Conditional distributions

Conditional expectation

In both discrete and continuous cases,

$$f(x|y) = \frac{f(x,y)}{f_Y(y)} \qquad f(y|x) = \frac{f(x,y)}{f_X(x)}$$

are conditional pmf/pdf's defining condition distributions: these conditional distributions define not only probabilities but conditional expected values as well

Extension of conditional pmf/pdf: in the multivariate case, a conditional pmf/pdf of $X_1, \ldots, X_k$ given $X_{k+1} = x_{k+1}, \ldots, X_n = x_n$ is

$$f(x_1 \ldots, x_{x_k}|x_{k+1}, \ldots, x_n) = \frac{f(x_1, \ldots, x_n)}{f(x_{k+1}, \ldots, x_n)}$$

i.e., joint pmf/pdf divided by the marginal pmf/pdf of conditioning variables

Definition: Suppose X, Y are jointly discrete or jointly continuous, $g(x,y)$ is a real-valued function, and x is such that $f(y|x)$ is defined. Then, the **conditional mean (or conditional expected value) of** $g(X,Y)$ **given that** $X = x$ is

Recall: $E[g(X,Y)] = \iint g(x,y) f_{X,Y}(x,y)\,dx\,dy$

$$\underbrace{E[g(X,Y)}|X = \underset{\text{given/fixed}}{x}] = E[g(X,Y)|x] = \begin{cases} \int g(x,y) f(y|\overset{\text{fixed}}{x}) dy & \text{continuous case} \\ \sum_y g(x,y) f(y|\underset{\text{fixed}}{x}) & \text{discrete case} \end{cases}$$

provided that

$$\underbrace{\int |g(x,y)| f(y|x) dy < \infty} \quad \text{or} \quad \underbrace{\sum_y |g(x,y)| f(y|x) < \infty}$$

Note: $E[g(X,Y)|X = x] = E[g(X,Y)|x]$ is a *function of* x

$E[g(X,Y)|Y=y]$ is a function of y.

Conditional distributions

Conditional expectation

The parallel definition holds for $\mathrm{E}[g(X,Y)|Y=y]$ too

The **conditional mean (or conditional expected value) of** $g(X,Y)$ **given that** $Y=y$ is

$$\overbrace{\mathrm{E}[g(X,Y)|Y=\underset{\uparrow\text{fixed}}{y}]}^{\text{is a function of } y} = \mathrm{E}[g(X,Y)|y] = \begin{cases} \displaystyle\int g(x,y)f(x|y)dx & \text{continuous case} \\ \displaystyle\sum_x g(x,y)f(x|y) & \text{discrete case} \end{cases}$$

provided that

$$\int |g(x,y)|f(x|y)dx < \infty \quad \text{or} \quad \sum_y |g(x,y)|f(x|y) < \infty$$

Mean and variance are the most common conditional expectations

1. Conditional mean $\mathrm{E}[X|y]$

$$\mathbb{E}(X|y) = \mathbb{E}\left(g(X,Y) \,\middle|\, Y=y\right) \qquad \boxed{g(X,Y) := X}$$

2. Conditional variance

$$\mathrm{Var}(X|y) = \mathrm{E}\left[\left(X - \mathrm{E}(X|y)\right)^2 |y\right] = \mathrm{E}[X^2|y] - (\mathrm{E}[X|y])^2$$

$$\mathrm{Var}(X) = \mathbb{E}\left[(X - \mathbb{E}(X))^2\right] = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$$

$$\mathrm{Var}(X|y) = \mathbb{E}\left[(X - \mathbb{E}(X|y))^2 \,\middle|\, y\right]$$

Conditional distributions

Conditional expectation: examples

Discrete Example: Recall

		x		
		1	2	3
	3	1/12	1/12	1/6
y	2	1/12	1/6	1/12
	1	1/6	1/12	1/12

y	$f(y\|X=1)$
3	1/4
2	1/4
1	1/2

y	$f(y\|X=2)$
3	1/4
2	1/2
1	1/4

y	$f(y\|X=3)$
3	1/2
2	1/4
1	1/4

$$
\begin{aligned}
\mathrm{E}[Y|X=1] &= \\
\mathrm{E}[Y|X=2] &= \frac{1}{4}(1)+\frac{1}{2}(2)+\frac{1}{4}(3) = 2 \\
\mathrm{E}[Y|X=3] &= \frac{1}{4}(1)+\frac{1}{4}(2)+\frac{1}{2}(3) = \frac{9}{4} \\
\mathrm{Var}(Y|X=1) &= \mathrm{E}[Y^2|X=1] - (\mathrm{E}[Y|X=1])^2 \\
&= \frac{1}{2}(1)^2+\frac{1}{4}(2)^2+\frac{1}{4}(3)^2 - \left(\frac{7}{4}\right)^2 = \frac{11}{16} \\
\mathrm{Var}(Y|X=2) &= \mathrm{E}[Y^2|X=2] - (\mathrm{E}[Y|X=2])^2 \\
&= \\
\mathrm{Var}(Y|X=3) &= \mathrm{E}[Y^2|X=3] - (\mathrm{E}[Y|X=3])^2 \\
&= \frac{1}{4}(1)^2+\frac{1}{4}(2)^2+\frac{1}{2}(3)^2 - \left(\frac{9}{4}\right)^2 = \frac{11}{16}
\end{aligned}
$$

Continuous Example: $f(x,y) = 1/x$, $0 < y < x < 1$.

We have seen $Y|X=x \sim \mathrm{Uni}(0,x)$

$$\mathbb{E}[\underbrace{Y|X=x}_{\sim \mathrm{Uni}(0,x)}] = x/2, \quad \mathrm{Var}[\underbrace{Y|X=x}_{\mathrm{Uni}(0,x)}] = x^2/12$$

Conditional distributions

Conditional expectations turned into random variables

Note again that $\mathrm{E}[g(X,Y)|X = x] = \mathrm{E}[g(X,Y)|x]$ can be thought of simply as some function of x, say

$$\boxed{m(x) = \mathrm{E}[g(X,Y)|x]}$$

This means that we can invent a random variable that is a function of X by defining

$$m(X) = \mathrm{E}[g(X,Y)|X]$$

is a new-random variable

random

Examples:

last Page

$\mathbb{E}(Y|x) = x/2$

$\mathrm{Var}(Y|x) = x^2/12$

Current Page

$\mathbb{E}(Y|X) = X/2$ R.V.

$\mathrm{Var}(Y|X) = X^2/12$ R.V.

Conditional distributions

Properties of Conditional Expectations

Conditional expectations have linearity properties like all other expectations

Result: For X, Y either jointly continuous or jointly discrete with $f_X(x) > 0$,

$$\mathrm{E}\big[ag(X,Y) + bh(X,Y) + c|\underbrace{X = x}_{\text{fixed}}\big] = a\mathrm{E}\big[g(X,Y)|X = \underset{\text{given/fixed}}{x}\big] + b\mathrm{E}\big[h(X,Y)|X = x\big] + c$$

$$\mathrm{E}\big[g(X)h(X,Y)|X = x\big] = g(x)\mathrm{E}\big[h(X,Y)|X = x\big]$$

hold, provided that the expectations above exist.

The result also holds for the random variable version of conditional expectations.

$\mathbb{E}(Y|X) =$ is a R.V. $\quad Var(Y|X)$ is a new-R.V.

Result 2: Provided that all the necessary conditional means given $X = x$ below exist for a set of x with probability 1, the equalities below hold with probability 1:

$$\mathrm{E}\big[ag(X,Y) + bh(X,Y) + c|X\big] = a\mathrm{E}\big[g(X,Y)|X\big] + b\mathrm{E}\big[h(X,Y)|X\big] + c$$

$$\mathrm{E}\big[g(X)h(X,Y)|X\big] = g(X)\mathrm{E}\big[h(X,Y)|X\big]$$

$$\int g(x)h(x,y)f(y|x)\,dy = g(x)\int h(x,y)f(y|x)\,dy = g(x)\,\mathbb{E}\big[h(X,Y)|X=x\big]$$

i.e., the quantities on the left- and right-hand sides above are random variables which are equal to each other with probability 1

Conditional distributions

Properties of Conditional Expectations (cont'd)

Two more properties of conditional expectations are next considered for the random variable version of conditional expectations:

1. ** $\mathrm{E}Y = \mathrm{E}[\overset{\text{R.V.}}{\mathrm{E}(Y|X)}]$
2. ** $\mathrm{Var}(Y) = \mathrm{E}[\mathrm{Var}(Y|X)] + \mathrm{Var}[\mathrm{E}(Y|X)]$

Theorem: If X and Y are two random variables and $\mathrm{E}Y$ exists, then

$$\boxed{\mathrm{E}Y = \mathrm{E}[\mathrm{E}(Y|X)]}$$

i.e., expected value of a conditional expectation is an unconditional expectation

Interpretation: Recall $\mathrm{E}(Y|X = x)$ is a function of x, say $m(x)$. Then,
fixed/given
$m(X) = \boxed{\mathrm{E}[Y|X]}$ is a r.v. (viewing X as a r.v. plugged into $m(x)$). So, the expected value of $m(X) = \mathrm{E}[Y|X]$ (i.e., "averaging out" X in $m(X)$) must be $\mathrm{E}Y$.

Proof (continuous case): By definition,

$$\underbrace{m(x) = \mathrm{E}(Y|X = x)} = \underbrace{\int_{-\infty}^{\infty} y f(y|x) dy} \quad (*)$$

and, by definition,

$$\underbrace{\mathrm{E}[\overset{m(X)}{\mathrm{E}(Y|X)}]} = \mathrm{E}[m(X)] = \int_{-\infty}^{\infty} m(x) \underbrace{f_X(x)} dx$$

$$\overset{(*)}{=} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} y f(y|x) dy \right] f_X(x) dx$$

$$\overset{(**)}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y \underbrace{f(y|x) f_X(x)} dy\, dx$$

$$= \underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f(x,y)\, dy\, dx} = \underbrace{\mathrm{E}(Y)}$$

(**) $f(y|x) = \dfrac{f(x,y)}{f(x)}$

Conditional distributions

Properties of Conditional Expectations (cont'd)

Theorem: If X and Y are two random variables where $\mathrm{Var}(Y)$ exists, then

**

$$\boxed{\mathrm{Var}(Y) = \mathrm{E}[\underbrace{\mathrm{Var}(Y|X)}_{\text{R.V.}}] + \mathrm{Var}[\underbrace{\mathrm{E}(Y|X)}_{\text{R.V.}}]}$$

i.e., unconditional variance is the mean of a conditional variance plus the variance of a conditional mean

Sometimes known as the EVVE formula

Proof: Let $m_2(x) = \mathrm{E}(Y^2|X = x)$ and $m_1(x) = \mathrm{E}(Y|X = x)$. Then,

$$\underbrace{m_2(x) - [m_1(x)]^2 = \mathrm{Var}(Y|X = x)}_{\text{define}}$$

and

$$m_2(X) = \underbrace{\mathrm{E}(Y^2|X)}_{\text{R.V.}}, \qquad m_1(X) = \underbrace{\mathrm{E}(Y|X)}_{\text{R.V.}}, \qquad m_2(X) - [m_1(X)]^2 = \mathrm{Var}(Y|X)$$

$$\boxed{\mathbb{E}(\underbrace{\mathbb{E}(Y^2|X)}_{m_2(X)}) = \mathbb{E}(Y^2)} =$$

on the last page

$$\boxed{\mathbb{E}[\underbrace{\mathbb{E}(Y|X)}_{m_1(X)}] = \mathbb{E}(Y)}$$

$$\begin{aligned}
\mathrm{Var}(Y) &= \mathrm{E}Y^2 - (\mathrm{E}Y)^2 \\
&= \mathrm{E}m_2(X) - [\mathrm{E}m_1(X)]^2 \\
&= \mathrm{E}m_2(X) - \mathrm{E}[m_1(X)]^2 + \mathrm{E}[m_1(X)]^2 - [\mathrm{E}m_1(X)]^2 \\
&= \mathrm{E}\left(m_2(X) - [m_1(X)]^2\right) + \mathrm{E}[m_1(X)]^2 - [\mathrm{E}m_1(X)]^2
\end{aligned}$$

$$\mathbb{E}(\mathrm{Var}(Y|X)) + \mathrm{Var}(\mathbb{E}(Y|X))$$

$$\mathbb{E}(m_1(X))^2 - (\mathbb{E}(m_1(X))^2 = \underbrace{\mathbb{E}\Big[\big(\underbrace{\mathbb{E}(Y|X)}_{\text{R.V.}}\big)^2\Big] - \Big(\mathbb{E}(\underbrace{\mathbb{E}(Y|X)}_{\text{R.V.}})\Big)^2}_{}$$

$$\mathrm{Var}(\text{R.V.}) = \mathrm{Var}(\mathbb{E}(Y|X))$$

Conditional distributions

Properties of Conditional Expectations: examples

Continuous Example: $f(x, y) = 1/x$, $0 < y < x < 1$

Recall that X ~uniform$(0, 1)$ and that, given $X = x \in (0, 1)$, $Y|X = x$ ~uniform$(0, x)$.
Starting with this information, find $\boxed{\mathrm{E}Y}$ and $\boxed{\mathrm{Var}(Y)}$.

$$\boxed{\mathbb{E}(Y|X) = X/2} \Rightarrow \underbrace{\mathbb{E}\big(\mathbb{E}(Y|X)\big)} = \mathbb{E}\left(\frac{X}{2}\right) = \boxed{\mathbb{E}(Y)}$$

$$= \frac{1}{2}\mathbb{E}(X) = 1/4$$

$$\mathrm{Var}(Y|X) = X^2/12 \Rightarrow \mathbb{E}\big[\mathrm{Var}(Y|X)\big] = \mathbb{E}\left[X^2/12\right] = 1/12\,\mathbb{E}(X^2) = 1/12 \int_0^1 x^2\,dx$$

$$= 1/12 \cdot 1/3 = 1/36$$

$$\to \mathrm{Var}\big(\mathbb{E}(Y|X)\big) = \mathrm{Var}(X/2) = 1/4\,\mathrm{Var}(X) = 1/4 \cdot 1/12$$

$$\mathrm{Var}(Y) = \mathbb{E}\big[\mathrm{Var}(Y|X)\big] + \mathrm{Var}\big[\mathbb{E}(Y|X)\big]$$

$$= \frac{1}{36} + \frac{1}{48} = 7/144$$

Discrete Example: Recall that we've already determined $\mathrm{E}Y = 2$ and $\mathrm{Var}(Y) = \frac{2}{3}$ from the joint pmf and we've found $\mathrm{E}(Y|X = x)$ and $\mathrm{Var}(Y|X = x)$

		x		
		1	2	3
y	3	1/12	1/12	1/6
	2	1/12	1/6	1/12
	1	1/6	1/12	1/12

$$\mathrm{E}(Y|X = x) = \begin{cases} 7/4 & x = 1 \\ 2 & x = 2 \\ 9/4 & x = 1 \end{cases} \quad \mathrm{Var}(Y|X = x) = \begin{cases} 11/16 & x = 1, 3 \\ 1/2 & x = 2 \end{cases}$$