PhD Prelim Exam
THEORY

(Majors and Co-majors)

Summer 2012
(Given on 7/12/12)

Part I (Includes Pages 1 and 2)

1. Let Ω be a nonempty set and let $\mathcal{D}$ be a collection of subsets of Ω, with the following three properties:

(a) $\Omega \in \mathcal{D}$;

(b) $A \in \mathcal{D}$ implies $A^c \in \mathcal{D}$;

(c) if $A_1, A_2, A_3, \ldots$ is an infinite sequence of disjoint subsets in $\mathcal{D}$ (i.e., $A_i \cap A_j = \emptyset$ for $i \neq j$) then $\cup_{i=1}^{\infty} A_i \in \mathcal{D}$;

Show that $A, B \in \mathcal{D}$ with $A \subseteq B$ implies $B \setminus A \in \mathcal{D}$.

2. What is the name for a collection $\mathcal{D}$ of subsets with properties as in Question 1?

3. Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two measure spaces. Let $f : \Omega_1 \to \Omega_2$ be $\langle \mathcal{F}_1, \mathcal{F}_2 \rangle$-measurable; define the measure $\mu_2 = \mu_1 f^{-1}$ on $(\Omega_2, \mathcal{F}_2)$; and let $h : \Omega_2 \to \mathbb{R}$ be nonnegative and $\langle \mathcal{F}_2, \mathcal{B}(\mathbb{R}) \rangle$-measurable (i.e., Borel measurable). Prove that:

$$\int_{\Omega_1} h \circ f(\omega_1)\, d\mu_1(\omega_1) = \int_{\Omega_2} h(\omega_2) \mu_2(\omega_2),$$

where $h \circ f : \Omega_1 \to \mathbb{R}$ is the functional composition (i.e. $h \circ f(\omega_1) = h(f(\omega_1))$, $\omega_1 \in \Omega_1$).
Hint: First consider simple functions h and then approximate general h with simple functions.

4. If μ_1, μ_2, μ_3 are σ-finite measures on a measurable space $(\Omega, \mathcal{F})$ such that $\mu_1 \ll \mu_2$ and $\mu_2 \ll \mu_3$, show that

$$\mu_1 \ll \mu_3 \quad \text{and} \quad \frac{d\mu_1}{d\mu_3} = \frac{d\mu_1}{d\mu_2}\frac{d\mu_2}{d\mu_3} \quad \text{a.e. } (\mu_3).$$

5. If μ_1, μ_2 are σ-finite measures on a measurable space $(\Omega, \mathcal{F})$ such that $\mu_1 \ll \mu_2$ and $\frac{d\mu_1}{d\mu_2} > 0$ a.e. (μ_2), show that

$$\mu_2 \ll \mu_1 \quad \text{and} \quad \frac{d\mu_2}{d\mu_1} = \frac{1}{\frac{d\mu_1}{d\mu_2}} \quad \text{a.e. } (\mu_1).$$

Questions on this page (Page 2) use terms defined in items (D.1)-(D.7) below.

For Questions 6,7,8: Define

(D.1) $\Omega_1 \equiv (0, \pi) \times (0, 2\pi) \times (-1, 1)$,

(D.2) $\mathcal{F}_1$ as the Borel σ-algebra on Ω_1,

(D.3) μ_1 as the Lebesgue measure on $(\Omega_1, \mathcal{F}_1)$,

(D.4) P as a probability measure on $(\Omega_1, \mathcal{F}_1)$ corresponding to the joint distribution of a random vector $(R, X_1, X_2) : \Omega_1 \to \Omega_1$ having the following four properties: R, X_1, X_2 are independent; R has a density g with respect to the Lebesgue measure on $((0, \pi), \mathcal{B}((0, \pi)))$; X_1 is uniformly distributed on $(0, 2\pi)$; and X_2 is uniformly distributed on $(-1, 1)$.

6. State the density $\frac{dP}{d\mu_1}(r, x_1, x_2) \equiv h_1(r, x_1, x_2)$, $(r, x_1, x_2) \in \Omega_1$, of the measure P with respect to μ_1 on $(\Omega_1, \mathcal{F}_1)$.

In addition to (D.1)-(D.4), for Questions 7,8: Define

(D.5) Ω_2 as the set of all 3×3 real-valued rotation matrices with corresponding σ-algebra $\mathcal{F}_2$,

(D.6) a $\langle \mathcal{F}_1, \mathcal{F}_2 \rangle$-measurable mapping $f : \Omega_1 \to \Omega_2$ given by

$$f(r, x_1, x_2) = \begin{bmatrix} \cos(r) & -u_3 \sin(r) & -u_2 \sin(r) \\ u_3 \sin(r) & \cos(r) & u_1 \sin(r) \\ u_2 \sin(r) & -u_1 \sin(r) & \cos(r) \end{bmatrix} + (1 - \cos(r)) \begin{bmatrix} u_1^2 & u_1 u_2 & u_1 u_3 \\ u_1 u_2 & u_2^2 & u_2 u_3 \\ u_1 u_3 & u_2 u_3 & u_3^2 \end{bmatrix}, \; (r, x_1, x_2) \in \Omega_1$$

with $u_1 = \sin(x_1) \sin(\arccos(x_2))$, $u_2 = \cos(x_1) \sin(\arccos(x_2))$, $u_3 = x_2$.

(D.7) For each $O \in \Omega_2$, there is a corresponding unique $(r, x_1, x_2) \in \Omega_1$ where $f(r, x_1, x_2) = O$; hence, the inverse f^{-1} maps an element $O \in \Omega_2$ to a single point $f^{-1}(O) = f^{-1} \circ f(r, x_1, x_2) = (r, x_1, x_2) \in \Omega_1$, for which $\cos(r) = [\text{trace}(O) - 1]/2$ holds.

7. The random matrix, defined by $O = f(R, X_1, X_2)$, has a distribution given by the induced probability measure $P_0 = Pf^{-1}$ on $(\Omega_2, \mathcal{F}_2)$; i.e., $P_0(A) = P(f^{-1}(A))$, $A \in \mathcal{F}_2$. On $(\Omega_2, \mathcal{F}_2)$, show that

$$\frac{dP_0}{d\mu_2}(O) = h_1(f^{-1}(O)), \quad O \in \Omega_2,$$

is the density of P_0 with respect to the measure $\mu_2 = \mu_1 f^{-1}$.
Hint: Use (D.7) and Question 3, recalling that h_1 is defined in Question 6.

In addition to (D.1)-(D.7), for Question 8: Let H denote the so-called Haar Measure on the measurable space $(\Omega_2, \mathcal{F}_2)$ of rotations for which it holds that $H(\Omega_2) = 1$ and

$$H(A) = \int_A 1 dH = \int_A k(f^{-1}(O)) d\mu_2, \quad A \in \mathcal{F}_2,$$

where $k(r, x_1, x_2) : \Omega_1 \to \mathbb{R}$ is given by $k(r, x_1, x_2) = [1 - \cos(r)]/[4\pi^2]$, $(r, x_1, x_2) \in \Omega_1$.

8. On $(\Omega_2, \mathcal{F}_2)$, carefully derive a density $\frac{dP_0}{dH}$ for the distribution P_0 of the random matrix $O = f(R, X_1, X_2)$ with respect to H and write this density in terms of trace(O).
Hint: Use Questions 4 and 5.

Part II

For all questions below: On a probability space $(\Omega, \mathcal{F}, P)$, let $X_1, X_2, \ldots$ be a stationary sequence of mean $\mathrm{E}(X_t) = 0$ random variables with $\mathrm{E}(X_t^2) < \infty$, where "stationary" means that for any collection of indices $i_1, \ldots, i_k \in \{1, 2, \ldots\}$ (i.e., $k \geq 1$ is arbitrary) and any shift $t \in \{0, 1, 2, \ldots\}$, it holds that $(X_{i_1}, \ldots, X_{i_k}) \stackrel{d}{=} (X_{i_1+t}, \ldots, X_{i_k+t})$; that is, $(X_{i_1}, \ldots, X_{i_k})$ has the same distribution as $(X_{i_1+t}, \ldots, X_{i_k+t})$. Consequently, from $k = 1$ and $t = 0$, we have $X_1 \stackrel{d}{=} X_2 \stackrel{d}{=} X_3 \stackrel{d}{=} X_4 \cdots$; from $k = 2$ and $t = 3$, we have $(X_1, X_4) \stackrel{d}{=} (X_2, X_5) \stackrel{d}{=} (X_3, X_6) \cdots$; and so on.

By stationarity, the autocovariance function $r(k) \equiv \mathrm{Cov}(X_t, X_{t+k})$, $k \geq 0$, as well as the mean $\mathrm{E}X_t = 0$ and variance $\mathrm{Var}(X_t) = r(0)$, do not depend on t. The variance of the sample mean $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ is then given by

$$n\mathrm{Var}(\bar{X}_n) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n}\right) r(k).$$

9. Let $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ denote the set of integers, let $\mathcal{F}_1$ denote its power set, and let μ be the counting measure on $(\mathbb{Z}, \mathcal{F}_1)$. Find a measurable function $f_n : \mathbb{Z} \to \mathbb{R}$ such that

$$\int_{\mathbb{Z}} f_n d\mu = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n}\right) r(k).$$

10. Assuming that $\sum_{k=-\infty}^{\infty} |r(k)| < \infty$, use the Dominated Convergence Theorem to prove that $n\mathrm{Var}(\bar{X}_n) \to \sum_{k=-\infty}^{\infty} r(k)$ as $n \to \infty$.

In addition, for Questions 11,12,13,14,15: Assuming that $\{X_i\}_{i\geq 1}$ is stationary with mean $E(X_1) = 0$, suppose further that $\mathrm{E}(X_1^4) < \infty$ and $\{X_i\}_{i\geq 1}$ is m-dependent for some fixed, integer $m \geq 1$. The term m-dependent means that any two random variables are independent whenever the (absolute) difference between their indices exceeds m (so X_i and X_j are independent if $|i - j| > m$).

11. Give an example of an m-dependent sequence $\{X_i\}_{i\geq 1}$ for which $r(m) \neq 0$.

12. Prove that $\bar{X}_n \to 0$ a.s. P as $n \to \infty$.
Hint: Divide $X_1, \ldots, X_n$ into $m + 1$ subseries of independent variables.

13. Using Holder's inequality, prove that $\mathrm{E}(\bar{X}_n^4) \leq C\frac{m^2}{n^2}\mathrm{E}(X_1^4)$ holds for any $n \geq 1$, using some real number $C > 0$ not depending on n, m or the distribution of $\{X_i\}_{i\geq 1}$.

In addition, for Questions 14,15: Using $X_1, \ldots, X_n$ and a block length b (where $1 \leq b < n$), suppose that we create consecutive data blocks $(X_1, \ldots, X_b)$, $(X_2, \ldots, X_{b+1})$, $\ldots$, $(X_{n-b+1}, \ldots, X_n)$ in order to estimate the variance $n\mathrm{Var}(\bar{X}_n)$ with $V_n = \frac{1}{n-b+1}\sum_{i=1}^{n-b+1} b(\bar{X}_{i,b})^2$, using the sample mean $\bar{X}_{i,b} = \frac{1}{b}\sum_{j=i}^{i+b-1} X_j$ of the ith block $(X_i, \ldots, X_{i-b+1})$, $i = 1, \ldots, n-b+1$.

14. Show that $\mathrm{Var}(V_n) \leq \frac{C_1}{n-b}$ for a constant $C_1 > 0$, not depending on b or n.

15. In addition to $b \to \infty$ as $n \to \infty$, specify conditions on the subsample length b so that $|V_n - n\mathrm{Var}(\bar{X}_n)| \xrightarrow{d} 0$ holds as $n \to \infty$ and prove this convergence in distribution.
Hint: Use Questions 10 and 14.

1. By (a)-(b), $\Omega^c = \emptyset \in \mathcal{D}$ and $B^c \in \mathcal{D}$. Then, $A_1 = A$, $A_2 = B^c$, $A_n = \emptyset$, $n \geq 3$, are disjoint sets in $\mathcal{D}$ so $\cup_{n=1}^{\infty} A_n = A \cup B^c \in \mathcal{D}$ by (c). By (b), $(A \cup B^c)^c = B \cap A^c \in \mathcal{D}$.

2. λ-system

3. Suppose first $h(\omega_2) = \sum_{i=1}^k c_i \mathbb{I}(\omega_2 \in A_i)$, $\omega_2 \in \Omega_2$, is a simple function, for $A_i \in \mathcal{F}_2$, $c_i \geq 0$, $i = 1, \ldots, k$. Then,

$$h(f(\omega_1)) = \sum_{i=1}^k c_i I(f(\omega_1) \in A_i) = \sum_{i=1}^k c_i I(\omega_1 \in f^{-1}(A_i)), \quad \omega_1 \in \Omega_1.$$

Hence, by definition,

$$\int_{\Omega_1} h(f(\omega_1)) d\mu_1(\omega_1) = \sum_{i=1}^k c_i \mu_1(f^{-1}(A_i)) = \sum_{i=1}^k c_i \mu_2(A_i) = \int_{\Omega_2} h(\omega_2) d\mu_2(\omega_2),$$

and the change of variables holds for simple functions. For general, non-negative h, we take a sequence of non-negative, simple functions $\phi_n : \Omega_2 \to \mathbb{R}$ such that $\phi_n \uparrow h$ as $n \to \infty$. Then, we've shown that $\int_{\Omega_1} \phi_n(f(\omega_1)) d\mu_1(\omega_1) = \int_{\Omega_2} \phi_n(\omega_2) d\mu_2(\omega_2)$ holds for all $n \geq 1$, while the following hold by definition of the integral of a non-negative, measurable function:

$$\begin{aligned} \int_{\Omega_1} h(f(\omega_1)) d\mu_1(\omega_1) &= \lim_{n\to\infty} \int_{\Omega_1} \phi_n(f(\omega_1)) d\mu_1(\omega_1), \\ \int_{\Omega_2} h(\omega_2) d\mu_2(\omega_2) &= \lim_{n\to\infty} \int_{\Omega_2} \phi_n(\omega_2) d\mu_2(\omega_2), \end{aligned}$$

because $\phi_n \circ f : \Omega_1 \to \mathbb{R}$ and $\phi_n : \Omega_2 \to \mathbb{R}$ are sequences of non-negative simple functions with $\phi_n \circ f \uparrow h \circ f$ and $\phi_n \uparrow h$.

4. $A \in \mathcal{F}$ with $\mu_3(A) = 0 \Rightarrow \mu_2(A) = 0 \Rightarrow \mu_1(A) = 0 \Rightarrow \mu_1 \ll \mu_3$.

Pick $A \in \mathcal{F}$, then by definition

$$\int_A \frac{d\mu_1}{d\mu_2} \frac{d\mu_2}{d\mu_3} d\mu_3 = \mathrm{E}_{\mu_2}\left(\mathbb{I}(A) \frac{d\mu_1}{d\mu_2}\right) = \mathrm{E}_{\mu_1}\left(\mathbb{I}(A)\right) = \mu_1(A).$$

Because R-N derivatives are unique (a.e.), we have $\frac{d\mu_1}{d\mu_2}\frac{d\mu_2}{d\mu_3} = \frac{d\mu_1}{d\mu_3}$ a.e. (μ_3).

5. If $A \in \mathcal{F}$ with $0 = \mu_1(A) = \int \mathbb{I}(A) \frac{d\mu_1}{d\mu_2} d\mu_2$, then $A_n = \{\omega : \frac{d\mu_1}{d\mu_2}(\omega) \geq \frac{1}{n}\}$ satisfies $\frac{1}{n}\mu_2(A_n \cap A) = \int \mathbb{I}(A)\mathbb{I}(A_n) d\mu_2 \leq \int \mathbb{I}(A) \frac{d\mu_1}{d\mu_2} d\mu_2 = 0$ for every $n \geq 1$. Hence, $\mu_2(A) = \mu_2(A \cap \{\omega : \frac{d\mu_1}{d\mu_2}(\omega) > 0\}) \leq \sum_{n=1}^{\infty} \mu_2(A \cap A_n) = 0$, so that $\mu_2 \ll \mu_1$.

Since $\frac{d\mu_1}{d\mu_1} = 1$ and $\frac{d\mu_1}{d\mu_2} > 0$ a.e. (μ_1), by Question 4 we have $1 = \frac{d\mu_1}{d\mu_2}\frac{d\mu_2}{d\mu_1}$ a.e. (μ_1) so that $\frac{d\mu_2}{d\mu_1} = 1/\frac{d\mu_1}{d\mu_2}$ a.e. (μ_1).

6. $\frac{dP}{d\mu_1}(r, x_1, x_r) = h_1(r, x_1, x_2) = g(r) \cdot \frac{1}{2\pi} \cdot \frac{1}{2}$, for $(r, x_1, x_2) \in \Omega_1$ (a.e. μ_1).

7. For $A \in \mathcal{F}_2$, by definition we have $P_0(A) = P(f^{-1}(A)) = \int_{\Omega_1} \mathbb{I}(f^{-1}(A)) h_1 d\mu_1$ using $\frac{dP}{d\mu_1} = h_1$ from Question 6. In the change of variables formula (Question 3), set $h = \mathbb{I}(f^{-1}(A))h_1 \circ f^{-1} : \Omega_2 \to \mathbb{R}$ (i.e., $h(O) = \mathbb{I}(f^{-1}(O) \in f^{-1}(A)) = \mathbb{I}(O \in A)h_1(f^{-1}(O))$, $O \in \Omega_2$). Then,

$$P_0(A) = \int_{\Omega_1} \mathbb{I}(f^{-1}(A))h_1 d\mu_1 = \int_{\Omega_1} h \circ f d\mu_1 = \int_{\Omega_2} h d\mu_2 = \int_{\Omega_2} \mathbb{I}(O \in A)h_1(f^{-1}(O))d\mu_2(O);$$

by the a.e. uniqueness of R-N derivatives, $\frac{dP_0}{d\mu_2}(O) = h_1(f^{-1}(O))$, $O \in \Omega_2$ (a.e. μ_2).

8. By the statement of the Haar measure, it holds (a.e. μ_1) that $\frac{dH}{d\mu_2}(O) = k(f^{-1}(O))$, $O \in \Omega_2$. Since $k(f^{-1}(O)) > 0$ for all $O \in \Omega_2$, by Question 5 we have $\frac{d\mu_2}{dH}(O) = 1/k(f^{-1}(O))$, $O \in \Omega_2$ (a.e. H). By Questions 4 and 7, we then have

$$\frac{dP_0}{dH}(O) = \frac{dP_0}{d\mu_2}(O) \cdot \frac{d\mu_2}{dH}(O) = \frac{h_1(f^{-1}(O))}{k(f^{-1}(O))} = 2\pi \frac{g[\arccos([\mathrm{trace}(O) - 1]/2)]}{3 - \mathrm{trace}(O)}, \quad O \in \Omega_2 \text{ (a.e. } H)$$

9. Define $f_n : \mathbb{Z} \to \mathbb{R}$ by

$$f_n(k) = \begin{cases} \left(1 - \frac{|k|}{n}\right) r(k) & \text{if } |k| \le n \\ 0 & \text{otherwise,} \end{cases} \qquad k \in \mathbb{Z}.$$

Then, $n\mathrm{Var}(\bar{X}_n) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n}\right) r(k) = \int_{\mathbb{Z}} f_n d\mu$ holds.

10. Define $g : \mathbb{Z} \to \mathbb{R}$ by $g(k) = 2|r(k)|$, $k \in \mathbb{Z}$. Then, $\int_{\mathbb{Z}} g d\mu = 2\sum_{k=-\infty}^{\infty} |r(k)| < \infty$; $|f_n(k)| \le g(k)$ for all $k \in \mathbb{Z}$ and all $n \ge 1$; and $f_n(k) \to f(k) = r(k)$ as $n \to \infty$ for each $k \in \mathbb{Z}$. By the Dominated Convergence Theorem,

$$\lim_{n\to\infty} n\mathrm{Var}(\bar{X}_n) = \lim_{n\to\infty} \int_{\mathbb{Z}} f_n d\mu = \int_{\mathbb{Z}} \lim_{n\to\infty} f_n d\mu = \int_{\mathbb{Z}} f d\mu = \sum_{k=-\infty}^{\infty} r(k).$$

11. Let $\{Z_t\}_{t\in\mathbb{Z}}$ be an iid collection of $N(0,1)$ variables and define $X_t = \sum_{j=0}^{m} Z_{t+j}$ for $t \ge 1$. Then, X_t is m-dependent and $r(m) = \mathrm{Cov}(X_t, X_{t+m}) = \mathrm{Cov}(Z_{t+m}, Z_{t+m}) = 1$.

12. Define an index set $S_k = \{k, k+(m+1), k+2(m+1), k+3(m+1), \ldots\}$ for $k = 1, \ldots, m+1$ and let $n_k = |S_k \cap \{1, \ldots, n\}|$. Note that $n/[(m+1)n_k] \to 1$ as $n \to \infty$ for each $k = 1, \ldots, m+1$. Since the random variables $\{X_j : j \in S_k\}$ are iid with mean zero (and $\mathrm{E}|X_j| < \infty$), by the SLLN, it holds that

$$\bar{X}_{n_k}^{(k)} \equiv \frac{1}{n_k} \sum_{j \in S_k \cap \{1,\ldots,n\}} X_j \to 0 \quad \text{a.s. } P$$

for each $k = 1, \ldots, m+1$. Hence,

$$\bar{X}_n = \sum_{k=1}^{m+1} \frac{n_k}{n} \bar{X}_{n_k}^{(k)} \to \sum_{k=1}^{m+1} \frac{1}{m+1} 0 = 0 \quad \text{a.s. } P.$$

13. Note that $\mathrm{E}(\bar{X}_n^4) \leq \frac{1}{n^4} 4! \sum_{1\leq i\leq j\leq k\leq \ell\leq n} |\mathrm{E}(X_iX_jX_kX_\ell)|$, where

$$\begin{aligned} |\mathrm{E}(X_iX_jX_kX_\ell)| &\leq [\mathrm{E}(X_iX_j)^2]^{1/2}[\mathrm{E}(X_kX_\ell)^2]^{1/2} \\ &\leq [\{\mathrm{E}(X_i^4)\mathrm{E}(X_j^4)\}^{1/2}]^{1/2}[\{\mathrm{E}(X_k^4)\mathrm{E}(X_\ell^4)\}^{1/2}]^{1/2} = \mathrm{E}(X_1^4) \end{aligned}$$

holds for all $1 \leq i \leq j \leq k \leq \ell$. From m-dependence, we have $\mathrm{E}(X_iX_jX_kX_\ell) = \mathrm{E}(X_i)\mathrm{E}(X_jX_kX_\ell) = 0$ if $j > i+m$ and $\mathrm{E}(X_iX_jX_kX_\ell) = \mathrm{E}(X_iX_jX_k)\mathrm{E}(X_\ell) = 0$ if $\ell > k+m$. Hence, letting $S_n = \{(i,j,k,\ell) : 1 \leq i \leq j \leq k \leq \ell \leq n, j \leq i+m, \ell \leq k+m\}$,

$$\mathrm{E}(\bar{X}_n^4) \leq \frac{24}{n^4} \sum_{(i,j,k,\ell)\in S_n} \mathrm{E}(X_1^4) = \frac{24}{n^4}\mathrm{E}(X_1^4)|S_n| \leq 4\cdot 24\mathrm{E}(X_1^4)\cdot\frac{m^2}{n^2}$$

since $|S_n| \leq n\cdot(m+1)\cdot n\cdot(m+1) \leq (2mn)^2$.

14. Let $n_b = n-b+1$. Then, $\mathrm{Var}(V_n) \leq \frac{2}{n_b^2}\sum_{1\leq i\leq j\leq n_b} b^2|\mathrm{Cov}(\bar{X}_{i,b}^2, \bar{X}_{j,b}^2)|$. Note that $\mathrm{Cov}(\bar{X}_{i,b}^2, \bar{X}_{j,b}^2) = 0$ if $j > i+m$ by m-dependence and, for any i, j, it holds that

$$|\mathrm{Cov}(\bar{X}_{i,b}^2, \bar{X}_{j,b}^2)| \leq [\mathrm{E}(\bar{X}_{i,b})^4]^{1/2}[\mathrm{E}(\bar{X}_{j,b})^4]^{1/2} = \mathrm{E}(\bar{X}_{1,b}^4) \leq C\mathrm{E}(X_1^4)\frac{m^2}{b^2}$$

by Question 13. Hence,

$$\mathrm{Var}(V_n) \leq 2Cb^2\mathrm{E}(X_1^4)\frac{n_b(m+1)}{n_b^2}\frac{m^2}{b^2} = \frac{C_1}{n_b} \leq \frac{C_1}{n-b},$$

for $C_1 = 2Cm^2(m+1)\mathrm{E}(X_1^4)$.

15. If $b\to\infty$ with $n-b\to\infty$ as $n\to\infty$, then $\mathrm{Var}(V_n)\to 0$ as $n\to\infty$ by Question 14. By stationarity and Question 10,

$$\mathrm{E}(V_n) = b\mathrm{E}(\bar{X}_{1,b}^2) = b\mathrm{Var}(\bar{X}_{1,b}) \to \sum_{k=-\infty}^{\infty} r(k)$$

as $b\to\infty$ (i.e., $n\to\infty$). Hence, if $b\to\infty$ with $n-b\to\infty$ as $n\to\infty$, then V_n is MSE-convergent to $\sum_{k=-\infty}^{\infty} r(k)$, so that $V_n \xrightarrow{p} \sum_{k=-\infty}^{\infty} r(k)$ as $n\to\infty$. By Question 10, we have $n\mathrm{Var}(\bar{X}_n) \to \sum_{k=-\infty}^{\infty} r(k)$ so that $V_n - n\mathrm{Var}(\bar{X}_n) \xrightarrow{p} 0$, implying $|V_n - n\mathrm{Var}(\bar{X}_n)| \xrightarrow{p} 0$ and then $|V_n - n\mathrm{Var}(\bar{X}_n)| \xrightarrow{d} 0$.

Part I

Let $X_1, \ldots, X_n$ be independent and identically distributed (iid) random variables with the marginal probability density function (pdf)

$$f_\theta(x) = \frac{1}{x\sqrt{2\pi\theta}} \exp\left\{-\frac{(\log x - \theta)^2}{2\theta}\right\}, \text{ for } x \in (0, \infty),$$

where $\theta \in (0, \infty)$ is an unknown parameter. ($\log X_1$ has a normal distribution with mean θ and variance θ.)

1. Show that there exists a unique maximum likelihood estimator (MLE) of θ, $\widehat{\theta}_n$, and give an explicit expression for $\widehat{\theta}_n$.
2. Find the asymptotic distribution of $\widehat{\theta}_n$ as $n \to \infty$.
3. Find an approximately size α likelihood ratio test (LRT) of $H_0 : \theta = \theta_0$ versus $H_a : \theta \neq \theta_0$, where θ_0 is a given positive number.
4. Show that there exists a uniformly minimum variance unbiased estimator (UMVUE) of $a\theta + b\theta^2$ for any given constants a and b. You do not need to give an absolutely explicit formula for the UMVUE.

Part II

Suppose that $X_1, \ldots, X_n$ are iid exponential random variables with the marginal pdf

$$f_\theta(x) = \frac{1}{\theta} \exp\left\{-\frac{x}{\theta}\right\} \text{ for } x \in \mathcal{X} \equiv (0, \infty),$$

and that one wishes to estimate $\theta \in (0, \infty)$. Consider the loss function $L(\theta, a) = (\theta - a)^2/\theta^2$ and the prior distribution G with the pdf

$$g_{\alpha,\beta}(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{-(\alpha+1)} \exp\left\{-\frac{\beta}{\theta}\right\} \text{ for } x \in \mathcal{X},$$

for some $\alpha, \beta \in (0, \infty)$, where $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$.

Note that $E(X_1|\theta) = \theta$, Var$(X_1|\theta) = \theta^2$, and for $\theta \sim g_{\alpha,\beta}(\theta)$ and $r < \alpha$,

$$E(\theta^r) = \beta^r \frac{\Gamma(\alpha - r)}{\Gamma(\alpha)}.$$

5. Find the posterior density function of θ given $X^n \equiv (X_1, \ldots X_n)$ in $\mathcal{X}^n$.
6. Find the Bayes estimator of θ, $\delta_{\alpha,\beta}(X^n)$.
7. Show that the risk function of $\delta_{\alpha,\beta}(X^n)$ is
$$\frac{1}{(n+1+\alpha)^2}\left[n + \left(1 + \alpha - \frac{\beta}{\theta}\right)^2\right]$$
and the Bayes risk of $\delta_{\alpha,\beta}(X^n)$ is
$$\frac{1}{n+1+\alpha}.$$
8. Find a minimax estimator of θ and argue carefully that your estimator is indeed minimax.

Part III

Let $\{X_n\}_{n\geq 1}$ be independent random variables such that for $n \geq 1$, X_n has the pdf

$$f_n(x) = \frac{n^\alpha}{2} \exp\{-n^\alpha |x|\},\ x \in \mathbb{R}^1 \equiv (-\infty, \infty),$$

where $\alpha \in (0, \infty)$ is a constant. Let $S_n = \sum_{j=1}^n X_j$ for $n \geq 1$.

9. Show that the sequence $\{X_n\}_{n\geq 1}$ satisfies the Lindeberg condition if and only if $\alpha \in (0, \frac{1}{2}]$.

   Hint: For $n \geq 1$, $E(X_n^2) = 2n^{-2\alpha}$.

10. For $\alpha \in (0, \frac{1}{2}]$, find constants a_n and b_n such that $(S_n - a_n)/b_n \xrightarrow{d} N(0, 1)$.

11. For $\alpha \in (\frac{1}{2}, \infty)$, show that S_n converges with probability one.

12. Let S be the random variable that is the almost sure limit of $\{S_n\}$ in Question 11. Find the characteristic function of S.

    Hint: For a constant $a \in \mathbb{R}^1$,

    $$\int_0^\infty \cos(au) \exp(-u) du = \frac{1}{1 + a^2}.$$

Part IV

Let h be a positive function defined on $\mathbb{R}^1$ such that $\int_a^b h(x)dx < \infty$ for any $a, b \in \mathbb{R}^1$ with $a < b$. Let $X_1, \ldots, X_n$ ($n \geq 2$) be iid random variables with the pdf

$$f_{\alpha,\beta}(x) = \begin{cases} c(\alpha, \beta)h(x) & \text{for } \alpha < x < \beta, \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha, \beta \in \mathbb{R}^1$, $\alpha < \beta$, and $c(\alpha, \beta) = \left(\int_\alpha^\beta h(x)dx\right)^{-1}$. Let $X_{(1)} = \min\{X_1, \ldots, X_n\}$ and $X_{(n)} = \max\{X_1, \ldots, X_n\}$.

13. Show that $(X_{(1)}, X_{(n)})$ is sufficient for (α, β).

14. Show that $(X_{(1)}, X_{(n)})$ is minimal sufficient for (α, β).

    Note: You may not use the result from the next question, Question 15.

15. Show that $(X_{(1)}, X_{(n)})$ is complete for (α, β).

    Hint: The joint pdf of $(X_{(1)}, X_{(n)})$ is given by

    $$f_{\alpha,\beta}(x, y) = n(n-1)[F_{\alpha,\beta}(y) - F_{\alpha,\beta}(x)]^{n-2} f_{\alpha,\beta}(x) f_{\alpha,\beta}(y) \quad \text{for } \alpha < x < y < \beta,$$

    where $F_{\alpha,\beta}$ is the cumulative distribution function (CDF) of X_1.

1. Let $Y_i = \log X_i$ for $i=1,\cdots,n$ and $T = \frac{1}{n}\sum_{i=1}^{n} Y_i^2$. The log-likelihood function is

$$L_n(\theta) = \sum_{i=1}^{n} \log f_\theta(X_i) = -\frac{n}{2}\log(2\pi) - \sum_{i=1}^{n} Y_i - \frac{n}{2}\log\theta - \frac{1}{2\theta}\sum_{i=1}^{n}(Y_i-\theta)^2$$

$$= -\frac{n}{2}\log(2\pi) - \frac{n}{2}\left(\log\theta + \frac{T}{\theta} + \theta\right).$$

Thus, $\frac{d}{d\theta}L_n(\theta) = -\frac{n}{2}\left(1 + \frac{1}{\theta} - \frac{T}{\theta^2}\right)$.

Solving the likelihood equation $\frac{d}{d\theta}L_n(\theta) = 0$ gives $\theta = \hat{\theta}_n$ $= \frac{-1+\sqrt{1+4T}}{2}$ (note that $\theta > 0$).

Since $\frac{d}{d\theta}L_n(\theta) = -\frac{n}{2\theta^2}(\theta^2+\theta-T) < 0$ for $\theta > \hat{\theta}_n$ ($\theta^2+\theta-T>0$ for $\theta > \hat{\theta}_n$) and $\frac{d}{d\theta}L_n(\theta) > 0$ for $\underset{(\theta>0)}{\theta < \hat{\theta}_n}$, $\hat{\theta}_n$ is the unique MLE of θ.

2. We have $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N\left(0, \frac{1}{I_1(\theta)}\right)$ as $n\to\infty$,

where $I_1(\theta) = -E_\theta\left(\frac{d^2}{d\theta^2}\log f_\theta(X_1)\right) = -E_\theta\left(\frac{1}{2\theta^2} - \frac{Y_1^2}{\theta^3}\right)$

$$= -\frac{1}{2\theta^2} + \frac{1}{\theta^3}E_\theta Y_1^2 = -\frac{1}{2\theta^2} + \frac{1}{\theta^3}(\theta^2+\theta) = \frac{2\theta+1}{2\theta^2}.$$

Thus, $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N\left(0, \frac{2\theta^2}{2\theta+1}\right)$.

3. An approximately size α LRT of $H_0: \theta = \theta_0$ versus $H_a: \theta \neq \theta_0$ is to reject H_0 if $\Lambda_n > \chi_1^2(1-\alpha)$, where $\chi_1^2(1-\alpha)$ is the $1-\alpha$ quantile of the χ_1^2 distribution, and

$$\Lambda_n = 2(L_n(\hat{\theta}_n) - L_n(\theta_0)) = n\cdot\left(\log\theta_0 + \theta_0 + \frac{T}{\theta_0} - \log\hat{\theta}_n - \hat{\theta}_n - \frac{T}{\hat{\theta}_n}\right), \quad \text{with } \hat{\theta}_n = \frac{-1+\sqrt{1+4T}}{2}.$$

4. Note that $f_\theta(x) = \exp\left[-\frac{1}{2}\log(2\pi\theta) - \frac{\theta}{2} - \frac{1}{2\theta}\cdot(\log x)^2\right]$.

Thus $\underline{X} = (X_1, \cdots, X_n)$ generates a one-dimensional exponential family and $T = \frac{1}{n}\sum_{i=1}^{n} Y_i^2$ is sufficient and complete for θ.

Since $Y_1 = \log X_1 \sim N(\theta, \theta)$, we have, for constants α, β,

$$E(\alpha Y_1 + \beta Y_1^2) = \alpha\theta + \beta(\theta^2 + \theta) = (\alpha+\beta)\theta + \beta\theta^2.$$

Let $\beta = b$ and $\alpha = a - b$. Then $E((a-b)Y_1 + bY_1^2) = a\theta + b\theta^2$.

By the Lehmann-Scheffé theorem, $E(((a-b)Y_1 + bY_1^2) \mid T)$ is a UMVUE of $a\theta + b\theta^2$.

5. The posterior density function of θ given $X^n = (X_1, \cdots, X_n)$ is

$$\frac{\prod_{i=1}^{n} f_\theta(X_i)\cdot g_{\alpha,\beta}(\theta)}{\int_0^\infty \prod_{i=1}^{n} f_\theta(X_i)\cdot g_{\alpha,\beta}(\theta)\,d\theta} = \frac{\theta^{-n}\cdot \exp\{-\frac{1}{\theta}\sum_{i=1}^{n}X_i\}\cdot\theta^{-(\alpha+1)}\cdot\exp\{-\frac{\beta}{\theta}\}}{\int_0^\infty \theta^{-n}\exp\{-\frac{1}{\theta}\sum_{i=1}^{n}X_i\}\cdot\theta^{-(\alpha+1)}\exp\{-\frac{\beta}{\theta}\}\,d\theta}$$

$$= \frac{(\beta + \sum_{i=1}^{n}X_i)^{\alpha+n}}{\Gamma(\alpha+n)}\cdot\theta^{-(\alpha+n+1)}\cdot\exp\left\{-\frac{1}{\theta}\left[\beta + \sum_{i=1}^{n}X_i\right]\right\},$$

where $\theta \in (0, \infty)$. (The posterior density of θ given X^n is $g_{\alpha+n,\, \beta+\sum_{i=1}^{n}X_i}(\theta)$.)

6. The Bayes estimator of θ is

$$\delta_{\alpha,\beta}(X^n) = \frac{E_{\theta|X^n}[\theta\cdot\frac{1}{\theta^2}|X^n]}{E_{\theta|X^n}[\frac{1}{\theta^2}|X^n]} = \frac{(\beta+\sum_{i=1}^n X_i)^{-1}\cdot\frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+n)}}{(\beta+\sum_{i=1}^n X_i)^{-2}\cdot\frac{\Gamma(\alpha+n+2)}{\Gamma(\alpha+n)}}$$

$$= \frac{\beta+\sum_{i=1}^n X_i}{\alpha+n+1}.$$

7. The risk function of $\delta_{\alpha,\beta}(X^n)$ is

$$R(\theta,\delta_{\alpha,\beta}(X^n)) = E_\theta\cdot\frac{(\theta-\delta_{\alpha,\beta}(X^n))^2}{\theta^2} = \frac{1}{\theta^2}E_\theta\cdot\left[\theta-\frac{\beta+\sum_{i=1}^n X_i}{\alpha+n+1}\right]^2$$

$$= \frac{1}{\theta^2}\cdot\left\{Var\left(\theta-\frac{\beta+\sum_{i=1}^n X_i}{\alpha+n+1}\right)+\left(\theta-\frac{\beta+nE_\theta X_1}{\alpha+n+1}\right)^2\right\}$$

$$= \frac{1}{\theta^2}\cdot\left\{\frac{n\cdot\theta^2}{(\alpha+n+1)^2}+\left(\theta-\frac{\beta+n\theta}{\alpha+n+1}\right)^2\right\}$$

$$= \frac{1}{(n+1+\alpha)^2}\cdot\left[n+\left(1+\alpha-\frac{\beta}{\theta}\right)^2\right].$$

The Bayes risk of $\delta_{\alpha,\beta}(X^n)$ is

$$R(G,\delta_{\alpha,\beta}(X^n)) = \int_0^\infty R(\theta,\delta_{\alpha,\beta}(X^n))\cdot g_{\alpha,\beta}(\theta)\,d\theta$$

$$= E_G\cdot\frac{1}{(n+1+\alpha)^2}\left[n+\left(1+\alpha-\frac{\beta}{\theta}\right)^2\right]$$

$$= \frac{1}{(n+1+\alpha)^2}\cdot\left[n+E_G\left(1+\alpha-\frac{\beta}{\theta}\right)^2\right]$$

$$= \frac{1}{(n+1+\alpha)^2}\cdot\left[n+(1+\alpha)^2-2(1+\alpha)\cdot\beta\cdot\beta^{-1}\cdot\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}+\beta^2\cdot\beta^{-2}\cdot\frac{\Gamma(\alpha+2)}{\Gamma(\alpha)}\right]$$

$$= \frac{1}{(n+1+\alpha)^2}(n+1+\alpha)$$

$$= \frac{1}{n+1+\alpha}.$$ So $R(G) = \frac{1}{n+1+\alpha}$ does not depend on β.

8. Let $\delta^*(X^n) = \frac{\sum_{i=1}^n X_i}{n+1}$. Then from the solution to Question 7,

$$R(\theta, \delta^*(X^n)) = \frac{1}{\theta^2} E_\theta \left[\theta - \frac{\sum_{i=1}^n X_i}{n+1}\right]^2 = \frac{1}{n+1}.$$

For $i=1,2,\cdots$, let G_i denote the prior distribution G with the pdf $g_{1/i,1}(\theta)$ ($\alpha = \frac{1}{i}$ and $\beta=1$). Then

$$R(G_i) = R(G_i, \delta_{1/i,1}(X^n)) = \frac{1}{n+1+1/i} \to \frac{1}{n+1} \quad \text{as } i \to \infty.$$

Thus, $\delta^*(X^n) = \frac{\sum_{i=1}^n X_i}{n+1}$ is a minimax estimator of θ.

9. Note that $EX_n^2 = \int_{-\infty}^{\infty} x^2 f_n(x)dx = \int_0^\infty x^2 \cdot 2f_n(x)dx$
(since $f_n(x) = f_n(-x)$)

$$= \int_0^\infty x^2 \cdot n^\alpha \exp\{-n^\alpha x\}dx$$

$$= n^{-2\alpha}\int_0^\infty t^2 \exp(-t)\cdot dt \qquad (t = n^\alpha \cdot x)$$

$$= 2n^{-2\alpha} < \infty.$$

Thus EX_n is finite and $EX_n = 0$ because $f_n(x) = f_n(-x)$ for $x \in R^1$.

Then $B_n^2 = Var\, S_n = \sum_{j=1}^n EX_j^2 = 2\sum_{j=1}^n j^{-2\alpha} \geq 2\sum_{j=1}^n \frac{1}{j}$ for $\alpha \in (0, \frac{1}{2}]$.

For every $\varepsilon > 0$, we have for $j = 1, \cdots, n$ and $\alpha \in (0, \frac{1}{2}]$,

$$E X_j^2 I(|X_j| > \varepsilon B_n) = 2\int_{\varepsilon B_n}^\infty x^2 f_j(x)dx$$

$$= 2\cdot j^{-2\alpha}\cdot \int_{\varepsilon j^\alpha \cdot B_n}^\infty t^2 \exp(-t)dt \qquad (t = j^\alpha \cdot x)$$

$$\leq 2\cdot j^{-2\alpha}\cdot \int_{\varepsilon B_n}^\infty t^2 \exp(-t)dt \quad (\text{since } j^\alpha \geq 1 \text{ for } \alpha \in (0, \tfrac{1}{2}])$$

Thus, $0 \le \frac{1}{B_n^2}\sum_{j=1}^{n} E\left(X_j^2 I(|X_j| > \varepsilon B_n)\right) \le \frac{1}{B_n^2}\cdot\sum_{j=1}^{n} 2\cdot j^{-2\alpha}\cdot \int_{\varepsilon B_n}^{\infty} t^2 \exp(-t)dt$

$= \int_{\varepsilon B_n}^{\infty} t^2 \exp(-t)dt \to 0$ as $n\to\infty$ because $\int_0^\infty t^2\exp(-t)dt = 2 < \infty$

and $\varepsilon B_n \ge 2\varepsilon \cdot \sum_{j=1}^{n}\frac{1}{j} \to \infty$ as $n \to \infty$.

Then $\frac{1}{B_n^2}\sum_{j=1}^{n} E(X_j^2 I(|X_j| > \varepsilon B_n)) \to 0$ as $n\to\infty$ and the Lindeberg condition holds for $\alpha \in (0, \frac{1}{2}]$.

For $\alpha \in (\frac{1}{2}, \infty)$, $B_n^2 = 2\sum_{j=1}^{n} j^{-2\alpha}$ converges to a positive number as $n\to\infty$. That is, $\lim_{n\to\infty} B_n^2 \equiv C_\alpha \in (0,\infty)$. As B_n increases in n, $\frac{1}{B_n^2}\sum_{j=1}^{n} E(X_j^2 I(|X_j| > B_n))$

$$\ge \frac{1}{C_\alpha}\cdot E(X_1^2 I(|X_1| > C_\alpha)) = \frac{2}{C_\alpha}\int_{C_\alpha}^{\infty} x^2 \exp(-x)dx > 0.$$

So the Lindeberg condition does not hold for $\alpha \in (\frac{1}{2}, \infty)$.

In summary, the sequence $\{X_n\}_{n\ge 1}$ satisfies the Lindeberg condition if and only if $\alpha \in (0, \frac{1}{2}]$.

10. By the Lindeberg central limit theorem, for $a_n = 0$ and $b_n = B_n = \sqrt{2\sum_{j=1}^{n} j^{-2\alpha}}$, $\frac{S_n - a_n}{b_n} = \frac{S_n}{B_n} \xrightarrow{d} N(0,1)$, where $\alpha \in (0, \frac{1}{2}]$.

11. Since $\{X_j\}_{j\geq 1}$ is a sequence of independent random variables, $EX_j = 0$ for $j \geq 1$, and $\sum_{j=1}^{\infty} Var(X_j) = 2\sum_{j=1}^{\infty} j^{-2\alpha} < \infty$ for $\alpha \in (\frac{1}{2}, \infty)$, by Kolmogorov's one-series theorem, S_n converges with probability one.

We may also apply Kolmogorov's three-series theorem by noting that for $\alpha \in (\frac{1}{2}, \infty)$,

(i) $\sum_{j=1}^{\infty} P(|X_j|>1) \leq \sum_{j=1}^{\infty} EX_j^2 = 2\sum_{j=1}^{\infty} j^{-2\alpha} < \infty$;

(ii) $\sum_{j=1}^{\infty} E(X_j 1_{\{|X_j|\leq 1\}}) = \sum_{j=1}^{\infty} 0 = 0$ converges;

(iii) $\sum_{j=1}^{\infty} Var(X_j 1_{\{|X_j|\leq 1\}}) = \sum_{j=1}^{\infty} E(X_j^2 1\{|X_j|\leq 1\})$

$$\leq \sum_{j=1}^{\infty} EX_j^2 = 2\sum_{j=1}^{\infty} j^{-2\alpha} < \infty.$$

Thus, S_n converges with probability one.

12. The characteristic function of X_j is

$$\phi_j(t) = E(e^{itX_j}) = E(\cos(tX_j))$$

(X_j is symmetrically distributed)

$$= \int_0^{\infty} \cos(tx) \cdot j^{\alpha} \exp(-j^{\alpha}x)\,dx$$

$$= \int_0^{\infty} \cos\left(\frac{t}{j^{\alpha}}u\right) \cdot \exp(-u)\,du \qquad (u = j^{\alpha}x)$$

$$= \frac{1}{1+\frac{t^2}{j^{2\alpha}}}.$$

Then $\phi_{S_n}(t) = E(e^{itS_n}) = \prod_{j=1}^{n} \phi_j(t) = \prod_{j=1}^{n} \frac{1}{1+\frac{t^2}{j^{2\alpha}}}$.

From Question 11, $\phi_S(t) = E(e^{itS}) = \lim_{n\to\infty} \phi_{S_n}(t) = \prod_{j=1}^{\infty} \frac{1}{1+\frac{t^2}{j^{2\alpha}}}$.

13. The joint pdf of $X_1, \cdots, X_n$ is

$$f_{\alpha,\beta}(\underline{x}) \equiv \prod_{i=1}^{n} f_{\alpha,\beta}(x_i) = [c(\alpha,\beta)]^n \cdot \prod_{i=1}^{n} h(x_i) \cdot I_{(\alpha,\beta)}(x_i)$$
$$= [c(\alpha,\beta)]^n \cdot \prod_{i=1}^{n} h(x_i) \cdot I_{(\alpha,\beta)}(x_{(1)}) \cdot I_{(\alpha,\beta)}(x_{(n)}),$$

where $x_{(1)} = \min\{x_1, \cdots, x_n\}$ and $x_{(n)} = \max\{x_1, \cdots, x_n\}$.

By the factorization theorem, $(X_{(1)}, X_{(n)})$ is sufficient for (α,β).

14. For any $\underline{x}$ and $\underline{y} \in \mathbb{R}^n$ such that $f_{\alpha,\beta}(\underline{x}) = f_{\alpha,\beta}(\underline{y}) \cdot k(\underline{x},\underline{y})$ for all $(\alpha,\beta) \in \mathbb{R}^2$ with $\alpha < \beta$, for some $k(\underline{x},\underline{y}) > 0$, we need to show that $(x_{(1)}, x_{(n)}) = (y_{(1)}, y_{(n)})$, where $y_{(1)} = \min\{y_1, \cdots, y_n\}$ and $y_{(n)} = \max\{y_1, \cdots, y_n\}$.

Note that $f_{\alpha,\beta}(\underline{x}) = f_{\alpha,\beta}(\underline{y}) \cdot k(\underline{x},\underline{y})$ implies that

$$I_{(\alpha,\beta)}(x_{(1)}) \cdot I_{(\alpha,\beta)}(x_{(n)}) \prod_{i=1}^{n} h(x_i) = I_{(\alpha,\beta)}(y_{(1)}) \cdot I_{(\alpha,\beta)}(y_{(n)}) \cdot \prod_{i=1}^{n} h(y_i) \cdot k(\underline{x},\underline{y}) \qquad (*)$$

If $x_{(1)} \neq y_{(1)}$, say $x_{(1)} < y_{(1)}$, then for $\alpha \in (x_{(1)}, y_{(1)})$ and $\beta > y_{(n)}$, $(*)$ cannot hold because its left side equals 0 ($I_{(\alpha,\beta)}(x_{(1)}) = 0$) while its right side is positive. Thus $x_{(1)} = y_{(1)}$. Similarly, $x_{(n)} = y_{(n)}$ and hence $(x_{(1)}, x_{(n)}) = (y_{(1)}, y_{(n)})$. This shows that $(X_{(1)}, X_{(n)})$ is minimal sufficient for (α,β).

15. For any measurable function of $(X_{(1)}, X_{(n)})$, $g(X_{(1)}, X_{(n)})$, such that $E_{\alpha,\beta}\, g(X_{(1)}, X_{(n)}) = 0$ for all $\alpha, \beta \in \mathbb{R}^1$ with $\alpha < \beta$, we need to show that $g(X_{(1)}, X_{(n)}) = 0$ a.s. $P_{\alpha,\beta}$, for all $\alpha, \beta \in \mathbb{R}^1$ with $\alpha < \beta$, where $P_{\alpha,\beta}$ denotes the distribution

of $(X_1, \cdots, X_n)$ with the joint pdf $\prod_{i=1}^{n} f_{\alpha,\beta}(x_i)$.

Note that for $\alpha < x \le y < \beta$, $F_{\alpha,\beta}(y) - F_{\alpha,\beta}(x) = c(\alpha,\beta)\int_x^y h(t)dt$.

Thus, $0 = E_{\alpha,\beta}\, g(X_{(1)}, X_{(n)})$

$$= n(n-1)\cdot[c(\alpha,\beta)]^n \cdot \int_\alpha^\beta \int_\alpha^y g(x,y)\cdot\left[\int_x^y h(t)dt\right]^{n-2} h(x)h(y)\,dx\,dy$$

$$\Rightarrow \int_\alpha^\beta \int_\alpha^y g(x,y)\left[\int_x^y h(t)dt\right]^{n-2} h(x)h(y)\,dx\,dy = 0 \qquad (**)$$

for all $\alpha, \beta \in \mathbb{R}^1$ with $\alpha < \beta$.

Differentiating with respect to β, $(**) \Rightarrow$

$$\int_\alpha^\beta g(x,\beta)\left[\int_x^\beta h(t)dt\right]^{n-2} h(x)\,h(\beta)\,dx = 0 \quad \text{a.s. } \mu, \quad (***)$$

where μ is the Lebesgue measure on $\mathbb{R}^1$.

Differentiating with respect to α, $(***) \Rightarrow$

$$g(\alpha,\beta)\left[\int_\alpha^\beta h(t)dt\right]^{n-2} h(\alpha)h(\beta) = 0 \quad \text{a.s. } \mu.$$

Thus, $g(\alpha,\beta) = 0$ a.s. μ because $h(x) > 0$ for $x \in \mathbb{R}^1$.

This shows that $g(X_{(1)}, X_{(n)}) = 0$ a.s. $P_{\alpha,\beta}$ for all $\alpha, \beta \in \mathbb{R}^1$ with $\alpha < \beta$ and thus $(X_{(1)}, X_{(n)})$ is complete for (α,β).

Part I

For $\theta \in \Theta = [-1, 1)$, consider a bivariate function

$$C_\theta(u,v) = \begin{cases} \frac{uv}{1-\theta(1-u)(1-v)} & \text{if } 0 \le u, v \le 1 \\ u & \text{if } 0 \le u \le 1,\ v > 1 \\ v & \text{if } 0 \le v \le 1,\ u > 1 \\ 1 & \text{if } u, v > 1 \\ 0 & \text{otherwise} \end{cases}$$

1. Show that $C_\theta(u,v)$ is a bivariate distribution function for each $\theta \in \Theta$.

2. From the bivariate distribution specified by $C_\theta(u,v)$, show that each marginal distribution is uniform$(0,1)$.

3. Let (U,V) be a random vector with $C_\theta(u,v)$ as its distribution function. Show that the conditional distribution function $P_\theta(U \le u|V = v)$ of U given $V = v \in (0,1)$ is $\frac{\partial C_\theta(u,v)}{\partial v}$.
Hint: you may consider $P_\theta(U \le u|v \le V \le v+h)$ as $h \to 0$.

Now let $\Phi_1(\cdot)$ and $\Phi_2(\cdot)$ be respectively the $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$ distribution functions.

4. Show that $C_\theta\{\Phi_1(x), \Phi_2(y)\}$ is a bivariate distribution function for $\mathbb{R}^2$.

Let (X,Y) be a bivariate random vector that has $C_\theta\{\Phi_1(x), \Phi_2(y)\}$ as its distribution function.

5. Derive the marginal distributions of X and Y.

6. Derive the conditional distribution function $P_\theta(X \le x|Y = y)$ of X given $Y = y \in \mathbb{R}$.
Hint: You may use the result in Question 3.

7. Is (X,Y) bivariate normally distributed? Justify your answer.

Part II

Let $X_1, \cdots, X_n$ be an independent and identically distributed sample from a distribution with the probability density function

$$f(x|\mu,\sigma) = \begin{cases} \frac{1}{\sigma}\sqrt{\frac{2}{\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} & \text{if } x > \mu \\ 0 & \text{otherwise,} \end{cases}$$

for parameters $\mu \in (-\infty, \infty)$ and $\sigma > 0$.

8. Show that $(X_1 - \mu)/\sigma$ has the same distribution as $|Z|$, where Z is a standard normal variable.

9. Identify a set of sufficient statistics for the unknown parameter (μ, σ^2).

10. Derive the maximum likelihood estimator (MLE) of (μ, σ^2). (Denote the corresponding MLEs for μ and for σ^2 as $\hat{\mu}_n$ and $\hat{\sigma}_n^2$ respectively.)

11. As $n \to \infty$, show that $\hat{\mu}_n \to \mu$ in probability and, using this, that $\hat{\sigma}_n^2 \to \sigma^2$ in probability as well.

12. Suppose that $\mu = 0$ is known. Derive the α-level ($\alpha \in (0,1)$) most powerful test for $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma = \sigma_1$ for two fixed parameter values $\sigma_0 > \sigma_1 > 0$.

13. Suppose that $\mu = 0$ is known and that $\sigma_0 > 0$ is a fixed, hypothesized parameter value. Let $R_n(\sigma_0)$ denote the ratio test statistic for $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma \neq \sigma_0$. Show that

$$\begin{aligned} -2\log R_n(\sigma_0) &= -n\log\left(\frac{\hat{\sigma}_n^2}{\sigma_0^2}\right) + n\left(\frac{\hat{\sigma}_n^2}{\sigma_0^2} - 1\right) \\ &= \frac{1}{2C_n^2}\left(\frac{\hat{\sigma}_n^2}{\sigma_0^2} - 1\right)^2, \end{aligned}$$

where $\hat{\sigma}_n^2 = n^{-1}\sum_{i=1}^n X_i^2$ and C_n is a value between 1 and $\hat{\sigma}_n^2/\sigma_0^2$.
Hint: For the second equality, consider a second order Taylor expansion of $\log(x)$, $x > 0$, around 1.

14. Using the result in Question 13, show that $-2\log R_n(\sigma_0)$ converges in distribution to χ_1^2 under $H_0 : \sigma = \sigma_0$ as $n \to \infty$.
Hint: Use the CLT and Slutsky's theorem, recalling χ_1^2 has mean 1 and variance 2.

Part I

For $\theta \in \Theta = [-1, 1)$, consider a bivariate function

$$C_\theta(u,v) = \begin{cases} \frac{uv}{1-\theta(1-u)(1-v)} & \text{if } 0 \le u, v \le 1 \\ u & \text{if } 0 \le u \le 1,\ v > 1 \\ v & \text{if } 0 \le v \le 1,\ u > 1 \\ 1 & \text{if } u, v > 1 \\ 0 & \text{otherwise} \end{cases}$$

1. (i) Can easily checked that for $u, v \in [0,1]$, $uv \le 1-\theta(1-u)(1-v)$ since $u+v-2uv \ge 0$ and $|\theta| \le 1$, so that $0 \le C_\theta(u,v) \le 1$.

(ii) Clearly $C_\theta(u,v)$ is continuous everywhere and hence right continuous wrt to each u and v.

(iii) The partial derivative wrt u is

$$\frac{\partial C_\theta(u,v)}{\partial u} = \frac{v(1-\theta(1-v))}{\{1-\theta(1-u)(1-v)\}^2} \ge 0.$$

A similar form is for the partial derivative wrt v. It can be checked that

$$C_\theta(u+h_1, v+h_2) - C_\theta(u+h_1, v) - C_\theta(u, v+h_2) + C_\theta(u,v) \ge 0.$$

(iv)

$$C_\theta(-\infty, v) = C_\theta(u, -\infty) = 0 \quad \text{and} \quad C_\theta(\infty, \infty) = 1.$$

So, C_θ is a distribution function.

2. The marginal cdf of U is given by $F(u) = C_\theta(u, \infty) = uI(u \in (0,1)) + I(u \ge 1)$, corresponding to the uniform$(0,1)$ distribution. So is the second marginal distribution by the argument.

3. For $v \in (0,1)$ and small $h > 0$, write

$$\begin{aligned} P_\theta(U \le u | v \le V \le v+h) &= \frac{P_\theta(U \le u, v \le V \le v+h)}{P_\theta(v \le V \le v+h)} \\ &= \frac{C_\theta(u, v+h) - C_\theta(u,v)}{h} \end{aligned}$$

using the continuity of $C_\theta(u,v)$ and the uniform$(0,1)$ distribution of V. Letting $h \to 0$, we have the desired claim.

4. Since Φ_i are distributions (the ranges are on-to $(0,1)$, continuous), and since $C_\theta(u,v)$ is a distribution supported on $(0,1)^2$, $C_\theta\{\Phi_1(x), \Phi_2(y)\}$ is a distribution by checking it is right continuous (in fact continuous everywhere) and other qualification for a CDF as we did in Problem 1.

5. $F_X(x) = C_\theta\{\Phi_1(x), \Phi_2(\infty)\} = C_\theta\{\Phi_1(x), 1\} = \Phi_1(x)$; similarly $F_Y(y) = \Phi_2(y)$.

6. For fixed $x, y \in \mathbb{R}$ and small $h > 0$, write

$$\begin{aligned}
& P_\theta(X \leq x | y \leq Y \leq y+h) \\
= & \frac{P_\theta(X \leq x, y \leq Y \leq y+h)}{P_\theta(y \leq Y \leq y+h)} \\
= & \frac{C_\theta(\Phi_1(x), \Phi_2(y+h)) - C_\theta(\Phi_1(x), \Phi_2(y))}{\Phi_2(y+h) - \Phi_2(y)} \\
= & \frac{C_\theta(\Phi_1(x), \Phi_2(y+h)) - C_\theta(\Phi_1(x), \Phi_2(y))}{h} \left[\frac{\Phi_2(y+h) - \Phi_2(y)}{h}\right]^{-1}
\end{aligned}$$

using the continuity of $C_\theta(\Phi_1(x), \Phi_2(y))$ and the normal distribution $\Phi_2(\cdot)$ of Y. Letting $h \to 0$, we have partial derivatives

$$\lim_{h \to 0} \frac{C_\theta(\Phi_1(x), \Phi_2(y+h)) - C_\theta(\Phi_1(x), \Phi_2(y))}{h} = C_\theta(\Phi_1(x), \Phi_2(y))\phi_2(y)$$

and

$$\lim_{h \to 0} \frac{\Phi_2(y+h) - \Phi_2(y)}{h} = \phi_2(y) > 0$$

where $\Phi_2'(y) = \phi_2(y) > 0$ because $\phi_2(y)$ is a normal density. Since $(\Phi_2(y+h) - \Phi_2(y))/h \to \phi_2(y) > 0$ as $h \to 0$, we have $[(\Phi_2(y+h) - \Phi_2(y))/h]^{-1} \to 1/\phi_2(y)$ as $h \to 0$. Hence, the conditional distribution is given by multiplying two limits

$$P_\theta(X \leq x | Y = y) = \lim_{h \to 0} P_\theta(X \leq x | y \leq Y \leq y+h) = C_\theta(\Phi_1(x), \Phi_2(y)).$$

7. No. If it were, the conditional distribution in Problem 6 would be a normal cdf, which it isn't.

Part II

Let $X_1, \cdots, X_n$ be independent and identically distributed sample from a distribution with the probability density function

$$f(x|\mu, \sigma) = \begin{cases} \frac{1}{\sigma}\sqrt{\frac{2}{\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} & \text{if } x > \mu \\ 0 & \text{otherwise} \end{cases}$$

8. For $z > 0$, the density of $|Z|$ is given by the derivative of $P(|Z| \leq z) = P(-z \leq Z \leq z) = \Phi(z) - \Phi(-z)$, where Φ is the standard normal distribution function. That is, the density of $|Z|$ is $2\phi(z)$, $z > 0$, where $\phi(\cdot)$ is the standard normal density with $\phi(z) = \phi(-z)$. Using a transformation, the density of $Y = (X_1 - \mu)/\sigma$ is the same, i.e., $2\phi(y)$ for $y > 0$.

9. The likelihood is proportional to

$$(\sigma^2)^{-n/2} \exp\{-\frac{\sum(X_i-\mu)^2}{2\sigma^2}\} I(\min\{X_i\} \geq \mu)$$

By the factorization theorem, a set of sufficient statistics is $(\min\{X_i\}, \sum X_i^2, \sum X_i)$.

10. The MLEs are $\hat{\mu}_n = \min\{X_i\}$ and $\hat{\sigma}_n^2 = n^{-1}\sum(X_i - \min\{X_i\})^2$.

11. Pick $\epsilon > 0$. Then,

$$\begin{aligned} P(|\min\{X_i\} - \mu| > \epsilon) &= P\left(\left|\min\frac{X_i-\mu}{\sigma}\right| > \epsilon/\sigma\right) \\ &= P(\min|Z_i| > \epsilon/\sigma) \\ &= [P(|Z_1| > \epsilon/\sigma)]^n \to 0 \end{aligned}$$

as $n \to \infty$. This establishes the first claim.

For the second claim, write

$$\begin{aligned} \hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \hat{\mu}_n)^2 &= \frac{1}{n}\sum_{i=1}^n [(X_i-\mu) - (\hat{\mu}_n - \mu)]^2 \\ &= \sigma^2\left[\frac{1}{n}\sum_{i=1}^n Z_i^2 - \frac{2}{\sigma}(\hat{\mu}_n-\mu)\frac{1}{n}\sum_{i=1}^n |Z_i| + \frac{1}{\sigma^2}(\hat{\mu}_n-\mu)^2\right]. \end{aligned}$$

where $Z_i = (X_i-\mu)/\sigma$. By the WLLN, $\frac{1}{n}\sum_{i=1}^n Z_i^2 \xrightarrow{p} E(Z_1^2) = 1$ and $\frac{1}{n}\sum_{i=1}^n |Z_i| \xrightarrow{p} E(|Z_1|) \in (0,\infty)$ as $n \to \infty$. Since $\hat{\mu}_n - \mu \xrightarrow{p} 0$, we have $\hat{\sigma}_n^2 \xrightarrow{p} \sigma^2$ by the continuous mapping theorem.

12. By the Neyman-Pearson lemma, the MP test has the form

$$\varphi(X_1,\ldots,X_n) = \begin{cases} 1 & \text{if } L(\sigma_0) > kL(\sigma_1) \\ \gamma & \text{if } L(\sigma_0) = kL(\sigma_1) \\ 0 & \text{if } L(\sigma_0) < kL(\sigma_1) \end{cases}$$

where $k > 0$ and $\gamma \in [0,1]$ are chosen so that

$$\alpha = E_{\sigma_0}[\varphi(X_1,\ldots,X_n)] = P_{\sigma_0}(L(\sigma_0) > kL(\sigma_1)) + \gamma P_{\sigma_0}(L(\sigma_0) = kL(\sigma_1)),$$

where the likelihood function is

$$L(\sigma) = \prod_{i=1}^n f(X_i|\sigma) = (\sigma^2 2/\pi)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n X_i^2}$$

for $\sigma > 0$ and $X_1, \ldots, X_n > 0$. For fixed $\sigma_0 > \sigma_1$ and $k > 0$,

$$L(\sigma_0) \overset{>}{\underset{<}{=}} kL(\sigma_1) \quad \text{iff} \quad \sum_{i=1}^{n} X_i^2 \overset{>}{\underset{<}{=}} K_1$$

where $K_1 \in \mathbb{R}$ is a constant, depending on σ_0, σ_1. So, the MP test has the form

$$\varphi(X_1, \ldots, X_n) = \begin{cases} 1 & \text{if } \sum_{i=1}^n X_i^2 > K_1 \\ \gamma & \text{if } \sum_{i=1}^n X_i^2 = K_1 \\ 0 & \text{if } \sum_{i=1}^n X_i^2 < K_1 \end{cases}$$

where K_1 is chosen to obtain size α. Since, under H_0, $\sum_{i=1}^n X_i^2/\sigma_0^2 = \sum_{i=1}^n Z_i^2 \sim \chi_n^2$ (because $Z_1 \sim N(0,1)$ and $Z_1^2 \sim \chi_1^2$), we can choose γ to be anything (say $\gamma = 1$) and need to pick K_1 so that

$$\alpha = E_{\sigma_0}[\varphi(X_1, \ldots, X_n)] = P_{\sigma_0}\left(\sum_{i=1}^n X_i^2 > K_1\right) = P(\chi_n^2 > K_1/\sigma_0^2).$$

Pick $K_1 = \sigma_0^2 \chi^2_{n;1-\alpha}$ where $P(\chi_n^2 < \chi^2_{n;1-\alpha}) = 1 - \alpha$.

13. The unrestricted MLE with a known $\mu = 0$ is $\hat{\sigma}_n^2 = n^{-1} \sum X_i^2$. The LRT statistic is then

$$R_n(\sigma_0) = \frac{L(\sigma_0)}{L(\hat{\sigma}_n)} = \left(\frac{\hat{\sigma}_n^2}{\sigma_0^2}\right)^{n/2} e^{-\frac{n}{2\sigma_0^2}\hat{\sigma}_n^2 + n/2}.$$

Then, setting $S_n = \hat{\sigma}_n^2/\sigma_0^2 = n^{-1}\sum_{i=1}^n Z_i^2$, we have

$$-2\log R_n(\sigma_0) = -n\log(S_n) + n(S_n - 1).$$

For $x > 0$, a second order Taylor expansion around 1 gives

$$\log(x) = \log(1) + \frac{1}{1}(x-1) - \frac{1}{2c^2}(x-1)^2$$

where c is some number between 1 and x. Hence, $\log(S_n) = (S_n - 1) - [2C_n^2]^{-1}(S_n - 1)^2$ where C_n is some number between 1 and S_n, so that

$$-2\log R_n(\sigma_0) = \frac{n}{2C_n^2}(S_n - 1)^2.$$

14. Since $E(Z_1^2) = 1$ and $S_n = n^{-1}\sum_{i=1}^n Z_i^2$, by the CLT, $\sqrt{n}(S_n - 1) \xrightarrow{d} N(0, \text{Var}(Z_1^2) = \text{Var}(\chi_1^2) = 2)$ as $n \to \infty$. Hence,

$$\frac{1}{\sqrt{2}}\sqrt{n}(S_n - 1) \xrightarrow{d} N(0,1)$$

so that

$$\frac{n}{2}(S_n - 1)^2 \xrightarrow{d} \chi_1^2$$

by the continuous mapping theorem. Also, since C_n in $-2\log R_n(\sigma_0) = C_n^{-2}2^{-1}n(S_n - 1)^2$ is between 1 and S_n, where $S_n \xrightarrow{p} 1$ by the WLLN, we have that $C_n \xrightarrow{p} 1$. By Slutsky's theorem,

$$-2\log R_n(\sigma_0) = \frac{n}{2C_n^2}(S_n - 1)^2 \xrightarrow{d} \chi_1^2$$

then follows as $n \to \infty$.

Part I: Let X be a random variable and n be an integer $n \geq 0$. Suppose given n, X is Binomial(n, p) where $p \in (0, 1)$. Now, assume n has a Poisson(λ) distribution where $\lambda > 0$. Note that if $n = 0$, then $X \equiv 0$, which is a degenerate random variable.

1. Find the moment generating function of X.
2. Using your result from (1), identify the distribution of X.
3. Find the mean and variance of X.

Part II: In this problem, suppose X is any random variable and W is a positive random variable.

4. Give an example where X and $-X$ have the same distribution.
5. Give an example of a W which has both of the following properties:
   - the random variable W has a probability distribution;
   - W and $\frac{1}{W}$ have the same distribution.
6. Assume W has the same distribution as $\frac{1}{W}$, and assume $E[W] < +\infty$. Prove that

$$E[W] \geq 1.$$

Part III: Suppose X_i is $N(\mu_i, 1)$ for $i = 1, 2$, and X_1 and X_2 are independent. Define

$$Y_1 = X_1 - X_2 \qquad \text{and} \qquad Y_2 = \frac{2}{3}X_1 + \frac{1}{3}X_2.$$

7. Find the joint distribution of $\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$.
8. Find the conditional distribution of Y_1 given Y_2.

Part IV: Suppose $Z_1, ..., Z_n$ are iid Beta$(2, 2)$ and $X_1, ..., X_n$ are iid Uniform$[0, 1]$ where the $Z's$ and $X's$ are also independent. Note that $E[Z_1] = E[X_1] = \frac{1}{2}$, $Var[Z_1] = \frac{1}{20}$, $Var[X_1] = \frac{1}{12}$, and the pdf of Beta(α, β) is $f(z) = \frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1}$ for $z \in [0, 1]$. Define

$$\bar{Z}_n = \frac{1}{n}\sum_{i=1}^{n} Z_i \qquad \text{and} \qquad \bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

9. Find
$$\lim_{n\to\infty} P\left\{\sum_{i=1}^{n} X_i > \sum_{i=1}^{n} Z_i + 2\sqrt{n}\right\}.$$
Explain your answer fully.

10. Let
$$W_i = \frac{X_i - \frac{1}{2}}{Z_i^2 + 2}, i = 1, ..., n.$$
Find normalizing constants a_n, b_n and a value σ^2 such that
$$a_n(\sum_{i=1}^{n} W_i - b_n) \to N(0, \sigma^2) \text{ in distribution, as } n \to \infty.$$
You should identify σ^2 with an expression in integral form and prove your assertions.

11. Let
$$V_n = \frac{\bar{Z}_n - \bar{X}_n}{1 - \bar{X}_n}.$$
Find normalizing constants a_n and b_n such that $a_n(V_n - b_n)$ has a non-degenerate limiting distribution as $n \to \infty$. Identify the limit distribution and prove your assertions.

Part I:

(1) $\phi_X(t) = E[e^{tX}] = E[E[e^{tX}/n]]$

Since $X/n \sim \text{Binomial}(n, p)$

$$\Rightarrow E[e^{tX}/n] = [(1-p) + pe^t]^n$$

and $n \sim \text{Poisson}(\lambda)$

$$\Rightarrow p(n) = \frac{e^{-\lambda} \cdot \lambda^n}{n!}$$

So $\phi_X(t) = E[\{(1-p)+pe^t\}^n] = \sum_{n=0}^{\infty} [(1-p)+pe^t]^n \cdot \frac{e^{-\lambda} \lambda^n}{n!}$

$$= e^{-\lambda} \sum_{n=0}^{\infty} \frac{\{\lambda[(1-p)+pe^t]\}^n}{n!} = e^{-\lambda} \cdot e^{\lambda[(1-p)+pe^t]}$$

$$= e^{\lambda p[e^t - 1]}$$

(2) $X \sim \text{Poisson}(\lambda p)$

(3) $E[X] = \lambda p$

$V[X] = \lambda p$.

Part II:

(4) Any symmetric distribution (symmetric about 0).
For instance, $X \sim N(0,1)$ and $-X$ is also $N(0,1)$.

(5) Define $X = \log W$, then $-X = \log \frac{1}{W}$.

Choose X to be a random variable which has a symmetric p.d.f.

Then $W = e^X$ and $\frac{1}{W} = e^{-X}$ should have the same distribution.

(6) If W and $\frac{1}{W}$ have the same distribution, then $E[W] = E[\frac{1}{W}]$.

By Cauchy-Schwarz inequality, for any variables u and v.

$$E|uv| \leq \{E[u^2]\}^{\frac{1}{2}} \{E[v^2]\}^{1/2}$$

$$1 = E\left[\sqrt{\tfrac{1}{W}} \cdot \sqrt{W}\right] \leq \left\{E\left[\tfrac{1}{W}\right]\right\}^{\frac{1}{2}} \left\{E[W]\right\}^{\frac{1}{2}}$$

$$= \{E[W]\}^{\frac{1}{2}} \{E[W]\}^{\frac{1}{2}}$$

$$= E[W]$$

So showed $E[W] \geq 1$.

Part III:

(7) $\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = A \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, where $A = \begin{bmatrix} 1 & -1 \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$.

Since $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, I_{2\times 2} \right)$,

$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = A \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim N\left(A \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, AA^T \right)$

i.e. $\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_1 - \mu_2 \\ \frac{2}{3}\mu_1 + \frac{1}{3}\mu_2 \end{bmatrix}, \begin{bmatrix} 2 & \frac{1}{3} \\ \frac{1}{3} & \frac{5}{9} \end{bmatrix} \right)$

(8) $Y_1 | Y_2 \sim N(\bar{\mu}, \bar{\Sigma})$

where $\bar{\mu} = (\mu_1 - \mu_2) + \frac{1}{3} \times \frac{9}{5} \times \left(y_2 - \left(\frac{2}{3}\mu_1 + \frac{1}{3}\mu_2 \right) \right) = \frac{3}{5}y_2 + \frac{3}{5}\mu_1 - \frac{6}{5}\mu_2$

$\bar{\Sigma} = 2 - \frac{1}{3} \times \frac{9}{5} \times \frac{1}{3} = \frac{9}{5}$

So $Y_1 | Y_2 \sim N\left(\frac{3}{5}y_2 + \frac{3}{5}\mu_1 - \frac{6}{5}\mu_2, \frac{9}{5} \right)$

Part IV :

(9) By the C.L.T.,

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}(X_i - Z_i)\right) \xrightarrow{\mathcal{L}} N(0, Var(X_i - Z_i))$$

where $Var(X_i - Z_i) = V(X_i) + V(Z_i) = \frac{1}{12} + \frac{1}{20} = \frac{2}{15}$

So $\frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} Z_i\right) \xrightarrow{\mathcal{L}} N\left(0, \frac{2}{15}\right)$

$$P\left(\sum_{i=1}^{n} X_i > \sum_{i=1}^{n} Z_i + 2\sqrt{n}\right) = P\left(\frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} Z_i\right) > 2\right)$$

So $\lim_{n\to\infty} P\left(\sum_{i=1}^{n} X_i > \sum_{i=1}^{n} Z_i + 2\sqrt{n}\right) = \lim_{n\to\infty} P\left(\frac{1}{\sqrt{n}}\left[\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} Z_i\right] > 2\right) =$

$= P(U > 2)$ where $U \sim N(0, \frac{2}{15})$

$= P\left(\frac{U}{\sqrt{\frac{2}{15}}} > \frac{2}{\sqrt{\frac{2}{15}}}\right) = 1 - \Phi(\sqrt{30})$

where $\Phi(\cdot)$ is the c.d.f. of $N(0,1)$.

(10) W_i is iid realizations, so

by the C.L.T., $\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} W_i - E[W_i]\right) \xrightarrow{\mathcal{L}} N(0, Var(W_i))$

$$E[W_i] = E\left[\frac{X_i - \frac{1}{2}}{Z_i^2 + 2}\right] = E\left[X_i - \frac{1}{2}\right] E\left[\frac{1}{Z_i^2 + 2}\right] = 0.$$

$$V[W_i] = E\left[\frac{(X_i - \frac{1}{2})^2}{(Z_i^2+2)^2}\right] = E\left[(X_i - \tfrac{1}{2})^2\right] E\left[\frac{1}{(Z_i^2+2)^2}\right]$$

$$= V(X_i) \cdot \int_0^1 \frac{1}{(z^2+2)^2} \cdot \frac{1}{B(2,2)} \cdot z(1-z)\, dz$$

$$= \frac{1}{12\, B(2,2)} \int_0^1 \frac{1}{(z^2+2)^2} z(1-z)\, dz \triangleq \alpha^2$$

So $\frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n} W_i - 0\right) \xrightarrow{\mathcal{L}} N(0, \alpha^2)$,

thus $a_n = \frac{1}{\sqrt{n}}$, $b_n = 0$, $\alpha^2 = \frac{1}{12\, B(2,2)} \int_0^1 \frac{z(1-z)}{(z^2+2)^2}\, dz$.

(11) By the C.L.T.,

$$\sqrt{n}\left(\begin{bmatrix}\bar{X}_n\\ \bar{Z}_n\end{bmatrix}-\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}\right) \xrightarrow{\mathcal{L}} N\left(0, \begin{bmatrix}\frac{1}{12} & 0\\ 0 & \frac{1}{20}\end{bmatrix}\right)$$

Define $g(x,z)=\frac{z-x}{1-x}$, by the Dealta method,

$$\sqrt{n}\left(g(\bar{X}_n,\bar{Z}_n)-g(\tfrac{1}{2},\tfrac{1}{2})\right) \xrightarrow{\mathcal{L}} N\left(0, \nabla^T g(\tfrac{1}{2},\tfrac{1}{2})\begin{bmatrix}\frac{1}{12} & 0\\ 0 & \frac{1}{20}\end{bmatrix}\nabla g(\tfrac{1}{2},\tfrac{1}{2})\right)$$

where $\nabla^T g(\frac{1}{2},\frac{1}{2}) = \left[\left.\frac{\partial g(x,z)}{\partial x}\right|_{(\frac{1}{2},\frac{1}{2})} \quad \left.\frac{\partial g(x,z)}{\partial z}\right|_{(\frac{1}{2},\frac{1}{2})}\right]$

$= \begin{bmatrix}-2 & 2\end{bmatrix}$

So $\sqrt{n}(V_n - 0) \xrightarrow{\mathcal{L}} N\left(0, \begin{bmatrix}-2 & 2\end{bmatrix}\begin{bmatrix}\frac{1}{12} & 0\\ 0 & \frac{1}{20}\end{bmatrix}\begin{bmatrix}-2\\ 2\end{bmatrix}\right)$

i.e. $\sqrt{n}(V_n-0) \xrightarrow{\mathcal{L}} N(0, \frac{8}{15})$.

thus $a_n=\sqrt{n}$, $b_n=0$, and the limit distribution is $N(0,\frac{8}{15})$.

Part I

Let $X_1, \cdots, X_n$ be independently and identically distributed random variables from the Poisson distribution with unknown mean $\eta > 0$. From this sample, we are interested in estimating $\theta = P(X_i = 0) = e^{-\eta}$. We consider the estimators

$$\hat{\theta}_1 = \frac{1}{n} \sum_{i=1}^{n} I(X_i = 0)$$

and

$$\hat{\theta}_2 = e^{-\bar{X}}$$

where $I(X_i = 0)$ takes the value one if $X_i = 0$ and takes the value zero otherwise, and $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$.

1. Find the limiting distribution of $\sqrt{n}\left(\hat{\theta}_1 - \theta\right)$ as $n \to \infty$.

2. Find the limiting distribution of $\sqrt{n}\left(\hat{\theta}_2 - \theta\right)$ as $n \to \infty$.

3. Compute the asymptotic relative efficiency of $\hat{\theta}_1$ with respect to $\hat{\theta}_2$. Which estimator do you prefer?

Part II

Let $X_1, \cdots, X_n$ be independently and identically distributed random variables from a distribution with density

$$f(x;\theta) = \begin{cases} \theta x^{\theta-1} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Define $Y = \max\{X_1, \cdots, X_n\}$. We want to test

$$H_0 : \theta = 1 \quad \text{versus } H_1 : \theta > 1$$

and will reject the null hypothesis when $Y > c$.

4. Find the power function for this test.

5. What choice of c will make the size of the test 0.05?

Part III

Let $X_1, \cdots, X_n$ (for $n > 2$) be a random sample from a normal distribution with mean μ and variance σ^2. Suppose that we are interested in estimating $\theta = \mu/\sigma$.

6. Find the maximum likelihood estimator of θ.

7. Find the value of γ such that γS^{-1} is unbiased for σ^{-1}, where

$$S^2 = (n-1)^{-1} \sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2 .$$

   [Hint: First show that the k-th moment of $\chi^2(\nu)$ distribution is $\Gamma(\nu/2 + k)2^k/\Gamma(\nu/2)$, provided that $\nu/2 + k > 0$. The $\chi^2(\nu)$ pdf is $(2^{\nu/2}\Gamma(\nu/2))^{-1} x^{(\nu/2)-1} \exp^{-x/2}$ for $x > 0$.]

8. Find the (uniformly) minimum variance unbiased estimator of θ.

Part IV

Let $(X_1, Y_1)', \cdots, (X_n, Y_n)'$ be a random sample from a bivariate normal distribution with mean $(\mu_x, \mu_y)'$ and variance-covariance matrix $\Sigma = \begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{pmatrix}$. We are interested in estimating $\theta = \mu_y$.

9. Assuming that parameters $\mu_x, \sigma_x^2, \sigma_{xy}, \sigma_y^2$ are known, find the maximum likelihood estimator (MLE) of θ and compute its variance.

10. If the other parameters are unknown, derive the MLE of θ. Is the estimator here or the estimator from question 9 more efficient? Explain.

Theory II (Co-Major) Phd Prelim Exam - Summer 2012 Solution

1. Since $\hat{\theta}_1 = n^{-1}\sum_{i=1}^{N} Z_i$ where $Z_1, \cdots, Z_n$ follow from independent Bernoulli distribution with $P(Z_i = 1) = \theta$, we can use the CLT to obtain $\sqrt{n}(\hat{\theta}_1 - \theta) \to N[0, \theta(1-\theta)]$.

2. Let $g(x) = e^{-x}$. Using Delta method, we have

$$\sqrt{n}\left\{g(\bar{x}) - g(\eta)\right\} \to N\left[0, \left\{g'(\eta)\right\}^2 V(x)\right]$$

which is equivalent to $\sqrt{n}(\hat{\theta}_2 - \theta) \to N[0, \eta e^{-2\eta}]$, where $\eta = -\log(\theta)$.

3. $RE(\hat{\theta}_1, \hat{\theta}_2) = V(\hat{\theta}_2)/V(\hat{\theta}_1) = \eta/(e^{\eta} - 1) < 1$. Thus, $\hat{\theta}_2$ is asymptotically more efficient than $\hat{\theta}_1$.

4. Power function: $P(\theta) = Pr(Y > c) = 1 - \{P(X_1 < c)\}^n = 1 - c^{n\theta}$, because the CDF of Beta$(\theta, 1)$ is $F(x) = x^{\theta}$ for $x \in (0, 1)$.

5. Solve $P(1) = 0.05$ for c. Thus, $c = (0.95)^{1/n}$.

6. The MLE for μ and σ^2 are $\bar{X}$ and $(n-1)S^2/n$, respectively, where $\bar{X} = n^{-1}\sum_{i=1}^{n} X_i$ and $S^2 = (n-1)^{-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$. Thus, the MLE for $\theta = \mu/\sigma$ is $\hat{\theta} = (\bar{X}/S)n^{1/2}/(n-1)^{1/2}$.

7. Because $(n-1)S^2/\sigma^2$ follows from $\chi^2(n-1)$ and the k-th moment of $\chi^2(n-1)$ is equal to $\Gamma[(n-1)/2 + k]2^k/\Gamma[(n-1)/2]$, provided that $(n-1)/2 + k > 0$. Thus,

$$\begin{aligned} E\left(S^{-1}\right) &= E\left\{\left[(n-1)S^2/\sigma^2\right]^{-1/2}\sigma^{-1}(n-1)^{1/2}\right\} \\ &= \sigma^{-1}(n-1)^{1/2}\Gamma\left[(n-2)/2\right]2^{-1/2}/\Gamma\left[(n-1)/2\right] \end{aligned}$$

and γS^{-1} is unbiased for σ^{-1} provided $n - 2 > 0$, where

$$\gamma = \left(\frac{n-1}{2}\right)^{-1/2}\frac{\Gamma\left[(n-1)/2\right]}{\Gamma\left[(n-2)/2\right]}.$$

8. The statistic $(\bar{X}, S^2)$ is the complete sufficient statistic for (μ, σ^2) and is unbiased for (μ, σ^2). Note also that $\bar{X}$ and S are independent and so $E(\bar{X}/S) = E(\bar{X})E(S^{-1})$. Therefore, the UMVUE for $\theta = \mu/\sigma$ is $\hat{\theta} = \gamma\bar{X}/S$ since it is unbiased and is a function of the complete sufficient statistics.

9. When other parameters are known, the likelihood for θ is

$$L(\theta) \propto \exp\left\{ -\frac{1}{2} \sum_{i=1}^{n} \begin{pmatrix} y_i - \theta \\ x_i - \mu_x \end{pmatrix}' \begin{pmatrix} \sigma_y^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_x^2 \end{pmatrix}^{-1} \begin{pmatrix} y_i - \theta \\ x_i - \mu_x \end{pmatrix} \right\}.$$

Thus, maximizing the likelihood is equivalent to minimizing

$$Q(\theta) = \sum_{i=1}^{n} \begin{pmatrix} y_i - \theta \\ x_i - \mu_x \end{pmatrix}' \begin{pmatrix} \sigma_y^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_x^2 \end{pmatrix}^{-1} \begin{pmatrix} y_i - \theta \\ x_i - \mu_x \end{pmatrix}$$

with respect to θ. Because we can write

$$Q(\theta) = \sum_{i=1}^{n} \left\{ y_i - \theta - \frac{\sigma_{xy}}{\sigma_x^2} (x_i - \mu_x) \right\}^2 \frac{1}{\sigma_y^2 - \sigma_{xy}^2/\sigma_x^2} + \sum_{i=1}^{n} (x_i - \mu_x)^2 \frac{1}{\sigma_x^2},$$

the MLE for θ that minimizes $Q(\theta)$ is

$$\hat{\theta} = \bar{y}_n + \frac{\sigma_{xy}}{\sigma_x^2} (\mu_x - \bar{x}_n)$$

where $(\bar{x}_n, \bar{y}_n) = n^{-1} \sum_{i=1}^{n} (x_i, y_i)$. The variance of $\hat{\theta}$ is $n^{-1}\sigma_y^2(1 - \rho^2)$ where $\rho = \sigma_{xy}/(\sigma_x \sigma_y)$.

10. When all the parameters are unknown, the maximum likelihood estimator of θ is $\bar{y}_n$ and its variance is $n^{-1}\sigma_y^2$, which is no smaller than the variance of the MLE computed in Problem # 9. Thus, the MLE of θ when the other parameters are known is more efficient.