

STAT 5430

Lec 26, M, Mar 31

MP & UMP testing → Homework 6 posted, due, M, Mar 31
Homework 7 posted, due, M, Apr 6
↑ testing

- Exam 2 is coming up (3 weeks away)
on W, April 16, 6:15-8:15 PM, 3rd floor seminar room
- No class on that W.
- I'll post:
 - ✓ study guide (sufficiency/completeness/tests)
 - practice exams
 - ✓ bring new 1 page (front/back)
formula sheet on exam 2 material
(I'll post one to use if you'd like)
 - can bring calculator & previous formula sheet for exam 1
 - I'll provide table of distributions /
STAT 542 facts on test as before

STAT 5430: Summary to date

Where we have been & where we are headed

- Completed
 - Introduction to Statistical Inference
 - Point Estimation
 - * MME/MLE
 - Criteria for Evaluating Point Estimators
 - * bias, variance, UMVUE, MSE
 - Elements of Decision Theory
 - * Minimax, finding Bayes estimators
 - Sufficiency and Point Estimation
 - * Factorization/Rao-Blackwell/Lehman-Scheffe Theorems
 - Hypothesis Testing I
 - * Size, Power, Most Powerful Tests, Uniformly Most Powerful Texts
- Next: Hypothesis Testing II
 - Likelihood Ratio Tests
 - Bayes Tests

Hypothesis Testing II

Background

We've seen a few approaches for creating “best” tests (i.e., optimal in power):

for a given size α

1. Neyman-Pearson Lemma

(“most powerful” tests of $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$)

2. Two methods for “uniformly most powerful” (UMP) tests

- Method I: Based on Neyman-Pearson Lemma

(may work for \mathbb{R}^p -valued parameters θ and tests of $H_0 : \theta \in \Theta_0 \subset \mathbb{R}^p$ vs $H_1 : \theta \notin \Theta_0$)

- Method II: Monotone Likelihood Ratio

(may work for real-valued $\theta \in \mathbb{R}$ and tests “ $H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$ ” or “ $H_0 : \theta \geq \theta_0$ vs $H_1 : \theta < \theta_0$ ”)

These “best tests” are designed for certain hypothesis structures (e.g., simple H_0 vs simple H_1) and UMP tests often don't exist outside of one-sided hypotheses (e.g., $H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$).

Nevertheless, these tests more broadly suggest that **good test rules involve comparing likelihoods**.

We next consider two further procedures for creating tests based on likelihoods:

- Likelihood Ratio Tests (LRT)
 - & Bayes Tests.
- } general testing procedures

Note: there are no general guarantees that LRT or Bayes tests are “optimal” in terms of power, but these tests are broadly applicable and can perform well in practice.

Hypothesis Testing II

Likelihood Ratio Tests

Definition: Let $f(\underline{x}|\theta)$, $\theta \in \Theta \subset \mathbb{R}^p$, be the joint pdf/pmf of $\underline{X} = (X_1, \dots, X_n)$ (the parameter θ can be vector-valued) and let Θ_0 be a nonempty proper subset of Θ . Then, the **likelihood ratio statistic** (LRS) for testing $H_0 : \theta \in \Theta_0 \subset \mathbb{R}^p$ vs $H_1 : \theta \notin \Theta_0$ is defined as

$$L(\theta) \equiv f(\underline{x}|\theta) \quad \text{likelihood}$$

$$\lambda(\underline{x}) = \frac{\max_{\theta \in \Theta_0} f(\underline{x}|\theta)}{\max_{\theta \in \Theta} f(\underline{x}|\theta)} \leftarrow \frac{\max L(\theta) \text{ over } \Theta_0}{\max L(\theta) \text{ over } \Theta}$$

Comments on interpreting $\lambda(\underline{x})$

1. if $\hat{\theta} \equiv \text{MLE of } \theta \text{ over entire } \Theta$ & $\tilde{\theta} \equiv \text{maximum of } f(\underline{x}|\theta) \text{ over } \theta \in \Theta_0$,

\uparrow maximizes $L(\theta)$ over Θ

here $L(\tilde{\theta}) \leq L(\hat{\theta})$

$$\lambda(\underline{x}) = \frac{f(\underline{x}|\tilde{\theta})}{f(\underline{x}|\hat{\theta})} = \frac{L(\tilde{\theta})}{L(\hat{\theta})} \in [0, 1]$$

2. if H_0 is true, roughly expect $\lambda(\underline{x})$ to be large " (ie, close to 1)

roughly, $\hat{\theta} \approx \theta \in \Theta_0$ & $\tilde{\theta} \approx \theta$ too

& so expect $\lambda(\underline{x}) \approx 1$

3. if H_0 is false, roughly expect $\lambda(\underline{x})$ to be small " (ie, close to 0)

roughly, $\hat{\theta} \approx \theta \notin \Theta_0$ if H_0 is false

& so expect $L(\tilde{\theta})$ to be small compared to $L(\hat{\theta})$

Definition: A size α **likelihood ratio test** (LRT) for testing $H_0 : \theta \in \Theta_0 \subset \mathbb{R}^p$ vs $H_1 : \theta \notin \Theta_0$ is defined as

$$\varphi(\underline{x}) = \begin{cases} 1 & \text{if } \lambda(\underline{x}) < k \\ \gamma & \text{if } \lambda(\underline{x}) = k \\ 0 & \text{if } \lambda(\underline{x}) > k \end{cases}$$

(reject H_0 if LRS $\lambda(\underline{x})$ is too small)

(don't reject if LRS $\lambda(\underline{x})$ is "big")

where $\gamma \in [0, 1]$, $0 \leq k \leq 1$ are constants determined by $\max_{\theta \in \Theta_0} E_{\theta} \varphi(\underline{X}) = \alpha$.

need test to have size α

Hypothesis Testing II

Likelihood Ratio Tests: Illustration

$$\frac{d \log L(\theta)}{d\theta} = \frac{\sum x_i}{\theta^2} - \frac{3n}{\theta} = 0$$

Example: Let X_1, \dots, X_n be iid $\text{Gamma}(\alpha = 3, \theta)$, $\theta > 0$. Find a size α LRT for $H_0: \theta = \theta_0$ vs $H_1: \theta \neq \theta_0$.

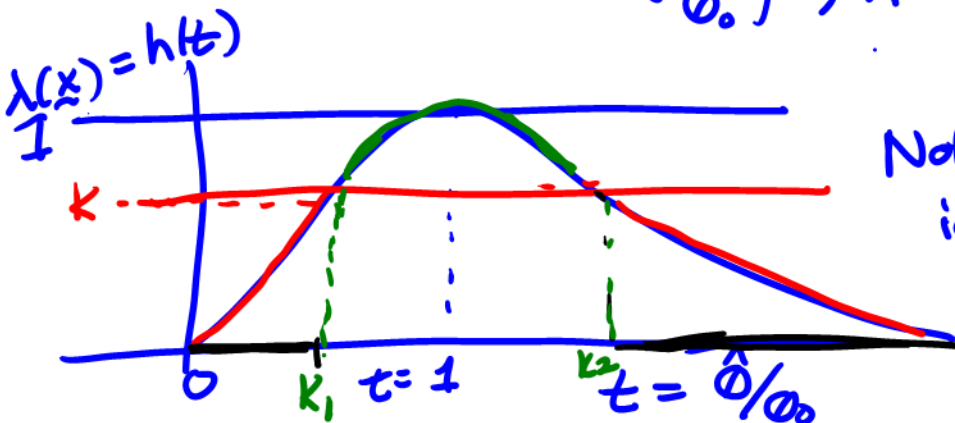
↑ θ_0 is fixed/given value ($\theta_0 = 1$)

Solution: $L(\theta) = f(\underline{x}|\theta) = \prod_{i=1}^n \left(\frac{x_i^2 e^{-x_i/\theta}}{2\theta^3} \right) = \frac{(\prod_{i=1}^n x_i^2) e^{-\sum x_i/\theta}}{2^n \theta^{3n}}, \theta > 0$

check $\hat{\theta} \equiv \text{MLE of } \theta$ is $\sum_{i=1}^n x_i / 3n = \bar{X}_n / 3$

$$\begin{aligned} \lambda(\underline{x}) &= \text{LRS of "H}_0: \theta = \theta_0 \text{ vs H}_1: \theta \neq \theta_0\text{"} \\ &= \frac{L(\theta_0)}{L(\hat{\theta})} = \frac{(\prod_{i=1}^n x_i^2) e^{-\sum x_i/\theta_0}}{2^n \theta_0^{3n}} \cdot \frac{2^n \hat{\theta}^{3n}}{(\prod_{i=1}^n x_i^2) e^{-\sum x_i/\hat{\theta}}} \\ &= e^{-3n\hat{\theta}/\theta_0} e^{3n} \left(\frac{\hat{\theta}}{\theta_0} \right)^{3n} \\ &= h\left(\frac{\hat{\theta}}{\theta_0}\right), \text{ where } h(t) = e^{-3nt + 3n} t^{3n}, t \geq 0. \end{aligned}$$

$\sum x_i = 3n\hat{\theta}$



Note: $h'(t) = 0$ at $t=1$
 i.e. $h(t)$ has a max at $t=1$
 $\& h(0) = 0 = \lim_{t \rightarrow \infty} h(t)$

$$h(t) > k \Leftrightarrow \hat{\theta}/\theta_0 \in (k_1, k_2) \quad \text{or} \quad h(t) < k \Leftrightarrow \frac{\hat{\theta}}{\theta_0} \notin (k_1, k_2)$$

Hence, the LRT is given by

$$\phi(\underline{x}) = \begin{cases} 1 & \text{if } \lambda(\underline{x}) < k \\ \gamma & \text{if } \lambda(\underline{x}) = k \\ 0 & \text{if } \lambda(\underline{x}) > k \end{cases} = \begin{cases} 1 & \text{if } \hat{\theta}/\hat{\sigma}_0 \notin (k_1, k_2) \\ \gamma & \text{if } \hat{\theta}/\hat{\sigma}_0 \in \{k_1, k_2\} \\ 0 & \text{if } \hat{\theta}/\hat{\sigma}_0 \in (k_1, k_2) \end{cases}$$

$\hat{\theta} \leftarrow \text{MLE}$