Functions of a random variable

Continuous r.v.s: the monotone case

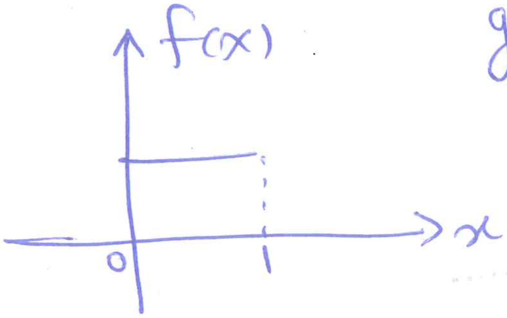
Theorem 2.1.5: *If X has pdf $f_X(x)$ and $Y = g(X)$ where $g(\cdot)$ has either a strictly positive or a strictly negative derivative on $\mathcal{X} = \{x \in \mathbb{R} : f_X(x) > 0\}$, then the pdf of Y has support $\mathcal{Y} = \{g(x) : x \in \mathcal{X}\}$ and is given by*

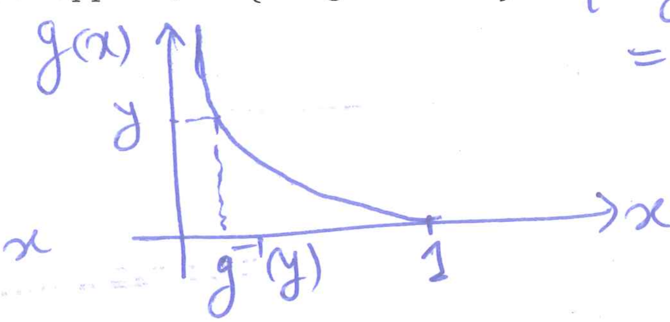
$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right| > 0 \quad \textit{for } y \in \mathcal{Y}; \qquad f_Y(y) = 0 \quad \textit{for } y \notin \mathcal{Y}$$

(This combines the two cases on last slide.)

Example: X has pdf $f_X(x) = 1$ for $0 < x < 1$ and $g(x) = -\log x$

Let $Y = g(X)$ so that Y has support $\mathcal{Y} = \{-\log x : x \in \mathcal{X}\} =$ $\{-\log x : 0 < x < 1\}$
$= (0, \infty)$





$\forall y \in \mathcal{Y}$, $y = -\log x \Rightarrow x = e^{-y} = g^{-1}(y)$

$\Rightarrow f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right|$

$= f_X(e^{-y}) \left| \frac{de^{-y}}{dy} \right| = 1 \left| -e^{-y} \right| = e^{-y}$ If $y > 0$

$f_Y(y) = \begin{cases} e^{-y} & \text{If } y > 0 \\ 0 & \text{ow} \end{cases}$

Functions of a random variable

Continuous r.v.s: the non-monotone case

The previous ideas on obtaining the pdf of $Y = g(X)$ can be extended as follows.

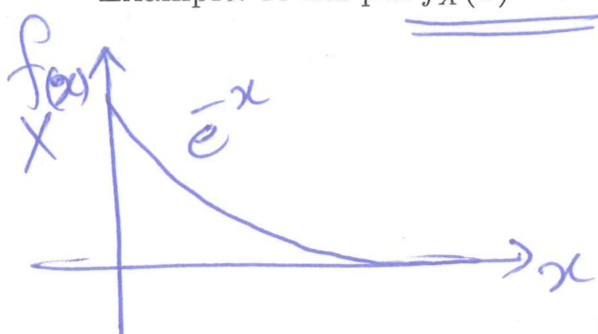
If g isn't monotone for all x, but there's a way to break up the support $\mathcal{X} = \{x : f_X(x) > 0\}$ into several intervals, on each of which g is strictly increasing or decreasing, then we simply add terms like

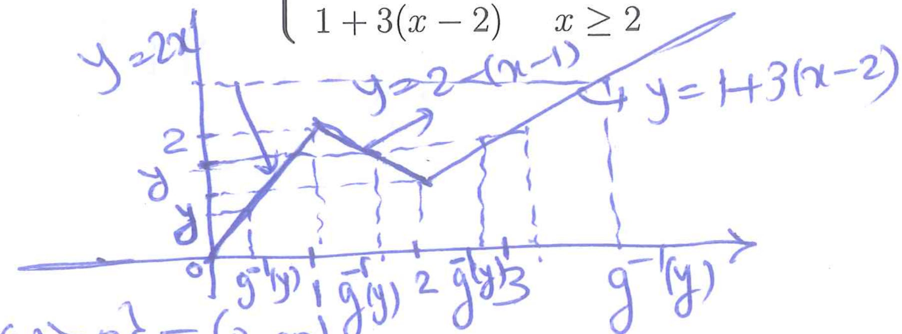
$$f_X(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right|$$

g is piece-wise monotone

to get the pdf $f_Y(y)$ on $\mathcal{Y} = \{g(x) : x \in \mathcal{X}\}$

Example: X has pdf $f_X(x) = e^{-x}$ for $x > 0$ and $g(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 2-(x-1) & 1 \le x \le 2 \\ 1+3(x-2) & x \ge 2 \end{cases}$





$\mathcal{Y} = \{g(x) : \text{where } f_X(x) > 0\} = (0, \infty)$

for $0 < y < 1 \Rightarrow g^{-1}(y) = y/2$ (1 piece from $y = 2x$)

for $1 \le y \le 2 \Rightarrow \begin{cases} \text{piece 1 as } g^{-1}(y) = y/2 \\ \text{piece 2 as } g^{-1}(y) = 3-y \\ \text{piece 3 as } g^{-1}(y) = \frac{(y-1)}{3} + 2 \end{cases}$

for $y > 2 \Rightarrow g^{-1}(y) = \frac{y-1}{3} + 2$

Finally,

pdf of Y is

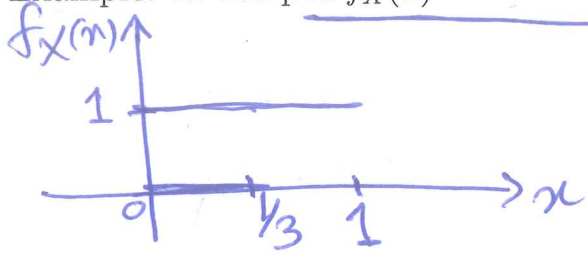
$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) = \left|\dfrac{dg^{-1}(y)}{dy}\right| = e^{-y/2}\left|\dfrac{1}{2}\right| & 0<y<1 \\ \text{add } \underline{\underline{3 \text{ pieces}}} \text{ of } f_X(g^{-1}(y))\left|\dfrac{dg^{-1}(y)}{dy}\right| & 1\le y\le 2 \\ = e^{-1/2}\left|1/2\right| + e^{-(3-y)}\left|-1\right| + e^{-\left(\frac{y-1}{3}\right)-2}\left|\dfrac{1}{3}\right| & \\ f_X(g^{-1}(y))\left|\dfrac{dg^{-1}(y)}{dy}\right| = e^{-\frac{(y-1)}{3}-2}\left|\dfrac{1}{3}\right| & y>2 \\ 0 & \text{ow} \end{cases}$$

Functions of a random variable

Continuous r.v.s: the non-monotone case

Note that unless g is strictly monotone (or at least there's a way to break up $\mathcal{X} = \{x : f_X(x) > 0\}$ into several intervals on each of which g is strictly increasing or decreasing), then X being a continuous r.v. does not necessarily imply that $Y = g(X)$ will be a continuous r.v.

Example: X has pdf $f_X(x) = 1$ for $0 < x < 1$ and $g(x) = \begin{cases} 0 & x < 1/3 \\ 1 & x \geq 1/3 \end{cases}$



Note that $Y = g(X)$ must be discrete.

$$\mathbb{P}(Y=0) = \mathbb{P}(g(X)=0) = \mathbb{P}(X < 1/3) = \int_{-\infty}^{1/3} f_X(x)\,dx$$

$$= \int_0^{1/3} 1\,dx = 1/3$$

$$\mathbb{P}(Y=1) = 2/3$$

$$P(Y=y) = f_Y(y) = \begin{cases} 1/3 & y=0 \\ 2/3 & y=1 \\ 0 & \text{ow} \end{cases}$$