STAT 5430
Lecture 07, W, Feb 5

- Homework 1 solution posted

practice on point estimation →

- Homework 2 is assigned in Canvas (due by next Monday, Feb 10, by midnight)

Office hours Mine: FM, 12-1 PM & by appointment
TA (Min-Yi): WR 11-12 in Snedecor 2404

Criteria for Evaluating Point Estimators

Comparing Unbiased Estimators: Relative Efficiency

Same basic idea as before: we like to compare *unbiased estimators* (U.E.'s) in terms of *variance* & "small variance" is a preferable property among U.E.'s

Definitions: Let T, T_1 and T_2 be U.E.'s of $\gamma(\theta)$. Then,

1. the **relative efficiency** of T_1 with respect to T_2 is defined as

$$\text{r.e.}(T_1, T_2, \theta) \equiv \frac{\text{Var}_\theta(T_2)}{\text{Var}_\theta(T_1)}$$

e.g. $\text{Var}_\theta(T_1) = \frac{1}{2}\text{Var}_\theta(T_2) \Rightarrow \text{r.e.}(T_1, T_2, \theta) = \frac{\text{Var}_\theta(T_2)}{\text{Var}_\theta(T_1)} = 2$

$\Rightarrow$ "T_1 is 2x as/more efficient than T_2"

2. T is called **efficient** if $\text{r.e.}(T_1, T, \theta) \le 1$ holds $\forall \theta \in \Theta$ & any other U.E. T_1

i.e. $\text{Var}_\theta(T) \le \text{Var}_\theta(T_1)$

"efficient $\equiv$ UMVUE" $\equiv$ best UE"

3. Let T be an efficient estimator & let T_1 be an U.E. of $\gamma(\theta)$. Then, the **efficiency** of T_1 is defined as

$$e_{T_1}(\theta) \equiv \text{r.e.}(T_1, T, \theta) = \frac{\text{Var}_\theta(T)}{\text{Var}_\theta(T_1)} \le 1.$$

T_1 compared to best UE T

Criteria for Evaluating Point Estimators

Comparing Unbiased Estimators: Relative Efficiency

Example: Let $X_1, \dots, X_n$ be iid UNIF$(0, \theta)$, $\theta > 0$. Consider two estimators

$$\text{U.E of } \theta \rightarrow T_1 \equiv \text{MME of } \theta \equiv 2\bar{X}_n$$

$$T_2 \equiv \text{MLE of } \theta \equiv X_{(n)} \leftarrow \max_{1\le i\le n} X_i$$

Check that: $E_\theta(T_1) = 2E_\theta(\bar{X}_n) = 2E_\theta(X_1) = 2\left(\frac{\theta}{2}\right) = \theta$

- the pdf of T_2 is $f_{T_2}(t) = \begin{cases} \frac{n}{\theta^n} t^{n-1} & 0 \le t \le \theta \\ 0 & \text{otherwise,} \end{cases}$ so that

$$E_\theta(T_2) = \frac{n}{n+1}\theta, \qquad \mathrm{Var}_\theta(T_2) = \frac{n}{(n+1)^2(n+2)}\theta^2 \leftarrow \mathrm{Var}_\theta(X_{(n)})$$

$$= \int_0^\theta t\, f_{T_2}(t|\theta)\,dt$$

$E_\theta(T_2) \neq \theta$

new estimator

- T_2 is not an U.E. of θ, but we can define an U.E. as $T_2' = \frac{n+1}{n} T_2$

$E_\theta(T_2') = \frac{n+1}{n} E_\theta(T_2) = \theta$ (so T_2' is UE of θ)

Compare T_1 & T_2' (both UE)

- $\mathrm{Var}_\theta(T_1) = \mathrm{Var}_\theta(2\bar{X}_n) = \frac{4}{n}\mathrm{Var}_\theta(X_1) = \frac{4}{n}\frac{\theta^2}{12} = \frac{\theta^2}{3n}$

$= 2^2\,\mathrm{Var}_\theta(\bar{X}_n) = 4\,\mathrm{Var}_\theta(\bar{X}_n) = 4 \cdot \frac{\mathrm{Var}_\theta(X_1)}{n}$ $\qquad \mathrm{Var}_\theta(X_1) = \frac{(\theta-0)^2}{12}$

- $\mathrm{Var}_\theta(T_2') = \left(\frac{n+1}{n}\right)^2 \mathrm{Var}_\theta(X_{(n)}) = \frac{\theta^2}{n(n+2)}$

- hence, r.e.$(T_1, T_2', \theta) = \frac{\mathrm{Var}_\theta(T_2')}{\mathrm{Var}_\theta(T_1)} = \frac{3}{n+2} \le 1$ for all $n \ge 1$

It can be shown that T_2' is UMVUE of θ here!

(can't use CRLB for this here.... CRLB not possible for UNIF$(0,\theta)$!)

Criteria for Evaluating Point Estimators

Comparing both Biased & Unbiased Estimators: MSE

Previously, we compared *unbiased* estimators in terms of variance, but how do we compare estimators when some are biased and others are biased?

Definition: For an estimator T of $\gamma(\theta)$, the **mean squared error (MSE)** of T is defined as

can use an estimator T that is UE or biased estimator

$$\mathrm{MSE}_\theta(T) \equiv \mathrm{E}_\theta\left\{[T-\gamma(\theta)]^2\right\}.$$

how close T is to target $\gamma(\theta)$ in terms of expected squared distance

Facts about MSE:

1. The MSE of an estimator T can always be decomposed as:

   compute as

   $$\mathrm{MSE}_\theta(T) = \underbrace{\mathrm{Var}_\theta(T)}_{\text{variance}} + \underbrace{[b_\theta(T)]^2}_{\text{squared biased}},$$

   where $b_\theta(T) = \mathrm{E}_\theta(T) - \gamma(\theta)$ is the bias of T.

2. If T is an U.E. of $\gamma(\theta)$, then $b_\theta(T) = E_\theta(T) - \gamma(\theta) = 0$

   & $MSE_\theta(T) = Var_\theta(T)$.

Criteria for Evaluating Point Estimators

Comparing both Biased & Unbiased Estimators: MSE

Example (continued). Let $X_1, \ldots, X_n$ be iid UNIF$(0, \theta)$, $\theta > 0$, and consider

$$\text{UE of } \theta \rightarrow T_1 \equiv \text{MME of } \theta \equiv 2\bar{X}_n \qquad b_\theta(T_1) = 0$$

$$T_2 \equiv \text{MLE of } \theta \equiv X_{(n)}$$

$$E_\theta(T_2) = E_\theta X_{(n)} = \frac{n}{n+1}\theta \Rightarrow b_\theta(T_2) = E_\theta(T_2) - \theta = \frac{n}{n+1}\theta - \theta = \frac{-\theta}{n+1}$$

$$MSE_\theta(T_2) = Var_\theta(T_2) + [b_\theta(T_2)]^2$$

$$= \frac{\theta^2 n}{(n+2)(n+1)^2} + \left[-\frac{\theta}{n+1}\right]^2 = \frac{2\theta^2}{(n+1)(n+2)}$$

$$MSE_\theta(T_1) = Var_\theta(T_1) + [b_\theta(T_1)]^2$$

$$= \theta^2/3n + 0^2 = \frac{\theta^2}{3n}$$

check that:

$$MSE_\theta(T_2) \underset{=}{<} MSE_\theta(T_1) \quad \begin{array}{l} \text{if } n > 2 \\ \text{if } n = 1 \text{ or } 2 \end{array}$$

check that

$$Var_\theta(T_2') = MSE_\theta(T_2') \underset{=}{<} MSE_\theta(T_2) \quad \begin{array}{l} \text{if } n > 1 \\ \text{if } n = 1 \end{array}$$

T_2' is version of T_2 that's an UE of θ