Multivariate distributions

Expectations of several random variables (cont'd)

Theoretical properties as before

e.g., if $a_1, \ldots, a_k, b \in \mathbb{R}$ and each $g_i : \mathbb{R}^n \to \mathbb{R}$

$$\mathrm{E}\left(\sum_{i=1}^{k} a_i g_i(X_1, \ldots, X_n) + b\right) = b + \sum_{i=1}^{k} a_i \mathrm{E} g_i(X_1, \ldots, X_n)$$

$$\mathbb{E}\left[\sum_{i=1}^{K} a_i g_i(X_1, \ldots, X_n) + b\right] = \sum_{i=1}^{K} a_i \mathbb{E}\left[g_i(X_1, \ldots, X_n)\right] + b$$

$$\mathbb{E}[\underbrace{a_1(X_1+X_2)+b}_{h(X_1,X_2)}] = \iint_{\mathbb{R}^2} h(x_1,x_2) f_{X_1,X_2}(x_1,x_2)\,dx_1 dx_2 = \iint [a_1(X_1+X_2)+b] f_{X_1,X_2}(x_1,x_2)\,dx_1 dx_2$$

$$= \underbrace{\iint a_1(X_1+X_2) f_{X_1,X_2}(x_1,x_2)\,dx_1 dx_2}_{a \iint h(x_1,x_2) f_{X_1,X_2}(x_1,x_2)\,dx_1 dx_2 \;=\; a\,\mathbb{E}(h(X_1,X_2))} + \underbrace{\iint b f_{X_1,X_2}(x_1,x_2)\,dx_1 dx_2}_{=b}$$

Marginal moments:

$$\mathrm{E}X_i = \mu_{X_i} = \begin{cases} \sum_{(x_1,\ldots,x_n)} x_i f(x_1, \ldots, x_n) & \text{discrete case} \\ \int \cdots \int x_i f(x_1, \ldots, x_n) dx_1 \ldots, dx_n & \text{continuous case} \end{cases}$$

$$\mathbb{E}X = \iint x f(x,y)\,dx\,dy \qquad \mathbb{E}Y = \iint y f(x,y)\,dx\,dy$$

Discrete Example: (X, Y) with joint pmf given in tabular form as

		x		
		1	2	3
y	3	1/12	1/12	1/6
	2	1/12	1/6	1/12
	1	1/6	1/12	1/12

1. $\mathbb{E}X = \sum_{(x,y)} x f_{X,Y}(x,y)$

$$\sum_y 1 f(1,y) + \sum_y 2 f(2,y) + \sum_y 3 f(3,y)$$

$$= 1\left[\tfrac{1}{12}+\tfrac{1}{12}+\tfrac{1}{6}\right] + 2\left[\tfrac{1}{12}+\tfrac{1}{6}+\tfrac{1}{12}\right] + 3\left[\tfrac{1}{6}+\tfrac{1}{12}+\tfrac{1}{12}\right] = 2$$

Can check $\mathbb{E}Y = 2$.

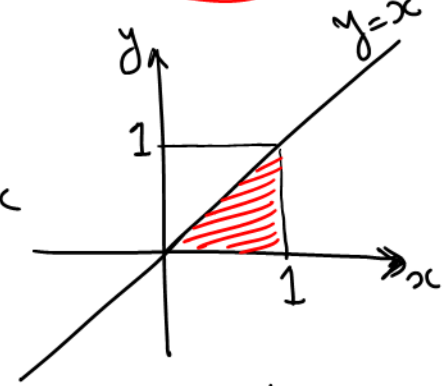
$$\mathbb{E}XY = \mathbb{E}(h(X,Y)) = \sum_x \sum_y h(x,y) f_{X,Y}(x,y) = \sum_x \sum_y xy\, f_{X,Y}(x,y)$$

$$= (1\times1\; f(1,1)) + (1\times2\; f(1,2)) + (1\times3\; f(1,3)) + \cdots = \boxed{50/12}$$

$$\int_y^1\left[\int_0^x \cancel{x}\,\frac{1}{\cancel{x}}\,dy\right]dx = \int_y^1 x\,dx = \frac{x^2}{2}\Big]_y^1 = \frac{1}{2} - y^2/2$$ which is NOT a Real-number, for $\mathbb{E}XY$

Multivariate distributions

Expectation: examples (cont'd)



Continuous example: pdf $f(x, y) = 1/x$ for $0 < y < x < 1$

(a) Find $\mathbb{E}[g(X,Y)] = \iint_{\mathbb{R}^2} g(x,y)\, f_{X,Y}(x,y)\, dy\, dx$

$$= \int_0^1\int_0^x g(x,y)\,\frac{1}{x}\,dy\,dx$$

(b) Use Part (a), to find $\mathbb{E}\underbrace{X}_{g(x,y)} = \int_0^1\left(\int_0^x \cancel{x}\,\frac{1}{\cancel{x}}\,dy\right)dx = \int_0^1 x\,dx = \frac{x^2}{2}\Big]_0^1 = 1/2$

(c) $$\mathbb{E}XY = \int_0^1\int_0^x \cancel{x}y\,\frac{1}{\cancel{x}}\,dy\,dx = \int_0^1\left[y^2/2\right)_0^x dx$$

$$= \int_0^1 \frac{x^2}{2}\,dx = \frac{1}{2}\int x^2\,dx = \frac{1}{6}x^3\Big]_0^1 = 1/6$$

Check $\mathbb{E}Y = 1/4$, $\boxed{\mathbb{E}X^2 = 1/3}$, $\boxed{\mathbb{E}Y^2 = 1/9}$

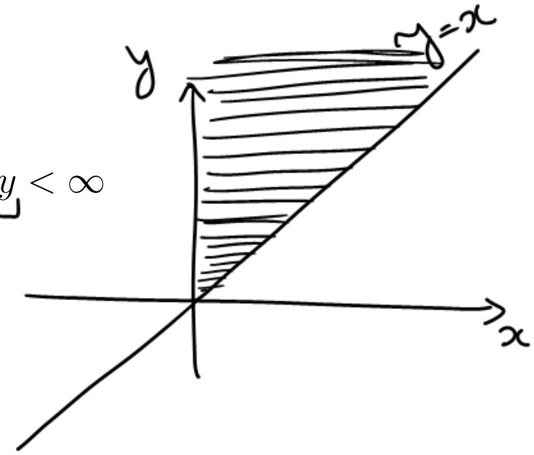
(d) Define $W = X - Y$. Find $\mathrm{Var}(W) = ?$

$$\begin{cases} \mathrm{Var}(W) = \mathbb{E}[W^2] - (\mathbb{E}W)^2 \\ \mathbb{E}W = \mathbb{E}(X-Y) = \mathbb{E}X - \mathbb{E}Y = 1/2 - 1/4 = 1/4 \\ \mathbb{E}W^2 = \mathbb{E}[(X-Y)^2] = \mathbb{E}[X^2 + Y^2 - 2XY] = \mathbb{E}[X^2] + \mathbb{E}[Y^2] - 2\mathbb{E}[XY] \\ \qquad = 1/3 + 1/9 - 2/6 = 1/9 \end{cases}$$

Multivariate distributions

Expectation: examples (cont'd)

Another continuous example: pdf $f(x,y) = e^{-y}$ for $0 < x < y < \infty$



$$\mathbb{E}\left[g(X,Y)\right] = \iint_{\mathbb{R}^2} g(x,y) \underbrace{f_{X,Y}(x,y)}\, dx\, dy$$

$$= \int_0^\infty \int_0^y g(x,y) e^{-y}\, dx\, dy$$

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx$$

check

$X \to g(X,Y)$

$$
\begin{aligned}
\mathrm{E}X &= \int_0^\infty \int_0^y x e^{-y} dx dy = \int_0^\infty \frac{1}{2} y^2 e^{-y} dy = \frac{\Gamma(3)}{2} = \frac{2!}{2} = 1 \\
\mathrm{E}Y &= \int_0^\infty \int_0^y y e^{-y} dx dy = \int_0^\infty y^2 e^{-y} dy = \Gamma(3) = 2! = 2 \\
\mathrm{E}XY &= \int_0^\infty \int_0^y xy e^{-y} dx dy = \int_0^\infty \frac{1}{2} y^3 e^{-y} dy = \frac{\Gamma(4)}{2} = \frac{3!}{2} = 3
\end{aligned}
$$

$f_X(x) = \int_x^\infty e^{-y} dy = e^{-x}$, $x > 0$; $\quad f_Y(y) = \int_0^y e^{-y} dx = y e^{-y}$, $y > 0$;

$$f_X(x) = \int_{\mathbb{R}} \underbrace{f_{X,Y}(x,y)}\, dy = \int_x^\infty e^{-y}\, dy = -e^{-y}\Big)_x^\infty = \cancel{-e^{-\infty}} - (-e^{-x})$$

$$= \underbrace{e^{-x}}_{f_X(x)} \quad \forall x > 0$$

Note: If $f_X(x) = \int_0^\infty e^{-y}\, dy = 1 \implies f_X(x) = 1 \ \forall x > 0$

$$\int_0^\infty f_X(x)\, dx = \int_0^\infty 1\, dx = \int_0^\infty dx \to \infty$$

$\therefore$

Multivariate distributions

Covariance

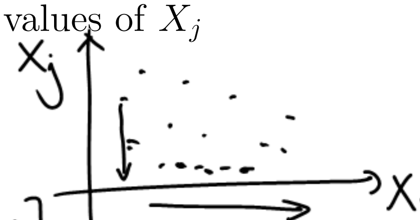
$Cov(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]$

Definition **Covariance of X_i and X_j** is

$$\mathrm{Cov}(X_i, X_j) = \sigma_{X_i,X_j} = \mathrm{E}\big[(X_i - \mathrm{E}X_i)(X_j - \mathrm{E}X_j)\big] = \mathrm{E}X_iX_j - (\mathrm{E}X_i)(\mathrm{E}X_j)$$

Interpretation:

- $\mathrm{Cov}(X_i, X_j) > 0$ means that larger (or smaller) than average values of X_i tend to occur with larger (or smaller) than average values of X_j
- $\mathrm{Cov}(X_i, X_j) < 0$ means that larger (or smaller) than average values of X_i tend to occur with smaller (or larger) than average values of X_j



Relationships:

1. $\mathrm{Cov}(X_i, X_i) = \mathrm{Var}(X_i)$ $\quad Cov(X_i, X_i) \overset{def}{=} \mathbb{E}[X_iX_i] - \mathbb{E}[X_i]\,\mathbb{E}[X_i] = \mathbb{E}[X_i^2] - (\mathbb{E}X_i)^2 = Var(X_i)$
2. $\mathrm{Cov}(X_i, X_j) = \mathrm{Cov}(X_j, X_i)$
3. $\mathrm{Cov}(aX_i + c, bX_j + d) = ab\mathrm{Cov}(X_i, X_j)$
4. $\mathrm{Var}(aX_i + bX_j + c) = a^2\mathrm{Var}(X_i) + b^2\mathrm{Var}(X_j) + 2ab\mathrm{Cov}(X_i, X_j)$

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$Var(Z) = Cov[Z,Z] = \mathbb{E}[Z^2] - (\mathbb{E}(Z))^2 = \mathbb{E}[(Z - \mathbb{E}Z)^2]$

Proof

$$\begin{aligned}
&\mathrm{Var}(\overset{Z}{aX_i + bX_j + c}) \\
&= \mathrm{E}\left(\Big[\underbrace{aX_i + bX_j + c - \mathrm{E}(aX_i + bX_j + c)}_{Z}\Big]^2\right) \\
&= \mathrm{E}\left(\Big[aX_i + bX_j + c - a\mathrm{E}X_i - b\mathrm{E}X_j - c\Big]^2\right) \\
&= \mathrm{E}\left(\Big[a(X_i - \mathrm{E}X_i) + b(X_j - \mathrm{E}X_j)\Big]^2\right) \\
&= \mathrm{E}\big(a^2(X_i - \mathrm{E}X_i)^2 + b^2(X_j - \mathrm{E}X_j)^2 + 2ab(X_i - \mathrm{E}X_i)(X_j - \mathrm{E}X_j)\big)
\end{aligned}$$

Multivariate distributions

Covariance (cont'd)

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5. more generally,

$$
\begin{aligned}
\operatorname{Cov}\left(c+\sum_{i=1}^{n} a_i X_i, d+\sum_{j=1}^{n} b_j X_j\right) &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j \operatorname{Cov}(X_i, X_j) \\
&= \sum_{i=1}^{n} a_i b_i \operatorname{Var}(X_i) + 2\sum_{1\le i<j\le n} a_i b_j \operatorname{Cov}(X_i, X_j)
\end{aligned}
$$

6. also, (4. above is a special case of the following)

$$
\begin{aligned}
\operatorname{Var}\left(c+\sum_{i=1}^{n} a_i X_i\right) &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \operatorname{Cov}(X_i, X_j) \\
&= \sum_{i=1}^{n} a_i^2 \operatorname{Var}(X_i) + 2\sum_{1\le i<j\le n} a_i a_j \operatorname{Cov}(X_i, X_j)
\end{aligned}
$$