Convergence concepts

Convergence in probability: weak law of large numbers

Theorem: Weak Law of Large Numbers (WLLN). Suppose $X_1, X_2, \ldots$ are iid having $\mathrm{E}X_1 = \mu$ and $\mathrm{Var}(X_1) = \sigma^2 < \infty$. Let $Y_n = \bar{X}_n = \sum_{i=1}^n X_i/n$. Then

$$Y_n = \bar{X}_n \xrightarrow{p} \mu \quad \text{as } n \to \infty.$$

Sample mean — mean of R.P.

Proof: Pick/fix $\epsilon > 0$. Then,

$$P(|Y_n - \mu| \geq \epsilon) = P(|\bar{X}_n - \mu| \geq \epsilon)$$

$$= \mathbb{P}(|\bar{X}_n - \mu|^2 \geq \varepsilon^2)$$

$$\leq \frac{\mathbb{E}[|\bar{X}_n - \mu|^2]}{\varepsilon^2} = \frac{\mathrm{Var}(\bar{X}_n)}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2}$$

$\mathbb{E}[\bar{X}_n] = \mu$

$\mathrm{Var}(\bar{X}_n) = \frac{\sigma^2}{n}$

Recall:

$\mathrm{Var}(W) = \mathbb{E}[(W - \mathbb{E}(W))^2]$

is a R.V.

Say, $W = \bar{X}_n$

$\underbrace{\mathrm{Var}(\bar{X}_n) = \mathbb{E}[(\bar{X}_n - \mu)^2]}$

$$\lim_{n\to\infty} \mathbb{P}(|Y_n - \mu| \geq \varepsilon) \leq \underbrace{\lim_{n\to\infty} \frac{\sigma^2}{n\varepsilon^2} = 0}$$

$$\boxed{Y_n \xrightarrow{P} \mu}$$

Examples:

1. $X_1, X_2, \ldots$ iid Bernoulli(p):

   $\mathbb{E}\bar{X}_n = \mu = p \overset{\text{WLLN}}{\Longrightarrow} \bar{X}_n \xrightarrow{P} p$

2. Let $X_1, X_2, \ldots$ iid with $\boxed{\mathrm{E}X_1^4 < \infty}$. Define $W_i = X_i^2$, $i \geq 1$

   $W_1, W_2, \ldots$

   $\bar{W}_n \longrightarrow \mathbb{E}[W_1] = \mathbb{E}[X_1^2]$

   Sample mean

   $$\Longrightarrow \boxed{\frac{1}{n}\sum_{i=1}^n W_i \xrightarrow{P} \mathbb{E}(X_1^2)}$$

   $\mathbb{E}W_i < \infty$

   $\mathrm{Var}(W_i) = \mathbb{E}[W_i^2] - (\mathbb{E}[W_i])^2$

   $= \mathbb{E}[X_i^4] - (\mathbb{E}[X_i^2])^2$

Convergence concepts

Convergence in distribution

Definition: Y_n **converges in distribution to** Y, denoted as $Y_n \xrightarrow{d} Y$ as $n \to \infty$, if

$$\boxed{\lim_{n\to\infty} F_{Y_n}(y) = F_Y(y)}$$

for **any** $y \in \mathbb{R}$ *at which* $F_Y(\cdot)$ *is continuous* (i.e., **not** all y)

- Concerns the limiting distribution of a sequence of r.v.s

  The distribution $F_Y(y)$ is Called
  the limiting dist.

- This is the most useful convergence concept for us:

  If $X_n \xrightarrow{d} X$ then assuming the cdf F_X of X is continuous at $a, b \in \mathbb{R}$,

  $$P(a < X_n \leq b) = F_{X_n}(b) - F_{X_n}(a) \to F_X(b) - F_X(a) = P(a < X \leq b)$$

  as $n \to \infty$

  For "large n" we can approximate probabilities of X_n with probabilities of X

  $X_n \xrightarrow{d} X_n \qquad X_n \Longrightarrow X$

- Also called weak convergence

Convergence concepts

Convergence in distribution: examples

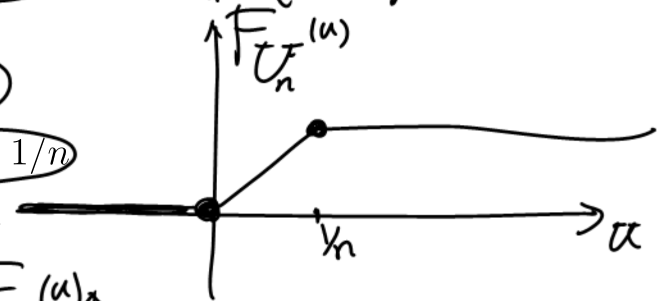
Example 1: $U_n \sim$Uniform$(0, 1/n)$. Show $U_n \xrightarrow{d} U$ where $U = 0$ is degenerate.

$\mathbb{P}(U=0)=1$

(Note, for this example, already showed that $U_n \xrightarrow{p} U$)

$\lim_{n\to\infty} F_{U_n}(u) = F_U(u)$

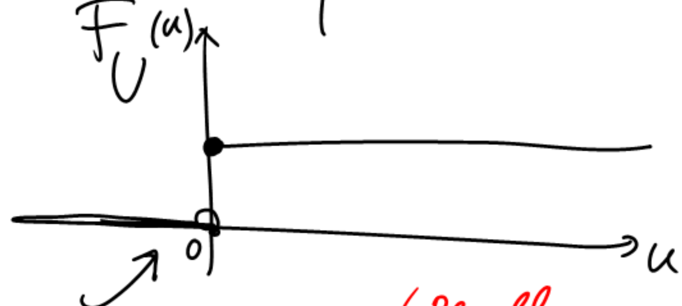
$$F_{U_n}(u) \overset{\text{def}}{=} P(U_n \le u) = \begin{cases} 0 & \text{if } u \le 0 \\ \underbrace{\left(\frac{u-0}{1/n-0}\right) = un}_{} & \text{if } 0 < u < 1/n \\ 1 & \text{if } u \ge 1/n \end{cases}$$



$\mathbb{P}(U=0)=1$

II

$$F_U(u) = P(U \le u) = \begin{cases} 0 & \text{if } u < 0 \\ 1 & \text{if } u \ge 0 \end{cases}$$



I

$$\lim_{n\to\infty} F_{U_n}(u) = \begin{cases} 0 & \text{if } u \le 0 \\ 1 & \text{if } u > 0 \end{cases}$$

Recall: $\lim_{n\to\infty} f(n) = \ell$

$\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \ge N$, $|f(n)-\ell| < \varepsilon$

to see this,

If $u \le 0 \Rightarrow F_n(u) = 0 \quad \forall n \in \mathbb{N} \Rightarrow \lim_{n\to\infty} F_n(u) = 0$

If $u > 0$, then for some N (depends on n) for which $u > \frac{1}{n}$ for all $n > N$ $\underbrace{\lim_{n\to\infty} F_n(u) = 1}_{n \ge N}$

$\Rightarrow U_n \xrightarrow{d} U$

Note: For this example:

$U_n \xrightarrow{d} U=0$ and we have seen befor $U_n \xrightarrow{P} U=0$

Convergence concepts

Convergence in distribution: examples

largest $X_1, \dots, X_n$

Example 2: $X_1, X_2, \dots$ iid Uniform(a, b) & let $Y_n = \max\{X_1, \dots, X_n\} = X_{(n:n)}$. $\equiv X_{(n)}$

Show $Y_n \xrightarrow{d} Y$ where $Y = b$ is degenerate.

$X_{(n)} \xrightarrow{d} Y$ where $\mathbb{P}(Y=b)=1$

$X_{n:n} = X_{(n)}$

$X_{k:n} = X_{(k)}$

$\lim_{n\to\infty} F_{Y_n}(y) = \lim_{n\to\infty} F_{X_{(n)}}(y) = F_Y(y)$ where $Y=b$

$$F_{Y_n}(y) = \underbrace{P(Y_n \le y)}_{\mathbb{P}(X_{(n)} \le y)} = [P(X_1 \le y)]^n = \begin{cases} 0 & \text{if } y \le a \\ \left(\frac{y-a}{b-a}\right)^n & \text{if } a < y < b \\ 1 & \text{if } y \ge b. \end{cases}$$

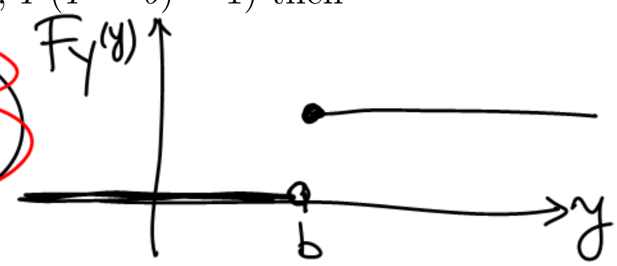
I

$$\lim_{n\to\infty} F_{Y_n}(y) = \begin{cases} 0 & \text{if } y < a \\ 0 & \text{if } a < y < b \\ 1 & \text{if } y \ge b. \end{cases}$$

If Y is a r.v. with a degenerate distribution at b (i.e., $P(Y = b) = 1$) then

II

$$F_Y(y) = \begin{cases} 0 & \text{if } y < b \\ 1 & \text{if } y \ge b. \end{cases}$$



For $Y \neq b$, $F_{Y_n}(y) \to F_Y(y)$ as $n \to \infty$

$\mathbb{P}(X_{(n)} \le x) = \mathbb{P}[\underbrace{\max(X_1, \dots, X_n)}_{} \le x] = \mathbb{P}(X_1 \le x, \dots, X_n \le x)$
$\overset{\text{ind}}{=} \mathbb{P}(X_1 \le x) \cdots \mathbb{P}(X_n \le x) \overset{\text{identically dis}}{=} [\mathbb{P}(X_1 \le x)]^n$

Convergence concepts

Convergence in distribution: examples (cont'd)

Example 3: $X_1, X_2, \dots$ iid Exponential(1) & let $Y_n = X_{(n:n)} - \log n$

$Y_n = X_{(n)} - \log n$

Step 1: Find $F_{Y_n}(y)$ (def of cdf)

$\mathbb{P}(X_{(n)} \le x) = [\mathbb{P}(X_1 \le x)]^n$

$$
\begin{aligned}
F_{Y_n}(y) &= P(X_{(n:n)} - \log n \le y) = \mathbb{P}(X_{(n)} - \log n \le y) \\
&= P(X_{(n:n)} \le y + \log n) = \mathbb{P}(X_{(n)} \le y + \log n) \\
&= [P(X_1 \le y + \log n)]^n \\
&= \begin{cases} 0 & \text{if } y \le -\log n \\ (1 - e^{-(y+\log n)})^n & \text{if } y > -\log n \end{cases}
\end{aligned}
$$

Fix $y \in \mathbb{R}$ and note

$$
\begin{aligned}
\lim_{n\to\infty} F_{Y_n}(y) &= \lim_{n\to\infty} (1 - e^{-(y+\log n)})^n \\
&= \lim_{n\to\infty} (1 - e^{-y} e^{-\log n})^n \\
&= \lim_{n\to\infty} (1 - e^{-y} n^{-1})^n \\
&= e^{-e^{-y}}
\end{aligned}
$$

$$\lim_{n\to\infty} F_{Y_n}(y) = \begin{cases} 0 & y \le -\log n \\ e^{-e^{-y}} & y > \log n \end{cases}$$

$\lim_{y\to\infty} e^{-e^{-y}} = 1$, $\lim_{y\to-\infty} e^{-e^{-y}} = 0$, $0 < e^{-e^{-y}} < 1 \quad \forall y \in \mathbb{R}$

using that $(1 + a/n)^n \to e^a$ for $a \in \mathbb{R}$

$F_Y(y) = e^{-e^{-y}}$, $-\infty < y < \infty$ is the cdf of Gumbel's extreme value distribution

$X_1, X_2, \dots$ are i.i.d EXP(1) $\Rightarrow X_{(n)} - \log n \Longrightarrow Y$ where $Y \sim$ Gumbel's dist.

$X_{(n)} - \log n \xrightarrow{d} Y$

Convergence concepts

Convergence in distribution: examples (cont'd)

Example 4: $X_1, X_2, \ldots$ iid $N(\mu, \sigma^2)$ & let $Y_n = \bar{X}_n$

$\overset{?}{\Longrightarrow} \quad X_n \xrightarrow{d} \mu$

(Recall: $X_1, X_2, \cdots$ are i.i.d $\mathbb{E}X_i = \mu$, $\operatorname{Var} X_i < \infty \Longrightarrow \bar{X}_n \xrightarrow{P} \mu$)

$$F_{Y_n}(y) \overset{\text{def}}{=} P(Y_n \le y) = \Phi\left(\frac{y-\mu}{\sigma/\sqrt{n}}\right)$$

$\mathbb{P}(Y_n \le y) = \mathbb{P}(X_n \le y)$
$= \mathbb{P}\left(\frac{X_n - \mu}{\sigma/\sqrt{n}} \le \frac{y-\mu}{\sigma/\sqrt{n}}\right)$
$= \Phi\left(\frac{y-\mu}{\sigma/\sqrt{n}}\right)$

I

$$\lim_{n\to\infty} F_{Y_n}(y) = \begin{cases} 0 & \text{if } y < \mu \\ 0.5 & \text{if } y = \mu \\ 1 & \text{if } y > \mu. \end{cases}$$

$F_{Y_n}(\mu) = .5$

$\lim_{n\to\infty} F_{Y_n}(\mu) \neq F_Y(\mu)$

If Y is a r.v. with a degenerate distribution at μ (i.e., $P(Y = \mu) = 1$) then

II

$$F_Y(y) = \begin{cases} 0 & \text{if } y < \mu \\ 1 & \text{if } y \ge \mu. \end{cases}$$

$1 \quad y > \mu$

$y = \mu \Rightarrow F_Y(\mu) = 1$