- Define the Jacobian

$$J = \det \underbrace{\begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}} = \det \underbrace{\begin{pmatrix} \frac{\partial u_1^{-1}(y_1,\ldots,y_n)}{\partial y_1} & \frac{\partial u_1^{-1}(y_1,\ldots,y_n)}{\partial y_2} & \cdots & \frac{\partial u_1^{-1}(y_1,\ldots,y_n)}{\partial y_n} \\ \frac{\partial u_2^{-1}(y_1,\ldots,y_n)}{\partial y_1} & \frac{\partial u_2^{-1}(y_1,\ldots,y_n)}{\partial y_2} & \cdots & \frac{\partial u_2^{-1}(y_1,\ldots,y_n)}{\partial y_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial u_n^{-1}(y_1,\ldots,y_n)}{\partial y_1} & \frac{\partial u_n^{-1}(y_1,\ldots,y_n)}{\partial y_2} & \cdots & \frac{\partial u_n^{-1}(y_1,\ldots,y_n)}{\partial y_n} \end{pmatrix}}$$

- If J is continuous and $J \neq 0$ over $\mathcal{B}$ (except possibly on a set with probability zero),

$$f_{Y_1,\ldots,Y_n}(y_1,\ldots,y_n) = f_{X_1,\ldots,X_n}(\boldsymbol{u}^{-1}(y_1,\ldots,y_n))|J|, \qquad (y_1,\ldots,y_n) \in \mathcal{B}$$

- Often only interested in one transformation $Y_1 = u_1(X_1,\ldots,X_n)$

  Then choose convenient definitions to fill out the transformation

  e.g. $Y_2 = X_2, \ldots, Y_n = X_n$

- If transformation is <u>not</u> <u>one-to-one</u>, then we partition $\mathcal{A}$ (the support of $(X_1,\ldots,X_n)$) into sets $\mathcal{A}_i$ where a transformation $\boldsymbol{Y} = \boldsymbol{u}_j(\boldsymbol{X})$ is one-to-one and then add pieces

**

$$f_{\boldsymbol{Y}}(\boldsymbol{y}) = \sum_{i=1}^{k} f_{\boldsymbol{X}}(\boldsymbol{u}_i^{-1}(\boldsymbol{y}))|J_i|$$

Multivariate transformations

Multivariate continuous case: example 1

- $X_1 \sim$Gamma(α_1, β) and $X_2 \sim$Gamma(α_2, β) are independent

$$f_{X_i}(x) = \frac{1}{\Gamma(\alpha_i)\beta^{\alpha_i}} x^{\alpha_i - 1} e^{-x/\beta}, \quad x > 0$$

- Transformation: $Y_1 = X_1 + X_2$ and $Y_2 = X_1/(X_1 + X_2)$

Y_1 and Y_2 are functions of X_1 and X_2.

- Inverse transformation:

- $\mathcal{A} = (0, \infty) \times (0, \infty)$ while $\mathcal{B} =$

- One-to-one transformation with

$$J =$$

Multivariate transformations

Multivariate continuous case: example 1 (cont'd)

- Joint pdf of Y_1, Y_2

$$
\begin{aligned}
f_{Y_1,Y_2}(y_1, y_2) &= f_{X_1,X_2}(y_1 y_2, y_1(1-y_2))|J| \\
&= f_{X_1}(y_1 y_2) \times f_{X_2}(y_1(1-y_2)) \times |J| \\
&= \frac{(y_1 y_2)^{\alpha_1 - 1} e^{-(y_1 y_2)/\beta}}{\Gamma(\alpha_1)\beta^{\alpha_1}} \times \frac{[y_1(1-y_2)]^{\alpha_2 - 1} e^{-[y_1(1-y_2)]/\beta}}{\Gamma(\alpha_2)\beta^{\alpha_2}} \times y_1 \\
&= \frac{y_1^{\alpha_1+\alpha_2-1} e^{-y_1/\beta}}{\beta^{\alpha_1+\alpha_2}} \times y_2^{\alpha_1-1}(1-y_2)^{\alpha_2-1} \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}
\end{aligned}
$$

$Y_1 \sim \text{Gamma}(\,,\,)$ $\quad$ $Y_2 \sim \text{Beta}(\,,\,)$

$\Rightarrow$ Y_1 and Y_2 are independent.

(a) X_1 and X_2 are independent

(b), Y_1 and Y_2 are function of X_1 and X_2

$$
\begin{cases} Y_1 = h(X_1, X_2) = X_1 + X_2 \\ Y_2 = W(X_1, X_2) = \dfrac{X_1}{X_1 + X_2} \end{cases}
$$

(c) Y_1 and Y_2 are independent.

Multivariate transformations

Multivariate continuous case: example 2

- $X_1 \sim N(0,1)$ and $X_2 \sim N(0,1)$ are independent

- Transformation: $Y_1 = X_1 + X_2$ and $\underbrace{Y_2 = X_2}_{\text{add}}$

- Inverse transformation:

- $\mathcal{A} = (-\infty, \infty) \times (-\infty, \infty)$ while $\mathcal{B} =$

- One-to-one transformation with

$$J =$$

Multivariate transformations

Multivariate continuous case: example 2 (cont'd)

- Joint pdf of Y_1, Y_2

$$
\begin{aligned}
f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(y_1-y_2, y_2)|J| &= f_{X_1}(y_1-y_2) \times f_{X_2}(y_2) \times 1 \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y_1-y_2)^2} \times \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y_2^2} \\
&= \frac{1}{2\pi} e^{-\left(\frac{y_1^2}{2} - y_1y_2 + y_2^2\right)} \\
&= \frac{1}{2\pi} e^{-\frac{y_1^2}{2}} e^{-y_2^2 + 2(y_1/2)y_2}
\end{aligned}
$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2\pi}} e^{-y_1^2/2} e^{-\left(y_2^2 - \frac{2y_1y_2}{2} + \frac{y_1^2}{4} - \frac{y_1^2}{4}\right)}}_{*}$$

(a) X_1 and X_2 are independent

(b) Y_1 and Y_2 are functions of X_1 and X_2

(c) Y_1 and Y_2 are NOT independent

Note: If X_1 and X_2 are independent then $h(X_1)$ and $W(X_2)$ are independent

If X_1 and X_2 are independent then $h(X_1, X_2)$ and $W(X_1, X_2)$ are not independent always.

- Marginal distribution of Y_1

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1,y_2)\, dy_2 = \boxed{\frac{1}{\sqrt{2\pi}} e^{-y_1^2/4}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\left(y_2 - \frac{y_1}{2}\right)^2} dy_2$$

$$\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2\pi}} e^{-y_1^2/4} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(\frac{1}{2})}} e^{\frac{-(y_2 - y_1/2)^2}{2(\frac{1}{2})}}}_{=1}$$

$\Rightarrow$ the density of normal $\left(\frac{y_1}{2}, \frac{1}{2}\right)$

Variance

Recall: $X \sim N(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

$\mu = y_1/2$, $\sigma^2 = 1/2$

$$f_X(x) = \frac{1}{\sqrt{2\pi(\frac{1}{2})}} e^{\frac{-(y_2 - y_1/2)^2}{2(1/2)}} = \frac{1}{\sqrt{2}} \left[\frac{1}{\sqrt{2\pi(1/2)}} e^{\frac{-(y_2 - y_1/2)^2}{2(1/2)}}\right]$$

Multivariate transformations

Multivariate continuous case: example 3 - convolutions

- Previous example (sum of two continuous r.v.s) is often of interest

- $(X_1, X_2) \sim f_{X_1,X_2}(x_1, x_2)$

- Transformation: $S = X_1 + X_2$ and $T = X_2$

- Inverse transformation: $X_1 = S - T$ and $X_2 = T$

- one-to-one transformation with

$$J = \det \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = 1$$

- Derive pdf of S

$$f_{S,T}(s,t) = f_{X_1,X_2}(\underbrace{s-t}_{x_1}, \underbrace{t}_{x_2}) \qquad f_S(s) = \int_{-\infty}^{\infty} f_{X_1,X_2}(s-t, t)dt$$

- If X_1, X_2 are independent then

Convolution

$f_{X_1} \circledast f_{X_2}$

$$f_S(s) = \int_{-\infty}^{\infty} f_{X_1}(s-t) f_{X_2}(t) dt$$

which is called the convolution formula

Multivariate transformations

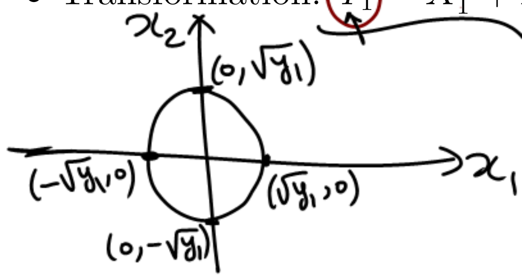
Multivariate continuous case: example 4 (not one-to-one)

Recall if the transformation is not one-to-one, break the support of $\boldsymbol{X}$ into subsets where the transformation is one-to-one (i.e., apply transformation on each piece and add the resulting density pieces)

- $X_1 \sim N(0,1)$ and $X_2 \sim N(0,1)$ are independent

$Y_1 \geq 0 \equiv Y_1 \in (0,\infty)$ $\quad Y_2 \in (-1,1)$

- Transformation: $\underbrace{Y_1 = X_1^2 + X_2^2}$ and $Y_2 = \underbrace{X_1/\sqrt{X_1^2+X_2^2}} = Y_2 = \dfrac{X_1}{\sqrt{Y_1}}$



- $\mathcal{A} = \underbrace{(-\infty,\infty) \times (-\infty,\infty)}$ while $\mathcal{B} = (0,\infty) \times (-1,1)$

$X_1 \qquad X_2$

- Inverse transformation:

$Y_1 = X_1^2 + X_2^2$

$Y_2 = \dfrac{X_1}{\sqrt{X_1^2+X_2^2}}$ $\longrightarrow$ $\underbrace{X_1 = Y_2\sqrt{Y_1}}_{X_1^2 = Y_2^2 Y_1}$

$X_2^2 = Y_1 - X_1^2$

$X_2 = \pm\sqrt{Y_1 - X_1^2}$

$= \pm\sqrt{Y_1 - Y_2^2 Y_1}$

$= \pm\sqrt{Y_1(1-Y_2^2)}$

- Piece I: $\underbrace{X_1 = \sqrt{Y_1}Y_2}$, $\underbrace{X_2 = \sqrt{Y_1(1-Y_2^2)}}$

  Piece II: $X_1 = \sqrt{Y_1}Y_2$, $X_2 = -\sqrt{Y_1(1-Y_2^2)}$

- Each transformation piece with

$$J_1 = \det\begin{pmatrix} \frac{y_2}{2\sqrt{y_1}} & \sqrt{y_1} \\ \frac{\sqrt{1-y_2^2}}{2\sqrt{y_1}} & \frac{\sqrt{y_1}}{2\sqrt{(1-y_2^2)}}(-2y_2) \end{pmatrix} = -\frac{y_2^2}{2\sqrt{1-y_2^2}} - \frac{\sqrt{1-y_2^2}}{2} = \frac{-1}{2\sqrt{1-y_2^2}}$$

$$J_2 = \det\begin{pmatrix} \frac{y_2}{2\sqrt{y_1}} & \sqrt{y_1} \\ -\frac{\sqrt{1-y_2^2}}{2\sqrt{y_1}} & -\frac{\sqrt{y_1}}{2\sqrt{(1-y_2^2)}}(-2y_2) \end{pmatrix} = \frac{\sqrt{1-y_2^2}}{2} + \frac{y_2^2}{2\sqrt{1-y_2^2}} = \frac{1}{2\sqrt{1-y_2^2}}$$

Multivariate transformations

Multivariate continuous case: example 4 (cont'd)

- Joint pdf of Y_1, Y_2

$$f_{Y_1,Y_2}(y_1, y_2)$$

$$** \quad = \underbrace{f_{X_1,X_2}\left(\overset{x_1}{\sqrt{y_1}y_2}, \overset{x_2>0}{\sqrt{y_1(1-y_2^2)}}\right)|J_1|}_{} + \underbrace{f_{X_1,X_2}\left(\overset{x_1}{\sqrt{y_1}y_2}, \overset{x_2<0}{-\sqrt{y_1(1-y_2^2)}}\right)|J_2|}_{}$$

$$= 2\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y_1y_2^2} \times \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y_1(1-y_2^2)} \times \frac{1}{2\sqrt{1-y_2^2}}$$

$$= \underbrace{\frac{1}{2\pi}\frac{1}{\sqrt{1-y_2^2}}e^{-\frac{1}{2}y_1}}_{} \quad , \; y_1 > 0, \; y_2 \in (-1,1)$$

- Marginal distribution of Y_1

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1, y_2)dy_2 = \frac{1}{2\pi}e^{-\frac{1}{2}y_1}\int_{-1}^{1}\frac{1}{\sqrt{1-y_2^2}}dy_2$$

$$= \frac{1}{2\pi}e^{-\frac{1}{2}y_1} \times \arcsin y_2\Big|_{-1}^{1}$$

$$= \frac{1}{2\pi}e^{-\frac{1}{2}y_1}\left(\underbrace{\pi/2 - (-\pi/2)}_{\pi}\right)$$

$$\Rightarrow Y_1 \sim \text{Exp}(2) \qquad = \frac{1}{2}e^{-\frac{1}{2}y_1} \quad \blacksquare$$