STAT 5430
Lecture 13, W, Feb 19

- No new homework!

- Solutions to Homeworks 1-3 posted.

- Exam 1 is scheduled for W, Feb 26 (two weeks)
  6:15-8:15 PM (Sned seminar room 3105)
  - No regular class on W, Feb 26
  - See Canvas for study guide, practice exams
  - can bring 1 page formula sheet (front/back) with anything on it
  - see Canvas for a "canned" sheet
  - I'll provide table with STAT 5420 distributions (see Canvas)

Large Sample Properties of Estimators

Asymptotic ~~Efficiency~~ Efficiency

Recall: For two *unbiased* estimators T and T^* of $\gamma(\theta)$, we compare the *variances* of the estimators to judge their *relative efficiency*, i.e. $RE(T, T^*, \theta) = \mathrm{Var}_\theta(T^*)/\mathrm{Var}_\theta(T)$. ie choose UE with smaller variance

Similarly, we compare large-sample variances of **asymptotically unbiased** estimators.

Definitions: Let $\{T_n^*\}$ and $\{T_n\}$ be asymptotically unbiased for $\gamma(\theta)$. Then, define

$$\lim_{n\to\infty} E_\theta(T_n^*) = \gamma(\theta) = \lim_{n\to\infty} E_\theta(T_n)$$

1. The **asymptotic relative efficiency** of $\{T_n\}$ with respect to $\{T_n^*\}$ at θ is defined as

$$ARE(T_n, T_n^*, \theta) = \lim_{n\to\infty} \frac{\mathrm{Var}_\theta(T_n^*)}{\mathrm{Var}_\theta(T_n)}, \quad \theta \in \Theta$$

compare by "limit of variance ratio"

2. $\{T_n^*\}$ is called **asymptotically efficient** if

$$ARE(T_n, T_n^*, \theta) \leq 1$$

holds $\forall \theta \in \Theta$ for any other $\{T_n\}$ that is asymptotically unbiased for $\gamma(\theta)$.

as $n \to \infty$, T_n^* has smallest variance out of asymp. U.E.

3. The **asymptotic efficiency** of $\{T_n\}$ is defined as

$$AE(T_n, \theta) \equiv ARE(T_n, T_n^*, \theta)$$

if $\{T_n^*\}$ is asymptotically efficient.

Large Sample Properties of Estimators

Asymptotic Effiency

Previous example. Let $X_1, X_2, \dots$ be iid uniform$(0, \theta)$, $\theta > 0$.

(from earlier work with MSE, etc)

Recall

- T_n = MME of θ based on $X_1, \dots, X_n = 2\bar{X}_n$

  (Aside: $\bar{X}_n \xrightarrow{P} E_\theta X_1 = \theta/2$ & $2\bar{X}_n \xrightarrow{P} 2 \cdot \theta/2 = \theta$)

- T_n^* = MLE of θ based on $X_1, \dots, X_n = \max_{1 \le i \le n} X_i = X_{(n)}$

- $\mathrm{E}_\theta T_n = \theta \qquad \mathrm{Var}_\theta(T_n) = \dfrac{3\theta^2}{n}$ ← looks like $\frac{1}{n}$

  MME is UE of θ & so is asymp. unbiased too.

- $\mathrm{E}_\theta T_n^* = \dfrac{n}{n+1}\theta \qquad \mathrm{Var}_\theta(T_n^*) = \dfrac{n\theta^2}{(n+1)^2(n+2)}$ ← looks like $\frac{1}{n^2}$

  The MLE is also <u>asymp. unbiased</u>: $\lim_{n\to\infty} E_\theta(T_n^*) = \theta$.

$$ARE(\underset{\text{MME}}{T_n}, \underset{\text{MLE}}{T_n^*}, \theta) = \lim_{n\to\infty} \frac{Var_\theta(T_n^*)}{Var_\theta(T_n)} = 0.$$

Large Sample Properties of Estimators

Asymptotic Properties of MLEs

Main Result: Let $X_1, X_2, \ldots, X_n$ be *iid* with common pmf/pdf $f(x|\theta)$. Let $\hat{\theta}_n \equiv$ MLE of $\theta \in \Theta \subset \mathbb{R}$ based on $X_1, X_2, \ldots, X_n$. Then, under Cramér-Rao-type regularity conditions: as $n \to \infty$,

(mild conditions allowing for CRLB to hold)

1. (Consistency) $\quad \hat{\theta}_n \xrightarrow{p} \theta$ as $n \to \infty$

we know distribution of MLE in large samples

2. (Asymp. normality) $\quad \sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N\left(0, \dfrac{1}{I_1(\theta)}\right)$, ← normal with mean 0 & variance $\frac{1}{I_1(\theta)}$

$$\text{where } I_1(\theta) = \mathrm{E}_\theta \left(\frac{d \log f(X_1|\theta)}{d\,\theta} \right)^2$$

$$= -E_\theta\left(\frac{d^2 \log f(X_1|\theta)}{d\theta^2}\right)$$

3. (Asymp. efficiency) $\quad$ The sequence of estimators $\{\hat{\theta}_n\}$ is asymptotically efficient. (for estimating θ).

Discussion: From property 2,

$E\hat{\theta}_n \approx \theta$ as $n \to \infty$

So, $\hat{\theta}_n$ is asymp unbiased

If T_n is any U.E of θ,

$$Var_\theta(T_n) \geq \underset{\text{for } \theta}{\text{CRLB}} = \frac{\left[\frac{d\theta}{d\theta}\right]^2}{nI_1(\theta)} = \frac{1}{nI_1(\theta)} \approx Var(\hat{\theta}_n)$$

$\Big[\ \sqrt{n}(\hat{\theta}_n - \theta) \approx Z \sim N\left(0, \frac{1}{I_1(\theta)}\right)$

$\to \hat{\theta}_n \approx \frac{1}{\sqrt{n}} Z + \theta \quad Var(\hat{\theta}) \approx \left(\frac{1}{\sqrt{n}}\right)^2 Var(Z)\ \Big] = \left(\frac{1}{\sqrt{n}}\right)^2 \cdot \frac{1}{I_1(\theta)}$

Large Sample Properties of Estimators

Asymptotic Normality: Delta Method

Remarks:

✓ Same as for CRLB

- The regularity conditions required for the validity of previous Theorem (i.e., asymptotic properties of MLEs) hold for distributions in the one-parameter exponential family: Binomial$(*, \theta)$, Negative Binomial$(*, \theta)$, Poisson(θ), $N(*, \theta)$, $N(\theta, *)$, Gamma$(*, \theta)$, Gamma$(\theta, *)$, etc., where $*$ indicates a given/known second parameter value.

- The following "Delta Method" result, combined with the Theorem above, is useful for finding the distribution of functions of $\hat{\theta}_n$, e.g., $g(\hat{\theta}_n)$. Because the result has wide applicability, we state it in a general form.

Delta Method: For a sequence $\{T_n\}$ of real-valued random variables, suppose it holds that

MLE case: $T_n = \hat{\theta}_n$
$a = \theta$
$\sigma^2(a) = \frac{1}{I_1(\theta)}$

$$\sqrt{n}(T_n - a) \xrightarrow{d} N\big(0, \sigma^2(a)\big) \quad \text{as } n \to \infty$$

for some $a \in \mathbb{R}$. Then for any function $g : \mathbb{R} \to \mathbb{R}$ that is *continuously differentiable at a* with derivative $g'(a) \neq 0$, it holds that

$$\sqrt{n}\Big(g(T_n) - g(a)\Big) \xrightarrow{d} N\Big(0, [g'(a)]^2\sigma^2(a)\Big) \quad \text{as } n \to \infty$$

$g'(a) \neq 0$ at centering value 'a'

Large Sample Properties of Estimators

Asymptotic Normality: Delta Method

$E_\theta(X_1)=\theta$

Example: Let $X_1, X_2, \ldots, X_n$ be iid Poisson(θ), $\theta > 0$.

1. Show that $T_n \equiv$MLE of $\gamma(\theta) = \theta/(1+\theta^2)$ is consistent.
2. Find the limiting distribution of $\sqrt{n}\big(T_n - \gamma(\theta)\big)$.

Solution: Check MLE of θ is $\hat{\theta}_n = \bar{X}_n$

So, $T_n = \gamma(\hat{\theta}_n)$ is MLE of $\gamma(\theta)$.

1. We know $\hat{\theta}_n \xrightarrow{P} \theta$ as $n\to\infty$ (MLE of θ is consistent!)

$\Rightarrow$ since $\gamma(\cdot)$ is continuous,

$T_n = \gamma(\hat{\theta}_n) \xrightarrow{P} \gamma(\theta)$ as $n\to\infty$ (continuous mapping theorem)

2. First, we know $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N(0, \frac{1}{I_1(\theta)})$ as $n\to\infty$. θ

$$I_1(\theta) = E_\theta\left[\left(\frac{d \log f(X_1|\theta)}{d\theta}\right)^2\right] = E_\theta\left[\left(\frac{d\log\left[\frac{e^{-\theta}\theta^{X_1}}{X_1!}\right]}{d\theta}\right)^2\right]$$

$$= E_\theta\left[\left(-1+\frac{X_1}{\theta}\right)^2\right] = \frac{E_\theta[(X_1-\theta)^2]}{\theta^2} = \frac{Var_\theta(X_1)}{\theta^2} = \frac{\theta}{\theta^2} = \frac{1}{\theta}$$

(also, $\sqrt{n}(\bar{X}_n - E_\theta(X_1)) = \sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N(0, Var_\theta(X_1)=\theta)$ as $n\to\infty$

By Delta Method,

$$\sqrt{n}(T_n - \gamma(\theta)) = \sqrt{n}(\gamma(\hat{\theta}_n) - \gamma(\theta)) \to N(0, \theta\cdot[\gamma'(\theta)]^2)$$

as $n\to\infty$ provided $\gamma'(\theta)\neq 0$

Check $\gamma'(\theta) = \frac{1-\theta^2}{(1+\theta^2)^2} \neq 0$ if $\theta \neq 1$ (Note $\theta > 0$) End of Exam Material

ASIDE:

(For $\theta = 1$, can show $\sqrt{n}(\gamma(\hat{\theta}_n) - \gamma(\theta)) \xrightarrow{d} N(0,0) \leftarrow 0$

or $\sqrt{n}(\gamma(\hat{\theta}_n) - \gamma(\theta)) \xrightarrow{P} 0$ as $n\to\infty$.)