Independence

Examples

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Example 1: sampling with replacement (Bernoulli trials)

Let $X_1, \ldots, X_n$ be iid Bernoulli(p) r.v.s & let $Y = \sum_{i=1}^n X_i$

$$M_{X_1+X_2+\cdots+X_n}(t) \overset{\text{def}}{=\!=} \mathbb{E}\left[e^{t(X_1+\cdots+X_n)}\right]$$

$$= \mathbb{E}\left[\underbrace{e^{tX_1} e^{tX_2} \cdots e^{tX_n}}\right]$$

$$\overset{X_1,\ldots,X_n \text{ are}}{\underset{\text{ind.}}{=\!=}} \mathbb{E}\left[e^{tX_1}\right] \cdots \mathbb{E}\left[e^{tX_n}\right]$$

$$\overset{X_1,X_2,\ldots,X_n \text{ are}}{\underset{\text{Ber}(p)}{=\!=}} \left[1-p+pe^t\right]^n$$

$$\Longrightarrow Y = \sum_{i=1}^n X_i \sim \text{Bin}(n,p) \Longrightarrow \begin{cases} \mathbb{E}(Y) = np \ \checkmark \\ Var(Y) = np(1-p) \ \checkmark \end{cases}$$

$$\mathbb{E}(Y) = \mathbb{E}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \mathbb{E}(X_i) = \sum_{i=1}^n p = np$$

$$Var(Y) = Var\left(\sum_{i=1}^n X_i\right) \underset{\text{are ind.}}{\overset{X_1,\ldots,X_n}{=\!=}} \sum_{i=1}^n Var(X_i) \underset{\text{are Ber}(p)}{\overset{X_1,\ldots,X_n}{=\!=}} \sum_{i=1}^n p(1-p)$$

$$= np(1-p)$$

Independence

Introduction

definition

Recall: Two events A, B are independent if $P(A \cap B) = P(A)P(B)$

We next want to extend the concept of independence to random variables

Definition: Random variables X and Y are **independent** if

$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B) \quad \text{for } \textbf{any } A, B \subset \mathbb{R}.$$

Some Characterizations of Independence

1. **Theorem:** In general, random variables X and Y are **independent** iff

   If and only If

   $$\mathbb{P}(X \le x, Y \le y) = \underbrace{F(x,y)}_{\mathbb{P}(X \le x)\cdot \mathbb{P}(Y \le y)} = F_X(x)F_Y(y) \quad \text{for any } x, y \in \mathbb{R}.$$

2. **Theorem:** Jointly discrete variables X and Y are **independent** iff

   $$\underbrace{P(X = x, Y = y)}_{\text{joint pmf}} = f(x,y) = f_X(x)f_Y(y) = \underbrace{P(X = x)P(Y = y)}_{\text{Product of marginal pmfs}} \quad \text{for any } x, y \in \mathbb{R}.$$

3. **Theorem:** Jointly continuous variables X and Y are **independent** iff (there exist versions of the pdfs $f(x,y)$, $f_X(x)$ and $f_Y(y)$ such that)

   $$f(x,y) = f_X(x)f_Y(y) \quad \text{for any } x, y \in \mathbb{R}.$$

   (The "there exist versions" business is needed because one can change densities at a few values (i.e., endpoints) without changing the integrals computed from them.)

Independence

Introduction (cont'd)

4. Furthermore, if X and Y are independent so that $f(x,y) = f_X(x)f_Y(y)$ then

$$f(x|y) = f_X(x) \quad \text{for any } x \in \mathbb{R}, \text{ any } y \text{ with } f_Y(y) > 0;$$

$$f(y|x) = f_Y(y) \quad \text{for any } y \in \mathbb{R}, \text{ any } x \text{ with } f_X(x) > 0.$$

$$f(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} \overset{\text{ind.}}{=} \frac{f_X(x)\cancel{f_Y(y)}}{\cancel{f_Y(y)}} = f_X(x)$$

In other words, under independence, the conditional distribution of Y given $X = x$ is the same as the marginal distribution of Y. This is another way to think of independence: given information about one variable, the distribution of the 2nd variable does not change.

Two uses of independence

1. examine a joint pdf to determine if the r.v.s are independent

2. assume independence and derive joint distribution from marginal distributions

Independence

Some Examples

Discrete example:

Are X and Y independent?

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$$

$$\mathbb{P}(X=x, Y=y) = \mathbb{P}(X=x)\cdot\mathbb{P}(Y=y)$$

		x: 1	2	3	
y	3	1/12	1/12	1/6	1/3
	2	1/12	1/6	1/12	1/3
	1	1/6	1/12	1/12	1/3
		1/3	1/3	1/3	

$\mathbb{P}(X=2, Y=1) = 1/12$ (*)

$\mathbb{P}(X=2)\cdot\mathbb{P}(Y=1) = \frac{1}{3}\,\frac{1}{3} = 1/9$ (**)

(*) & (**) $\Longrightarrow$ X and Y are NOT independent.

Continuous example (the familiar one): $f(x,y) = 1/x$, $0 < y < x < 1$.

Are X and Y independent?

$Y \mid X \sim \mathrm{Uni}(0,x) \Longrightarrow f(y|x) = \frac{1}{x} \quad \forall\ 0<y<x$

$f_Y(y) = \begin{cases} -\log y & 0<y<1 \\ 0 & \text{ow} \end{cases} \qquad f_Y(y) = -\log y \quad 0<y<1$

$\Longrightarrow f(y|x) \neq f_Y(y) \Longrightarrow$ X and Y are NOT independent.

Another continuous example: $f(x,y) = 2y$, $0 < x < 1$, $0 < y < 1$.

Are X and Y independent?

$$f_X(x) = \int_0^1 f_{X,Y}(x,y)\,dy = \int_0^1 2y\,dy = y^2\Big]_0^1 = 1$$

$\Longrightarrow$ I: $f_X(x) = \begin{cases} 0 & \text{If } x \notin (0,1) \\ 1 & \text{If } x \in (0,1) \end{cases}$

$$f_Y(y) = \int_0^1 f(x,y)\,dx = \int_0^1 2y\,dx = 2yx\Big]_0^1 = 2y \quad \forall\ 0<y<1$$

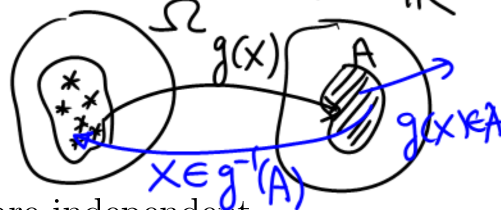
II: $f_Y(y) = \begin{cases} 2y & 0<y<1 \\ 0 & \text{ow} \end{cases}$

$2y = f_{X,Y}(x,y) = f_X(x) f_Y(y) = 1\cdot 2y \quad \forall (x,y) \in (0,1)\times(0,1)$

$\Longrightarrow$ X and Y are independent.

$g(X)$ is R.V.

$g(X): \Omega \longrightarrow \mathbb{R}$



Independence

Some important results & facts

1. **Theorem:** If X, Y are independent then $g(X)$ and $h(Y)$ are independent.

new r.v. new r.v.

Proof: $\mathbb{P}(\underbrace{g(X)\in A} \text{ and } \underbrace{h(Y)\in B})$ $\quad g^{-1}(A) \equiv \{x\in\mathbb{R},\ g(x)\in A\}$ ← set

$= \mathbb{P}(X\in g^{-1}(A) \text{ and } Y\in h^{-1}(B))$

$\underset{\text{are ind.}}{\overset{X \text{ and } Y}{=}} \mathbb{P}(X\in g^{-1}(A))\,\mathbb{P}(Y\in h^{-1}(B))$

$= \mathbb{P}(g(X)\in A)\cdot\mathbb{P}(h(Y)\in B)$ ✓

ex: X and Y are independent then e^{-X} and $\sin Y$ are independent.

2. **Theorem:** If X, Y are independent and the mean of $g(X)h(Y)$ exists, then

$$\mathrm{E}\underbrace{g(X)h(Y)}_{Z(X,Y)} = \mathrm{E}g(X)\mathrm{E}h(Y)$$

$\mathbb{E}(XY) = \mathbb{E}(X)\,\mathbb{E}(Y)$

$\mathbb{E}(Z(X,Y)) \overset{\text{def}}{=} \iint z(x,y)\, f_{XY}(x,y)\,dx\,dy \underset{\text{X and Y are ind.}}{=} \iint g(x)h(y) f_X(x) f_Y(y)\,dx\,dy$

$= \int g(x) f_X(x)\,dx \int h(y) f_Y(y)\,dy = \mathbb{E}[g(X)]\,\mathbb{E}[h(Y)].$

3. **Corollary:** If X, Y are independent and EXY exists, then

$$\mathrm{Cov}(X, Y) = 0$$

Proof: $\mathrm{Cov}(X,Y) \overset{\text{def}}{=} \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \underset{\text{are ind.}}{\overset{X \text{ and } Y}{=}} \mathbb{E}[X]\mathbb{E}[Y] - \mathbb{E}[X]\mathbb{E}[Y] = 0$

So independent $\Rightarrow$ 0 correlation. But, it is easy enough to find examples showing that 0 correlation does NOT imply independence.

Independence

Some important results & facts

4. **Corollary:** If X, Y are independent and $\mathrm{E}X^2, \mathrm{E}Y^2$ exist, then

$$\mathrm{Var}(aX + bY) = \mathrm{Var}(aX) + \mathrm{Var}(bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$$

$$\mathrm{Var}(aX+bY) = \mathrm{Var}(aX) + \mathrm{Var}(bX) + 2ab\,\underbrace{\mathrm{Cov}(X,Y)}_{=0}$$

In general

$$\mathrm{Var}\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 \mathrm{Var}(X_i) \quad \text{If } \underbrace{X_1, X_2, \dots, X_n \text{ are independent}}$$

5. **Corollary:** If X, Y are independent and X has mgf $M_X(t)$ and Y has $M_Y(t)$ then

$$\rightarrow \boxed{M_{X+Y}(t) = M_X(t)M_Y(t)}$$

holds for any t where $M_X(t)$ and $M_Y(t)$ exist.

(Note $M_X(t)$ must exist in some $(-h_1, h_1)$ and $M_Y(t)$ must exist in some $(-h_2, h_2)$, so that both $M_X(t)$ and $M_Y(t)$ exist in some neighborhood of zero.)

$$M_{X+Y}(t) \overset{\text{def}}{=} \mathbb{E}\left[e^{t(X+Y)}\right] = \mathbb{E}\left[e^{tX} \cdot e^{tY}\right] \xrightarrow[\text{indepdnt}]{X \,\&\, Y \text{ are}} \mathbb{E}\left[e^{tX}\right] \cdot \mathbb{E}\left[e^{tY}\right] = M_X(t)\, M_Y(t).$$

$$M_{X_1+X_2+\cdots+X_n}(t) = M_{X_1}(t) \cdot M_{X_2}(t) \cdots M_{X_n}(t) \quad \text{If } \underbrace{X_1, X_2, \dots, X_n \text{ are independent}}$$

6. **Theorem:** Suppose X has mgf $M_X(t)$ and Y has $M_Y(t)$. Then, X, Y are independent iff

def ↗ X and Y are indepedent ↗

$$M_{X,Y}(t_1, t_2) = \mathrm{E}e^{t_1X + t_2Y} = \mathrm{E}e^{t_1X}\mathrm{E}e^{t_2Y} = M_X(t_1)M_Y(t_2)$$

two arguments t_1, t_2

holds for all $t_1, t_2 \in (-h, h)$ in some neighborhood of zero.

$$M_{X,Y}(t_1,t_2) = \frac{1}{(1-t_1)(1-t_1-t_2)} \qquad t_1 < 1, \quad t_1 + t_2 < 1$$

$$M_X(t_1) = M_{X,Y}(t_1, 0) = \frac{1}{(1-t_1)^2} \Longrightarrow \text{X and Y are } \underline{\text{NOT}} \text{ independent.}$$

$$M_Y(t_2) = M_{X,Y}(0, t_2) = \frac{1}{(1-t_2)}$$

$$M_{X,Y}(t_1,t_2) = e^{\frac{t_1^2+t_2^2}{2}} \qquad M_X(t_1) = e^{t_1^2/2} \qquad M_Y(t_2) = e^{t_2^2/2}$$

Independence

Multivariate case

Definition: Random variables $X_1, \ldots, X_n$ are **independent** if

$$P(X_1 \in A_1, \ldots, X_n \in A_n) = \prod_{i=1}^{n} P(X_i \in A_i) \quad \text{for any } A_1, \ldots, A_n \subset \mathbb{R}$$

Characterizations

1. $X_1, \ldots, X_n$ are **independent** iff
$$F(x_1, \ldots, x_n) = P(X_1 \leq x_1, \ldots, X_n \leq x_n) = \prod_{i=1}^{n} F_{X_i}(x_i) \quad \text{for any } x_1, \ldots, x_n \in \mathbb{R}$$
2. $X_1, \ldots, X_n$ are **independent** iff
$$f(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i) \quad \text{for any } x_1, \ldots, x_n \in \mathbb{R}$$

Important Properties: If $X_1, \ldots, X_n$ are independent r.v.s, then

1. $g_1(X_1), \ldots, g_n(X_n)$ are independent r.v.s
2. it holds that $\mathrm{E}\left[\prod_{i=1}^{n} g_i(X_i)\right] = \prod_{i=1}^{n} \mathrm{E}\left[g_i(X_i)\right]$
3. $\mathrm{Cov}(X_i, X_j) = 0$ for $i \neq j$
4. $\mathrm{Var}\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 \mathrm{Var}(X_i)$
5. if the mgf $M_{X_i}(t)$ exists for each $i = 1, \ldots, n$, then the mgf of $X_1 + \cdots + X_n$ is
$$M_{X_1+\cdots+X_n}(t) = \prod_{i=1}^{n} M_{X_i}(t)$$
for all t where each $M_{X_i}(t)$ exits.
6. MGF characterization: $X_1, \ldots, X_n$ are independent iff
$$M_{X_1,\ldots,X_n}(t_1, \ldots, t_n) = \prod_{i=1}^{n} M_{X_i}(t_i)$$