Ph.D. PRELIMINARY EXAMINATION

March 18, 2003

PART I: Theory
(Co-majors)

1. Suppose X is a continuous random variable.

   (a) Show that $P(X^2 \leq x) = P(X^2 \leq x | X > 0)$ if X has the same distribution as $-X$.

   (b) Show that X_n^2 converges in distribution to X^2 whenever $\{X_n\}$ is a sequence of continuous random variables that converges in distribution to X.

2. Suppose X_{ij} has a normal distribution with mean μ_i and variance σ^2 for $i = 1, 2$ and $j = 1, 2, \ldots, n$. Furthermore suppose all $2n$ random variables are independent and that σ^2 is known. Consider the statistic

$$W = n\sum_{i=1}^{2}(\bar{X}_i - \tilde{X}_i)^2/\sigma^2 \text{ where, for } i = 1, 2;\ \bar{X}_i = \sum_{j=1}^{n} X_{ij}/n \text{ and } \tilde{X}_i = \begin{cases} \bar{X}_i & \text{if } \bar{X}_1 \leq \bar{X}_2 \\ (\bar{X}_1 + \bar{X}_2)/2 & \text{if } \bar{X}_1 > \bar{X}_2 \end{cases}.$$

   This statistic can be used to test $H_0 : \mu_1 \leq \mu_2$ against $H_A : \mu_1 > \mu_2$.

   (a) Using 1(a) or otherwise, show that the distribution of W, when $\mu_1 = \mu_2$, is

$$P(W \leq w) = \begin{cases} 0 & \text{if } w < 0 \\ \frac{1}{2} + \frac{1}{2}P(\chi^2(1) \leq w) & \text{if } w \geq 0 \end{cases}$$

   where $\chi^2(1)$ denotes a chi-squared random variable with 1 degree of freedom.

   (b) Find the probability that $W > 3.84$ when $\mu_1 = \mu_2$. [Note: 3.84 is approximately the 0.95 quantile of a chi-squared distribution with 1 degree of freedom.]

   (c) Find the mean of W when $\mu_1 = \mu_2$.

   (d) Find the variance of W when $\mu_1 = \mu_2$.

3. Suppose the conditions of part 2 hold except that σ^2 is unknown. For the case $\mu_1 = \mu_2$, find the distribution of

$$V = \frac{\sigma^2}{S^2}W \text{ where } S^2 = \frac{1}{2(n-1)}\sum_{i=1}^{2}\sum_{j=1}^{n}(X_{ij} - \bar{X}_i)^2$$

   and W is as defined in part 2.

4. Now suppose X_{ij} has density $\frac{1}{\sigma}f\left(\frac{x-\mu_i}{\sigma}\right)$ for $i = 1, 2$ and $j = 1, 2, \ldots, n$ where $f(x)$ denotes the probability density function of an unspecified continuous distribution with mean 0 and variance 1. Furthermore suppose V and S are as defined in part 3, all $2n$ random variables are independent, and σ^2/S^2 converges in probability to 1 as $n \to \infty$. Derive the limiting distribution of V as $n \to \infty$ for the case $\mu_1 = \mu_2$.

1. Suppose X is a continuous random variable.

   (a) Show that $P(X^2 \le x) = P(X^2 \le x | X > 0)$ if X has the same distribution as $-X$.

$$
\begin{aligned}
P(X^2 \le x) &= P(X^2 \le x|X<0)\cdot P(X<0) + P(X^2 \le x|X>0)\cdot P(X>0) \\
&= P[(-X)^2 \le x|(-X)<0]\cdot P(X<0) + P(X^2 \le x|X>0)\cdot P(X>0) \\
&= P(X^2 \le x|X>0)\cdot P(X<0) + P(X^2 \le x|X>0)\cdot P(X>0) \\
&= P(X^2 \le x|X>0)\cdot [P(X<0)+P(X>0)] \\
&= P(X^2 \le x|X>0)
\end{aligned}
$$

   (b) Show that X_n^2 converges in distribution to X^2 whenever $\{X_n\}$ is a sequence of continuous random variables that converges in distribution to X.

$$
\begin{aligned}
\lim_{n\to\infty} P(X_n^2 \le t) &= \lim_{n\to\infty} P(-\sqrt{t} \le X_n \le \sqrt{t}) = \lim_{n\to\infty} P(X_n \le \sqrt{t}) - \lim_{n\to\infty} P(X_n \le -\sqrt{t}) \\
&= P(X \le \sqrt{t}) - P(X \le -\sqrt{t}) = P(-\sqrt{t} \le X \le \sqrt{t}) = P(X^2 \le t)
\end{aligned}
$$

2. Suppose X_{ij} has a normal distribution with mean μ_i and variance σ^2 for $i = 1,2$ and $j = 1,2,\ldots,n$. Furthermore suppose all $2n$ random variables are independent and that σ^2 is known. Consider the statistic

$$W = n\sum_{i=1}^{2}(\bar{X}_i - \tilde{X}_i)^2/\sigma^2 \text{ where, for } i=1,2;\ \bar{X}_i = \sum_{j=1}^{n} X_{ij}/n \text{ and } \tilde{X}_i = \begin{cases} \bar{X}_i & \text{if } \bar{X}_1 \le \bar{X}_2 \\ (\bar{X}_1+\bar{X}_2)/2 & \text{if } \bar{X}_1 > \bar{X}_2 \end{cases}.$$

This statistic can be used to test $H_0 : \mu_1 \le \mu_2$ against $H_A : \mu_1 > \mu_2$.

   (a) Using 1(a) or otherwise, show that the distribution of W, when $\mu_1 = \mu_2$, is

$$P(W \le w) = \begin{cases} 0 & \text{if } w < 0 \\ \frac{1}{2} + \frac{1}{2}P(\chi^2(1) \le w) & \text{if } w \ge 0 \end{cases}$$

   where $\chi^2(1)$ denotes a chi-squared random variable with 1 degree of freedom.
   First note that $W = 0$ when $\bar{X}_1 \le \bar{X}_2$, and

$$
\begin{aligned}
W &= n[\{\bar{X}_1 - (\bar{X}_1+\bar{X}_2)/2\}^2 + \{\bar{X}_2 - (\bar{X}_1+\bar{X}_2)/2\}^2]/\sigma^2 \\
&= \frac{n}{2\sigma^2}(\bar{X}_1 - \bar{X}_2)^2
\end{aligned}
$$

   when $\bar{X}_1 > \bar{X}_2$. Thus we have

$$
\begin{aligned}
P(W \le w) &= P(W \le w|\bar{X}_1 \le \bar{X}_2)\cdot P(\bar{X}_1 \le \bar{X}_2) + P(W \le w|\bar{X}_1 > \bar{X}_2)\cdot P(\bar{X}_1 > \bar{X}_2) \\
&= P(W \le w|\bar{X}_1 \le \bar{X}_2)\cdot 0.5 + P(W \le w|\bar{X}_1 > \bar{X}_2)\cdot 0.5 \\
&= P(0 \le w|\bar{X}_1 \le \bar{X}_2)\cdot 0.5 + P(\frac{n}{2\sigma^2}(\bar{X}_1 - \bar{X}_2)^2 \le w|\bar{X}_1 > \bar{X}_2)\cdot 0.5 \\
&= 1(w \ge 0)\cdot 0.5 + P(\frac{n}{2\sigma^2}(\bar{X}_1 - \bar{X}_2)^2 \le w|\sqrt{\frac{n}{2\sigma^2}}(\bar{X}_1 - \bar{X}_2) > 0)\cdot 0.5 \\
&= 1(w \ge 0)\cdot 0.5 + P(\frac{n}{2\sigma^2}(\bar{X}_1 - \bar{X}_2)^2 \le w)\cdot 0.5
\end{aligned}
$$

where the last equality follows from part 1(a).

Now note that $\frac{n}{2\sigma^2}(\bar{X}_1 - \bar{X}_2)^2$ *has a chi-squared distribution with 1 d.f. because* $\bar{X}_1 - \bar{X}_2 \sim N(0, 2\sigma^2/n)$. *Thus* W *is an equal mixture of a distribution degenerate at 0 and a chi-squared distribution with 1 d.f. when* $\mu_1 = \mu_2$.

(b) Find the probability that $W > 3.84$ when $\mu_1 = \mu_2$. [Note: 3.84 is approximately the 0.95 quantile of a chi-squared distribution with 1 degree of freedom.]

Let U *denote a chi-squared random variable with 1 d.f.*

$$P(W > 3.84) = 1 - P(W \leq 3.84) = 1 - [0.5 + 0.5 \cdot P(U \leq 3.84)] \approx 1 - [0.5 + 0.5 \cdot (0.95)] = 0.025$$

(c) Find the mean of W when $\mu_1 = \mu_2$.

Note that W *has the same distribution as* $B \cdot U$ *where* B *is a Bernoulli random variable with success probability 0.5 independent of* U*, a chi-squared random variable with 1 d.f. Thus*

$$E(W) = E(BU) = E(B)E(U) = (0.5) \cdot (1) = 0.5.$$

(d) Find the variance of W when $\mu_1 = \mu_2$.

$$\mathrm{Var}(W) = E(W^2) - E(W)^2 = E(B^2)E(U^2) - 0.25 = (0.5) \cdot (2 + 1) - 0.25 = 1.25.$$

3. Suppose the conditions of part 2 hold except that σ^2 is unknown. For the case $\mu_1 = \mu_2$, find the distribution of

$$V = \frac{\sigma^2}{S^2} W \text{ where } S^2 = \frac{1}{2(n-1)} \sum_{i=1}^{2} \sum_{j=1}^{n} (X_{ij} - \bar{X}_i)^2$$

and W is as defined in part 2.

The same argument used in part 2(a) implies that the distribution of V *is an equal mixture of a distribution degenerate at 0 and the distribution of*

$$\frac{n}{2S^2}(\bar{X}_1 - \bar{X}_2)^2 = \frac{\frac{n}{2\sigma^2}(\bar{X}_1 - \bar{X}_2)^2}{S^2/\sigma^2}.$$

As argued previously, the numerator is a chi-squared random variable with 1 d.f. The denominator is a chi-squared random variable divided by its degrees of freedom $(2n - 2)$*. Furthermore, by the independence of the sample mean and sample variance for normal samples, the denominator is independent of the numerator. Thus the distribution of* V *is an equal mixture of a distribution degenerate at 0 and an* F*-distribution with 1 and* $2n - 2$ *d.f. when* $\mu_1 = \mu_2$.

4. Now suppose X_{ij} has density $\frac{1}{\sigma}f\left(\frac{x-\mu_i}{\sigma}\right)$ for $i = 1, 2$ and $j = 1, 2, \ldots, n$ where $f(x)$ denotes the probability density function of an unspecified continuous distribution with mean 0 and variance 1. Furthermore suppose V and S are as defined in part 3, all $2n$ random variables are independent, and σ^2/S^2 converges in probability to 1 as $n \to \infty$. Derive the limiting distribution of V as $n \to \infty$ for the case $\mu_1 = \mu_2$.

(The following is not the most direct proof, but it is one proof that uses only facts contained in the first 5 chapters of Casella and Berger.)

The same argument used in part 2(a) implies that the distribution of V is an equal mixture of a distribution degenerate at 0 and the distribution of

$$\frac{n}{2S^2}(\bar{X}_1 - \bar{X}_2)^2 = \frac{\sigma^2}{S^2}\left(\frac{\sqrt{n}}{\sigma}\frac{1}{n}\sum_{j=1}^{n}\frac{X_{1j} - X_{2j}}{\sqrt{2}}\right)^2 = \frac{\sigma^2}{S^2}(\sqrt{n}\bar{Y}/\sigma)^2$$

where $\bar{Y} = \frac{1}{n}\sum_{j=1}^{n} Y_j$ *and* $Y_j = \frac{X_{1j}-X_{2j}}{\sqrt{2}}$ *for* $j = 1, \ldots, n$.

By the Central Limit Theorem, $\sqrt{n}\bar{Y}/\sigma$ *converges in distribution to* $N(0,1)$. *Thus* $(\sqrt{n}\bar{Y}/\sigma)^2$ *converges in distribution to a chi-squared distribution with 1 d.f. by part 1(b). Because* σ^2/S^2 *converges in probability to 1, Slutsky's Theorem implies that*

$$\frac{\sigma^2}{S^2}(\sqrt{n}\bar{Y}/\sigma)^2 = \frac{n}{2S^2}(\bar{X}_1 - \bar{X}_2)^2$$

converges in distribution to a chi-squared distribution with 1 d.f. Thus the asymptotic distribution of V is an equal mixture of a distribution degenerate at 0 and a chi-squared distribution with 1 d.f. when $\mu_1 = \mu_2$.

Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ $(n \geq 3)$ be iid observations from a bivariate normal distribution with unknown parameters μ_X, μ_Y, $\sigma_X^2 > 0$, $\sigma_Y^2 > 0$ and $-1 < \rho < 1$, with probability density function

$$f(x,y;\mu_X,\mu_Y,\sigma_X^2,\sigma_Y^2,\rho)$$
$$= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right)\right)$$

for $-\infty < x < \infty$ and $-\infty < y < \infty$.

1. Find the conditional distribution of Y_i given $X_i = x$.
2. Suppose that Y is the response variable and X is the explanatory variable. Show that Y_i and X_i have the relationship
$$Y_i = \beta + \gamma X_i + \varepsilon_i,$$
where β and γ are constants, ε_i is normally distributed, and X_i and ε_i are independent. Identify β and γ in terms of the original parameters.
3. Let $\delta^2 = \sigma_Y^2(1-\rho^2)$. Using Parts 1 and 2, write down the likelihood function of $(\beta, \gamma, \mu_X, \sigma_X^2, \delta^2)$ at the observed data.
4. Find the maximum likelihood estimator of γ (as defined in Part 2).
5. Give a test of size α $(0 < \alpha < 1)$ for $H_0 : \gamma \leq 0$ vs $H_1 : \gamma > 0$. (No detail in derivation is required here.)
6. Explain why your test in Part 5 indeed has size α. (Again, no detail in derivation is required.)

Suppose that from now on we are interested in interval estimation for $\mu_W = \mu_X - \mu_Y$. Let $W_i = X_i - Y_i$, $1 \leq i \leq n$.

1. Find a pivotal quantity in terms of μ_W and the first two sample moments of $W_1, \ldots, W_n$.
2. Construct a $1-\alpha$ $(0 < \alpha < 1)$ two-sided confidence interval for μ_W (with the shortest length) using the pivotal quantity.
3. Suppose that it is known $\rho = 0$ and $\sigma_X^2 = \sigma_Y^2$. Give an alternative $1-\alpha$ two-sided confidence interval for μ_W.
4. Compare the two confidence intervals when $\rho = 0$ and $\sigma_X^2 = \sigma_Y^2$.

542-543 (II) Solution

1. Following a standard calculation, we have

$$Y_i \mid X_i = x \sim N\left(\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X),\ (1-\rho^2)\sigma_Y^2\right).$$

2. Let $\beta = \mu_Y - \rho\frac{\sigma_Y}{\sigma_X}\mu_X$, $\gamma = \rho\frac{\sigma_X}{\sigma_Y}$. Then

$$Y_i - (\beta - \gamma X_i)\Big|_{X_i = x} \sim N(0, (1-\rho^2)\sigma_Y^2).$$

Since the conditional distribution does not depend on x, we know $\varepsilon_i \triangleq Y_i - (\beta - \gamma X_i)$ is independent of X_i.

3. From Parts 1 and 2, the joint pdf of (X_i, Y_i) can be rewritten as

$$\frac{1}{\sqrt{2\pi}\,\sigma_X} e^{-\frac{1}{2\sigma_X^2}(x-\mu_X)^2} \cdot \frac{1}{\sqrt{2\pi\sigma_Y^2(1-\rho^2)}} e^{-\frac{1}{2(1-\rho^2)\sigma_Y^2}(y-\beta-\gamma x)^2}.$$

Thus

$$L(\beta, \gamma, \mu_X, \sigma_X^2, \delta^2) = \left(\frac{1}{\sqrt{2\pi}\,\sigma_X}\right)^n \left(\frac{1}{\sqrt{2\pi\,\cancel{\ }\,\delta^2}}\right)^n \exp\left\{-\frac{1}{2\sigma_X^2}\sum_{i=1}^n (x_i-\mu_X)^2 - \frac{1}{2\delta^2}\sum_{i=1}^n (y_i-\beta-\gamma x_i)^2\right\}.$$

4. To maximize the likelihood, as far as β and γ are concerned, we need to minimize $\sum_{i=1}^n (Y_i - \beta - \gamma X_i)^2$, which gives the familiar LSE:

$$\hat{\gamma} = \frac{\Sigma(X_i - \bar{X}_n)(Y_i - \bar{Y}_n)}{\Sigma(X_i - \bar{X}_n)^2}, \quad \hat{\beta} = \bar{Y}_n - \hat{\gamma}\bar{X}_n.$$

5. Let $T = \dfrac{\hat{\gamma}}{\sqrt{\dfrac{S^2}{\sum_{i=1}^n (X_i - \bar{X}_n)^2}}}$, where $S^2 = \frac{1}{n-2}\sum_{i=1}^n (Y_i - \hat{\beta} - \hat{\gamma}X_i)^2$.

A familiar test is $\phi(\underset{\sim}{x}, \underset{\sim}{y}) = \begin{cases} 1 & \text{when } T > t_{n-2,\alpha} \\ 0 & \text{otherwise.} \end{cases}$

6. Conditional on $\underset{\sim}{X}$, we know $\hat{\gamma}$ has a normal distribution, S^2 basically has a χ^2 distribution, and $\hat{\gamma}$ and S^2 are independent, and then under H_0,

$$\frac{\hat{\gamma}-\gamma}{\sqrt{\frac{S^2}{\Sigma(X_i-\bar{X}_n)^2}}} \sim t_{n-2}.$$

Since the conditional distribution does not depend on $\underset{\sim}{X}$, the unconditional distribution is also t_{n-2}. Let $\theta=(\beta, \gamma, \sigma_X^2, \mu_X, \delta^2)$.

For $\gamma \le 0$,

$$P_\theta(T \ge t_{n-2,\alpha}) = P_\theta\left(\frac{\hat{\gamma}-\gamma}{\sqrt{\frac{S^2}{\Sigma(X_i-\bar{X}_n)^2}}} \ge \frac{-\gamma}{\sqrt{\frac{S^2}{\Sigma(X_i-\bar{X}_n)^2}}} + t_{n-2,\alpha}\right)$$

$$\le P_\theta\left(\frac{\hat{\gamma}-\gamma}{\sqrt{\frac{S^2}{\Sigma(X_i-\bar{X}_n)^2}}} \ge t_{n-2,\alpha}\right) = \alpha,$$

with equality when $\gamma=0$.

7. Let $U = \dfrac{\bar{W}-\mu_W}{\sqrt{\frac{1}{n}S_W^2}}$, where $\bar{W}$ and S_W^2 are the sample mean and sample variances of $W_1, \dots, W_n$, respectively. Note that $U \sim t_{n-1}$. So U is a pivotal quantity.

8. Based on the fact that t distribution is unimodal and symmetric about 0, a U-based $1-\alpha$ two-sided CI for μ_W with the shortest length is

$$\bar{W} - t_{n-1,\alpha/2}\cdot\frac{S_W}{\sqrt{n}} \le \mu_W \le \bar{W} + t_{n-1,\alpha/2}\frac{S_W}{\sqrt{n}}.$$

9. $$V = \frac{\bar{X}_n - \bar{Y}_n - \mu_w}{\sqrt{\frac{2}{n} S_p^2}} \sim t_{2n-2}, \quad \text{where}$$

$$S_p^2 = \frac{S_X^2 + S_Y^2}{2n-2}.$$ So V is a pivotal quantity.

Based on V, we have an alternative CI for μ_w:

$$\bar{X}_n - \bar{Y}_n - t_{2n-2,\alpha/2} \cdot \frac{S_p}{\sqrt{n/2}} \le \mu_w \le \bar{X}_n - \bar{Y}_n + t_{2n-2,\alpha/2} \cdot \frac{S_p}{\sqrt{n/2}}.$$

10. Both confidence intervals have the same center.

Note $E S_p^2 = \sigma^2$ (the common variance)

$$E S_w^2 = Var(X_1 - Y_1) = 2\sigma^2.$$

So we expect $\frac{S_p}{\sqrt{n/2}}$ to be close to $\frac{S_w}{\sqrt{n}}$.

Then we would prefer the CI in Part 9 because of the larger degrees of freedom in the t distribution (which yields a shorter length of the CI).

Suppose that $(X_1, \ldots, X_n)$ is a random sample from the distribution with probability density function f_θ, where $\theta > 0$ and

$$f_\theta(x) = \frac{\theta}{2} e^{-\theta|x|}, \qquad -\infty < x < \infty.$$

For this question, you may use the following facts without proving them.

- $E(|X_1|) = 1/\theta$ and $Var(|X_1|) = 1/\theta^2$.
- $\sum_{i=1}^n |X_i|$ is complete for θ.
- Suppose that the distribution of X is $Gamma(a, b)$, that is, X has probability density function
$$f(x) = \frac{1}{\Gamma(a)b^a} x^{a-1} e^{-x/b}.$$
Then the moment generating function for X is
$$M(t) = (1 - bt)^{-a}, \qquad t < 1/b.$$

(a) Show that $\sum_{i=1}^n |X_i|$ is sufficient for θ.

(b) Find the UMVUE for $1/\theta$.

(c) Find the MLE for $1/\theta$.

(d) Let T_n be the MLE of $1/\theta$. Find the limiting distribution for $\sqrt{n}(T_n - 1/\theta)$.

(e) Now suppose that $\theta \geq 1$ and we want to test
$$H_0 : \theta = 1 \text{ vs } H_a : \theta > 1.$$
Show that the size α likelihood ratio test has a rejection region of the form
$$\left\{ (x_1, \ldots, x_n) : \frac{\sum_{i=1}^n |x_i|}{n} < c \right\},$$
where $0 < \alpha < 1$ and the constant c is determined by α. Explain how to determine c based on α without giving an explicit expression for c.

(f) For part (e), find an approximate value for c based on the limiting distribution of $\sqrt{n}(T_n - 1/\theta)$. Express the approximate value for c in terms of n, α and Φ^{-1}, the inverse function of the standard normal cummulative distribution function.

(g) For part (e), express c in terms of α and the inverse of the cummulative distribution function of some common distribution. The distribution needs to be identified.

(a)

$$f_\theta(x_1, \dots, x_n) = \prod_{i=1}^{n} f_\theta(x_i)$$

$$= \left(\frac{\theta}{2}\right)^n \cdot e^{-\theta \cdot \sum_{i=1}^{n} |x_i|}$$

By Factorization Theorem, $\sum_{i=1}^{n} |X_i|$ is sufficient for θ.

(b) Let $T = \sum_{i=1}^{n} |X_i|$, then

$$E(T) = n \cdot E(|X_1|) = \frac{n}{\theta} \quad \text{and} \quad E\left(\frac{T}{n}\right) = \frac{1}{\theta}.$$

Since T is complete and sufficient for θ

and $\frac{T}{n}$ is a function of T with $E\left(\frac{T}{n}\right) = \frac{1}{\theta}$,

$\frac{T}{n}$ is the UMVUE for $\frac{1}{\theta}$.

(c)

$$l(\theta) = \log f_\theta(x_1, \dots x_n) = n \log \theta - n \log 2 - \theta \cdot \sum_{i=1}^{n} |x_i|$$

$$l'(\theta) = \frac{n}{\theta} - \sum_{i=1}^{n} |x_i|$$

$$= \frac{\sum_{i=1}^{n} |x_i|}{\theta} \left(\frac{n}{\sum_{i=1}^{n} |x_i|} - \theta \right).$$

542-543(III) Solutions

$$\ell'(\theta) \begin{cases} < 0 & \text{if} \quad 0 < \theta < n/\sum_{i=1}^{n}|x_i| \\ = 0 & \text{if} \quad \theta = n/\sum_{i=1}^{n}|x_i| \\ > 0 & \text{if} \quad \theta > n/\sum_{i=1}^{n}|x_i| \end{cases}$$

$\ell(\theta)$ is maximized at $\theta = n/\sum_{i=1}^{n}|x_i|$

$\Rightarrow$ MLE for θ is $\dfrac{n}{\sum_{i=1}^{n}|X_i|}$

$\Rightarrow$ MLE for $\frac{1}{\theta}$ is $\dfrac{\sum_{i=1}^{n}|X_i|}{n}$

(d) By CLT,

$$\frac{\sqrt{n}\left(\frac{\sum_{i=1}^{n}|X_i|}{n} - E|X_1|\right)}{\sqrt{Var(|X_1|)}} \xrightarrow{\mathcal{D}} N(0,1).$$

$$\Rightarrow \quad \frac{\sqrt{n}\left(T_n - \frac{1}{\theta}\right)}{\frac{1}{\theta}} \xrightarrow{\mathcal{D}} N(0,1)$$

(e) The likelihood ratio

$$\lambda(x_1, \dots, x_n) = \frac{\left(\frac{1}{2}\right)^n \cdot e^{-\sum_{i=1}^n |x_i|}}{\left(\frac{\hat{\theta}}{2}\right)^n \cdot e^{-\hat{\theta} \cdot \sum_{i=1}^n |x_i|}},$$

where $\hat{\theta}(\underset{\sim}{x})$ is the maximizer of $\ell(\theta) = \log f_\theta(x_1, \dots x_n)$ over $[1, \infty)$. From the calculation in part (c),

$$\hat{\theta}(\underset{\sim}{x}) = \begin{cases} \frac{n}{\sum_{i=1}^n |x_i|} & \text{if } \frac{n}{\sum_{i=1}^n |x_i|} \geq 1 \\ 1 & \text{if } \frac{n}{\sum_{i=1}^n |x_i|} < 1 \end{cases}, \quad \text{so}$$

$$\lambda(x_1, \dots, x_n) = \begin{cases} e^{n - \sum_{i=1}^n |x_i|} \left(\frac{\sum_{i=1}^n |x_i|}{n}\right)^n & \text{if } \frac{n}{\sum_{i=1}^n |x_i|} \geq 1 \\ 1 & \text{if } \frac{n}{\sum_{i=1}^n |x_i|} < 1 \end{cases}$$

and

$$\lambda(X_1, \dots, X_n) = \begin{cases} e^{n(1 - T_n + \log T_n)} & \text{if } T_n \leq 1 \\ 1 & \text{if } T_n > 1 \end{cases}$$

where

$$T_n = \frac{\sum_{i=1}^n |X_i|}{n}$$

The function $g(t) = 1 - t + \log t$ is increasing

on $(0, 1)$. since $g'(t) = -1 + \frac{1}{t} > 0$ for $0 < t < 1$.

Therefore, the LRT rejects H_0 if

$$\lambda(X_1, --, X_n) < k$$

$$\Longleftrightarrow \quad T_n < c.$$

To make the test size α, we choose c

so that $$P_{\theta=1}\left(T_n < c\right) = \alpha.$$

(f). From part (d), when $\theta = 1$.

$$\sqrt{n}\,(T_n - 1) \xrightarrow{\mathcal{D}} N(0, 1)$$

$$P_{\theta=1}(T_n < c) = P_{\theta=1}\left(\sqrt{n}(T_n - 1) < \sqrt{n}(c-1)\right)$$

$$\approx \Phi\left(\sqrt{n}(c-1)\right)$$

$$\Phi\left(\sqrt{n}(c-1)\right) \approx \alpha$$

$$\Rightarrow c \approx 1 + \frac{1}{\sqrt{n}} \cdot \Phi^{-1}(\alpha)$$

(g) We will first identify the distribution for $|X_i|$.

$$P(|X_i| \le t) = \begin{cases} \int_{-t}^{t} \frac{\theta}{2} e^{-\theta|x|}\, dx, & \text{if } t > 0 \\ 0 & \text{if } t \le 0. \end{cases}$$

$$= \begin{cases} \int_{0}^{t} \theta\, e^{-\theta|x|}\, dx, & \text{if } t > 0 \\ 0 & \text{if } t \le 0 \end{cases}$$

$$= \int_{-\infty}^{t} g(x)\, dx,$$

where $g(x) = \begin{cases} \theta e^{-\theta|x|} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$

Since g is the probability density function for $|X_i|$, $|X_i| \sim \text{Gamma}(1, \frac{1}{\theta})$.

The moment generating function for $x \cdot n T_n$ is

$$E\left(e^{t(\lambda n T_n)}\right) = E\left(e^{t\lambda\left(\sum_{i=1}^{n}|X_i|\right)}\right)$$

$$= \left(E\, e^{t\lambda|X_1|}\right)^n$$

$$= \left(\frac{1}{1-\frac{1}{\theta}\cdot t\lambda}\right)^n$$

$$= \left(\frac{1}{1-t\left(\frac{\lambda}{\theta}\right)}\right)^n,$$

So $\lambda n T_n \sim \text{Gamma}\left(n, \frac{\lambda}{\theta}\right)$

Take $\lambda = 2\theta$, then the distribution for $2\theta n T_n$ is $\text{Gamma}(n, 2)$, which is χ^2_{2n}, the Chi-square distribution with degrees of freedom $2n$. Let F be the cummulative distribution function for χ^2_{2n}.

Then

$$P_{\theta=1}\left(T_n < c\right) = \alpha$$

$$\Rightarrow \quad F(2nc) = \alpha, \quad c = \frac{F^{-1}(\alpha)}{2n}.$$