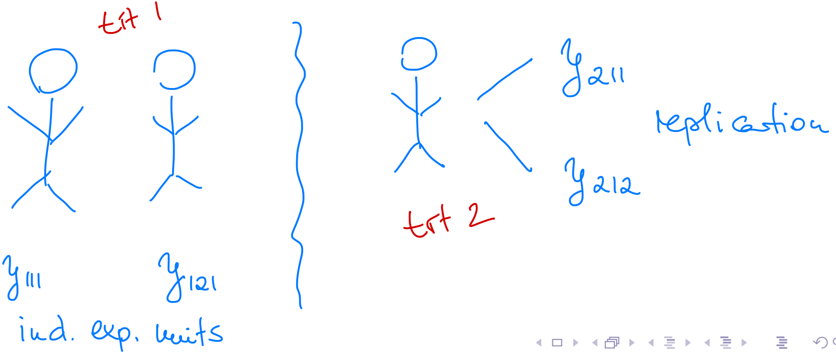
2 trt, 3 experimental units, 4 obs. units

$$\boldsymbol{y} = \begin{array}{c@{}l} \left[\begin{array}{c} y_{111} \\ y_{121} \\ \hline y_{211} \\ y_{212} \end{array}\right] & \begin{array}{l} \text{trt 1} \\ \\ \\ \text{trt 2} \end{array} \end{array}, \quad \boldsymbol{X} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{Z} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$



First Example

$$\boldsymbol{y} = \begin{bmatrix} y_{111} \\ y_{121} \\ y_{211} \\ y_{212} \end{bmatrix}, \quad \boldsymbol{X} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{Z} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{X}_1 = \boldsymbol{1}, \qquad \boldsymbol{X}_2 = \boldsymbol{X}, \qquad \boldsymbol{X}_3 = \boldsymbol{Z}$$

intercept only

intercept + trt

intercept + trt + random effect

First Example

$$\boldsymbol{y} = \begin{bmatrix} y_{111} \\ y_{121} \\ y_{211} \\ y_{212} \end{bmatrix}, \quad \boldsymbol{X} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{Z} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{X}_1 = \boldsymbol{1}, \qquad \boldsymbol{X}_2 = \boldsymbol{X}, \qquad \boldsymbol{X}_3 = \boldsymbol{Z}$$

Recall that

$\hat{y}_{(j+1)} - \hat{Y}_{(j)}$

$$SS(j+1|j) = \boldsymbol{y}^\top(\boldsymbol{P}_{j+1} - \boldsymbol{P}_j)\boldsymbol{y} = ||\underline{\boldsymbol{P}_{j+1}}\boldsymbol{y} - \underline{\boldsymbol{P}_j}\boldsymbol{y}||^2.$$

First Example

$$\boldsymbol{y} = \begin{bmatrix} y_{111} \\ y_{121} \\ y_{211} \\ y_{212} \end{bmatrix}, \quad \boldsymbol{X} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{Z} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{X}_1 = \boldsymbol{1}, \qquad \boldsymbol{X}_2 = \boldsymbol{X}, \qquad \boldsymbol{X}_3 = \boldsymbol{Z}$$

overall average

$$\boldsymbol{P}_1\,\boldsymbol{y} = \begin{bmatrix} \bar{y}_{\cdots} \\ \bar{y}_{\cdots} \\ \bar{y}_{\cdots} \\ \bar{y}_{\cdots} \end{bmatrix}, \quad \boldsymbol{P}_2\,\boldsymbol{y} = \begin{bmatrix} \bar{y}_{1\cdot 1} \\ \bar{y}_{1\cdot 1} \\ \bar{y}_{21\cdot} \\ \bar{y}_{21\cdot} \end{bmatrix}, \quad \boldsymbol{P}_3\,\boldsymbol{y} = \begin{bmatrix} y_{111} \\ y_{121} \\ \bar{y}_{21\cdot} \\ \bar{y}_{21\cdot} \end{bmatrix}$$

$$
\boldsymbol{P}_1\,\boldsymbol{y} = \begin{bmatrix} \bar{y}_{\cdots} \\ \bar{y}_{\cdots} \\ \bar{y}_{\cdots} \\ \bar{y}_{\cdots} \end{bmatrix}, \quad \boldsymbol{P}_2\,\boldsymbol{y} = \begin{bmatrix} \bar{y}_{1\cdot 1} \\ \bar{y}_{1\cdot 1} \\ \bar{y}_{21\cdot} \\ \bar{y}_{21\cdot} \end{bmatrix}, \quad \boldsymbol{P}_3\,\boldsymbol{y} = \begin{bmatrix} y_{111} \\ y_{121} \\ \bar{y}_{21\cdot} \\ \bar{y}_{21\cdot} \end{bmatrix}
$$

Thus,

$\hat{y}_{(2)} - \hat{y}_{(1)}$

$$
\begin{aligned}
SS_{trt} &= \boldsymbol{y}^\top(\boldsymbol{P}_2 - \boldsymbol{P}_1)\boldsymbol{y} = ||\boldsymbol{P}_2\,\boldsymbol{y} - \boldsymbol{P}_1\,\boldsymbol{y}||^2 \\
&= (\bar{y}_{1\cdot 1} - \bar{y}_{\cdots})^2 + (\bar{y}_{1\cdot 1} - \bar{y}_{\cdots})^2 + (\bar{y}_{21\cdot} - \bar{y}_{\cdots})^2 + (\bar{y}_{21\cdot} - \bar{y}_{\cdots})^2 \\
&= 2(\bar{y}_{1\cdot 1} - \bar{y}_{\cdots})^2 + 2(\bar{y}_{21\cdot} - \bar{y}_{\cdots})^2 = (\bar{y}_{1\cdot 1} - \bar{y}_{21\cdot})^2,
\end{aligned}
$$

where the last line follows from

$$
\bar{y}_{1\cdot 1} - \bar{y}_{\cdots} = \bar{y}_{1\cdot 1} - (\bar{y}_{1\cdot 1} + \bar{y}_{21\cdot})/2 = (\bar{y}_{1\cdot 1} - \bar{y}_{21\cdot})/2
$$

and

$$
\bar{y}_{21\cdot} - \bar{y}_{\cdots} = \bar{y}_{21\cdot} - (\bar{y}_{1\cdot 1} + \bar{y}_{21\cdot})/2 = -(\bar{y}_{1\cdot 1} - \bar{y}_{21\cdot})/2.
$$

Deriving the other sums of squares similarly and noting that $r_1 = 1$, $r_2 = 2$, and $r_3 = 3$ so that the degrees of freedom for each sum of squares is 1, we have

- $$\begin{aligned} MS_{trt} &= \boldsymbol{y}^\top(\boldsymbol{P}_2 - \boldsymbol{P}_1)\boldsymbol{y} = 2(\overline{y}_{1\cdot 1} - \overline{y}_{\dots})^2 + 2(\overline{y}_{21\cdot} - \overline{y}_{\dots})^2 \\ &= (\overline{y}_{1\cdot 1} - \overline{y}_{21\cdot})^2 \end{aligned}$$
- $$\begin{aligned} MS_{xu(trt)} &= \boldsymbol{y}^\top(\boldsymbol{P}_3 - \boldsymbol{P}_2)\boldsymbol{y} = (y_{111} - \overline{y}_{1\cdot 1})^2 + (y_{121} - \overline{y}_{1\cdot 1})^2 \\ &= \frac{1}{2}(y_{111} - y_{121})^2 \end{aligned}$$
- $$\begin{aligned} MS_{ou(xu,trt)} &= \boldsymbol{y}^\top(\boldsymbol{I} - \boldsymbol{P}_3)\boldsymbol{y} = (y_{211} - \overline{y}_{21\cdot})^2 + (y_{212} - \overline{y}_{21\cdot})^2 \\ &= \frac{1}{2}(y_{211} - y_{212})^2. \end{aligned}$$

$\bar{y}_{1\cdot 1}$ write out in terms of model:

$\tau_1 + \bar{u}_{1\cdot} + \bar{e}_{1\cdot 1}$

$$
\begin{aligned}
E(MS_{trt}) &= E(\bar{y}_{1\cdot 1} - \bar{y}_{21\cdot})^2 \\
&= E(\tau_1 - \tau_2 + \bar{u}_{1\cdot} - u_{21} + \bar{e}_{1\cdot 1} - \bar{e}_{21\cdot})^2 \\
&= (\tau_1 - \tau_2)^2 + \mathrm{Var}(\bar{u}_{1\cdot}) + \mathrm{Var}(u_{21}) + \mathrm{Var}(\bar{e}_{1\cdot 1}) \\
&\quad + \mathrm{Var}(\bar{e}_{21\cdot}) \\
&= (\tau_1 - \tau_2)^2 + \frac{\sigma_u^2}{2} + \sigma_u^2 + \frac{\sigma_e^2}{2} + \frac{\sigma_e^2}{2} \\
&= (\tau_1 - \tau_2)^2 + 1.5\sigma_u^2 + \sigma_e^2
\end{aligned}
$$

$\mathrm{Var}(u_{ij}) = \sigma_u^2$

balanced m

$$
\begin{aligned}
\mathrm{E}(MS_{xu(trt)}) &= \tfrac{1}{2}\mathrm{E}(y_{111} - y_{121})^2 \\
&= \tfrac{1}{2}\mathrm{E}(u_{11} - u_{12} + e_{111} - e_{121})^2 \\
&= \tfrac{1}{2}(2\sigma_u^2 + 2\sigma_e^2) \\
&= \sigma_u^2 + \sigma_e^2 \\
\mathrm{E}(MS_{ou(xu,trt)}) &= \tfrac{1}{2}\mathrm{E}(y_{211} - y_{212})^2 \\
&= \tfrac{1}{2}\mathrm{E}(e_{211} - e_{212})^2 \\
&= \sigma_e^2
\end{aligned}
$$

SOURCE	EMS
trt	$(\tau_1 - \tau_2)^2 + 1.5\sigma_u^2 + \sigma_e^2$
$xu(trt)$	$\sigma_u^2 + \sigma_e^2$
$ou(xu, trt)$	σ_e^2

$$F_1 = \frac{\frac{MS(trt)}{E(MS(trt))}}{\frac{MS(xu(trt))}{E(MS(xu(trt)))}}$$

With some nontrivial work, it can be shown that

$$F = \left(\frac{MS_{trt}}{1.5\sigma_u^2 + \sigma_e^2}\right) \Big/ \left(\frac{MS_{xu(trt)}}{\sigma_u^2 + \sigma_e^2}\right) \sim F_{1,1}\left(\frac{(\tau_1 - \tau_2)^2}{3\sigma_u^2 + 2\sigma_e^2}\right).$$

now we have a scaled F- distribution

The test statistic that we used to test

$$H_0 : \tau_1 = \cdots = \tau_t$$

in the balanced case is not F distributed in this unbalanced case.

$$\frac{MS_{trt}}{MS_{xu(trt)}} \sim \frac{1.5\sigma_u^2 + \sigma_e^2}{\sigma_u^2 + \sigma_e^2} F_{1,1}\left(\frac{(\tau_1 - \tau_2)^2}{3\sigma_u^2 + 2\sigma_e^2}\right)$$

A Statistic with an Approximate F Distribution

- We'd like our denominator to be an unbiased estimator of $1.5\sigma_u^2 + \sigma_e^2$ in this case.

- Consider $1.5MS_{xu(trt)} - 0.5MS_{ou(xu,trt)}$
  The expectation is

$$1.5(\sigma_u^2 + \sigma_e^2) - 0.5\sigma_e^2 = 1.5\sigma_u^2 + \sigma_e^2.$$

- The ratio

$$\frac{MS_{trt}}{1.5MS_{xu(trt)} - 0.5MS_{ou(xu,trt)}}$$

can be used as an approximate F statistic with 1 numerator DF and a denominator DF obtained using the Cochran-Satterthwaite method.

Chapter 13

- The Cochran-Satterthwaite method will be explained in the next set of notes.
- We should not expect this approximate F-test to be reliable in this case because of our pitifully small dataset.

Best Linear Unbiased Estimates in this First Example

What do the BLUEs of the treatment means look like in this case? Recall

$$\boldsymbol{\beta} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad \boldsymbol{X} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{Z} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$
\begin{aligned}
\boldsymbol{\Sigma} &= \mathrm{Var}(\boldsymbol{y}) = \boldsymbol{Z}\boldsymbol{G}\boldsymbol{Z}^\top + \boldsymbol{R} = \sigma_u^2 \boldsymbol{Z}\boldsymbol{Z}^\top + \sigma_e^2 \boldsymbol{I} \\
&= \sigma_u^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} + \sigma_e^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \sigma_u^2 & 0 & 0 & 0 \\ 0 & \sigma_u^2 & 0 & 0 \\ 0 & 0 & \sigma_u^2 & \sigma_u^2 \\ 0 & 0 & \sigma_u^2 & \sigma_u^2 \end{bmatrix} + \begin{bmatrix} \sigma_e^2 & 0 & 0 & 0 \\ 0 & \sigma_e^2 & 0 & 0 \\ 0 & 0 & \sigma_e^2 & 0 \\ 0 & 0 & 0 & \sigma_e^2 \end{bmatrix} \\
&= \begin{bmatrix} \sigma_u^2 + \sigma_e^2 & 0 & 0 & 0 \\ 0 & \sigma_u^2 + \sigma_e^2 & 0 & 0 \\ 0 & 0 & \sigma_u^2 + \sigma_e^2 & \sigma_u^2 \\ 0 & 0 & \sigma_u^2 & \sigma_u^2 + \sigma_e^2 \end{bmatrix}
\end{aligned}
$$

It follows that

$$\widehat{\boldsymbol{\beta}}_{\boldsymbol{\Sigma}} = (\boldsymbol{X}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{X})^- \boldsymbol{X}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{y}$$

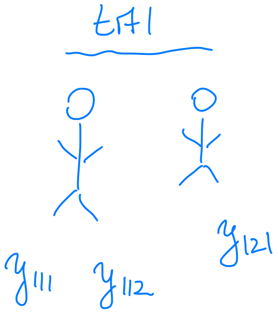
$P_2 y$

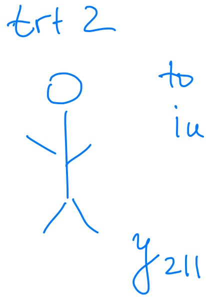
$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \boldsymbol{y} = \begin{bmatrix} \overline{y}_{1\cdot 1} \\ \overline{y}_{21\cdot} \end{bmatrix}$$

Fortunately, this is a linear estimator that does not depend on unknown variance components.

4 obs. units, 3 exp. units, 2 treatment

$$\boldsymbol{y} = \begin{bmatrix} y_{111} \\ y_{112} \\ y_{121} \\ y_{211} \end{bmatrix}, \quad \boldsymbol{X} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{Z} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$





note the diff. to first example in terms of replications for each trt.

In this case, it can be shown that

$$
\begin{aligned}
\widehat{\boldsymbol{\beta}}_{\boldsymbol{\Sigma}} &= (\boldsymbol{X}^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^-\boldsymbol{X}^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{y} \\
&= \begin{array}{c} \text{trt1} \\[2ex] \text{trt2} \end{array}
\begin{bmatrix} \frac{\sigma_e^2+\sigma_u^2}{3\sigma_e^2+4\sigma_u^2} & \frac{\sigma_e^2+\sigma_u^2}{3\sigma_e^2+4\sigma_u^2} & \frac{\sigma_e^2+2\sigma_u^2}{3\sigma_e^2+4\sigma_u^2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} y_{111} \\ y_{112} \\ y_{121} \\ y_{211} \end{bmatrix} \\
&= \begin{bmatrix} \frac{2\sigma_e^2+2\sigma_u^2}{3\sigma_e^2+4\sigma_u^2}\,\overline{y}_{11\cdot} + \frac{\sigma_e^2+2\sigma_u^2}{3\sigma_e^2+4\sigma_u^2}\,y_{121} \\ y_{211} \end{bmatrix}.
\end{aligned}
$$

It can be shown that the weights on $\overline{y}_{11\cdot}$ and y_{121} are

$$\frac{\frac{1}{\text{Var}(\overline{y}_{11\cdot})}}{\frac{1}{\text{Var}(\overline{y}_{11\cdot})}+\frac{1}{\text{Var}(y_{121})}} \text{ and } \frac{\frac{1}{\text{Var}(y_{121})}}{\frac{1}{\text{Var}(\overline{y}_{11\cdot})}+\frac{1}{\text{Var}(y_{121})}}, \text{ respectively.}$$

=

This is a special case of a more general phenomenon: the BLUE is a weighted average of independent linear unbiased estimators with weights for the linear unbiased estimators proportional to the inverse variances of the linear unbiased estimators.

// end lecture 26
3-31-25