14. Linear Mixed-Effects Models for Data from Split-Plot Experiments

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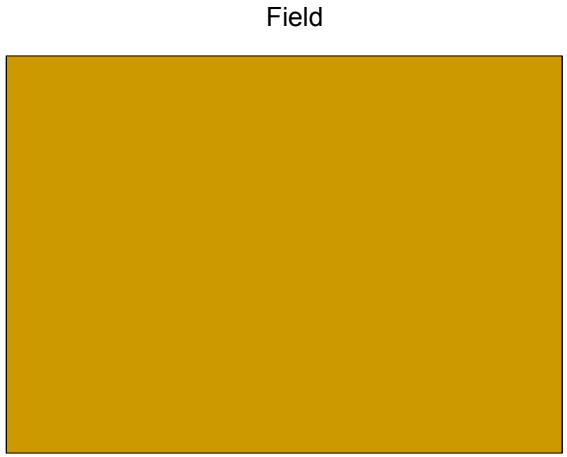
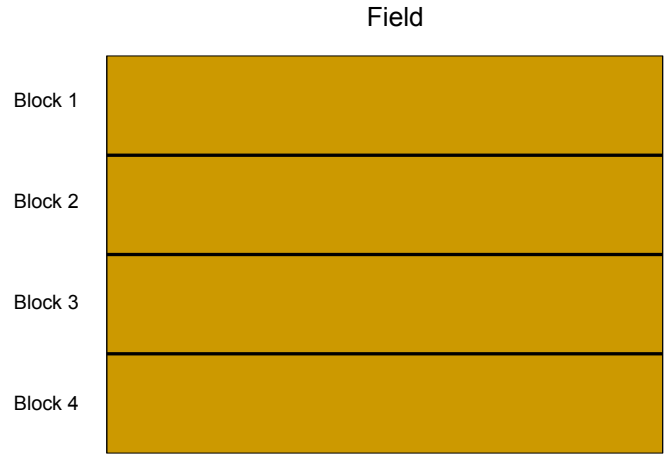
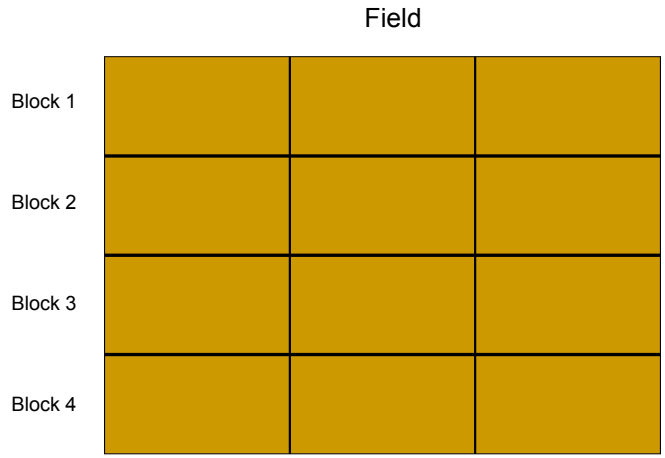
Start with a field, next partition field into blocks, then partition blocks into plots







Randomly Assign Genotypes to Plots within Blocks

Whole plot trt: genotype : A, B, C

Field

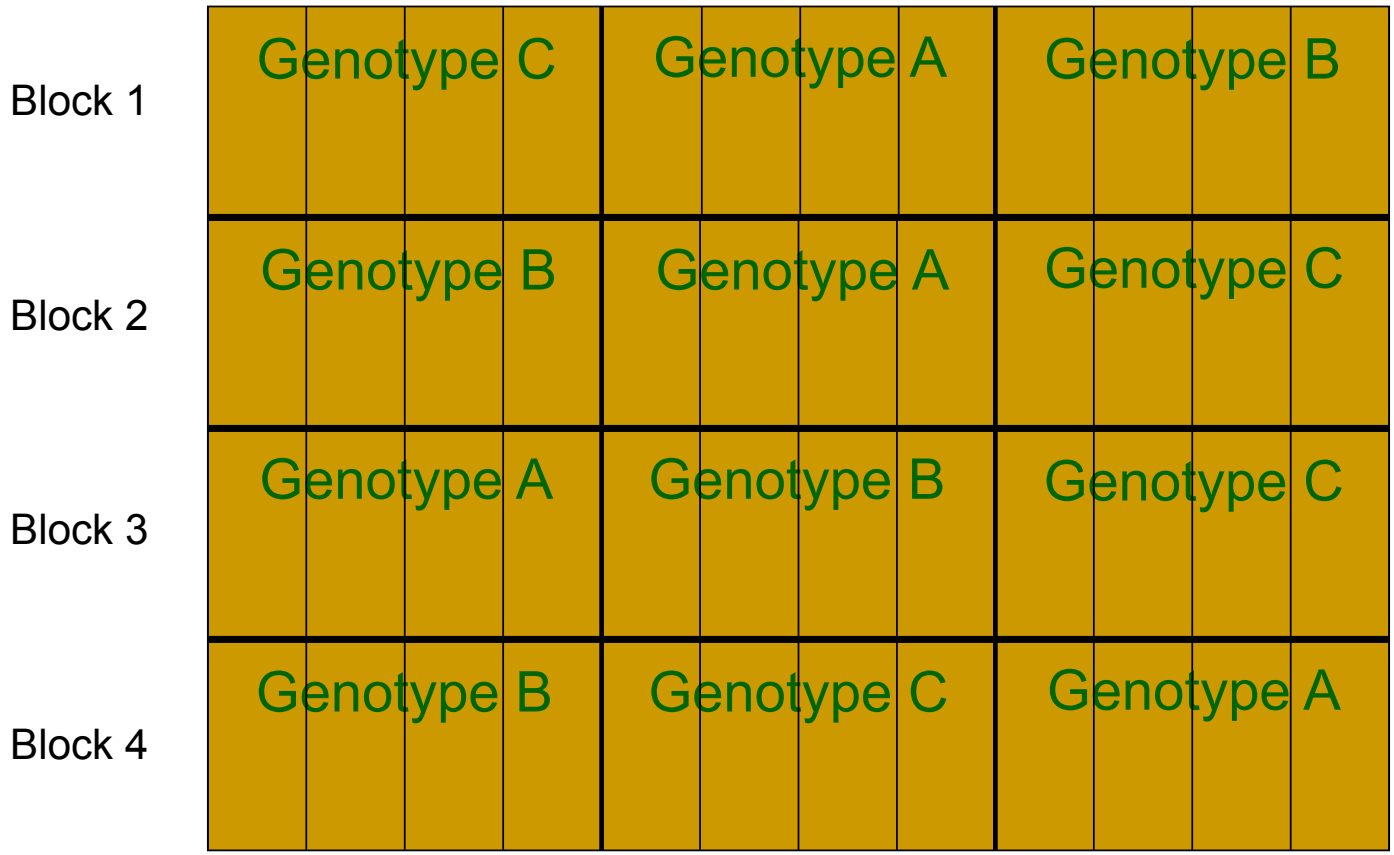
Block 1	Genotype C	Genotype A	Genotype B
Block 2	Genotype B	Genotype A	Genotype C
Block 3	Genotype A	Genotype B	Genotype C
Block 4	Genotype B	Genotype C	Genotype A

4 replic. of each genotype

Partition Each Whole Plot into Split Plots

Second factor: fertilizer – 4 levels 0, 50, 100, 150

Field



Randomly Assign Fertilizer Amounts within Split Plots

randomly assigning fertilizer within each whole plot exp. unit

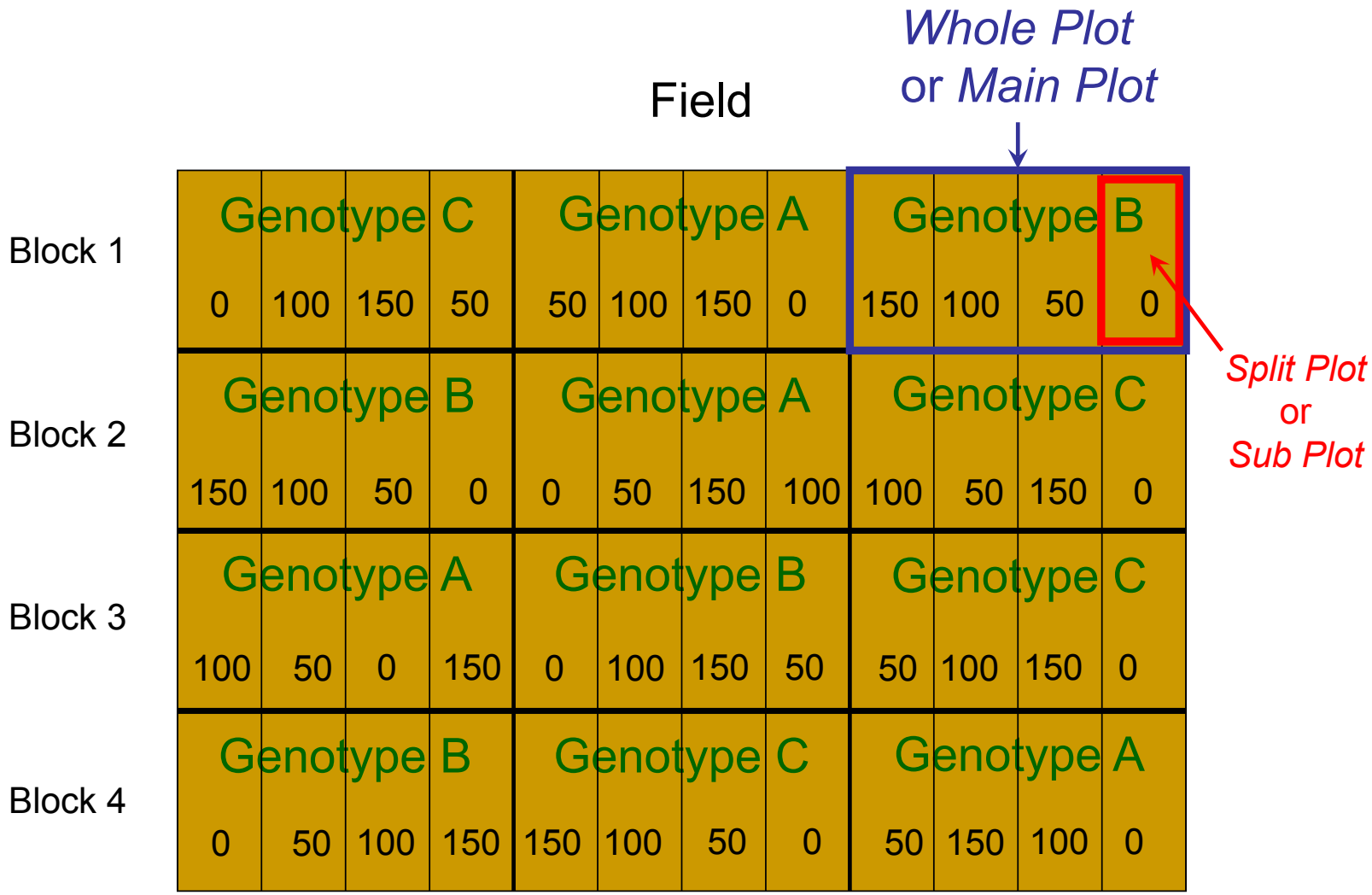
Field

Block 1	Genotype C				Genotype A				Genotype B			
	0	100	150	50	50	100	150	0	150	100	50	0
Block 2	Genotype B				Genotype A				Genotype C			
	150	100	50	0	0	50	150	100	100	50	150	0
Block 3	Genotype A				Genotype B				Genotype C			
	100	50	0	150	0	100	150	50	50	100	150	0
Block 4	Genotype B				Genotype C				Genotype A			
	0	50	100	150	150	100	50	0	50	150	100	0

12 whole plot exp. units

4 subplots within each whole plot exp. unit

An Example Split-Plot Experiment



- This experiment has two factors: genotype and fertilizer amount.
- Genotype has levels A, B, and C.
- Fertilizer has levels 0, 50, 100, 150 lbs. N / acre.
- Genotype is called the *whole-plot* (or *main-plot*) factor because its levels are randomly assigned to whole plots (main plots).
- Fertilizer is called the *split-plot* factor because its levels are randomly assigned to split plots within each whole plot.

Experimental Units in Split-Plot Designs

- Whole plots are the *whole-plot experimental units* because the levels of the whole-plot factor (genotype) are randomly assigned to whole plots.
- The split-plots are the *split-plot experimental units* because the levels of the split-plot factor (amount of fertilizer) are randomly assigned to split plots within each whole plot.
- Thus, we have two different sizes of experimental units in split-plot experimental designs.

Same Treatment Structure in an RCBD

Field

Block 1	B 100	B 0	A 0	C 100	B 150	C 50	A 50	A 150	C 150	B 50	C 0	A 100
Block 2	A 150	A 0	C 50	A 50	B 100	B 50	C 100	C 0	A 100	C 150	B 150	B 0
Block 3	C 0	A 0	A 100	B 100	B 50	B 0	A 150	C 50	A 50	C 150	C 100	B 150
Block 4	B 0	C 150	B 50	A 150	C 100	A 0	B 150	C 50	B 100	C 0	A 100	A 50

Same Treatment Structure in an CRD

Field

B 50	B 0	A 150	B 100	A 100	C 150	A 50	B 0	A 50	C 100	C 0	C 100
A 50	A 0	C 50	B 50	B 150	B 50	A 0	C 0	A 100	C 50	B 150	B 0
C 0	A 0	A 100	A 150	A 0	B 0	A 150	B 150	A 50	B 150	C 100	A 100
B 50	B 100	B 100	C 150	C 100	C 50	A 150	C 50	C 150	C 0	C 150	B 100

Why Use a Split-Plot Design?

- Split-plot designs usually arise because logistical constraints make a CRD or RCBD impractical.
- For example, it may be easier to change from one fertilizer level to another as a tractor drives through a field, while it may be more difficult to change from planting one genotype to planting another.
- In the engineering literature, split-plot designs are sometimes called designs with *hard-to-change* factors.

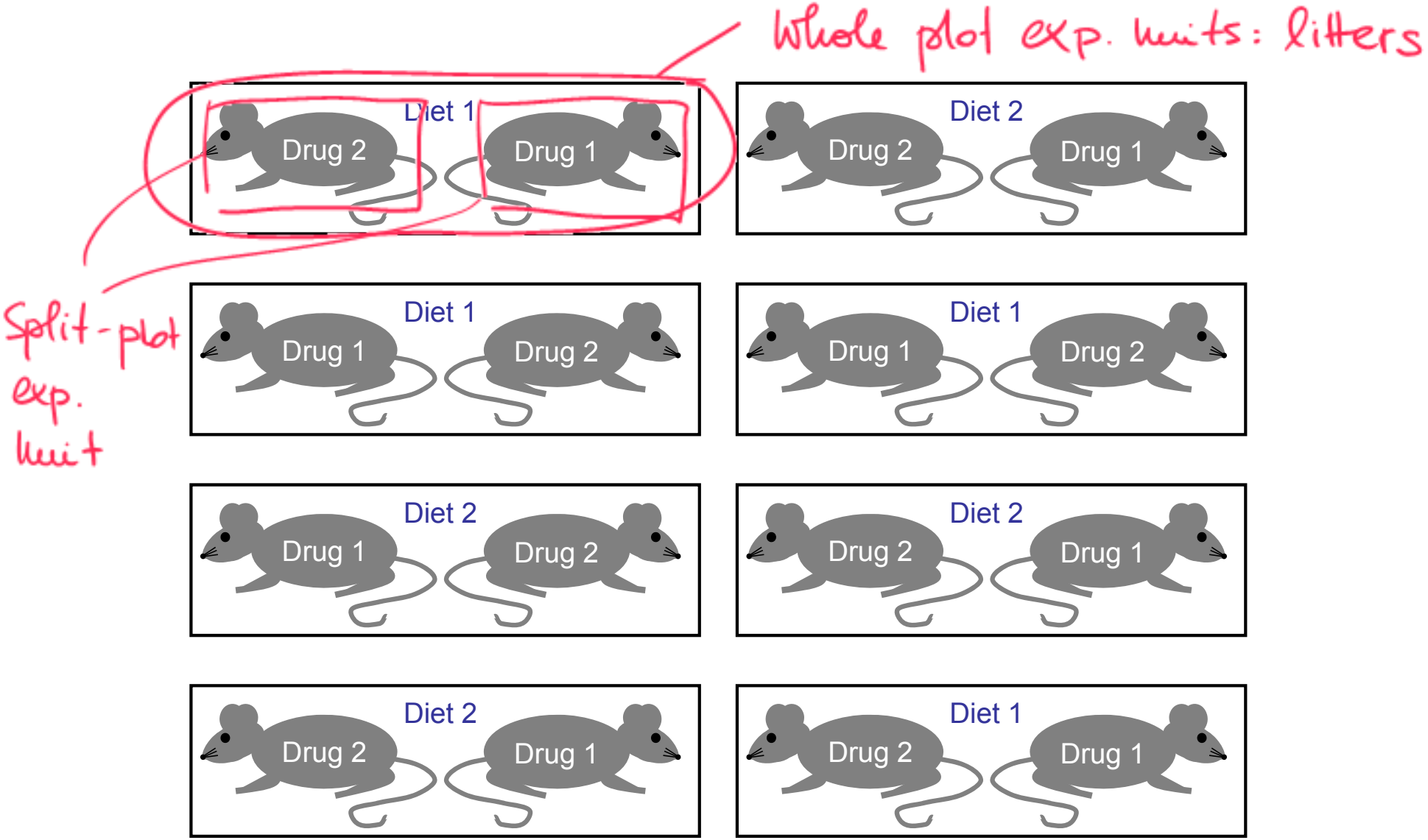
Recognizing Designs with Split-Plot Structures

- Many variations on split-plot designs are used for practical reasons.
- Examples include split-split-plot designs and split-block designs, but the names of these designs are not so important.
- Pay close attention to the experimental unit to which the levels of each factor are randomly assigned to recognize split-plot-like design structures.

Split-plot designs may not involve plots of land.

- Suppose eight pairs of mice from eight litters are housed in eight cages so that each cage holds two mice from the same litter.
- Suppose diets 1 and 2 are randomly assigned to the litters with four litters per diet.
- Within each cage, suppose drugs 1 and 2 are randomly assigned to the mice with one mouse per drug.

A Split-Plot Experimental Design



- Diet is the whole-plot treatment factor.
- Litters are the whole-plot experiment units.
- Drug is the split-plot treatment factor.
- Mice are the split-plot experiment units.

Mouse example:

indep.
Σ & 8 litters

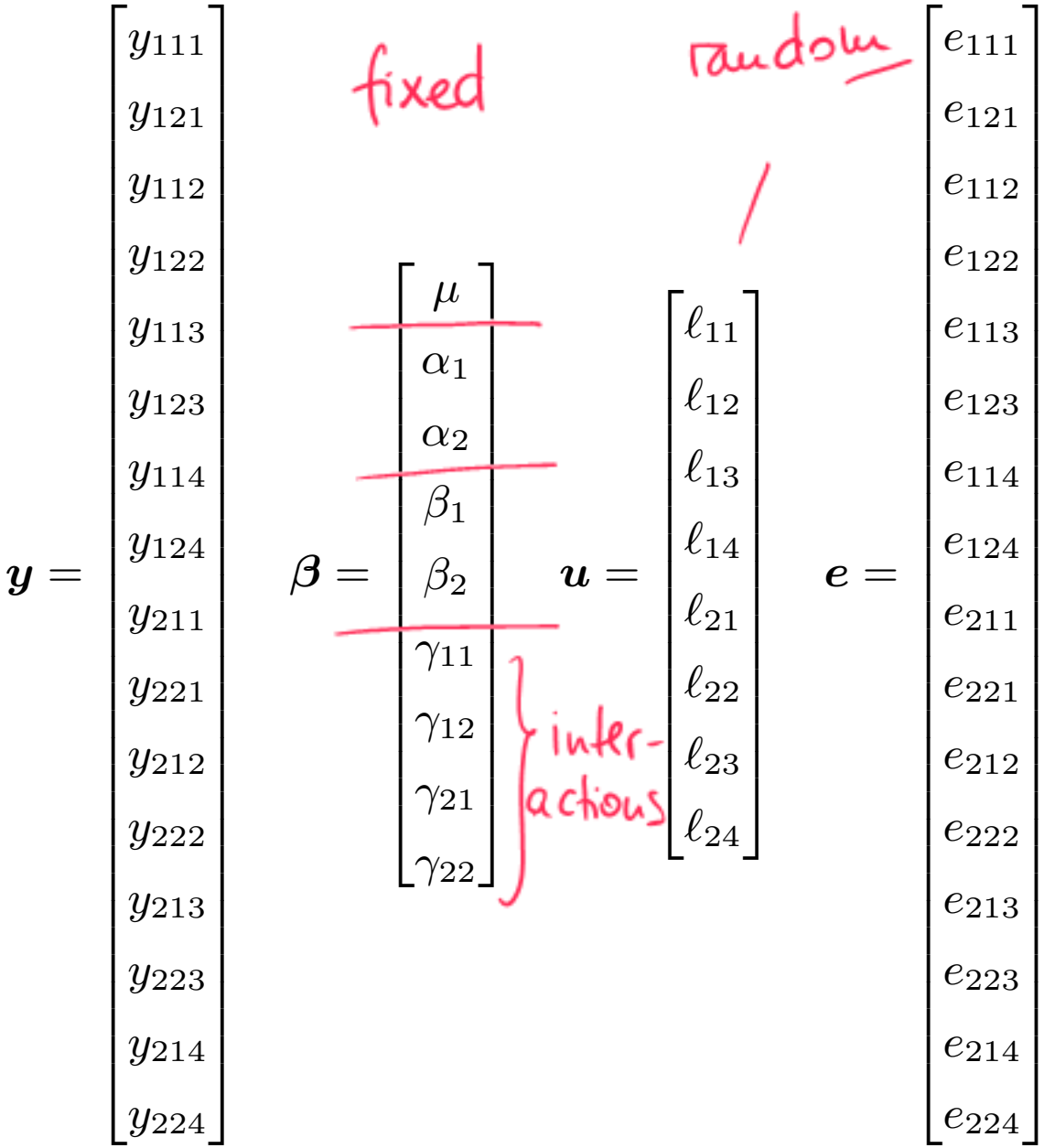
Diet $i = 1, 2$, Drug $j = 1, 2$, Litter $k = 1, 2, 3, 4$ (within each Diet i)

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \ell_{ik} + e_{ijk} \ (i = 1, 2;\ j = 1, 2; k = 1, ..., 4)$$

$\mu + \alpha_i + \beta_j + \gamma_{ij} =$ mean for Diet i and Drug j

$\ell_{ik} =$ random litter effect $=$ whole-plot exp. unit random effect

$e_{ijk} =$ random error effect $=$ split-plot exp. unit random effect


fixed
random
$\boldsymbol{y} =$
y_{111} y_{121} y_{112} y_{122} y_{113} y_{123} y_{114} y_{124} y_{211} y_{221} y_{212} y_{222} y_{213} y_{223} y_{214} y_{224}
$\boldsymbol{\beta} =$
μ
α_1 α_2
β_1 β_2
γ_{11} γ_{12} γ_{21} γ_{22}
inter-actions
$\boldsymbol{u} =$
ℓ_{11} ℓ_{12} ℓ_{13} ℓ_{14} ℓ_{21} ℓ_{22} ℓ_{23} ℓ_{24}
$\boldsymbol{e} =$
e_{111} e_{121} e_{112} e_{122} e_{113} e_{123} e_{114} e_{124} e_{211} e_{221} e_{212} e_{222} e_{213} e_{223} e_{214} e_{224}

Overall mean μ

reflects two diets

reflects drug

inter-actions

$$\boldsymbol{X} = \left[\underset{16\times 1}{\mathbf{1}}, \underset{2\times 2}{\boldsymbol{I}} \otimes \underset{8\times 1}{\mathbf{1}}, \underset{8\times 1}{\mathbf{1}} \otimes \underset{2\times 2}{\boldsymbol{I}} \; \underset{2\times 2}{\boldsymbol{I}} \otimes \underset{4\times 1}{\mathbf{1}} \otimes \underset{2\times 2}{\boldsymbol{I}} \right]$$

8 indep. litters

$$\boldsymbol{Z} = \underset{8\times 8}{\boldsymbol{I}} \otimes \underset{2\times 1}{\mathbf{1}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$ two mice within each litter

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{Z}\boldsymbol{u} + \boldsymbol{e}$$

$$\begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{e} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}, \begin{bmatrix} \sigma^2_\ell \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \sigma^2_e \boldsymbol{I} \end{bmatrix} = \begin{bmatrix} \boldsymbol{G} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{R} \end{bmatrix} \right)$$

$$\begin{aligned} \text{Var}(\boldsymbol{Zu}) &= \boldsymbol{ZGZ}^\top = \sigma^2_\ell \boldsymbol{ZZ}^\top \\ &= \sigma^2_\ell \left[\underset{8\times 8}{\boldsymbol{I}} \otimes \underset{2\times 1}{\boldsymbol{1}} \right] \left[\underset{8\times 8}{\boldsymbol{I}} \otimes \underset{2\times 1}{\boldsymbol{1}} \right]^\top \\ &= \sigma^2_\ell \left[\underset{8\times 8}{\boldsymbol{I}} \otimes \underset{2\times 2}{\boldsymbol{11}^\top} \right] \\ &= \text{Block Diagonal with blocks: } \begin{bmatrix} \sigma^2_\ell & \sigma^2_\ell \\ \sigma^2_\ell & \sigma^2_\ell \end{bmatrix} \end{aligned}$$

$$\mathrm{Var}(\boldsymbol{y}) = \boldsymbol{ZGZ}^\top + \boldsymbol{R} = \sigma^2_\ell \underset{8\times 8}{\boldsymbol{I}} \otimes \underset{2\times 2}{\boldsymbol{1}\boldsymbol{1}}^\top + \sigma^2_e \boldsymbol{I}$$

$= $ Block Diagonal with blocks: $\begin{bmatrix} \sigma^2_\ell + \sigma^2_e & \sigma^2_\ell \\ \sigma^2_\ell & \sigma^2_\ell + \sigma^2_e \end{bmatrix}$

end lecture 28

4-4-25