We can partition

litter random effect

$$\begin{aligned} \boldsymbol{Z} &= [\underbrace{\underset{4\times 4}{\boldsymbol{I}} \otimes \underset{8\times 1}{\mathbf{1}}}, \underbrace{\underset{16\times 16}{\boldsymbol{I}} \otimes \underset{2\times 1}{\mathbf{1}}}] \\ &= [\boldsymbol{Z}_\ell, \boldsymbol{Z}_a]. \end{aligned}$$

mouse effect

We have

$$\begin{aligned} \boldsymbol{Zu} &= [\boldsymbol{Z}_\ell, \boldsymbol{Z}_a] \begin{bmatrix} \boldsymbol{\ell} \\ \boldsymbol{a} \end{bmatrix} \\ &= \boldsymbol{Z}_\ell \boldsymbol{\ell} + \boldsymbol{Z}_a \boldsymbol{a} \end{aligned}$$

and

end lecture 22
3-14-25

$$\begin{aligned}
\text{Var}(\boldsymbol{Zu}) &= \boldsymbol{ZGZ}^\top \\
&= [\boldsymbol{Z}_\ell, \boldsymbol{Z}_a] \begin{bmatrix} \sigma^2_\ell \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \sigma^2_a \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{Z}_\ell^\top \\ \boldsymbol{Z}_a^\top \end{bmatrix} \\
&= \boldsymbol{Z}_\ell(\sigma^2_\ell \boldsymbol{I})\boldsymbol{Z}_\ell^\top + \boldsymbol{Z}_a(\sigma^2_a \boldsymbol{I})\boldsymbol{Z}_a^\top \\
&= \sigma^2_\ell \boldsymbol{Z}_\ell \boldsymbol{Z}_\ell^\top + \sigma^2_a \boldsymbol{Z}_a \boldsymbol{Z}_a^\top \\
&= \sigma^2_\ell \underset{4\times 4}{\boldsymbol{I}} \otimes \underset{8\times 8}{\boldsymbol{1}\boldsymbol{1}^\top} + \sigma^2_a \underset{16\times 16}{\boldsymbol{I}} \otimes \underset{2\times 2}{\boldsymbol{1}\boldsymbol{1}^\top}.
\end{aligned}$$

assume: ℓ and a are independent

We usually assume that all random effects and random errors are mutually independent and that the errors (like the effects within each factor) are identically distributed:

$$\begin{bmatrix} \boldsymbol{\ell} \\ \boldsymbol{a} \\ \boldsymbol{e} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \sigma_\ell^2 \boldsymbol{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \sigma_a^2 \boldsymbol{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \sigma_e^2 \boldsymbol{I} \end{bmatrix} \right).$$

The unknown variance parameters $\sigma_\ell^2, \sigma_a^2, \sigma_e^2 \in \mathbb{R}^+$ are called *variance components*.

In this case, we have $\boldsymbol{R} = \text{Var}(\boldsymbol{e}) = \sigma_e^2 \boldsymbol{I}$.

Thus,

$$\begin{aligned} \text{Var}(\boldsymbol{y}) &= \boldsymbol{Z}\boldsymbol{G}\boldsymbol{Z}^\top + \boldsymbol{R} \\ &= \sigma_\ell^2 \boldsymbol{Z}_\ell \boldsymbol{Z}_\ell^\top + \sigma_a^2 \boldsymbol{Z}_a \boldsymbol{Z}_a^\top + \sigma_e^2 \boldsymbol{I}. \end{aligned}$$

This is a block diagonal matrix with a block as follows.

just 1 letter: next slide

(To get a block to fit on one slide, let $\ell = \sigma_\ell^2, a = \sigma_a^2, e = \sigma_e^2$).

8 x 8 matrix

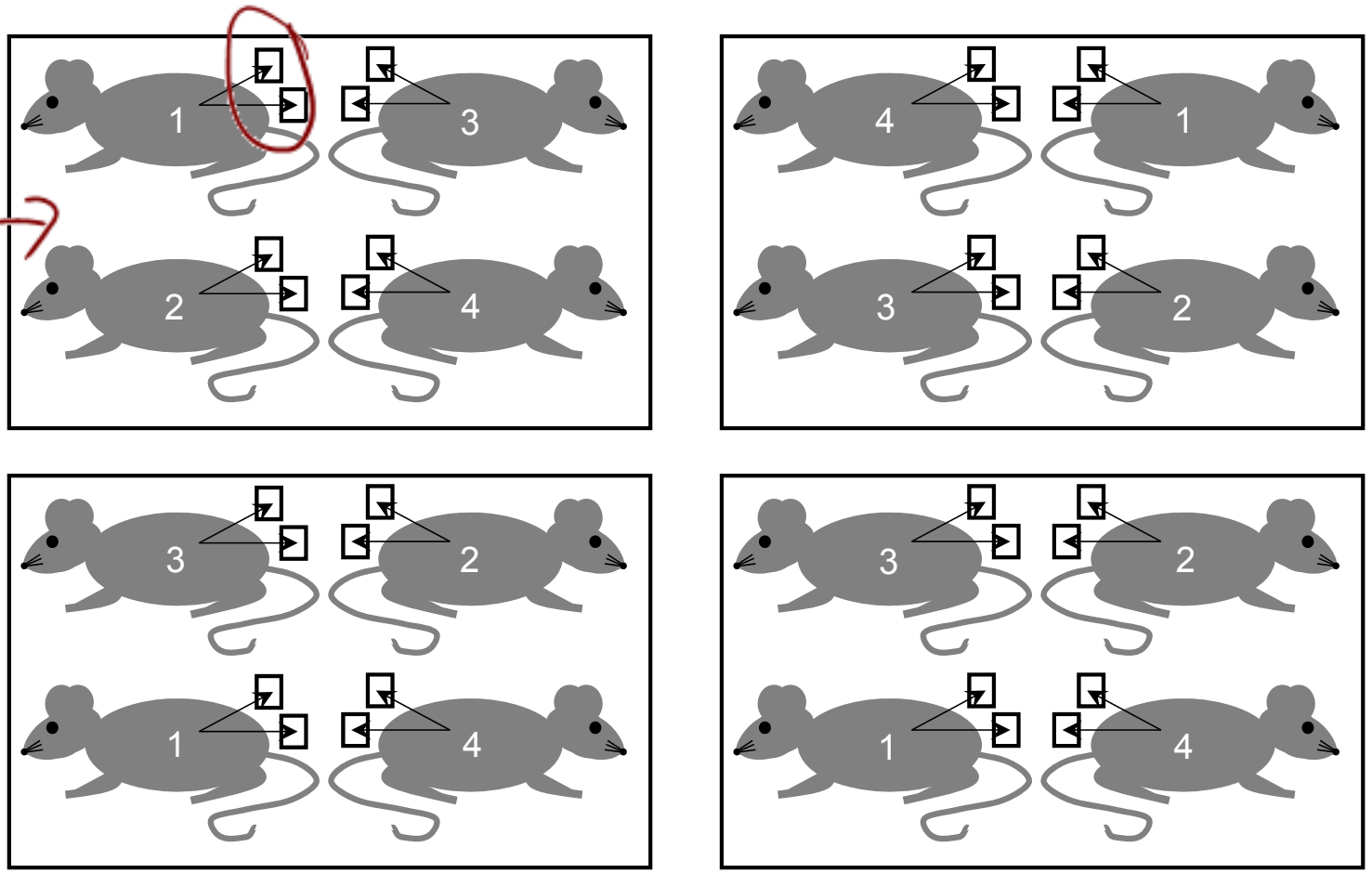
$Var(y_{111}) = Var(y_{112})$: mouse 1, trt 1, in litter 1 : 2 obs

$Cov(y_{111}, y_{112})$ = Cov between the 2 obs. from the same mouse

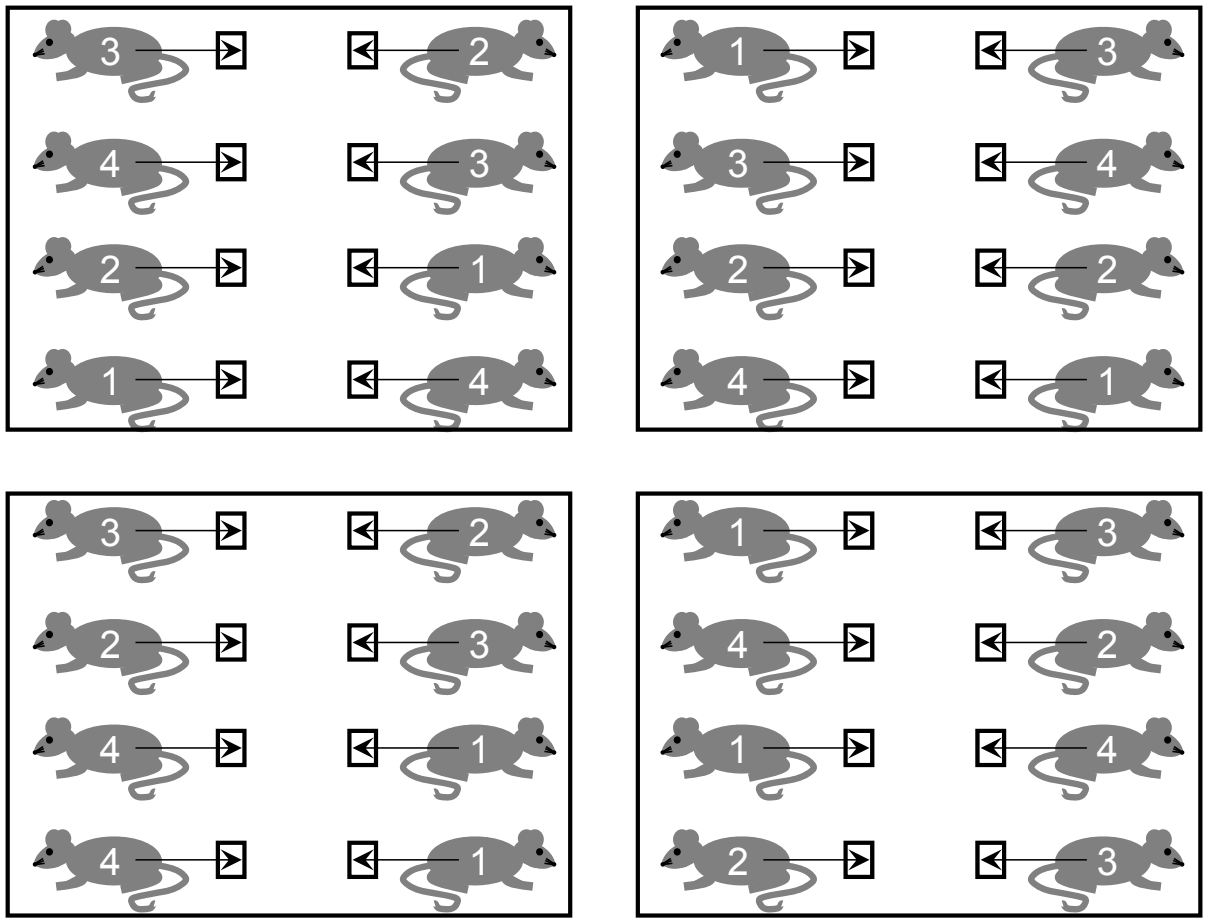
$\ell+a+e$	$\ell+a$	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ
$\ell+a$	$\ell+a+e$	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ
ℓ	ℓ	$\ell+a+e$	$\ell+a$	ℓ	ℓ	ℓ	ℓ
ℓ	ℓ	$\ell+a$	$\ell+a+e$	ℓ	ℓ	ℓ	ℓ
ℓ	ℓ	ℓ	ℓ	$\ell+a+e$	$\ell+a$	ℓ	ℓ
ℓ	ℓ	ℓ	ℓ	$\ell+a$	$\ell+a+e$	ℓ	ℓ
ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	$\ell+a+e$	$\ell+a$
ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	$\ell+a$	$\ell+a+e$

Cov between mouse 1 and the remaining 3 mice in here this is a single block on the main diagonal of $Var(y)$ litter 1

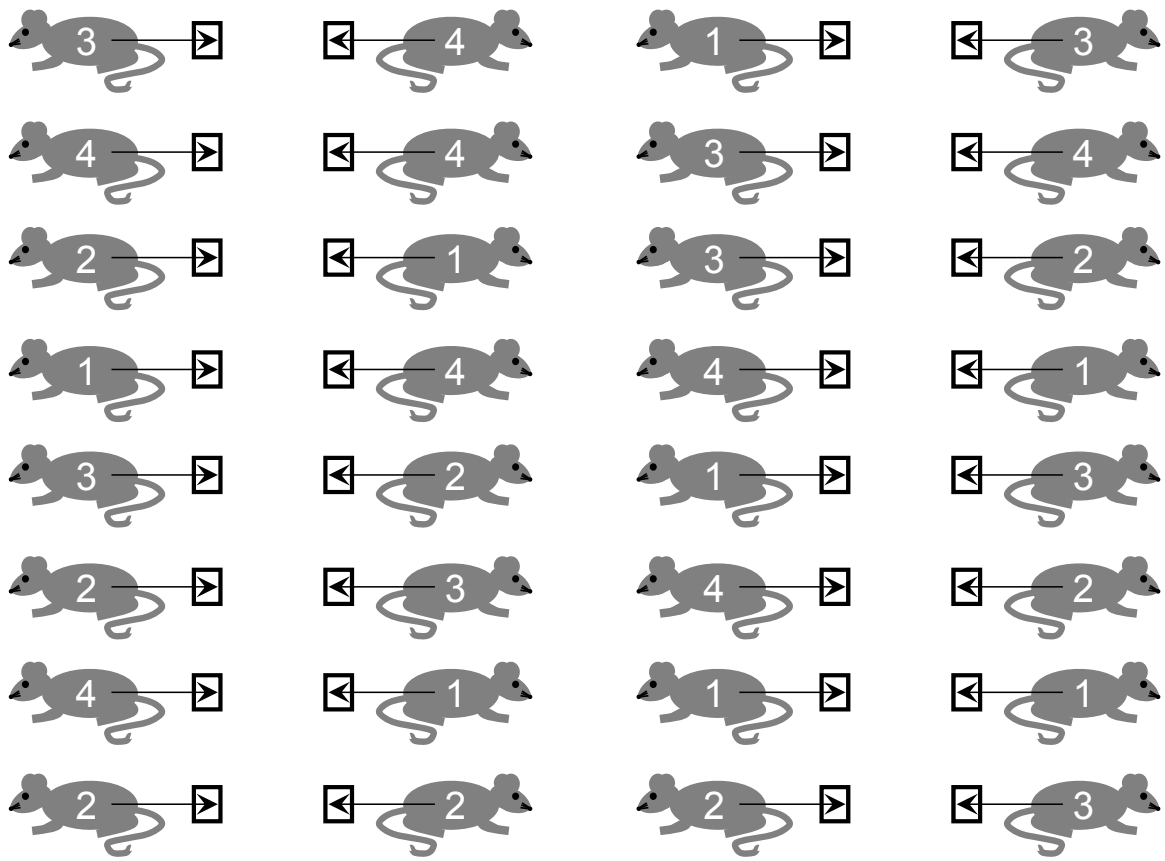
Random Effects Specify the Correlation Structure



Without the mouse random effects, our model would correspond to an RCBD with 2 mice per treatment per litter.



With no random effects, our model would correspond to a CRD with 8 mice per treatment.



Review of Experimental Design Terminology

Experiment – An investigation in which the investigator applies some treatments to experimental units and then observes the effect of the treatments on the experimental units by measuring one or more response variables.

Treatment – a condition or set of conditions applied to experimental units in an experiment.

Experimental Unit – the physical entity to which a treatment is randomly assigned and independently applied.

Response Variable – a characteristic of an experimental unit that is measured after treatment and analyzed to assess the effects of treatments on experimental units.

Observational Unit – the unit on which a response variable is measured.

There is often a one-to-one correspondence between experimental units and observational units, but that is not always true.

In our example involving plant heights and soil moisture levels, pots were the experimental units because soil moisture levels were randomly assigned to pots.

Seedlings were the observational units because the response was measured separately for each seedling.

Whenever there is more than one observational unit for an experimental unit or whenever the response is measured multiple times for an experimental unit, we say we have *multiple observations per experiment unit*.

This scenario is also referred to as *subsampling* or *pseudo-replication*.

Whenever an experiment involves multiple observations per experimental unit, it is important to include a random effect for each experimental unit.

Without a random effect for each experimental unit, a one-to-one correspondence between observations and experimental units is assumed.

Including random effects in a model is one way to account for a lack of independence among observations that might be expected based on the design of an experiment.

Completely Randomized Design (CRD) – experimental design in which, for given number of experiment units per treatment, all possible assignments of treatments to experimental units are equally likely.

Block – a group of experimental units that, prior to treatment, are expected to be more like one another (with respect to one or more response variables) than experimental units in general.

Randomized Complete Block Design (RCBD) – experimental design in which separate and completely randomized treatment assignments are made for each of multiple blocks in such a way that all treatments have at least one experimental unit in each block.

Blocking – grouping similar experimental units together and assigning different treatments within such groups of experimental units.

The experiment involving muscle samples from mice used blocking.

Each litter was a block in that experiment.

Each mouse was an experimental unit.

Each muscle sample was an observational unit.