when no interaction is present the additive model is a

5. Analysis of Two-Factor Experiments Based on Additive Models

simple & convenient way to model the data

An Example Two-Factor Experiment

Researchers were interested in studying the effects of 2 diets (low fiber, high fiber) and 3 drugs (D1, D2, D3) on weight gained by Yorkshire hogs. A total of 12 hogs were assigned to the 6 diet-drug combinations using a balanced and completely randomized experimental design. Hogs were housed in individual pens, injected with their assigned drugs once per week, and fed their assigned diets for a 6-week period. The amount of weight gained during the 6-week period was recorded for each hog.

As discussed in the previous set of slides, this experiment involves the two factors: Diet (with levels low fiber and high fiber) and Drug (with levels D1, D2, and D3).

The Additive Model

When factors do not interact, it makes sense to consider the *additive* model:

$$y_{ijk} = \mu + \alpha_i + \beta_j + \epsilon_{ijk} \quad (i = 1, 2;\ j = 1, 2, 3;\ k = 1, 2) \quad \text{where}$$

$\mu, \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3$ are unknown real-valued parameters and

$$\epsilon_{111}, \epsilon_{112}, \epsilon_{121}, \epsilon_{122}, \epsilon_{131}, \epsilon_{132}, \epsilon_{211}, \epsilon_{212}, \epsilon_{221}, \epsilon_{222}, \epsilon_{231}, \epsilon_{232}$$

$$\stackrel{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2)$$

for some unknown $\sigma^2 > 0$.

Table of Treatments and Means for the Additive Model

Treatment	Diet	Drug	Mean	
1	1	1	$\mu + \alpha_1 + \beta_1$	μ_{11}
2	1	2	$\mu + \alpha_1 + \beta_2$	
3	1	3	$\mu + \alpha_1 + \beta_3$	μ_{13}
4	2	1	$\mu + \alpha_2 + \beta_1$	
5	2	2	$\mu + \alpha_2 + \beta_2$	
6	2	3	$\mu + \alpha_2 + \beta_3$	

Diet 1 = Low Fiber, Diet 2 = High Fiber

Drug 1 = D1, Drug 2 = D2, Drug 3 = D3

Additive Model in Matrix and Vector Form

$$
\begin{bmatrix} y_{111} \\ y_{112} \\ y_{121} \\ y_{122} \\ y_{131} \\ y_{132} \\ y_{211} \\ y_{212} \\ y_{221} \\ y_{222} \\ y_{231} \\ y_{232} \end{bmatrix}
=
\begin{bmatrix}
1 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}
+
\begin{bmatrix} \epsilon_{111} \\ \epsilon_{112} \\ \epsilon_{121} \\ \epsilon_{122} \\ \epsilon_{131} \\ \epsilon_{132} \\ \epsilon_{211} \\ \epsilon_{212} \\ \epsilon_{221} \\ \epsilon_{222} \\ \epsilon_{231} \\ \epsilon_{232} \end{bmatrix}
$$

$$
\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}, \quad \boldsymbol{\epsilon} \sim \mathcal{N}(\boldsymbol{0}, \sigma^2 \boldsymbol{I})
$$

Copyright ©2020 Dan Nettleton (Iowa State University) with modifications by Ulrike Genschel

Additive Model in Matrix and Vector Form

$\hat{y}_{11k} = \hat{\mu} + \hat{\alpha}_1 + \hat{\beta}_1$

$\hat{y}_{ijk} = \hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j$

$$
\begin{bmatrix} y_{111} \\ y_{112} \\ y_{121} \\ y_{122} \\ y_{131} \\ y_{132} \\ y_{211} \\ y_{212} \\ y_{221} \\ y_{222} \\ y_{231} \\ y_{232} \end{bmatrix}
=
\begin{bmatrix} \mu + \alpha_1 + \beta_1 \\ \mu + \alpha_1 + \beta_1 \\ \mu + \alpha_1 + \beta_2 \\ \mu + \alpha_1 + \beta_2 \\ \mu + \alpha_1 + \beta_3 \\ \mu + \alpha_1 + \beta_3 \\ \mu + \alpha_2 + \beta_1 \\ \mu + \alpha_2 + \beta_1 \\ \mu + \alpha_2 + \beta_2 \\ \mu + \alpha_2 + \beta_2 \\ \mu + \alpha_2 + \beta_3 \\ \mu + \alpha_2 + \beta_3 \end{bmatrix}
+
\begin{bmatrix} \epsilon_{111} \\ \epsilon_{112} \\ \epsilon_{121} \\ \epsilon_{122} \\ \epsilon_{131} \\ \epsilon_{132} \\ \epsilon_{211} \\ \epsilon_{212} \\ \epsilon_{221} \\ \epsilon_{222} \\ \epsilon_{231} \\ \epsilon_{232} \end{bmatrix}
$$

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}, \quad \boldsymbol{\epsilon} \sim \mathcal{N}(\boldsymbol{0}, \sigma^2 \boldsymbol{I})$$

Copyright ©2020 Dan Nettleton (Iowa State University) with modifications by Ulrike Genschel

Table of Cell Means for the Additive Model

- All Interactions are Zero for the Additive Model.
- The simple effect of Diet is $\alpha_1 - \alpha_2$ for all levels of Drug.
- The simple effect of Drug j vs. Drug j' is $\beta_j - \beta_{j'}$ regardless of Diet.

marginal mean difference for diet : $\bar{\mu}_{1\cdot} - \bar{\mu}_{2\cdot}$

	Drug 1	Drug 2	Drug 3
Diet 1	$\mu + \alpha_1 + \beta_1$	$\mu + \alpha_1 + \beta_2$	$\mu + \alpha_1 + \beta_3$
Diet 2	$\mu + \alpha_2 + \beta_1$	$\mu + \alpha_2 + \beta_2$	$\mu + \alpha_2 + \beta_3$

$= \alpha_1 - \alpha_2$

for estimating α_1 & α_2 we will use all available data (across all 3 levels of drug)

Marginal Means for the Additive Model

difference is $\alpha_1 - \alpha_2$

	Drug 1	Drug 2	Drug 3	
Diet 1	$\mu + \alpha_1 + \beta_1$	$\mu + \alpha_1 + \beta_2$	$\mu + \alpha_1 + \beta_3$	$\mu + \alpha_1 + \bar{\beta}.$
Diet 2	$\mu + \alpha_2 + \beta_1$	$\mu + \alpha_2 + \beta_2$	$\mu + \alpha_2 + \beta_3$	$\mu + \alpha_2 + \bar{\beta}.$
	$\mu + \bar{\alpha}. + \beta_1$	$\mu + \bar{\alpha}. + \beta_2$	$\mu + \bar{\alpha}. + \beta_3$	$\mu + \bar{\alpha}. + \bar{\beta}.$

$\beta_1 - \beta_2$ $\beta_2 - \beta_3$

averaging over drug

averaging over diet

The main effect of Diet is

$$(\mu + \alpha_1 + \bar{\beta}_.) - (\mu + \alpha_2 + \bar{\beta}_.) = \alpha_1 - \alpha_2$$

	Drug 1	Drug 2	Drug 3	
Diet 1	$\mu + \alpha_1 + \beta_1$	$\mu + \alpha_1 + \beta_2$	$\mu + \alpha_1 + \beta_3$	$\mu + \alpha_1 + \bar{\beta}_.$
Diet 2	$\mu + \alpha_2 + \beta_1$	$\mu + \alpha_2 + \beta_2$	$\mu + \alpha_2 + \beta_3$	$\mu + \alpha_2 + \bar{\beta}_.$
	$\mu + \bar{\alpha}_. + \beta_1$	$\mu + \bar{\alpha}_. + \beta_2$	$\mu + \bar{\alpha}_. + \beta_3$	$\mu + \bar{\alpha}_. + \bar{\beta}_.$

Copyright ©2020 Dan Nettleton (Iowa State University) with modifications by Ulrike Genschel

Simple Effects = Main Effects for the Additive Model

The main effect for Drug 1 vs. Drug 2 is

$$(\mu + \bar{\alpha}. + \beta_1) - (\mu + \bar{\alpha}. + \beta_2) = \beta_1 - \beta_2$$

	Drug 1	Drug 2	Drug 3	
Diet 1	$\mu + \alpha_1 + \beta_1$	$\mu + \alpha_1 + \beta_2$	$\mu + \alpha_1 + \beta_3$	$\mu + \alpha_1 + \bar{\beta}.$
Diet 2	$\mu + \alpha_2 + \beta_1$	$\mu + \alpha_2 + \beta_2$	$\mu + \alpha_2 + \beta_3$	$\mu + \alpha_2 + \bar{\beta}.$
	$\mu + \bar{\alpha}. + \beta_1$	$\mu + \bar{\alpha}. + \beta_2$	$\mu + \bar{\alpha}. + \beta_3$	$\mu + \bar{\alpha}. + \bar{\beta}.$

Copyright ©2020 Dan Nettleton (Iowa State University) with modifications by Ulrike Genschel

Tests for Main Effects in the Additive Model

No Diet main effect $\iff \alpha_1 = \alpha_2$

No Drug main effects $\iff \beta_1 = \beta_2 = \beta_3$

	Drug 1	Drug 2	Drug 3	
Diet 1	$\mu + \alpha_1 + \beta_1$	$\mu + \alpha_1 + \beta_2$	$\mu + \alpha_1 + \beta_3$	$\mu + \alpha_1 + \bar{\beta}.$
Diet 2	$\mu + \alpha_2 + \beta_1$	$\mu + \alpha_2 + \beta_2$	$\mu + \alpha_2 + \beta_3$	$\mu + \alpha_2 + \bar{\beta}.$
	$\mu + \bar{\alpha}. + \beta_1$	$\mu + \bar{\alpha}. + \beta_2$	$\mu + \bar{\alpha}. + \beta_3$	$\mu + \bar{\alpha}. + \bar{\beta}.$

Copyright ©2020 Dan Nettleton (Iowa State University) with modifications by Ulrike Genschel

H_0: No Diet Main Effect $(\alpha_1 = \alpha_2)$

$\alpha_1 - \alpha_2$

$$\begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

Copyright ©2020 Dan Nettleton (Iowa State University) with modifications by Ulrike Genschel

H_0: No Drug Main Effects $(\beta_1 = \beta_2 = \beta_3)$

$\beta_1 - \beta_2$

$$C = \begin{pmatrix} c_1^T \\ c_2^T \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\beta_2 - \beta_3$

Copyright ©2020 Dan Nettleton (Iowa State University) with modifications by Ulrike Genschel

LSMEANS are OLS estimators of the quantities in the margins below.

	Drug 1	Drug 2	Drug 3	
Diet 1	$\mu+\alpha_1+\beta_1$	$\mu+\alpha_1+\beta_2$	$\mu+\alpha_1+\beta_3$	$\boxed{\mu+\alpha_1+\bar{\beta}.}$
Diet 2	$\mu+\alpha_2+\beta_1$	$\mu+\alpha_2+\beta_2$	$\mu+\alpha_2+\beta_3$	$\mu+\alpha_2+\bar{\beta}.$
	$\mu+\bar{\alpha}.+\beta_1$	$\mu+\bar{\alpha}.+\beta_2$	$\mu+\bar{\alpha}.+\beta_3$	$\mu+\bar{\alpha}.+\bar{\beta}.$

Copyright ©2020 Dan Nettleton (Iowa State University) with modifications by Ulrike Genschel

For example, the LSMEAN for Diet 1 is

$$\boldsymbol{c}^\top\widehat{\boldsymbol{\beta}} = \left[1, 1, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right] \begin{bmatrix} \widehat{\mu} \\ \widehat{\alpha}_1 \\ \widehat{\alpha}_2 \\ \widehat{\beta}_1 \\ \widehat{\beta}_2 \\ \widehat{\beta}_3 \end{bmatrix} = \underline{\widehat{\mu}} + \underline{\widehat{\alpha}_1} + \boxed{\frac{\widehat{\beta}_1 + \widehat{\beta}_2 + \widehat{\beta}_3}{3}},$$

estimate of marginal mean $\bar{\mu}_{1\cdot}$

where $\widehat{\beta}$ is any solution to the Normal Equations.

Although $\widehat{\beta}$ will depend on which of infinitely many solutions to the Normal Equations is used, $\boldsymbol{c}^\top\widehat{\boldsymbol{\beta}}$ will be the same for all solutions.

SAS Code for the Additive Model

```
proc import datafile='C:\dietdrug.txt'
   dbms=TAB replace out=d;
run;

proc mixed;
  class diet drug;
  model weightgain=diet drug;
  lsmeans diet drug / cl;
  estimate 'diet effect' diet 1 -1 / cl;
  estimate 'drug 1 - drug 2' drug 1 -1  0;
  estimate 'drug 1 - drug 3' drug 1  0 -1;
  estimate 'drug 2 - drug 3' drug 0  1 -1;
  contrast 'drug main effects' drug 1 -1  0,
                               drug 1  0 -1;
run;
```

confidence limits

use comma to seperate several comparisons

Fitting the Additive Model in R

```
d=read.delim("http://.../S510/dietdrug.txt")

d$diet=factor(d$diet)

d$drug=factor(d$drug)

o=lm(weightgain~diet+drug,data=d)
```

`coef(o)` is $\hat{\boldsymbol{\beta}}_R$.

`vcov(o)` is $\widehat{\text{Var}}(\hat{\boldsymbol{\beta}}_R) = \hat{\sigma}^2(\boldsymbol{X}_R^\top \boldsymbol{X}_R)^{-1}$.

`o$df` is $n - r$.

this model matrix is full rank
(the X R sets up internally)

The Additive Model Matrix is Not Full-Column Rank

$$
\begin{bmatrix} y_{111} \\ y_{112} \\ y_{121} \\ y_{122} \\ y_{131} \\ y_{132} \\ y_{211} \\ y_{212} \\ y_{221} \\ y_{222} \\ y_{231} \\ y_{232} \end{bmatrix}
=
\begin{bmatrix}
1 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}
+
\begin{bmatrix} \epsilon_{111} \\ \epsilon_{112} \\ \epsilon_{121} \\ \epsilon_{122} \\ \epsilon_{131} \\ \epsilon_{132} \\ \epsilon_{211} \\ \epsilon_{212} \\ \epsilon_{221} \\ \epsilon_{222} \\ \epsilon_{231} \\ \epsilon_{232} \end{bmatrix}
$$

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}, \quad \boldsymbol{\epsilon} \sim N(\boldsymbol{0}, \sigma^2 \boldsymbol{I})$$

The Full-Rank Formulation Used by R

$$\begin{bmatrix} y_{111} \\ y_{112} \\ y_{121} \\ y_{122} \\ y_{131} \\ y_{132} \\ y_{211} \\ y_{212} \\ y_{221} \\ y_{222} \\ y_{231} \\ y_{232} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_2 \\ \beta_2 \\ \beta_3 \end{bmatrix} + \begin{bmatrix} \epsilon_{111} \\ \epsilon_{112} \\ \epsilon_{121} \\ \epsilon_{122} \\ \epsilon_{131} \\ \epsilon_{132} \\ \epsilon_{211} \\ \epsilon_{212} \\ \epsilon_{221} \\ \epsilon_{222} \\ \epsilon_{231} \\ \epsilon_{232} \end{bmatrix}$$

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}, \quad \boldsymbol{\epsilon} \sim \mathcal{N}(\boldsymbol{0}, \sigma^2 \boldsymbol{I})$$

The Full-Rank Formulation Used by R

$$
\begin{bmatrix} y_{111} \\ y_{112} \\ y_{121} \\ y_{122} \\ y_{131} \\ y_{132} \\ y_{211} \\ y_{212} \\ y_{221} \\ y_{222} \\ y_{231} \\ y_{232} \end{bmatrix}
=
\begin{bmatrix} \mu \\ \mu \\ \mu+\beta_2 \\ \mu+\beta_2 \\ \mu+\beta_3 \\ \mu+\beta_3 \\ \mu+\alpha_2 \\ \mu+\alpha_2 \\ \mu+\alpha_2+\beta_2 \\ \mu+\alpha_2+\beta_2 \\ \mu+\alpha_2+\beta_3 \\ \mu+\alpha_2+\beta_3 \end{bmatrix}
+
\begin{bmatrix} \epsilon_{111} \\ \epsilon_{112} \\ \epsilon_{121} \\ \epsilon_{122} \\ \epsilon_{131} \\ \epsilon_{132} \\ \epsilon_{211} \\ \epsilon_{212} \\ \epsilon_{221} \\ \epsilon_{222} \\ \epsilon_{231} \\ \epsilon_{232} \end{bmatrix}
$$

mean diet 1 drug 1

β_2 reflects the difference between $\bar{y}_{ijk}$ when going from drug 1 to drug 2

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}, \quad \boldsymbol{\epsilon} \sim \mathcal{N}(\boldsymbol{0}, \sigma^2 \boldsymbol{I})$$

Copyright ©2020 Dan Nettleton (Iowa State University) with modifications by Ulrike Genschel

Table of Means for the R Full-Rank Formulation

	Drug 1	Drug 2	Drug 3	
Diet 1	μ	$\mu + \beta_2$	$\mu + \beta_3$	$\mu + \frac{\beta_2+\beta_3}{3}$
Diet 2	$\mu + \alpha_2$	$\mu + \alpha_2 + \beta_2$	$\mu + \alpha_2 + \beta_3$	$\mu + \alpha_2 + \frac{\beta_2+\beta_3}{3}$
	$\mu + \frac{\alpha_2}{2}$	$\mu + \frac{\alpha_2}{2} + \beta_2$	$\mu + \frac{\alpha_2}{2} + \beta_3$	$\mu + \frac{\alpha_2}{2} + \frac{\beta_2+\beta_3}{3}$

Main Effects in the R Additive Model Formulation

No Diet main effect $\Longleftrightarrow \alpha_2 = 0$

this differs from the set-up using the less-than-full rank model matrix

end lecture 10 2-12-25

No Drug main effects $\Longleftrightarrow \beta_2 = \beta_3 = 0$

	Drug 1	Drug 2	Drug 3	
Diet 1	μ	$\mu+\beta_2$	$\mu+\beta_3$	$\mu+\frac{\beta_2+\beta_3}{3}$
Diet 2	$\mu+\alpha_2$	$\mu+\alpha_2+\beta_2$	$\mu+\alpha_2+\beta_3$	$\mu+\alpha_2+\frac{\beta_2+\beta_3}{3}$
	$\mu+\frac{\alpha_2}{2}$	$\mu+\frac{\alpha_2}{2}+\beta_2$	$\mu+\frac{\alpha_2}{2}+\beta_3$	$\mu+\frac{\alpha_2}{2}+\frac{\beta_2+\beta_3}{3}$

all the same when $\beta_2=\beta_3=0$