Why does $SS_{Blocks \times Fert} + SS_{Blocks \times Geno \times Fert} = SS_{Error}$?

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- There are no terms in our model corresponding to $Block \times Fert$ combinations; thus, there is no reason to devote a separate line of our ANOVA table to $Block \times Fert$.

- Also, it can be shown that

$$\mathrm{E}(MS_{Blocks \times Fert}) = \mathrm{E}(MS_{Blocks \times Geno \times Fert}) = \sigma_e^2$$

Thus, it makes sense to estimate σ_e^2 with an inverse variance weighted average of independent unbiased estimators:

For this slide only, let

is needed to calculate the weights in the weight average

$$1 = Blocks \times Fert \text{ and } 2 = Blocks \times Geno \times Fert.$$

For $\ell = 1, 2$, $MS_\ell \sim \frac{E(MS_\ell)}{df_\ell}\chi^2_{df_\ell} \implies \text{Var}(MS_\ell) = 2\sigma_e^4/df_\ell.$

$$\frac{\text{Var}^{-1}(MS_1)MS_1 + \text{Var}^{-1}(MS_2)MS_2}{\text{Var}^{-1}(MS_1) + \text{Var}^{-1}(MS_2)} = \frac{\frac{df_1}{2\sigma_e^4}MS_1 + \frac{df_2}{2\sigma_e^4}MS_2}{\frac{df_1}{2\sigma_e^4} + \frac{df_2}{2\sigma_e^4}}$$

$$= \frac{df_1 MS_1 + df_2 MS_2}{df_1 + df_2}$$

$$= \boxed{\frac{SS_1 + SS_2}{df_1 + df_2}}$$

MSE

Thus, we combine the $Blocks \times Fert$ and $Blocks \times Geno \times Fert$ lines of the ANOVA table and label the resulting line as $Error$.

$$SS_{Blocks \times Fert} + SS_{Blocks \times Geno \times Fert} = SS_{Error}$$

$$df_{Blocks \times Fert} + df_{Blocks \times Geno \times Fert} = df_{Error}$$

$$MS_{Error} = SS_{Error} / df_{Error}$$

$$\mathrm{E}(MS_{Error}) = \sigma_e^2$$

Now let's look at the ANOVA table and the analyses that can be done with it in more detail.

For greater generality, let

here $w = 3$ Genotype: A, B, C

- $w =$ the number of levels of the whole-plot treatment factor,

here $s = 4$ 0, 50, 100, 150

- $s =$ the number of levels of the split-plot treatment factor, and
- $b =$ the number of blocks. $b = 4$

ANOVA Table for the Traditional Split-Plot Design

Source	DF
$Blocks$	$b-1$
$Genotypes$	$w-1$
$Blocks \times Geno$	$(b-1)(w-1)$
$Fert$	$s-1$
$Geno \times Fert$	$(w-1)(s-1)$
$Blocks \times Fert$	$(b-1)(s-1)$
$+Blocks \times Geno \times Fert$	$+(b-1)(w-1)(s-1)$
$C.Total$	$bws-1$

Combine

ANOVA Table for the Traditional Split-Plot Design

Source	DF
$Blocks$	$b-1$
$Genotypes$	$w-1$
$Blocks \times Geno$	$(b-1)(w-1)$
$Fert$	$s-1$
$Geno \times Fert$	$(w-1)(s-1)$
$Error$	$w(b-1)(s-1)$
$C.Total$	$bws-1$

ANOVA Table Sums of Squares

block : $k=1,\ldots,b$

$i=1,\ldots,w$

i: genotype

y_{ijk}

j = fertilizer $j=1,\ldots s$

Source		Sum of Squares
$Block$	$(b-1)$	$\sum_{i=1}^{w}\sum_{j=1}^{s}\sum_{k=1}^{b}(\bar{y}_{\cdot\cdot k}-\bar{y}_{\cdots})^2$
$Geno$	$(w-1)$	$\sum_{i=1}^{w}\sum_{j=1}^{s}\sum_{k=1}^{b}(\bar{y}_{i\cdot\cdot}-\bar{y}_{\cdots})^2$
$Block \times Geno$	$(b-1)(w-1)$	$\sum_{i=1}^{w}\sum_{j=1}^{s}\sum_{k=1}^{b}(\bar{y}_{i\cdot k}-\bar{y}_{i\cdot\cdot}-\bar{y}_{\cdot\cdot k}+\bar{y}_{\cdots})^2$
$Fert$		$\sum_{i=1}^{w}\sum_{j=1}^{s}\sum_{k=1}^{b}(\bar{y}_{\cdot j\cdot}-\bar{y}_{\cdots})^2$
$Geno \times Fert$		$\sum_{i=1}^{w}\sum_{j=1}^{s}\sum_{k=1}^{b}(\bar{y}_{ij\cdot}-\bar{y}_{i\cdot\cdot}-\bar{y}_{\cdot j\cdot}+\bar{y}_{\cdots})^2$
$Error$		$\sum_{i=1}^{w}\sum_{j=1}^{s}\sum_{k=1}^{b}(y_{ijk}-\bar{y}_{i\cdot k}-\bar{y}_{ij\cdot}+\bar{y}_{i\cdot\cdot})^2$
$C.Total$		$\sum_{i=1}^{w}\sum_{j=1}^{s}\sum_{k=1}^{b}(y_{ijk}-\bar{y}_{\cdots})^2$

Simplified ANOVA Table Sums of Squares

$$\frac{\text{Sum of squares}}{\text{df}} = MS$$

next derive

$E(MS)$

Source	Sum of Squares
$Block$	$ws\sum_{k=1}^{b}(\bar{y}_{\cdot\cdot k} - \bar{y}_{\cdot\cdot\cdot})^2$
$Geno$	$sb\sum_{i=1}^{w}(\bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot\cdot\cdot})^2$
$Block \times Geno$	$s\sum_{i=1}^{w}\sum_{k=1}^{b}(\bar{y}_{i\cdot k} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot\cdot k} + \bar{y}_{\cdot\cdot\cdot})^2$
$Fert$	$wb\sum_{j=1}^{s}(\bar{y}_{\cdot j\cdot} - \bar{y}_{\cdot\cdot\cdot})^2$
$Geno \times Fert$	$b\sum_{i=1}^{w}\sum_{j=1}^{s}(\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdot\cdot\cdot})^2$
$Error$	$\sum_{i=1}^{w}\sum_{j=1}^{s}\sum_{k=1}^{b}(y_{ijk} - \bar{y}_{i\cdot k} - \bar{y}_{ij\cdot} + \bar{y}_{i\cdot\cdot})^2$
$C.Total$	$\sum_{i=1}^{w}\sum_{j=1}^{s}\sum_{k=1}^{b}(y_{ijk} - \bar{y}_{\cdot\cdot\cdot})^2$

$$E(MS_{Geno}) = \boxed{\frac{sb}{w-1}\sum_{i=1}^{w} E(\bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot\cdot\cdot})^2}$$

df

$\bar{y}_{i\cdot\cdot} = \bar{\mu}_{i\cdot} + \bar{w}_{i\cdot} + \bar{e}_{i\cdot\cdot}$

same $\bar{y}_{\cdot\cdot\cdot}$

note that b_k will cancel itself

$$= \frac{sb}{w-1}\sum_{i=1}^{w} E(\bar{\mu}_{i\cdot} - \bar{\mu}_{\cdot\cdot} + \bar{w}_{i\cdot} - \bar{w}_{\cdot\cdot} + \bar{e}_{i\cdot\cdot} - \bar{e}_{\cdot\cdot\cdot})^2$$

due to $E(w_{ik}) = E(e_{ijk}) = 0$ all crossproducts

$$= sb\left\{\frac{\sum_{i=1}^{w}(\bar{\mu}_{i\cdot} - \bar{\mu}_{\cdot\cdot})^2}{w-1} + \underbrace{\boxed{E\left[\frac{\sum_{i=1}^{w}(\bar{w}_{i\cdot} - \bar{w}_{\cdot\cdot})^2}{w-1}\right]}}_{=\sigma_w^2/b} + E\left[\frac{\sum_{i=1}^{w}(\bar{e}_{i\cdot\cdot} - \bar{e}_{\cdot\cdot\cdot})^2}{w-1}\right]\right\}$$

involving either $\bar{w}_{i\cdot}, \bar{w}_{\cdot\cdot}, \bar{e}_{i\cdot\cdot}, \bar{e}_{\cdot\cdot\cdot}$ will disappear

$$= sb\frac{\sum_{i=1}^{w}(\bar{\mu}_{i\cdot} - \bar{\mu}_{\cdot\cdot})^2}{w-1} + s\cancel{b}\frac{\sigma_w^2}{\cancel{b}} + \cancel{sb}\frac{\sigma_e^2}{\cancel{sb}}$$

$$= sb\frac{\sum_{i=1}^{w}(\bar{\mu}_{i\cdot} - \bar{\mu}_{\cdot\cdot})^2}{w-1} + s\sigma_w^2 + \sigma_e^2$$

$$\mathrm{E}(MS_{Block\times Geno}) = \frac{s}{(w-1)(b-1)}\sum_{i=1}^{w}\sum_{k=1}^{b}\mathrm{E}(\bar{y}_{i\cdot k} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot\cdot k} + \bar{y}_{\cdot\cdot\cdot})^2$$

$$= \frac{s}{(w-1)(b-1)}\sum_{i=1}^{w}\sum_{k=1}^{b}\mathrm{E}(w_{ik} - \bar{w}_{i\cdot} - \bar{w}_{\cdot k} + \bar{w}_{\cdot\cdot} + \bar{e}_{i\cdot k} - \bar{e}_{i\cdot\cdot} - \bar{e}_{\cdot\cdot k} + \bar{e}_{\cdot\cdot\cdot})^2$$

$$= \frac{s}{(w-1)(b-1)}\mathrm{E}\left[\sum_{i=1}^{w}\sum_{k=1}^{b}(w_{ik} - \bar{w}_{i\cdot})^2 - 2\sum_{i=1}^{w}\sum_{k=1}^{b}(w_{ik} - \bar{w}_{i\cdot})(\bar{w}_{\cdot k} - \bar{w}_{\cdot\cdot}) + \sum_{i=1}^{w}\sum_{k=1}^{b}(\bar{w}_{\cdot k} - \bar{w}_{\cdot\cdot})^2 + e^2 \text{ sum}\right]$$

$$= \frac{s}{(w-1)(b-1)}\mathrm{E}\left[\sum_{i=1}^{w}\sum_{k=1}^{b}(w_{ik} - \bar{w}_{i\cdot})^2 - w\sum_{k=1}^{b}(\bar{w}_{\cdot k} - \bar{w}_{\cdot\cdot})^2 + e^2 \text{ sum}\right]$$

$$= \frac{s}{(w-1)(b-1)}[w(b-1)\sigma_w^2 - w(b-1)\sigma_w^2/w + \mathrm{E}(e^2 \text{ sum})]$$

It can be shown that

$$\mathrm{E}(e^2 \text{ sum}) = \mathrm{E}\left[\sum_{i=1}^{w}\sum_{k=1}^{b}(\bar{e}_{i\cdot k} - \bar{e}_{i\cdot\cdot} - \bar{e}_{\cdot\cdot k} + \bar{e}_{\cdot\cdot\cdot})^2\right]$$

$$= \frac{(w-1)(b-1)}{s}\sigma_e^2.$$

Putting it all together yields

$$\mathrm{E}(MS_{Block \times Geno}) = s\sigma_w^2 + \sigma_e^2.$$

Source	Expected Mean Squares
$Block$	
$Geno$	$s\sigma_w^2 + \sigma_e^2 + \frac{sb}{w-1}\sum_{i=1}^{w}(\bar{\mu}_{i\cdot} - \bar{\mu}_{\cdot\cdot})^2$
$Block \times Geno$	$s\sigma_w^2 + \sigma_e^2$
$Fert$	
$Geno \times Fert$	
$Error$	

The Test for Whole-Plot Factor Main Effects

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end lecture
4-9-25

To test for genotype main effects, i.e.,

$$\mathrm{H}_0 : \bar{\mu}_{1\cdot} = \cdots = \bar{\mu}_{w\cdot} \iff \mathrm{H}_0 : \frac{sb}{w-1}\sum_{i=1}^{w}(\bar{\mu}_{i\cdot} - \bar{\mu}_{\cdot\cdot})^2 = 0,$$

compare $\frac{MS_{Geno}}{MS_{Block \times Geno}}$ to a central F distribution with $w-1$ and $(w-1)(b-1)$ degrees of freedom.