overall goal: statistical inference for fixed and random effects (variance components)

12. The ANOVA Approach to the Analysis of Linear Mixed-Effects Models

new concept: Expected Mean Squares

why? depending on what we are going to test

$\dfrac{\qquad\qquad}{MSE} = F$ does not need to be the overall MSE

We begin with a relatively simple special case. Suppose

fixed overall mean random effect of j^{th} exp. unit (trt i)

$$y_{ijk} = \mu + \tau_i + u_{ij} + e_{ijk}, \quad (i = 1, \ldots, t;\ j = 1, \ldots, n;\ \underline{k = 1, \ldots, m})$$

denote ith trt effect # of obs. per experimental unit

$$\underline{\boldsymbol{\beta}} = (\mu, \tau_1, \ldots, \tau_t)^\top, \ \underline{\boldsymbol{u}} = (u_{11}, u_{12}, \ldots, u_{tn})^\top, \underline{\boldsymbol{e}} = (e_{111}, e_{112}, \ldots, e_{tnm})^\top$$

$N = t \cdot n \cdot m$

$\boldsymbol{\beta} \in \mathbb{R}^{t+1}$, an unknown parameter vector,

$$\begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{e} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \sigma_u^2 \boldsymbol{I} & \mathbf{0} \\ \mathbf{0} & \sigma_e^2 \boldsymbol{I} \end{bmatrix} \right), \text{ where}$$

$\sigma_u^2, \sigma_e^2 \in \mathbb{R}^+$ are unknown variance components.

- This is the standard model for a CRD with t treatments, n experimental units per treatment, and m observations per experimental unit.

- We can write the model as $\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{Z}\boldsymbol{u} + \boldsymbol{e}$, where

$$\boldsymbol{X} = [\underbrace{\mathbf{1}_{tnm\times 1}}_{\mu},\ \underbrace{\boldsymbol{I}_{t\times t} \otimes \mathbf{1}_{nm\times 1}}_{\text{represent } \tau_i\ (i=1,\dots,t)}] \quad \text{and} \quad \boldsymbol{Z} = [\overbrace{\boldsymbol{I}_{tn\times tn}}^{t\cdot n \text{ experim. units}} \otimes \underbrace{\mathbf{1}_{m\times 1}}_{\substack{\text{accounts for}\\ \text{replications within}\\ \text{experimental units}}}].$$

Special Case of $t = n = m = 2$

$t \cdot n \cdot m = 2 \cdot 2 \cdot 2 = \underline{\underline{8 \text{ obs.}}}$

$$\boldsymbol{X} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \mu \\ \tau_1 \\ \tau_2 \end{bmatrix}, \quad \boldsymbol{Z} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{u} = \begin{bmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \end{bmatrix}$$

Connection to ANOVA

same
Overall mean for all 8 obs.
intercept only model not accounting for trt or random effects

Let $\boldsymbol{X}_1 = \mathbf{1}_{tnm\times 1}$, $\boldsymbol{X}_2 = [\mathbf{1}_{tnm\times 1},\ \underbrace{\boldsymbol{I}_{t\times t} \otimes \mathbf{1}_{nm\times 1}}_{\text{trt effect}}]$, and $\boldsymbol{X}_3 = [\boldsymbol{I}_{tn\times tn} \otimes \mathbf{1}_{m\times 1}]$.

Note that $\mathcal{C}(\boldsymbol{X}_1) \subset \mathcal{C}(\boldsymbol{X}_2) \subset \mathcal{C}(\boldsymbol{X}_3)$, $\boldsymbol{X} = \boldsymbol{X}_2$, and $\boldsymbol{Z} = \boldsymbol{X}_3$.

As usual, let $\boldsymbol{P}_j = \boldsymbol{P}_{\boldsymbol{X}_j} = \boldsymbol{X}_j(\boldsymbol{X}_j^\top \boldsymbol{X}_j)^- \boldsymbol{X}_j^\top$ for $j = 1, 2, 3$.

X_2 accounts for trt effect in addition to intercept

$X_3 = Z$ account for random effect in addition to intercept & trt

An ANOVA Table

Sum of Squares	Degrees of Freedom
$\boldsymbol{y}^\top(\boldsymbol{P}_2 - \boldsymbol{P}_1)\boldsymbol{y}$	$\text{rank}(\boldsymbol{X}_2) - \text{rank}(\boldsymbol{X}_1) = t - 1$
$\boldsymbol{y}^\top(\boldsymbol{P}_3 - \boldsymbol{P}_2)\boldsymbol{y}$	$\text{rank}(\boldsymbol{X}_3) - \text{rank}(\boldsymbol{X}_2) = tn - t$
$\boldsymbol{y}^\top(\boldsymbol{I} - \boldsymbol{P}_3)\boldsymbol{y}$	$\text{rank}(I) - \text{rank}(\boldsymbol{X}_3) = tnm - tn$
$\boldsymbol{y}^\top(\boldsymbol{I} - \boldsymbol{P}_1)\boldsymbol{y}$	$\text{rank}(I) - \text{rank}(\boldsymbol{X}_1) = tnm - 1$

Shortcut for Obtaining DF from Source

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Source	DF
$treatments$	$t-1$
exp. units nested within $exp.units(treatments)$ trt	$(n-1)t$
obs. units are nested within $obs.units(exp.units, treatments)$ exp. units which are nested within trt	$(m-1)nt$
$c.total$	$tnm-1$