# Functions of a random variable

Determining distributions: continuous case

For a continuous r.v. $X$, the r.v. $Y = g(X)$ will typically (but not always) be continuous. $F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(g(X) \le y) = \mathbb{P}(\{x \in \mathcal{X} : g(x) \le y\})$

To determine the distribution of $Y$, one can try either of two approaches:

1. compute the cdf $F_Y(\cdot)$ of $Y$ as

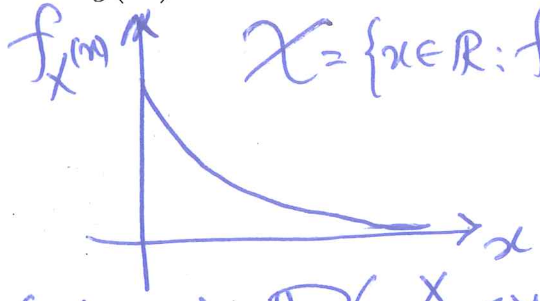
$$F_Y(y) = P(Y \le y) = P(g(X) \le y) = \int_{\{x \in \mathbb{R} : g(x) \le y\}} f_X(x)dx$$

   This is a general approach, but its success depends on computing the integral.

2. compute the pdf $f_Y(\cdot)$ directly through a transformation technique which is *only* valid if the function $g$ is monotone or "piecewise monotone."

Example 1. Let $X$ have pdf $f_X(x) = e^{-x}$, $x > 0$.

Let $Y = g(X) = e^X$. Note $Y$ has support $\mathcal{Y} = \{e^x : x \in \mathcal{X}\} = \{e^x : x > 0\} = (1, \infty)$




$\mathcal{X} = \{x \in \mathbb{R} : f_X(x) > 0\} = (0, \infty)$

$\mathbb{P}(Y \le y) = \mathbb{P}(e^X \le y) = 0$ for $y \le 0$.

$\mathbb{P}(Y \le y) = \mathbb{P}(e^X \le y) = \mathbb{P}(X \le \log y) = \int_{-\infty}^{\log y} e^{-x} dx$ ($y > 0$)

$= -e^{-x}\Big|_0^{\log y} = 1 - \frac{1}{y}$ for $y > 0 \Rightarrow F_Y(y) = \begin{cases} 0 & \text{If } y \le 1 \\ 1 - \frac{1}{y} & \text{If } y > 1 \end{cases}$

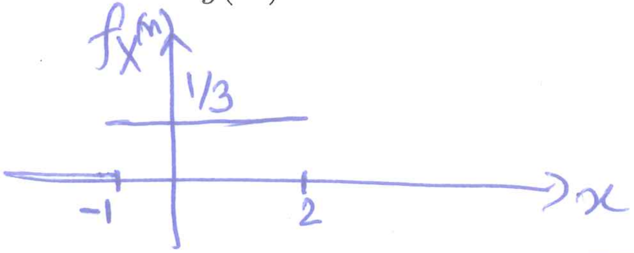
$f_Y(y) = \frac{dF_Y(y)}{dy} = \begin{cases} 0 & \text{If } y \le 1 \\ y^{-2} & \text{If } y > 1 \end{cases}$ → Support of $Y$ is $\mathcal{Y} = \{y : f_Y(y) > 0\} = (1, \infty)$

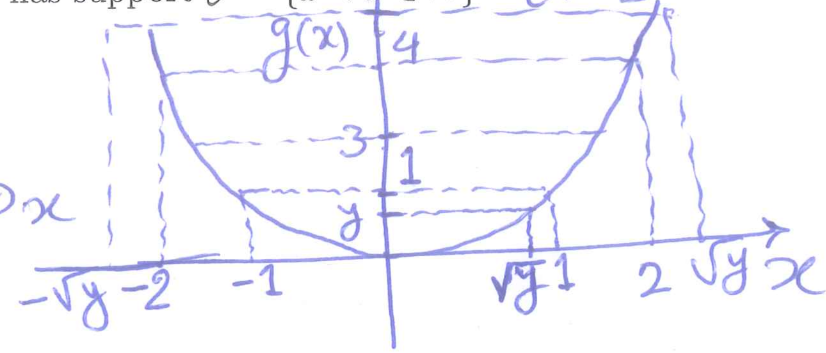
# Functions of a random variable

## Determining distributions: continuous case (cont'd)

Example 2. Let $X$ have pdf $f_X(x) = 1/3$, $-1 < x < 2$. $X \sim \text{Uni}(-1,2)$

Let $Y = g(X) = X^2$. Note $Y$ has support $\mathcal{Y} = \{x^2 : x \in \mathcal{X}\} = [0,4]$




$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(X^2 \le y) = \begin{cases} 0 & y < 0 \\ \mathbb{P}(-\sqrt{y} \le X \le \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\,dx & y \ge 0 \end{cases}$$

$$= \begin{cases} 0 & y < 0 \\ \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{3}\,dx = \frac{2\sqrt{y}}{3} & 0 \le y < 1 \\ \int_{-1}^{\sqrt{y}} \frac{1}{3}\,dx = \frac{\sqrt{y}+1}{3} & 1 \le y < 4 \\ \int_{-1}^{2} \frac{dx}{3} = 1 & y \ge 4 \end{cases}$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \begin{cases} \frac{1}{3} y^{-1/2} & 0 < y < 1 \\ \frac{1}{6} y^{-1/3} & 1 \le y < 4 \\ 0 & \text{ow} \end{cases}$$

Approach ② Find pdf of $Y=g(x)$ directly

## Functions of a random variable

### Continuous r.v.s: the monotone case

Recall the support $\mathcal{X} = \{x \in \mathbb{R} : f_X(x) > 0\}$. Consider $Y = g(X)$.

Additionally, suppose $g(\cdot)$ has a strictly *positive* derivative. Then,

- $g$ is strictly (monotone) increasing ($u < v$ in $\mathcal{X}$ iff $g(u) < g(v)$)
- $Y$ will have support $\mathcal{Y} = \{g(x) : x \in \mathcal{X}\} = \{g(x): f_X(x) > 0\}$
- given $y$, there's a unique $x$ where $g(x) = y$ or $g^{-1}(y) = x$

  Note: $\{x \in \mathcal{X} : g(x) \le y\} = \{x \in \mathcal{X} : g^{-1}(g(x)) \le g^{-1}(y)\}$
  $= \{x \in \mathcal{X} : x \le g^{-1}(y)\}$

- the cdf of $Y$ is  Pick $y \in \mathcal{Y}$

$$F_Y(y) = P(Y \le y) = P(g(X) \le y) = P(X \le g^{-1}(y)) = F_X[g^{-1}(y)] = \int_{-\infty}^{g^{-1}(y)} f_X(x)dx$$

and the pdf of $Y$ at $y \in \mathcal{Y}$ is

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{dF_X[g^{-1}(y)]}{dy} = f_X(g^{-1}(y))\underbrace{\frac{dg^{-1}(y)}{dy}}_{>0} > 0$$

- $g$ increasing $\Rightarrow g^{-1}(y)$ increasing $\Rightarrow \frac{dg^{-1}(y)}{dy} > 0$

If $g(\cdot)$ has a strictly *negative* derivative, then $g$ is strictly *decreasing*; $Y$ has cdf

$$F_Y(y) = P(Y \le y) = P(g(X) \le y) = P(X \ge g^{-1}(y)) = 1 - F_X[g^{-1}(y)] = \int_{g^{-1}(y)}^{\infty} f_X(x)dx$$

and the pdf of $Y$ at $y \in \mathcal{Y}$ is

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{d\{1 - F_X[g^{-1}(y)]\}}{dy} = -f_X(g^{-1}(y))\frac{dg^{-1}(y)}{dy} = \underbrace{f_X(g^{-1}(y))}_{>0}\Bigg[-\underbrace{\frac{dg^{-1}(y)}{dy}}_{<0}\Bigg] > 0$$

where $g^{-1}(y)$ is strictly decreasing so $\frac{dg^{-1}(y)}{dy} < 0$.

## Functions of a random variable

Continuous r.v.s: the monotone case

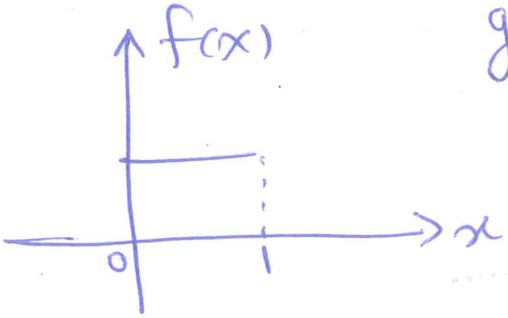
**Theorem 2.1.5**: *If $X$ has pdf $f_X(x)$ and $Y = g(X)$ where $g(\cdot)$ has either a strictly positive or a strictly negative derivative on $\mathcal{X} = \{x \in \mathbb{R} : f_X(x) > 0\}$, then the pdf of $Y$ has support $\mathcal{Y} = \{g(x) : x \in \mathcal{X}\}$ and is given by*

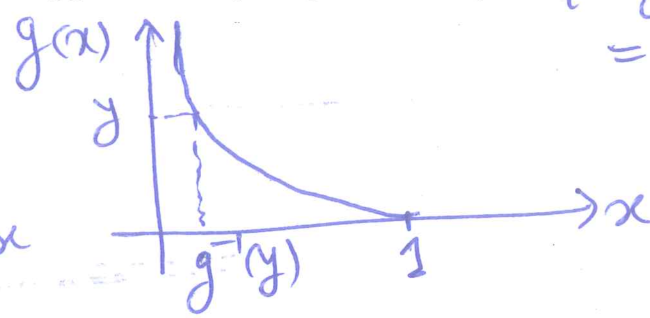
$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right| > 0 \quad \text{for } y \in \mathcal{Y}; \qquad f_Y(y) = 0 \quad \text{for } y \notin \mathcal{Y}$$

(This combines the two cases on last slide.)

Example: $X$ has pdf $f_X(x) = 1$ for $0 < x < 1$ and $g(x) = -\log x$

Let $Y = g(X)$ so that $Y$ has support $\mathcal{Y} = \{-\log x : x \in \mathcal{X}\} =$ $\{-\log x : 0 < x < 1\}$ $= (0, \infty)$




$\forall y \in \mathcal{Y}$, $y = -\log x \Rightarrow x = e^{-y} = g^{-1}(y)$

$\Rightarrow f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right|$

$= f_X(e^{-y}) \left| \frac{de^{-y}}{dy} \right| = 1 \left| -e^{-y} \right| = e^{-y}$ If $y > 0$

$f_Y(y) = \begin{cases} e^{-y} & \text{If } y > 0 \\ 0 & \text{ow} \end{cases}$