Common univariate distributions

Continuous distributions: Gamma

$X \sim$Gamma(α, β) $\quad \alpha > 0, \beta > 0$

α: Shape Parameter Controls the overal form of the distribution (how peaked or skewed)
β: Scale Parameters Control the dispersion or spread

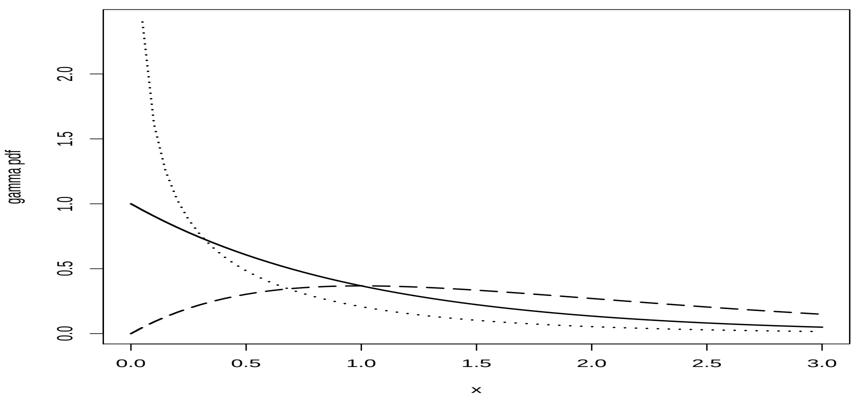
- pdf given by

$$f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \quad 0 < x < \infty$$

- Motivation: flexible family for positive quantities

- $\alpha > 0$ is shape parameter.

  ($\alpha < 1$ density unbounded near $x = 0$, $\alpha > 1$ density is zero at $x = 0$)



- $\beta > 0$ is scale parameter. $\quad$ i.e., if $X \sim$Gamma(α, β) then $Z = \frac{X}{\beta} \sim$Gamma$(\alpha, 1)$

- $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$, $\alpha > 0$, is the gamma function, which ensures that $f_X(x)$ is a density

  Some properties

  1. $\Gamma(1 + \alpha) = \alpha\Gamma(\alpha)$ for $\alpha > 0$
  2. $\Gamma(\alpha) = (\alpha - 1)!$ for integer $\alpha \geq 1$
  3. $\Gamma(1/2) = \sqrt{\pi}$

Common univariate distributions

Continuous distributions: Gamma (cont'd)

$X \sim$Gamma(α, β) $\quad \alpha > 0, \beta > 0$

- $\mathrm{E}X^r = \beta^r \Gamma(\alpha + r)/\Gamma(\alpha)$ for $r > 0$

Proof: $\quad \mathrm{E}X^r = \dfrac{1}{\Gamma(\alpha)\beta^\alpha} \displaystyle\int_0^\infty x^{r+\alpha-1} e^{-x/\beta} dx = \dfrac{\Gamma(r+\alpha)\beta^{r+\alpha}}{\Gamma(\alpha)\beta^\alpha}$

$$r=1 \Rightarrow \mathbb{E}(X) = \frac{\Gamma(\alpha+1)\beta^{\alpha+1}}{\Gamma(\alpha)\beta^\alpha} = \alpha\beta$$

- Mean: $\mathrm{E}X = \alpha\beta$

$$r=2 \Rightarrow \mathbb{E}X^2 = \cdots$$

- Variance: $\mathrm{Var}(X) = \mathrm{E}X^2 - [\mathrm{E}X]^2 = \alpha\beta^2$

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

- mgf: $M_X(t) = \mathrm{E}e^{tX} = (1 - \beta t)^{-\alpha}$, $t < 1/\beta$

$$M_X(t) \overset{\text{def}}{=} \mathbb{E}\left[e^{tX}\right] = \int_0^\infty e^{tx} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx$$

Recall: $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha}\int_0^\infty x^{\alpha-1} e^{-x\left[\frac{1}{\beta}-t\right]} dx \quad \left(\frac{1}{\beta}-t\right)>0 \quad \left(\frac{1}{\beta}-t\right)^\alpha$$

$x\left(\frac{1}{\beta}-t\right)=y$

$dx\left(\frac{1}{\beta}-t\right)=dy$

$$\frac{1}{\Gamma(\alpha)\beta^\alpha}\int_0^\infty \left[\frac{y}{\left(\frac{1}{\beta}-t\right)}\right]^{\alpha-1} e^{-y} \frac{dy}{\left(\frac{1}{\beta}-t\right)}$$

$$= \frac{(1-\beta t)^\alpha}{\beta^\alpha} \qquad \frac{1}{\left(\frac{1}{\beta}-t\right)^\alpha} = \frac{\beta^\alpha}{(1-\beta t)^\alpha}$$

$$= \frac{1}{\cancel{\Gamma(\alpha)}\beta^\alpha\left(\frac{1}{\beta}-t\right)^\alpha}\underbrace{\int_0^\infty y^{\alpha-1} e^{-y} dy}_{\cancel{\Gamma(\alpha)}}$$

$$\circledast = \frac{\beta^\alpha}{(1-\beta t)^\alpha \cancel{\beta^\alpha}} = (1-\beta t)^{-\alpha} \Longrightarrow M_X(t) = (1-\beta t)^{-\alpha} \text{ for } t < \frac{1}{\beta}$$

- Relationship of gamma and Poisson cdfs for integer α:

HW4

$$F_X(x|\alpha,\beta) = P(Y \geq \alpha) \quad \text{where } Y \sim \text{Poisson}(x/\beta)$$

FYI

Common univariate distributions

Continuous distributions: Gamma

Special Cases

- Chi-squared: χ^2_p =Gamma($\alpha = p/2, \beta = 2$), integer $p > 0$ "degree of freedom" parameter

- Exponential: Exp(β) =Gamma($\alpha = 1, \beta$)

$$f_X(x) = \frac{1}{\beta}e^{-x/\beta}, \quad 0 < x < \infty$$

  used in models for failure times

  (memoryless: $P(X > s + t|X > t) = P(X > s)$)

- Weibull: If $X \sim$Exp(β) and $\gamma > 0$, then $\underbrace{W = X^{1/\gamma} \sim \text{Weibull}(\gamma, \beta)}$

$$f_W(w) = \frac{\gamma}{\beta}w^{\gamma-1}e^{-w^\gamma/\beta}, \quad 0 < w < \infty$$

  (This is important in 5330)

  general failure time distribution

- Inverse-Gamma: If $X \sim$Gamma(α, β) then $Y = 1/X$ has the inverse gamma distribution

$$\underbrace{f_Y(y) = \frac{1}{\Gamma(\alpha)\beta^\alpha}\left(\frac{1}{y}\right)^{1+\alpha} e^{-\frac{1}{\beta y}}, \quad 0 < y < \infty}$$

Common univariate distributions

Continuous distributions: Beta

$X \sim \text{Beta}(\alpha, \beta) \quad \alpha > 0, \beta > 0$

$\alpha < 1 \Rightarrow \alpha - 1 < 0,\ x \uparrow 0 \quad f_X(x) \to \infty$

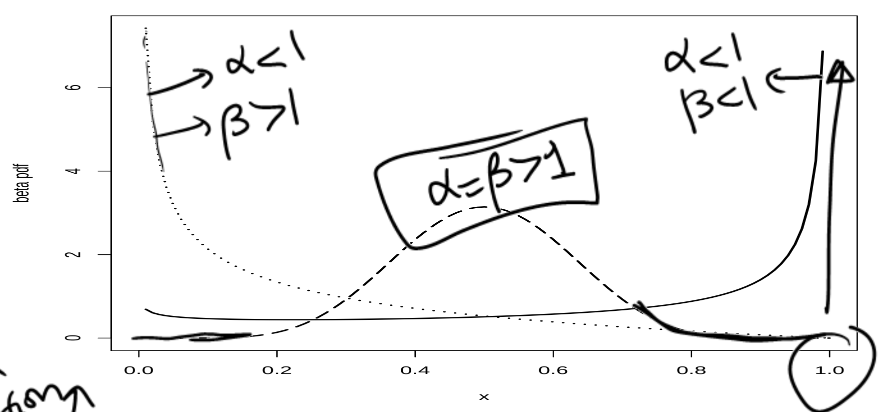
- pdf given by

$$f_X(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad \boxed{0 < x < 1}$$

- Motivation: flexible family, often for modeling quantities as proportions

- $\alpha, \beta > 0$ are both shape parameters

  1. α determines shape near $x = 0$
     ($\alpha < 1$ density unbounded near $x = 0$, $\alpha > 1$ density is zero at $x = 0$)
  2. β determines shape near $x = 1$
     ($\beta < 1$ density unbounded near $x = 1$, $\beta > 1$ density is zero at $x = 1$)
  3. $\alpha = \beta$ gives a symmetric distribution



Beta function

- $B(\alpha, \beta) = \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}$ is the beta function

- The distribution can be moved from the range $(0, 1)$ to any finite range (l, u) by taking $Y = l + X(u - l)$

$Y \sim \text{Beta}(l, u)$ $\quad X \sim \text{Beta}(0,1)$

Recall: $\Gamma(y+1) = y\Gamma(y) \quad \forall y>0 \qquad \Gamma(y+1) = \int_0^\infty x^{y+1} e^{-x}\, dx$

Common univariate distributions

Continuous distributions: Beta (cont'd)

$X \sim$Beta$(\alpha, \beta) \quad \alpha > 0, \beta > 0$

- $\mathrm{E}X^r = = \dfrac{B(\alpha+r,\beta)}{B(\alpha,\beta)} = \dfrac{\Gamma(\alpha+\beta)\Gamma(\alpha+r)}{\Gamma(\alpha+\beta+r)\Gamma(\alpha)}$ for $r > 0$

Proof: $\quad \mathrm{E}X^r = \dfrac{1}{B(\alpha,\beta)} \displaystyle\int_0^1 x^{r+\alpha-1}(1-x)^{\beta-1}dx$

$$\mathbb{E}X^r = \int_0^1 x^r \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\, dx$$

$$= \int_0^1 \frac{x^{r+\alpha-1}}{B(\alpha,\beta)}(1-x)^{\beta-1}\, dx$$

$$= \frac{1}{B(\alpha,\beta)} \int_0^1 \boxed{x^{(r+\alpha)-1}(1-x)^{\beta-1} \frac{B(r+\alpha,\beta)}{B(r+\alpha,\beta)}}\, dx$$

$Y \sim$ density of a Beta$(r+\alpha, \beta)$

$$= \frac{B(r+\alpha,\beta)}{B(\alpha,\beta)} \underbrace{\int_0^1 f_Y(y)dy}_{=1} = \frac{\Gamma(r+\alpha)\cancel{\Gamma(\beta)}}{\Gamma(r+\alpha+\beta)} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cancel{\Gamma(\beta)}}$$

$$= \boxed{\frac{\Gamma(r+\alpha)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\alpha+\beta+r)}} \; (*) \to \mathbb{E}X^r$$

- Mean: $\mathrm{E}X = \dfrac{\alpha}{\alpha+\beta}$

$$r=1 \Rightarrow \mathbb{E}[X] = \frac{\Gamma(\alpha+1)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\alpha+\beta+1)} = \frac{\alpha\cancel{\Gamma(\alpha)}\cancel{\Gamma(\alpha+\beta)}}{(\alpha+\beta)\cancel{\Gamma(\alpha+\beta)}\cancel{\Gamma(\alpha)}}$$

- Variance: $\mathrm{Var}(X) = \mathrm{E}X^2 - [\mathrm{E}X]^2 = \left(\dfrac{\alpha}{\alpha+\beta}\right)\left(\dfrac{\beta}{\alpha+\beta}\right)\left(\dfrac{1}{\alpha+\beta+1}\right)$

$$r=2 \Rightarrow \mathbb{E}[X^2] \overset{(*)}{\underset{r=2}{=}} \frac{\Gamma(\alpha+2)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\alpha+\beta+2)} = \cdots$$

- Related Distribution: $U \sim$Uniform$(0,1)$ if $\alpha = \beta = 1$

$$f_X(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

Beta(α,β)

$$B(1,1) = \frac{\Gamma(1)\cancel{\Gamma(1)}}{\cancel{\Gamma(2)}} = 1$$

$$\overset{\alpha=\beta=1}{=} \begin{cases} 1 \quad \cancel{x^0}(1-x)^0 & \text{for } 0<x<1 \\ 0 & \text{ow} \end{cases}$$

$$M_X(t) \overset{def}{=} \int_0^1 \frac{e^{tx} x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$$

Common univariate distributions

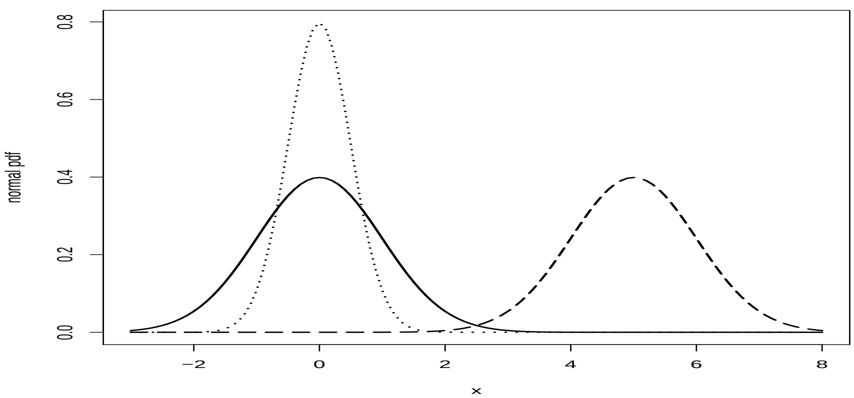
Continuous distributions: Normal (Gaussian) distributon

$X \sim N(\mu, \sigma^2) \quad -\infty < \mu < \infty, \sigma > 0$

- pdf given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty$$

$\Rightarrow \mathbb{E} X = \mu$
$\Rightarrow \mathrm{Var}\, X = \sigma^2$



- Motivation: single most important distribution
  - widely used & analytically tractable
  - bell-shaped density seems to occur naturally
  - Central Limit Theorem (normal distribution is extremely relevant in large samples; more later)

- $\mu \in \mathbb{R}$ is the mean $\mathrm{E}X$ of the distribution

- $\sigma^2 = \mathrm{Var}(X)$ is the variance of the distribution; σ is the standard deviation

- Many properties of the normal distribution can be most easily derived using the $N(0, 1)$ or **standard normal distribution**
  1. If $X \sim N(\mu, \sigma^2)$ then $Z = (X - \mu)/\sigma \sim N(0, 1)$
  2. If $Z \sim N(0, 1)$, then $X = a + bZ \sim N(\mu = a, \sigma^2 = b^2)$ for $a, b \in \mathbb{R}$