STAT 5430

Lec 22, F, Mar 14

Basu's theorem & Intro to testing → Homework 5 posted, due M, Mar 24 (after break)

Hypothesis Testing I

Finding best test for simple H_0 vs simple H_1!

Most Powerful Tests (Simple vs Simple Hypotheses)

Let $f(\underset{\sim}{x}|\theta)$, $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$, $\theta \in \Theta$, be the joint pdf/pmf of $\underset{\sim}{X} = (X_1, \ldots, X_n)$. We want to test the hypothesis

$$H_0 : \theta = \theta_0 \quad \text{vs} \quad H_1 : \theta = \theta_1 \qquad \text{where } \theta_0, \theta_1 \in \Theta,\ \theta_0 \neq \theta_1.$$

given parameters

"best test"

Definition: A test function $\varphi(\underset{\sim}{x})$ is called a **most powerful** (MP) test of size α if

1. $\mathrm{E}_{\theta_0}\varphi(\underset{\sim}{X}) = \alpha$. ← prob of Type I error at θ_0 is set to $\alpha \in [0,1]$

   ← $1 - E_{\theta_1}\varphi(\underset{\sim}{X}) \leq 1 - E_{\theta_1}\bar{\varphi}(\underset{\sim}{X})$
2. $\mathrm{E}_{\theta_1}\varphi(\underset{\sim}{X}) \geq \mathrm{E}_{\theta_1}\bar{\varphi}(\underset{\sim}{X})$ holds for any other test rule $\bar{\varphi}(\underset{\sim}{x})$ with $\mathrm{E}_{\theta_0}\bar{\varphi}(\underset{\sim}{X}) \leq \alpha$.

   $\Rightarrow$ prob of Type II error at θ_1 or $1 - E_{\theta_1}\varphi(\underset{\sim}{X})$ $\leq$ Prob of Type II error at θ_1 or $1 - E_{\theta_1}\bar{\varphi}(\underset{\sim}{X})$

A MP test does exist at least for simple H_0 vs simple H_1, as described below.

Theorem: (Neyman-Pearson Lemma) Let $f(\underset{\sim}{x}|\theta)$, $\theta \in \Theta$, be the joint pdf/pmf of $X_1, \ldots, X_n$. Then for testing $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$, a MP test of size α exists for all $\alpha \in [0, 1]$ and is given by

$$\varphi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \ f(\underset{\sim}{x}|\theta_1) > kf(\underset{\sim}{x}|\theta_0) \\ \gamma & \text{if } \ f(\underset{\sim}{x}|\theta_1) = kf(\underset{\sim}{x}|\theta_0) \\ 0 & \text{if } \ f(\underset{\sim}{x}|\theta_1) < kf(\underset{\sim}{x}|\theta_0) \end{cases} \tag{4}$$

$L(\theta) = f(\underset{\sim}{x}|\theta)$ likelihood function

where $\gamma \in [0, 1]$ and $0 \leq k \leq \infty$ are constants satisfying

$$\mathrm{E}_{\theta_0}\varphi(\underset{\sim}{X}) = \alpha. \tag{5}$$

← pick k & γ to get size α at θ_0

$\frac{L(\theta_1)}{L(\theta_0)}$ $>$ k reject H_0 w prob 1

$=$ k reject H_0 w. prob γ

$<$ k don't reject H_0

form makes MP test!

Hypothesis Testing I

Most Powerful Tests, cont'd

Remarks on Neyman-Pearson Lemma:

- Let $L(\theta) \equiv f(\underset{\sim}{x}|\theta)$, the likelihood function at θ. Then the MP test in (4) rejects H_0 for all $\underset{\sim}{x}$ such that the likelihood $L(\theta_1)$ of θ_1 is "large" compared to the likelihood of $L(\theta_0)$ of θ_0; the "largeness" factor k is determined by (5).

- In general, the choice of (γ, k) satisfying (5) is <u>not</u> unique.

- If the distribution of $f(\underset{\sim}{x}|\theta_1)/f(\underset{\sim}{x}|\theta_0)$ under θ_0 is continuous, then we may set $\gamma = 0$.

i.e. We usually only are concerned with γ for discrete data!
(for continuous data, take $\gamma = 0$)

Hypothesis Testing I

$EX_1 = 3\theta$

Illustration of Most Powerful Test (Continuous Case)

known

Example: Let $X_1, \ldots, X_n$ be a random sample from Gamma$(\alpha = 3, \theta)$, $\theta > 0$. Find a MP test of size α for $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$ (where $0 < \theta_0 < \theta_1$).

Solution: $f(\underset{\sim}{x}|\theta)$ = joint density of $\underset{\sim}{x}$

$$= \prod_{i=1}^{n} \left(\frac{x_i^2 e^{-x_i/\theta}}{2\theta^3} \right) = \left(\prod_{i=1}^{n} x_i^2\right)(2\theta^3)^{-n} e^{-\sum_{i=1}^{n} x_i/\theta}$$

By the theorem, the MP test of size is given by

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & f(\underset{\sim}{x}|\theta_1) > k f(\underset{\sim}{x}|\theta_0) \\ \gamma & = \\ 0 & < \end{cases}$$

where $E_{\theta_0}\phi(\underset{\sim}{x}) = \alpha$.

Note: $f(\underset{\sim}{x}|\theta_1) \gtreqless k f(\underset{\sim}{x}|\theta_0)$

$$\Leftrightarrow (2\theta_1^3)^{-n} \cancel{\left(\prod_{i=1}^{n} x_i^2\right)} e^{-\sum x_i/\theta_1} \gtreqless k (2\theta_0^3)^{-n} \cancel{\left(\prod_{i=1}^{n} x_i^2\right)} e^{-\sum x_i/\theta_0}$$

$$\Leftrightarrow -n\log(2\theta_1^3) - \sum_{i=1}^{n} x_i/\theta_1 \gtreqless \log k + -n\log(2\theta_0^3) - \sum_{i=1}^{n} x_i/\theta_0$$

$$\Leftrightarrow \left(\sum_{i=1}^{n} x_i\right)\underbrace{\left(\frac{1}{\theta_0} - \frac{1}{\theta_1}\right)}_{\substack{>0 \\ \text{since } \theta_0 < \theta_1}} \gtreqless \underbrace{\log k - n\log(2\theta_0^3) + n\log(2\theta_1^3)}_{\equiv k_1}$$

$$\Leftrightarrow \left(\sum_{i=1}^{n} x_i\right) \gtreqless \left(\frac{1}{\theta_0} - \frac{1}{\theta_1}\right)^{-1} k_1 \equiv k_2$$

The MP test (of size α) is

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \sum_{i=1}^n X_i > k_2 \\ r & \text{if } \sum_{i=1}^n X_i = k_2 \\ 0 & \text{if } \sum_{i=1}^n X_i < k_2 \end{cases}$$

where $0 \le r \le 1$ & k_2 are chosen so that $E_{\theta_0}\phi(\underset{\sim}{X}) = \alpha$.

Note:

$$\alpha = E_{\theta_0}\phi(\underset{\sim}{X}) = 1 \cdot P_{\theta_0}\left(\sum_{i=1}^n X_i > k_2\right) + \overset{\swarrow \text{ pick } r=0}{r}\underbrace{P_{\theta_0}\left(\sum_{i=1}^n X_i = k_2\right)}_{=0 \text{ since } \underset{\sim}{X} \text{ are continuous}}$$

$\sum_{i=1}^n X_i \sim \text{gamma}(3n, \theta_0)$

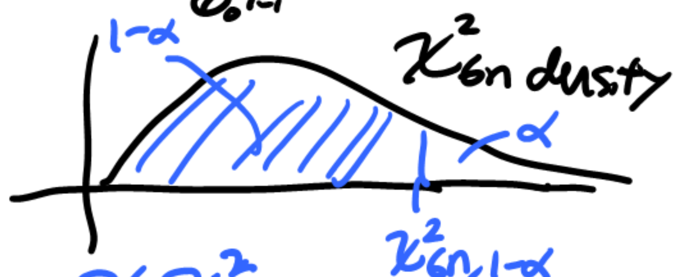
$$= P_{\theta_0}\left(\sum_{i=1}^n X_i > k_2\right)$$

$$= P_{\theta_0}\left(\frac{2\sum_{i=1}^n X_i}{\theta_0} > \frac{2k_2}{\theta_0}\right)$$

$$= P\Big(\chi^2_{6n} > \underbrace{\frac{2k_2}{\theta_0}}_{\chi^2_{6n,1-\alpha}}\Big)$$

Note: $\frac{2X_1}{\theta_0} \sim \text{gamma}(3,2) \sim \chi^2_6$

so $\frac{2}{\theta_0}\sum_{i=1}^n X_i \sim \chi^2_{6n}$



$P(\chi^2_{6n} \le \chi^2_{6n,1-\alpha}) = 1-\alpha$

So, $k_2 = \frac{\theta_0}{2}\chi^2_{6n,1-\alpha}$

So the MP test (size α) of $H_0: \theta = \theta_0$ vs $H_1: \theta = \theta_1$ $(\theta_1 > \theta_0)$

is $\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \sum_{i=1}^n X_i > \frac{\theta_0}{2}\chi^2_{6n,1-\alpha} \\ 0 & \text{o.w.} \end{cases}$