STAT 5430

Lec 23, M, Mar 24

- Basu's theorem & Intro to testing → Homework 5 posted, due M, Mar 24 ←
  I'll try to fix gradescope / send email
- MP & UMP testing → Homework 6 posted, due, M, Mar 31
- Exam 2 coming up!
  two + weeks away

Hypothesis Testing I

Illustration of Most Powerful Test (Discrete Case)

discrete case

Example: Let $X_1, \dots, X_n$ be iid Binomial$(2, p)$, $0 < p < 1$. Find a MP test of size α for $H_0 : p = p_0$ vs $H_1 : p = p_1$ (where $0 < p_1 < p_0 < 1$) when

(i) $p_0 = 0.6$, $n = 10$, $\alpha = 0.0003$

(ii) $p_0 = 0.6$, $n = 10$, $\alpha = 0.001$

(iii) p_0, α, n are arbitrary (but fixed)

Solution: joint pmf of $X_1, \dots, X_n$ is

$$f(\underset{\sim}{x}|p) = \prod_{i=1}^{n}\left[\binom{2}{x_i} p^{x_i}(1-p)^{2-x_i}\right]$$

$$= \cancel{\left(\prod_{i=1}^{n}\binom{2}{x_i}\right)} p^{\sum_{i=1}^n x_i}(1-p)^{2n-\sum_{i=1}^n x_i}$$

$$f(\underset{\sim}{x}|p_1) \gtreqless k f(\underset{\sim}{x}|p_0)$$

$$\Leftrightarrow p_1^{\sum x_i}(1-p_1)^{2n-\sum x_i} \gtreqless k\, p_0^{\sum x_i}(1-p_0)^{2n-\sum x_i}$$

$$\Leftrightarrow (\textstyle\sum x_i)\log p_1 + (2n-\sum x_i)\log(1-p_1) \gtreqless \log(k) + (\sum x_i)\log p_0 + (2n-\sum x_i)\log(1-p_0)$$

$$\Leftrightarrow \left(\sum_{i=1}^n x_i\right)\overbrace{\log\Bigg[\underbrace{\frac{p_1(1-p_0)}{p_0(1-p_1)}}_{<1 \text{ since } 0<p_1<p_0}\Bigg]}^{<0} \gtreqless \underbrace{\log(k) + \log\left[\left(\frac{1-p_0}{1-p_1}\right)^{2n}\right]}_{\equiv k_1}$$

$$\Leftrightarrow \left(\sum_{i=1}^n x_i\right) \lesseqgtr k_1\left[\log\left(\frac{p_1(1-p_0)}{p_0(1-p_1)}\right)\right]^{-1} \equiv k_2$$

Hence, a MP test of size α for
$H_0: p=p_0$ vs $H_1: p=p_1$ $(p_1<p_0)$ is

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \sum_{i=1}^n x_i < k_2 \\ r & \text{if } \sum_{i=1}^n x_i = k_2 \\ 0 & \text{if } \sum_{i=1}^n x_i > k_2 \end{cases}$$

where $r \in [0,1]$ & k_2 are chosen so that

$$\alpha = E_{p_0}\phi(\underset{\sim}{x}) = P_{p_0}\left(\sum_{i=1}^n x_i < k_2\right) + r P_{p_0}\left(\sum_{i=1}^n x_i = k_2\right) \leftarrow (**)$$

If the parameter is p_0, $\sum_{i=1}^n X_i \sim \text{Binomial}(2n, p_0)$

For $p_0=0.6$, $n=10$, we have
$$\begin{cases} P_{0.6}\left(\sum_{i=1}^n x_i < 5\right) = 0.0003 \\ P_{0.6}\left(\sum_{i=1}^n x_i < 6\right) = 0.0016 \end{cases}$$

(a) $\alpha = 0.0003$, $n=10$, $p_0=0.6$

pick $k_2=5$, $r=0$ so that $(**)$ holds with $\alpha = 0.0003$

So the MP test of size 0.0003 is

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \sum_{i=1}^n x_i < 5 \\ 0 & \text{o.w} \end{cases} = \begin{cases} 1 & \text{if } \sum_{i=1}^n x_i \le 4 \\ 0 & \text{o.w} \end{cases}$$

(b) $\alpha = 0.001$, $n=10$, $p_0=0.6$

$$\underset{\alpha}{0.001} = \underbrace{P_{0.6}\left(\sum_{i=1}^n x_i < 5\right)}_{0.0003} + r \underbrace{P_{0.6}\left(\sum_{i=1}^n x_i = 5\right)}_{P_{0.6}(\sum x_i < 6) - P_{0.6}(\sum x_i < 5) = 0.0016 - 0.0003 = 0.0013}$$

$\Rightarrow r = 7/13$

So, MP test of size $\alpha = 0.001$ is

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} x_i < 5 \\ 7/13 & \text{if } \sum_{i=1}^{n} x_i = 5 \\ 0 & \text{if } \sum_{i=1}^{n} x_i > 5 \end{cases}$$

(c) general n, p_0, α pick k_2 so that

$$P_{p_0}\left(\sum_{i=1}^{n} X_i < k_2\right) \le \alpha < P_{p_0}\left(\sum_{i=1}^{n} X_i < k_2 + 1\right)$$

& take $$\gamma = \frac{\alpha - P_{p_0}\left(\sum_{i=1}^{n} X_i < k_2\right)}{P_{p_0}\left(\sum_{i=1}^{n} X_i = k_2\right)}$$