Multivariate transformations

Multivariate continuous case

The final technique for determining distributions of transformed random variables:

if we transform continuous random variables, using a one-to-one continuously differentiable transformation, we can *directly* find the pdf of the new random variables

Set-up

- Suppose continuous $(X_1, \ldots, X_n)$ has joint pdf $f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)$ with support $\mathcal{A} = \{(x_1, \ldots, x_n) : f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) > 0\}$

- Transformation:

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix}_{n\times 1}$$

$$\boldsymbol{Y} = \boldsymbol{u}(\boldsymbol{X}) \quad \text{or} \quad \begin{aligned} Y_1 &= u_1(X_1, \ldots, X_n) \\ Y_2 &= u_2(X_1, \ldots, X_n) \\ &\vdots \\ Y_n &= u_n(X_1, \ldots, X_n) \end{aligned}$$

$$u(\cdot) = \begin{pmatrix} u_1(\cdot) \\ u_2(\cdot) \\ \vdots \\ u_n(\cdot) \end{pmatrix}_{n\times 1}$$

with $\mathcal{B}$=support of $f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n)$

$$\mathcal{B} = \{(y_1, \ldots, y_n) : f_{Y_1,Y_2,\ldots,Y_n}(y_1, \ldots, y_n) > 0\}$$

- Assume transformation is one-to-one with inverse functions

$$x_i = u_i^{-1}(y_1, \ldots, y_n), \quad i = 1, \ldots, n$$

- Define the Jacobian

$$J = \det \underbrace{\begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}} = \det \underbrace{\begin{pmatrix} \frac{\partial u_1^{-1}(y_1,\ldots,y_n)}{\partial y_1} & \frac{\partial u_1^{-1}(y_1,\ldots,y_n)}{\partial y_2} & \cdots & \frac{\partial u_1^{-1}(y_1,\ldots,y_n)}{\partial y_n} \\ \frac{\partial u_2^{-1}(y_1,\ldots,y_n)}{\partial y_1} & \frac{\partial u_2^{-1}(y_1,\ldots,y_n)}{\partial y_2} & \cdots & \frac{\partial u_2^{-1}(y_1,\ldots,y_n)}{\partial y_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial u_n^{-1}(y_1,\ldots,y_n)}{\partial y_1} & \frac{\partial u_n^{-1}(y_1,\ldots,y_n)}{\partial y_2} & \cdots & \frac{\partial u_n^{-1}(y_1,\ldots,y_n)}{\partial y_n} \end{pmatrix}}$$

number

- If J is continuous and $J \neq 0$ over $\mathcal{B}$ (except possibly on a set with probability zero),

$$f_{Y_1,\ldots,Y_n}(y_1,\ldots,y_n) = f_{X_1,\ldots,X_n}(\boldsymbol{u}^{-1}(y_1,\ldots,y_n))|J| \qquad (y_1,\ldots,y_n) \in \mathcal{B}$$

$$= f_{X_1,X_2,\ldots,X_n}\left(u_1^{-1}(y_1,\ldots,y_n),\ldots,u_n^{-1}(y_1,y_2,\ldots,y_n)\right)$$

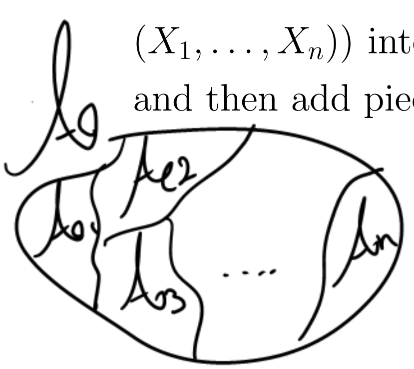
- Often only interested in one transformation $Y_1 = u_1(X_1,\ldots,X_n)$

  Then choose convenient definitions to fill out the transformation

  e.g. $Y_2 = X_2, \ldots, Y_n = X_n$

need n Variables $Y_1,\ldots,Y_n$ from n Variables $X_1,\ldots,X_n$

- If transformation is not one-to-one, then we partition $\mathcal{A}$ (the support of $(X_1,\ldots,X_n)$) into sets $\mathcal{A}_i$ where a transformation $\boldsymbol{Y} = \boldsymbol{u}_j(\boldsymbol{X})$ is one-to-one and then add pieces



$$f_{\boldsymbol{Y}}(\boldsymbol{y}) = \sum_{i=1}^{k} f_{\boldsymbol{X}}(\boldsymbol{u}_i^{-1}(\boldsymbol{y}))|J_i|$$

$(x_1, x_2) \in (0,\infty) \times (0,\infty)$

Multivariate transformations

Multivariate continuous case: example 1

- $X_1 \sim$Gamma(α_1, β) and $X_2 \sim$Gamma(α_2, β) are independent

$$f_{X_i}(x) = \frac{1}{\Gamma(\alpha_i)\beta^{\alpha_i}} x^{\alpha_i - 1} e^{-x/\beta}, \quad x > 0$$

$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1)\, f_{X_2}(x_2)$$

$Y_1 = U_1(X_1, X_2)$
$Y_2 = U_2(X_1, X_2)$

Step1:

- Transformation: $Y_1 = X_1 + X_2$ and $Y_2 = X_1/(X_1 + X_2)$

$$\begin{cases} Y_1 = X_1 + X_2 \\ Y_2 = \dfrac{X_1}{X_1 + X_2} \end{cases} \Rightarrow X_1 = Y_1 Y_2 \Rightarrow X_2 = Y_1 - Y_1 Y_2 \Rightarrow \begin{cases} X_1 = Y_1 Y_2 = U_1^{-1}(Y_1, Y_2) \\ X_2 = Y_1 - Y_1 Y_2 = U_2^{-1}(Y_1, Y_2) \end{cases}$$

- Inverse transformation:

Step2:

- $\mathcal{A} = (0,\infty) \times (0,\infty)$ while $\mathcal{B} =$

$$\mathcal{B} = \{(y_1, y_2) : f_{Y_1,Y_2}(y_1,y_2) > 0\} = \{(y_1,y_2) \in (0,\infty) \times (0,1) : f_{Y_1,Y_2}(y_1,y_2) > 0\}$$

$$\mathcal{B} = (0,\infty) \times (0,1)$$

Step3:

- One-to-one transformation with

$$\begin{cases} X_1 = Y_1 Y_2 \\ X_2 = Y_1 - Y_1 Y_2 \end{cases}$$

$$J = \det \begin{pmatrix} \frac{\partial X_1}{\partial Y_1} & \frac{\partial X_1}{\partial Y_2} \\ \frac{\partial X_2}{\partial Y_1} & \frac{\partial X_2}{\partial Y_2} \end{pmatrix} = \det \begin{pmatrix} Y_2 & Y_1 \\ 1 - Y_2 & -Y_1 \end{pmatrix}$$

$$= -Y_2 Y_1 - Y_1(1 - Y_2)$$

$$= -\cancel{Y_2 Y_1} - Y_1 + \cancel{Y_1 Y_2} = -Y_1$$

- Joint pdf of Y_1, Y_2

Step4:

$$\begin{cases} X_1 = Y_1 Y_2 \\ X_2 = Y_1 - Y_1 Y_2 \end{cases} \Rightarrow f_{X_1,X_2}(x_1, x_2) \equiv f_{X_1,X_2}(y_1 y_2, y_1 - y_1 y_2)$$

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(y_1 y_2, y_1(1-y_2))|J|$$

X₁ and X₂ are ind.

$$= f_{X_1}(\underbrace{y_1 y_2}_{x_1}) \times f_{X_2}(\underbrace{y_1(1-y_2)}_{x_2}) \times |J|$$

$X_1 \sim \text{Gamma}(\alpha_1, \beta)$

$$= \frac{(y_1 y_2)^{\alpha_1 - 1} e^{-(y_1 y_2)/\beta}}{\Gamma(\alpha_1)\beta^{\alpha_1}} \times \frac{[y_1(1-y_2)]^{\alpha_2 - 1} e^{-[y_1(1-y_2)]/\beta}}{\Gamma(\alpha_2)\beta^{\alpha_2}} \times y_1$$

Note:

$$= \frac{y_1^{\alpha_1+\alpha_2-1} e^{-y_1/\beta}}{\Gamma(\alpha_1+\alpha_2)\beta^{\alpha_1+\alpha_2}} \times y_2^{\alpha_1-1}(1-y_2)^{\alpha_2-1} \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \frac{\Gamma(\alpha_1+\alpha_2)}{1}$$

① $Y_1 \sim \text{Gamma}(\alpha_1+\alpha_2, \beta)$

② $Y_2 \sim \text{Beta}(\alpha_1, \alpha_2)$

$\mathcal{S} = \underbrace{(0,\infty)}_{y_1} \times (0,1)$

③ $f_{Y_1,Y_2}(y_1, y_2) = f_{Y_1}(y_1) f_{Y_2}(y_2)$ Y_1 and Y_2 are independent

Multivariate transformations

Multivariate continuous case: example 2

- $X_1 \sim N(0,1)$ and $X_2 \sim N(0,1)$ are independent

- Transformation: $Y_1 = X_1 + X_2$ and $Y_2 = X_2$

Step1

- Inverse transformation:

$$Y_1 = X_1 + X_2 \implies X_1 = Y_1 - Y_2$$
$$Y_2 = X_2 \implies X_2 = Y_2$$

Step2

- $\mathcal{A} = (-\infty, \infty) \times (-\infty, \infty)$ while $\mathcal{B} = (-\infty,\infty) \times (-\infty,\infty)$

Step3:

- One-to-one transformation with

$$J = \det \begin{pmatrix} \frac{\partial X_1}{\partial Y_1} & \frac{\partial X_1}{\partial Y_2} \\ \frac{\partial X_2}{\partial Y_1} & \frac{\partial X_2}{\partial Y_2} \end{pmatrix} = \det \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = 1 - 0 = 1$$

Multivariate transformations

Multivariate continuous case: example 2 (cont'd)

- Joint pdf of Y_1, Y_2

$$
\begin{aligned}
f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(\underbrace{y_1-y_2}_{x_1}, \underbrace{y_2}_{x_2})|\overset{=1}{J}| &= f_{X_1}(y_1-y_2)\times f_{X_2}(y_2)\times 1 \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(y_1-y_2)^2}\times\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y_2^2} \\
&= \frac{1}{2\pi}e^{-\left(\frac{y_1^2}{2}-y_1y_2+y_2^2\right)} \\
&= \boxed{\frac{1}{2\pi}e^{-\frac{y_1^2}{2}}e^{-y_2^2+2(y_1/2)y_2}}
\end{aligned}
$$

- Marginal distribution of Y_1

$$
f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1,y_2)dy_2 = \frac{1}{\sqrt{2\pi}}e^{-y_1^2/4}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\left(y_2-\frac{y_1}{2}\right)^2}dy_2
$$