Expected values

Some properties

Theorem 2.2.5: *Suppose X is a r.v. such that* $\mathrm{E}|g_1(X)| < \infty$ *and* $\mathrm{E}|g_2(X)| < \infty$ *and let $a, b, c \in \mathbb{R}$ be fixed constants. Then,*

1. $\mathrm{E}\Big[ag_1(X) + b\Big] = a\mathrm{E}g_1(X) + b$

2. $\mathrm{E}\Big[ag_1(X) + bg_2(X) + c\Big] = a\mathrm{E}g_1(X) + b\mathrm{E}g_2(X) + c$

3. *If $g_1(x) \geq a$ for all x, then* $\mathrm{E}g_1(X) \geq a$

4. *If $g_1(x) \leq b$ for all x, then* $\mathrm{E}g_1(X) \leq b$

5. *If $g_1(x) \geq g_2(x)$ for all x, then* $\mathrm{E}g_1(X) \geq \mathrm{E}g_2(X)$

$$\mathbb{E}\Big[\underbrace{a_1 g_1(X) + a_2 g_2(X) + c}_{h(X)}\Big] \overset{def}{=} \int_{-\infty}^{\infty} h(x) f_X(x)\,dx = \int_{-\infty}^{\infty} \Big\{\overbrace{a_1 g_1(x) + a_2 g_2(x)}^{h(x)} + c\Big\} f_X(x)\,dx$$

$$= a_1 \int_{-\infty}^{\infty} g_1(x) f_X(x)\,dx + a_2 \int_{-\infty}^{\infty} g_2(x) f_X(x)\,dx + c \int_{-\infty}^{\infty} f_X(x)\,dx = a_1 \mathbb{E}(g_1(X)) + a_2 \mathbb{E}(g_2(X)) + c$$

$$E(h(X)) = \int_{\mathbb{R}} h(x) f_X(x)\,dx$$

$$E(X) = \int_{\mathbb{R}} x f_X(x)\,dx$$

(5) $g_1(x) \geq g_2(x)$ $\forall x$, $f_X(x) > 0$ $\Longrightarrow \int_{-\infty}^{\infty} g_1(x) f_X(x) \geq \int_{-\infty}^{\infty} g_2(x) f_X(x) \Rightarrow E(g_1(X)) \geq E(g_2(X))$

Expectations are also invariant under transformation:

If $\underbrace{Y = g(X)}_{y = g(x)}$, then

$$\mathrm{E}Y \overset{def}{=} \sum_y y f_Y(y) dy = \sum_y y\, \mathbb{P}(Y = y)$$
$$= \sum_x g(x) f_X(x) \cancel{dx} = \mathrm{E}g(X)$$

(In the continuous case, replace sums with integrals)

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$$

$Y = g(X)$
$E(Y)$

Expected values

Variance

An important instance of this E$g(X)$ notion comes using $g(X) = [X - \mathrm{E}X]^2$

Definition: The **variance** of a random variable X, denoted Var(X) or σ_X^2, is

$$\mathrm{Var}(X) = \sigma_X^2 = \mathrm{E}[X - \mathrm{E}X]^2, = \mathbb{E}\left[(X - \mathbb{E}(X))^2\right]$$

the expected squared distance between X and its mean EX

Two important "variance" facts:

1. $\mathrm{Var}(a + bX) = b^2\mathrm{Var}(X)$ for any real numbers a, b
2. $\mathrm{Var}(X) = \mathrm{E}X^2 - [\mathrm{E}X]^2$

Proof: $\mathrm{Var}(X) \overset{def}{=} \mathbb{E}\left[(X - \underbrace{\mathbb{E}(X)}_{\mu})^2\right] \overset{\mathbb{E}(X)=\mu}{=\!=\!=}$

$$= \mathbb{E}\left[X^2 + \mu^2 - 2\mu X\right] = \mathbb{E}\left[X^2\right] + \mu^2 - \overbrace{2\mu\, \underbrace{\mathbb{E}(X)}_{\mu}}^{2\mu^2}$$

$$= \mathbb{E}\left[X^2\right] - \mu^2 = \mathbb{E}\left[X^2\right] - (\mathbb{E}(X))^2.$$

Example: X is uniform$(0, 1)$. Find Var(X) and Var(Y) for $Y = 1 + 3X^2$.

Expected values

Other moments and distributional summaries

Moments are an important summary of a distribution

1. $\mu = \mu_X = \mathrm{E}X$ is often called the mean

2. $\mu'_n = \mathrm{E}X^n$ is the nth moment provided $\mathrm{E}X^n$ exists, i.e.,

$$\sum_x |g(x)| f_X(x) < \infty \quad \text{(discrete case)}$$

$$\int_{-\infty}^{\infty} |g(x)| f_X(x) dx < \infty \quad \text{(continuous case)}$$

3. $\mu_n = \mathrm{E}[(X-\mu)^n]$ is the nth central moment provided $\mathrm{E}X^n$ exists
   (a) $\mathrm{Var}(X) = \sigma_X^2 = \mathrm{E}[(X-\mu)^2] = \mu_2$ is the variance
   (b) $\sigma_X = \sqrt{\mathrm{Var}(X)}$ is the standard deviation
   (c) μ_3 is skewness (i.e., measures distributional balance around μ)
   (d) μ_4 is kurtosis (i.e., measure of how long the distributional tails are)

Regarding moments:

1. If $\mathrm{E}X^r$ exists for some $r > 0$ then $\mathrm{E}X^s$ exists for $0 \leq s \leq r$

2. If $\mathrm{E}X^r$ does not exist for some $r > 0$, then $\mathrm{E}X^s$ will not exist for $s > r$

3. $\mathrm{E}X^2$ exists if and only if $\mathrm{Var}(X)$ exists

4. For $r > 0$, the existence of $\mathrm{E}X^r$ is a matter of the distribution of X not having "heavy tails" (i.e., X doesn't assume "large" values with "large" probability)