# Multivariate distributions

Moment generating functions

*Definition:* The **joint moment generating function** of $(X_1, \ldots, X_n)$ is

$$M_{X_1,\ldots,X_n}(t_1, \ldots, t_n) \stackrel{\text{def}}{=} \mathrm{E}e^{t_1 X_1 + \cdots + t_n X_n}, \quad t_1, \ldots, t_n \in \mathbb{R}$$

if the expectation exists for all $-h < t_1, \ldots, t_n < h$ for some $h > 0$

- Joint mgf can provide univariate mgfs

$$M_{X_i}(t_i) = M_{X_1,\ldots,X_n}(t_1 = 0, \ldots, t_{i-1} = 0, t_i, t_{i+1} = 0, \ldots, t_n = 0)$$

- Applications as before (more later):
  - characterizes distributions

    e.g., if $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_n)$ have the same mgfs, then these vectors have the same distribution

  - transformations

    e.g., mgf of $(a_1 X_1, \ldots, a_n X_n)$ is $M_{X_1,\ldots,X_n}(a_1 t_1, \ldots, a_n t_n)$

  - convergence (later)

  - moments, e.g.

$$q=1, \; r=1 \quad \mathbb{E}(XY) = \frac{\partial^2}{\partial t_1 \partial t_2} M_{X,Y}(t_1,t_2)\Big|_{(t_1,t_2)=(0,0)}$$

$$\mathrm{E}(X_i^q X_j^r X_k^s) = \frac{\partial^{q+r+s}}{\partial t_i^q \partial t_j^r \partial t_k^s} M_{X_1,\ldots,X_n}(t_1, \ldots, t_n)\Big|_{(t_1,\ldots,t_n)=(0,\ldots,0)}$$

$(X,Y)\sim$ Multinomial $\left(n, \frac{e^{t_1}p_1}{S}, \frac{e^{t_2}p_2}{S}\right)$, $f(x,y)=\frac{n!}{x!\,y!\,(n-x-y)!}\left(\frac{e^{t_1}p_1}{S}\right)^x\left(\frac{e^{t_2}p_2}{S}\right)^y\left(1-\frac{e^{t_1}p_1}{S}-\frac{e^{t_2}p_2}{S}\right)^{n-x-y}$

$1-\frac{e^{t_1}p_1}{S}-\frac{e^{t_2}p_2}{S} = \frac{S-e^{t_1}p_1-e^{t_2}p_2}{S} = \cancel{e^{t_1}p_1}+\cancel{e^{t_2}p_2}+(1-p_1-p_2)-\cancel{e^{t_1}p_1}-\cancel{e^{t_2}p_2}$

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Moment generating functions: example

In $n$ trials

$X := \#$ of outcomes "a"

$Y := \#$ of outcomes "b"

$n-x-y = \#$ of outcomes "c"

Let $(X,Y)\sim$Multinomial$(n,p_1,p_2)$, so the joint pmf of $(X,Y)$ is

$\sum_x\sum_y f(x,y)=1$

$$f(x,y)=\frac{n!}{x!y!(n-x-y)!}p_1^x p_2^y (1-p_1-p_2)^{n-x-y}$$

$p_1+p_2+(1-p_1-p_2)=1$

for $0\le x, y, x+y\le n$

$$M_{X,Y}(t_1,t_2)\overset{\text{def}}{=}\mathrm{E}e^{t_1X+t_2Y} = \mathbb{E}(h(X,Y)) = \boxed{\sum_x\sum_y h(x,y)\,f_X(x,y)}$$

$x+y\le n$

$0\le y\le n-x$

$$=\sum_{x=0}^{n}\sum_{y=0}^{n-x}\underbrace{e^{t_1x+t_2y}}_{h(x,y)}\underbrace{\frac{n!}{x!y!(n-x-y)!}p_1^x p_2^y(1-p_1-p_2)^{n-x-y}}_{f_{X,Y}(x,y)}$$

$$=\sum_{x=0}^{n}\sum_{y=0}^{n-x}\frac{n!}{x!y!(n-x-y)!}(e^{t_1}p_1)^x(e^{t_2}p_2)^y(1-p_1-p_2)^{n-x-y}$$

$S$

$e^{t_1}p_1+e^{t_2}p_2+(1-p_1-p_2)\neq 1$

$\frac{e^{t_1}p_1}{S}+\frac{e^{t_2}p_2}{S}+\frac{1-p_1-p_2}{S}=1$

$$=\sum_{x=0}^{n}\sum_{y=0}^{n-x}\frac{n!}{x!\,y!\,(n-x-y)!}\left(\frac{e^{t_1}p_1}{S}\right)^x\left(\frac{e^{t_2}p_2}{S}\right)^y S^n\,\frac{(1-p_1-p_2)^{n-x-y}}{S^{n-x-y}}$$

$S := e^{t_1}p_1+e^{t_2}p_2+(1-p_1-p_2)$

$$=S^n\underbrace{\sum_x\sum_y\frac{n!}{x!\,y!\,(n-x-y)!}\left(\frac{e^{t_1}p_1}{S}\right)^x\left(\frac{e^{t_2}p_2}{S}\right)^y\left(\frac{1-p_1-p_2}{S}\right)^{n-x-y}}_{\text{pmf of }\left(n,\frac{e^{t_1}p_1}{S},\frac{e^{t_2}p_2}{S}\right)}$$

$$=\boxed{S^n=\left[e^{t_1}p_1+e^{t_2}p_2+(1-p_1-p_2)\right]^n}$$

$$1=\frac{e^{t_1}p_1}{S}+\frac{e^{t_2}p_2}{S}+\frac{1-p_1-p_2}{S}=\boxed{\frac{e^{t_1}p_1+e^{t_2}p_2+(1-p_1-p_2)}{S}}$$

$$M_{X,Y}(t_1,t_2)=S^n=\left[e^{t_1}p_1+e^{t_2}p_2+1-p_1-p_2\right]^n$$

Main ideas was to Consider

$$S^n\left(\frac{p_1e^{t_1}}{S}\right)^x\left(\frac{p_1e^{t_2}}{S}\right)^y\left(\frac{1-p_1-p_2}{S}\right)^{n-x-y}$$

$$S:=p_1e^{t_1}+p_2e^{t_2}+1-p_1-p_2$$

$$M_X(t_1)=M_{X,Y}(t_1,0)=\left(e^{t_1}p_1+e^{\cancel{0}}p_2+1-p_1-p_2\right)^n$$

$$=\left(e^{t_1}p_1+\cancel{p_2}+1-p_1-\cancel{p_2}\right)^n$$

$$=\left(e^{t_1}p_1+1-p_1\right)^n \Longrightarrow$$

$$\boxed{M_Y(t_2)=M_{X,Y}(0,t_2)=\left(e^{t_2}p_2+1-p_2\right)^n}$$

$$\mathrm{Cov}(X,Y)=\mathbb{E}XY-\underbrace{\mathbb{E}X}_{np_1}\,\underbrace{\mathbb{E}Y}_{np_2}$$

$$\mathbb{E}XY=\frac{\partial^2}{\partial t_1\,\partial t_2}M_{X,Y}(t_1,t_2)\Big|_{(t_1,t_2)=(0,0)}=\boxed{n(n-1)p_1p_2}$$

$$\mathrm{Cov}(X,Y)=n(n-1)p_1p_2-n^2p_1p_2=-np_1p_2$$

# Conditional distributions

Introduction

Recall $P(A|B)$ is the probability that $A$ occurs given that $B$ occurs:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{for } P(B) > 0$$

We want to apply this idea to random variables

Already seen one example: truncated distributions, e.g, fix $x_0$

$$P(\underbrace{\boxed{X \le x}}_{A}|\overbrace{\boxed{X > x_0}}^{B}) = \frac{P(\overbrace{X \le x}^{A}, \overbrace{X > x_0}^{B})}{P(\underbrace{X > x_0}_{B})} = \frac{P(x_0 < X \le x)}{P(X > x_0)} = \frac{F(x) - F(x_0)}{1 - F(x_0)} \quad x > x_0$$

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

*Definitions:* Conditional distribution (given general event $A$)

Suppose we observe $A$ with $P(A) > 0$

1. **conditional cdf**

$$F(x|A) = P(X \le x|A) = \frac{P(A, X \le x)}{P(A)}$$

$$F_X(x) = \mathbb{P}(X \le x)$$

$$F_X(x|A) = \mathbb{P}(X \le x|A)$$

2. **conditional pmf/pdf**

$$f(x|A) = P(X = x|A) = \frac{P(A, X = x)}{P(A)}, \quad x \in \mathbb{R} \qquad \text{pmf if } X \text{ is discrete}$$

$$f(x|A) = \frac{dF(x|A)}{dx}, \quad x \in \mathbb{R} \qquad \text{pdf if } X \text{ is continuous}$$

For bivariate $(X, Y)$, we're interested in conditional pmf/pdf $f(x|Y = y)$