PhD Prelim Exam
THEORY
(Majors and Co-majors)

Summer 2011
(Given on 7/7/11)

Part I

1. Define the following terms:

   a) σ-algebra,

   b) measure,

   c) measure space,

   d) probability space,

   e) random variable,

   f) cdf on $I\!R$,

   g) Borel σ-algebra on $I\!R$.

2. Let $G : I\!R \to I\!R$ be nondecreasing, i.e., $x_1 \le x_2 \Rightarrow G(x_1) \le G(x_2)$. Define $G(-\infty) \equiv \lim_{x\to-\infty} G(x)$, $G(\infty) = \lim_{x\to\infty} G(x)$, and for $a \in I\!R$, $G(a+) = \lim_{x\downarrow a} G(x)$, $G(a-) = \lim_{x\uparrow a} G(x)$. Let

$$\mathcal{C} \equiv \left\{I : I = (a, b],\ a, b \in I\!R \text{ or } (-\infty, a],\ a \in I\!R \text{ or } (b, \infty),\ b \in I\!R\right\}$$

For any $I \in \mathcal{C}$, define

$$\mu_G(I) = \begin{cases} G(b+) - G(a+) & \text{if } I = (a, b],\ a, b \in I\!R \\ G(a+) - G(-\infty) & \text{if } I = (-\infty, a],\ a \in I\!R \\ G(\infty) - G(b+) & \text{if } I = (b, \infty),\ b \in I\!R\ . \end{cases}$$

For any $A \subset I\!R$, define

$$\mu_G^*(A) \equiv \inf\left\{\sum_{j=1}^{\infty} \mu_G(I_j) = I_j \in \mathcal{C},\ \bigcup_{j=1}^{\infty} I_j \supset A\right\}.$$

Let

$$\mathcal{M}_{\mu_G^*} \equiv \left\{A : A \subset I\!R,\ \mu_G^*(E) = \mu_G^*(E \cap A) + \mu_G^*(E \cap A^c)\ \forall\ E \subset I\!R\right\}.$$

It is known that $(I\!R, \mathcal{M}_{\mu_G^*}, \mu_G^*)$ is a measure space and the class $\mathcal{C} \subset \mathcal{M}_{\mu_G^*}$.

   a) Show that $\mathcal{M}_{\mu_G^*} \supset \mathcal{B}(I\!R)$, the Borel σ-algebra of $I\!R$.

   b) Show that if $A \in \mathcal{M}_{\mu_G^*}$, $\mu_G^*(A) = 0$ and $B \subset A$, then $B \in \mathcal{M}_{\mu_G^*}$.

   c) Let F be a cdf on $I\!R$. That is, $F(\cdot)$ is nondecreasing, $F(x+) = F(x)$ for all x in $I\!R$, $F(-\infty) = 0$, $F(\infty) = 1$. Suppose there exists a countable set $D \subset I\!R$ such that $\sum_{x\in D}(F(x+) - F(x-)) = 1$. Show that $\mathcal{M}\mu_F^* \equiv P(I\!R)$, the power set of $I\!R$. Give an example of such an F.

Part II

Recall that every $0 \le u < 1$ can be written in terms of its binary expansion as $\sum_{n=1}^{\infty} \frac{\delta_n(u)}{2^n}$ where $\delta_n(u) = 0$ or 1 by defining $\delta_1(u) = 0$ if $0 \le u < \frac{1}{2}$ and $= 1$ if $\frac{1}{2} \le u < 1$, $\delta_2(u) = \delta_1(u_1)$ where $u_1 = 2\left(u - \frac{\delta_1(u)}{2}\right)$ and $\delta_3(u) = \delta_1(u_2)$ where $u_2 = 2^2\left(u - \frac{\delta_1(u)}{2} - \frac{\delta_2(u)}{2^2}\right)$ and so on. Let U be a uniform $[0,1]$ random variable and consider the random variables $\delta_n(U)$, $n \ge 1$.

3. Show that $\{\delta_n(U)\}_{n\ge1}$ are iid Bernoulli (1/2) random variables.

4. Let F be a cdf on $I\!R$. Let $F^{-1}(x) \equiv \inf\{y : F(y) \ge x\}$ for $0 \le x \le 1$.

- **a)** Show that for any $0 \le x \le 1$ and $y \in I\!R$,
$$F(y) \ge x \text{ if and only if } F^{-1}(x) \le y\ .$$
- **b)** Show also that $F^{-1}(x)$ is Borel measureable and $X \equiv F^{-1}(U)$ has cdf F.

5. Given a sequence of cdf's $\{F_k\}_{k\ge1}$ on $I\!R$, show that there exist functions ϕ_k on $[0,1]$ such that $\{X_k \equiv \phi_k(U)\}_{k\ge1}$ are *independent* and for each k, X_k has cdf F_k.

(**Hint:** The set $N \equiv \{1,2,3,\ldots\}$ can be written as $\bigcup_{i=1}^{\infty} A_i$, where A_i are disjoint subsets of N such that for each i, A_i is infinite.)

Part III

For each $n \ge 1$, let X_n be a random variable with the Binomial (n, p_n) distribution. Suppose that as $n \to \infty$, $p_n \to 0$ and $np_n \to \lambda$ for some $0 < \lambda < \infty$.

6. Show that there exist $\{\pi_i\}_{i\ge0}$ such that
$$\sum_{i=0}^{\infty} |p_{n,i} - \pi_i|^2 \to 0 \text{ as } n \to \infty$$
where $p_{n,i} \equiv Pr(X_n = i)$, $i \ge 0$, $n \ge 1$.

7. Does the existence result in (6) hold if $\lambda = 0$?

8. State the Lindeberg-Feller CLT. Suppose $a_n \to \infty$ but $\dfrac{a_n - np_n}{\sqrt{np_n(1-p_n)}} \to x$, for $-\infty < x < \infty$. Evaluate
$$\lim_{n\to\infty} P(X_n \le a_n)\ .$$
Justify your steps.

(**Hint:** Use Polya's Theorem: If F_n, F are cdf on $I\!R$, $F_n \xrightarrow{d} F$ and F is continuous then $\sup_{x\in I\!R} |F_n(x) - F(x)| \to 0$.)

Solution to Theory I Statistics Ph.D Prelim
Summer 2011

Part I

Problem 1 Bookwork

Problem 2

a) $\mathcal{M}_{\mu_G^*} \supset \mathcal{C}$ by given information.

Also $\mathcal{M}_{\mu_G^*}$ is a σ-algebra (a fact that is given)

So $\mathcal{M}_{\mu_G^*} \supset \sigma(\mathcal{C})$, The σ-algebra generated by $\mathcal{C}$.

But $\sigma(\mathcal{C}) \equiv \mathcal{B}(\mathbb{R})$, the Borel σ-algebra of $\mathbb{R}$.

b) Let $B \subset A$ and $\mu_G^*(A) = 0$. Then for any $E \subset \Omega$,

$\mu_G^*(E \cap B) \leq \mu_G^*(E \cap A) \leq \mu_G^*(A) = 0$.

Also $\mu_G^*(E) \geq \mu_G^*(E \cap B^c)$

So $\mu_G^*(E) \geq \mu_G^*(E \cap B^c) + \mu_G^*(E \cap B)$. ①

By subadditivity of $\mu_G^*(\cdot)$,

$\mu_G^*(E) \leq \mu_G^*(E \cap B^c) + \mu_G^*(E \cap B)$ ②

Now (1) + (2) $\Rightarrow$ $\mu_G^*(E) = \mu_G^*(E \cap B^c) + \mu_G^*(E \cap B)$

for any $E \subset \Omega$. So $B \in \mathcal{M}_{\mu_G^*}$.

c) Since $\mu_F^*(\{x\}) = \cancel{F(x)}\ F(x+) - F(x-)$ $\forall x \in R$,

and D is countable and μ_F^* is countably additive

Part I Problem 2 part c (Contd)

$$\mu_F^*(D) = \sum_{x \in D} \mu_F^*(\{x\}) = \sum_{x \in D} (F(x+) - F(x-)) = 1.$$

Since $\mu_F^*(R) = 1$, it follows that $\mu_F^*(D^c) = 0$.

So by (b) for any $B \subset R$, $B \cap D^c \in \mathcal{M}_{\mu_F^*}$

Since $B \cap D$ is atmost countable, $B \cap D \in \mathcal{M}_{\mu_F^*}$.

Thus $B = (B \cap D) \cup (B \cap D^c) \in \mathcal{M}_{\mu_F^*}$.

Thus $\mathcal{P}(R) \subset \mathcal{M}_{\mu_F^*}$. But $\mathcal{M}_{\mu_F^*} \subset \mathcal{P}(R)$

and so $\mathcal{M}_{\mu_F^*} = \mathcal{P}(R)$.

Examples of such F includes any discrete cdf such as those of Binomial (n, p), Poisson (λ), negative binomial etc.

Part II

Problem 3 Let $\delta_i \equiv \delta_i(U)$, $i \geq 1$, where U is a uniform $[0, 1]$ r.v.

Then
$$P(\delta_1 = 1) = P(\tfrac{1}{2} < U \leq 1) = \tfrac{1}{2}$$
$$P(\delta_1 = 0) = P(0 < U < \tfrac{1}{2}) = \tfrac{1}{2}.$$

Further, for any $i_1, i_2, \ldots, i_k \in \{0, 1\}$, $k < \infty$

$$P(\delta_1 = i_1, \delta_2 = i_2, \ldots, \delta_k = i_k)$$
$$= P(U \in \text{an interval } I_{i_1, i_2, \ldots, i_k} \text{ determined by } i_1, i_2, \ldots, i_k)$$

$= \text{length of } I_{i_1, i_2, \ldots i_k} = \frac{1}{2^k}$

$= P(\delta_1 = i_1)\, P(\delta_2 = i_2) \ldots P(\delta_k = i_k)$

This implies that $\{\delta_i\}_{i \geq 1}$ are iid Ber$(\frac{1}{2})$.

Problem 4 a) Fix $0 \leq x_0 \leq 1$, $y_0 \in R$. Then, by definition of infimum

① $F(y_0) \geq x_0 \implies F^{-1}(x_0) \leq y_0$.

② Conversely, if $F^{-1}(x_0) \leq y_0$, by definition of infimum,

$\exists\ y_n \downarrow F^{-1}(x_0)$ such that $F(y_n) \geq x_0$

By the right continuity of F on R,

$$\lim_{n \to \infty} F(y_n) = F(F^{-1}(x_0))$$

So $F(F^{-1}(x_0)) \geq x_0$

Since by assumption $y_0 \geq F^{-1}(x_0)$, it follows that $F(y_0) \geq x_0$

So for any $0 \leq x_0 \leq 1$, $y_0 \in R$

$$F(y_0) \geq x_0 \iff F^{-1}(x_0) \leq y_0$$

b) Since U is uniform $[0,1]$, for any x in R

$$\begin{aligned} P(X \leq x) &= P(F^{-1}(U) \leq x) \\ &= P(U \leq F(x)) \quad \text{(by part(a))} \\ &= F(x) \end{aligned}$$

Part II

Problem 5

Let $N = \{1, 2, 3, \cdots \}$

Let $\{p_k : k \geq 1\}$ be the set of prime integers

$\{2, 3, 5, 7, 11, 13, 17, 19, \cdots \}$

Let $A_k \equiv \{p_k^j, \ j = 1, 2 \cdots \}, \ k \geq 1.$

Let $\tilde{A}_1 = A_1 \cup \{1\}$, $\tilde{A}_k = A_k$ for $k \geq 2$

Then $N = \bigcup_{i \geq 1} \tilde{A}_i$, each $\tilde{A}_i$ is infinite

Let $$U_k \equiv \sum_{j \in \tilde{A}_k} \frac{\delta_j(U)}{2^j}, \quad k \geq 1.$$

Then, $\{U_k\}_{k \geq 1}$ are iid uniform $[0,1]$ r.v.

Let $X_k = F_k^{-1}(U_k), k \geq 1$ Then $\{X_k\}_{k \geq 1}$ are independent with X_k having cdf F_k (by Problem 4, part b))

Part III

Problem 6 For $n \geq 1$, $0 \leq i \leq n$,

$$p_{ni} = P(X_n = i) = \binom{n}{i} p_n^i (1-p_n)^{n-i}$$

since $X_n \sim Bin(n, p_n)$.

Now $$\binom{n}{i} p_n^i (1-p_n)^{n-i} = \frac{(n(n-1) \cdots (n-i+1) p_n^i)}{i!} \left(1 - \frac{np_n}{n}\right)^{n-i}$$

This converges as $n \to \infty$ to $$\frac{\lambda^i e^{-\lambda}}{i!} \equiv \pi_i, \text{ say}.$$

Since $\sum_{i=0}^{\infty} \pi_i = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = 1$,

$\forall \epsilon > 0, \exists n_\epsilon \ni \sum_{i > n_\epsilon} \pi_i < \epsilon$.

For fixed n_ϵ,

$$\lim_{n\to\infty} \sum_{i \le n_\epsilon} |p_{ni} - p_i| = 0.$$

Also $\sum_{i > n_\epsilon} p_{ni} = 1 - \sum_{i \le n_\epsilon} p_{ni}$.

So $\lim_{n\to\infty} \sum_{i > n_\epsilon} p_{ni} = 1 - \sum_{i \le n_\epsilon} \pi_i < \epsilon$.

Hence

$$\lim_{n\to\infty} \sum_{i=0}^{\infty} |p_{ni} - \pi_i|^2 \le \lim_{n\to\infty} \sum_{i=0}^{\infty} |p_{ni} - \pi_i|$$

$$= \lim_{n\to\infty} \left(\sum_{i \le n_\epsilon} + \sum_{i > n_\epsilon} \right)$$

$$\le \lim_{n\to\infty} \sum_{i \le n_\epsilon} |p_{ni} - p| + \lim_{n\to\infty} \sum_{i > n_\epsilon} p_{ni} + \cancel{\lim} \sum_{i > n_\epsilon} \pi_i$$

$$= 0 + \epsilon + \epsilon = 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary the claim follows.

Problem 7 Yes, since here $\pi_0 = 1$, $\pi_i = 0$ for $i \ge 1$.
Same proof works.

Problem 8 By the Lindeberg Feller CLT applied to

$\{\delta_{ni} \ 1 \le i \le n\}_{n \ge 1}$ where $\{\delta_{ni} : 1 \le i \le n\}$ are iid

Bernoulli (p_n)

It follows that if $X_n = \sum_1^n \delta_{ni}$ then

$$\frac{X_n - n p_n}{\sqrt{n p_n(1-p_n)}} = \frac{\sum_{i=1}^n (\delta_{ni} - p_n)}{\sqrt{\sum_{i=1}^n Var(\delta_{ni} - p_n)}} \xrightarrow{d} N(0,1)$$

(Lindeberg Feller CLT applies since δ_{ni} are bounded by 1 r.v)

So $P(X_n \le a_n) = P\left(\frac{X_n - np_n}{\sqrt{np_n(1-p_n)}} \le \frac{a_n - np_n}{\sqrt{np_n(1-p_n)}} \right)$

$\longrightarrow \Phi(x)$, Φ being the normal cdf

& as $\frac{a_n - np_n}{\sqrt{np_n(1-p_n)}} \to x$, and $\Phi(\cdot)$ is continuous.

Here one uses Polya's theorem.

If F_n, F are cdfs on $\mathbb{R}$ and $F_n \xrightarrow{d} F$
and F is continuous then $\sup_{x \in \mathbb{R}} |F_n(x) - F(x)| \longrightarrow 0$.

Part I

Let $X_i, i = 1, 2, \ldots, n$, $n \geq 2$, be i.i.d. random variables with the translated Weibull distribution with probability density

$$f(x; \theta, \alpha, \kappa) = \frac{\kappa}{\alpha}\left(\frac{x-\theta}{\alpha}\right)^{\kappa-1} \exp\left\{-\left(\frac{x-\theta}{\alpha}\right)^{\kappa}\right\} I_{(\theta,\infty)}(x),$$

where $\theta \in \mathcal{R}$ is the location parameter, $\alpha > 0$ is the scale parameter, $\kappa > 0$ is the shape parameter, and $I_{(a,b)}(x)$ is the indicator function with value 1 if $x \in (a, b)$ and 0 otherwise.

1. Consider the special case $\theta = 1$ and $\kappa = 0.5$. Write the density function in the canonical form of an exponential family $f(x; \eta)$, and find the natural parameter space for η. Write down the condition for a natural exponential family to be of full rank, and determine if this natural exponential family is of full rank. Find the Fisher information in X_1 for η.

2. Consider the special case $\kappa = 1$. Find a minimal sufficient statistic for (θ, α) based on $(X_1, X_2, \ldots, X_n)$ and prove that it is minimal sufficient.

3. Consider the special case $\kappa = 0.5$. Either find the MLE of (θ, α) based on $(X_1, X_2, \ldots, X_n)$ or say why such an estimator does not exist. Then either identify the largest region D such that $\forall(\theta, \alpha) \in D$ the distribution is Fisher information regular or say why no such parameter vectors exist.

4. Consider the special case $\kappa = 1$. Let $\lambda = 1/\alpha$. Under the squared error loss function $L((\hat{\theta}, \hat{\lambda}), (\theta, \lambda)) = (\theta - \hat{\theta})^2 + (\lambda - \hat{\lambda})^2$, find the Bayes estimator for (θ, λ) with respect to the prior distribution with probability density
$$G(\theta, \lambda) = I_{(0,1)}(\theta) \exp\{-\lambda\} I_{(0,\infty)}(\lambda)$$
based on $(X_1, X_2, \ldots, X_n)$.

Part II

Consider the estimation of $p \in (0,1)$ based on $X \sim \text{binomial}(16, p)$ using the squared error loss function, and the decision rule $\delta(x) = ax + b$.

5. Find the risk function for $\delta(x)$ and find a and b which make $\delta(x)$ an equalizer rule.

6. Is $\delta(x)$ in question 5 a minimax rule? If your answer is yes, prove your claim. If your answer is no, give a rule which has smaller maximum risk.

7. Consider the general case $X \sim \text{binomial}(n, p)$. Suppose we believe p is near zero and use $Beta(1, 10)$ as the prior distribution on p. Let $\delta_0(x)$ be the Bayes decision rule with respect to this prior distribution and $R(n)$ be the maximum risk of the minimax rule, prove that as $n \to \infty$,

$$\sup_p R(p, \delta_0)/R(n) \to 1.$$

Part III

Let X_1 and X_2 be two independent random variables having exponential distributions with $E[X_1] = 1/\lambda_1$ and $E[X_2] = 1/\lambda_2$. Let $Y = \lambda_1 X_1 - \lambda_2 X_2$.

8. Find the characteristic function of Y.

9. Show that Y has the standard Laplace distribution, i.e., that its probability density function is given by

$$f(y) = \frac{1}{2}\exp\{-|y|\}.$$

Part IV

Let $\{X_n\}_{n\geq 1}$ be a sequence of independent random variables with

$$P(X_n = n) = P(X_n = -n) = p_n$$

and $P(X_n = 0) = 1 - 2p_n$ for $p_n \in (0,1)$.

10. Give an example of $\{p_n\}_{n\geq 1}$ for which $\{X_n\}_{n\geq 1}$ satisfies the Lindeberg condition.

11. Find constants $\{a_n\}_{n\geq 1}$ (depending upon $\{p_n\}_{n\geq 1}$) such that

$$\frac{\sum_{j=1}^{n} X_j}{a_n} \xrightarrow{d} N(0,1).$$

Theory II

Solution Stat Ph.D Prelim 2011

1. $f(x;\alpha) = \frac{1}{2\alpha}\left(\frac{x-1}{\alpha}\right)^{-\frac{1}{2}} e^{-\left(\frac{x-1}{\alpha}\right)^{\frac{1}{2}}} I_{(1,\infty)}(x)$

$= \exp\left\{-\sqrt{x-1}\cdot\frac{1}{\sqrt{\alpha}} + \log\frac{1}{\sqrt{\alpha}}\right\}\cdot\frac{1}{2\sqrt{x-1}} I_{(1,\infty)}(x)$

Let $\eta = \frac{1}{\sqrt{\alpha}}$, we have the canonical form

$f(x;\eta) = \exp\{-\eta\sqrt{x-1} + \log\eta\}\cdot\frac{1}{2\sqrt{x-1}} I_{(1,\infty)}(x)$

The natural para. space is $(0,\infty)$.

It is full rank.

$I(\eta) = \frac{d^2(-\log\eta)}{d\eta^2} = \frac{1}{\eta^2}$

2. The min. suff. stat is $(X_{(1)}, \sum_{i=1}^{n} X_i)$

Need to show $f_\theta(\underset{\sim}{X};\theta,\alpha) = f(\underset{\sim}{Y};\theta,\alpha)\cdot K(\underset{\sim}{X},\underset{\sim}{Y})$ $\forall\theta$

implies $X_{(1)} = Y_{(1)}$, $\sum_{i=1}^{n} X_i = \sum_{i=1}^{n} Y_i$

If $X_{(1)} \neq Y_{(1)}$ then $\exists\,\theta$ btwn $X_{(1)}$ and $Y_{(1)}$ s.t. one side is 0 and the other side is not. Thus $X_{(1)} = Y_{(1)}$.

For $\theta < X_{(1)} = Y_{(1)}$, $\frac{f(\underset{\sim}{X};\theta,\alpha)}{f(\underset{\sim}{Y};\theta,\alpha)} = e^{-\frac{1}{\alpha}(\sum X_i - \sum Y_i)}$

is free of $\alpha \Rightarrow \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} Y_i$

So the min. suff. stat is $(X_{(1)}, \sum_{i=1}^{n} X_i)$

3. For fixed α, $f(\underset{\sim}{X}; \theta, \alpha)$ is a positive increasing function of θ for $\theta \in (-\infty, X_{(1)})$ and 0 for $\theta \in [X_{(1)}, \infty)$, so MLE does not exist. There is also no (θ, α) for which the distribution is Fisher information regular.

4. Under the squared error loss, the Bayes estimator is the posterior mean.

$$f(\underset{\sim}{X} | \theta, \lambda) = \lambda^n \exp\{-\lambda(\Sigma X_i - n\theta)\} I_{(0,\infty)}(X_{(1)})$$

$$G(\theta, \lambda) = e^{-\lambda} I_{(0,\infty)}(\lambda) \; I_{(0,1)}(\theta)$$

For $X_{(1)} \geq 0$

$$f(\theta, \lambda | \underset{\sim}{X}) = \frac{f(\underset{\sim}{X}|\theta,\lambda) G(\theta,\lambda)}{\int_0^\infty \int_0^1 f(\underset{\sim}{X}|\theta,\lambda) G(\theta,\lambda)\, d\theta\, d\lambda} = \frac{1}{C(\underset{\sim}{X})} f(\underset{\sim}{X}|\theta,\lambda) G(\theta,\lambda)$$

For $X_{(1)} \geq 1$

$$C(\underset{\sim}{X}) = \int_0^1 \int_0^\infty e^{-\lambda(\Sigma X_i - n\theta + 1)} \lambda^n \, d\lambda \, d\theta$$

$$= \int_0^1 \frac{\Gamma(n+1)}{(\Sigma X_i - n\theta + 1)^{n+1}} d\theta$$

$$= \Gamma(n+1) \left(\frac{1}{(\Sigma X_i - n + 1)^n} - \frac{1}{(\Sigma X_i + 1)^n} \right)$$

For $X_{(1)} \in (0,1)$

$$C(\underset{\sim}{X}) = \Gamma(n+1) \left(\frac{1}{(\Sigma X_i - n X_{(1)} + 1)^n} - \frac{1}{(\Sigma X_i + 1)^n} \right)$$

For $X_{(1)} \leq 0$, the posterior distribution is not defined.

For $X_{(1)} \geq 1$, $\hat{\theta} = E[\theta | X] = \int_0^\infty \int_0^1 \theta \lambda^n e^{-\lambda\{\Sigma X_i - n\theta + 1\}} d\theta\, d\lambda / C(X)$

$$= \int_0^\infty \lambda^n e^{-\lambda\{\Sigma X_i + 1\}} \int_0^1 \theta e^{\lambda n \theta} d\theta \, d\lambda / C(X)$$

$$= \int_0^\infty \lambda^n e^{-\lambda\{\Sigma X_i + 1\}} \left(\frac{e^{\lambda n}}{\lambda n} - \frac{e^{\lambda n}}{(\lambda n)^2} + \frac{1}{(\lambda n)^2} \right) d\lambda / C(X)$$

$$= \int_0^\infty \frac{1}{n}\lambda^{n-1} e^{-\lambda(\Sigma X_i - n+1)} - \frac{1}{n^2}\lambda^{n-2} e^{-\lambda(\Sigma X_i - n+1)} + \frac{1}{n^2}\lambda^{n-2} e^{-\lambda(\Sigma X_i+1)} \, d\lambda / C(X)$$

$$= \left(\frac{1}{n}\frac{\Gamma(n)}{(\Sigma X_i - n+1)^n} - \frac{1}{n^2}\frac{\Gamma(n-1)}{(\Sigma X_i - n+1)^{n-1}} + \frac{1}{n^2}\frac{\Gamma(n-1)}{(\Sigma X_i+1)^{n-1}}\right) / C(X)$$

$$② \hat{\lambda} = E[\lambda | X] = \int_0^1 \int_0^\infty \lambda^{n+1} e^{-\lambda(\Sigma X_i - n\theta+1)} d\lambda \, d\theta / C(X)$$

$$= \int_0^1 \frac{\Gamma(n+2)}{(\Sigma X_i - n\theta+1)^{n+2}} d\theta / C(X)$$

$$= \frac{\Gamma(n+2)}{n(n+1)} \left(\frac{1}{(\Sigma X_i - n+1)^{n+1}} - \frac{1}{(\Sigma X_i+1)^{n+1}}\right) / C(X)$$

$$= \left(\frac{\Gamma(n)}{(\Sigma X_i - n+1)^{n+1}} - \frac{\Gamma(n)}{(\Sigma X_i+1)^{n+1}}\right) / C(X)$$

$$= \frac{1}{n}\left(\frac{1}{(\Sigma X_i - n+1)^{n+1}} - \frac{1}{(\Sigma X_i+1)^{n+1}}\right)\left(\frac{1}{(\Sigma X_i - n+1)^n} - \frac{1}{(\Sigma X_i+1)^n}\right)^{-1}$$

For $X_{(1)} \in [0,1)$, through similar calculation we have

$$\hat{\theta} = \frac{1}{C(X)}\left(\frac{1}{n}\frac{\Gamma(n)}{(\Sigma X_i - nX_{(1)}+1)^n} - \frac{1}{n^2}\frac{\Gamma(n-1)}{(\Sigma X_i - nX_{(1)}+1)^{n-1}} + \frac{1}{n^2}\frac{\Gamma(n-1)}{(\Sigma X_i+1)^{n-1}}\right)$$

$$\hat{\lambda} = \frac{1}{n}\left(\frac{1}{(\Sigma X_i - nX_{(1)}+1)^{n+1}} - \frac{1}{(\Sigma X_i+1)^{n+1}}\right)\left(\frac{1}{(\Sigma X_i - nX_{(1)}+1)^n} - \frac{1}{(\Sigma X_i+1)^n}\right)^{-1}$$

5. $R(p,\delta) = E_p\,(aX+b-p)^2$

$= p^2(240a^2-32a+1)+p(16a^2+32ab-2b)+b^2$

For δ to be equalizer rule, we must have

$$\begin{cases} 240a^2-32a+1=0 \\ 16a^2+32ab-2b=0 \end{cases} \Rightarrow \begin{cases} a=0.05 \\ b=0.1 \end{cases}$$

6. Since $\delta(X)=0.05X+0.1$ is an equalizer rule, only need to show it is Bayes wrt some G.

Let $G \sim Beta(2,2)$, then $E[p|X] = \frac{X+2}{16+2+2} = 0.05X+0.1$

i.e., $\delta(X)$ is Bayes wrt $G \sim Beta(2,2)$

7. $\delta_0(X) = E[p|X] = \frac{X+1}{n+1+10} = \frac{X+1}{n+11}$

$R(p,\delta_0) = E\left[\left(\frac{X+1}{n+11}-p\right)^2\right] = \frac{1}{(n+11)^2}\left(np(1-p)+(1-11p)^2\right)$

$= \frac{(121-n)p^2+(n-22)p+1}{(n+11)^2}$

For n large, the numerator is maximized at $p=\frac{n-22}{2n-242} \in (0,1)$,

w/ $\sup_p R(p,\delta_0) = \frac{1}{(n+11)^2}\left(1+\frac{(n-22)^2}{4(n-121)}\right) = \frac{1}{4n}+o\left(\frac{1}{n}\right)$ as $n\to\infty$

Using arguments similar to Q6, we have $\delta_n(X) = \frac{X+\frac{\sqrt{n}}{2}}{n+\sqrt{n}}$

w/ $R(n) = \left(\frac{\sqrt{n}}{2(n+\sqrt{n})}\right)^2 = \frac{1}{4n}+o\left(\frac{1}{n}\right)$ as $n\to\infty$

Thus $\sup_p R(p,\delta_0)/R(n) \longrightarrow 1$ as $n\to\infty$.

8. $\phi_{X_1}(t) = \frac{\lambda_1}{\lambda_1 - it}$ $\phi_{X_2}(t) = \frac{\lambda_2}{\lambda_2 - it}$

$$\phi_Y(t) = \phi_{\lambda_1 X_1 - \lambda_2 X_2}(t) = \frac{\lambda_1}{\lambda_1 - i\lambda_1 t} \cdot \frac{\lambda_2}{\lambda_2 + i\lambda_2 t} = \frac{1}{1+t^2}$$

9. Let $\tilde{Y}$ be a r.v. w/ standard Laplace distribution

$$\phi_{\tilde{Y}}(t) = E[e^{it\tilde{Y}}]$$
$$= \int_{-\infty}^{\infty} e^{ity} \frac{1}{2} e^{-|y|} dy$$
$$= \int_{\infty}^{0} \frac{1}{2} e^{ity+y} dy + \int_0^{\infty} \frac{1}{2} e^{ity-y} dy$$
$$= \frac{1}{2}\left(\frac{e^{(it+1)y}}{it+1}\Bigg|_{-\infty}^{0} + \frac{e^{(it-1)y}}{it-1}\Bigg|_{0}^{\infty} \right)$$
$$= \frac{1}{1+t^2} = \phi_Y(t)$$

10. Lindeberg condition $L_n = \frac{1}{B_n^2} \sum_{j=1}^{n} E[X_j^2 I[|X_j| > \varepsilon B_n]] \to 0$ as $n\to\infty$

where $B_n^2 = \sum_{j=1}^{n} EX_j^2 = 2\sum_{j=1}^{n} j^2 P_j$

Take $P_j = \frac{1}{\sqrt{j}}$ for example, $B_n^2 = 2\sum_{j=1}^{n} j^2 j^{-\frac{1}{2}} = 2\sum_{j=1}^{n} j^{\frac{3}{2}} \sim \frac{4}{5} n^{\frac{5}{2}}$

$$L_n = \frac{1}{\frac{4}{5}n^{\frac{5}{2}}} \sum_{j=1}^{n} j^2 P(j > \varepsilon B_n) = \frac{5}{4n^{\frac{5}{2}}} \sum_{j=1}^{n} j^2 j^{-\frac{1}{2}} 1\{j > \varepsilon B_n\} = \frac{5}{4n^{\frac{5}{2}}} \sum_{[\varepsilon B_n]}^{n} j^{\frac{3}{2}} \to 0$$

as $n\to\infty$

11. By Lindeberg CLT, Let $a_n = B_n = \sqrt{\frac{4}{5}}\, n^{\frac{5}{4}}$

$$\frac{\sum_{j=1}^{n} X_j}{a_n} \xrightarrow{d} N(0,1)$$

Part I

Let $\boldsymbol{X} \sim N_p(\boldsymbol{0}, \boldsymbol{I}_p)$, where $\boldsymbol{0} \in \mathbb{R}^p$, and $\boldsymbol{I}_p$ is the $p \times p$ identity matrix. Further, let Y be independent of $\boldsymbol{X}$ and distributed as χ^2_p (*i.e.*, the χ^2-distribution with p degrees of freedom).

1. Find the (joint) probability density of $\boldsymbol{Z} = \sqrt{Y}\frac{\boldsymbol{X}}{\|\boldsymbol{X}\|}$.

Part II

Let $X_0, X_1, X_2, \ldots$ be a sequence of dependent integer-valued random variables such that the conditional distribution of the random variable X_i given $X_0, X_1, X_2, \ldots, X_{i-1}$ is the same as the conditional distribution of X_i given X_{i-1}, for any $i \geq 1$. Suppose that $X_0 \equiv 0$.

2. Show that for any $n > i$, the conditional distribution of X_i given $X_{i+1}, X_{i+2}, \ldots, X_n$ is the same as the conditional distribution of X_i given X_{i+1}.

3. Now, suppose that all X_i take values in {0,1,2}. Suppose that

$$\begin{aligned}
P(X_i = 0 | X_{i-1} = 0) &= 1 - p \\
P(X_i = 1 | X_{i-1} = 0) &= p \\
P(X_i = 0 | X_{i-1} = 1) &= 1 - q \\
P(X_i = 2 | X_{i-1} = 1) &= q \\
P(X_i = 0 | X_{i-1} = 2) &= 1 - l \\
P(X_i = 2 | X_{i-1} = 2) &= l
\end{aligned}$$

Recall that $X_0 \equiv 0$. Let $Y = \min\{i > 0 : X_i = 0\}$. Calculate $E(Y)$.

Part III

Consider a joint probability mass function $f(x, y)$ where

$$f(x,y) \propto \begin{cases} \exp\{\alpha(x+y) + \rho xy + \rho(1-x)(1-y)\} & (x,y) \in \{(0,0),(0,1),(1,0),(1,1)\} \\ 0 & \text{otherwise .} \end{cases}$$

Note that the parameter vector $(\alpha, \rho) \in \mathbb{R}^2$.

4. Is the above mass function in the regular exponential family?

Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be iid with the bivariate pmf $f(x, y)$.

5. Calculate the expected value of the number of (X_i, Y_i), $i = 1, 2, \ldots, n$ for which $X_i = Y_i$.

6. Provide a (joint) minimal sufficient statistic for the parameter (α, ρ).

7. Find the MLE of (α, ρ), call this $(\hat{\alpha}, \hat{\rho})$.

8. Assume $\rho = 0$. Under this restriction, find the MLE of α, call this $\tilde{\alpha}$.

9. Provide a likelihood ratio test of approximate size 0.05 for testing $H_0 : \rho = 0$. You may assume that n is large.

Part I :

1. Let $\underset{\sim}{X} \sim N_p(\underset{\sim}{0}, I)$.

$Y \sim \chi^2_p$

ie, $f(y) \propto \exp\left(-\frac{y}{2}\right) \cdot y^{\frac{p}{2}-1}, \quad y > 0.$

Let $\underset{\sim}{z} = \sqrt{Y} \dfrac{\underset{\sim}{X}}{\|\underset{\sim}{X}\|}$ and $u = \dfrac{\|\underset{\sim}{X}\|}{\sqrt{Y}}$.

ie, $\varphi(\underset{\sim}{X}, Y) = \left(\dfrac{\sqrt{Y}\,\underset{\sim}{X}}{\|\underset{\sim}{X}\|}, \dfrac{\|\underset{\sim}{X}\|}{\sqrt{Y}}\right)$.

$\Rightarrow \varphi(\underset{\sim}{z}, u) = (u\underset{\sim}{z}, \underset{\sim}{z}'\underset{\sim}{z})$.

Then $J_\psi = u^{p-1}$.

So $f(z, u) \propto u^{p-1} \exp\left\{-(u^2+1)\dfrac{\|\underset{\sim}{z}\|^2}{2}\right\} \|\underset{\sim}{z}\|^p$.

Integrating out u yields $f(\underset{\sim}{z}) = \int_0^\infty f(\underset{\sim}{z}, u)\, du$

Thus $f(\underset{\sim}{z}) \propto \exp\left\{\dfrac{\underset{\sim}{z}'\underset{\sim}{z}}{2}\right\}$; $\underset{\sim}{z} \in \mathbb{R}^p$. So, $\underset{\sim}{z} \sim N(\underset{\sim}{0}, I)$

since the integral w.r.t. u is like a gamma density integrated out.

Thus $\underset{\sim}{z} \sim N(\underset{\sim}{0}, I)$.

Part II:

2. $X_0, X_1, X_2, \ldots$ are realizations from a Markov Chain.

$$P(X_i \mid X_{i+1}, X_{i+2}, \ldots, X_n) = \frac{P(X_i, X_{i+1}, X_{i+2}, \ldots, X_n)}{P(X_{i+1}, X_{i+2}, \ldots, X_n)}$$

$$= \sum_{x_1, x_2, \ldots, x_{i-1}} P(x_1, x_2, \ldots, x_n) \Big/ \sum_{x_1, x_2, \ldots, x_i} P(x_1, x_2, \ldots, x_n).$$

$$= \frac{\sum_{x_1, x_2, \ldots, x_{i-1}} P(X_1 \mid X_0) P(X_2 \mid X_1) \ldots P(X_i \mid X_{i-1}) P(X_{i+1} \mid X_i) \ldots P(X_n \mid X_{n-1})}{\sum_{x_1, x_2, \ldots, x_i} P(X_1 \mid X_0) P(X_2 \mid X_1) \ldots P(X_i \mid X_{i-1}) P(X_{i+1} \mid X_i) \ldots P(X_n \mid X_{n-1})}$$

$$= \frac{\sum_{x_{i-1}} P(X_{i-1}) P(X_i \mid X_{i-1}) P(X_{i+1} \mid X_i)}{\sum_{x_{i-1}} \sum_{x_i} P(X_i \mid X_{i-1}) P(X_{i-1}) P(X_{i+1} \mid X_i)}$$

$$= \frac{P(X_{i+1} \mid X_i) . P(X_i)}{P(X_{i+1})} = P(X_i \mid X_{i+1}).$$

3. Let Y be the return to state 0. Then,

$$p_k(x) = \begin{cases} 1-p, & x=1 \\ p(1-q), & x=2 \\ pq\,\ell^{x-3}(1-\ell), & x \geq 3. \end{cases}$$

Then, $E(Y) = 1.(1-p) + 2.\,p(1-q) + \sum_{x=3}^{\infty} x\,pq\,\ell^{x-3}(1-\ell).$

$$= 1-p+2p-2pq + pq(1-\ell)\left[\sum_{x=3}^{\infty} \ell^{x-3}(x-2) + 2\sum_{x=3}^{\infty} 2\ell^{x-3}\right]$$

$$= 1+p-2pq+pq(1-\ell)\left[\frac{d}{d\ell}\sum_{x=3}^{\infty} \ell^{x-2} + \frac{2}{1-\ell}\right]$$

$$= 1 + p - 2pq + pq(1-\ell)\left[\frac{d}{d\ell}\left(\frac{1}{1-\ell} - 1\right) + \frac{2}{1-\ell}\right).$$

$$= 1 + p - 2pq + pq(1-\ell)\left[\frac{+1}{(1-\ell)^2} + \frac{2}{1-\ell}\right]$$

$$= 1 + p - 2pq + pq(1-\ell)\left[\frac{1 + 2(1-\ell)}{(1-\ell)^2}\right] = 1 + p - 2pq + pq\,\frac{(3-2\ell)}{1-\ell}.$$

Part III :

4. Yes, since $\exp\{\alpha(x+y) + \rho(xy + \overline{1-x}.\overline{1-y}) - \ln(e^{\rho} + e^{2\alpha+\rho} + 2e^{\alpha})\}$ and the statespace does not depend on the parameter.
So, we have a 2-parameter REF.

5. $f(x,y) \propto \exp\{\alpha(x+y) + \rho(xy + (1-x)(1-y))\}$.

$E(XY + (1-X)(1-Y)) = E[2XY + (1-X-Y)]$

Note that $f(x,y) = \dfrac{1}{e^{\rho} + 2e^{\alpha} + e^{2\alpha+\rho}} \exp\{\alpha(X+Y) + \rho(XY + \overline{1-X}.\overline{1-Y})\}$

Now, $XY + \overline{1-X}.\overline{1-Y} = 1$ w.p. $\dfrac{e^{\rho} + e^{2\alpha+\rho}}{e^{\rho} + 2e^{\alpha} + e^{2\alpha+\rho}}$

0 o.w.

So $E\left[\sum_{i=1}^{n}(X_iY_i + \overline{1-X_i}.(1-Y_i))\right] = n.\dfrac{e^{\rho} + e^{2\alpha+\rho}}{e^{\rho} + 2e^{\alpha} + e^{2\alpha+\rho}}$.

6. The minimal joint sufficient statistic for (ρ, α) is

$$\left[\sum_{i=1}^{n}\{X_iY_i + (1-X_i)(1-Y_i)\},\ \sum_{i=1}^{n}(X_i + Y_i)\right].$$

It is easy to see that this is sufficient. Since it is of the same dimension as the parameter set, it is minimal also.

7. The loglikelihood is given by

$$\ell = -n\ln\left(2e^{\alpha}+e^{\rho}+e^{2\alpha+\rho}\right)+\alpha\sum_{i=1}^{n}(X_i+Y_i)+\rho\sum_{i=1}^{n}\left(X_iY_i+\overline{1-X_i}\;\overline{1-Y_i}\right)$$

$$= -n\ln\left(2e^{\alpha}+e^{\rho}+e^{2\alpha+\rho}\right)+\alpha m+\rho K \quad \text{where } m=\sum_{i=1}^{n}(X_i+Y_i)$$

$$K=\sum_{i=1}^{n}\left[X_iY_i+\overline{1-X_i}\;\overline{1-Y_i}\right]$$

$$\frac{\partial \ell}{\partial \alpha}=0 \Rightarrow -\frac{2ne^{\alpha}+2ne^{2\alpha+\rho}}{2e^{\alpha}+e^{\rho}+e^{2\alpha+\rho}}+m=0$$

$$\Rightarrow \frac{2ne^{\alpha}\left(1+e^{\alpha+\rho}\right)}{2e^{\alpha}+e^{\rho}+e^{2\alpha+\rho}}=m.$$

$$\frac{\partial \ell}{\partial \rho}=0 \Rightarrow \frac{-n\left(e^{\rho}+e^{2\alpha+\rho}\right)}{2e^{\alpha}+e^{\rho}+e^{2\alpha+\rho}}+K=0 \Rightarrow K=\frac{ne^{\rho}\left(1+e^{2\alpha}\right)}{2e^{\alpha}+e^{\rho}+e^{2\alpha+\rho}}.$$

Now, let $a=e^{\alpha}$; $b=e^{\rho}$.

$$\Rightarrow \frac{2na(1+ab)}{2a+b+a^2b}=m. \quad \text{and} \quad \frac{nb(1+a^2)}{2a+b+a^2b}=k.$$

$$\Rightarrow \frac{2\cancel{n}a(1+ab)}{m}=\frac{\cancel{n}b(1+a^2)}{k} \Rightarrow 2ka(1+ab)=mb(1+a^2)$$

$$\Rightarrow 2ka+2ka^2b=mb+ma^2b.$$

$$\Rightarrow b=2ka/(m+ma^2-2ka^2).$$

Also, $2na(1+ab)=m(2a+b+a^2b)$

So, $$2na\left(1+\frac{2Ka^2}{m+ma^2-2ka^2}\right)=m\left(2a+\frac{2ka(1+a^2)}{m+ma^2-2ka^2}\right)$$

$$\Rightarrow \cancel{2}n\cancel{a}\left(m+ma^2-2ka^2+2ka^2\right)=\cancel{2}m\cancel{a}\left(m+ma^2-2ka^2+2k+2ka^2\right)$$

$$\Rightarrow n(m+ma^2)=m(m+ma^2+2k).$$

$$\Rightarrow n(1+a^2)=m(1+a^2)+2k-ka^2=m(1+a^2)-k(a^2-1).$$

$$\Rightarrow a^2n+n-a^2m-m+a^2k-k=0. \Rightarrow a=\sqrt{\frac{K+m-n}{n+K-m}}$$, only (+)ve root being valid.

So, $$a=\sqrt{\sum_{i=1}^{n}X_iY_i\Big/\left(\sum_{i=1}^{n}X_iY_i-\sum_{i=1}^{n}X_i-\sum_{i=1}^{n}Y_i+n\right)}.\quad b=\frac{2Ka}{m+ma^2-2Ka^2}.$$ Solve for $(\hat{\alpha},\hat{\rho})$

8. Assume $\rho = 0$. Under this restriction the MLE for α is sol^n to: $e^{\alpha} = \sum_{i=1}^{n}(X_i+Y_i) \Big/ \left[2n - \sum_{i=1}^{n}(X_i+Y_i)\right]$; $\Rightarrow \tilde{\alpha} = \ln\left[\dfrac{\sum_{i=1}^{n}(X_i+Y_i)}{2n - \sum_{i=1}^{n}(X_i+Y_i)}\right]$

9. The likelihood ratio test for testing $H_0: \rho = 0$.

Under H_0; let $a_0 = e^{\tilde{\alpha}}$.

Then $\sup\limits_{\theta \in \Theta_0} l(\theta) = -n \ln\left[2a_0 + a_0^2 + 1\right] + \sum_{i=1}^{n}(X_i+Y_i)\ln a_0$.

$\sup\limits_{\theta \in \Theta_0 \cup \Theta_1} l(\theta) = -n \ln\left[2a_1 + b_1 + a_1^2 b_1\right] + \ln a_1\, m + \ln b_1\, k$

where a_1, b_1 are the a and b of the part (7).

Since $-2\left[\sup\limits_{\Theta_0} l(\theta) - \sup\limits_{\Theta_0 \cup \Theta_1} l(\theta)\right] \xrightarrow{d} \chi^2_{\text{full-reduced}} \equiv \chi^2_1$.

we have

$$-2\left[-n\ln(2a_0 + a_0^2 + 1) + m\ln a_0 + n\ln(2a_1 + b_1 + a_1^2 b_1) - m\ln a_1 - k\ln b_1\right]$$

$$= -2\left[n\left\{\ln(2a_1 + b_1 + a_1^2 b_1) - \ln(2a_0 + a_0^2 + 1)\right\} - m\ln a_1 - k\ln b_1 + m\ln a_0\right]$$

$$= -2\left[\ln\left(\frac{2a_1 + b_1 + a_1^2 b_1}{2a_0 + a_0^2 + 1}\right)^n + \ln\left(\frac{a_0^m}{a_1^m\, b_1^k}\right)\right]$$

$$= 2\ln\frac{(2a_0 + a_0^2 + 1)^n a_1^m b_1^k}{(2a_1 + b_1 + a_1^2 b_1)^n a_0^m}.$$

So, we need to compare

$$\tau = \frac{(2a_0 + a_0^2 + 1)^n a_1^m b_1^k}{(2a_1 + a_1^2 b_1 + b_1)^n a_0^m} > \exp\left\{\frac{\chi^2_{1;1-\alpha}}{2}\right\}.$$

where $\alpha = 0.05$.

Part I

Let $X_1, X_2, X_3, \ldots$ be an infinite sequence of independent and identically distributed Bernoulli(p) random variables for $p \in (0,1)$. For some $\lambda > 0$, let n be a Poisson(λ) random variable, which is independent of the Bernoulli sequence $\{X_i\}_{i=1}^{\infty}$. Let

$$S_n = \sum_{i=1}^{n} X_i$$

which is the random variable of interest here.

1. Derive $E(S_n)$ and $Var(S_n)$.
2. Derive the moment generating function (MGF) of S_n.
3. Use the MGF obtained above to verify your answers in 1.
4. Establish using the MGF that

$$\frac{S_n - E(S_n)}{\sqrt{\lambda}} \longrightarrow N(0, \eta^2) \quad \text{in distribution, as} \quad \lambda \longrightarrow \infty$$

   and identify η^2.

Part II

The hazard function of a random variable X is defined as

$$h_X(t) = \lim_{\delta \to 0} \frac{P(t \le X < t + \delta | X \ge t)}{\delta} .$$

5. Suppose that X is a continuous random variable with density f and a cdf F. Obtain an expression of $h_X(t)$ in terms of F and f.

Let $X \sim \text{Exp}(\beta)$ such that $E(X) = \beta$. Define $Y = X^{1/\gamma}$ for a $\gamma > 0$.

6. Derive the density function of Y.
7. Derive the hazard function of Y.

Part III

Let $Z \sim N(0,1)$.

8. Use the approach that proves the Chebyshev inequality to show that

$$P(|Z| \geq t) \leq \sqrt{\frac{2}{\pi}} \frac{1}{t} e^{-t^2/2}.$$

9. Use integration by parts to establish that

$$P(|Z| \geq t) \geq \sqrt{\frac{2}{\pi}} \frac{t}{1+t^2} e^{-t^2/2} .$$

You may like to know that

$$P(|Z| \geq t) = \frac{2}{\sqrt{2\pi}} \int_t^\infty \frac{1}{1+z^2} e^{-z^2/2} dz + \frac{2}{\sqrt{2\pi}} \int_t^\infty \frac{z^2}{1+z^2} e^{-z^2/2} dz .$$

Part IV

Let U be a uniform$(0,1)$ random variable, and V be a continuous random variable with density taking positive values only on $(0,1)$. For $(u,v) \in (0,1)^2$, suppose that the conditional cdf of V given $U = u$ is

$$\left(\frac{v}{u+v-uv}\right)^2 .$$

10. Identify the joint cdf of (U, V) for $(u,v) \in (0,1)^2$.

11. Identify the marginal cdf of V.

Part I

1. Derive $E(S_n)$ and $Var(S_n)$;
   $E(S_n) = E(E(S_n|n)) = E(np) = \lambda p$; $Var(S_n) = E\{Var(S_n|n)\} + Var\{E(S_n|n)\} = E(np(1-p)) + Var(np) = \lambda p$.

2. Derive the moment generating function (MGF) of S_n;
   Let $\theta = 1 - p + pe^t$.

$$M_{S_n}(t) = E(e^{t\sum_{i=1}^n X_i}) = E((1-p+pe^t)^n) = \sum_{k=0}^{\infty} \theta^k \lambda^k e^{-\lambda}/k! = e^{-\lambda+\theta)\lambda} = e^{-p\lambda(1-e^t)}.$$

3. $M'_{S_n}(t) = p\lambda e^t e^{p\lambda(e^t-1)}$. So, $M'(0) = p\lambda$.
   $M''_{S_n}(t) = p\lambda e^{p\lambda(e^t-1)}\{e^t + p\lambda e^{2t}\}$. So, $M''_{S_n}(0) = p\lambda + (p\lambda)^2$. Hence, $Var(S_n) = M''_{S_n}(0) - \{M'_{S_n}(0)\}^2 = p\lambda$.

4. The MGF of $\frac{S_n - E(S_n)}{\sqrt{\lambda}}$ is

$$\begin{aligned} e^{-\sqrt{\lambda}pt} M_{S_n}(t/\sqrt{\lambda}) &= \exp(-\sqrt{\lambda}pt)\exp\{p\lambda\{t/\sqrt{\lambda} + t^2/\{2\lambda\} + o(1/\lambda)\} \\ &= \exp(pt^2/2 + o(1)) \to \exp(pt^2/2) \quad \text{as} \quad \lambda \to \infty. \end{aligned}$$

   $\eta^2 = p$.

Part II

5.
$$h_X(t) = \lim_{\delta \to 0} \frac{F_X(t+\delta) - F_X(t)}{\{1 - F_X(t)\}\delta} = f_X(t)/\{1 - F_X(t)\}.$$

6.
$$f_Y(t) = \frac{\gamma}{\beta} y^{\gamma-1} e^{-y^\gamma/\beta} .$$

7.
$$F_Y(t) = 1 - e^{-t^\gamma/\beta}$$

$$h_Y(t) = f_Y(t)/\{1 - F_Y(t)\} = \frac{\gamma}{\beta} t^{\gamma-1} .$$

Part III

8.

$$P(|Z| \geq t) = \frac{2}{\sqrt{2\pi}} \int_t^\infty e^{-z^2/2} dz \leq \frac{2}{\sqrt{2\pi}} \int_t^\infty \frac{z}{t} e^{-z^2/2} dz = \sqrt{\frac{2}{\pi}} \frac{e^{-t^2/2}}{t}.$$

9.

$$P(|Z| \geq t) = \frac{2}{\sqrt{2\pi}} \int_t^\infty \frac{1}{1+z^2} e^{-z^2/2} dz + \frac{2}{\sqrt{2\pi}} \int_t^\infty \frac{z^2}{1+z^2} e^{-z^2/2} dz.$$

The second integral, by integration by parts, is

$$\int_t^\infty \frac{z^2}{1+z^2} e^{-z^2/2} dz = \frac{t}{1+t^2} e^{-t^2/2} + \int_t^\infty \frac{1-z^2}{(1+z^2)^2} e^{-z^2/2} dz.$$

As

$$\frac{1}{1+z^2} + \frac{1-z^2}{(1+z^2)^2} \geq 0,$$

$$P(|Z| \geq t) \geq \frac{2}{\sqrt{2\pi}} \frac{t}{1+t^2} e^{-t^2/2}.$$

Part IV

10. As U is uniform, the joint cdf for $(u, v) \in (0,1)^2$

$$F(u, v) = \int_0^u \left(\frac{v}{t + v - tv} \right)^2 dt = \frac{vu}{(u + v - uv)^2}.$$

For other values of (u, v), the distribution will be trivial.

11. The marginal cdf of V is $F(1, v) = v$, hence it is uniform$(0, 1)$.

You may take the following facts as given:

Let W be a Gamma(α, β) random variable with the following pdf

$$f(w) = \frac{1}{\beta^\alpha \Gamma(\alpha)} e^{-\frac{w}{\beta}} w^{\alpha-1}, \; w \geq 0 \; .$$

Then $\mathrm{E}(W) = \alpha\beta$ and $\mathrm{Var}(W) = \alpha\beta^2$. If $c > 0$ is a constant then $cW \sim$ Gamma$(\alpha, c\beta)$. Also if $W_1, W_2, \ldots, W_n$ are iid Gamma(α, β), then $\sum_{i=1}^{n} W_i \sim$ Gamma$(n\alpha, \beta)$.

For the following questions, suppose $X_1, X_2, \ldots, X_n$, $n > 1$ are a random sample from the following probability density function

$$f_\theta(x) = \begin{cases} \sqrt{\frac{2\theta}{\pi}}\, e^{-\frac{\theta x^2}{2}} & x \geq 0 \\ 0 & \text{otherwise} \; , \end{cases}$$

where $\theta > 0$ is unknown.

1. Show that $\sum_{i=1}^{n} X_i^2 \sim$ Gamma$\left(\frac{n}{2}, \frac{2}{\theta}\right)$.

2. Find a uniformly minimum variance unbiased estimator (UMVUE) of $\frac{1}{\theta}$.

3. Find a uniformly minimum variance unbiased estimator (UMVUE) of θ when $n > 2$.

4. Show that a method of moments estimator of θ is

$$\tilde{\theta}_n = 2/(\pi \bar{X}^2),$$

where $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$.

5. Find the limiting distribution of $\sqrt{n}(\tilde{\theta}_n - \theta)$ as $n \to \infty$.

6. Show that the maximum likelihood estimator of θ is

$$\hat{\theta}_n = \frac{n}{\sum_{i=1}^{n} X_i^2}.$$

7. Find the limiting distribution of $\sqrt{n}(\hat{\theta}_n - \theta)$ as $n \to \infty$.

8. For fixed $\theta_0 > 0$, find the likelihood ratio test statistic $\lambda_{\theta_0}(X_1, X_2, \ldots, X_n)$ for testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$.

9. Let a prior on θ be Gamma(α, β) with α, β known. Consider the loss function $L(\gamma(\theta), t) = \frac{(t-\gamma(\theta))^2}{\gamma(\theta)}$ for estimating $\gamma(\theta)$, a real valued function of θ. Find the Bayes estimator of $\sqrt{\theta}$ under this loss function.

10. Now let $n = 10$. Invert the likelihood ratio test obtained in (8) to find a 95% confidence interval for θ. (You **don't** need to provide any numerical solution.)

1. Let $y_i = x_i^2$. Then the Jacobian of the transformation is $\frac{1}{2\sqrt{y_i}}$ and so the pdf of y_i is

$$f(y_i) = \sqrt{\frac{\theta}{2\pi}}\, e^{-\frac{\theta y_i}{2}}\, y_i^{\frac{1}{2}-1} .$$

So, $Y_i \sim \text{Gamma}\left(\frac{1}{2}, \frac{2}{\theta}\right)$ and hence $\sum_{i=1}^{n} X_i^2 \equiv \sum_{i=1}^{n} Y_i \sim \text{Gamma}\left(\frac{n}{2}, \frac{2}{\theta}\right)$.

2. Since $\{f_\theta(x), \theta > 0\}$ is an exponential family, we know that $\sum_{i=1}^{n} X_i^2$ is complete and sufficient. From (1) we know that $\sum_{i=1}^{n} X_i^2 \sim \text{Gamma}\left(\frac{n}{2}, \frac{2}{\theta}\right)$, so $E\left(\sum_{i=1}^{n} X_i^2\right) = \frac{n}{\theta}$, i.e., $E\left(\frac{\sum X_i^2}{n}\right) = \frac{1}{\theta}$, which implies that $\frac{\sum X_i^2}{n}$ is a UMVUE of $\frac{1}{\theta}$.

3. Since $n > 2$, we have

$$E\left(\frac{1}{\sum X_i^2}\right) = \frac{1}{\frac{2}{\theta}(\frac{n}{2} - 1)} = \frac{\theta}{n-2} .$$

So, $\frac{n-2}{\sum X_i^2}$ is UMVUE of θ.

4. Note that $E(X_1) = \sqrt{\frac{2}{\pi\theta}}$. So MOM of θ is obtained as the solution of the following equation

$$\sqrt{\frac{2}{\pi\theta}} = \bar{X} .$$

So, $\tilde{\theta}_n = \frac{2}{\pi \bar{X}^2}$.

5. Since $E(X_i^2) = \frac{1}{\theta}$, we have $V(X_1) = \frac{1}{\theta} - \frac{2}{\pi\theta} = (1 - \frac{2}{\pi})\frac{1}{\theta}$. So by CLT, we have

$$\sqrt{n}\left(\bar{X} - \sqrt{\frac{2}{\pi\theta}}\right) \to N\left(0, \left(1 - \frac{2}{\pi}\right)\frac{1}{\theta}\right) .$$

Let $g(x) = \frac{2}{\pi x^2}$. So $g'(x) = \frac{-4}{\pi x^3}$ and by the Delta method we have

$$\sqrt{n}(\tilde{\theta}_n - \theta) \to N\left(0, 2(\pi - 2)\theta^2\right) .$$

6. The likelihood function is given by

$$L(\theta) = \left(\frac{2\theta}{\pi}\right)^{\frac{n}{2}} e^{-\frac{\theta}{2}\sum_{i=1}^{n} x_i^2} .$$

The log likelihood function is

$$\ell(\theta) = \frac{n}{2}\log(2\theta) - \frac{n}{2}\log\pi - \frac{\theta}{2}\sum_{i=1}^{n} x_i^2 .$$

Since $\ell''(\theta) = -\frac{n}{2\theta^2} < 0$, by solving $\ell'(\theta) = 0$, we get $\hat{\theta}_n = \frac{n}{\sum x_i^2}$.

7. Note that $\ell''(\theta) = -\frac{n}{2\theta^2}$. So the Fisher information number is $I_n(\theta) = \frac{n}{2\theta^2}$. It now follows from the properties of MLE that

$$\sqrt{n}(\hat{\theta}_n - \theta) \to N(0, 2\theta^2) \ .$$

8. The likelihood ratio test statistic for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$ is given by

$$\begin{aligned}
\lambda_{\theta_0}(x_1, x_2, \ldots, x_n) &= \frac{(\frac{2\theta_0}{\pi})^{\frac{n}{2}} \ \exp\left\{ -\frac{\theta_0}{2} \sum_{i=1}^{n} x_i^2 \right\}}{(\frac{2\hat{\theta}_n}{\pi})^{\frac{n}{2}} \ \exp\left\{ -\frac{\hat{\theta}_n}{2} \sum_{i=1}^{n} x_i^2 \right\}} \\
&= \left(\frac{\theta_0}{\hat{\theta}_n}\right)^{\frac{n}{2}} \ \exp\left\{ -\frac{n}{2}\left(\frac{\theta_0}{\hat{\theta}_n} - 1\right) \right\} \\
&= \left(\frac{\theta_0 \sum x_i^2}{n}\right)^{\frac{n}{2}} \ \exp\left\{ -\frac{n}{2}\left(\frac{\theta_0 \sum x_i^2}{n} - 1\right) \right\} .
\end{aligned}$$

9. The posterior density is given by

$$\begin{aligned}
\pi(\theta|x) &\propto \theta^{\frac{n}{2}} \ e^{-\frac{\theta}{2}\sum x_i^2} \ \theta^{\alpha-1} e^{-\frac{\theta}{\beta}} \\
&= \theta^{(\frac{n}{2}+\alpha)-1} \ e^{-\theta\left(\frac{\sum x_i^2}{2} + \frac{1}{\beta}\right)} .
\end{aligned}$$

The posterior risk for estimating $\sqrt{\theta}$ is

$$\begin{aligned}
& c \int_0^\infty \frac{(t - \sqrt{\theta})^2}{\sqrt{\theta}} \ \theta^{(\frac{n}{2}+\alpha)-1} \ e^{-\theta\left(\frac{\sum x_i^2}{2} + \frac{1}{\beta}\right)} d\theta \\
&= ct^2 \frac{\Gamma(\frac{n-1}{2} + \alpha)}{\left(\frac{\sum x_i^2}{2} + \frac{1}{\beta}\right)^{\frac{n-1}{2}+\alpha}} - 2t + c.c_1
\end{aligned}$$

where c and c_1 are constants given by

$$c = \frac{\left(\frac{\sum x_i^2}{2} + \frac{1}{\beta}\right)^{\frac{n}{2}+\alpha}}{\Gamma(\frac{n}{2} + \alpha)} \text{ and } c_1 = \frac{\Gamma(\frac{n+1}{2} + \alpha)}{\left(\frac{\sum x_i^2}{2} + \frac{1}{\beta}\right)^{\frac{n+1}{2}+\alpha}} .$$

So the Bayes estimator of $\sqrt{\theta}$ is

$$\begin{aligned}
\frac{\left(\frac{\sum x_i^2}{2} + \frac{1}{\beta}\right)^{\frac{n-1}{2}+\alpha}}{c\Gamma(\frac{n-1}{2} + \alpha)} &= \frac{\left(\frac{\sum x_i^2}{2} + \frac{1}{\beta}\right)^{\frac{n-1}{2}+\alpha}}{\Gamma(\frac{n-1}{2} + \alpha)} \cdot \frac{\Gamma(\frac{n}{2} + \alpha)}{\left(\frac{\sum x_i^2}{2} + \frac{1}{\beta}\right)^{\frac{n}{2}+\alpha}} \\
&= \frac{\Gamma(\frac{n}{2} + \alpha)}{\Gamma(\frac{n-1}{2} + \alpha)} \cdot \frac{1}{\sqrt{\frac{\sum x_i^2}{2} + \frac{1}{\beta}}} .
\end{aligned}$$

10. From (8) we know that a size α LR test is given by

$$\left\{x : (\theta_0 \sum_{i=1}^{n} x_i^2)^{\frac{n}{2}} \ \exp\left\{ - \frac{\theta_0 \sum_{i=1}^{n} x_i^2}{2} \right\} < k\right\}$$

where k is such that

$$P_{\theta_0}\Big((\theta_0 \sum_{i=1}^{n} X_i^2)^{\frac{n}{2}} \ \exp\left\{-\frac{\theta_0 \sum_{i=1}^{n} X_i^2}{2}\right\} < k\Big) = \alpha \ .$$

Since $g(x) = x^{\frac{n}{2}} e^{-\frac{x}{2}}$ is a concave function in x and under $H_0 : \theta = \theta_0$, $\theta_0 \sum_{i=1}^{n} X_i^2 \sim$ Gamma$(\frac{n}{2}, 2) \equiv \chi_n^2$, the size α LR test is given by

$$\phi(x) = \begin{cases} 1 & \text{if } \theta_0 \sum_{i=1}^{n} x_i^2 < \chi^2_{n,1-\alpha+\alpha_1} \text{ or } \theta_0 \sum_{i=1}^{n} x_i^2 > \chi^2_{n,\alpha_1} \\ 0 & \text{otherwise} \end{cases}$$

where $\chi^2_{n,\alpha}$ is the upper α quantile of χ_n^2 distribution, i.e., $P(\chi_n^2 > \chi^2_{n,\alpha}) = \alpha$ and $\alpha_1 \in (0, \alpha)$ is chosen such that

$$(\chi^2_{n,1-\alpha+\alpha_1})^{\frac{n}{2}} \ e^{-\frac{\chi^2_{n,1-\alpha+\alpha_1}}{2}} = (\chi^2_{n,\alpha_1})^{\frac{n}{2}} \ e^{-\frac{\chi^2_{n,\alpha_1}}{2}} \ . \tag{1}$$

Inverting the above size α test, we get a $(1 - \alpha)$ confidence interval of θ as follows

$$\left\{\theta : \chi^2_{n,1-\alpha+\alpha_1} < \theta \sum_{i=1}^{n} x_i^2 < \chi^2_{n,\alpha_1}\right\}$$

where α_1 is chosen satisfying (1). Since $n = 10$ and $\alpha = 0.05$, we find solution of (1) numerically to be $\alpha_1 \approx 0.01654686$ and hence a 95% confidence interval for θ is given by

$$\left\{\theta : 3.516 < \theta \sum_{i=1}^{n} x_i^2 < 21.729\right\}.$$