STAT 5430

Lec 25, F, Mar 28

MP & UMP testing → Honework 6 posted, due, M, Mar 31

Homework 5 solutions to be posted

- Exam 2 is coming up (3 weeks away) on W, April 16, 6:15-8:15 PM, 3rd floor seminar room
- No class on that W.
- I'll post: ✓ study guide (sufficiency/completeness/tests)
  - practice exams
  - ✓ bring new 1 page (front/back) formula sheet on exam 2 material (I'll post one to use if you'd like)
  - can bring calculator & previous formula sheet for exam 1
  - I'll provide table of distributions / STAT 542 facts on test as before

Recap: so we have MP tests for simple H_0 & simple H_1

Hypothesis Testing I

Uniformly Most Powerful (UMP) Tests

Definition: Let $f(\underset{\sim}{x}|\theta)$, $\theta \in \Theta \subset \mathbb{R}^p$, be the joint pdf/pmf of $\underset{\sim}{X} = (X_1, \ldots, X_n)$ and let Θ_0 be a nonempty proper subset of Θ. Then, a test rule $\varphi(\underset{\sim}{x})$ for testing $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \notin \Theta_0$ is called a **uniformly most powerful** (UMP) test of size α if

"best test" in this case when either H_0 or H_1 is composite

1. $\max_{\theta \in \Theta_0} \mathrm{E}_\theta \varphi(\underset{\sim}{X}) = \alpha$ ← right size
2. it holds that $\mathrm{E}_\theta \varphi(\underset{\sim}{X}) \geq \mathrm{E}_\theta \tilde{\varphi}(\underset{\sim}{X})$ for all $\theta \notin \Theta_0$, given any other test rule $\tilde{\varphi}(\underset{\sim}{x})$ with $\max_{\theta \in \Theta_0} \mathrm{E}_\theta \tilde{\varphi}(\underset{\sim}{X}) \leq \alpha$.

↑ any other θ under H_1

(UMP test DON'T always exist... but often DO exist if H_1 is "one-sided", e.g. $H_1: \mu > 1$ or $H_1: \mu \leq 2$)

Two General Methods of Finding UMP Tests

1. **Method I:** Based on Neyman-Pearson Lemma ← start here
2. **Method II:** Using Monotone Likelihood Ratio (MLR) property ← today

Method I (Neyman-Pearson Lemma-based)

To find a UMP size α test for $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \notin \Theta_0$,

← carefully pick/fix one parameter from Θ_0

1. first fix one $\theta_0 \in \Theta_0$ (*suitably*) and also $\theta_1 \notin \Theta_0$
2. then use the Neyman-Pearson lemma to find a MP size α test $\varphi(\underset{\sim}{x})$ for $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$, where
   (a) $\varphi(\underset{\sim}{x})$ does not depend on $\theta_1 \notin \Theta_0$ and ← would have gotten same test $\varphi(\underset{\sim}{x})$ for any chosen $\theta_1 \notin \Theta_0$ placed in $H_1: \theta = \theta_1$
   (b) $\max_{\theta \in \Theta_0} \mathrm{E}_\theta \varphi(\underset{\sim}{X}) = \alpha$ ← right size

Then $\varphi(\underset{\sim}{x})$ is a UMP size α test for $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \notin \Theta_0$.

(Homework 6 has the "proof" of this.)

Note: $\phi(\underset{\sim}{x})$ has "right size" $\max_{\theta \in \Theta_0} E_\theta \phi(\underset{\sim}{X}) = \alpha$

but to be UMP/best test here would need:

For any other test $\tilde{\phi}(\underset{\sim}{X})$ with $\sup_{\theta \in \Theta_0} E_\theta \tilde{\phi}(X) \leq \alpha$

and any $\theta_1 \notin \Theta_0$, need $E_{\theta_1} \phi(\underset{\sim}{X}) \geq E_{\theta_1} \tilde{\phi}(\underset{\sim}{X})$.

But by design, $\phi(\underset{\sim}{X})$ is MP test of size α for

$H_0: \theta = \theta_0$ vs $H_1: \theta = \theta_1$ & $E_{\theta_0} \tilde{\phi}(X) \leq \alpha$ holds

so $E_{\theta_1} \phi(\underset{\sim}{X}) \geq E_{\theta_1} \tilde{\phi}(\underset{\sim}{x})$ must be true

Hypothesis Testing I

Finding UMP Tests (Method II)

Method II for UMP Tests uses Monotone Likelihood Ratio, defined below

Definition: Let $f(\underset{\sim}{x}|\theta)$, $\theta \in \Theta \subset \mathbb{R}$, be the joint pdf/pmf of $\underset{\sim}{X} = (X_1, \ldots, X_n)$ (note that the parameter θ is real-valued). Then, $\{f(\underset{\sim}{x}|\theta) : \theta \in \Theta\}$ is said to have **monotone likelihood ratio** (MLR) in a real-valued statistic $T = t(\underset{\sim}{X})$ if: for any $\theta_1 < \theta_2$, there exists a nondecreasing function $g_{\theta_1,\theta_2}(\cdot) : \mathbb{R} \to [0, \infty]$ such that

likelihood

$$\frac{\overset{L(\theta_2)}{f(\underset{\sim}{x}|\theta_2)}}{\underset{L(\theta_1)}{f(\underset{\sim}{x}|\theta_1)}} = g_{\theta_1,\theta_2}\Big(\underbrace{t(\underset{\sim}{x})}_{\text{statistic}}\Big), \quad \text{for all } \underset{\sim}{x} \in \{\underset{\sim}{y} : f(\underset{\sim}{y}|\theta_1) + f(\underset{\sim}{y}|\theta_2) > 0\}$$

data where $f(\underset{\sim}{x}|\theta_1)>0$ or $f(\underset{\sim}{x}|\theta_2)>0$ or both

Here $a/b = +\infty$ if $a > 0, b = 0$

Example: Let $f(\underset{\sim}{x}|\theta) = c(\theta)h(\underset{\sim}{x})\exp[q_1(\theta)t_1(\underset{\sim}{x})]$, $\theta \in \Theta \subset \mathbb{R}$ be the joint pdf/pmf of $X_1, \ldots, X_n$ (i.e., in the exponential family) and $q_1(\theta)$ is nondecreasing. Show that $\{f(\underset{\sim}{x}|\theta) : \theta \in \Theta\}$ has MLR in $t_1(\underset{\sim}{x})$.

Solution: Fix $\theta_1 < \theta_2$.

Note: $\{\underset{\sim}{y} : f(\underset{\sim}{y}|\theta_i) > 0\} = \{\underset{\sim}{y} : h(\underset{\sim}{y}) > 0\}$ for $i = 1, 2$

$\Rightarrow B \equiv \{\underset{\sim}{y} : f(\underset{\sim}{y}|\theta_1) + f(\underset{\sim}{y}|\theta_2) > 0\} = \{\underset{\sim}{y} : h(\underset{\sim}{y}) > 0\}$

Pick $\underset{\sim}{x} \in B$.

$$\frac{f(\underset{\sim}{x}|\theta_2)}{f(\underset{\sim}{x}|\theta_1)} = \frac{c(\theta_2)}{c(\theta_1)}\exp\Big[\underbrace{\{q_1(\theta_2) - q_1(\theta_1)\}}_{\geq 0 \text{ since } \theta_1<\theta_2} t_1(\underset{\sim}{x})\Big]$$

$$= g_{\theta_1,\theta_2}(t_1(\underset{\sim}{x}))$$

where $g_{\theta_1,\theta_2}(t) = \frac{c(\theta_2)}{c(\theta_1)}\exp[\{q_1(\theta_2) - q_1(\theta_1)\}t]$ is non-decreasing in t as $t\uparrow$

$\therefore$ MLR in $t_1(\underset{\sim}{x})$

Note: if $q_1(\theta)$ is nonincreasing, check that MLR in $-t_1(\underset{\sim}{x})$

$$\frac{f(\underset{\sim}{x}|\theta_2)}{f(\underset{\sim}{x}|\theta_1)} = g_{\theta_1,\theta_2}(-t_1(\underset{\sim}{x}))$$

where $g_{\theta_1,\theta_2}(t) = \frac{c(\theta_2)}{c(\theta_1)} \exp[\underbrace{\{q_1(\theta_1) - q_1(\theta_2)\}}_{\geq 0 \text{ by } \theta_1 < \theta_2} t]$

is non-decreasing in t as $t \uparrow \infty$

Hypothesis Testing I

Finding UMP Tests (Method II)

↙ based on MLR

Context: Method II of finding a UMP test applies only when $\Theta \subset \mathbb{R}$ (i.e., parameter θ is real-valued) *and* the testing problem is of the form "$H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$" or "$H_0 : \theta \geq \theta_0$ vs $H_1 : \theta < \theta_0$"

↑ only can do 1-sided H_0

Theorem (Method II using MLR): Let $f(\underset{\sim}{x}|\theta)$, $\theta \in \Theta \subset \mathbb{R}$, be the joint pdf/pmf of $X_1, \ldots, X_n$. Assume that $\{f(\underset{\sim}{x}|\theta) : \theta \in \Theta\}$ has MLR in $T = t(\underset{\sim}{x})$. Then,

1. a size α UMP test for $H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$ ($\theta_0 \in \Theta$ fixed) is given by

$$\varphi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } t(\underset{\sim}{x}) > k \\ \gamma & \text{if } t(\underset{\sim}{x}) = k \\ 0 & \text{if } t(\underset{\sim}{x}) < k \end{cases}$$

where $\gamma \in [0, 1]$ and $-\infty \leq k \leq \infty$ are constants satisfying $\mathrm{E}_{\theta_0}\varphi(\underset{\sim}{X}) = \alpha$.

↑ calibrate test at θ_0

2. a size α UMP test for $H_0 : \theta \geq \theta_0$ vs $H_1 : \theta < \theta_0$ ($\theta_0 \in \Theta$ fixed) is given by

$$\varphi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } t(\underset{\sim}{x}) < k \\ \gamma & \text{if } t(\underset{\sim}{x}) = k \\ 0 & \text{if } t(\underset{\sim}{x}) > k \end{cases}$$

where $\gamma \in [0, 1]$ and $-\infty \leq k \leq \infty$ are constants satisfying $\mathrm{E}_{\theta_0}\varphi(\underset{\sim}{X}) = \alpha$. ← calibrate test at θ_0

Hypothesis Testing I

Illustration of Finding UMP Test (Method II)

↙ support depends on θ

Example: Let $X_1, \ldots, X_n$ be a random sample from uniform$(0,\theta)$, $\theta > 0$. Find a UMP test of size α for $H_0 : \theta \le \theta_0$ vs $H_1 : \theta > \theta_0$ (where $\theta_0 > 0$ is fixed).

i.e. $\theta_0 = 1$

Solution: joint pdf of $X_1, \ldots, X_n$ is given by

$$f(\underset{\sim}{x} \mid \theta) = \theta^{-n} I(0 < x_{(1)} \le x_{(n)} \le \theta) \quad \text{using indicator}$$

Fix $\theta_2 > \theta_1$. $B \equiv \{ \underset{\sim}{y} : \underbrace{f(\underset{\sim}{y} \mid \theta_1)}_{0 < y_{(1)} \le y_{(n)} \le \theta_1} + \underbrace{f(\underset{\sim}{y} \mid \theta_2)}_{0 < y_{(1)} \le y_{(n)} \le \theta_2} > 0 \}$

$$= \{ \underset{\sim}{y} : 0 < y_{(1)} \le y_{(n)} \le \theta_2 \}$$

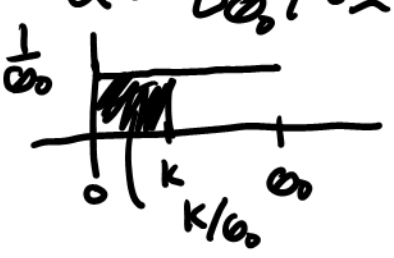
Fix $\underset{\sim}{x} \in B$, $\dfrac{f(\underset{\sim}{x} \mid \theta_2)}{f(\underset{\sim}{x} \mid \theta_1)} = \begin{cases} \theta_2^{-n} / \theta_1^{-n} & \text{if } x_{(n)} \le \theta_1 \\ \dfrac{\theta_2^{-n}}{0} = +\infty & \text{if } \theta_1 < x_{(n)} \le \theta_2 \end{cases}$

is non-decreasing function of $X_{(n)}$

$\Rightarrow \{ f(\underset{\sim}{x} \mid \theta) : \theta > 0 \}$ has MLR in $t(\underset{\sim}{x}) \equiv X_{(n)}$

$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } X_{(n)} > k \\ \gamma = 0 & \text{if } X_{(n)} = k \\ 0 & \text{if } X_{(n)} < k \end{cases}$ is a size α UMP test, provided that $E_{\theta_0} \phi(\underset{\sim}{x}) = \alpha$.

$$\alpha = E_{\theta_0} \phi(\underset{\sim}{x}) = P_{\theta_0}(X_{(n)} > k) = 1 - P_{\theta_0}(X_{(n)} \le k)$$
$$= 1 - [P_{\theta_0}(X_1 \le k)]^n$$
$$= 1 - [k/\theta_0]^n$$



$$\Rightarrow k = \theta_0 (1-\alpha)^{1/n}$$

$$P_{\theta_0}(X_{(n)} \le k) = P_{\theta_0}(\text{all } X_1, \dots, X_n \le k)$$

$$= P_{\theta_0}(X_1 \le k, X_2 \le k, \dots, X_n \le k)$$

$$= [P_{\theta_0}(X_1 \le k)]^n$$