13. The Cochran-Satterthwaite Approximation for Linear Combinations of Mean Squares

Suppose $M_1, ..., M_k$ are independent mean squares and that

$$\frac{d_i M_i}{\mathrm{E}(M_i)} \sim \chi^2_{d_i} \qquad \forall\, i = 1, \ldots, k.$$

It follows that

$$\mathrm{E}\left[\frac{d_i M_i}{\mathrm{E}(M_i)}\right] = d_i, \;\; \mathrm{Var}\left[\frac{d_i M_i}{\mathrm{E}(M_i)}\right] = 2d_i, \text{ and } \; M_i \sim \boxed{\frac{\mathrm{E}(M_i)}{d_i}} \chi^2_{d_i}$$

for all $i = 1, \ldots, k$.

Scaled $\chi^2_{d_i}$

Consider the random variable

Chapt 12 : 1.5 MS(xu(trt)) − 0.5 MSE

$$M = a_1 M_1 + a_2 M_2 + \cdots + a_k M_k, \quad (1)$$

where $a_1, a_2, \ldots, a_k$ are known constants in $\mathbb{R}$.

Note that M is a linear combination of scaled χ^2 random variables.

The Cochran-Satterthwaite approximation works by assuming that M is approximately distributed as a scaled χ^2, just like each of the variables in the linear combination.

$$\frac{dM}{\mathrm{E}(M)} \overset{approx.}{\sim} \chi^2_d \iff M \overset{approx.}{\sim} \frac{\mathrm{E}(M)}{d}\chi^2_d. \quad (2)$$

What choice for d makes the approximation most reasonable?

The Cochran-Satterthwaite formula for the approximate degrees of freedom associated with the linear combination of mean squares defined by M is

$$d = \frac{M^2}{\sum_{i=1}^{k} a_i^2 M_i^2 / d_i} = \frac{\left(\sum_{i=1}^{k} a_i M_i\right)^2}{\sum_{i=1}^{k} a_i^2 M_i^2 / d_i}.$$

Why?

Consider the definition/approximation of M in (1) and (2):

- $M = a_1 M_1 + a_2 M_2 + \cdots + a_k M_k$ and (1)

- $M \overset{approx.}{\sim} \frac{\mathrm{E}(M)}{d} \chi^2_d.$ (2)

- Calculate $\mathrm{Var}(M)$ using M as defined in (1) and also using its approximation given in (2)

- Both variances are functions of d. We equate both variances and solve for d.

- The details are shown on the next two slides.

from (2) on slide 5

$$
\begin{aligned}
\mathrm{Var}(M) &\approx \left(\frac{\mathrm{E}(M)}{d}\right)^2 \mathrm{Var}\left(\chi^2_d\right) \\
&= \left(\frac{\mathrm{E}(M)}{d}\right)^2 (2d) \\
&= \frac{2\left[\mathrm{E}(M)\right]^2}{d} \\
&\approx \frac{2M^2}{d}.
\end{aligned}
$$

slide 2

And

$$\begin{aligned}
\text{Var}(M) &= a_1^2\text{Var}(M_1) + \cdots + a_k^2\text{Var}(M_K) \\
&= a_1^2\left[\frac{\text{E}(M_1)}{d_1}\right]^2 2d_1 + \cdots + a_k^2\left[\frac{\text{E}(M_k)}{d_k}\right]^2 2d_k \\
&= 2\sum_{i=1}^k \frac{a_i^2[\text{E}(M_i)]^2}{d_i} \\
&\approx 2\sum_{i=1}^k a_i^2 M_i^2/d_i.
\end{aligned}$$

Equating these two variance approximations yields

$$\frac{2M^2}{d} = 2\sum_{i=1}^{k} a_i^2 M_i^2 / d_i$$

and solving for d yields

$$d = \frac{M^2}{\sum_{i=1}^{k} a_i^2 M_i^2 / d_i} = \frac{\left(\sum_{i=1}^{k} a_i M_i\right)^2}{\sum_{i=1}^{k} a_i^2 M_i^2 / d_i}.$$

Recall the first example from the last slide set.

account for trt 2

exp. unit 1

exp. unit 3

$$\boldsymbol{y} = \begin{bmatrix} y_{111} \\ y_{121} \\ y_{211} \\ y_{212} \end{bmatrix} \begin{array}{l} \text{trt 1} \\ \\ \\ \text{trt 2} \end{array}, \quad \boldsymbol{X} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{Z} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

exp. unit 2

$$\boldsymbol{X}_1 = \boldsymbol{1}, \qquad \boldsymbol{X}_2 = \boldsymbol{X}, \qquad \boldsymbol{X}_3 = \boldsymbol{Z}$$

$$\boldsymbol{y}^\top(\boldsymbol{P}_2 - \boldsymbol{P}_1)\boldsymbol{y} + \boldsymbol{y}^\top(\boldsymbol{P}_3 - \boldsymbol{P}_2)\boldsymbol{y} + \boldsymbol{y}^\top(\boldsymbol{I} - \boldsymbol{P}_3)\boldsymbol{y} = \boldsymbol{y}^\top(\boldsymbol{I} - \boldsymbol{P}_1)\boldsymbol{y}$$

Expected Mean Squares

SOURCE	EMS
trt	$1.5\sigma_u^2 + \sigma_e^2 + (\tau_1 - \tau_2)^2$
$xu(trt)$	$\sigma_u^2 + \sigma_e^2$
$ou(xu, trt)$	σ_e^2

$$
\begin{aligned}
\mathrm{E}(1.5MS_{xu(trt)} - 0.5MS_{ou(xu,trt)}) &= 1.5(\sigma_u^2 + \sigma_e^2) - 0.5\sigma_e^2 \\
&= 1.5\sigma_u^2 + \sigma_e^2
\end{aligned}
$$

An Approximate F Test

The statistic

$$F = \frac{MS_{trt}}{1.5MS_{xu(trt)} - 0.5MS_{ou(xu,trt)}}$$

is approximately F distributed with 1 numerator degree of freedom and denominator degrees of freedom approximated by the Cochran-Satterthwaite Method:

from SAS, R,

$$d = \frac{(1.5MS_{xu(trt)} - 0.5MS_{ou(xu,trt)})^2}{(1.5)^2\left[MS_{xu(trt)}\right]^2 + (-0.5)^2\left[MS_{ou(xu,trt)}\right]^2}.$$

SAS Code for Example

```
data d;
   input trt xu y;
   cards;
1 1 6.4
1 2 4.2
2 1 1.5
2 1 0.9
;
run;
```

SAS Code for Example

Proc Glimmix or Proc Mixed

random int / subject = xu

does the same

Methods of Moments

```
proc mixed method=type1;
  class trt xu;
  model y=trt / ddfm=satterthwaite;
  random xu(trt);
run;
```

random effects

tells SAS that xu are nested within treatment

```
The Mixed Procedure

                  Model Information

Data Set                          WORK.D
Dependent Variable                y
Covariance Structure              Variance Components
Estimation Method                 Type 1
Residual Variance Method          Factor
Fixed Effects SE Method           Model-Based
Degrees of Freedom Method         Satterthwaite


             Class Level Information

Class    Levels    Values

trt           2    1 2
xu            2    1 2
```

Dimensions

Covariance Parameters	2
Columns in X	3
Columns in Z	3
Subjects	1
Max Obs Per Subject	4

σ_u^2 σ_e^2

SAS recognizes the 3 distinct exp. units

Number of Observations

Number of Observations Read	4
Number of Observations Used	4
Number of Observations Not Used	0

Type 1 Analysis of Variance

End Lecture 27: 04/02/25

Source	DF	Sum of Squares	Mean Square
trt	1	16.810000	16.810000
xu(trt)	1	2.420000	2.420000
Residual	1	0.180000	0.180000

Source	Expected Mean Square	Error Term
trt	Var(Residual)+1.5 Var(xu(trt))+Q(trt)	1.5 MS(xu(trt)) - 0.5 MS(Residual)
xu(trt)	Var(Residual)+Var(xu(trt))	MS(Residual)
Residual	Var(Residual)	.