STAT 5430

Lec 27, W, Apr 2

- Homework 7 posted, due, M, Apr 7
  ↑ testing
- Exam 2 is coming up (2 weeks away) on W, April 16, 6:15-8:15 PM, 3rd floor seminar room
- No class on that W.
- I'll post: study guide (sufficiency/completeness/tests)
  - practice exams
  - bring new 1 page (front/back) formula sheet on exam 2 material (I'll post one to use if you'd like)
  - can bring calculator & previous formula sheet for exam 1
  - I'll provide table of distributions / STAT 542 facts on test as before

STAT 5430: Summary to date
Where we have been
& where we are headed

- Completed

  - Introduction to Statistical Inference

  - Point Estimation

    * MME/MLE

  - Criteria for Evaluating Point Estimators

    * bias, variance, UMVUE, MSE

  - Elements of Decision Theory

    * Minimax, finding Bayes estimators

  - Sufficiency and Point Estimation

    * Factorization/Rao-Blackwell/Lehman-Scheffe Theorems

  - Hypothesis Testing I

    * Size, Power, Most Powerful Tests, Uniformly Most Powerful Texts

- Next: Hypothesis Testing II

  - Likelihood Ratio Tests

  - Bayes Tests

Hypothesis Testing II
Background

We've seen a few approaches for creating "best" tests (i.e., optimal in power):

1. Neyman-Pearson Lemma

   for a given size α

   ("most powerful" tests of $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$)

2. Two methods for "uniformly most powerful" (UMP) tests

   - Method I: Based on Neyman-Pearson Lemma

     (may work for $\mathbb{R}^p$-valued parameters θ and tests of $H_0 : \theta \in \Theta_0 \subset \mathbb{R}^p$ vs $H_1 : \theta \notin \Theta_0$)

   - Method II: Monotone Likelihood Ratio

     (may work for real-valued $\theta \in \mathbb{R}$ and tests "$H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$" or "$H_0 : \theta \geq \theta_0$ vs $H_1 : \theta < \theta_0$")

These "best tests" are designed for certain hypothesis structures (e.g., simple H_0 vs simple H_1) and UMP tests often don't exist outside of one-sided hypotheses (e.g., $H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$).

Nevertheless, these tests more broadly suggest that **good test rules involve comparing likelihoods**.

We next consider two further procedures for creating tests based on likelihoods:

- Likelihood Ratio Tests (LRT) } *general testing procedures*
- & Bayes Tests.

Note: there are no general guarantees that LRT or Bayes tests are "optimal" in terms of power, but these tests are broadly applicable and can perform well in practice.

Hypothesis Testing II
Likelihood Ratio Tests

Definition: Let $f(x|\theta)$, $\theta \in \Theta \subset \mathbb{R}^p$, be the joint pdf/pmf of $X = (X_1, \ldots, X_n)$ (the parameter θ can be vector-valued) and let Θ_0 be a nonempty proper subset of Θ. Then, the **likelihood ratio statistic** (LRS) for testing $H_0 : \theta \in \Theta_0 \subset \mathbb{R}^p$ vs $H_1 : \theta \notin \Theta_0$ is defined as

$$L(\theta) \equiv f(x|\theta)$$
likelihood

$$\lambda(x) = \frac{\max\limits_{\theta \in \Theta_0} f(x|\theta)}{\max\limits_{\theta \in \Theta} f(x|\theta)}.$$

← max $L(\theta)$ over $\textcircled{H}_0$

← max $L(\theta)$ over $\textcircled{H}$

test statistic function of data x

← maximizes $L(\theta)$ over $\textcircled{H}_0$

Comments on interpreting $\lambda(x)$

1. if $\hat{\theta} \equiv$ MLE of θ over entire Θ & $\tilde{\theta} \equiv$ maximum of $f(x|\theta)$ over $\theta \in \Theta_0$,

maximizes $L(\theta)$ over $\textcircled{H}$

here $L(\tilde{\theta}) \leq L(\hat{\theta})$

$$\lambda(x) = \frac{f(x|\tilde{\theta})}{f(x|\hat{\theta})} = \frac{L(\tilde{\theta})}{L(\hat{\theta})} \in [0,1]$$

2. if $\underline{H_0}$ is true, roughly expect $\lambda(x)$ to be "large" (i.e., close to 1)

roughly, $\hat{\theta} \approx \theta \in \textcircled{H}_0$ & $\tilde{\theta} \approx \theta$ too

& so expect $\lambda(x) \approx 1$

3. if $\underline{H_0}$ is false, roughly expect $\lambda(x)$ to be "small" (i.e., close to 0)

roughly, $\tilde{\theta} \approx \theta \notin \textcircled{H}_0$ if H_0 is false

& so expect $L(\tilde{\theta})$ to be small compared to $L(\hat{\theta})$

Definition: A size α **likelihood ratio test** (LRT) for testing $H_0 : \theta \in \Theta_0 \subset \mathbb{R}^p$ vs $H_1 : \theta \notin \Theta_0$ is defined as

$$\varphi(x) = \begin{cases} 1 & \text{if } \lambda(x) < k \\ \gamma & \text{if } \lambda(x) = k \\ 0 & \text{if } \lambda(x) > k \end{cases}$$

(reject H_0 if LRS $\lambda(x)$ is too small)

(don't reject if LRS $\lambda(x)$ is "big")

where $\gamma \in [0, 1]$, $0 \leq k \leq 1$ are constants determined by $\max_{\theta \in \Theta_0} E_\theta \varphi(X) = \alpha$.

need test to have size α

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$$\frac{d \log L(\theta)}{d\theta} = \frac{\Sigma x_i}{\theta^2} - \frac{3n}{\theta} = 0$$

Example: Let $X_1, \ldots, X_n$ be iid Gamma$(\alpha = 3, \theta)$, $\theta > 0$. Find a size α LRT for $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$.

↑ θ_0 is fixed/given value $(\theta_0 = 1)$

Solution: $L(\theta) \equiv f(\underline{x}|\theta) = \prod_{i=1}^{n} \left(\frac{x_i^2 e^{-x_i/\theta}}{2\theta^3} \right) = \frac{\left(\prod_{i=1}^{n} x_i^2\right) e^{-\Sigma x_i/\theta}}{2^n \theta^{3n}}, \; \theta > 0$

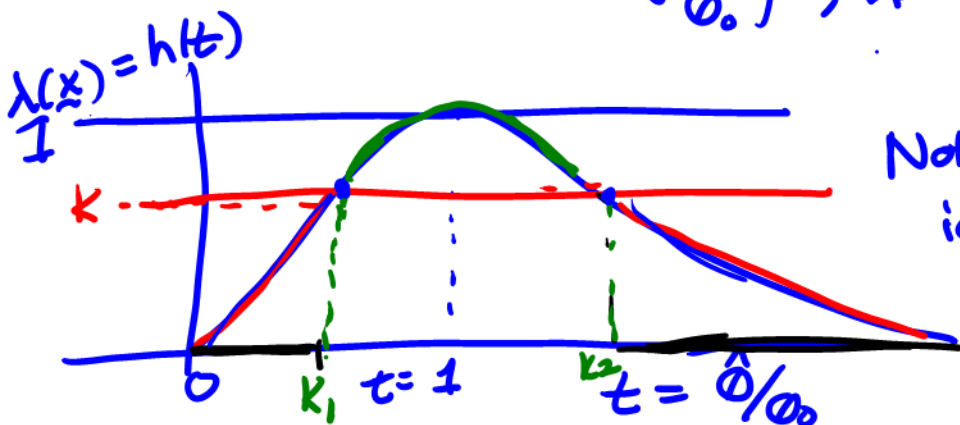check $\hat{\theta} = MLE$ of θ is $\frac{\sum_{i=1}^{n} x_i}{3n} = \bar{X}_n/3$

$\lambda(\underline{x}) = LRS$ of "$H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$"

$e^{-3n\hat{\theta}/\theta_0}$

$\Sigma x_i = 3n\hat{\theta}$

$= \frac{L(\theta_0)}{L(\hat{\theta})} = \dfrac{\left(\prod_{i=1}^{n} x_i^2\right) e^{-\Sigma x_i/\theta_0} \big/ 2^n \theta_0^{3n}}{\left(\prod_{i=1}^{n} x_i^2\right) e^{-\Sigma x_i/\hat{\theta}} \big/ 2^n (\hat{\theta})^{3n}}$

e^{-3n}

$= e^{-3n\hat{\theta}/\theta_0} \; e^{3n} \left(\frac{\hat{\theta}}{\theta_0} \right)^{3n}$

$= h\left(\frac{\hat{\theta}}{\theta_0} \right)$, where $h(t) = e^{-3nt + 3n} \, t^{3n}$, $t \geq 0$.

Note: $h'(t) = 0$ at $t = 1$
ie $h(t)$ has a max at $t = 1$
& $h(0) = 0 = \lim_{t \to \infty} h(t)$

$\lambda(\underline{x}) = h(t)$

$h(t) > k \iff \frac{\hat{\theta}}{\theta_0} \in (k_1, k_2)$ or $h(t) < k \iff \frac{\hat{\theta}}{\theta_0} \notin (k_1, k_2)$

Hence, the LRT is given by

$$\phi(\underline{x}) = \begin{cases} 1 & \text{if } \lambda(\underline{x}) < k \\ \gamma & \text{if } \lambda(\underline{x}) = k \\ 0 & \text{if } \lambda(\underline{x}) > k \end{cases} = \begin{cases} 1 & \text{if } \overset{\text{MLE}}{\hat{\theta}}/\theta_0 \notin [k_1, k_2] \\ \gamma & \text{if } \hat{\theta}/\theta_0 \in \{k_1, k_2\} \\ "0" 0 & \text{if } \hat{\theta}/\theta_0 \in (k_1, k_2) \end{cases}$$

where k_1, k_2 so that $E_{\theta_0}\phi(\underline{x}) = \alpha$, $h(k_1) = h(k_2)$

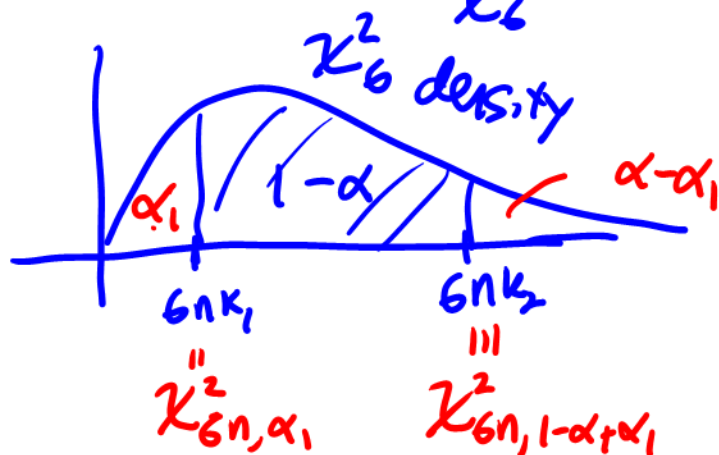ie $E_{\theta_0}\phi(x) = P_{\theta_0}\left(\hat{\theta}/\theta_0 \notin [k_1, k_2]\right) = \alpha$

$\Rightarrow 1 - \alpha = P_{\theta_0}\left(\overset{6n}{k_1} \le \overset{6n}{\hat{\theta}}\overset{6n}{\le} k_2\right)$

$\qquad = P_{\theta_0}\left(6nk_1 \le \underbrace{\sum_{i=1}^{n}\frac{2X_i}{\theta_0}}_{\chi_6^2} \le 6nk_2\right)$

$\hat{\theta} = \sum_{i=1}^{n} X_i / 3n$

$\dfrac{2X_i}{\theta_0} \sim \chi^2_{(2\alpha)} \sim \chi^2_6 \quad \alpha = 3$

$\dfrac{6n\hat{\theta}}{\theta_0} = \sum_{i=1}^{n}\dfrac{2X_i}{\theta_0} \sim \chi^2_{6n}$

χ^2_6 density

α_1 $\quad 1-\alpha \quad \alpha - \alpha_1$

$6nk_1 \qquad 6nk_2$

$\chi^2_{6n,\alpha_1} \qquad \chi^2_{6n, 1-\alpha+\alpha_1}$

for some $\alpha_1 < \alpha$
so that

$$h\left(\dfrac{\chi^2_{6n,\alpha_1}}{6n}\right) = h\left(\dfrac{\chi^2_{6n, 1-\alpha+\alpha_1}}{6n}\right)$$

Hence, the size α LRT is

$$\phi(\underline{x}) = \begin{cases} 1 & \text{if } \dfrac{\hat{\theta}}{\theta_0} < \dfrac{\chi^2_{6n,\alpha_1}}{6n} \text{ or } \dfrac{\hat{\theta}}{\theta_0} > \dfrac{\chi^2_{6n, 1-\alpha+\alpha_1}}{6n} \\ 0 & \text{o.w} \end{cases}$$

$\leftarrow$ scale

Example: Let $X_1, \ldots, X_n$ be iid Exponential(θ, ν), $\theta > 0$, $\nu \in \mathbb{R}$ with common pdf
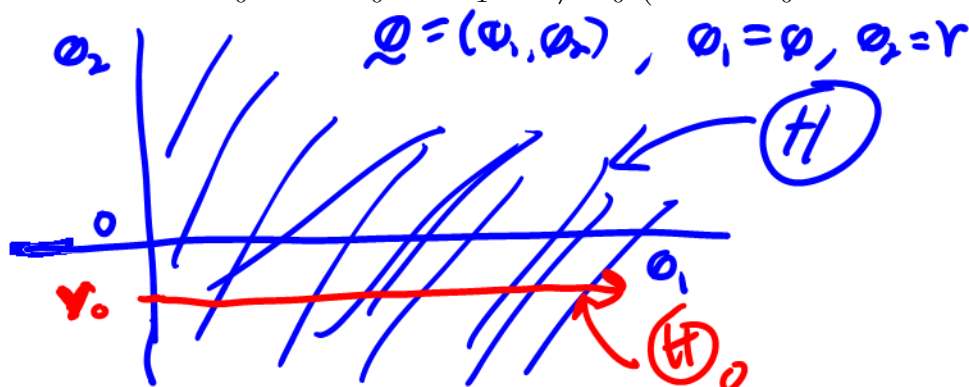
$\nwarrow$ location

use indicators
to write
likelihoods

$$f(x|\theta, \nu) = \begin{cases} \frac{1}{\theta} e^{-(x-\nu)/\theta} & \text{if } x \geq \nu \\ 0 & \text{otherwise} \end{cases}$$

e.g. $\dfrac{X_i - \nu}{\theta} \sim Exp(1)$

Find a size α LRT for $H_0 : \nu = \nu_0$ vs $H_1 : \nu \neq \nu_0$ (where $\nu_0 \in \mathbb{R}$ is fixed).

$\underset{\sim}{\theta} = (\theta_1, \theta_2)$, $\theta_1 = \varphi$, $\theta_2 = \nu$

θ_2 ... $\textcircled{H}$

0

ν_0 ... θ_1

$\textcircled{H}_0$

Hence, $f(\underset{\sim}{x}|\underset{\sim}{\theta}) \equiv$ joint pdf of $X_1, \ldots, X_n$

$$= \prod_{i=1}^{n} \left[\frac{1}{\theta_1} e^{-(X_i - \theta_2)/\theta_1} I(X_i \geq \theta_2) \right]$$

$$= \frac{1}{\theta_1^n} e^{-\Sigma(X_i - \theta_2)/\theta_1} I(\min X_i \geq \theta_2)$$

check $\hat{\underset{\sim}{\theta}} = (\hat{\theta}_1, \hat{\theta}_2) = \left(\dfrac{\Sigma(X_i - \min X_i)}{n}, \min X_i \right) \leftarrow$ MLE over $\textcircled{H}$

& $\max\limits_{\substack{\theta_1 > 0 \\ \theta_2 = \nu_0}} f(\underset{\sim}{x}|\underset{\sim}{\theta})$ $\quad \left(\dfrac{\Sigma(X_i - \nu_0)}{n}, \nu_0 \right) \leftarrow$ MLE over $\textcircled{H}_0$

$\|$ $\quad\quad \uparrow \tilde{\theta}_1 \quad \uparrow \tilde{\theta}_2$

$f(\underset{\sim}{x}|(\tilde{\theta}_1, \tilde{\theta}_2))$ where $\hat{} \rightarrow$ e^{-n}

LRS

$$\lambda(\underset{\sim}{x}) = \frac{f(\underset{\sim}{x}|(\tilde{\theta}_1, \tilde{\theta}_2))}{f(\underset{\sim}{x}|(\hat{\theta}_1, \hat{\theta}_2))} = \frac{\left(\frac{1}{\tilde{\theta}_1}\right)^n \left(e^{-\frac{\Sigma(X_i - \tilde{\theta}_2)}{\tilde{\theta}_1}} I(\min X_i \geq \tilde{\theta}_2)\right)}{\left(\frac{1}{\hat{\theta}_1}\right)^n \left(e^{-\Sigma(X_i - \hat{\theta}_2)/\hat{\theta}_1} I(\min X_i \geq \hat{\theta}_2)\right)}$$

e^{-n} ... $\nearrow \min X_i$ ... $\swarrow 1$

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$$\frac{\sum(X_i - \tilde{\theta}_2)}{n} = \tilde{\theta}_1 \quad \uparrow v_0 \qquad n\left(\frac{\sum(X_i - \tilde{\theta}_2)}{\tilde{\theta}_1 \, n}\right) = n\frac{\tilde{\theta}_1}{\tilde{\theta}_1} = n$$

$$\lambda(\underline{X}) = \left(\frac{\hat{\theta}_1}{\tilde{\theta}_1}\right)^n I\left(\min X_i \geq \tilde{\theta}_2\right)$$
$$\uparrow v_0$$

(Under $H_0 : v = v_0$ expect all data
$\min X_i \geq v_0 \Rightarrow$ if observe,
that $\min X_i < v_0 \Rightarrow \lambda(\underline{X}) = 0$
$\Rightarrow$ reject H_0)

A size α LRT is

$$\phi(\underline{x}) = \begin{cases} 1 & \text{if } \lambda(\underline{x}) < k \\ \gamma \geq 0 & \text{if } \lambda(\underline{x}) = k \\ 0 & \text{if } \lambda(\underline{x}) > k \end{cases}$$

where $0 \leq k, \gamma \leq 1$
so that
$$\max_{\theta \in \Theta_0} E\phi(\underline{x}) = \alpha.$$

So,

$$\alpha = \max_{\substack{\theta_1 > 0 \\ \theta_2 = v_0}} P_{\theta}\left(\underbrace{\left(\frac{\hat{\theta}}{\tilde{\theta}}\right)^n \overbrace{I(\min X_i \geq v_0)}^{\text{Under } H_0 : v = v_0 \text{ this is always } 1} < k}_{\lambda(\underline{x})} \right)$$

$$= \max_{\substack{\theta_1 > 0 \\ \theta_2 = v_0}} P_{\theta}\left(\left(\frac{\hat{\theta}}{\tilde{\theta}}\right) < k^{1/n} \right)$$

$$= \max_{\substack{\theta_1 > 0 \\ \theta_2 = v_0}} P_{\theta}\left(\frac{\sum(X_i - \min X_i)}{\sum(X_i - v_0)} < k^{1/n} \right)$$

$$= P\left(\frac{\sum(Y_i - \min Y_i)}{\sum_{i=1}^{n} Y_i} < k^{1/n} \right) \qquad \| \quad \frac{\sum_{i=1}^{n}(Y_i - \min Y_i)}{\sum_{i=1}^{n} Y_i}$$

↑ doesn't depend
on $\theta_1 > 0, v_0$

where $Y_i = X_i - v_0 \sim Exp(1)$

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