Cheatsheet - Mathematical Methods for QF

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1. Probability

Definitions

- Change of variable for Random Variables, given that $y = y(x)$: $g(y) = \frac{p(x)}{|dy/dx|}$
- Moments of a distribution: $\mu_l = E[X^l] = \int x^l p(x)dx$
- Variance: $\sigma^2 = E[(X-\mu)^2] = E[X^2] - E[X]^2$
- Skewness - asymmetry parameter: $S = E[(\frac{X-\mu}{\sigma})^3]$
- Kurtosis - measure of "tail weights": $\kappa = E[(\frac{X-\mu}{\sigma})^4] - 3$
- Covariance: $Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] = E[XY] - \mu_X\mu_Y$
- Correlation: $\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X\sigma_Y}$
- If X and Y are independent then $cov(X,Y) = 0$, but not the other way around

Common distributions

- Uniform distribution:
  - $p(x) = \frac{1}{b-a}$
  - $\mu = \frac{a+b}{2}$
  - $\sigma^2 = \frac{(b-a)^2}{12}$
- Binomial distribution:
  - $p(k,n,p) = \binom{n}{k}p^k q^{n-k}$
  - $\mu = np$
  - $\sigma^2 = np(1-p)$
- Gaussian distribution $X \sim N(\mu, \sigma^2)$:
  - $p(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
  - $p(z) = \frac{1}{\sqrt{2\pi}}e^{-z^2/2}$, $z = \frac{x-\mu}{\sigma}$
  - $I(a) = \int e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$
  - $\frac{-dI(a)}{da} = \int x^2 e^{-ax^2}dx = 0.5\frac{\pi}{a^3}$
  - $f(\lambda) = E[e^{\lambda X}] = e^{\lambda\mu + \lambda^2\sigma^2/2}$
  - $F(z) = \phi(\frac{x-\mu}{\sigma}) = P(Z < z) = \phi(z) = \int_{-\infty}^{z} e^{-z'^2/2}dz'$
- Lognormal distribution:
  - $r \sim N(\mu, \sigma^2)$, $r = log(x)$ and $X \sim Lognormal(\mu, \sigma^2)$
  - $g(x) = \frac{1}{\sqrt{2\pi\sigma^2}x}e^{-\frac{(logx-\mu)^2}{2\sigma^2}}$
  - $E[X] = e^{\mu+\sigma^2}$
  - $var(X) = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$
- Poisson distribution:
  - $p(k,\lambda t) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$
  - $E[X] = \lambda$, $E[X^2] = \lambda + \lambda^2$
  - $var(X) = \lambda$
  - $Binomial(k,n,p) \to Poisson(k,\lambda)$ as $n \to \infty$ $p \to 0$ and $np = \lambda$
- Cauchy distribution - fat tails:
  - $p(x_0, \gamma) = \frac{1}{\pi\gamma(1+(\frac{x-x_0}{\gamma})^2)}$
  - Extreme events are highly likely to occur

Characteristic function

- $\tilde{f}(t) = E[e^{itX}]$
- $E[X^l] = (-i)^l \frac{d^l \tilde{f}(t)}{dt^l}|_{t=0}$
- If $Y = X_1 + X_2$ then $\tilde{p}(t) = \tilde{p}_1(t)\tilde{p}_2(t)$
- If $X \sim N(\mu, \sigma^2)$ and $\tilde{p}(t) = e^{-\sigma^2 t^2/2 + i\mu t}$
- If $Y = X_1 + X_2 + \cdots + X_N$ is a Gaussian given that X_i is Gaussian
- $\tilde{p}(t) = \tilde{p}_1(t)\tilde{p}_2(t)\ldots\tilde{p}_N(t) = e^{-\hat{\sigma}^2 t^2/2 + i\hat{\mu}t}$, then $\hat{\sigma}^2 = \sum \sigma_i^2$ and $\hat{\mu} = \sum \mu_i$. $Y \sim N(\mu N, \sigma^2 N)$

Cumulant expansion

- $K(t) = log(E[e^{itX}]) = log(\tilde{f}(t))$
- $K(t) = \sum \frac{(it)^l}{l!}C_l$
- $C_n = (-i^n)\frac{d^n log\tilde{f}(t)}{dt^n}$
- $C1 =< X >, C_2 =< X^2 > - < X >^2$ and for Gaussian RVs: $C_3, C4, \cdots = 0$
- If RVs are independent: $C_n(X_1 + X_2 + \cdots + X_N) = C_n(X_1) + C_n(X_2) + \cdots + C_n(X_n)$

2. Discrete-time stochastic processes

Random walk model

- No dependence on past history. Stationarity and RVs are IID.
- $S_T = z_1 + z_2 + \cdots + z_T$, $E[z_t] = 0$, $E[z_t^2] = 1$ and $E[z_t z_t'] = 0, t \neq t'$
- $E[S_T] = 0$ and $var(S_T) = T$

Generalized random walk model

- $r_t = \mu + \sigma z_t$
- $X_T = r_1 + r_2 + \cdots + r_T = \sum_{t=1}^{T} r_t$, $E[X_T] = \mu T$ and $var(X_T) = T\sigma^2$

ARMA(p,q) model

$$R_t = c_o + c_1 R_{t-1} + \cdots + c_p R_{t-p} + \sigma z_t + \phi_1 z_{t-1} + \cdots + \phi_q z_{t-q}$$

- Past observations determine the likelihood of future outcomes
- Stationarity: probability distributions don't change over time
- Weak stationarity: $E[X_t]$ and $E[X_t^2]$ remain constant

3. Time series models

About autocorrelation estimators

- $\gamma_k = Cov(r_t, r_{t-k})$
- $\gamma_0 = var(r_t)$
- $\rho_k = Corr(r_t, r_{t-k}) = \frac{\gamma_k}{\gamma_0}$
- $var(r_t^{(2)})) = var(r_t + r_{t-1}) = 2var(r_t)(1+\rho_1)$
- $\hat{\gamma}_k = \frac{\sum_{t=k+1}^{k+T}(r_t - \hat{\mu})(r_{t-k} - \mu')}{T-1}$
- $\sqrt{T}(\hat{\rho}_k - \rho_k) \sim N(0,1)$

Testing the random walk - variance ratio

- $r_t^{(2)} = r_t + r_{t-1} = log(\frac{P_t}{P_{t-2}})$
- $r_t^{(q)} = \sum_{i=1}^{q} r_{t-i+1} = log(P_t/P_{t-q})$
- If returns are uncorrelated: $var(r_t^{(q)}) = qvar(r_t)$
- $VR(q) = \frac{var(r_t^{(q)})}{var(r_t)} = 1 + 2\sum_{k=1}^{q-1}(1 - \frac{k}{q})\rho_k$.
- If there's autocorrelation, then $VR(q)$ deviates from 1.

Order and model selection

- ACF plot = correlation coefficient of a given time series and "k" lags of itself over successive time intervals.
- PACF plot = coefficient of last lagged term in a linear regression. Captures a "direct" correlation btw time series and a lagged version of itself.
- AR order - check PACF plot.
- MA order - check ACF plot.

Solving the AR(1) model

$$R_t = c_0 + c_1 R_{t-1} + \sigma z_t$$

- $E[R_t] = \frac{c_0}{1-c_1} = \mu$ and $\lambda = -c1$
- Mean reversion: $R_t - \mu = -\lambda(R_{t-1} - \mu) + \sigma z_t$
- $\gamma_0 = var(R_t) = \frac{\sigma^2}{1-\lambda^2}$
- If $|\lambda| < 1$, then shocks die off over time
- $\gamma_k = Cov(R_t, R_{t-k}) = (-\lambda)^k\gamma_0$

Forecasting the AR(1) model

- $Y_t = \frac{R_t - \mu}{\sigma}$
- $Y_{t+1} = z_{t+1} - \lambda Y_t$, up to "t" information is known.
- $E[Y_{t+1}|I_t] = -\lambda Y_t$ and $E[Y_{t+2}|I_t] = \lambda^2 Y_t$
- The optimal forecast is the conditional mean: $f_{t,h} = E[X_{t+h}|I_t]$
- $e_{t+h} = X_{t+h} - f_{t,h}$, $MSE = E[e_{t+h}^2]$
- Forecast uncertainty: model setup and parameters estimation.

Random walk on a binomial tree

$$S_{t+1} = \begin{cases} S_t u, & p \\ S_t d, & (1-p) \\ d = 1/u \end{cases}$$

$$Prob(S_t = S_0 u^k d^{t-k}) = \binom{t}{k} p^k (1-p)^{t-k}$$

Lognormal notation

$$r_t = a + bx_t = \begin{cases} log(u), & p \\ log(d), & (1-p) \\ d = 1/u \end{cases}$$

- $x_t \sim Bernoulli(p)$
- $a = log(d)$, $b = log(\mu) - log(d)$
- $\mu = E[r_t] = a + pb$, $\sigma^2 = var(r_t) = b^2 p(1-p)$
- $log(u) = \mu + \sigma\sqrt{\frac{1-p}{p}}$
- $log(d) = \mu - \sigma\sqrt{\frac{p}{1-p}}$

Gambler's ruin

- Prob. of success: p
- Initial capital: $x_0 > 0$
- House assets or desired limit: a
- Stop when: break house ($x_f = a$) or lose capital ($x_f = 0$)
- Probability of ruin: $Q_x = pQ_{x+1} + qQ_{x-1}$
- Boundary conditions: $Q_0 = 1$ and $Q_a = 0$
- $Q_x = \frac{(q/p)^a - (q/p)^x}{(q/p)^a - 1}$
- For $p = q = 0.5$, then: $E[gain/step] = p - q = 0$
- For different stake b: $Q_x = pQ_{x+b} + qQ_{x-b}$
  - For $p \neq q$: $Q_x = \frac{(q/p)^{a/b} - (q/p)^{x/b}}{(q/p)^{a/b} - 1}$
  - Larger bets = lower ruin probability = higher chance of success
- If appetite is unbounded ($a \to \infty$) then:
  - $p \leq q$: $Q_x = 1$; certain ruin in unfavorable odds
  - $p > q$: $Q_x = (q/p)^x$
- Expected duration of the game:
  - $D_x = pD_{x+1} + qD_{x-1} + 1$
  - $D_x = \begin{cases} x(a-x), & p = q \\ \frac{x}{q-p} - \frac{a}{q-p}\frac{1-(q/p)^x}{1-(q/p)^a}, & p \neq q \end{cases}$

4. Continous-time stochastic processes

Brownian motion

- Discrete Random Walk: $B_{1,T} = \sum_{t_0+1}^{t_0+T} z_t$
- Scaling the random walk:
  - $\Delta t = T/n$, $\lambda = \sqrt{\Delta t} = \sqrt{T/n}$ and $\epsilon_t = \lambda z_t$.
  - $B_{\Delta t,T} = \sum_{t=1}^{n} \epsilon_t = \sqrt{\Delta t}\sum_{t=1}^{n} z_t$
  - $E[B_{\Delta t,T}] = 0$ and $var(B_{\Delta t,T}) = T$
  - $\lim_{\Delta t \to 0} B_{\Delta t,T} \sim N(0,T)$
- Finite process:
  - $X(t1,t2) = B(t2) - B(t1)$
  - $X \sim N(0, t2 - t1)$
- Infinitesimal process:
  - $dB_t \sim N(0, dt)$
  - $B(T) = B(0) + \int_0^T dB_t$
- General lognormal process: $log(\frac{S_{t2}}{S_{t1}}) = \int_{t1}^{t2} \mu(t)dt + \int_{t1}^{t2} \sigma(t)dB_t$

Ito process

$$dX_t = a.dt + b.dB_t$$

- $E[dB_t] = 0$, $E[dB_t^2] = dt$, $E[dB_t^3] = 0$, $E[dB_t^4] = 3(dt)^2$
- $E[dX_t] = adt$, $E[dX_t^2] = a^2(dt)^2 + b^2dt$
- $var(dX_t) = b^2dt$, $var(dX_t^2) = 2b^4(dt)^2 + O(dt^3)$

Ito's lemma

If X is an Ito process, then $F = f(X)$ is also an Ito process:

$$dF = \frac{\partial F}{\partial t}.dt + \frac{\partial F}{\partial X}.dX + \frac{b^2}{2}.\frac{\partial^2 F}{\partial X^2}.dt$$

- Heuristic: $(dB_t)^2 \to dt$ and $dX_t^2 \to b^2dt$

Ito processes

- Brownian motion with drift
  - $dS_t = \mu dt + \sigma dB_t$
  - $S_T = S_0 + \mu T + \sigma(B_T - B_0)$
- Geometric Brownian motion with drift
  - $dS_t = \mu S_t dt + \sigma S_t dB_t$
  - $dlog(S_t) = (\mu - \sigma^2)dt + \sigma dB_t$
  - $S_T = S_0 e^{(\mu - \sigma^2/2)T + \sigma(B_T - B_0)}$
- Ornstein-Uhlenbeck process - mean reversion
  - $dS_t = \lambda(\bar{S} - S_t)dt + \sigma dB_t$
- Cox-Ingersoll-Ross process - interest rates
  - $d\rho_t = \lambda(\bar{\rho} - \rho_t)dt + \sigma\sqrt{\rho_t}dB_t$
  - $F = \sqrt{\rho}$
  - $dF = (\frac{4\lambda\bar{\sigma} - \sigma^2}{8F} - 0.5\lambda F)dt + 0.5\sigma dB_t$

Black-Scholes equation

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2}.\frac{\partial^2 V}{\partial S^2} - rV = 0$$

- $V = V(t,S)$, $dS = \mu S dt + \sigma S dB_t$
- $dV = \frac{\partial V}{\partial t}.dt + \frac{\partial V}{\partial S}.dS + \frac{\sigma^2 S^2}{2}.\frac{\partial^2 V}{\partial S^2}.dt$
- Heuristic derivation:
  - Let $\pi = V - \Delta S$, then: $d\pi = dV - \Delta dS$
  - $d\pi = (\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}.\frac{\partial^2 V}{\partial S^2}).dt + (\frac{\partial V}{\partial S} - \Delta)dS$
  - To vanish stochastic term: $\Delta = \frac{\partial V}{\partial S}$
  - Risk-free portfolio: $d\pi = r\pi dt = r(V - \Delta S)dt$
- BSE with dividend yield:
  - $d\pi = (rV - \Delta S(r - D))dt$
  - $\frac{\partial V}{\partial t} + (r-D)S\frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2}.\frac{\partial^2 V}{\partial S^2} - rV = 0$
- Currency options:
  - $\frac{\partial V}{\partial t} + (r - r^*)S\frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2}.\frac{\partial^2 V}{\partial S^2} - rV = 0$
  - Foreign interest rate: r^*
- Commodity options:
  - $\frac{\partial V}{\partial t} + (r - q - D)S\frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2}.\frac{\partial^2 V}{\partial S^2} - rV = 0$
  - Cost of carry: q
- Options on futures:
  - $F = e^{r(T-t)}S$
  - $\frac{\partial V}{\partial t} + \frac{\sigma^2 F^2}{2}.\frac{\partial^2 V}{\partial F^2} - rV = 0$

5. Fixed income modeling

One factor model

$$\pi = q_1 V_1 + q_2 V_2$$

- Instantaneous spot rate: $dy = a.dt + b.dB_t$
- $dV_1 = (\frac{\partial V_1}{\partial t} + \frac{b^2}{2}.\frac{\partial^2 V}{\partial y^2}).dt + \frac{\partial V_1}{\partial y}.dy$
- To have a riskless portfolio:
  - $\frac{q_1}{q_2} = \frac{-\frac{\partial V_2}{\partial y}}{\frac{\partial V_1}{\partial y}}$
  - $d\pi = q_1 dV_1 + q_2 dV_2 = y\pi dt$
- All maturities satisfy the same PDE:
  - $\frac{\partial V_i}{\partial t} + \frac{b^2}{2}.\frac{\partial^2 V}{\partial S^2} - yV_i - f(t,y)\frac{\partial V_i}{\partial y} = 0$
  - Boundary condition for Zero-Coupon bond: $V_i(T,y) = 1$
- Market price of risk η:
  - Extra deterministic rate of return per unit of randomness

- Identical for all bonds - unobservable
- $\eta = \frac{a+f(t,y)}{b}$
- $\frac{dV-yVdt}{b\frac{\partial V}{\partial y}} = \frac{a+f(t,y)}{b}dt + dB_t$

Interest rate models

Vasicek model - mean reversion model

- $dy = \alpha(\bar{y} - y)dt + \sigma dB_t$
- $\eta = c_0 + c_1 y$
- $-f(t,y) = \alpha'(\bar{y'} - y)$
- $y(t) = y_0 e^{-\alpha t} + \bar{y}(1 - e^{-\alpha t}) + \sigma e^{-\alpha t}.\int_0^t e^{\alpha S} dBs$
- $E[y(t)] = y_0 + (\$bary - y_0)(1 - e^{-\alpha t})$
- $Var(y(t)) = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t})$
- Bond pricing PDE: $\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}.\frac{\partial^2 V}{\partial y^2} - yV + \alpha(\bar{y} - y)\frac{\partial V}{\partial y} = 0$
- Ansatz approach: try $V(t,y) = e^{f(t)-yg(t)}$ and apply boundary conditions.

6. Continuous-time finance

$$\frac{\partial p}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2} + \mu\frac{\partial p}{\partial x} = 0$$

- Solution: $X_t \sim N(\mu t, \sigma^2 t)$
- Backward equation - Fokker plank:
  - $\frac{\partial p}{\partial t_0} + \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x_0^2} + \mu\frac{\partial p}{\partial x_0} = 0$
  - Solution: $X_t \sim N(\mu(T - t_0), \sigma^2(T - t_0))$

Diffusion equation:

$$\frac{\partial p}{\partial t} - \frac{1}{2}\frac{\partial^2 p}{\partial z^2} = 0$$

- Pure Brownian motion: $\mu = 0$ and $\sigma = 1$
- One solution: $p_0(z,t) \sim N(0,t)$
- General solution:
  - $p(z, t = 0) = f(z)$
  - $p(z,t) = \int p_0(z - w, t)f(w)dw = \frac{1}{\sqrt{2\pi t}}\int e^{-(z-w)^2/(2t)}f(w)dw$

Special functions

- $f(s) = max(S - K, 0) = \frac{|S-K|+S-K}{2}$
- $\frac{df(S)}{dS} = \theta(S - K) = \begin{cases} 1, & S > K \\ 0, & o.w. \end{cases}$
- $\frac{df^2(S)}{dS^2} = \delta(S - K) = \begin{cases} 0, & S \neq K \\ \infty, & S = K \end{cases}$
- $\delta(x) = \lim_{t\to 0}\frac{e^{-x^2/(2t)}}{\sqrt{2\pi t}} = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$
- $\int_{-\infty}^{\infty}\delta(x)dx = 1$
- $\int_{-\infty}^{\infty}\delta(x)f(x)dx = f(0)$
- $\int_{-\infty}^{\infty}\delta(x - y)dx = f(y)$

Survival probabilities and barriers

$$\begin{cases} P_S(z,t) = \frac{1}{\sqrt{2\pi\sigma^2 t}}.(e^{-(z-\mu t-z_0)^2/(2\sigma^2 t)} \\ -Ce^{-(z-\mu t+z_0-2z^*)^2/(2\sigma^2 t)}) \\ C = e^{-2\mu(z_0-z^*)/\sigma^2} \\ P_S(t) = \int_{z^*}^{\infty} P_S(Z,t)dz \\ = \phi(\frac{\mu t+z_0-z^*}{\sqrt{\sigma^2 t}}) - e^{-2\mu(z_0-z^*)/\sigma^2}\phi(\frac{\mu t-z_0+z^*}{\sqrt{\sigma^2 t}}) \end{cases}$$

- Boundary condition at barrier: $P_S(z^*, t) = 0$

Black-Scholes Equation solutions

- $dS = \mu S dt + \sigma S dB_t$
- $V(S,t) = e^{-r(T-t)}E_t[V(S_T, T)]$
- Risk neutral pricing: replace μ by r.
- For a call option:
  - $V(S,t) = e^{-r(T-t)}\int p(S_T, T, S, t)V(S_T, T)dS_T$
  - $V(S,t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}}\int e^{-(x-x')^2\frac{e^{-r(T-t)}}{2\sigma^2(T-t)})}f(x')dx'$
  - $f(x') = g(S') = max(S' - K, 0)$
  - $V(S,t) = S\phi(d_+) - Ke^{-r(T-t)}\phi(d_-)$
  - $d_{\pm} = \frac{log(S/Ke^{-r(T-t)})}{\sigma\sqrt{T-t}} \pm 0.5\sigma\sqrt{T - t}$

American perpetual put

$$\frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

- Time independent: $\frac{\partial V}{\partial t} = 0$
- Boundary condition: $V(\hat{S}) = K - \hat{S}$
- Try solution: $V(S) = S^\alpha$
- To maximize V: $\hat{S} = \frac{K}{1+\sigma^2/(2r)}$
- $V(s) = \frac{K\sigma^2/2r}{1+\sigma^2/2r}(\frac{S}{K}(1 + \sigma^2/2r))^{-2r/\sigma^2}$

Measures and martingales

- An Ito process is a martingale iff it has zero drift.
- $E_t[X_{t'}] = X_t,\ t < t'$ then $E[dX_t] = 0$
- Discounted price process $F = e^{-rt}S$
- S follows a Geometric Brownian Motion
- dF is a martingale process iff $\mu = r$

Risk-neutral pricing

- Replace μ by r
- $d(log(S_t)) = (r - \sigma^2/2)dt + \sigma dB_t^Q$
- $log(S_T/S_0) \sim N((r - \sigma^2/2)T, \sigma^2 T)$
- Use risk neutral probabilities: $X_t = e^{-r(T-t)}E_t^Q[X_T]$
- All traded (non-arbitrage) assets have discounted price processes that are martingales.

Ito processes in higher dimensions

$dX_i = a_i(t, X_1, X_2, \ldots)dt + b_i(t, X_1, X_2, \ldots)dB_i$

- $dF = \frac{\partial F}{\partial t}dt + \sum\frac{\partial F}{\partial X_i}dX_i + \frac{1}{2}\sum\frac{\partial^2 F}{\partial X_i\partial X_j}dX_i dX_j$
- For 2 variables: $dF = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial X}dX + \frac{\partial F}{\partial Y}dY + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}dX^2 + \frac{1}{2}\frac{\partial^2 F}{\partial Y^2}dY^2 + \frac{\partial^2 F}{\partial X\partial Y}dXdY$
- Ito's lemma: $dF = \frac{\partial F}{\partial t}dt + \sum\frac{\partial F}{\partial X_i}dX_i + \frac{1}{2}\sum\rho_{ij}b_i b_j\frac{\partial^2 F}{\partial X_i\partial X_j}dt$
- Heuristics:
  - $(dB_i)^2 \to dt$
  - $dB_i dB_j \to \rho_{ij}dt$
  - $(dX_i)^2 \to b_i^2 dt$
  - $dX_i dX_j \to \rho_{ij}b_i b_j dt$

7. Linear algebra of asset pricing

- Payoff matrix: $A[s \times n] : R^n \to R^s$
- Portfolio vector: $x[n \times 1]$
- Payoff vector: $b[s \times 1] = Ax$
- Given a payoff, solve portfolio: $x = A^{-1}b$
- $Im(T)[s \times 1] = \{T(\bar{v}) : \bar{v}\ \epsilon\ V\}$
- $Ker(T)[n \times 1] = \{\bar{v}\ \epsilon\ V : T(\bar{v}) = 0\}$
- Complete market:
  - Every payoff can be generated by some portfolio
  - $Im(A)$ = Full space of payoffs
  - $rank(A) = s$
  - Unique solution for x, if: $n = s \to \exists! A^{-1}$
  - If $n > s$, then drop $n - s$ redundant securities
  - $rank(A) = rank(A^T) \leq min(n, s)$
  - $rank(AB) \leq min(rank(A), rank(B))$
  - $rank(AA^T) = rank(A)$
- Incomplete market:
  - Some payoffs can't be replicated
  - $Im(A) \subset R^s$
  - Redundant securities:
    - Payoffs are linearly dependent
    - $dim(Ker(A)) > 0$
    - Not unique solutions
    - $z \in Ker(A) \to A(x + cz) = Ax + cAz = Ax$
- Market value:
  - Vector of prices: $S[n \times 1]$
  - $MV = \sum S_i x_i = S^T x = S[x]$
- Arbitrage I - something for nothing: $V = S^T x \leq 0$ and later $Ax \geq 0$
- Arbitrage II - nothing from something: $Ax = 0$ and $S^T x \neq 0$, $rank(A) < s$
- Law of one price: $Ax1 = Ax2 \to S^T x1 = S^T x2$

Arrow-Debreu securities

- $Ax = e_j = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{pmatrix}$ and $x = A^{-1}e_j$
- State prices - price of a 1 payoff in each state: $\psi[s \times 1] \in R^s > 0$
- If $A\tilde{x} = 0 \rightarrow S^T x = 0$
- Price for a security w/ positive payoff b: $\psi^T b > 0$
- More generally: $S = A^T \psi$
- No arbitrage: $\exists \psi > 0$
- Arbitrage, if every solution has $\psi \leq 0$
- Pseudo-inverse matrix:
  - $M = (AA^T)^{-1}A$, only works if $n > s$ and $rank(A) = s$
  - $\psi = MS$. Check that: $\psi = MS \geq 0$ and $S = A^T \psi$
- If market is incomplete:
  - $\psi = \psi_0 + xw$
  - $Ker(A^T) = Ker(AA^T) = w$
- To avoid arbitrage, find $\psi > 0,\ s.t.\ A^T\psi = S$
  - No solution = arbitrage
  - 1 solution = complete market
  - Multiple solutions = incomplete market

Asset pricing duality

- $Ax = b$, $A^T\psi = S$
- $S^T x = \psi^T b \rightarrow S = A^T \psi$
- $Ker(A^T) \perp Im(A)$ and $Im(A^T) \perp Ker(A)$
- If $Ax = 0 \rightarrow S^T x = 0$ and $A^T \psi = S = 0 \rightarrow \psi^T b = 0$
- State pricing: $S_b = \{\psi^T b : A^T \psi = S; \psi \in R^s, \psi > 0\}$
- Replication pricing: $S_b = S^T x = \{S^T x : x \in R^n; Ax = b\}$
  - $S_{min} < Sb < S_{max}$
  - $S_{min} = max\{S^T x : Ax \leq b\}$
  - $S_{max} = min\{S^T x : Ax \geq b\}$

Absence of arbitrage - FTAP

- Type 1: If $Ax = b \geq 0$ then $S^T x > 0 \equiv \psi^T b > 0$
- Type 2: If $Ax = 0$ then $S^T x = 0$
- Farkas' lemma = arbitrage or not:
  - $A^T \psi = S \rightarrow \psi \geq 0$ has a solution, or
  - $Ax \geq 0 \rightarrow S^T x < 0$ has a solution

8. Optimization

Critical points

- Taylor's theorem:
  - $f(x) - f(x_0) \approx 0.5 f''(x_0)(x - x_0)^2$
  - Subject to: $f'(x_0) = 0$ and $f''(x_0) \neq 0$
  - $f(x) - f(x_0) \approx 0.5(x - x_0)^T Q(x - x_0)$
- Local/global minima and maxima:
  - Hessian: $(Q_f)_{i,j} = \frac{\partial^2 f}{\partial x_i \partial x_j}$
  - If all $\lambda > 0$: convex and minimum
  - If all $\lambda < 0$: concave and maximum
  - If mixed: saddle points
  - $\lambda = 0$: flat directions
  - $A\bar{v} = \lambda \bar{v}$
  - For a 2x2 matrix:
    - $\lambda_1 \lambda_2 = detQ = ac - bd$
    - $\lambda_1 + \lambda_2 = trQ$

Lagrange multipliers

The maximum along a path is tangent to a contour line.

$$L(x, y, \lambda) = h(x, y) - \lambda(g(x, y) - c)$$

- The extrema occurs when:
  - $\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial lambda}$
  - $\Delta h = \lambda \Delta g$

Minimum variance portfolio

- $\mu_P = \mu^T w$ subject to $\sum w_i = 1$
- $\sigma_P^2 = w^T C w = \sum w_i^2 \sigma_i^2 + 2 \sum_{i<j} w_i w_j \sigma_i \sigma_j \rho_{ij}$
- Covariance matrix - C:
  - Symmetric
  - Positive definite: $w^T C w > 0,\ \forall w,\ \lambda > 0$
- Orthogonal portfolios = uncorrelated
- Sharpe ratio = $\frac{r - r_f}{\sigma}$
- $L(w, l) = 0.5 w^T C w + l(1 - \iota^T w)$
- $w_{min} = \frac{C^{-1}\iota}{\iota^T C^{-1} \iota}$
- $\sigma_{min}^2 = l = \frac{1}{\iota^T C^{-1} \iota}$

Risk and return portfolio

Minimize risk given a level of return.

- $L(w, l) = 0.5 w^T C w + l(1 - \iota^T w) + m(\mu_P - \mu^T w)$
- $M = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$
  - $a = \iota^T C^{-1} \iota$
  - $b = \mu^T C^{-1} \iota$
  - $c = \mu^T C^{-1} \mu$
- $(l\ m)^T = M^{-1}(1\ \mu_P)^T$
- $\sigma_P^2 = \frac{a\mu_P^2 - 2b\mu_P + c}{ac - b^2}$