Convergence concepts

Convergence in distribution via MGFs

MGFs can also be used to show convergence in distribution

Recall earlier result: Suppose

1. $Y_1, Y_2, \ldots$ are a sequence of r.v.s, each having mgf $M_{Y_n}(t)$;
2. $\lim_{n\to\infty} M_{Y_n}(t) = M_Y(t)$ holds for any t in some neighborhood of 0;
3. and $M_Y(t)$ is the legitimate mgf of a r.v. Y.

Then, $Y_n \xrightarrow{d} Y$.

This result is important for establishing the main convergence in distribution result: the central limit theorem (to follow shortly)

Another useful result in connection to mgfs: if $r_n = o(1/n)$ is a remainder term, then Recall: $\lim_{n\to\infty}(1+c/n)^n = e^c$

$$\lim_{n\to\infty}\left(1+\frac{c}{n}+r_n\right)^{nb} = \lim_{n\to\infty}\left(1+\frac{c}{n}+o\left(\frac{1}{n}\right)\right)^{nb} = e^{cb}, \quad c, b \in \mathbb{R}$$ $\lim_{n\to\infty}\frac{s_n}{t_n} = 0$

Notation: for two generic sequences s_n, t_n, "little o" notation $\boxed{s_n = o(t_n)}$ means $s_n/t_n \to 0$ as $n \to \infty$. In particular, $r_n = o\left(\frac{1}{n}\right)$ means r_n is smaller than $1/n$ and

$$r_n/(1/n) = nr_n \to 0 \quad \text{as } n\to\infty.$$

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Convergence in distribution via MGFs: examples

Example 1: weak law of large numbers revisited

Let $X_1, X_2, \ldots$ be iid Bernoulli(p) & let $Y_n = \bar{X}_n = S_n/n$ where $S_n = \sum_{i=1}^n X_i$

WLLN $\bar{X}_n \xrightarrow{P} p = EX_1$, $\bar{X}_n \xrightarrow{d} p$ as $n \to \infty$

$S_n = \sum_{i=1}^n X_i \sim \text{Bin}(n,p)$

$$\begin{aligned}
M_{Y_n}(t) \overset{\text{def of } M}{=} \mathrm{E}e^{tY_n} \overset{\text{def of } Y_n}{=} \mathrm{E}e^{n^{-1}tS_n} &= M_{S_n}(t/n) \\
&= \left[pe^{t/n} + (1-p)\right]^n \\
&= \left[p\left(1 + \frac{(t/n)}{1!} + \frac{(t/n)^2}{2!} + \cdots\right) + (1-p)\right]^n \\
&= \left[1 + \frac{pt}{n} + o\left(\frac{1}{n}\right)\right]^n \longrightarrow e^{tp} \text{ as } n \to \infty.
\end{aligned}$$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $e^{t/n} = \sum_{k=0}^{\infty} \frac{(t/n)^k}{k!}$

$\dfrac{\sum_{k=2}^{\infty} \frac{(t/n)^k}{k!}}{\frac{1}{n}} \underset{n\to\infty}{\to} 0$

$\boxed{e^{pt}} = M_Y(t) = \mathbb{E}\left[\boxed{e^{tY}}\right] \Rightarrow Y = p$ with Prob. 1

$M_{Y_n}(t) \longrightarrow M_Y(t)$ where $Y = p$ with Prob. 1.

$\Rightarrow \boxed{Y_n \xrightarrow{d} Y} \equiv \boxed{Y_n \xrightarrow{d} p}$

Example 2: Suppose $X_n \sim$Binomial$(n, \lambda/n)$. Then, $X_n \xrightarrow{d} Y$, $Y \sim$Possion(λ)

$$M_{X_n}(t) = \left(\frac{\lambda}{n}e^t + (1 - \lambda/n)\right)^n = \left(1 + \frac{1}{n}\lambda(e^t - 1)\right)^n \longrightarrow e^{\lambda(e^t-1)}$$

But, $e^{\lambda(e^t-1)}$ is just $M_Y(t)$ where $Y \sim \text{Poisson}(\lambda)$.

Recall: $\lim_{n\to\infty}\left(1+\frac{c}{n}\right)^n = e^c$

$\Rightarrow X_n \xrightarrow{d} Y$ as $n \to \infty$.

$\boldsymbol{X}_1 = (X_{11}, \dots, X_{1k})$
$\boldsymbol{X}_2 = (X_{21}, \dots, X_{2k})$
$\vdots$

Convergence concepts

Central limit theorem (CLT)

CLT: If $\boldsymbol{X}_1, \boldsymbol{X}_2, \dots,$ are iid random vectors in $\mathbb{R}^k$ with mean $\mathrm{E}\boldsymbol{X}_i = \boldsymbol{\mu} \in \mathbb{R}^k$ and $\mathrm{Var}(\boldsymbol{X}_i) = \boldsymbol{\Sigma}$, then

$$\boxed{\sqrt{n}}\left(\bar{\boldsymbol{X}}_n - \boldsymbol{\mu}\right) \xrightarrow{d} MVN_k(\mathbf{0}, \boldsymbol{\Sigma})$$

as $n \to \infty$, where $\bar{\boldsymbol{X}}_n = \sum_{i=1}^n \boldsymbol{X}_i / n$.

Interpretation:

iid sums/averages have approximately normal distribution for large n, regardless of the population distribution of $\boldsymbol{X}_i$ as long as the variance exists

- We say **asymptotic or limiting distribution** of $\sqrt{n}\left(\bar{\boldsymbol{X}}_n - \boldsymbol{\mu}\right)$ is $MVN_k(\mathbf{0}, \boldsymbol{\Sigma})$

- We sometimes say that **asymptotic distribution** of $\bar{\boldsymbol{X}}_n$ is $MVN_k(\boldsymbol{\mu}, \frac{1}{n}\boldsymbol{\Sigma})$, or

$$\sqrt{n}(\bar{\boldsymbol{X}}_n - \boldsymbol{\mu}) \xrightarrow{d} MVN_k(0, \Sigma)$$

$$\bar{\boldsymbol{X}}_n \overset{a}{\sim} MVN_k(\boldsymbol{\mu}, \frac{1}{n}\boldsymbol{\Sigma})$$

- CLT can apply to dependent r.v.s too (as long as not too dependent)

  CLT can apply to non-iid r.v.s too (but not if one r.v. dominates others)

Convergence concepts

Central limit theorem (CLT): examples

Example 1: Example: $X_1, X_2, \ldots$, iid Uniform$(0, 1)$

$X_i \sim Uni(0,1) \Rightarrow \mathbb{E}X_i = 1/2 \quad Var\, X_i = 1/12$

$\sqrt{n}(\overline{X}_n - \frac{1}{2}) \longrightarrow N(0, \frac{1}{12})$ as $n \to \infty$

$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$

Example 2: Example: $X_1, X_2, \ldots$, iid Bernoulli(p)

$\mathbb{E}X_i = p, \ Var\, X_i = p(1-p)$

By CLT $\sqrt{n}(\overline{X}_n - p) \longrightarrow N(0, p(1-p))$ as $n \to \infty$

$\overline{X}_n \overset{a}{\sim} N(p, \frac{p(1-p)}{n})$

Convergence concepts

Central limit theorem (CLT): proof

Proof of CLT in the 1-dimensional case assuming MGF of X_1 exists

$\sqrt{n}(\bar{X}_n-\mu) \xrightarrow{d} N(0,\sigma^2)$

$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\underbrace{(X_i-\mu)}_{Y_i} = \frac{1}{\sqrt{n}}\left[\sum_{i=1}^{n}X_i - n\mu\right] = \frac{n}{\sqrt{n}}\left[\frac{\sum X_i}{n}-\mu\right] = \sqrt{n}(\bar{X}-\mu)$

1. Write

$$Z_n = \sqrt{n}\left(\bar{X}_n-\mu\right) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} Y_i$$

where $Y_i = X_i - \mu$ are iid with mean 0 and variance σ^2

2. Write mgf

$$M_{Z_n}(t) \overset{\text{def of MGF}}{=} \mathrm{E}e^{tZ_n} \overset{\text{def of } Z_n=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}Y_i}{=} \mathrm{E}e^{\sum_{i=1}^{n}(tY_i/\sqrt{n})} = \mathrm{E}\prod_{i=1}^{n} e^{(tY_i/\sqrt{n})}$$

$$\overset{Y_i\text{'s are independent}}{=} \prod_{i=1}^{n}\mathrm{E}e^{tY_i/\sqrt{n}}$$

$$\overset{\text{def of MGF}}{=} \prod_{i=1}^{n} M_{Y_i}(t/\sqrt{n})$$

$$= \left[M_{Y_1}(t/\sqrt{n})\right]^n$$

3. Expand $M_{Y_1}(t/\sqrt{n})$ in a Taylor series around 0

$$M_{Y_1}(t/\sqrt{n}) = M_{Y_1}(0) + \frac{t/\sqrt{n}}{1!}\underbrace{M'_{Y_1}(0)}_{=0} + \frac{(t/\sqrt{n})^2}{2!}M''_{Y_1}(0) + \sum_{k=3}^{\infty}\frac{(t/\sqrt{n})^k}{k!}M^{(k)}_{Y_1}(0)$$

$$= 1 + \boxed{\frac{t^2\sigma^2}{2}\frac{1}{n}} + o\left(\frac{1}{n}\right)$$

using

$$M_{Y_1}(0) = 1, \qquad M'_{Y_1}(0) = \mathrm{E}Y_1 = 0, \qquad M''_{Y_1}(0) = \mathrm{E}Y_1^2 = \mathrm{Var}(Y_1) = \sigma^2$$

$\Rightarrow \lim_{n\to\infty}\left[M_{Y_1}\left(\frac{t}{\sqrt{n}}\right)\right]^n = \lim_{n\to\infty}\left[1+\frac{\sigma^2t^2}{2n}+o\left(\frac{1}{n}\right)\right]^n = e^{\frac{\sigma^2t^2}{2}}$ is the MGF of $N(0,\sigma^2)$

$\sqrt{n}(\bar{X}_n-\mu) \to \boxed{N(0,\sigma^2)}$