STAT 5430

Lec 21, W, Mar 12

Basu's theorem & Intro to testing $\rightarrow$ Homework 5 posted, due M, Mar 24 (after break)

Hypothesis Testing I

Terminology: Error Probabilities

$\to \phi(\underset{\sim}{x}) \in [0,1]$

Remark: For any **general** test function, the same holds true: $\phi(\cdot)$,

1. Prob. of a type I error at $\underset{\sim}{\theta}$ ($\underset{\sim}{\theta} \in \Theta_0$) $= P_{\underset{\sim}{\theta}}(\text{reject } H_0) = E_{\underset{\sim}{\theta}}\phi(\underset{\sim}{X})$

   ↑ key result

2. Prob. of a type II error at $\underset{\sim}{\theta}$ ($\underset{\sim}{\theta} \notin \Theta_0$) $= P_{\underset{\sim}{\theta}}(\text{fail to reject } H_0) = 1 - E_{\underset{\sim}{\theta}}\phi(\underset{\sim}{X})$.

Same as for simple test rules ($\phi(\underset{\sim}{x}) = 0$ or 1)

e.g. $X \sim \text{Binomial}(2, \theta)$, $0<\theta<1$

general test rule $\phi(x) = \begin{cases} 1 & \text{if } x=0 \\ \frac{1}{2} & \text{if } x=1 \\ 0 & \text{if } x=2 \end{cases} \Rightarrow E_\theta \phi(X) = 1 \cdot P_\theta(X=0) + \frac{1}{2} P_\theta(X=1) + 0\, P_\theta(X=2) = P_\theta(\text{reject } H_0)$

Definition: Let $\phi(\underset{\sim}{X})$ be a test rule for testing $H_0 : \underset{\sim}{\theta} \in \Theta_0$ vs $H_1 : \underset{\sim}{\theta} \notin \Theta_0$,

1. $\underbrace{\max_{\underset{\sim}{\theta} \in \Theta_0} E_{\underset{\sim}{\theta}}\phi(\underset{\sim}{X})}$ is called **the size** or **the level** of $\phi(\underset{\sim}{X})$

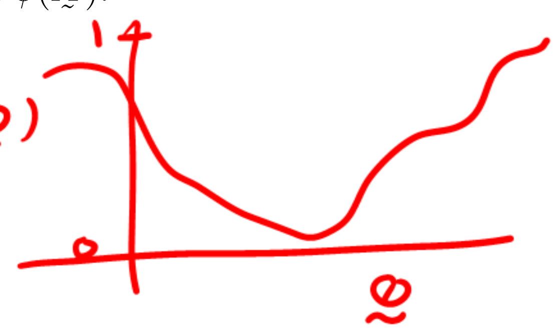
   max Prob of Type I error for $\underset{\sim}{\theta} \in \Theta_0$
   max $P_\theta(\text{reject } H_0)$, $\underset{\sim}{\theta} \in \Theta_0$

2. $\Pi_\phi(\underset{\sim}{\theta}) = E_{\underset{\sim}{\theta}}\phi(\underset{\sim}{X})$ is called the **power function** of $\phi(\underset{\sim}{X})$.

power function

depends on ϕ

& depends on data generating parameter



Note: For $\underset{\sim}{\theta} \in \Theta_0$, $\Pi_\phi(\underset{\sim}{\theta}) = \mathrm{E}_{\underset{\sim}{\theta}}\phi(\underset{\sim}{X}) =$ probability of type I error
$= P_{\underset{\sim}{\theta}}(\text{reject } H_0)$, $\underset{\sim}{\theta} \in \Theta_0$

For $\underset{\sim}{\theta} \notin \Theta_0$, probability of type II error $= 1 - \Pi_\phi(\underset{\sim}{\theta}) = 1 - \mathrm{E}_{\underset{\sim}{\theta}}\phi(\underset{\sim}{X})$
$= 1 - P_{\underset{\sim}{\theta}}(\text{reject } H_0)$
$= P_\theta(\text{don't reject } H_0)$, $\underset{\sim}{\theta} \notin \Theta_0$

Want
$E_{\underset{\sim}{\theta}}\phi(\underset{\sim}{X}) = \Pi_\phi(\underset{\sim}{\theta})$ to be small for any $\underset{\sim}{\theta} \in \Theta_0$
& $E_{\underset{\sim}{\theta}}\phi(\underset{\sim}{X}) = \Pi_\phi(\underset{\sim}{\theta})$ to be large for any $\underset{\sim}{\theta} \notin \Theta_0$

Hypothesis Testing I

Illustration of Error Probabilities/Size/Power

$\leftarrow E_\theta X_1 = \theta$

Example: Let $X_1, \ldots, X_n$ be iid Exponential(θ), $\theta > 0$. Let

$\bar{X}_n$ estimates θ

$$\varphi(\underset{\sim}{x}) = \begin{cases} 0 & \text{if } \bar{X}_n \geq 1 \\ 1 & \text{if } \bar{X}_n < 1 \end{cases}$$

be a test rule for $H_0 : \theta \geq 1$ vs. $H_1 : \theta < 1$. Find

$\Theta_0 \equiv [1, \infty)$ ← parameter values under H_0

(i) the probability of Type I error at $\theta = 1$ (if $n = 5$)

(ii) the size of $\varphi(\cdot)$

(iii) the probability of Type II error at $\theta = 1/3$ (if $n = 5$)

(iv) the power function of $\varphi(\cdot)$

Solution: (iv) $\Pi_\phi(\theta) = E_\theta \phi(\underset{\sim}{X}) = P_\theta(\bar{X}_n < 1)$

$= P_\theta\left(\sum_{i=1}^n X_i < n\right)$ $\quad \sum_{i=1}^n X_i \sim \text{gamma}(n, \theta)$

$= P_\theta\left(\frac{2\sum_{i=1}^n X_i}{\theta} < \frac{2n}{\theta}\right)$ $\quad \frac{2\sum_{i=1}^n X_i}{\theta} \sim \text{gamma}(n, 2) \sim \chi^2_{2n}$

$= F_{2n}\left(\frac{2n}{\theta}\right)$, $F_{2n}(\cdot) \equiv$ cdf of χ^2_{2n}

(i) "prob of Type I error at $\theta = 1$"

$= E_{\theta=1} \phi(\underset{\sim}{X}) = \Pi_\phi(1) = F_{2n}\left(\frac{2n}{1}\right)$, $n = 5$

$= F_{10}(10) = 0.560$

(ii) size of $\phi(\cdot) = \max\limits_{\theta \in \Theta_0} \underbrace{E_\theta \phi(\underset{\sim}{X})}_{\Pi_\phi(\theta)} = \max\limits_{\theta \geq 1} F_{2n}\left(\frac{2n}{\theta}\right)$ ← $\theta \uparrow 1$, $F_{2n}\left(\frac{2n}{\theta}\right) \uparrow$ (decreasing function of θ)

$\Theta_0 \equiv [1, \infty)$ $\quad \underbrace{\Pi_\phi(\theta)}_{P_\theta(\text{reject } H_0)}$ $= F_{2n}(2n)$

(iii) "prob of Type II error at $\theta = \frac{1}{3}$"

$= P_{\theta = \frac{1}{3}}(\text{don't reject } H_0) = 1 - E_{\theta=\frac{1}{3}} \phi(\underset{\sim}{X}) = 1 - F_{2n}\left(\frac{2n}{1/3}\right)$

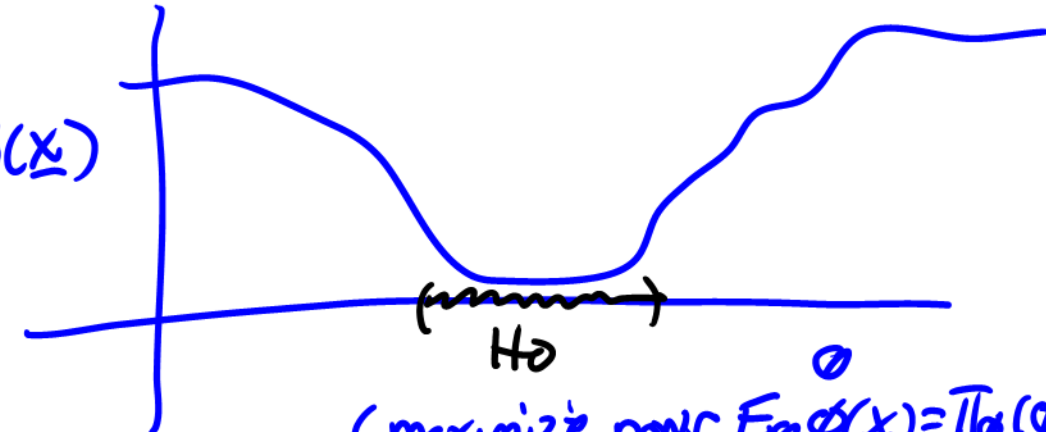
$= 0.001$, if $n = 5$

Hypothesis Testing I

Error Probabilities/Size/Power

In general, it is not possible to minimize the probability of both types of errors, simultaneously (for a given sample size).

$\Pi_\phi(\theta)$
$= E_\theta \phi(\underset{\sim}{X})$



(maximize power $E_\theta \phi(\underset{\sim}{X}) = \Pi_\phi(\underset{\sim}{\theta})$ for $\theta \notin \Theta_0$)

One can at best minimize the probability of one type of error (Type II) while fixing the other error probability (Type I) at a given level.

$\max_{\theta \in \Theta_0} P_\theta(\text{reject } H_0) = \max_{\theta \in \Theta_0} \Pi_\phi(\underset{\sim}{\theta}) \equiv \text{size} \leq \alpha$

Note: Minimizing the probability of Type II error $1 - E_{\underset{\sim}{\theta}}(\phi(\underset{\sim}{X}))$ for any $\underset{\sim}{\theta} \notin \Theta_0$ is the same as maximizing power function $\Pi_\phi(\underset{\sim}{\theta}) = E_{\underset{\sim}{\theta}}\phi(\underset{\sim}{X})$ for any $\underset{\sim}{\theta} \notin \Theta_0$, while maintaining $\underbrace{\max_{\underset{\sim}{\theta} \in \Theta_0} E_{\underset{\sim}{\theta}}\phi(\underset{\sim}{X})}_{\text{size}} \leq \alpha$ for some given $\alpha \in [0, 1]$

Hypothesis Testing I

Most Powerful Tests (Simple vs Simple Hypotheses)

Finding best test for simple H_0 vs simple H_1!

Let $f(\underset{\sim}{x}|\theta)$, $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$, $\theta \in \Theta$, be the joint pdf/pmf of $\underset{\sim}{X} = (X_1, \ldots, X_n)$. We want to test the hypothesis

$$H_0 : \theta = \theta_0 \quad \text{vs} \quad H_1 : \theta = \theta_1 \qquad \text{where } \theta_0, \theta_1 \in \Theta,\ \theta_0 \neq \theta_1.$$

Definition: A test function $\varphi(\underset{\sim}{x})$ is called a **most powerful** (MP) test of size α if

1. $\mathrm{E}_{\theta_0}\varphi(\underset{\sim}{X}) = \alpha$.
2. $\mathrm{E}_{\theta_1}\varphi(\underset{\sim}{X}) \geq \mathrm{E}_{\theta_1}\bar{\varphi}(\underset{\sim}{X})$ holds for any other test rule $\bar{\varphi}(\underset{\sim}{x})$ with $\mathrm{E}_{\theta_0}\bar{\varphi}(\underset{\sim}{X}) \leq \alpha$.

A MP test does exist at least for simple H_0 vs simple H_1, as described below.

Theorem: (Neyman-Pearson Lemma) Let $f(\underset{\sim}{x}|\theta)$, $\theta \in \Theta$, be the joint pdf/pmf of $X_1, \ldots, X_n$. Then for testing $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$, a MP test of size α exists for all $\alpha \in [0, 1]$ and is given by

$$\varphi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } f(\underset{\sim}{x}|\theta_1) > k f(\underset{\sim}{x}|\theta_0) \\ \gamma & \text{if } f(\underset{\sim}{x}|\theta_1) = k f(\underset{\sim}{x}|\theta_0) \\ 0 & \text{if } f(\underset{\sim}{x}|\theta_1) < k f(\underset{\sim}{x}|\theta_0) \end{cases} \tag{4}$$

$L(\theta) = f(\underset{\sim}{x}|\theta)$ likelihood function

where $\gamma \in [0, 1]$ and $0 \leq k \leq \infty$ are constants satisfying

$$\mathrm{E}_{\theta_0}\varphi(\underset{\sim}{X}) = \alpha. \tag{5}$$