Multivariate Normal Distribution

Independence results

Independence $\Rightarrow$ zero covariance

Zero covariance + MVN $\Rightarrow$ independence

Result 4: Suppose $\boldsymbol{X} \sim MVN_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and consider a partition of $\boldsymbol{X}, \boldsymbol{\mu}, \boldsymbol{\Sigma}$ as

$$\boldsymbol{X} = \begin{pmatrix} \boldsymbol{X}^{(1)} \\ \boldsymbol{X}^{(2)} \end{pmatrix} \qquad \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}^{(1)} \\ \boldsymbol{\mu}^{(2)} \end{pmatrix} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}$$

where $\boldsymbol{X}^{(1)}$ is $p \times 1$ and $\boldsymbol{X}^{(2)}$ is $(k-p) \times 1$. Then, $\boldsymbol{X}^{(1)}$ and $\boldsymbol{X}^{(2)}$ are independent if and only if $\text{Cov}(\boldsymbol{X}^{(1)}, \boldsymbol{X}^{(2)}) = \boldsymbol{\Sigma}_{12} = \boldsymbol{0}$.

$\Longleftrightarrow$ $X^{(1)}$ and $X^{(2)}$ are indepedent $\Rightarrow$ $Cov(X^{(1)}, X^{(2)}) = 0$ ✓

$\Longleftarrow$

Proof: Assume $\text{Cov}(\boldsymbol{X}^{(1)}, \boldsymbol{X}^{(2)}) = \boldsymbol{\Sigma}_{12} = \boldsymbol{0}$. Write $\xrightarrow{?}$ $X^{(1)}$ and $X^{(2)}$ are independent.

+ X is MVN

$$\begin{pmatrix} Y_1 \\ \vdots \\ Y_k \end{pmatrix} \boldsymbol{Y} = \begin{pmatrix} \boldsymbol{Y}^{(1)} \\ \boldsymbol{Y}^{(2)} \end{pmatrix} \in \mathbb{R}^k \qquad \boldsymbol{t} = \begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} = \begin{pmatrix} \boldsymbol{t}_1 \\ \boldsymbol{t}_2 \end{pmatrix} \in \mathbb{R}^k \quad (\boldsymbol{t}_1 \in \mathbb{R}^p, \boldsymbol{t}_2 \in \mathbb{R}^{k-p})$$

where $\boldsymbol{Y}^{(1)} \sim MVN_p(\boldsymbol{\mu}^{(1)}, \boldsymbol{\Sigma}_{11})$ and $\boldsymbol{Y}^{(2)} \sim MVN_{k-p}(\boldsymbol{\mu}^{(2)}, \boldsymbol{\Sigma}_{22})$ are independent. Then, the mgf of $\boldsymbol{Y}$ at $\boldsymbol{t}$ is

$$\begin{aligned} M_{\boldsymbol{Y}} = \text{E}e^{\boldsymbol{t}'\boldsymbol{Y}} = \text{E}e^{\boldsymbol{t}_1'\boldsymbol{Y}^{(1)} + \boldsymbol{t}_2'\boldsymbol{Y}^{(2)}} = \text{E}e^{\boldsymbol{t}_1'\boldsymbol{Y}^{(1)}} e^{\boldsymbol{t}_2'\boldsymbol{Y}^{(2)}} &= \text{E}e^{\boldsymbol{t}_1'\boldsymbol{Y}^{(1)}} \text{E}e^{\boldsymbol{t}_2'\boldsymbol{Y}^{(2)}} \\ &= M_{\boldsymbol{Y}_1}(\boldsymbol{t}_1) M_{\boldsymbol{Y}_2}(\boldsymbol{t}_2) \\ &= e^{\boldsymbol{t}_1'\boldsymbol{\mu}^{(1)} + \frac{1}{2}\boldsymbol{t}_1'\boldsymbol{\Sigma}_{11}\boldsymbol{t}_1} e^{\boldsymbol{t}_2'\boldsymbol{\mu}^{(2)} + \frac{1}{2}\boldsymbol{t}_2'\boldsymbol{\Sigma}_{22}\boldsymbol{t}_2} \end{aligned}$$

last results

$Y^{(1)}$ and $Y^{(2)}$ are indepdent

$t_1' = (t_1 \dots t_p)$

$t_2' = (t_{p+1}, \dots, t_k)$

$= e^{t'\mu + \frac{t'\Sigma t}{2}}$

$= M_X(t)$ Since $X \sim MVN(\mu, \Sigma)$

(1) $t'\mu = t_1'\mu^{(1)} + t_2'\mu^{(2)}$

$t_1'\Sigma_{11}t_1 + t_2'\Sigma_{22}t_2$

$= t' \boxed{\begin{pmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} \end{pmatrix}} t = t'\Sigma t$

$\Rightarrow$ X and Y have the same distribution $\Rightarrow$ $(X^{(1)}, X^{(2)})$ have the same dist. $(Y^{(1)}$ and $Y^{(2)})$

If $Cov(X^{(1)}, X^{(2)}) = 0$

Multivariate Normal Distribution

Independence results (cont'd)

Sometimes we begin from $X_1, \ldots, X_k$ are independent & $X_i \sim N(\mu_i, \sigma_i^2)$

In which case

- $\boldsymbol{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_k \end{pmatrix} \sim MVN_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k)'$ and $\boldsymbol{\Sigma} = \text{diag}(\sigma_1^2, \ldots, \sigma_k^2)$

$\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{pmatrix}$ has the same dist. $\begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_k \end{pmatrix} + \begin{pmatrix} \sigma_1^2 & & 0 \\ & \sigma_2 & \\ 0 & \ddots & \sigma_k^2 \end{pmatrix} \begin{pmatrix} Z_1 \\ \vdots \\ Z_k \end{pmatrix}$

- $\sum_{i=1}^k a_i X_i$ and $\sum_{j=1}^k b_j X_j$ are normal

$\begin{pmatrix} \sum a_i X_i \\ \sum b_j X_j \end{pmatrix} \sim MVN$

- $\text{Cov}(\sum_{i=1}^k a_i X_i, \sum_{j=1}^k b_j X_j) = \underbrace{\sum_{i=1}^k \sum_{j=1}^k a_i b_j \text{Cov}(X_i, X_j)} = \sum_{i=1}^k a_i b_i \sigma_i^2$

$\text{Cov}(X_i, X_j) = 0$ If $i \neq j$
$= \sigma_i^2$ If $i = j$

- $\sum_{i=1}^k a_i X_i$ and $\sum_{j=1}^k b_j X_j$ are independent iff $\sum_{i=1}^k a_i b_i \sigma_i^2 = 0$

$\sum_{i=1}^{k} a_i b_i \sigma_i^2 = 0 + MVN \Longrightarrow$ independent

- if all $\sigma_i^2 = \sigma^2$ then $\sum_{i=1}^k a_i X_i$ and $\sum_{j=1}^k b_j X_j$ are independent iff $\sum_{i=1}^k a_i b_i = 0$

Multivariate Normal Distribution

Not every MVN variable has a joint pdf

Example: Let Z_1, Z_2 be independent standard normals and define

$X_1 = 1Z_1 - Z_2 + 3$
$X_2 = 2Z_1 - 2Z_2 + 0$
$= \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

$$X_1 = Z_1 - Z_2 + 3 \qquad X_2 = 2Z_1 - 2Z_2$$

$\mathrm{Var}(AZ + b) = A\,\mathrm{Var}(Z)A'$

Then,

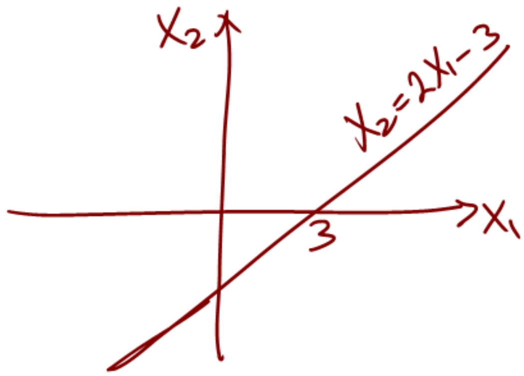
$$\boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \underbrace{\underbrace{\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}}_{A} \underbrace{\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}}_{Z} + \boxed{\begin{pmatrix} 3 \\ 0 \end{pmatrix}}}$$

and

$$\mathrm{E}\mathbf{X} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \checkmark \qquad \mathrm{Var}(\boldsymbol{X}) = \underbrace{\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}}_{A'} = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} = \Sigma$$

Here Σ^{-1} does not exist
$\det(\Sigma) = 0$

Note what's going on in this example: $\boldsymbol{X}$ is not genuinely 2-dimensional and actually assumes all of its values on a hyperplane (i.e., a line) in $\mathbb{R}^2$



$X_1 = \underbrace{Z_1 - Z_2} + 3$
$\rightarrow X_2 = 2Z_1 - 2Z_2 = 2(Z_1 - Z_2)$
$= 2(X_1 - 3) = \underbrace{2X_1 - 6}$
$\boxed{X_2 = 2X_1 - 6}$

We should be tipped off to this fact here because $\mathrm{Var}(\boldsymbol{X}) = \boldsymbol{\Sigma}$ has rank $1 < 2$ and an earlier lemma says that, since $\mathrm{Var}(\boldsymbol{X})$ is singular (not positive-definite), then $\boldsymbol{X}$ be must be confined to a lower dimensional hyperplane with probability 1.

(X_1, X_2) are in a line in $\mathbb{R}^2$

So, this makes clear that not every k-variate multivariate normal distribution can have a joint pdf in $\mathbb{R}^k$: only those with non-singular $\boldsymbol{\Sigma}$ can have joint densities

Multivariate Normal Distribution

The multivariate normal pdf

Result 5: If $\boldsymbol{X} \sim MVN_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with non-singular $\boldsymbol{\Sigma}$, then the pdf is given by

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \left(\frac{1}{(2\pi)^k \det(\boldsymbol{\Sigma})}\right)^{1/2} e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})'\overset{A'A}{\boldsymbol{\Sigma}^{-1}}(\boldsymbol{x}-\boldsymbol{\mu})}, \quad \boldsymbol{x} = (x_1, \ldots, x_k)' \in \mathbb{R}^k$$

$X \sim N(\mu, \sigma^2) \qquad X = \mu + \sigma Z$

Derivation:

1. For $Z_1, \ldots, Z_k$ iid $N(0,1)$, the pdf of $\boldsymbol{Z} = (Z_1, \ldots, Z_k)'$ is

$$f_{\boldsymbol{Z}}(\boldsymbol{z}) = (2\pi)^{-k/2} e^{-\frac{1}{2}\sum_{i=1}^k z_i^2} = (2\pi)^{-k/2} e^{-\frac{1}{2}\boldsymbol{z}'\boldsymbol{z}}, \quad \boldsymbol{z} = (z_1, \ldots, z_k)' \in \mathbb{R}^k$$

$$f_{Z_1,\ldots,Z_k}(z_1 \cdots z_k) = f_{Z_1}(z_1) \cdots f_{Z_k}(z_k)$$

$$f_Z(\boldsymbol{z}) = (*)\, e^{-\frac{1}{2} z'z}$$

2. $\boldsymbol{P}'\boldsymbol{P} = \boldsymbol{\Sigma} > 0$ implies $\boxed{\boldsymbol{A} = (\boldsymbol{P}')^{-1} = (\boldsymbol{P}^{-1})'}$ exists so that the transformation $\mathbf{X} = \boldsymbol{\mu} + \boldsymbol{P}'\boldsymbol{Z}$ has an inverse transformation $\boldsymbol{A}(\mathbf{X} - \boldsymbol{\mu}) = \boldsymbol{Z}$

$X \sim MVN_k(\mu, \Sigma)$

$$\begin{aligned} f_{\boldsymbol{X}}(\boldsymbol{x}) = f_{\boldsymbol{Z}}(\underbrace{\boldsymbol{A}(\boldsymbol{x}-\boldsymbol{\mu})}_{Z})|J| &= f_{\boldsymbol{Z}}(\boldsymbol{A}(\boldsymbol{x}-\boldsymbol{\mu}))|\det(\boldsymbol{A})| \\ &= (2\pi)^{-k/2}|\det(\boldsymbol{A})| e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{A}'\boldsymbol{A}(\boldsymbol{x}-\boldsymbol{\mu})} \end{aligned}$$

$$e^{-\frac{1}{2}(A(x-\mu))'(A(x-\mu))} \qquad (*)$$

$$(x-\mu)'A'A(x-\mu)$$

Note. $(AB)' = B'A'$

$$(x-\mu)'\Sigma^{-1}(x-\mu)$$

3. Then need some linear algebra

$$\boldsymbol{\Sigma}^{-1} = (\boldsymbol{P}'\boldsymbol{P})^{-1} = \boldsymbol{P}^{-1}(\boldsymbol{P}')^{-1} = \boldsymbol{P}^{-1}\boldsymbol{A} = \boldsymbol{A}'\boldsymbol{A}$$

and $1 = \det(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1}) = \det(\boldsymbol{\Sigma})\det(\boldsymbol{\Sigma}^{-1})$

$$\frac{1}{\det(\boldsymbol{\Sigma})} = \det(\boldsymbol{\Sigma}^{-1}) = \det(\boldsymbol{A}'\boldsymbol{A}) = \det(\boldsymbol{A}')\det(\boldsymbol{A}) = |\det(\boldsymbol{A})|^2$$

$$\det|A| = \sqrt{\frac{1}{\det \Sigma}}$$