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Algebraic Solution of an LP

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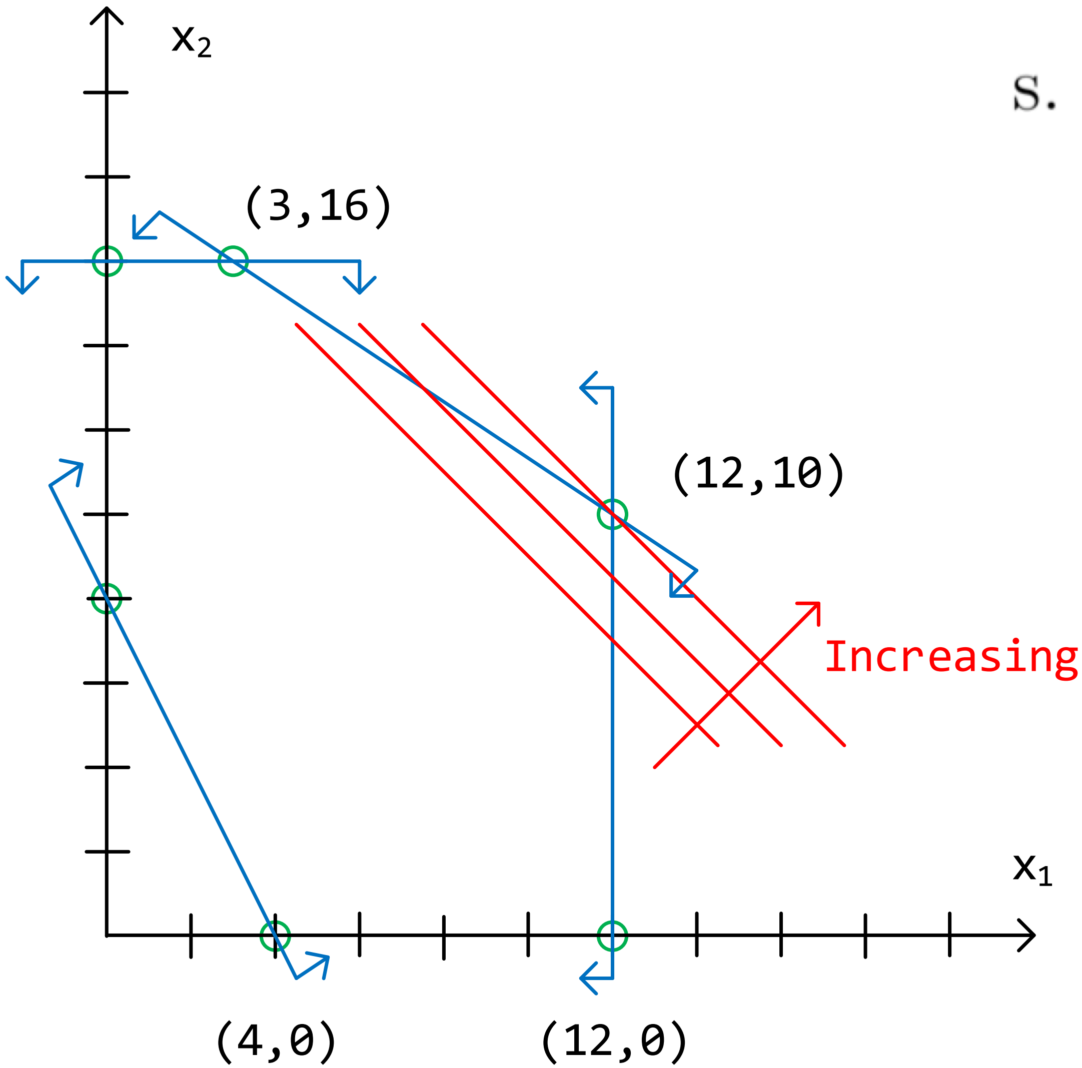


$$\max_{x_1,x_2} \quad x_1 + x_2$$

$$\text{s. t.} \quad \frac{2}{3}x_1 + x_2 \leq 18$$

$$2x_1 + x_2 \geq 8$$

$$x_1 \leq 12$$

$$x_2 \leq 16$$

$$x_1, x_2 \geq 0.$$

LP_expl_algebraic.py

```
from cvxopt.modeling import variable,op
x_1 = variable()
x_2 = variable()
c1 = ( (2/3)*x_1+x_2 <= 18 )
c2 = ( 2*x_1+x_2 >= 8 )
c3 = ( x_1 <= 12 )
c4 = ( x_2 <= 16 )
c5 = ( x_1>=0 )
c6 = ( x_2>=0 )
lp1 = op(-x_1-x_2, [c1,c2,c3,c4,c5,c6])
lp1.solve()
lp1.status
print(x_1.value)
print(x_2.value)
print(lp1.objective.value())
```

```
'optimal'
[ 1.20e+01]
[ 1.00e+01]
[-2.20e+01]
```

So, optimal objective function value is 22.

Recall: Geometrical solution

Focusing on bounded feasible problems, we conclude indicating that the optimal solution needs to be an extreme point of the feasible region, which is a polytope in the n-dimensional space.

This is indeed one statement of the fundamental theorem of linear programing.

Fundamental Theorem of Linear programing

Algebraic view

To gain algebraic insight, we transform Problem (1) to standard form:

$$\min_{x_1,x_2,x_3,x_4,x_5,x_6} \quad z = -x_1 - x_2 \tag{3a}$$

$$\text{s. t.} \quad \frac{2}{3}x_1 + x_2 + x_3 = 18 \tag{3b}$$

$$2x_1 + x_2 - x_4 = 8 \tag{3c}$$

$$x_1 + x_5 = 12 \tag{3d}$$

$$x_2 + x_6 = 16 \tag{3e}$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0, \tag{3f}$$

Matrix notation

$$\min_{x_1,x_2} \quad z = \overset{\mathbf{c^T}}{\begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}} \overset{\mathbf{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}} \tag{4a}$$

$$\text{s. t.} \quad \overset{\mathbf{A}}{\begin{bmatrix} \frac{2}{3} & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}} \overset{\mathbf{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}} = \overset{\mathbf{b}}{\begin{bmatrix} 18 \\ 8 \\ 12 \\ 16 \end{bmatrix}}. \tag{4b}$$

Recall: **X≥0**

Algebraic view

Considering the feasible region of this problem from an algebraic viewpoint, i.e., equation (4b),

a way to obtain feasible solutions (i.e., points in the feasible region) is selecting a number of columns equal to the number of rows,

solve the system for the corresponding variables, and assign 0 to the remaining variables.

Solutions obtained this way are called *basic solutions*.

Algebraic view

For instance, let us select columns 3, 4, 5 and 6, and as a result, variables x_3, x_4, x_5 and x_6. Then, the system to be solved is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 18 \\ 8 \\ 12 \\ 16 \end{bmatrix},$$

That is x_1 and x_2 are set to zero.

We pick up the green

$$\begin{array}{cc} \qquad\mathbf{N} & \qquad\qquad\mathbf{B} \end{array}$$

$$\left[\begin{array}{cc|cccc} \frac{2}{3} & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \\ \hline x_3 \\ x_4 \\ x_5 \\ x_6 \end{array}\right] = \begin{bmatrix} 18 \\ 8 \\ 12 \\ 16 \end{bmatrix}^{\mathbf{b}}.$$

$\mathbf{x_N}$: *non-basic variables*

$\mathbf{x_B}$: *basic variables*

$$(x_1, x_2) = (0, 0)$$

$$
\begin{array}{ccccc}
\mathbf{B} & \mathbf{x_B} & & \mathbf{b} \\
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} &
\begin{bmatrix} x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} & = &
\begin{bmatrix} 18 \\ 8 \\ 12 \\ 16 \end{bmatrix}
\end{array}
$$

Algebraic view

and its solution is

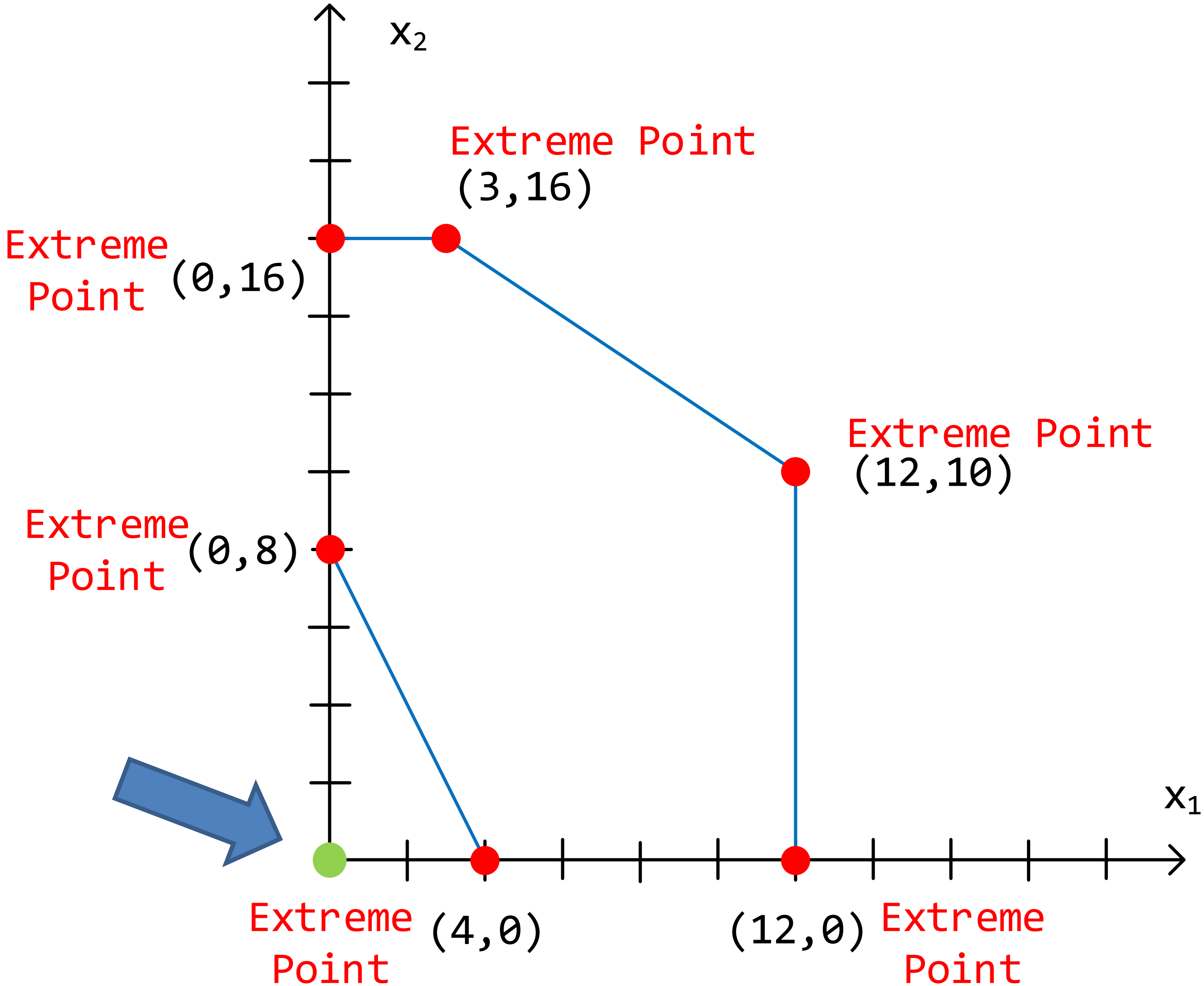
$$(x_3, x_4, x_5, x_6) = (18, -8, 12, 16)$$

and

$$(x_1, x_2) = (0, 0).$$

If all variables of a basic solution are non-negative, the basic solution is called *basic feasible solution.* This is a basic infeasible solution.

BFS



x_2
Extreme Point
(3,16)
Extreme
Point
(0,16)
Extreme Point
(12,10)
Extreme
Point
(0,8)
x_1
Extreme
Point
(4,0)
(12,0)
Extreme
Point

Let's do another selection

$x_B=(x_1,x_2,x_4,x_5)^T$

$$\begin{bmatrix} \frac{2}{3} & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 18 \\ 8 \\ 12 \\ 16 \end{bmatrix}.$$

$x_N=(x_3,x_6)^T$

$+$

$$(x_3, x_6) = (0, 0)$$

$$\underbrace{\begin{bmatrix} \frac{2}{3} & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_4 \\ x_5 \end{bmatrix}}_{\mathbf{x_B}} = \underbrace{\begin{bmatrix} 18 \\ 8 \\ 12 \\ 16 \end{bmatrix}}_{\mathbf{b}}$$

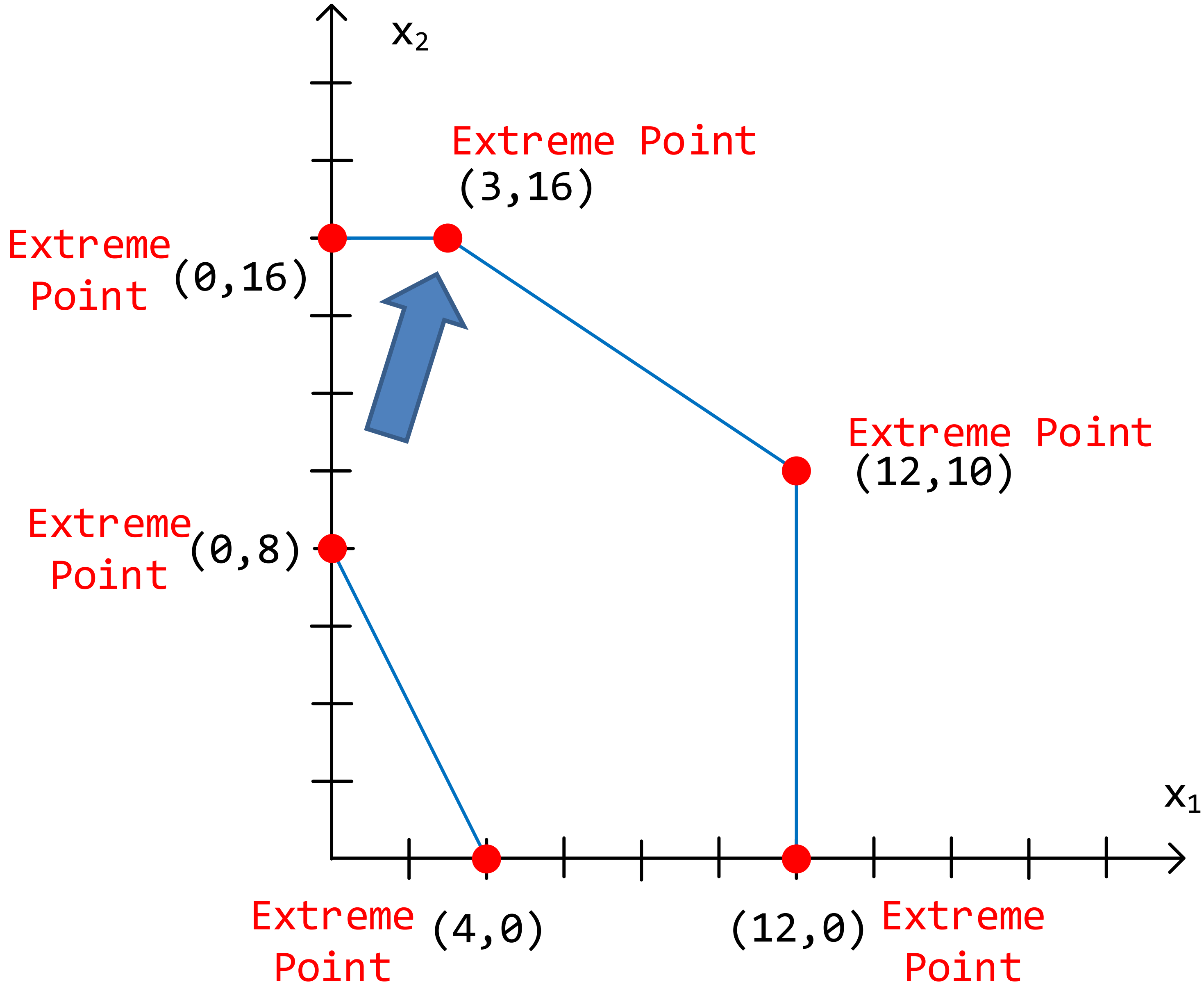
and its solution is

$$(x_1, x_2, x_4, x_5) = (3, 16, 14, 9)$$

and

$$(x_3, x_6) = (0, 0).$$

If all variables of a basic solution are non-negative, the basic solution is called *basic feasible solution*. This is a basic feasible solution.


x2
Extreme Point
(3,16)
Extreme
Point
(0,16)
Extreme Point
(12,10)
Extreme
Point
(0,8)
x1
Extreme
Point
(4,0)
(12,0)
Extreme
Point

Since the number of combinations of 6 columns taken 4 at a time is 15 (i.e., $\begin{pmatrix} 6 \\ 2 \end{pmatrix} = \frac{6!}{4!(6-4)!} = 15$), the table below provides all alternatives for this example:

#	Columns	Solution	Objective	x_1	x_2
1	1-2-3-4	$12, 16, -6, 32^{(2)}$	-	-	-
2	1-2-3-5	$-4, 16, 14/3, 16^{(2)}$	-	-	-
3	1-2-3-6	$12, -16, 26, 32^{(2)}$	-	-	-
4	1-2-4-5	$3, 16, 14, 9$	-19	3	16
5	1-2-4-6	$12, 10, 26, 6$	-22	12	10
6	1-2-5-6	$-15/2, 23, 39/2, -7^{(2)}$	-	-	-
7	1-3-4-5	singular$^{(1)}$	-	-	-
8	1-3-4-6	$12, 10, 16, 16$	-12	12	0
9	1-3-5-6	$4, 46/3, 8, 16$	-4	4	0
10	1-4-5-6	$27, 46, -15, 16^{(2)}$	-	-	-
11	2-3-4-5	16,2,8,12	-16	0	16
12	2-3-4-6	singular$^{(1)}$	-	-	-
13	2-3-5-6	$8, 10, 12, 8$	-8	0	8
14	2-4-5-6	$18, 10, 12, -2^{(2)}$	-	-	-
15	3-4-5-6	$18, -8, 12, 16^{(2)}$	-	-	-

$^{(1)}$ Submatrix B is singular

$^{(2)}$ Infeasible solution: non-negativity violated

We observe that out of the 15 combinations,

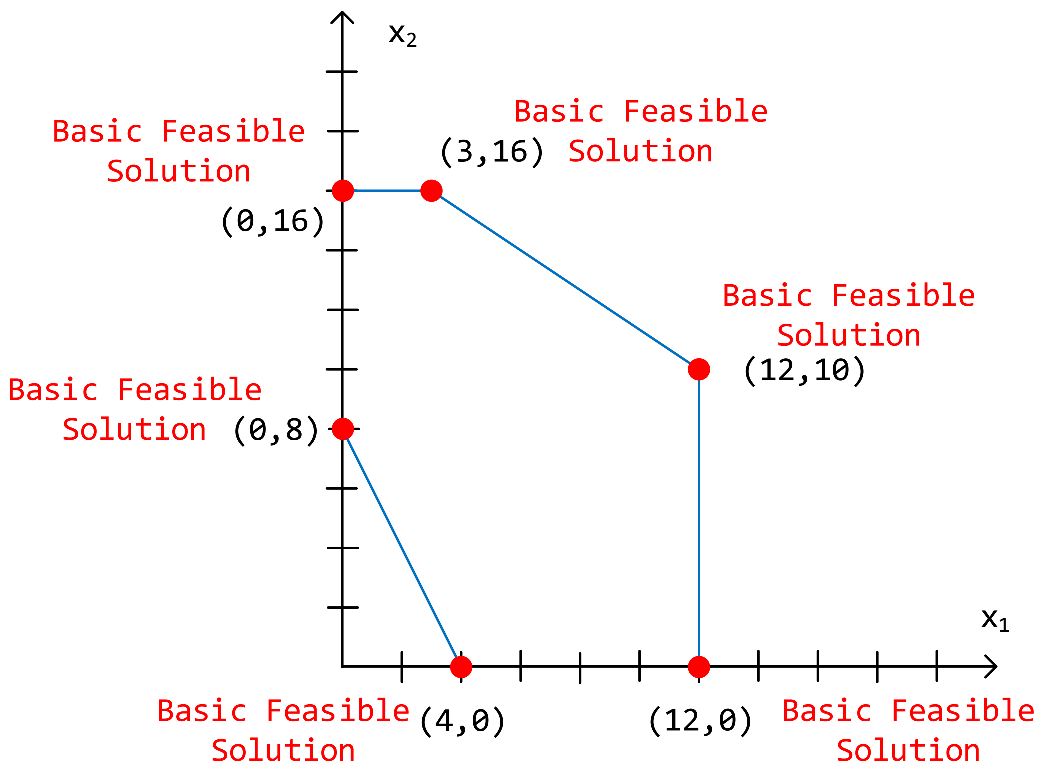
2 combinations involve singular matrices (combinations 1-3-4-5 and 2-3-4-6),

7 combinations result in infeasible solutions (combinations 1-2-3-4, 1-2-3-5, 1-2-3-6, 1-2-5-6, 1-4-5-6, 2-4-5-6 and 3-4-5-6) and

6 combinations result in feasible solutions (combinations 1-2-4-5, 1-2-4-6, 1-3-4-6, 1-3-5-6, 2-3-4-5, 2-3-5-6).

A careful inspection of these 6 basic feasible solutions reveal that they correspond to the corners of the polygon in the figure below (electricity production problem.

Specifically, this figure identifies these basic feasible solutions that correspond to extreme points.



“Each EP of the feasible region is a BFS and vice versa”

This is indeed a relevant fact:

each extreme point of the polytope describing from a geometrical viewpoint (canonical form) the feasible region of an LP problem is

a basic feasible solution of the system of equations describing from an algebraic viewpoint (standard form) such feasible region.

Optimal solution of LP is a BFS

Based on the above observations,

the main LP theorem says that if an LP problem has a solution,

such solution is a basic feasible solution of the linear systems of equations describing algebraically its feasibility region, and

an extreme point of the polytope that describes such feasible region from a geometrical viewpoint.

An idea for minimization …

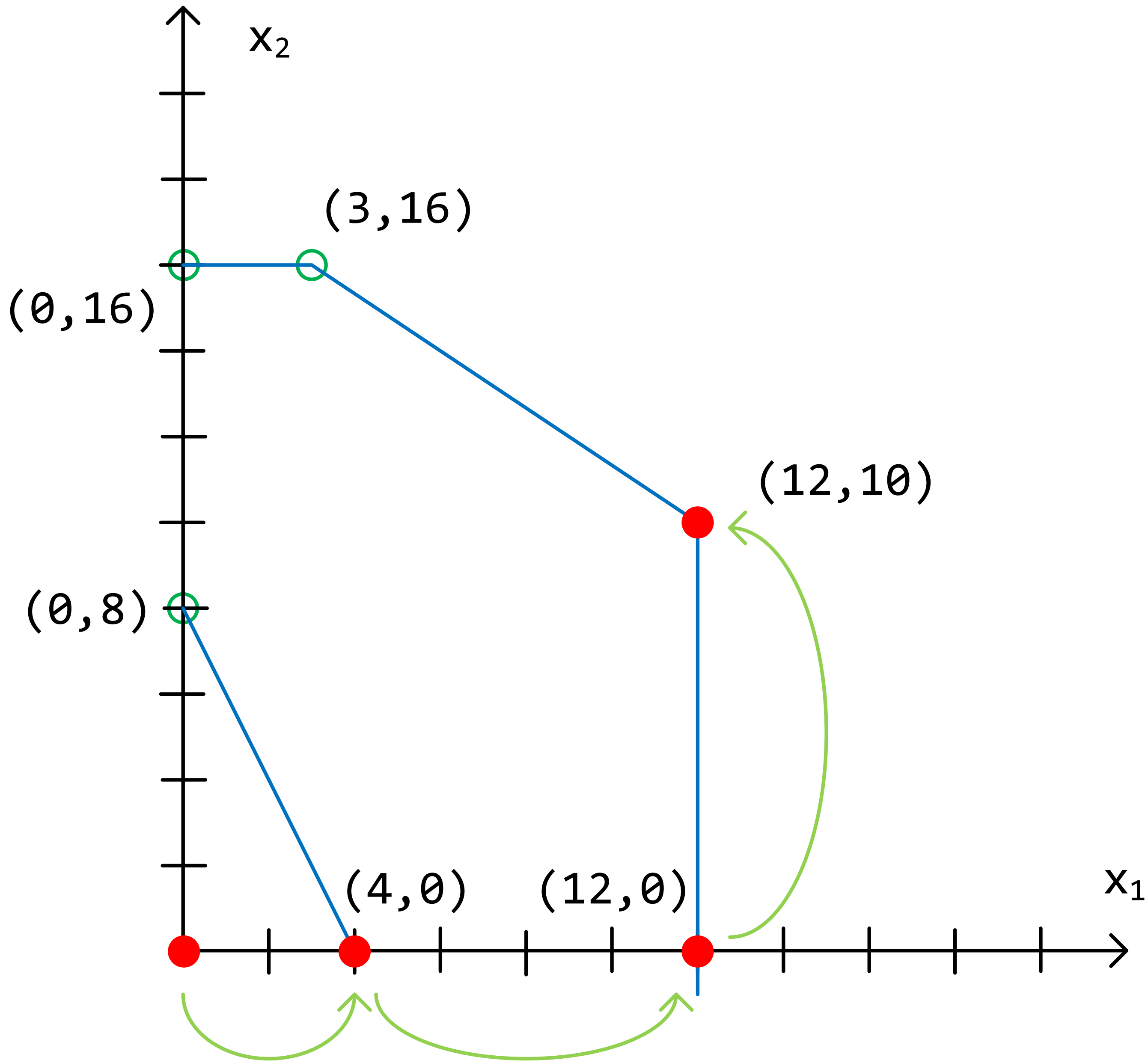
Therefore, a possible solution strategy is

to locate a basic feasible solution, and then,

moving from basic feasible solution to basic feasible solution while decreasing the objective function

until no further decrease is possible.

In fact, the Simplex algorithm follows this strategy.


x2
(3,16)
(0,16)
(12,10)
(0,8)
(4,0)
(12,0)
x1

Algebra of finding a BFS

Another example

Maximize $z = 2x_1 + 3x_2$

subject to

$$\begin{aligned} x_1 - 2x_2 + x_3 \qquad\qquad &= 4 \\ 2x_1 + x_2 \qquad + x_4 \qquad &= 18 \\ x_2 \qquad\qquad + x_5 &= 10 \\ x_1, x_2, x_3, x_4, x_5 \geq 0 & \end{aligned}$$

Adjacent Extreme Points (EP)

- Note that the difference between any two adjacent EPs is in one basic and non-basic variable.
- This idea is utilized in the Simplex algorithm to jump from one BFS (EP) to another BFS (EP).