STAT 5430
Lecture 14, F, Feb 21

- No new homework!
- Solutions to Homeworks 1-3 posted.
- Exam 1 is scheduled for W, Feb 26 (two weeks)
  6:15-8:15 PM (Sned seminar room 3105)
  - No regular class on W, Feb 26
  - See Canvas for study guide, practice exams
  - can bring 1 page formula sheet (front/back) with anything on it
  - see Canvas for a "canned" sheet
  - I'll provide table with STAT 5420 distributions (see Canvas)

STAT 5430: Summary to date
Where we have been
& where we are headed

- Completed
  - Introduction to Statistical Inference
  - Point Estimation
    - MME/MLE
  - Criteria for Evaluating Point Estimators
    - bias, variance, UMVUE, MSE
  - Elements of Decision Theory
    - Minimax, finding Bayes estimators

- Next: Sufficiency and Point Estimation
  - Sufficiency/Data Reduction
  - Factorization Theorem
  - Rao-Blackwell Theorem
  - Completeness/Lehman-Scheffe Theorem/UMVUE
  - Exponential Families

Sufficiency and Point Estimation (Chapter 6)

Sufficiency as Data Reduction

Definition: Let $X_1, \ldots, X_n$ be r.v.'s with joint pdf/pmf $f(\underset{\sim}{x}|\underset{\sim}{\theta})$, $\underset{\sim}{\theta} \in \Theta \subset \mathbb{R}^p$ and let $\underset{\sim}{S} \equiv (S_1, \ldots, S_k)$ be a vector of estimators. Then, $\underset{\sim}{S}$ is called (jointly) **sufficient** for $\underset{\sim}{\theta}$ if the conditional distribution of $(X_1, \ldots, X_n)$ given $\underset{\sim}{S}$ does *not* depend on $\underset{\sim}{\theta}$.

Example: Let $X_1, \ldots, X_n$ be iid Geometric(θ), $0 < \theta < 1$. Show that $S \equiv X_1 + \cdots + X_n$ is sufficient for θ.

Solution: conditional pmf of $(X_1, \ldots, X_n)$ given $S = s$ is

$$P_\theta(X_1 = x_1, \ldots, X_n = x_n \mid S = s) = \frac{P_\theta(X_1 = x_1, \ldots, X_n = x_n, S = s)}{P_\theta(S = s)}$$

$$= \begin{cases} \dfrac{P_\theta(X_1 = x_1, \ldots, X_n = x_n, S = s)}{P_\theta(S = s)} & \text{if } x_1 + x_2 + \cdots x_n = s \\ 0 & \text{o.w.} \end{cases}$$

↑ Neg-Binomial(n, θ)
\# of trials until "n" successes

$$= \begin{cases} \dfrac{P_\theta(X_1 = x_1, \ldots, X_n = x_n)}{\binom{s-1}{n-1}\theta^n (1-\theta)^{s-n}} & \text{if } x_1 + x_2 + \cdots x_n = s \\ 0 & \text{o.w.} \end{cases}$$

$$= \begin{cases} \dfrac{P_\theta(X_1 = x_1) \cdots P_\theta(X_n = x_n)}{\binom{s-1}{n-1}\theta^n (1-\theta)^{s-n}} & \text{if } x_1 + \cdots x_n = s \\ 0 & \text{o.w} \end{cases}$$

$$= \begin{cases} \dfrac{\prod_{i=1}^{n}\left[\theta(1-\theta)^{x_i-1}\right]}{\binom{s-1}{n-1}\theta^n(1-\theta)^{s-n}} & \text{if } x_1+\cdots+x_n=s \\ 0 & \text{o.w} \end{cases}$$

$$= \begin{cases} \dfrac{\theta^n(1-\theta)^{s-n}}{\binom{s-1}{n-1}\theta^n(1-\theta)^{s-n}} & \text{if } x_1+\cdots+x_n=s \\ 0 & \text{o.w} \end{cases}$$

$$= \begin{cases} 1/\binom{s-1}{n-1} & \text{if } x_1+\cdots+x_n=s \\ 0 & \text{o.w} \end{cases}$$

free of θ! $\Rightarrow$ S is sufficient for θ.

Sufficiency and Point Estimation

Factorization Theorem

Remarks on Sufficiency:

Recall in the definition: $p = \#$ of parameters, $k = \#$ of statistics

$k = p$: eg. last example Geometric(θ), $p = 1 = k$

$k > p$: eg. $X_1, \ldots, X_n$ iid UNIF($\theta, \theta+1$) $\Rightarrow \underset{\sim}{S} = (\min X_i, \max X_i)$ sufficient for θ; $p=1$

$k < p$: eg. $n=1$, $X_1 \sim N(\mu, \sigma^2)$, $p=2$, but X_1 is sufficient, $k=1$

Factorization Theorem: Let $X_1, \ldots, X_n$ be r.v.'s with joint pdf/pmf $f(\underset{\sim}{x}|\underset{\sim}{\theta})$, $\underset{\sim}{\theta} \in \Theta \subset \mathbb{R}^p$ and let $\underset{\sim}{S} = (S_1, \ldots, S_k)$ be a vector of estimators. Then, $\underset{\sim}{S}$ is sufficient for $\underset{\sim}{\theta}$ *if and only if* there exist functions $g(\underset{\sim}{S}, \underset{\sim}{\theta})$ and $h(\underset{\sim}{x})$ such that $h(\underset{\sim}{x})$ does NOT depend on $\underset{\sim}{\theta}$ and

$$\underbrace{f(\underset{\sim}{x}|\underset{\sim}{\theta})}_{X_1 \ldots X_n \text{ data pdf/pmf}} = g(\underset{\sim}{S}, \underset{\sim}{\theta})h(\underset{\sim}{x}) \quad \text{for all } \underset{\sim}{x} \text{ and all } \underset{\sim}{\theta}$$

$\underset{\sim}{S}$ & $\underset{\sim}{\theta}$ are "linked" inside $f(\underset{\sim}{x}|\underset{\sim}{\theta})$

Example: Let $X_1, \ldots, X_n$ be iid Negative-Binomial(r, θ), $0 < \theta < 1$ (known integer $r \geq 1$). Show that $S = X_1 + \cdots + X_n$ is sufficient for θ. (last time Geo(θ) ~ Neg-Binom($r=1, \theta$))

Solution: joint pmf of $X_1, \ldots, X_n$ is

$$f(\underset{\sim}{x}|\theta) = \prod_{i=1}^{n} f(x_i|\theta) = \prod_{i=1}^{n}\left[\binom{x_i-1}{r-1}\theta^r(1-\theta)^{x_i-r}\, \underset{\text{Indicator}}{I}\{x_i \in A_r\}\right]$$

where $A_r = \{r, r+1, r+2, \ldots\}$

$$= \underbrace{\theta^{nr}(1-\theta)^{\sum x_i - nr}}_{g(\sum_{i=1}^n x_i, \theta)} \underbrace{\left\{\prod_{i=1}^{n}\left[\binom{x_i-1}{r-1} I\{x_i \in A_r\}\right]\right\}}_{h(\underset{\sim}{x})} \quad \forall \underset{\sim}{x},\ \forall\, 0<\theta<1$$

where $g(s, \theta) = \theta^{nr}(1-\theta)^{s-nr}$

Hence, by factorization theorem, $S = \sum_{i=1}^{n} X_i$ is sufficient

Sufficiency and Point Estimation

Factorization Theorem, cont'd

$\underset{\sim}{1} = (1,\dots,1)'$

Example: Suppose $(X_1,\dots,X_n) \sim MVN(\mu \cdot \underset{\sim}{1}, \sigma^2 \cdot A)$ where $\mu \in \mathbb{R}$, $\sigma^2 > 0$ and A is a known $n \times n$ positive definite matrix. Find a sufficient statistic for (μ, σ^2).

Solution: joint pdf of $(X_1,\dots,X_n)$ is

$$f(\underset{\sim}{x} \mid \mu, \sigma^2) = \frac{1}{(\sigma^2 2\pi)^{n/2}} \frac{1}{[\det(A)]^{1/2}} \exp\left[-\frac{1}{2\sigma^2}(\underset{\sim}{x}-\mu\underset{\sim}{1})'A^{-1}(\underset{\sim}{x}-\mu\underset{\sim}{1})\right]$$

$$= \underbrace{\frac{1}{\sigma^n} \exp\left[\frac{-1}{2\sigma^2}\left[\underset{\sim}{x}'A^{-1}\underset{\sim}{x} + 2\mu\underset{\sim}{x}'A^{-1}\underset{\sim}{1} + \mu^2\underset{\sim}{1}'A^{-1}\underset{\sim}{1}\right]\right]}_{g(\underset{\sim}{x}'A^{-1}\underset{\sim}{x},\ \underset{\sim}{x}'A^{-1}\underset{\sim}{1},\ \mu,\ \sigma^2)} \underbrace{\frac{1}{(2\pi)^{n/2}} \frac{1}{[\det(A)]^{1/2}} I(\underset{\sim}{x} \in \mathbb{R}^n)}_{h(\underset{\sim}{x})}$$

Hence, by Factorization Theorem,

$\underset{\sim}{S} = (\underset{\sim}{x}'A^{-1}\underset{\sim}{x},\ \underset{\sim}{x}'A^{-1}\underset{\sim}{1})$ are sufficient for (μ, σ^2)

Remarks:

1. The choice of $g(\underset{\sim}{S}, \underset{\sim}{\theta})$ and $h(\underset{\sim}{x})$ is not unique.

2. Any 1-to-1 function of a sufficient statistic is also sufficient.

   Example: In last example, suppose $A = I_{n \times n}$.