Sampling from the Normal Distribution

Review of random samples

- Recall that $X_1, \ldots, X_n$ iid with pdf $f_X(x)$ means

$$f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_X(x_i)$$

- A function $T = T(X_1, \ldots, X_n)$ of the random variables is a **statistic**

R.V.

$E(T)$ $\quad$ $Var(T)$ $\quad$ distribution of T

- Previous results for random samples with mean $\mathrm{E}X_1$ and variance $\mathrm{Var}(X_1)$:

The sample mean $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ has

$$\mathrm{E}(\bar{X}_n) = \mathrm{E}X_1 = \mu \qquad \mathrm{Var}(\bar{X}_n) = \frac{\mathrm{Var}(X_1)}{n} = \frac{\sigma^2}{n}$$

(i.i.d)

The sample variance $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$ has

$$\mathrm{E}(S^2) = \mathrm{Var}(X_1) = \sigma^2 \qquad \mathrm{Var}(S^2) = \frac{1}{n}\left(\mathrm{E}(X_1-\mu)^4 - \frac{n-3}{n-1}\sigma^4\right)$$

- If the distribution of the X_i's is normal (i.e., $X_1, \ldots, X_n$ iid $\sim N(\mu, \sigma^2)$), then we can derive the *exact* distribution of $\bar{X}_n, S^2$ *for any n*

Sampling from the Normal Distribution

Preliminary results/facts

1. If $Z \sim N(0, 1)$ then $Z^2 \sim \chi^2_1$.

2. If $X \sim N(\mu, \sigma^2)$ then $(X - \mu)^2/\sigma^2 \sim \chi^2_1$

3. If $Y_1, \ldots, Y_n$ are independent r.v.s where $Y_i \sim \chi^2_{\nu_i}$, then $Y = \sum_{i=1}^n Y_i \sim \chi^2_{\sum_{i=1}^n \nu_i}$

Proof: We've basically seen this already: use mgf technique for sums

$$M_Y(t) = \mathrm{E}e^{tY} = \mathrm{E}e^{t\sum_{i=1}^n Y_i} = \mathrm{E}\prod_{i=1}^n e^{tY_i} = \prod_{i=1}^n \mathrm{E}e^{tY_i} = \prod_{i=1}^n M_{Y_i}(t)$$

4. If $X_1, \ldots, X_n$ are iid $N(\mu, \sigma^2)$ then $\sum_{i=1}^n \left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi^2_n$

Sampling from the Normal Distribution

Preliminary results/facts (cont'd)

5. If $X_1, \ldots, X_n$ are independent and $X_i \sim N(\mu_i, \sigma_i^2)$ then
$Y = b + \sum_{i=1}^n a_i X_i \sim N\left(b + \sum_{i=1}^n a_i \mu_i, \sum_{i=1}^n a_i^2 \sigma_i^2\right)$

Y is a linear combination of independent normal $X_1, \ldots, X_n$

$\Rightarrow Y \sim N$

$$\mathbb{E}(Y) = \mathbb{E}\left(b + \sum_{i=1}^n a_i X_i\right) = b + \sum_{i=1}^n a_i \mathbb{E}X_i = b + \sum_{i=1}^n a_i \mu_i$$

$$Var(Y) = Var\left(b + \sum_{i=1}^n a_i X_i\right) \xrightarrow[X_i\text{'s are indp.}]{} \sum_{i=1}^n a_i^2 Var(X_i) = \sum_{i=1}^n a_i^2 \sigma_i^2$$

6. If $X_1, \ldots, X_n$ are iid $N(\mu, \sigma^2)$ then $\bar{X}_n \sim N(\mu, \sigma^2/n)$

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i \sim N(\mu, \sigma^2/n)$$

7. If $X_1, \ldots, X_n$ are iid $N(\mu, \sigma^2)$ then $n\frac{(\bar{X}_n - \mu)^2}{\sigma^2} \sim \chi_1^2$

$$\bar{X}_n \sim N(\mu, \sigma^2/n) \Rightarrow \frac{\bar{X} - \mu}{\sqrt{\sigma^2/n}} \sim N(0,1)$$

$$\left(\frac{\bar{X} - \mu}{\sqrt{\sigma^2/n}}\right)^2 = \boxed{\frac{\bar{X} - \mu}{\frac{\sigma^2}{n}}} \sim \chi_1^2$$

$$= n\frac{\bar{X} - \mu}{\sigma^2} \sim \chi_1^2$$

Sampling from the Normal Distribution

Joint distribution of $\bar{X}$ and S^2

Result: If $X_1, \dots, X_n$ are iid $N(\mu, \sigma^2)$ then

(a) $\bar{X}_n \sim N(\mu, \sigma^2/n)$

(b) $(n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$

(c) $\bar{X}_n$ and S^2 are independent.

gotta show

$\bar{X}_n$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$

If I show that $\bar{X}_n$ and $Y = \begin{pmatrix} X_1 - \bar{X}_n \\ X_2 - \bar{X}_n \\ \vdots \\ X_n - \bar{X}_n \end{pmatrix}$ are independent $\Rightarrow \bar{X}_n$ and S^2 are independent since S^2 is a function of $\begin{pmatrix} X_1 - \bar{X}_n \\ \vdots \\ X_n - \bar{X}_n \end{pmatrix}$.

Proof: First, recall since X_i's are iid & normal, then $\boldsymbol{X} = (X_1, \dots, X_n)'$ is MVN_n

1. Then, for $\boldsymbol{Y} = (Y_1, \dots, Y_n)'$ where $Y_i = X_i - \bar{X}_n$,

$$\boldsymbol{W} = \begin{pmatrix} \boldsymbol{Y} \\ \bar{X}_n \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \\ \bar{X}_n \end{pmatrix} = \begin{pmatrix} X_1 - \bar{X}_n \\ X_2 - \bar{X}_n \\ \vdots \\ X_n - \bar{X}_n \\ \bar{X}_n \end{pmatrix}$$

$\to Y$

2. Also,

$$\mathrm{Cov}(\boldsymbol{Y}, \bar{X}_n) = \begin{pmatrix} \mathrm{Cov}(Y_1, \bar{X}_n) \\ \mathrm{Cov}(Y_2, \bar{X}_n) \\ \vdots \\ \mathrm{Cov}(Y_n, \bar{X}_n) \end{pmatrix} = \begin{pmatrix} \mathrm{Cov}(X_1 - \bar{X}_n, \bar{X}_n) \\ \mathrm{Cov}(X_2 - \bar{X}_n, \bar{X}_n) \\ \vdots \\ \mathrm{Cov}(X_n - \bar{X}_n, \bar{X}_n) \end{pmatrix}$$

where

$$\begin{aligned} \mathrm{Cov}(X_i - \bar{X}_n, \bar{X}_n) &= \mathrm{Cov}(X_i, \bar{X}_n) - \mathrm{Var}(\bar{X}_n) \\ &= \frac{1}{n}\sum_{j=1}^{n} \mathrm{Cov}(X_j, X_i) - \frac{\sigma^2}{n} = \frac{\sigma^2}{n} - \frac{\sigma^2}{n} = 0 \end{aligned}$$

fixed

$$= \begin{cases} \mathrm{Cov}(X_j, X_i) = \sigma^2 & \text{If } j = i \\ 0 & \text{If } j \neq i \end{cases}$$

3. So $\underline{\underline{\bar{X}_n \text{ is independent of } \boldsymbol{Y}}}$ (why?)

that is, $\bar{X}_n$ is independent of $Y_i = X_i - \bar{X}_n$, $i = 1, \ldots, n$

$$\begin{pmatrix} Y \\ \overline{X}_n \end{pmatrix} \sim MVN + Cov(Y, \overline{X}_n) = 0 \Rightarrow Y \text{ and } \overline{X}_n \text{ are independent} \equiv \begin{pmatrix} X_1 - \overline{X}_n \\ \vdots \\ X_n - \overline{X}_n \end{pmatrix} \text{ and } \overline{X}_n \text{ are independent.}$$

$$S^2 = f(X_1 - \overline{X}_n, \ldots, X_n - \overline{X}_n)$$

4. Consequently, $\underline{\underline{\bar{X}_n}}$ is $\underline{\underline{\text{independent}}}$ of $S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2$ $\quad f(y_1, y_2, \ldots, y_n) = \frac{1}{n-1}\sum_{i=1}^n y_i^2$

To find the distribution of S^2: first write

$$\boxed{\frac{(n-1)S^2}{\sigma^2}} \overset{\text{def of } S^2}{=} \frac{1}{\sigma^2}\sum_{i=1}^n (X_i - \bar{X}_n)^2$$

$$= \frac{1}{\sigma^2}\sum_{i=1}^n \Big([X_i - \mu] - [\bar{X}_n - \mu]\Big)^2$$

$(a-b)^2 = a^2 + b^2 - 2ab$

$$= \frac{1}{\sigma^2}\left(\sum_{i=1}^n [X_i - \mu]^2 + n[\bar{X}_n - \mu]^2 - 2[\bar{X}_n - \mu]\sum_{i=1}^n [X_i - \mu]\right)$$

$= -2n(\bar{X}_n - \mu)^2$

$\sum_{i=1}^n (X_i - \mu) = \sum_{i=1}^n X_i - n\mu = n\bar{X} - n\mu = n(\bar{X} - \mu)$, $\quad = n(\bar{X}_n - \mu)$

$$= \boxed{\frac{1}{\sigma^2}}\left(\sum_{i=1}^n [X_i - \mu]^2 - n[\bar{X}_n - \mu]^2\right)$$

$$= \sum_{i=1}^n \left(\frac{X_i - \mu}{\sigma}\right)^2 - \left(\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}\right)^2$$

$$\frac{(n-1)S^2}{\sigma^2} = \sum_{i=1}^n \left(\frac{X_i - \mu}{\sigma}\right)^2 - \left(\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}\right)^2$$

that is,

$$\underbrace{\sum_{i=1}^n \left(\frac{X_i - \mu}{\sigma}\right)^2}_{U} = \underbrace{\left(\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}\right)^2}_{V} + \underbrace{\frac{(n-1)S^2}{\sigma^2}}_{W}$$

Note:

ⓐ $U \sim \chi^2_n \Rightarrow M_U(t) = (1-2t)^{-n/2}$

ⓑ $V \sim \chi^2_1 \Rightarrow M_V(t) = (1-2t)^{-1/2}$

$M_W(t) = \dfrac{(1-2t)^{-n/2}}{(1-2t)^{-1/2}} = (1-2t)^{-n/2+1/2} = (1-2t)^{\frac{-(n-1)}{2}}$ $\quad W \sim \chi^2_{n-1}$

ⓒ V and W are independent $\Rightarrow M_U(t) = M_V(t)M_W(t)$

$M_U(t) = \mathbb{E}[e^{tU}] = \mathbb{E}[e^{t(V+W)}] = \mathbb{E}[e^{tV}e^{tW}] \overset{V \text{ and } W \text{ are ind.}}{=} \underbrace{\mathbb{E}[e^{tV}]\mathbb{E}[e^{tW}]}_{= M_V(t)M_W(t)}$

Sampling from the Normal Distribution

Joint distribution of $\bar{X}$ and S^2: more comments

- $\bar{X}_n$ and S^2 are independent only for the normal distribution

- Many other proofs for the joint distribution of $\bar{X}$ and S^2 exist

Skip

- An alternative "proof" of independence of $\bar{X}$ and S^2

First, let $Z_i = (X_i - \mu)/\sigma$ for $i = 1, \ldots, n$ (i.e. Z_i's iid $N(0,1)$) and note

$$\sqrt{n}\bar{Z}_n = \sqrt{n}\frac{(\bar{X}_n - \mu)}{\sigma}, \qquad \sum_{i=1}^{n} Z_i^2 - n(\bar{Z}_n)^2 = \sum_{i=1}^{n}(Z_i - \bar{Z}_n)^2 = \frac{(n-1)S^2}{\sigma^2}$$

Then, introduce a special transformation as follows:

$$\begin{aligned}
Y_1 &= \frac{1}{\sqrt{2}}Z_1 - \frac{1}{\sqrt{2}}Z_2 \\
Y_2 &= \frac{1}{\sqrt{6}}Z_1 + \frac{1}{\sqrt{6}}Z_2 - \frac{2}{\sqrt{6}}Z_3 \\
&\vdots \\
Y_k &= \frac{1}{\sqrt{k(k+1)}}Z_1 + \frac{1}{\sqrt{k(k+1)}}Z_2 + \cdots + \frac{1}{\sqrt{k(k+1)}}Z_k - \frac{k}{\sqrt{k(k+1)}}Z_{k+1} \\
&\vdots \\
Y_{n-1} &= \frac{1}{\sqrt{(n-1)n}}Z_1 + \frac{1}{\sqrt{(n-1)n}}Z_2 + \cdots + \frac{1}{\sqrt{(n-1)n}}Z_{n-1} - \frac{n-1}{\sqrt{(n-1)n}}Z_n \\
Y_n &= \frac{1}{\sqrt{n}}Z_1 + \frac{1}{\sqrt{n}}Z_2 + \cdots + \frac{1}{\sqrt{n}}Z_n
\end{aligned}$$

Sampling from the Normal Distribution

Joint distribution of $\bar{X}$ and S^2 (alternative proof, cont'd)

In matrix terms

$$\mathbf{Y} = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} = \mathbf{AZ} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & \cdots & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & 0 \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \end{pmatrix}, \qquad \mathbf{Z} = \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix}$$

Important properties of matrix $\boldsymbol{A}$:

- rows of $\mathbf{A}$ are orthonormal

$$\sum_{j=1}^{n} a_{ij}^2 = 1 \qquad\qquad \sum_{j=1}^{n} a_{ij}a_{kj} = 0, \quad k \neq i$$

- This implies $\mathbf{AA}' = I_{n\times n}$

- $1 = \det(I_{n\times n}) = \det(\mathbf{AA}') = \det(\mathbf{A})\det(\mathbf{A}') = [\det(\mathbf{A})]^2 = [\det(\mathbf{A}')]^2$

Sampling from the Normal Distribution

Joint distribution of $\bar{X}$ and S^2 (alternative proof, cont'd)

Carry out the multivariate transformation from $\mathbf{Z}$ to $\mathbf{Y}$

- Inverse transformation is $\mathbf{z} = \mathbf{A}'\mathbf{y}$

$$f(y_1, \ldots, y_n) = f_{Z_1,\ldots,Z_n}(z_1, \ldots, z_n)|J|\Big|_{\mathbf{z}=\mathbf{A}'\mathbf{y}}$$

- Note

  1. $J = \det(\mathbf{A}')$ so $|J| = |\det(\mathbf{A}')| = 1$

  2. $f_{Z_1,\ldots,Z_n}(z_1, \ldots, z_n) = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum_{i=1}^n z_i^2}$

  3. Also,

$$\sum_{i=1}^n z_i^2 = \mathbf{z}'\mathbf{z} = \mathbf{y}'\mathbf{A}'\mathbf{A}\mathbf{y} = \mathbf{y}'\mathbf{y} = \sum_{i=1}^n y_i^2$$

- Hence,

$$f(y_1, \ldots, y_n) = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum_{i=1}^n y_i^2} = \prod_{i=1}^n \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y_i^2}$$

  Now conclude:

  1. So $Y_1, \ldots, Y_n$ are iid $N(0, 1)$

  2. $Y_n = \sqrt{n}\bar{Z}_n = \sqrt{n}\frac{(\bar{X}_n - \mu)}{\sigma} \sim N(0, 1)$ so $\bar{X}_n \sim N(\mu, \sigma^2/n)$

  3. $\sum_{i=1}^{n-1} Y_i^2 \sim \chi^2_{n-1}$ and

$$\sum_{i=1}^{n-1} Y_i^2 = \sum_{i=1}^n Y_i^2 - Y_n^2 = \sum_{i=1}^n Z_i^2 - n(\bar{Z}_n)^2 = \sum_{i=1}^n (Z_i - \bar{Z}_n)^2 = \frac{(n-1)S^2}{\sigma^2}$$

  4. Y_n independent of $Y_1, \ldots, Y_{n-1}$ implies $\underline{\bar{X}_n \text{ independent of } S^2}$ ∎

Sampling from the Normal Distribution

Derived distributions: Student's t

Let $Z \sim \underline{N(0,1)}$ and $V \sim \chi^2_\nu$ be $\boxed{\text{independent}}$ r.v.s, then the r.v.

$$T = \frac{Z}{\sqrt{V/\nu}}$$

$Z \to N(0,1)$

are indep.

has a **Student's t distribution** with ν degrees of freedom, denoted $T \sim t_\nu$.

T has a pdf which can be derived as follows.

- Transformation: $T = Z/\sqrt{V/\nu}$ and $W = V$
- Inverse transformation: $Z = T\sqrt{W/\nu}$ and $V = W$

$$\longrightarrow \begin{pmatrix} \frac{\partial z}{\partial t} & \frac{\partial z}{\partial w} \\ \frac{\partial v}{\partial t} & \frac{\partial v}{\partial w} \end{pmatrix} = \begin{pmatrix} (w/\nu)^{1/2} & tw^{-1/2}2^{-1}\nu^{-1/2} \\ 0 & 1 \end{pmatrix} \quad \Rightarrow \quad |J| = (w/\nu)^{1/2}$$

Hence, for $-\infty < t < \infty$, $w > 0$,

$$\begin{aligned} f_{T,W}(t,w) &= f_Z(z) f_V(v) |J| \Big|_{z=t(w/\nu)^{1/2}, v=w} \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2(w/\nu)} \frac{1}{\Gamma(\nu/2)2^{\nu/2}} w^{\nu/2-1} e^{-w/2} (w/\nu)^{1/2} \end{aligned}$$

- Integrate out W: let $\alpha = (1+\nu)/2$, $\beta = 2/(1+t^2/\nu)$

$$\begin{aligned} f_T(t) = \int_0^\infty f_{T,W}(t,w)dw &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\nu}} \frac{1}{\Gamma(\nu/2)2^{\nu/2}} \int_0^\infty w^{(1+\nu)/2-1} e^{-w(1+t^2/\nu)/2} dw \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\nu}} \frac{1}{\Gamma(\nu/2)2^{\nu/2}} \cdot \Gamma(\alpha)\beta^\alpha \int_0^\infty \frac{w^{\alpha-1} e^{-w/\beta}}{\Gamma(\alpha)\beta^\alpha} dw \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\nu}} \frac{1}{\Gamma(\nu/2)2^{\nu/2}} \cdot \Gamma((1+\nu)/2) \left(\frac{2}{1+t^2/\nu}\right)^{(1+\nu)/2} \\ &= \boxed{\frac{\Gamma((1+\nu)/2)}{\Gamma(\nu/2)} \frac{1}{\sqrt{\pi\nu}} \left(\frac{1}{1+t^2/\nu}\right)^{(1+\nu)/2}} \end{aligned}$$

density function of t_ν

$f_T(t) = f_T(-t)$

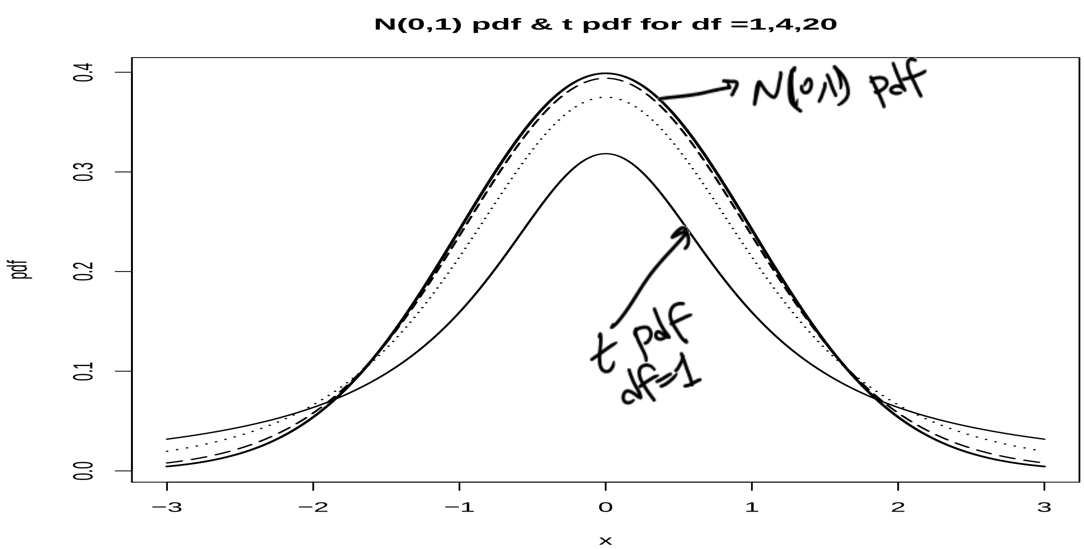
Sampling from the Normal Distribution

Derived distributions: Student's t

Properties

1. t density is symmetric around zero
2. $\mathrm{E}T = 0$ for $\nu > 1$
3. $\mathrm{Var}(T) = \mathrm{E}T^2 = \nu \mathrm{E}(Z^2 V^{-1}) = \nu \mathrm{E}Z^2 \cdot \mathrm{E}(V^{-1}) = \frac{\nu}{\nu - 2}$ for $\nu > 2$

   $\mathrm{Var}(T) = \infty$ for $\nu \le 2$
4. $\nu = 1$ gives Cauchy distribution $\boxed{f(t) = \frac{1}{\pi}(1 + t^2)^{-1}}$
5. as $\nu \to \infty$, t-distribution converges to $N(0, 1)$



Motivating application: $X_1, \ldots, X_n$ iid $N(\mu, \sigma^2)$

- $Z = \sqrt{n}(\bar{X}_n - \mu)/\sigma \sim N(0, 1)$ and $V = (n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$ are independent
- it then follows that

$$t = \frac{\overset{N(0,1)}{Z}}{\sqrt{\underbrace{V}_{\chi^2_{n-1}}/(n-1)}} = \frac{\sqrt{n}(\bar{X}_n - \mu)/\sigma}{S/\sigma} = \frac{\sqrt{n}(\bar{X}_n - \mu)}{S} \sim \boxed{t_{n-1}}$$