Functions of a random variable

Probability Integral Transform

This is a famous (and for some purposes very useful) transformations connected with continuous cdfs

$$F(x) = \int_{-\infty}^{x} f(t)dt, \quad t \in \mathbb{R}.$$

Result: If X has a continuous cdf $F(\cdot)$ then the random variable $Y = F(X)$ is uniformly distributed on $(0, 1)$, i.e., Y has

$$\text{pdf } f_Y(y) = \begin{cases} 1 & 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases} \qquad \text{cdf } F_Y(y) = \begin{cases} 0 & y \leq 0 \\ y & 0 \leq y \leq 1 \\ 1 & y \geq 1 \end{cases}$$

Proof. We'll suppose that the cdf $F(\cdot)$ is *strictly increasing* on $(-\infty, \infty)$.
(The result holds also for general, continuous $F(\cdot)$ but the proof is more intricate.)

Expected values

Definitions

- May be interested in a distributional summary rather than the entire distribution

- Expected value of a random variable is its "probability-weighted average"

- *Definition:* The **expected value** or **mean** of a random variable $g(X)$, denoted by $\mathrm{E}g(X)$ or $\mathrm{E}[g(X)]$ or $\mathrm{E}(g(X))$, is

$$\mathrm{E}g(X) = \sum_x g(x) f_X(x) \qquad \text{(discrete case)}$$

$$\mathrm{E}g(X) = \int_{-\infty}^{\infty} g(x) f_X(x) dx \qquad \text{(continuous case)}$$

provided that

$$\sum_x |g(x)| f_X(x) < \infty \qquad \text{(in discrete case)}$$

$$\int_{-\infty}^{\infty} |g(x)| f_X(x) dx < \infty \quad \text{(in continuous case)}$$

We say that the expected value or mean $\mathrm{E}g(X)$ does *not* exist if

$$\sum_x |g(x)| f_X(x) = \infty \qquad \text{(in discrete case)}$$

$$\int_{-\infty}^{\infty} |g(x)| f_X(x) dx = \infty \qquad \text{(in continuous case)}$$

Expected values

Examples

Examples:

1. Random seating of ten people around a table: $X = \#$ seats between A & B.

2. Toss a coin with $P(\text{'T' on toss } i) = p$. Supposing coin flips are independent, let Y =toss on which 1st 'T' is observed so that

$$P(Y = y) = \begin{cases} (1-p)^{y-1}p & y = 1, 2, 3, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Expected values

Examples (cont'd)

3. X is uniform$(0,1)$, $f_X(x) = 1$, $\boxed{0 < x < 1}$

$$E(X) = \int_{\mathbb{R}} x f(x)\,dx = \int_0^1 x \cdot 1 \; dx = \frac{x^2}{2}\Big|_0^1 = 1/2$$

$$X \sim EXP(\lambda) \quad f_X(x) = \lambda e^{-\lambda x} \;\; \forall x > 0 \to E(X) = \frac{1}{\lambda},\; Var(X) = \frac{1}{\lambda^2}$$

$$f_X(x) = \frac{1}{\beta} e^{-x/\beta} \;\; \forall x > 0 \to E(X) = \beta,\; Var(X) = \beta^2$$

4. Exponential r.v. X with pdf $f_X(x) = \frac{1}{2} e^{-x/2}$, $x > 0$

$$\mathrm{E}X = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^\infty x \frac{1}{2} e^{-x/2}\,dx = \underbrace{-xe^{-x/2}}_{uv}\Big|_0^\infty - \int_0^\infty -e^{-x/2}\,dx$$

$$u = x \qquad \tfrac{1}{2}e^{-x/2}\,dx = dv$$

$$-xe^{-x/2}\Big|_0^\infty = 0$$

$$= -\left[+2e^{-x/2}\right]_0^\infty + [0 + 2] = 2$$

i.e., integrate by parts: $\int u dv = uv - \int v du$ where $\boxed{u = x}$, $dv = \frac{1}{2} e^{-x/2} dx$ (so $du = 1dx$, $v = -e^{-x/2}$)

$$E(X) = 2$$

$$E(X^2) = \int_0^\infty x^2 \frac{1}{2} e^{-x/2}\,dx \xlongequal{u = x^2,\; \frac{1}{2}e^{-x/2}dx = dv} uv - \int v\,du = \cdots \overset{?}{=} 8$$

$g(X)$

$$Var(X) \overset{\text{later}}{=} E(X^2) - \left(E(X)\right)^2 = 8 - (2)^2 = 4$$

Another way: $\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha - 1}\,dx$

$$E(X) = \int_0^\infty x \left(\frac{1}{2} e^{-x/2}\right) dx \xlongequal[dx = 2dy]{x/2 = y} \int_0^\infty (2y)\left(\frac{1}{\cancel{2}} e^{-y}\right) \cancel{2}\,dy$$

$$= 2\int_0^\infty y e^{-y}\,dy = 2\,\underbrace{\Gamma(2)}_{1} = 2$$

$$\Gamma(n) = (n-1)!$$

$E(X^2) = \int_0^\infty x^2 \left(\frac{1}{2} e^{-x/2} dx\right) \xrightarrow[x^2 = 4y^2]{x/2 = y} \int_0^\infty (4y^2)\left(\frac{1}{\cancel{2}} e^{-y}\right) \cancel{2} dy$

$= 4 \underbrace{\int_0^\infty y^2 e^{-y} dy}_{\Gamma(3) = 2} = 8$

$\Gamma(3) = (3-1)! = 2!$

Expected values

Examples (cont'd)

5. Cauchy r.v. X with pdf $f_X(x) = \dfrac{1}{\pi(1+x^2)}$, $x \in \mathbb{R}$

$$\int_{-\infty}^{\infty} |x| f_X(x)dx = \int_{-\infty}^{\infty} \frac{|x|}{\pi(1+x^2)}dx = \frac{2}{\pi}\int_0^\infty \frac{x}{1+x^2}dx = \left[\frac{1}{\pi}\log(1+x^2)\right]\Big|_0^\infty = \infty$$

Hence, the mean EX of a Cauchy r.v. does not exist

6. Consider a jury pool consisting of 100 people (10 black, 90 white) from which 12 are randomly chosen.

   What is the expected number of black jurors?

$P(A) = \dfrac{n(A)}{n(S)}$

X = # of black jurors

range of $X = \{0, 1, 2, \cdots, 10\}$

Pmf of $X = \mathbb{P}(X = x) = \dfrac{\binom{10}{x}\binom{90}{12-x}}{\binom{100}{12}}$

$E(X) = \sum_{x=0}^{10} \dfrac{x\binom{10}{x}\binom{90}{12-x}}{\binom{100}{12}} \underset{\text{later}}{=} 1.2$

Expected values

Some properties

Theorem 2.2.5: *Suppose X is a r.v. such that* $\mathrm{E}|g_1(X)| < \infty$ *and* $\mathrm{E}|g_2(X)| < \infty$ *and let* $a, b, c \in \mathbb{R}$ *be fixed constants. Then,*

1. $\mathrm{E}\Big[ag_1(X) + b\Big] = a\mathrm{E}g_1(X) + b$
2. $\mathrm{E}\Big[ag_1(X) + bg_2(X) + c\Big] = a\mathrm{E}g_1(X) + b\mathrm{E}g_2(X) + c$
3. *If* $g_1(x) \geq a$ *for all* x*, then* $\mathrm{E}g_1(X) \geq a$
4. *If* $g_1(x) \leq b$ *for all* x*, then* $\mathrm{E}g_1(X) \leq b$
5. *If* $g_1(x) \geq g_2(x)$ *for all* x*, then* $\mathrm{E}g_1(X) \geq \mathrm{E}g_2(X)$

$$E\Big[\underbrace{a_1 g_1(X) + a_2 g_2(X) + c}_{h(X)}\Big] \overset{def}{=} \int_{-\infty}^{\infty} h(x) f_X(x)\, dx = \int_{-\infty}^{\infty} \Big\{\overbrace{a_1 g_1(x) + a_2 g_2(x)}^{h(x)} + c\Big\} f_X(x)\, dx$$

$$= a_1 \int_{-\infty}^{\infty} g_1(x) f_X(x)\, dx + a_2 \int_{-\infty}^{\infty} g_2(x) f_X(x)\, dx + c \int_{-\infty}^{\infty} f_X(x)\, dx = a_1 E(g_1(X)) + a_2 E(g_2(X)) + c$$

$$E(h(X)) = \int_{\mathbb{R}} h(x) f_X(x)\, dx$$

$$E(X) = \int_{\mathbb{R}} x f_X(x)\, dx$$

(5) $g_1(x) \geq g_2(x)$ $\forall x$, $f_X(x) > 0$ $\Longrightarrow \int_{-\infty}^{\infty} g_1(x) f_X(x) \geq \int_{-\infty}^{\infty} g_2(x) f_X(x) \Rightarrow E(g_1(X)) \geq E(g_2(X))$

Expectations are also invariant under transformation:

If $\underbrace{Y = g(X)}_{y = g(x)}$, then

$$\begin{aligned} \mathrm{E}Y &\overset{def}{=} \sum_y y f_Y(y) dy = \sum_y y \mathbb{P}(Y = y) \\ &= \sum_x g(x) f_X(x) \cancel{dx} = \mathrm{E}g(X) \end{aligned}$$

(In the continuous case, replace sums with integrals)

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\, dy = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

$Y = g(X)$
$E(Y)$

Expected values

Variance

An important instance of this $\mathrm{E}g(X)$ notion comes using $g(X) = [X - \mathrm{E}X]^2$

Definition: The **variance** of a random variable X, denoted $\mathrm{Var}(X)$ or σ_X^2, is

$$\mathrm{Var}(X) = \sigma_X^2 = \mathrm{E}[X - \mathrm{E}X]^2, = \mathbb{E}\left[(X - E(X))^2\right]$$

the expected squared distance between X and its mean $\mathrm{E}X$

Two important "variance" facts:

1. $\mathrm{Var}(a + bX) = b^2\mathrm{Var}(X)$ for any real numbers a, b

2. $\mathrm{Var}(X) = \mathrm{E}X^2 - [\mathrm{E}X]^2$

Proof: $\mathrm{Var}(X) \stackrel{def}{=} \mathbb{E}\left[(X - \underbrace{E(X)}_{\mu})^2\right] \quad \underline{E(X) = \mu}$

$$= \mathbb{E}\left[X^2 + \mu^2 - 2\mu X\right] = E\left[X^2\right] + \mu^2 - \underbrace{2\mu\, \mathbb{E}(X)}_{2\mu^2}, \quad \mathbb{E}(X) = \mu$$

$$= E\left[X^2\right] - \mu^2 = E\left[X^2\right] - \left(E(X)\right)^2.$$

Example: X is uniform$(0, 1)$. Find $\mathrm{Var}(X)$ and $\mathrm{Var}(Y)$ for $Y = 1 + 3X^2$.

Expected values

Other moments and distributional summaries

Moments are an important summary of a distribution

1. $\mu = \mu_X = \mathrm{E}X$ is often called the mean

2. $\mu'_n = \mathrm{E}X^n$ is the nth moment provided $\mathrm{E}X^n$ exists, i.e.,

$$\sum_x |g(x)| f_X(x) < \infty \quad \text{(discrete case)}$$

$$\int_{-\infty}^{\infty} |g(x)| f_X(x) dx < \infty \quad \text{(continuous case)}$$

3. $\mu_n = \mathrm{E}[(X-\mu)^n]$ is the nth central moment provided $\mathrm{E}X^n$ exists

   (a) $\mathrm{Var}(X) = \sigma_X^2 = \mathrm{E}[(X-\mu)^2] = \mu_2$ is the variance

   (b) $\sigma_X = \sqrt{\mathrm{Var}(X)}$ is the standard deviation

   (c) μ_3 is skewness (i.e., measures distributional balance around μ)

   (d) μ_4 is kurtosis (i.e., measure of how long the distributional tails are)

Regarding moments:

1. If $\mathrm{E}X^r$ exists for some $r > 0$ then $\mathrm{E}X^s$ exists for $0 \leq s \leq r$

2. If $\mathrm{E}X^r$ does not exist for some $r > 0$, then $\mathrm{E}X^s$ will not exist for $s > r$

3. $\mathrm{E}X^2$ exists if and only if $\mathrm{Var}(X)$ exists

4. For $r > 0$, the existence of $\mathrm{E}X^r$ is a matter of the distribution of X not having "heavy tails" (i.e., X doesn't assume "large" values with "large" probability)