Functions of a random variable

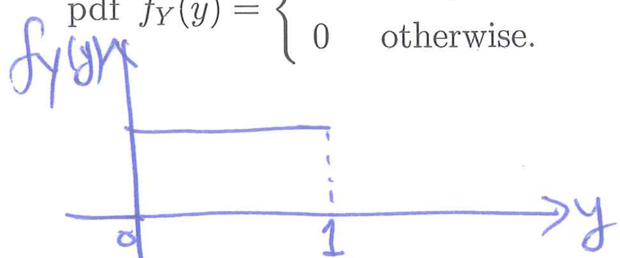
Probability Integral Transform (PIT)

This is a famous (and for some purposes very useful) transformations connected with continuous cdfs

$$F(x) = \int_{-\infty}^{x} f(t)dt, \quad t \in \mathbb{R}.$$

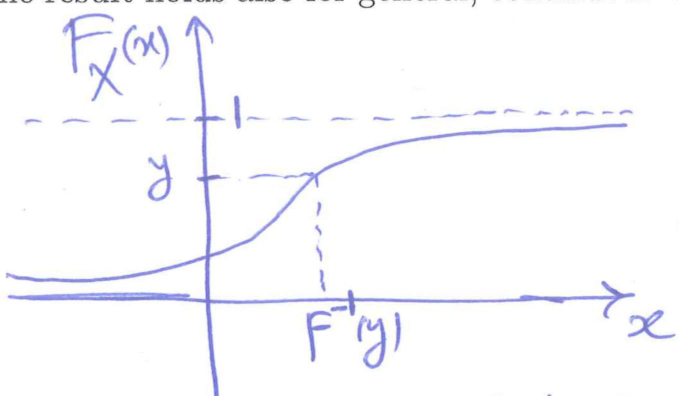
Result: If X has a continuous cdf $F(\cdot)$ then the random variable $Y = F(X)$ is uniformly distributed on $(0,1)$, i.e., Y has

$$\text{pdf } f_Y(y) = \begin{cases} 1 & 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases} \qquad \text{cdf } F_Y(y) = \begin{cases} 0 & y \le 0 \\ y & 0 \le y \le 1 \\ 1 & y \ge 1 \end{cases}$$



$Y = F(x) \Leftrightarrow x = F^{-1}(y)$

Proof. We'll suppose that the cdf $F(\cdot)$ is *strictly increasing* on $(-\infty, \infty)$.
(The result holds also for general, continuous $F(\cdot)$ but the proof is more intricate.)



$y \le 0 \Rightarrow \mathbb{P}(Y \le y) = \mathbb{P}(F(x) \le y) = 0$

$y \ge 1 \quad \mathbb{P}(Y \le y) = \mathbb{P}(F(X) \le y) = 1$

$0 < y < 1 \Longrightarrow \mathbb{P}(Y \le y) = \mathbb{P}(F(X) \le y)$
$= \mathbb{P}(F^{-1}(F(X)) \le F^{-1}(y))$
$= \mathbb{P}(X \le F^{-1}(y))$
$\underset{\text{def}}{=} F(F^{-1}(y)) = y$

Expected values

Definitions

- May be interested in a distributional summary rather than the entire distribution

- Expected value of a random variable is its "probability-weighted average"

- *Definition:* The **expected value** or **mean** of a random variable $g(X)$, denoted by $\mathrm{E}g(X)$ or $\mathrm{E}[g(X)]$ or $\mathrm{E}(g(X))$, is

$$\mathrm{E}g(X) = \sum_x g(x) f_X(x) \qquad \text{(discrete case)}$$

$$\mathrm{E}g(X) = \int_{-\infty}^{\infty} g(x) f_X(x) dx \qquad \text{(continuous case)}$$

provided that

$$\left.\begin{aligned} &\sum_x |g(x)| f_X(x) < \infty \qquad \text{(in discrete case)} \\ &\int_{-\infty}^{\infty} |g(x)| f_X(x) dx < \infty \qquad \text{(in continuous case)} \end{aligned}\right\} \quad (*)$$

need/want "$E[g(X)]$" to be real/finite number so that we require $(*)$ by definition

We say that the expected value or mean $\mathrm{E}g(X)$ does *not* exist if

$$\sum_x |g(x)| f_X(x) = \infty \qquad \text{(in discrete case)}$$

$$\int_{-\infty}^{\infty} |g(x)| f_X(x) dx = \infty \qquad \text{(in continuous case)}$$

Expected values

Examples

Examples:

1. Random seating of ten people around a table: $X = \#$ seats between A & B.

$X=x$	$f(x)$	$E(X)$ $\sum x f(x)$	$\lvert X-1\rvert$	$E\lvert X-1\rvert = \sum \lvert X-1\rvert f(x)$
0	2/9	0	1	2/9
1	2/9	2/9	0	0
2	2/9	4/9	1	2/9
3	2/9	6/9	2	4/9
4	1/9	4/9	3	3/9
		16/9		11/9

$E(X)$ $\quad$ $E\lvert X-1\rvert$

2. Toss a coin with P('T' on toss i) $= p$. Supposing coin flips are independent, let Y =toss on which 1st 'T' is observed so that

$$P(Y=y) = \begin{cases} (1-p)^{y-1}p & y = 1, 2, 3, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

$$E(Y) = \sum_{y=1}^{\infty} y\, f_Y(y) = \sum_{y=1}^{\infty} y(1-p)^{y-1}p = p\sum_{y=0}^{\infty} y(1-p)^{y-1}$$

$$= \frac{p}{p^2} = \frac{1}{p} \quad \text{for } 0<p<1$$

Note:

$$\frac{1}{p^2} = \sum_{y=0}^{\infty} y(1-p)^{y-1} \quad (\text{why?})$$

$0<p<1$