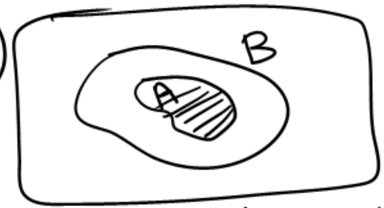
(c) $A \subset B \Rightarrow P(A) \leq P(B)$

$B = A \cup (B \cap A^c) \Rightarrow P(B) = P(A) + P(\underset{\geq 0}{\underline{B \cap A^c}})$



$\Rightarrow P(B) \geq P(A).$

Introduction to Probability

Properties of probability functions (cont'd)

Bonferroni's Inequality: $P(A \cap B) \geq P(A) + P(B) - 1$ (I)

Proof: $P(A \cap B) = \boxed{P(A) + P(B) - P(A \cup B)}$, $\quad P(A \cup B) \leq 1$

$\boxed{-P(A \cup B) \geq -1}$ II

$\geq P(A) + P(B) - 1$

Theorem 1.2.11. If P is a probability function, then

(a) $P(A) = \sum_{i=1}^{\infty} P(A \cap C_i)$ for any partition $C_1, C_2, \ldots \in \mathcal{B}$ (i.e., disjoint C_i's & $\bigcup_{i=1}^{\infty} C_i = S$)

(b) $P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i)$ for any sets $A_1, A_2, \ldots \in \mathcal{B}$

Proof of (b)

Define $A_1^* = A_1$, $A_2^* = A_2 / A_1$, $A_3^* = A_3 \setminus A_1 \cup A_2$, $\ldots$, $A_k^* = A_k / \bigcup_{i=1}^{k-1} A_i$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\bigcup_{i=1}^{\infty} A_i^*\right) = \sum_{i=1}^{\infty} P(A_i^*) \underset{A_i^* \subseteq A_i}{\leq} \sum_{i=1}^{\infty} P(A_i).$$

Part a: $A = A \cap S$

$= A \cap \left(\bigcup_{i=1}^{\infty} C_i\right)$

$= \bigcup_{i=1}^{\infty} (A \cap C_i)$ disjoint

$\Rightarrow P\left(\bigcup_{i=1}^{\infty} (A \cap C_i)\right) = \sum_{i=1}^{\infty} P(A \cap C_i)$

Principle of Inclusion-Exclusion: For any sets $A_1, \ldots, A_n$,

$$\begin{aligned} P\left(\bigcup_{i=1}^{n} A_i\right) &= \sum_{k=1}^{n} (-1)^{k-1} \left(\sum_{1 \leq i_1 < \cdots < i_k \leq n} P(A_{i_1} \cap \cdots \cap A_{i_k}) \right) \\ &= \sum_{i=1}^{n} P(A_i) - \sum_{1 \leq i < j \leq n}^{n} P(A_i \cap A_j) + \cdots + (-1)^{n-1} P\left(\bigcap_{i=1}^{n} A_i\right) \end{aligned}$$

This generalizes $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ and is proven by induction.

Introduction to Probability

Probability: the equally likely outcome case

- Sample space is *finite* $S = \{s_1, \ldots, s_N\}$ and all outcomes are equally likely

  e.g., coin toss, die toss, random sampling

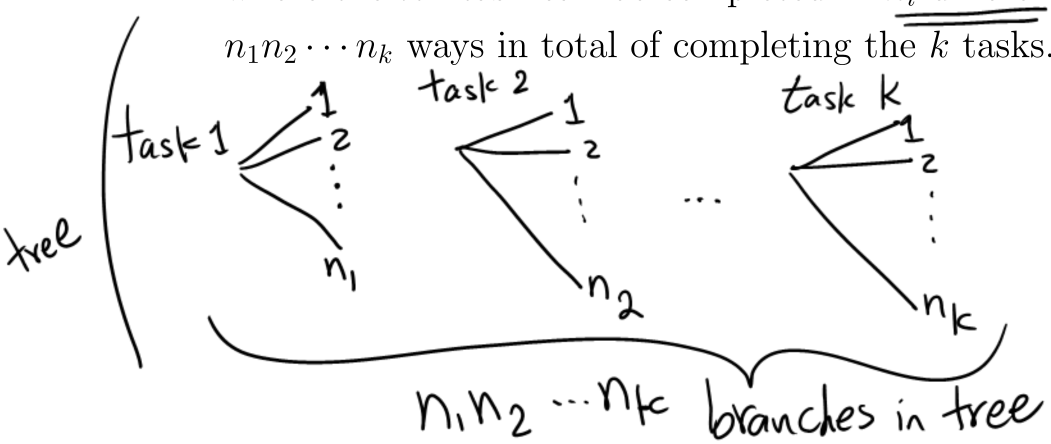
- Hence, $P(\{s_i\}) = 1/N$ for each $i = 1, \ldots, N$ and

$$P(A) = \sum_{s_i \in A} P(\{s_i\}) = \sum_{s_i \in A} \frac{1}{N} = \frac{\#\text{ elements in } A}{\#\text{ elements in } S}$$

- Application of this probability model sometimes requires *enumerating* or counting the number of possible outcomes of an experiment (each equally likely)

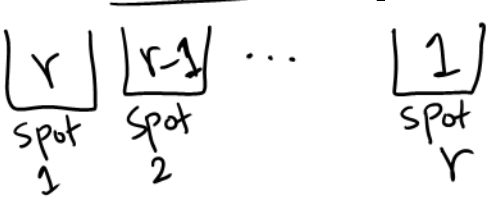
- There are basically 4 counting techniques (combinatorics)

  1. **Fundamental Theorem of Counting:** If there are separate k tasks, where the ith task can be completed in n_i different ways, then there are $n_1 n_2 \cdots n_k$ ways in total of completing the k tasks.



  2. **Permuations/ordered arrangements I:**

     r objects can be placed in $r! = r(r-1)(r-2)\ldots(1)$ ordered arrangements



Introduction to Probability

Probability: the equally likely outcome case (cont'd)

3. **Permuations/ordered arrangements II:** When selecting r objects from n objects (without replacement), then the number of ordered arrangements possible is

$$n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

$\underset{\text{spot } 1}{\boxed{n}}\ \underset{\text{spot } 2}{\boxed{n-1}} \cdots \underset{\text{spot } r}{\boxed{n-r+1}}$ $\quad n(n-1)\cdots(n-r+1)$

Note: # of ordered arrangments with replacement $\quad \underset{1}{\boxed{n}}\ \underset{2}{\boxed{n}} \cdots \underset{r}{\boxed{n}} = n^r$

4. **Combinations/unordered selections:** The number of ways to choose r objects from n objects (without replacement), where the ordering doesn't matter, is

$$\binom{n}{r} \equiv \frac{n!}{r! \cdot (n-r)!}$$

Why? Choose ordered arrangments

$\underset{1}{\boxed{n}}\ \underset{2}{\boxed{n-1}} \cdots \underset{r}{\boxed{n-r+1}} \Rightarrow \frac{n!}{(n-r)!}$

+ divide out # ways to arrange r objects $(r!)$

So $\frac{n!}{r!\,(n-r)!}$

- Table listing the number of ways to select r objects from a group of n

	objects chosen without replacement	objects chosen with replacement
ordered	$\frac{n!}{(n-r)!}$	n^r
unordered	$\binom{n}{r} = \frac{n!}{r! \cdot (n-r)!}$	$\binom{n+r-1}{r}$

Introduction to Probability

Example: the equally likely outcome case

- lotto games often require a player to pick r from among the first n integers
- e.g., Minnesota Lottery "Gopher 5": pick $r = 5$ numbers from $n = 47$
- Number of possibilities

  1. if the order matters and no repetition is allowed

  $$(47)(46)(45)(44)(43) = 184,072,680$$

  $$\frac{n!}{(n-r)!}$$

  2. if the order matters and repetition is allowed

  $n = 47$, $r = 5$

  $$(47)(47)(47)(47)(47) = 47^5 = 229,345,007$$

  n^r

  3. if the order doesn't matter and no repetition is allowed

  $$\frac{(47)(46)(45)(44)(43)}{(5)(4)(3)(2)(1)} = \binom{47}{5} = 1,533,939$$

  $\binom{n}{r}$

- In the true Gopher 5 lotto (case 3 above), what is the probability that one lotto ticket matches 4 out of the 5 lottery numbers drawn?

$$\frac{\binom{5}{4}\binom{42}{1}}{\binom{47}{5}}$$

\# of ways to check 1 from the remaining 42 "wrong" number.

ticket possible

.00014

Conditional probability and independence

Defining conditional probability

- So far all probabilities are with respect to S, i.e., $P(A)$ of some event $A \subset S$

- Knowledge of some kind might affect our opinion concerning $P(A)$
  - Roll two six-sided die (36 equally likely outcomes)
  - let event A = first die is 1 $\quad P(A) = \frac{6}{36} = \frac{1}{6}$
  - what is $P(A)$?
  - let event B = sum is 3 $\quad \{(1,2), (2,1)\}$ $\quad P(B) = 2/36 = \frac{1}{18}$
  - we might give a different guess for $P(A)$ if we know B has occurred

- $P(A|B)$ denotes the conditional probability of A given B occurs

$\boxed{P(A|B) = 1/2}$ $\quad P(\text{"1st roll is 1"} \mid (1,2),(2,1)) = 1/2$

- In conditioning, B can be thought of as the updated sample space

- Note: In a sense, all probability is conditional.
  The notation $P(A)$ can be interpreted as shorthand for $P(A|S)$

$P(A) = P(A|S)$

Conditional probability and independence

Formal definition of conditional probability

- *Definition:* If A, B are events in S with $P(B) > 0$ then

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- In conditioning, B can be thought of as the updated sample space i.e., not all of S is relevant since we know B has occurred

- This is actually a "semi"-formal definition (i.e., a more general, but technical, definition of conditional probability exists using Borel fields; see STAT 642)

Conditional Probability and independence

Conditional probability function

$P(\cdot|B)$ is a probability function that satisfies the usual axioms and properties

$P: S \longrightarrow [0,1]$

Axioms:

- $P(A|B) \geq 0$ for all events A

  $P(A) \geq 0$

- $P(B|B) = 1$ (B is the updated sample space)

  $P(S) = 1$

- If $A_1, A_2, \ldots$ are pairwise disjoint events, then

$$P\left(\bigcup_{i=1}^{\infty} A_i \middle| B\right) = \sum_{i=1}^{\infty} P(A_i|B)$$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Some properties:

- $P(A^c|B) = 1 - P(A|B)$
- $P(A_1 \cup A_2|B) = P(A_1|B) + P(A_2|B) - P(A_1 \cap A_2|B)$

Conditional Probability and independence

Example of conditional probability

Roll two six-sided die (36 equally likely outcomes) (1st roll, 2nd roll)

$$S = \left\{ \begin{array}{ccc} (1,1), (1,2), & \cdots, & (1,6) \\ \vdots & \ddots & \\ (6,1), (6,2) & \cdots & (6,6) \end{array} \right\}$$

Events: A = first die is 1; B = sum is 3; C = sum is 7

Calculations:

$P(A) = 6/36 = 1/6$

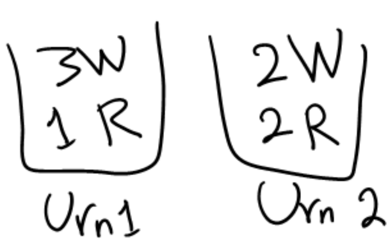
$P(B) = 2/36$

$P(C) = 6/36$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/36}{2/36} = 1/2$$

$$P(C|B) = \frac{P(C \cap B)}{P(B)} = 0$$

Note: $p(A|B) = \frac{p(A \cap B)}{p(B)} \Longrightarrow p(A \cap B) = p(A|B)\,p(B)$



Question: Select 1 ball randomly from Urn 1 & place into Urn 2; Then select 1 ball from Urn 2. What is the probability that the ball selected from Urn 2 is red?

$$P(2\text{nd } R) = P(\underbrace{1^{st}\, W \cap 2^{nd}\text{ Red}} \text{ or } \underbrace{1^{st}\text{ Red} \cap 2^{nd}\text{ Red}})$$

$$= \underbrace{P(1^{st}\, W \cap 2^{nd}\text{ Red})} + P\underbrace{(1^{st}\text{ Red} \cap 2^{nd}\text{ Red})}$$

$$= P(2\text{nd Red} \mid 1\text{st } W)\underbrace{P(1\text{st } W)} + P(2\text{nd Red} \mid 1\text{st Red}).\, P(1\text{st Red})$$

$$\left(3/4 \quad 2/5\right) + \left(1/4 \quad 3/5\right) = \frac{9}{20}$$