STAT 5430

Lec 35, W, Apr 23

- Homework 8 is assigned & due M, Apr 28 by midnight
  ↑ 2nd to last homework
  (practice on inverting tests & pivotal quantities for making CIs)
- Exam 2 solutions & grading key posted

Interval Estimation I

Variance Stabilizing Transformations ← another way to get asymptotic pivot

Definition: Let $X_1, \dots, X_n$ be iid random variables with common pdf/pmf $f(x|\theta)$, where $\theta \in \Theta \subset \mathbb{R}$ (real-valued). Let $\hat{\theta}_n$ be an estimator of θ based on $X_1, \dots, X_n$ such that

↖ e.g. $\hat{\theta}_n$ could be sample mean or MLE

$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N\left(0, \sigma^2(\theta)\right) \quad \text{as } n \to \infty$$

← not asymptotic pivot! (variance depends on θ)

for all $\theta \in \Theta$. Then, a function $g : \mathbb{R} \to \mathbb{R}$ is called a **variance stabilizing transformation** (VST) for $\{\hat{\theta}_n\}$ if

$$g'(\theta) \cdot \sigma(\theta) = 1 \quad \text{holds for all } \theta \in \Theta,$$

where g' denotes the derivative of g.

Remark: If g is a VST for $\{\hat{\theta}_n\}$, then by the Delta Method,

$$\sqrt{n}\left(g(\hat{\theta}_n) - g(\theta)\right) \xrightarrow{d} N\left(0, \sigma^2(\theta)[g'(\theta)]^2 = \ 1\right) \quad \text{as } n \to \infty, \forall\theta,$$

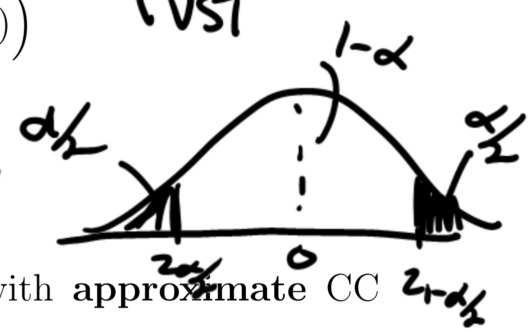
↑ $[g'(\theta)\sigma(\theta)]^2 = [1]^2 = 1$ ↑ VST

implying that

$$Q_n(X_1, \dots, X_n, \theta) \equiv \sqrt{n}\left(g(\hat{\theta}_n) - g(\theta)\right)$$

is asymptotically pivotal with $Q \equiv Z \sim N(0,1)$.

$$P(\underbrace{a}_{z_{\alpha/2}} \leq \underbrace{Q}_{Z} \leq \underbrace{b}_{z_{1-\alpha/2}}) = 1-\alpha$$



Then, a large-sample confidence region/interval for θ, with **approximate** CC $(1-\alpha)$, is

$$\begin{aligned} C_{\underset{\sim}{X}} &= \left\{\theta : \theta \in \Theta,\ z_{\alpha/2} \leq Q_n(X_1, \dots, X_n, \theta) \leq z_{1-\alpha/2}\right\} \\ &= \left\{\theta : \theta \in \Theta,\ z_{\alpha/2} \leq \sqrt{n}\left(g(\hat{\theta}_n) - g(\theta)\right) \leq z_{1-\alpha/2}\right\} \end{aligned}$$

sample means $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, $X_1, \dots, X_n$ iid $E_\theta X_1$ & $Var_\theta(X_1)$

(1) CLT:

$$\sqrt{n}\left(\bar{X}_n - E_\theta(X_1)\right) \xrightarrow{d} N\left(0, Var_\theta(X_1)\right) \text{ as } n \to \infty$$

(2) MLE $\hat{\theta}_n$

$$\sqrt{n}\left(\hat{\theta}_n - \theta\right) \xrightarrow{d} N\left(0, \frac{1}{I_1(\theta)}\right) \text{ as } n \to \infty$$

Interval Estimation I

Variance Stabilizing Transformations (VSTs): Illustration

Example 1: Let $X_1, \ldots, X_n$ be iid Poisson(θ), $\theta > 0$. Find a VST based on the MLE/MME $\hat{\theta}_n = \bar{X}_n$ and find a corresponding large-sample CI for θ with approximate C.C. $1 - \alpha$.

By CLT, $(E_\theta X_1 = \theta, Var_\theta(X_1) = \theta)$

$$\sqrt{n}(\bar{X}_n - \theta) \xrightarrow{d} N(0, Var_\theta(X_1) = \theta) \text{ as } n \to \infty.$$

g is VST if $g'(\theta) \cdot \sqrt{Var_\theta(X_1)} = g'(\theta)\sqrt{\theta} = 1$

or $g'(\theta) = 1/\sqrt{\theta}$ for $\theta > 0$.

So, $g(\theta) = \int \frac{1}{\sqrt{\theta}} d\theta$ anti-derivative of $1/\sqrt{\theta} = \theta^{-1/2}$

$$= 2\sqrt{\theta} + \underset{\uparrow \text{ take } C=0}{C}$$

We have

$$\underbrace{\sqrt{n}(g(\bar{X}_n) - g(\theta))}_{Q_n(X_1, \ldots, X_n, \theta)} = \sqrt{n}(2\sqrt{\bar{X}_n} - 2\sqrt{\theta}) \xrightarrow{d} \underset{Q}{N(0,1)}$$

A large sample CI for θ (with approx. C.C. $1-\alpha$) is

$$C_{\underset{\sim}{X}} = \left\{\theta > 0 : z_{\alpha/2} \le \sqrt{n}(2\sqrt{\bar{X}_n} - 2\sqrt{\theta}) \le z_{1-\alpha/2}\right\}$$

$$= \left\{\theta > 0 : \sqrt{\bar{X}_n} - \frac{z_{1-\alpha/2}}{2\sqrt{n}} \le \sqrt{\theta} \le \sqrt{\bar{X}_n} - \frac{z_{\alpha/2}}{2\sqrt{n}}\right\}$$

$$= \left[\left(\sqrt{\bar{X}_n} - \frac{z_{1-\alpha/2}}{2\sqrt{n}}\right)^2, \left(\sqrt{\bar{X}_n} - z_{\alpha/2}/2\sqrt{n}\right)^2\right]$$

Interval Estimation I

Variance Stabilizing Transformations (VSTs): Illustration

Example 2: Let $X_1, \ldots, X_n$ be iid Geometric(θ), $0 < \theta < 1$. Find a VST based on the MLE/MME $\hat{\theta}_n = 1/\bar{X}_n$ and find a corresponding large-sample CI for θ with approximate C.C. $1 - \alpha$.

recall: $\mathrm{Var}(\sqrt{n}\,\hat\theta_n) \approx \cancel{\frac{1}{n}}\frac{1}{I_1(\theta)}$
$\approx$ CRLB

Solution: By MLE properties,

$$\sqrt{n}(\hat\theta_n - \theta) \xrightarrow{d} N\left(0, \frac{1}{I_1(\theta)} = \theta^2(1-\theta)\right) \text{ as } n \to \infty.$$

$$\text{Recall } I_1(\theta) = -E_\theta\left[\frac{d^2 \log[\theta(1-\theta)^{X_1-1}]}{d\theta}\right] = \frac{1}{\theta^2(1-\theta)}$$

$g(\theta)$ is VST for $\{\hat\theta_n\}$ if

$$g'(\theta)\sqrt{\theta^2(1-\theta)} = 1, \quad \text{for } 0<\theta<1$$

$$\text{if } g'(\theta) = \frac{1}{\sqrt{\theta^2(1-\theta)}}, \quad \text{for } 0<\theta<1$$

$$\text{if } g(\theta) = \int \frac{1}{\sqrt{\theta^2(1-\theta)}}\,d\theta$$

$$\int \frac{1}{1+\theta^2}\,d\theta = \arctan(\theta)$$

calculus book

$$= \log\left(\frac{1-\sqrt{1-\theta}}{1+\sqrt{1-\theta}}\right) + C$$

set C to 0

Hence, a large-sample CI for θ with approximate C.C. $1-\alpha$ is

$$C_{\underline{X}} = \{0<\theta<1 : z_{\alpha/2} \le \sqrt{n}(g(\hat\theta_n) - g(\theta)) \le z_{1-\alpha/2}\}$$

$$= \left\{0<\theta<1 : z_{\alpha/2} \le \sqrt{n}\left[\log\left(\frac{1-\sqrt{1-\hat\theta_n}}{1+\sqrt{1-\hat\theta_n}}\right) - \log\left(\frac{1-\sqrt{1-\theta}}{1+\sqrt{1-\theta}}\right)\right] \le z_{1-\alpha/2}\right\}$$

ASIDE on pivots

Result: T is statistic with continuous cdf $F(t|\theta)$, $t \in \mathbb{R}$.

Then, $Q(T,\theta) \equiv F(T|\theta) \sim UNIF(0,1)$.

Make CI for θ:

$$\left\{\theta : \frac{\alpha}{2} \le Q(T,\theta) \equiv F(T|\theta) \le 1-\frac{\alpha}{2}\right\}$$

Problem 9.11 : Test $H_0: \theta=\theta_0$ using

$$\phi(\underset{\sim}{X}) = \begin{cases} 0 & \frac{\alpha}{2} \le F(T|\theta_0) \le 1-\frac{\alpha}{2} \\ 1 & \text{o.w} \end{cases}$$

$$A(\theta_0) = \left\{T : \frac{\alpha}{2} \le F(T|\theta_0) \le 1-\frac{\alpha}{2}\right\}$$

$$\begin{aligned} C_T &\equiv \{\theta_0 : T \in A(\theta_0)\} \\ &= \left\{\theta_0 : \frac{\alpha}{2} \le F(T|\theta_0) \le 1-\frac{\alpha}{2}\right\} \end{aligned}$$

$$P_{\theta_0}(\theta_0 \in C_T) = P_{\theta_0}\Big(\frac{\alpha}{2} \le \underbrace{F(T|\theta_0)}_{\text{UNIF}} \le 1-\frac{\alpha}{2}\Big) = 1-\alpha$$