Representation of Finite groups

- Some Solutions to Exercises

Contents

1 Generalities on linear representations

No exercises in this section.

2 Character Theory

Exercise 2.1 One can realize $\chi + \chi'$ as the character of $\rho \oplus \rho'$. Pick $s \in G$. Suppose

$$(e_1, \dots, e_m), \quad (e'_1, \dots, e'_n)$$

are eigenbasis for ρ, ρ' with eigenvalues (λ_i, λ'_j are $\mathbb{C}$-valued functions of G)

$$(\lambda_1, \dots, \lambda_m), \quad (\lambda'_1, \dots, \lambda'_n)$$

then one can define a basis of $\rho \oplus \rho'$ by

$$(e''_1, \dots, e''_{m+n}) := (e_1, \dots, e_m, e'_1, \dots, e'_n)$$

with corresponding eigenvalues

$$(\lambda''_1, \dots, \lambda''_{m+n}) := (\lambda_1, \dots, \lambda_m, \lambda'_1, \dots, \lambda'_n)$$

Now we compute :

$$(\chi + \chi')^2_\sigma = \mathrm{Tr}(\rho \oplus \rho') = \sum_{i \le j} \lambda''_i \lambda''_j = \left(\sum_{i \le m < j} + \sum_{i \le j \le m} + \sum_{m < i \le j} \right) \lambda''_i \lambda''_j = \chi\chi' + \chi^2_\sigma + \chi'^2_\sigma$$

We omit the similar computation for alternating squares.

Exercise 2.2 Let $(e_i)_{i \in X}$ be a basis of X, suppose for each $s \in G$ that

$$\rho_s(e_i) = \sum_{j \in X} r_{ji}(s) e_j$$

then we have

$$\chi = \sum_i r_{ii}$$

Since ρ is permutation representation of the group action G on X, we get

$$r_{ji}(s) = \delta_{si,j}$$

From this observation, we get

$$\chi(s) = \sum_i r_{ii}(s) = \sum_i \delta_{si,i}$$

This is the number of elements in X fixed by s.

Exercise 2.3 The idea is to define $\rho'_s(x') = x' \circ \rho_s^{-1}$. In a more informal way :

$$\rho' = \circ \rho^{-1}$$

We have the following calculation :

$$< \rho'_s(x'), \rho_s x > = < x' \rho_s^{-1}, \rho_s x > = < x', x >$$

This shows existence of ρ'; uniqueness follows from nondegeneracy of $<,>$.
To compute the character χ' of ρ', take an eigenbasis e_i of ρ with values λ_i define $e'_i \in V'$ by

$$< e'_i, e_j > = \delta_{ij}$$

then we have

$$< \rho'_s(e'_i), e_j > = < e'_i, \rho_s^{-1} e_j > = < \lambda_i^* e'_i, e_j >$$

so by non-degeneracy of $<,>$, we see that $\rho'_s(e'_i) = \lambda_i^* e'_i$, hence

$$\chi' = \sum_i \lambda_i^* = (\sum_i \lambda_i)^* = \chi^*$$

Exercise 2.4 The fact that it is a representation is straightfoward. Let us show that $\rho, \rho'_1 \otimes \rho_2$ are isomorphic representations, where ρ'_1 the contragredient of ρ_1 in **Exercise 2.3** .
Suppose $(d_1, \dots, d_m), (e_1, \dots, e_n)$ are good basis for ρ_1, ρ_2 with eigenvalues $(\lambda_1, \dots, \lambda_m), (\mu_1, \dots, \mu_m)$.

- Define basis $(d'_1, \dots, d'_m)$ for ρ'_1 as in **Exercise 2.3** ; it has eigenvalues $(\lambda_1^*, \dots, \lambda_m^*)$. Now we get a good basis $(d'_i \otimes e_j)$ for $\rho'_1 \otimes \rho_2$ with eigenvalues $\lambda_i^* \mu_j$.
- Define basis $(f_{11}, f_{21}, \dots, f_{mn})$ of ρ by

  $$f_{ij}(d_k) = \delta_{ik} e_j$$

  then we have

  $$\rho_s(f_{ij})(d_k) = (\rho_{2,s} f_{ij} \rho_{1,s}^{-1})(d_k) = \lambda_k^* \mu_j \delta_{ik} e_j = \lambda_k^* \mu_j f_{ij}(d_k) = \begin{cases} 0, & \text{if } k \neq i \\ \lambda_i^* \mu_j f_{ij}(d_k), & \text{if } k = i \end{cases}$$

  so we get $\rho_s(f_{ij})(d_k) = \lambda_i^* \mu_j f_{ij}(d_k)$, hence f_{ij} has eigenvalue $\lambda_i^* \mu_j$.

Now define a linear map between the representation spaces of ρ and $\rho'_1 \otimes \rho_2$:

$$T: \; f_{ij} \longmapsto d'_i \otimes e_j$$

This clearly defines an isomorphism of representations.

Exercise 2.5 The number of times ρ contains 1 is given by

$$(\rho|1) = \frac{1}{g} \sum_{t \in G} \chi(t)$$

by using Theorem 4 and the fact that the character of 1 is the constant function with value 1.

Exercise 2.6

(a) By decomposing the representation by restriction to orbits, it suffices to show :

> If G acts transitively on X, then ρ decomposes as $1 \oplus \psi$, and that the corresponding this decomposition, we have a decomposition of χ into $1 + \psi$ such that $(\psi|1) = 0$.

This follows directly from the computation

$$(\chi|1) = \frac{1}{g}\sum_{t\in G}\chi(t) = \frac{1}{g}\sum_{t\in G}|X_t| = \frac{1}{g}\sum_{x\in X}|G_x| = \frac{1}{g}\sum_{x\in X}\frac{g}{|Gx|} = \frac{1}{g}\sum_{x\in X}\frac{g}{|X|} = 1$$

(the first equality is by **Exercise 2.5** , the second by **Exercise 2.2** , the third by a counting argument, the fourth by orbit-stabilizer, the fifth by the assumption that G acts transitively.)

(b) Identify this representation with the tensor product $\rho \otimes \rho$.

(c) Equivalence between (i)-(iii) is already established in the hint.
The hint also said that (iii) is equivalent to $(\psi^2|1) = 1$.
Since ψ is real-valued (since χ, 1 are), we get $(\psi^2|1) = (\psi|\psi)$.
These observations establishes the equivalence between (iii) and (iv).

Exercise 2.7 Suppose χ is one of such with dimension 1, let χ_i be the irreducible characters of G of dimension n_i, we have the following observation :

$$(\chi, \chi_i) = \frac{1}{g}\sum_{t\in G}\chi(t)\chi_i(t^{-1}) = \frac{nn_i}{g}n_i$$

Take $\chi_i = 1$, we see that n/g is a non-negative integer; this will suffice.

Exercise 2.8 In the canonical decomposition of V given by

$$V = \bigoplus_i V_i$$

choose a decomposition of V_i as

$$V_i = \bigoplus_j W_{i,j}$$

where each $W_{i,j}$ is isomorphic to W_i, then we also have canonical injections :

$$\alpha_{i,j} : \ W_i \xrightarrow{\sim} W_{i,j} \longrightarrow V_i \longrightarrow V$$

and projection maps

$$\rho_{i,j} : \ V \longrightarrow V_i \longrightarrow W_{i,j} \xrightarrow{\sim} W_i$$

Notice that these maps are all morphisms of representations.

(a) Assume $h \neq 0$. By assumption, h is a morphism of representations.
Take the subrepresentation $\mathrm{Ker}(h)$, we see that h is injective.
Compose h with the projection maps $\rho_{i,j}$, we see that as a vector space, we have

$$H_i = \bigoplus_j \mathrm{Span}(\alpha_{i,j})$$

From this, we see that H_i has the required dimension.

(b) Since G acts trivially on each $\mathrm{Span}(\alpha_{i,j})$ (the G-structure on H_i given in **Exercise 2.4**), the above decomposition is readily a direct sum decomposition. Restriction of F to each $\mathrm{Span}(\alpha_{i,j}) \otimes W_i$ induce an isomorphism (of representations) to $W_{i,j}$.

(c) Since G acts trivially on each $\mathrm{Span}(\alpha_{i,j})$ (as remarked in (b)), H_i is a trivial G-space, hence any direct sum decomposition H_i as direct sum of lines (as vector spaces), is readily a direct sum decomposition of trivial representations.

Exercise 2.9 Recall the maps $\alpha_{i,j}$ in **Exercise 2.8** . The image of the evaluation morphism is

$$\mathrm{Span}(\alpha_{i,j}(e_\alpha)) = V_{i,\alpha}$$

so we are done already.

Exercise 2.10 Since $x \in V_i$, we have

$$x = p_i(x) = (\sum_\alpha p_{\alpha\alpha})(x) = (\sum_\alpha p_{\alpha 1}p_{1\alpha})(x) = \sum_\alpha p_{\alpha 1}(x_1^\alpha) \in \sum_\alpha W(x_1^\alpha)$$

It is easy to see that $V(x)$ admits another description :

$$V(x) = \mathrm{Span}(\rho_t(x))_{t\in G}$$

By the direct description of $p_{1\alpha}$, we get

$$x_1^\alpha = p_{1\alpha}(x) = \frac{n}{g}\sum_{t\in G} r_{\alpha 1}(t^{-1})\rho_t(x) \in \mathrm{Span}(\rho_t(x))_{t\in G} = V(x)$$

These two observations establishes the identity

$$V(x) = \sum_\alpha W(x_1^\alpha)$$

3 Subgroups, Products, Induced Representations

Exercise 3.1 Each ρ_t is an automorphism of representation, so are homotheties, hence any one dimensional subspace is a subrepresentation.

Exercise 3.2

(a) Since ρ_s is a morphism for $s \in C$, we have by Schur's lemma that ρ_s is a homothety.
Since the eigenvalues of ρ_s lies on the unit circle, we get $|\chi(s)| = n$ consequencely.

(b) By orthogonality of characters and (a) that

$$g = \sum_{t\in G} |\chi(s)|^2 \geq \sum_{t\in C} |\chi(s)|^2 = cn^2$$

(c) Recall that each element in G has finite order. Define $\xi = \exp(2\pi i/g)$.
By (a), for each $t \in C$, there exists an integer μ_s such that

$$\rho_s = \xi^{\mu_s}$$

Let $d = \gcd(\mu_s)_{s \in C}$, then there exists integers a_s such that

$$d = \sum_{s \in C} a_s \mu_s$$

Define an element t in C by

$$t = \prod_{s \in C} s^{a_s}$$

then we get $\rho_t = \xi^d$. Since ρ is faithful, we see that t generates C.

Exercise 3.3 A character from an abelian group is just a homomorphism from G to $\mathbb{C}^*$. The group structure on $\mathbb{C}^*$ then endows $\hat{G}$ with a structure of an abelian group.
To check that the map into the double dual of G is an injection, notice that

$$\chi(x) = \chi(y) \text{ for all } \chi \in \hat{G} \text{ iff } \chi(xy^{-1}) = 1 \text{ for all } \chi \in \hat{G} \text{ iff } x = y$$

This will suffice. For reasons of cardinality, this map is a bijection.

Exercise 3.4 Use the hint and Example 1,3 in that subsection.

Exercise 3.5 We already know that W can be identified as the space of elements in V that vanishes off H. Notice that the explicit description of the action of G on V implies that $\rho_s W$ is the space of elements in V that that vanishes off Hs^{-1}; from this observation, the conditions of the definition of an induced representation is easily checked.

Exercise 3.6 The idea is to calculate the characters of ρ and $\theta \otimes r_K$. Let $(u, v) \in H \times K$, we get

$$\chi_\rho(uv) = \sum_{\substack{t \in K \\ t^{-1}uvt \in H}} \chi_\theta(t^{-1}uvt)$$

We pause for a bit to consider what does this summation does.

- Suppose $v = 1$, then we have for all $t \in K$ that $t^{-1}uvt = u$, and that

$$\{t \in K : t^{-1}uvt \in H\} = K$$

- Suppose $v \neq 1$, then we have for all $t \in K$ that $t^{-1}uvt \notin H$, so

$$\{t \in K : t^{-1}uvt \in H\} = \emptyset$$

From these two observations, we get

$$\chi_\rho(uv) = \sum_{\substack{t \in K \\ t^{-1}uvt \in H}} \chi_\theta(t^{-1}uvt) = \begin{cases} k\chi_\theta(u) & \text{if } v = 1 \\ 0 & \text{if } v \neq 1 \end{cases}$$

On the other hand, we know that

$$\chi_{\theta \otimes r_K}(uv) = \chi_\theta(u)\chi_{r_K}(v) = \begin{cases} k\chi_\theta(u) & \text{if } v = 1 \\ 0 & \text{if } v \neq 1 \end{cases}$$

so ρ is isomorphic to $\theta \otimes r_K$.

4 Compact Groups

No exercises in this section.

5 Examples

nah.

6 The Group Algebra

Exercise 6.1 Let us show (i) implies (ii). Define the nontrivial proper submodule V as given in the hint, suppose it has a complementary summand W. Take some nontrivial element

$$x = \sum_{t \in G} a_t t \in W, \quad \sum_{t \in G} a_t \neq 0$$

then the element $\sum_{s \in G} sx$ is non-trivial and lies in the intersection of W and V.

Exercise 6.2 The formula is bilinear in each argument, so it suffices to treat the case where $u, v \in G$. By definition, we get

$$< u, v >= g\delta_{uv^{-1}}$$

so we only need to show

$$\delta_{uv^{-1}} = \frac{1}{g} \sum_i n_i \chi_i(uv^{-1})$$

which is clear by the character theory of the regular representation of G.

Exercise 6.3

(a) Since U is finite and contains G, the first assertion is clear. For the identity

$$\mathrm{Tr}_{W_i}(\rho_i(s^{-1})u_i)^* = \mathrm{Tr}_{W_i}(u_i'\rho_i(s))$$

recall that we may choose an eigenbasis for W_i with respect to ρ_i, so we may safely say that $\mathrm{Tr}_{W_i}(\rho_i(s^{-1})u_i)$ is the sum of eigenvalues. Observe also that

$$(\rho_i(s^{-1})u_i)(u_i'\rho_i(s)) = 1$$

so we deduce the identity by using an argument like

$$\mathrm{Tr}_{W_i}(\rho_i(s^{-1})u_i)^* = \sum \lambda_j^* = \sum \lambda_j^{-1} = \mathrm{Tr}_{W_i}(u_i'\rho_i(s))$$

The identity

$$\mathrm{Tr}_{W_i}(u_i'\rho_i(s)) = \mathrm{Tr}_{W_i}(\rho_i(s)u_i')$$

is clear by noticing that $u_i'\rho_i(s), \rho_i(s)u_i'$ are conjugates.

(b) By (a) and **Exercise 6.1**, we see that

$$\sum_{t \in G} |u(t)|^2 = \sum_{t \in G} u(t)u'(t^{-1}) = \frac{1}{g} < u, u' >= \frac{1}{g} \sum_{i=1}^{h} n_i \mathrm{Tr}_{W_i}(uu') = \frac{1}{g} \sum_{i=1}^{h} n_i^2 = 1$$

(c) Obvious.

(d) Take $U = \mathbb{Z}[G]$.

Exercise 6.4 By the computation

$$\omega_k(p_i) = \frac{1}{n_k} \sum_{t \in G} \frac{n_i}{g} \chi_i(t^{-1}) \chi_k(t) = \delta_{ik}$$

we see that the image of the elements p_i under the isomorphism described in proposition 13 form a basis of $\mathbb{C}^h$, so p_i form a basis, and it also follows from the computations

$$\omega_k(p_i p_j) = \omega_k(p_i)\omega_k(p_j) = \delta_{ik}\delta_{jk} = \delta_{ij}\delta_{ik} = \begin{cases} \omega_k(p_i) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\omega_k(\sum_i p_i) = \sum_i \delta_{ik} = 1 = \omega_k(1)$$

that the other required identities are verified.

Exercise 6.5 Let ν be a homomorphism from the center of $\mathbb{C}[G]$ to $\mathbb{C}$. **Exercise 6.4** says

$$\begin{cases} \nu(p_i) \in \{0, 1\} \\ \sum_i \nu(p_i) = 1 \end{cases}$$

It is easy to deduce from this that ν is just one of the ω_i.

Exercise 6.6 Firstly, since

$$\bigoplus_i \mathbb{Z}e_i \subseteq \text{Cent.}(\mathbb{C}[G])$$

we get $\bigoplus_i \mathbb{Z}e_i \subseteq \text{Cent.}(\mathbb{Z}[G])$. For the reverse, choose some

$$u = \sum_{t \in G} u(t)t \in \text{Cent.}(\mathbb{Z}[G])$$

Suppose $t, t' \in G$ lie in the same conjugacy class, say,

$$st = t's$$

for some $s \in G$, then by $us = su$, we get $u(t) = u(t')$. This observation shows $u \in \bigoplus_i \mathbb{Z}e_i$.

Exercise 6.7 By the hint, we are done by applying the triangle inequality.

Exercise 6.8 (For each nonnegative integer N, we define $\xi_N := \exp(2\pi i/N)$.)
Let $f(x)$ be the minimal polynomial of a over $\mathbb{Q}$, then $f(x) \in \mathbb{Z}[x]$ by integrality.
In order to show $|A| \leq 1$, it suffices to show the following :

Claim : The conjugates of a (roots of $f(x)$) over Q all have length ≤ 1.

Since λ_i are roots of unities, may take an integer $N \gg 0$ such that $\lambda_i \in \{\xi_N^0, \ldots, \xi_N^{N-1}\}$.
We have the following diagram of field extensions :

$$\mathbb{Q}(\xi_N) \text{ --- } \mathbb{Q}(a) \text{ --- } \mathbb{Q}$$

Recall that the Galois group of the cyclotomic extension $\mathbb{Q}(\xi_N)/\mathbb{Q}$ is given by

$$\text{Gal}(\mathbb{Q}(\xi_N)/\mathbb{Q}) = \{\theta_d : \xi_N \mapsto \xi_N^d | (d, N) = 1\} \simeq (\mathbb{Z}/N\mathbb{Z})^\times$$

Define a polynomial

$$F(x) := \prod_{\sigma\in \mathrm{Gal}(\mathbb{Q}(\xi_N)/\mathbb{Q})} (x-\sigma(a)) = \prod_{d:(d,N)=1} (x-\theta_d(a))$$

then $F(x)\in\mathbb{Q}[x]$ and $f(x)|F(x)$ by the property of minimal polynomials.
By the definition of θ_d, we have the following calculation :

$$\theta_d(a) = \theta_d(\frac{1}{n}(\sum_{i=1}^{h}\lambda_i)) = \frac{1}{n}(\sum_{i=1}^{h}\lambda_i^d)$$

so $|\theta_d(a)|\leq 1$, so the roots of $F(x)$ (hence those of $f(x)$) all have length ≤ 1; this proves the claim.
By the definition of $f(x)$, we see that the constant term of $f(x)$ is $\pm A$, so we get $A\in\mathbb{Z}$.
Since $|A|\leq 1$, $A\in\{0,\pm 1\}$. If $A=0$, $a=0$; If $A=\pm 1$, $|a|=1$ by our claim, and hence $\lambda_i = a$ for all i.

Exercise 6.9 Since the e_i lies in the center of the group algebra, it follows immediately that

$$\frac{c(s)}{n}\chi(s)$$

are all algebraic integers (for each $s\in G$). For the second assertion, we claim that

$$\frac{1}{n}\chi(s)$$

is an algebraic integer; this follows from the fact that $\chi(s)/n$ is a $\mathbb{Z}$-combination of the algebraic integers $\chi(s)$ and $c(s)\chi(s)/n$ by using the assumption that $(c(s),n)=1$.
By **Exercise 6.8** , the eigenvalues of ρ_s are all the same given $\chi(s)\neq 0$, so ρ_s is a homothety.

Exercise 6.10 By the character theory of the regular representation of G, we get

$$1+\sum_{\chi\neq 1}\chi(1)\chi(s) = 0$$

whenever $s\neq 1$. Notice that suppose every irreducible character of G satisfies

$$\text{Either } \chi(s)=0 \text{ or } p|\chi(1)$$

then we see that

$$-1 = \sum_{\chi\neq 1}\chi(1)\chi(s) = \sum_{\substack{\chi\neq 1\\ p|\chi(1)}}\chi(1)\chi(s)$$

This formula then exhibits $1/p$ as an algebraic integer in view of Proposition 15.
For the second assertion, notice that given ρ,χ satisfying the condition, we see that

$$\chi(s)\neq 0 \text{ and } (\chi(1),c(s))=1$$

so in view of **Exercise 6.9**, $\rho(s)$ is just a homothety.
For the last part, notice first that since χ isn't trivial, $N\neq G$. Next, since

$$sN\in \text{Cent.}(G/N) \text{ iff the commutator } [st] \text{ with any } t\in G \text{ lies in G iff } \rho([st])=1 \text{ for all } t\in G$$

we see that the last assertion is clearly true by the observation that $\rho(s)$ is a homothety.