longrightarrow X$ is a reflection. ∗

Proof. Suppose we are given a diagram of the form

$$\begin{tikzcd} X' \arrow[rr] \arrow[dr, "f"'] & & X \arrow[dl, dashed] \\ & Y & \end{tikzcd}$$

with $Y \in \mathscr{A}''$, want to show existence and uniqueness of the dashed arrow completing the diagram.
Take injective envelope $0 \longrightarrow Y \longrightarrow E$, complete it to an exact sequence

$$0 \longrightarrow Y \longrightarrow E \longrightarrow Z \longrightarrow 0$$

By general properties of $\mathscr{A}'$ and $\mathscr{A}''$, we get $E \in \mathscr{A}'$ and $Z \in \mathscr{A}'$.
Construct the following commutative diagram with exact rows :

$$\begin{tikzcd} 0 \arrow[r] & X' \arrow[r, "i"] \arrow[d, "f"] & X \arrow[r, "p"] \arrow[d, dashed, "g"] & X'' \arrow[r] \arrow[d, dashed, "h"] & 0 \\ 0 \arrow[r] & Y \arrow[r, "j"] & E \arrow[r, "q"] & Z \arrow[r] & 0 \end{tikzcd}$$

Since X'' is torsion, we get $h = 0$, so g factors through j, so we get a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, dashed, "g'"'] \arrow[dr, "g"] & \\ Y \arrow[rr, "j"] & & E \end{tikzcd}$$

Using monicity of j, we have $f = g'i$; this shows existence. Uniqueness is **Exercise 12.1** . ■

Exercise 12.1 Show uniqueness (Hint : Suppose there are two solutions, show that their difference factors through p. Now use the fact that X″ is a torsion object.) $*$

Recall We have inclusion of categories :

$$\mathcal{A}'' \longrightarrow \mathcal{A}' \longrightarrow \mathcal{A}$$

and we have shown that the left adjoint T of the inclusion $\mathcal{A}' \longrightarrow \mathcal{A}$ exists, and is explicitly defined in **Definition 9.4** (using **Exercise 9.2**).
In the next proposition, we will use T to construct a left adjoint S of the inclusion $\mathcal{A}'' \longrightarrow \mathcal{A}'$.
A crucial fact we will use is that the canonical morphisms $\mathrm{X} \longrightarrow \mathrm{T(X)}$ are all epis. $*$

Proposition 12.2 (*Construction Theorem*) For every $\mathrm{X} \in \mathcal{A}'$, there exists a monic

$$0 \longrightarrow \mathrm{X} \longrightarrow \mathrm{S(X)}$$

with $\mathrm{S(X)} \in \mathcal{A}''$ such that this morphism is a reflection. $*$

Proof. Again, take an injective envelope of X, and extend it to an exact sequence

$$0 \longrightarrow \mathrm{X} \longrightarrow \mathrm{E} \longrightarrow \mathrm{Y} \longrightarrow 0$$

We don't know if Y is mono, but we may consider its reflection in $\mathcal{A}'$. Take exact sequence

$$0 \longrightarrow \mathrm{Y}' \longrightarrow \mathrm{Y} \longrightarrow \mathrm{T(Y)} \longrightarrow 0$$

we can construct a commutative diagram with exact columns and exact bottom two rows

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{X} \arrow[r, dashed] \arrow[d, equal] & \mathrm{K} \arrow[r, dashed] \arrow[d] & \mathrm{Y}' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{X} \arrow[r] \arrow[d] & \mathrm{E} \arrow[r] \arrow[d] & \mathrm{Y} \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & \mathrm{T(Y)} \arrow[r, equal] \arrow[d] & \mathrm{T(Y)} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

By a diagrammatic lemma (dual to that in the proof of **Lemma 11.1**), the top row is exact.
Now we show $0 \longrightarrow \mathrm{X} \longrightarrow \mathrm{K}$ is the reflection of X by using the recognition theorem above.
First, Y′ is the kernel of $\mathrm{Y} \longrightarrow \mathrm{T(Y)}$, so it is torsion, by **Proposition 10.2** .
Next, $\mathrm{E} \in \mathcal{A}''$ by **Exercise 9.4**. Since $\mathrm{T(Y)} \in \mathcal{A}'$, we see that K is a pure subobject of E, so we conclude from **Proposition 9.7** that $\mathrm{K} \in \mathcal{A}''$. ■

13 Absolutely Pure Objects form an Abelian Category

Summary We show in this section that under the setting of section 12, the category $\mathcal{A}''$ is abelian. This shows that the category of left exact functors $[\mathcal{A}, \mathsf{Ab}]_{\mathrm{L}}$ is abelian.

Recall We have constructed left adjoint to the inclusions, pictured schematically as

$$\mathcal{A}'' \underset{\mathrm{S}}{\overset{\subseteq}{\rightleftarrows}} \mathcal{A}' \underset{\mathrm{T}}{\overset{\subseteq}{\rightleftarrows}} \mathcal{A}$$

and the canonical maps associated to T (resp. S) evaluated at objects are all epis (resp. monics).

Notation When we think of a morphism in $\mathcal{A}''$, it is often that we always think of it as a morphism in $\mathcal{A}$. We know that we can take (co-)kernels, cokernels in $\mathcal{A}$, but it is not yet know if $\mathcal{A}''$ admits (co-)kernel; even if it does, it might not coincide with the (co-)kernel taken in the bigger category (that is, it might not commute with the inclusion functor).[6]

To clarify the situation, we say for a given morphism in $\mathcal{A}$ the morphism is an $\mathcal{A}''$*-morphism* if it is a morphism in $\mathcal{A}''$. An $\mathcal{A}''$-morphism is an $\mathcal{A}''$*-monic* (resp. $\mathcal{A}''$*-epi*) if it is a monic (resp. epi) in $\mathcal{A}''$. An $\mathcal{A}''$*-kernel* (resp. $\mathcal{A}''$*-cokernel*) of an $\mathcal{A}''$-morphism is the kernel (resp. cokernel) of the morphism in $\mathcal{A}''$. Similarly, a monic (resp. epi) in $\mathcal{A}''$ is $\mathcal{A}''$*-normal* if it is normal in $\mathcal{A}''$. ∗

Exercise 13.1 Show that $\mathcal{A}''$ admits zero objects, is additive, admits finite (co-)products. ∗

Lemma 13.1 (*Kernels*) The $\mathcal{A}$-kernel of an $\mathcal{A}''$-morphism lies in $\mathcal{A}''$. ∗

Proof. Let $\mathrm{X} \xrightarrow{f} \mathrm{Y}$ be an $\mathcal{A}''$-morphism, we have $\mathcal{A}$-exact sequences :

$$0 \longrightarrow \mathrm{Ker}_{\mathcal{A}}(f) \longrightarrow \mathrm{X} \longrightarrow \mathrm{Im}_{\mathcal{A}}(f) \longrightarrow 0\ , \qquad 0 \longrightarrow \mathrm{Im}_{\mathcal{A}}(f) \longrightarrow \mathrm{Y}$$

Therefore, $\mathrm{Ker}_{\mathcal{A}}(f) \in \mathcal{A}''$ by using the general property of $\mathcal{A}'$ and **Proposition 9.7** . ■

Corollary Given an $\mathcal{A}''$-morphism, its $\mathcal{A}''$-kernel is its $\mathcal{A}$-kernel;
it is $\mathcal{A}''$-monic iff it is $\mathcal{A}$-monic. ∗

Proof. This is true by using the fact that ST is the left adjoint of the inclusion. ■

Lemma 13.2 (*Cokernels*) .
The $\mathcal{A}''$-cokernel of an $\mathcal{A}''$-morphism is defined by applying ST to the $\mathcal{A}$-cokernel. ∗

Proof. This is true by using the fact that ST is the left adjoint of the inclusion. ■

Corollary An $\mathcal{A}''$-morphism is epi iff its $\mathcal{A}$-cokernel is torsion. ∗

Proof. This is true by using **Exercise 10.1** . ■

[6] One useful example to keep in mind is how (co-)kernels of a sheaf (for example, of abelian groups) is defined. Our case shares some sort of resemblence to this example - sheafification is a reflection.

Aside One can translate universal properties into the language of representable functors.
In this sense, **Lemma 13.2** and the corollary to **Lemma 13.1** , can be verified by making sense of

$$\varprojlim_i \mathrm{Hom}_{\mathcal{A}''}(X, \alpha(i)) \simeq \varprojlim_i \mathrm{Hom}_{\mathcal{A}''}(STX, \alpha(i)) \simeq \varprojlim_i \mathrm{Hom}_{\mathcal{A}}(X, \alpha(i))$$
$$\simeq \mathrm{Hom}_{\mathcal{A}}(X, \varprojlim_i \alpha(i)) \simeq \mathrm{Hom}_{\mathcal{A}''}(X, \varprojlim_i \alpha(i))$$

$$\varprojlim_i \mathrm{Hom}_{\mathcal{A}''}(\alpha(i), X) \simeq \varprojlim_i \mathrm{Hom}_{\mathcal{A}''}(ST\alpha(i), X) \simeq \varprojlim_i \mathrm{Hom}_{\mathcal{A}}(\alpha(i), X)$$
$$\simeq \mathrm{Hom}_{\mathcal{A}}(\varinjlim_i \alpha(i), X) \simeq \mathrm{Hom}_{\mathcal{A}''}\left(ST\left(\varinjlim_i \alpha(i)\right), X\right)$$

and interpret (co-)kernels as (co-)limits. *

Lemma 13.3 (*Monics are Normal*) $\mathcal{A}''$-monics are $\mathcal{A}''$-normal. *

Proof. We've seen that $\mathcal{A}''$-monics are $\mathcal{A}''$-morphisms that are also $\mathcal{A}$-monics.
Take one of such - that is, an $\mathcal{A}$-exact sequence with objects lying in $\mathcal{A}''$:

$$0 \longrightarrow X \longrightarrow Y$$

Extend it to an $\mathcal{A}$-exact sequence :

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

then we see that $Z \in \mathcal{A}'$ by absolute purity of X, so $ST(Z) \simeq S(Z)$. We get the following :

$$\mathrm{Ker}_{\mathcal{A}''}(\mathrm{Cok}_{\mathcal{A}''}(\ X \longrightarrow Y\ )) \simeq \mathrm{Ker}_{\mathcal{A}}(\ Y \longrightarrow ST(Z)\ ) \simeq \mathrm{Ker}_{\mathcal{A}}(\ Y \longrightarrow S(Z)\ )$$

Since $Z \longrightarrow S(Z)$ is monic, we get

$$\mathrm{Ker}_{\mathcal{A}}(\ Y \longrightarrow S(Z)\ ) \simeq \mathrm{Ker}_{\mathcal{A}}(\ Y \longrightarrow Z\ ) \simeq \mathrm{Ker}_{\mathcal{A}}(\mathrm{Cok}_{\mathcal{A}}(\ X \longrightarrow Y\ ))$$

Since monics in $\mathcal{A}$ are normal (being abelian), we get finally that

$$\mathrm{Ker}_{\mathcal{A}}(\mathrm{Cok}_{\mathcal{A}}(\ X \longrightarrow Y\ )) \simeq (\ X \longrightarrow Y\ )$$

This shows our assertion. ■

Lemma 13.4 (*Epis are Normal*) $\mathcal{A}''$-epis are $\mathcal{A}''$-normal. *

Proof. Suppose we are given an $\mathcal{A}''$-epi $X \xrightarrow{f} Y$. Take exact sequences in $\mathcal{A}$:

$$X \longrightarrow \mathrm{Im}_{\mathcal{A}}(f) \longrightarrow 0\ , \qquad 0 \longrightarrow \mathrm{Im}_{\mathcal{A}}(f) \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

Then Z is torsion (Notice that $\mathrm{Cok}_{\mathcal{A}''}(\ X \longrightarrow Y\ ) \simeq \mathrm{Cok}_{\mathcal{A}}(\ \mathrm{Im}_{\mathcal{A}}(f) \longrightarrow Y\ ) \simeq Z$).
By **Proposition 12.1** , we have $Y \simeq S(\mathrm{Im}_{\mathcal{A}}(f))$ (Notice that $\mathrm{Im}_{\mathcal{A}}(f) \in \mathcal{A}'$).
Firstly, we have

$$\mathrm{Cok}_{\mathcal{A}''}(\mathrm{Ker}_{\mathcal{A}''}(\ X \longrightarrow Y\ )) \simeq \mathrm{Cok}_{\mathcal{A}''}(\mathrm{Ker}_{\mathcal{A}}(\ X \longrightarrow Y\ ))$$

Since epis in $\mathcal{A}$ are normal, we have

$$\mathrm{Cok}_{\mathcal{A}}(\mathrm{Ker}_{\mathcal{A}}(\ X \longrightarrow Y\ )) \simeq (\ X \longrightarrow \mathrm{Im}_{\mathcal{A}}(f)\ )$$

By the description of an $\mathcal{A}''$-cokernel given in **Corollary** of **Lemma 13.2** , we have

$$\mathrm{Cok}_{\mathcal{A}''}(\mathrm{Ker}_{\mathcal{A}}(\ X \longrightarrow Y\ )) = (\ X \longrightarrow \mathrm{Im}_{\mathcal{A}}(f) \longrightarrow \mathrm{ST}(\mathrm{Im}_{\mathcal{A}}(f))\ )$$

We have seen that $\mathrm{Im}_{\mathcal{A}}(f) \in \mathcal{A}'$, so we get

$$\mathrm{ST}(\mathrm{Im}_{\mathcal{A}}(f)) \simeq \mathrm{S}(\mathrm{Im}_{\mathcal{A}}(f)) \simeq Y$$

This shows that

$$\mathrm{Cok}_{\mathcal{A}''}(\mathrm{Ker}_{\mathcal{A}''}(\ X \longrightarrow Y\ )) \simeq (\ X \longrightarrow Y\ )$$

This proves the assertion. ■

Theorem 13.5 (*Absolutely Pure Objects form an Abelian Category*) $\mathcal{A}''$ is abelian. ∗

14 Proof of the General Embedding Theorem

In this section, $\mathcal{A}$ is a small abelian category.

Summary Recall that we have the fully faithful embedding functor (section 7)

$$\mathrm{H} :\ \mathcal{A}^{\mathrm{op}} \longrightarrow [\mathcal{A}, \mathsf{Ab}]_{\mathrm{L}}$$

and we know $[\mathcal{A}, \mathsf{Ab}]_{\mathrm{L}}$ is abelian. The final steps of the proof of the general theorem are as follows : (compare the paragraph about defects in section 7)

- We can embed $\mathcal{A}$ into ${[\mathcal{A}, \mathsf{Ab}]_{\mathrm{L}}}^{\mathrm{op}}$ fully faithfully; it turns out that this functor is exact.
- We show that ${[\mathcal{A}, \mathsf{Ab}]_{\mathrm{L}}}^{\mathrm{op}}$ is fully abelian, by using the weak embedding theorem; this amounts to show that $[\mathcal{A}, \mathsf{Ab}]_{\mathrm{L}}$ is complete, admitting an injective cogenerator.

Theorem 14.1 $[\mathcal{A}, \mathsf{Ab}]_{\mathrm{L}}$ is complete and admits an injective cogenerator. ∗

Proof. By **Exercise 14.1** , **Exercise 14.2** , this category is complete, admits injective envelopes and admits a generator G. Need an injective cogenerator. Define

$$G^{+} = \prod_{Q \in \mathcal{Q}(G)} Q$$

(Notice that $\mathcal{Q}(G)$ is small by **Proposition 5.2** , **Exercise 9.2**), take an injective envelope $0 \longrightarrow G^{+} \longrightarrow E$. We claim that E is an injective cogenerator.

Given a non-zero morphism $A \xrightarrow{f} B$, want to find $B \xrightarrow{g} E$ such that

$$A \xrightarrow{f} B \xrightarrow{g} E$$

is not 0. Since G is a generator, may take some morphism $G \xrightarrow{h} A$ such that

$$G \xrightarrow{h} A \xrightarrow{f} B$$

is not 0. Factor this map through its image, we have an epi-mono factorization

$$\begin{tikzcd} G \arrow[rr, "fh"] \arrow[dr, "p"'] & & B \\ & I \arrow[ur, "i"'] & \end{tikzcd}$$

Notice $I \in \mathscr{Q}(G)$, get a monic $m : I \longrightarrow G^+ \longrightarrow E$. Get a factorization

$$\begin{tikzcd} I \arrow[r, "i"] \arrow[d] \arrow[dr, "m"] & B \arrow[d, dashed, "g"] \\ G^+ \arrow[r] & E \end{tikzcd}$$

Notice that $0 \neq mp = gip = gfh$. Therefore, $gf \neq 0$. ■

Remark Notice in the above proof, essentiality isn't used. It fact, we only used the fact that the category has enough injectives. ∗

Exercise 14.1 Show that $[\mathscr{A}, \mathsf{Ab}]_L$ is closed under products and essential extensions.
(Hint : For products, consider pointwise; For essential extensions, use **Exercise 9.3**)
Hence this category is complete and admits injective envelopes. ∗

Exercise 14.2 Show that $[\mathscr{A}, \mathsf{Ab}]_L$ admits a generator. (Hint : The category $[\mathscr{A}, \mathsf{Ab}]$ has a generator G defined in **Proposition 3.1** . Now use completeness of $[\mathscr{A}, \mathsf{Ab}]_L$.) ∗

Theorem 14.2 The embedding functor $H : \mathscr{A}^{op} \longrightarrow [\mathscr{A}, \mathsf{Ab}]_L$ is full, faithful, exact. ∗

Proof. Only need exactness. Suppose given an exact sequence

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

Let E be the injective cogenerator, then E is exact by **Exercise 11.1** , get

$$0 \longrightarrow E(X') \longrightarrow E(X) \longrightarrow E(X'') \longrightarrow 0$$

Yoneda lemma says that this is isomorphic to the result by applying H and then $\mathrm{Hom}(-, E)$.
By **Exercise 6.4** (and the fact that E is a cogenerator), we have exactness of

$$H(X) \longrightarrow H(X') \longrightarrow 0$$

We already know that H is left exact, so we are done. ■

Proof of the General Embedding Theorem (***Theorem 2.2*** *).* .
Since an exact full abelian subcategory of a fully abelian category is fully abelian, we are done. ■