STAT 5430

Lec 34, M, Apr 21

Homework 8 is assigned & due M, Apr 28 by midnight

2nd to last homework

(practice on inverting tests & pivotal quantities for making CIs)

Interval Estimation I

Asymptotically Pivotal Quantities: Illustration

("large n + LRS + χ^2-approximation = asymptotic pivot")

Example 2: Let $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ be independent random variables where $X_1, \ldots, X_n$ are iid Exponential(θ), $\theta > 0$, and $Y_1, \ldots, Y_n$ are iid Exponential(λ), $\lambda > 0$. Find a large-sample confidence region for (θ, λ), with approximate C.C. $1-\alpha$, based on a likelihood ratio statistic. (LRS)

Want a large-sample confidence for (θ,λ) based on LRS.

Use LRS $\lambda(\underset{\sim}{X},\underset{\sim}{Y})$ for testing

$H_0: \theta=\theta_0, \lambda=\lambda_0$ vs H_1: not H_0 ($\lambda\neq\lambda_0$ or $\theta\neq\theta_0$ or both)

Under H_0: $\underbrace{-2\log\lambda(\underset{\sim}{X},\underset{\sim}{Y})}_{\substack{Q_n(\underset{\sim}{X},\underset{\sim}{Y},\lambda_0,\theta_0)\\ \text{asymp. pivot}}} \xrightarrow{d} \underbrace{\chi^2_2}_{Q \text{ limit}}$ as $n\to\infty$

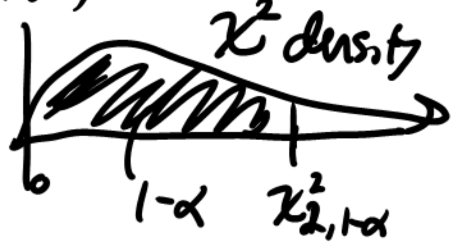
$Q \equiv \chi^2_2$

Note that $(\hat\theta_n, \hat\lambda_n) \equiv$ MLE's of θ & λ are $\hat\theta_n = \bar X_n$ & $\hat\lambda_n = \bar Y_n$

$$\lambda(\underset{\sim}{X},\underset{\sim}{Y}) = \frac{\left\{\prod_{i=1}^n \frac{1}{\theta_0} e^{-X_i/\theta_0}\right\}\left\{\prod_{i=1}^n \frac{1}{\lambda_0} e^{-Y_i/\lambda_0}\right\}}{\left\{\prod_{i=1}^n \frac{1}{\hat\theta_n} e^{-X_i/\hat\theta_n}\right\}\left\{\prod_{i=1}^n \frac{1}{\hat\lambda_n} e^{-Y_i/\hat\lambda_n}\right\}}$$

$$= e^{2n}\left(\frac{\hat\theta_n}{\theta_0}\right)^n\left(\frac{\hat\lambda_n}{\lambda_0}\right)^n \exp\left[-n\left(\frac{\hat\theta_n}{\theta_0}+\frac{\hat\lambda_n}{\lambda_0}\right)\right] \leftarrow (*)$$

Again $Q\equiv\chi^2_2$, $P(\underset{a}{0} \le \chi^2_2 \le \underset{b}{\chi^2_{2,1-\alpha}}) = 1-\alpha$



$C_{\underset{\sim}{X}\underset{\sim}{Y}} = \{(\theta_0,\lambda_0)\in(0,\infty)\times(0,\infty):$
$\underbrace{0 \le -2\log\lambda(\underset{\sim}{X},\underset{\sim}{Y}) \le \chi^2_{2,1-\alpha}}\}$

$$= \left\{ (\omega_0, \lambda_0) \in (0,\infty) \times (0,\infty) : e^{-\frac{1}{2}\chi^2_{2,1-\alpha}} \le \lambda(\underset{\sim}{X}, \underset{\sim}{Y}) \le 1 \right\}$$

$\uparrow$ (*)

$\uparrow$ "take (ω_0, λ_0) having a large likelihood ratio statistic"

Interval Estimation I

Variance Stabilizing Transformations ← another way to get asymptotic pivot

Definition: Let $X_1, \ldots, X_n$ be iid random variables with common pdf/pmf $f(x|\theta)$, where $\theta \in \Theta \subset \mathbb{R}$ (real-valued). Let $\hat{\theta}_n$ be an estimator of θ based on $X_1, \ldots, X_n$ such that

e.g. $\hat{\theta}_n$ could be sample mean or MLE

$$\Rightarrow \quad \sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N\left(0, \sigma^2(\theta)\right) \quad \text{as } n \to \infty$$

← not asymptotic pivot! (variance depends on θ)

for all $\theta \in \Theta$. Then, a function $g : \mathbb{R} \to \mathbb{R}$ is called a **variance stabilizing transformation** (VST) for $\{\hat{\theta}_n\}$ if

$$g'(\theta) \cdot \sigma(\theta) = 1 \quad \text{holds for all } \theta \in \Theta,$$

where g' denotes the derivative of g.

Remark: If g is a VST for $\{\hat{\theta}_n\}$, then by the Delta Method,

$$\sqrt{n}\left(g(\hat{\theta}_n) - g(\theta)\right) \xrightarrow{d} N\left(0, \sigma^2(\theta)[g'(\theta)]^2 = 1\right) \quad \text{as } n \to \infty, \forall \theta,$$

$[g'(\theta)\sigma(\theta)]^2 = [1]^2 = 1$

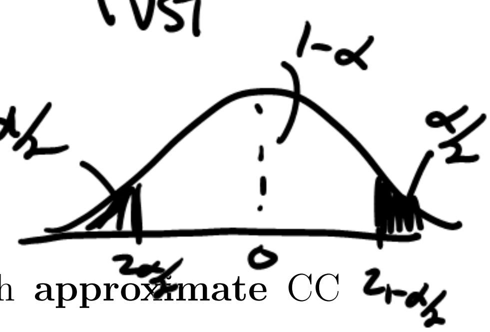
VST

implying that

$$Q_n(X_1, \ldots, X_n, \theta) \equiv \sqrt{n}\left(g(\hat{\theta}_n) - g(\theta)\right)$$

is asymptotically pivotal with $Q \equiv Z \sim N(0, 1)$.

$$P(\underset{z_{\alpha/2}}{a} \le \underset{Z}{Q} \le \underset{z_{1-\alpha/2}}{b}) = 1 - \alpha$$



Then, a large-sample confidence region/interval for θ, with **approximate** CC $(1 - \alpha)$, is

$$\begin{aligned} C_{\underset{\sim}{X}} &= \left\{\theta : \theta \in \Theta,\ z_{\alpha/2} \le Q_n(X_1, \ldots, X_n, \theta) \le z_{1-\alpha/2}\right\} \\ &= \left\{\theta : \theta \in \Theta,\ z_{\alpha/2} \le \sqrt{n}\left(g(\hat{\theta}_n) - g(\theta)\right) \le z_{1-\alpha/2}\right\} \quad \leftarrow \end{aligned}$$

Interval Estimation I

Variance Stabilizing Transformations (VSTs): Illustration

Example 1: Let $X_1, \ldots, X_n$ be iid Poisson(θ), $\theta > 0$. Find a VST based on the MLE/MME $\hat{\theta}_n = \bar{X}_n$ and find a corresponding large-sample CI for θ with approximate C.C. $1 - \alpha$.

By CLT, $(E_\theta X_1 = \theta, Var_\theta(X_1) = \theta)$

$$\sqrt{n}(\bar{X}_n - \theta) \xrightarrow{d} N(0, Var_\theta(X_1) = \theta) \text{ as } n \to \infty.$$

g is VST if $g'(\theta) \cdot \sqrt{Var_\theta(X_1)} = g'(\theta)\sqrt{\theta} = 1$

or $g'(\theta) = 1/\sqrt{\theta}$ for $\theta > 0$.

So, $g(\theta) = \int \frac{1}{\sqrt{\theta}} d\theta$ anti-derivative of $1/\sqrt{\theta} = \theta^{-1/2}$

$= 2\sqrt{\theta} + C$

take $C = 0$

We have

$$\underbrace{\sqrt{n}\left(g(\bar{X}_n) - g(\theta)\right)}_{Q_n(X_1, \ldots, X_n, \theta)} = \sqrt{n}\left(2\sqrt{\bar{X}_n} - 2\sqrt{\theta}\right) \xrightarrow{d} \underbrace{N(0,1)}_{Q}$$