STAT 5430

Lec 39, F , May 2

- Homework 9 is assigned & due Sunday, May 4
  but you can submit this on Monday, May 5
- Lecture on M, May 5
- Lecture/review on W, May 7
- No class on F, May 9
- Final Exam on Tuesday, May 13, 7:30-9:30 PM

see Canvas

- Comprehensive - but focus on material since Exam 2 (interval estimation)
- Formula sheet for new material/interval & 2 formula sheets previous material
  (3 sheets (front/back) each total)
- Practice Exams

Interval Estimation II

Bayes Intervals

Definition: A **highest posterior density** (HPD) credible set of level $(1-\alpha)$ is a set of the form

← pick "cut-off" C

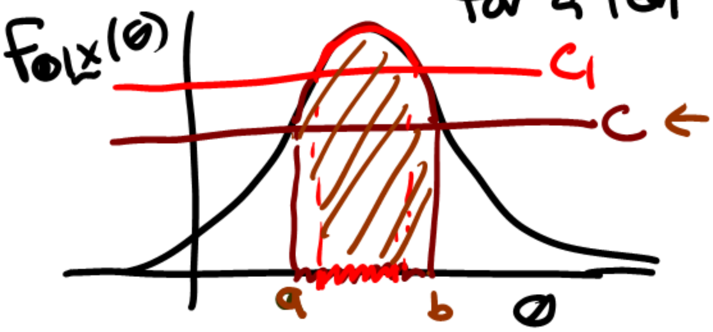
$$C_{\underset{\sim}{x}} = \{\underset{\sim}{\theta} \in \Theta : f_{\underset{\sim}{\theta}|\underset{\sim}{x}}(\underset{\sim}{\theta}) \geq C\}, \text{ for } \textbf{some } C > 0$$

such that $\underbrace{P(\underset{\sim}{\theta} \in C_{\underset{\sim}{x}}|\underset{\sim}{X} = \underset{\sim}{x})}_{\text{posterior prob}} = 1-\alpha, \ \forall \underset{\sim}{x}$.

has "right" post. prob to be a $(1-\alpha)$ credibility set

Discussion: Why do this?

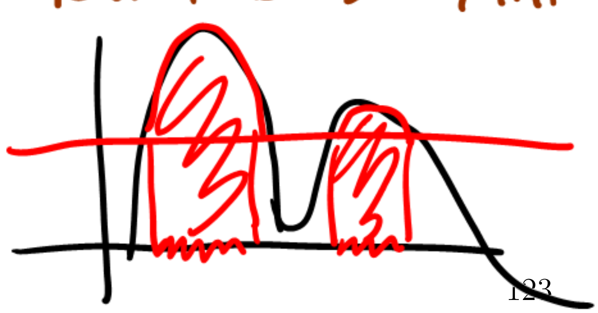
Consider posterior density $f_{\theta|\underset{\sim}{x}}(\theta)$ for a real-valued θ, which is unimodal



pick c so that $\int_a^b f_{\theta|\underset{\sim}{x}}(\theta)\,d\theta = 1-\alpha \Rightarrow$ gives the HPD credible set as $C_{\underset{\sim}{x}} = [a,b]$

Note: Where $f_{\theta|\underset{\sim}{x}}(\theta)$ is "high", want to "pack in" an area of $1-\alpha$ over a short region of θ.

So, HPD credible sets achieve $(1-\alpha)$ posterior coverage but tend to be small/informative sets for θ (guesses for θ)



Interval Estimation II

Bayes HPD Intervals: Illustration

Example: Let $X_1, \ldots, X_n$ be iid $N(\theta, \sigma^2)$ with $\theta \in \mathbb{R}$ and known $\sigma^2 > 0$. Suppose a prior distribution for θ is $N(\mu, \tau^2)$ for some known $\mu \in \mathbb{R}, \tau^2 > 0$.

$\Rightarrow$ posterior distribution of θ given $\underset{\sim}{X}$ is

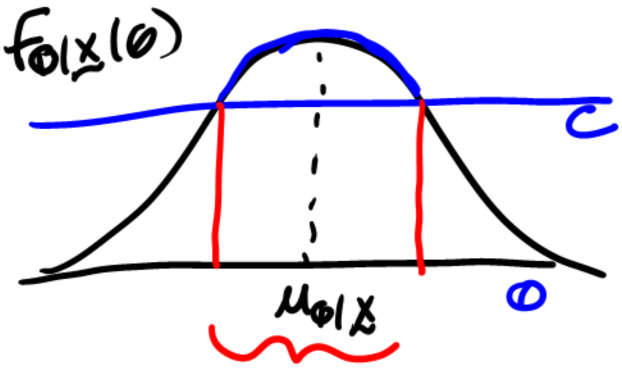
$$\theta \mid \underset{\sim}{X} \sim \text{Normal}\left(\mu_{\theta|\underset{\sim}{x}}, \sigma^2_{\theta|\underset{\sim}{x}}\right)$$

unimodal

depend on $\underset{\sim}{X}, n, \mu, \tau^2$
(done this before)

Find a $(1-\alpha)$ HPD credible set for θ:

$$C_{\underset{\sim}{x}} = \{\theta : f_{\theta|\underset{\sim}{x}}(\theta) \geq c\} = \left\{\theta : \frac{1}{\sqrt{2\pi}\,\sigma_{\theta|\underset{\sim}{x}}} e^{-\frac{(\theta - \mu_{\theta|x})^2}{2\sigma^2_{\theta|\underset{\sim}{x}}}} \geq c\right\}$$

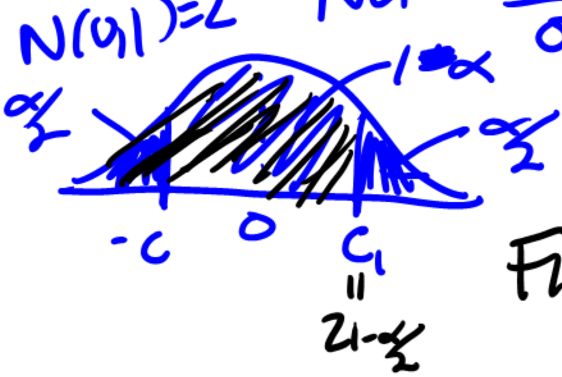


$$= \left\{\theta : \left|\frac{\theta - \mu_{\theta|\underset{\sim}{x}}}{\sigma_{\theta|\underset{\sim}{x}}}\right| \leq c_1\right\}$$

Need to pick c_1 (or c) so that

$$1-\alpha = P(\theta \in C_{\underset{\sim}{x}} \mid \underset{\sim}{X}) = P\left(\left|\frac{\theta - \mu_{\theta|\underset{\sim}{x}}}{\sigma_{\theta|\underset{\sim}{x}}}\right| \leq c_1 \,\Big|\, \underset{\sim}{X}\right)$$

Note: $\frac{\theta - \mu_{\theta|x}}{\sigma_{\theta|\underset{\sim}{x}}} \mid \underset{\sim}{X} \sim \underset{\equiv Z}{N(0,1)}$ $= P(|Z| \leq c_1)$



pick $c_1 \equiv z_{1-\alpha/2}$.

Finally, HPD credible set for θ is

$$\left\{\theta : \left|\frac{\theta - \mu_{\theta|x}}{\sigma_{\theta|x}}\right| \leq z_{1-\alpha/2}\right\}$$

Interval Estimation II

Evaluating Interval Estimators

Remark 1: For two interval estimators $I_C = [L_C(\underset{\sim}{X}), U_C(\underset{\sim}{X})]$ and $I_D = [L_D(\underset{\sim}{X}), U_D(\underset{\sim}{X})]$ with the same C.C. $1-\alpha$, then I_D is **preferred** to I_C if

most important

$$\mathrm{E}_\theta[\text{length } I_D] \leq \mathrm{E}_\theta[\text{length } I_C], \quad \forall \theta \in \Theta$$

better interval estimator (for a given C.C $1-\alpha$) has shorter expected length (more informative)

Remark 2: For confidence regions $C_{\underset{\sim}{X}}$ and $D_{\underset{\sim}{X}}$ for $\underset{\sim}{\theta} \in \Theta \subset \mathbb{R}^p$ with the same C.C. $1-\alpha$, $D_{\underset{\sim}{X}}$ is preferred to $C_{\underset{\sim}{X}}$ if

$$\mathrm{E}_{\underset{\sim}{\theta}}[\text{volume } D_{\underset{\sim}{X}}] \leq \mathrm{E}_{\underset{\sim}{\theta}}[\text{volume } C_{\underset{\sim}{X}}], \quad \forall \underset{\sim}{\theta} \in \Theta.$$

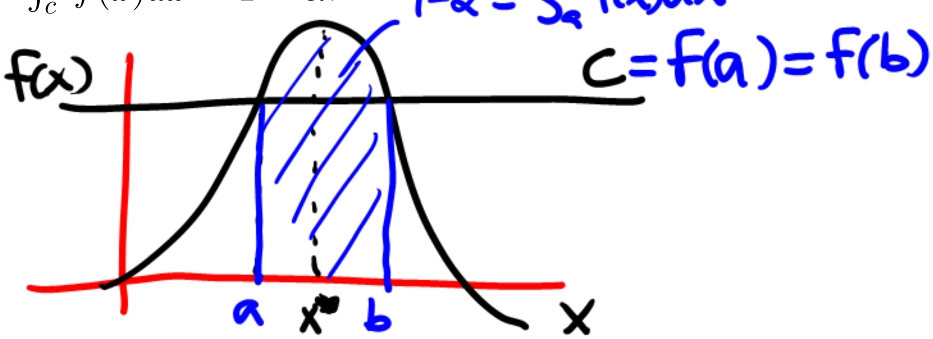
For finding CIs with short length

Theorem on Interval Lengths (Theorem 9.3.1 [CB]): Let $f(x)$, $x \in \mathbb{R}$, be a unimodal pdf. If an interval $[a, b]$ satisfies

1. $\int_a^b f(x)dx = 1-\alpha$
2. $f(a) = f(b) > 0$
3. $a \leq x^* \leq b$, where x^* is the mode/peak of $f(x)$,

then $[a, b]$ has smallest (shortest) length out of all possible intervals $[c, d]$ satisfying $\int_c^d f(x)dx = 1-\alpha$.

$1-\alpha = \int_a^b f(x)dx$



(same principle as HPD Bayes intervals)

Interval Estimation II

Evaluating Interval Estimators: Illustration

Example: Let $X_1, \dots, X_n$ iid $N(\mu, \sigma^2)$, $\sigma^2 > 0$, $\mu \in \mathbb{R}$ both unknown. Find a CI to estimate μ of the form

$$I_{a,b} = \left[\overline{X}_n - \frac{bS}{\sqrt{n}},\ \overline{X}_n - \frac{aS}{\sqrt{n}}\right]$$

sample mean

$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2$ sample variance

$S = \sqrt{S^2}$

(where $-\infty < a < b < \infty$ and S^2 is the sample variance) such that $P_{\mu,\sigma^2}(\mu \in I_{a,b}) = 1 - \alpha$, $\forall\ \mu, \sigma^2$ and such that the expected length $\mathrm{E}_{\mu,\sigma^2}(I_{a,b})$ is as short as possible, $\forall\ \mu, \sigma^2$.

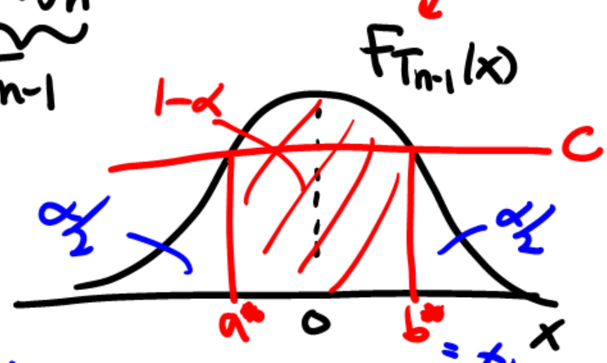
Solution: $1-\alpha = P_{\mu,\sigma^2}(\mu \in I_{a,b})$

$$= P_{\mu,\sigma^2}\left(\overline{X}_n - \frac{bS}{\sqrt{n}} \le \mu \le \overline{X}_n - \frac{aS}{\sqrt{n}}\right)$$

$\frac{\overline{X}_n - \mu}{S/\sqrt{n}} \sim T_{n-1}$

$$= P_{\mu,\sigma^2}\Big(a \le \underbrace{\frac{\overline{X}_n - \mu}{S/\sqrt{n}}}_{T_{n-1}} \le b\Big)$$

$$= \int_a^b f_{T_{n-1}}(x)\,dx \quad \text{①}$$



Note: expected length $\mathrm{E}_{\mu,\sigma^2}(I_{a,b}) = \frac{b-a}{\sqrt{n}}\mathrm{E}_{\mu,\sigma^2}(S)$ ← ②

Hence, of all intervals $I_{a,b}$ satisfying ①, I_{a^*,b^*} minimizes ② where a^* & b^* satisfy

① $\int_{a^*}^{b^*} f_{T_{n-1}}(x)\,dx = 1-\alpha$, ② $f_{T_{n-1}}(a^*) = f_{T_{n-1}}(b^*)$

& ③ $a^* \le 0 \le b^*$.

So, $a^* = -b^*$, $b^* = t_{1-\frac{\alpha}{2}, n-1}$

Interval Estimation II

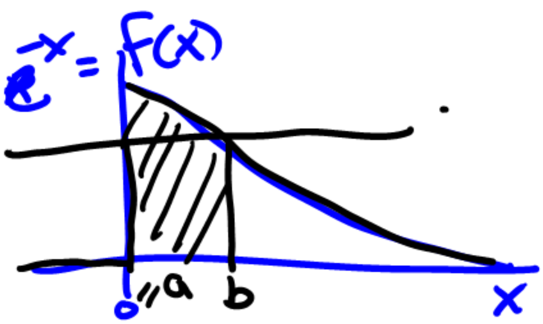
Evaluating Interval Estimators

Remark 3: The main idea of the theorem on interval lengths (Theorem 9.3.1) above is to pack in "area" under a density curve and use this to get <u>short</u> CIs. This concept can apply if Theorem 9.3.1 doesn't.

Example: Let $X \sim$Exponential$(\frac{1}{\beta})$, $\beta > 0$. Find a CI for β of the form $[\frac{a}{X}, \frac{b}{X}]$, $b > a \geq 0$ such that the C.C. is $1 - \alpha$ and the length $(b - a)$ is as short as possible.

<u>Solution</u> Note: $\frac{X}{1/\beta} = \beta X \sim \text{Exponential}(1)$

So, $1-\alpha = P_\beta\left(\beta \in \left[\frac{a}{X}, \frac{b}{X}\right]\right) = P_\beta(a \le \underbrace{\beta X}_{Exp(1)} \le b)$



$= \int_a^b e^{-x}\,dx$

So, pick $a=0$

$1-\alpha = e^{-a} - e^{-b} = 1 - e^{-b}$

$\Rightarrow b = -\log(\alpha)$