STAT 5430

Lec 20, M, Mar 10

sufficiency & completeness → Homework 4 posted, due M, Mar 10

Basu's theorem & Intro to testing → Homework 5 posted, due M, Mar 24 (after break)

STAT 5430: Summary to date
Where we have been
& where we are headed

- Completed
  - Introduction to Statistical Inference
  - Point Estimation
    - MME/MLE
  - Criteria for Evaluating Point Estimators
    - bias, variance, UMVUE, MSE
  - Elements of Decision Theory
    - Minimax, finding Bayes estimators
  - Sufficiency and Point Estimation
    - Factorization Theorem, Rao-Blackwell Theorem, Completeness, Lehman-Scheffe Theorem/UMVUE

- Next: Hypothesis Testing I
  - General Concepts: hypotheses, size, power
  - Most Powerful Tests/Neyman-Pearson Lemma
  - Uniformly Most Powerful Texts/Monotone Likelihood Ratios

Hypothesis Testing I

Terminology

Definitions

1. A (statistical) hypothesis is a statement about a population parameter.

   "claim"

2. The two complementary hypotheses in a testing problem are called the **null hypothesis** (denoted H_0) and the **alternative hypothesis** (denoted H_1).

   "starting claim" H_0

   "counter-claim" (what you want to show) H_1 or H_a

Example : $\underset{\sim}{\theta} \equiv (\theta_1, \theta_2) \rightarrow$ average blood sugar level of a group of patients before and after taking a new drug

before, after

H_0 : no effect $\leftrightarrow$ $H_0 : \theta_1 = \theta_2$ (no drug effect)

H_1 : effective $\leftrightarrow$ $H_a : \theta_1 \neq \theta_2$ or $\theta_1 > \theta_2$ (drug has effect)

H_0 or H_1

Definition: If a statistical hypothesis H completely specifies the distribution of $(X_1, \ldots, X_n)$, then it is **simple**; otherwise H is called **composite**.

e.g. $X_1, \ldots, X_n$ iid $N(\mu, 1)$, $\mu \in \mathbb{R}$

$H_0 : \mu = 0$ (simple hypothesis, we know $X_i \sim N(0,1)$)

$H_0 : \mu = 1$ (simple hypothesis)

$H_0 : \mu \leq 0$ (composite hypothesis)

$H_0 : \mu \neq 0$ (composite hypothesis)

$H_1 : \mu > 1$ (composite hypothesis)

$H_1 : \mu = 2$ (simple hypothesis)

e.g. $X_1, \ldots, X_n$ iid $N(\mu, \sigma^2)$

$H_0 : \mu = 0$ (composite)

Hypothesis Testing I

Terminology: Test Functions

Definition: Let $\mathcal{X}$ be the set of all possible values of $\underset{\sim}{X} = (X_1, \dots, X_n)$. Then, a **test function** or a **test rule** $\phi(X_1, \dots, X_n)$ is a function from $\mathcal{X}$ into $[0,1]$ with the interpretation that if $\underset{\sim}{X} = (X_1, \dots, X_n)$ is observed then H_0 is rejected with probability $\phi(\underset{\sim}{X}) \equiv \phi(X_1, \dots, X_n)$.

data $\underset{\sim}{X} \rightarrow \phi(\underset{\sim}{X}) \in [0,1] \Rightarrow$ "coin flip" $Y \sim \text{Bernoulli}(\phi(\underset{\sim}{X}))$

$\Rightarrow$ if $\begin{cases} Y = 1 & (\text{w. prob } \phi(\underset{\sim}{X})), \text{ reject } H_0 \\ Y = 0 & (\text{w. prob } 1 - \phi(\underset{\sim}{X})), \text{ don't reject } H_0 \text{ (accept } H_0) \end{cases}$

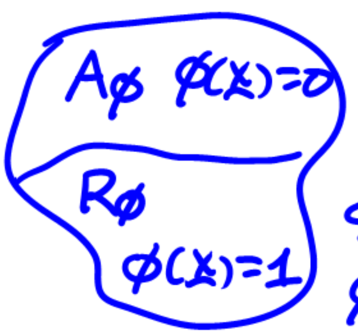
e.g. $\underset{\sim}{X} \rightarrow \phi(\underset{\sim}{X}) = 3/7$ reject H_0 w. prob $3/7$, don't reject H_0 w. prob $4/7$

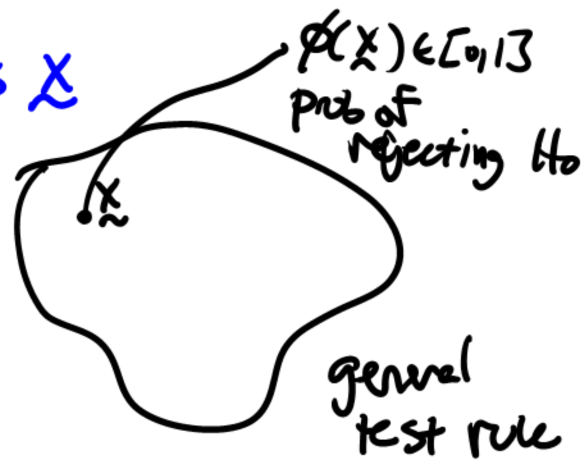
Note: $\phi(\underset{\sim}{X}) = 1 \Rightarrow$ reject H_0
$\phi(\underset{\sim}{X}) = 0 \Rightarrow$ don't reject H_0

Definition: If $\phi(\underset{\sim}{X}) \in \{0,1\}$, $\forall \underset{\sim}{X} \in \mathcal{X}$, then

1. $\phi(\underset{\sim}{X})$ is called a **simple test function** (rule). ⇐ two possibilities reject H_0 ($\phi(\underset{\sim}{X}) = 1$) or don't reject H_0 $\phi(\underset{\sim}{X}) = 0$)
2. $R_\phi = \{\underset{\sim}{X} : \phi(\underset{\sim}{X}) = 1\}$ is called the **rejection region** of $\phi(\underset{\sim}{X})$.
3. $A_\phi = \{\underset{\sim}{X} : \phi(\underset{\sim}{X}) = 0\}$ is called the **acceptance region** of $\phi(\underset{\sim}{X})$.

Little Picture: all possible data outcomes $\underset{\sim}{X}$





Hypothesis Testing I

Terminology: Error Probabilities

"accept" ≡ don't reject Ho

		action based on $\phi(\cdot)$	
		fail to reject H_0	reject H_0
the state of nature	H_0 is true	OK	Type I error
	H_0 is false	Type II error	OK

Note: Suppose $\phi(\underset{\sim}{X}) \in \{0, 1\}$ is a **simple** test rule for testing

$$H_0 : \underset{\sim}{\theta} \in \underbrace{\Theta_0}_{\text{some subset}} \text{ vs. } H_1 : \underset{\sim}{\theta} \notin \Theta_0 \text{ for some subset } \Theta_0 \subset \Theta. \text{ Then,}$$

(some parameter $\underset{\sim}{\theta}$ must generate data $\underset{\sim}{X}$)

1. for any $\underset{\sim}{\theta} \in \Theta_0$, the probability of type I error at $\underset{\sim}{\theta}$

Ho is true

key step

$$P_{\underset{\sim}{\theta}}(\text{reject } H_0) \;=\; P_{\underset{\sim}{\theta}}(\underset{\sim}{X} = (X_1, \ldots, X_n) \in R_\phi) \;=\; \mathrm{E}_{\underset{\sim}{\theta}}\phi(\underset{\sim}{X}),$$

$$\text{since } \phi(\underset{\sim}{X}) = \begin{cases} 1 & \text{if } \underset{\sim}{X} \in R_\phi \\ 0 & \text{if } \underset{\sim}{X} \notin R_\phi \quad \leftarrow \text{if } \underset{\sim}{X} \in A_\phi \end{cases} \Rightarrow \; E_{\underset{\sim}{\theta}}\phi(\underset{\sim}{X}) = 1 \cdot P_{\underset{\sim}{\theta}}(\underset{\sim}{X} \in R_\phi) + 0 \cdot P_{\underset{\sim}{\theta}}(\underset{\sim}{X} \in A_\phi)$$

$$= P_{\underset{\sim}{\theta}}(\underset{\sim}{X} \in R_\phi) = P_{\underset{\sim}{\theta}}(\text{reject Ho}) \text{ for } \underset{\sim}{\theta} \in \Theta_0$$

2. and for any $\underset{\sim}{\theta} \notin \Theta_0$, the probability of type II error at $\underset{\sim}{\theta}$ is

Ho is false

$$P_{\underset{\sim}{\theta}}(\underbrace{\text{fail to reject } H_0}_{\text{don't reject Ho}}) \;=\; P_{\underset{\sim}{\theta}}(\underset{\sim}{X} \in A_\phi) \;=\; 1 - P_{\underset{\sim}{\theta}}(\underset{\sim}{X} \in R_\phi) \;=\; 1 - E_{\underset{\sim}{\theta}}\phi(\underset{\sim}{X})$$

$$= 1 - P_{\underset{\sim}{\theta}}(\text{reject Ho})$$

$$\underset{\sim}{\theta} \xrightarrow{\text{generates}} \underset{\sim}{X} \xrightarrow{\text{rule}} \phi(\underset{\sim}{X}) \qquad E_{\underset{\sim}{\theta}}\phi(\underset{\sim}{X}) = P_{\underset{\sim}{\theta}}(\text{reject Ho})$$

Want $E_{\underset{\sim}{\theta}}\phi(\underset{\sim}{X})$ to be **small** whenever $\underset{\sim}{\theta}$ satisfies Ho

$E_{\underset{\sim}{\theta}}\phi(\underset{\sim}{X})$ to be **large** whenever $\underset{\sim}{\theta}$ doesn't satisfy Ho

Hypothesis Testing I

Terminology: Error Probabilities

$\to \phi(\underset{\sim}{X}) \in [0,1]$

Remark: For any **general** test function, the same holds true: $\phi(\cdot)$,

1. Prob. of a type I error at $\underset{\sim}{\theta}$ ($\underset{\sim}{\theta} \in \Theta_0$) $=$ $P_{\underset{\sim}{\theta}}$(reject H_0) $=$ $E_{\underset{\sim}{\theta}}\phi(\underset{\sim}{X})$

2. Prob. of a type II error at $\underset{\sim}{\theta}$ ($\underset{\sim}{\theta} \notin \Theta_0$) $=$ $P_{\underset{\sim}{\theta}}$(fail to reject H_0) $=$ $1 - E_{\underset{\sim}{\theta}}\phi(\underset{\sim}{X})$.

Same as for simple test rules ($\phi(\underset{\sim}{X}) = 0$ or 1)

e.g. $X \sim \text{Binomial}(2, \theta)$, $0<\theta<1$

general test rule
$$\phi(X) = \begin{cases} 1 & \text{if } x=0 \\ \frac{1}{2} & \text{if } x=1 \\ 0 & \text{if } x=2 \end{cases} \Rightarrow E_\theta\phi(X) = 1 \cdot P_\theta(X=0) + \frac{1}{2}P_\theta(X=1) + 0\, P_\theta(X=2) = P_\theta(\text{reject } H_0)$$

Definition: Let $\phi(\underset{\sim}{X})$ be a test rule for testing $H_0 : \underset{\sim}{\theta} \in \Theta_0$ vs $H_1 : \underset{\sim}{\theta} \notin \Theta_0$,

1. $\max_{\underset{\sim}{\theta} \in \Theta_0} E_{\underset{\sim}{\theta}}\phi(\underset{\sim}{X})$ is called **the size** or **the level** of $\phi(\underset{\sim}{X})$

2. $\Pi_\phi(\underset{\sim}{\theta}) = E_{\underset{\sim}{\theta}}\phi(\underset{\sim}{X})$ is called the **power function** of $\phi(\underset{\sim}{X})$.

Note: For $\underset{\sim}{\theta} \in \Theta_0$, $\Pi_\phi(\underset{\sim}{\theta}) = \mathrm{E}_{\underset{\sim}{\theta}}\phi(\underset{\sim}{X}) =$ probability of type I error

For $\underset{\sim}{\theta} \notin \Theta_0$, probability of type II error $= 1 - \Pi_\phi(\underset{\sim}{\theta}) = 1 - \mathrm{E}_{\underset{\sim}{\theta}}\phi(\underset{\sim}{X})$