# PhD Prelim Exam
# THEORY

**Summer 2005**
**(Given on 7/21/05)**

Let $\{X_n\}_{n\geq 1}$ and $\{Y_n\}_{n\geq 1}$ be sequences of random variables defined over a probability space $(\Omega, \mathcal{F}, P)$.

1. Suppose $X_n$ is AN$(\mu, \sigma_n^2)$ for a sequence of strictly positive constants $\{\sigma_n^2\}_{n\geq 1}$, and at each $n$ $Y_n$ is distributed as follows: $Y_n = 0$ with probability $1 - n^{-1}$ and $Y_n = n$ with probability $n^{-1}$.

   Show that $X_n + Y_n$ is AN$(\mu, \sigma_n^2)$ too.

2. Suppose $\mu_n = E(X_n)$ for each $n \geq 1$. Consider estimation of a parameter $\theta$ by $X_n$. Answer with justification the following statements:

   (a) If $X_n$ is consistent for $\theta$, must $X_n$ be asymptotically unbiased for estimation of $\theta$?

   (b) If $X_n$ is asymptotically unbiased for estimation of $\theta$, must $X_n$ be consistent estimator for $\theta$ ?

   (c) If $X_n$ is asymptotically unbiased and $Var(X_n) \to 0$ as $n \to \infty$, must $X_n$ be consistent ?

3. $\{X_n\}$ is said to be uniformly integrable if

$$\lim_{c\to\infty} \sup_n E\{|X_n| I(|X_n| > c)\} = 0.$$

   Establish the following statement:

   If there exists a random variable $Y \in L^1(\Omega, \mathcal{F}, P)$ such that $|X_n| \leq Y$, then $\{X_n\}$ is uniformly integrable.

4. Let $\{X_i\}_{i=1}^n$ be the underlying random variables of interests from a probability space $(\Omega, \mathcal{F}, \mathcal{P}_\theta)$ where $\theta$ is an unknown parameter taking value in an open set $\Theta \subset R^d$ for a positive integer $d$.

   Due to measurement errors, $\{X_i\}_{i=1}^n$ are not directly observable. What we observe are $\{Y_i\}_{i=1}^n$:

$$Y_i = X_i + \epsilon_i$$

   where $\{\epsilon_i\}_{i=1}^n$ are independent and identically distributed (IID) measurement errors (MEs) with zero mean and a **known** finite variance $\sigma_e^2$, and are mutually independent of $\{X_i\}_{i=1}^n$.

   Let $\mathcal{P}_\theta$ be the Gamma family of distributions with probability density function

$$f_\theta(x) = \frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha} I(0 < x < \infty)$$

   where $\theta = (\alpha, \beta) \in \Theta = (0,\infty) \times (0,\infty)$, which has a moment generating function $(1 - \beta t)^{-\alpha} I(t < \beta^{-1})$.

   Even we have a full parametric model for the measurement error $\epsilon$, it can be difficult to apply the maximum likelihood estimation, especially when the ME is not from the same parametric family as $X_i$. An approach commonly used in the context of measurement error is the method of moments, which is semiparametric as we do not have to assume a parametric model for the ME except its first two moments.

   (a) Derive a consistent estimator, say $\hat{\theta} = (\hat{\alpha}, \hat{\beta})$, of $\theta$ based on the method of moment and show that $\hat{\theta}$ is a consistent estimator of $\theta$.

   (b) Show that $\sqrt{n}(\hat{\alpha} - \alpha) \xrightarrow{d} N(0, v^2)$ and give an expression for $v^2$.

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Theory I Major (SOLUTIONS) 

A brief Outlines of Solutions (Marking of these questions will be up to the fine details):

(1). We want to show $\frac{X_n-\mu+Y_n}{\sigma_n} \xrightarrow{d} N(0,1)$. From Slutsky Theorem, we only need to show $Y_n/\sigma_n = o_p(1)$. For any $\epsilon > 0$, $P(|Y_n| > \epsilon\sigma_n) \leq P(Y_n = n) = n^{-1} \to 0$, hence $Y_n/Sigma_n = o_p(1)$.

(2) (i) Not necessarily. For instance, if we let $X_n = \mu + Y_n$ where $Y_n$ is as defined in (1). The same derivation in (1) leads to $Y_n = o_p(1)$ too. Note that $E(Y_n) = 1$. Hence $X_n \xrightarrow{p} \mu$ (consistent). But, $E(X_n) = \mu + 1$ for all $n$. So, it is asy. biased.

(ii). Not necessarily. Again use the same definition of $X_n$ as in (2)(i). Then, $X_n$ is unbiased for $\mu + 1$, but does not $\xrightarrow{p} \mu + 1$ as $X_n \xrightarrow{p} \mu$ instead.

(iii). It is true. As it will be converging in mean square error, which implies convergence in probability.

(3)

(i) As $E\{|X_n|I(|X_n| > c)\} \leq E\{|Y|I(|Y| > c)\}$ for each $n$, $\sup_n E\{|X_n|I(|X_n| > c)\} \leq E\{|Y|I(|Y| > c)\}$ which converges to zero as $c \to \infty$ as $Y$ is integrable. Hence $X_n$ is uniformally integrable (u.i.).

(ii) Note that $X_n$ is u.i. iff (a) $\sup_n E(|X_n|) < \infty$ and (b) $\forall\ \epsilon > 0$, $\exists \delta(\epsilon) > 0$ s.t. $\forall E \in \mathcal{F}$, if $P(E) < \delta(\epsilon)$ then $\int_E |X_n| dP < \epsilon$ for every $n$.

As $X_n$ and $Y_n$ are both u.i., $\sup_n E(|X_n + Y_n|) \leq \sup_n E(|X_n|) + \sup_n E(|X_n|) < \infty$. Also, $\forall \epsilon > 0$, $\exists\ \delta_1$ and $\delta_2$ s.t. if $P(E) < \delta_1 or \delta_2$ then $\int_E |X_n| dr < \epsilon/2$ or $\int_E |Y_n| dr < \epsilon/2$ for every $n$. Choose $\delta = \min \delta_1, \delta_2$ property (b) can be established for $X_n + Y_n$. Hence $X_n + Y_n$ is u.i.

The case for $X_n + Y_1$ can be argued exactly the same way.

(4)

(i) The MM estimator is easily established by solving a system of equations:

$$\bar{Y} = \alpha\beta \quad \text{and} \quad S_Y^2 = \alpha\beta^2 + \sigma_\epsilon^2$$

where $\bar{Y}_k = n^{-1} \sum Y_i^k$ and $S_y^2 = n^{-1} \sum (Y_i - \bar{Y})^2 = \bar{Y}_2 - \bar{Y}_i^2$.

$\hat{\alpha} = \bar{Y}^2/(S_y^2 - \sigma_\epsilon^2)$ and $\hat{\beta} = (S_y^2 - \sigma_\epsilon^2)/\bar{Y}$.

The WLLN leads to $\bar{Y} \xrightarrow{p} \alpha\beta$ and $S_y^2 - \sigma_\epsilon^2 \xrightarrow{p} \alpha\beta^2$. Then, apply the mapping theorem with mappings $g_1(x, y) = x^2/y$ and $g_2(x, y) = y/x$ respectively, and lead to the consistency of the MM estimators.

(ii) Let $\mu_{yk} = E(Y^k)$ and note that $\sqrt{n}\left(\bar{Y}_1 - \mu_{y1}), \bar{Y}_2 - \mu_{y2}\right)^T \xrightarrow{d} N_2((0,0)^T, \Sigma)$ where

$$\Sigma = \begin{pmatrix} \mu_{y2} - \mu_{y1}^2 & \mu_{y3} - \mu_{y1}\mu_{y2} \\ \mu_{y3} - \mu_{y1}\mu_{y2} & \mu_{y4} - \mu_{y2}^2 \end{pmatrix}. \tag{1}$$

Note that $\hat{\alpha} = \frac{\bar{Y}_1^2}{\bar{Y}_2 - \bar{Y}_1^2 - \sigma_\epsilon^2} = g(\bar{Y}_1, \bar{Y}_2)$, where $g(y_1, y_2) = y_1^2/(y_2 - y_1^2 - \sigma_\epsilon^2)$ and $g_1'(y_1, y_2) = \frac{\partial g}{\partial y_1} = \frac{2y_1y_2}{(y_2 - y_1^2 - \sigma_\epsilon^2)^2}$ and $g_2'(y_1, y_2) = \frac{\partial g}{\partial y_2} = -\frac{y_1^2}{(y_2 - y_1^2 - \sigma_\epsilon^2)^2}$. By Taylor expansion,

$$\hat{\alpha} = g(\mu_{y1}, \mu_{y2}) + \frac{\partial g}{\partial y_1}(\mu_{y1}, \mu_{y2})(\bar{Y}_1 - \mu_{y1}) + \frac{\partial g}{\partial y_2}(\mu_{y1}, \mu_{y2})(\bar{Y}_2 - \mu_{y2}) + O_p(n^{-1})$$

by checking on the second partial derivatives which are all bounded. Hence

$$\sqrt{n}(\hat{\alpha} - \alpha) = g_1'(\mu_{y1}, \mu_{y2})\sqrt{n}(\bar{Y}_1 - \mu_{y1}) + g_2'(\mu_{y1}, \mu_{y2})\sqrt{n}(\bar{Y}_2 - \mu_{y2}) + O_p(n^{-1/2})$$

where

$$g_1'(\mu_{y1}, \mu_{y2}) = \frac{2(1+\beta)}{\beta} \quad \text{and} \quad g_2'(\mu_{y1}, \mu_{y2}) = -2/\beta^2.$$

From Slutsky and the Cramer-Wold device, $\sqrt{n}(\hat{\alpha} - \alpha) \xrightarrow{d} N(0, v^2)$ where

$$v = 4\beta^{-2}((1+\beta), -\beta^{-1})\Sigma((1+\beta), -\beta^{-1})^T.$$

1. (a) Consider the random experiment of performing $n$ Bernouilli trials with the $i$th trial resulting in a 1 with probability $p_i$ and 0 with probability $(1-p_i)$. Assume that $n$ is finite and the $n$ trials are independent and that the outcomes of all the trials are noted. Formulate a probability model $(\Omega, \mathcal{F}, P)$ for this experiment where $\Omega$ is the sample space, $\mathcal{F}$ is the $\sigma$-algebra of events and $P$ is the probability distribution.

   (b) Suppose a countable number of such Bernouilli trials are performed with a given set $\{p_i\}_{i\geq 1}$ of probabilities and independently. Formulate a probability model for this experiment. Identify an event on this experiment that is decidable only by the data on all but a finite number of the trials.

   (c) For the model in (b), let $Y_n$ be the proportion of 1's in the first $n$ trials. Suppose $\lim_n \frac{1}{n}\sum_1^n p_i = p$ exists. Prove that $Y_n \to p$ w.p.1 by using the Borel-Cantelli lemma and by bounding $P(|Y_n - \frac{1}{n}\sum_1^n p_i| > \epsilon)$ appropriately using 4th moments.

2. Let $(S_i, \mathcal{F}_i)$ $\quad i = 1,2$ be two measurable spaces. Let $\lambda$ be a probability measure on $(S_1, \mathcal{F}_1)$. Let $Q : S_1 \times \mathcal{F}_2 \to [0,1]$ be such that for each $\theta$ in $S_1, Q(\theta, \cdot)$ is a probability measure on $(S_2, \mathcal{F}_2)$ and for each $A$ in $\mathcal{F}_2, Q(\cdot, A) : S_1 \to [0,1]$ is $\mathcal{F}_1$ measureable. Let $\mathcal{C} \equiv \{A_1 \times A_2 : A_i \in \mathcal{F}_i \quad i = 1,2\}$ be the collection of all measureable rectangles. Let $P : \mathcal{C} \to [0,1]$ be defined by $P(A_1 \times A_2) = \int_{A_1} Q(\theta, A_2)\lambda(d\theta)$. Assume the fact that $P$ is countably additive on $\mathcal{C}$.

   (a) State a theorem that guarantees that $P$ can be extended to be a probability measure on $(\Omega \equiv S_1 \times S_2, \mathcal{F} \equiv \mathcal{F}_1 \times \mathcal{F}_2)$ where $\mathcal{F}_1 \times \mathcal{F}_2$ is defined as the $\sigma$-algebra generated by the class $\mathcal{C}$.

   (b) Now for $\omega = (s_1, s_2)$, let $\theta(\omega) = s_1,\ X(\omega) = s_2$. Think of $\theta$ as the parameter, $X$ as the data, $\lambda$ as the prior and $Q(\theta, \cdot)$ as the distribution of data $X$ given the parameter $\theta$.

      i. Determine $\nu(\cdot) \equiv P(X \in \cdot)$ in terms of $Q(\cdot, \cdot)$ and $\lambda$.

      ii. Now fix $A_1$ in $\mathcal{F}_1$. Show that the set function

      $$\mu_{A_1}(\cdot) \equiv P(\theta \in A_1, X \in \cdot)$$

      from $\mathcal{F}_2$ to $[0,1]$ is a measure and it is dominated by $\nu$.

      iii. Conclude that there exists a function $\mathbf{Q}(x, A_1)$ such that for any $A_2$ in $S_2$

      $$P(\theta \in A_1, X \in A_2) = \int_{A_2} \mathbf{Q}(x, A_1)\nu(dx).$$

      Call $\mathbf{Q}(x, \cdot)$ the posterior distribution of $\theta$ given $X = x$.

   (c) Consider the special case $S_1 = S_2 = \Re \qquad \mathcal{F}_1 = \mathcal{F}_2 = \mathcal{B}(\Re), \lambda(\cdot) \sim N(0,1) \qquad Q(\theta, \cdot) \sim N(\theta, \cdot)$. Find $\mathbf{Q}(x, \cdot)$.

3. (a) State Scheffe's theorem and prove it using DCT.

   (b) Let
$$X_n \sim N(\mu_n, \sigma_n^2), X \sim N(\mu, \sigma^2).$$
   Let $\lim_n \mu_n = \mu, \lim_n \sigma_n^2 = \sigma^2, \mu \in \Re,\ \ \sigma^2 \in (0, \infty)$. Let $h : \Re \to \Re$ be bounded Borel measureable. Show that $\lim_n Eh(X_n) = Eh(X)$.

4. (a) State the Gilvenko-Cantelli theorem on the sequence of empirical distribution functions based on a sequence $\{X_i\}_{i \geq 1}$ of pairwise iid random variables.

   (b) Evaluate
$$\lim_{n \to \infty} \int_0^\infty \int_0^\infty \int_0^\infty e^{-\sum_1^n x_i} (\Pi_{i=1}^n x_i) \sin\left(\frac{\sum_{i=1}^n \sum_{j=1}^n x_i^2 x_j}{(\sum_1^n x_i^4)^2}\right) dx_1 dx_2 \ldots dx_n$$

   i. State a version of the CLT.

   ii. Let $\{X_i\}_{i \geq 1}$ be iid with $EX_1 = 0, EX_1^2 = 1$. Let $\{\epsilon_i\}_{i \geq 1}$ be a sequence of random variables such that $\frac{1}{\sqrt{n}} \sum_{i=1}^n \epsilon_i \to 0$ in probability. Let $Y_i = X_i + \epsilon_i$. Show that $\lim_{n \to \infty} P\left(\sum_{i=1}^n Y_i \leq x\sqrt{n} + logn)\right)$ exists for all $x$ in $\Re$ and evaluate the limit.

Solution to Theory II Stat Ph.D. Theory Prelim Qn (by K.B. Athreya)
Summer 2005.

1. a) $\Omega \equiv \{0,1\}^n \equiv \{\omega : \omega = (\delta_1, \delta_2, \ldots, \delta_n),\ \delta_i = 0 \text{ or } 1\}$

$\mathcal{F} \equiv \mathcal{P}(\Omega)$ the power set

$$P\{\omega\} = \prod_{i=1}^{n} p_i^{\delta_i}(1-p_i)^{1-\delta_i}, \text{ if } \omega = (\delta_1, \delta_2, \ldots \delta_n)$$

$$P(A) \equiv \sum_{\omega \in A} P\{\omega\} \text{ for any } A \in \mathcal{F}.$$

b) $\Omega \equiv \{\omega : \omega = (\delta_1, \delta_2, \ldots\ldots),\ \delta_i \in \{0,1\}\}$

Let $\mathcal{F}_0 \equiv \{A : A = \{\omega : (\delta_1, \delta_2, \ldots \delta_n) \in A_n\}$ for some $n < \infty$ and $A_n \in \{0,1\}^n\}$

Let $\mathcal{F} \equiv \sigma(\mathcal{F}_0)$, the $\sigma$-algebra generated by $\mathcal{F}_0$.

Let $P$ be the $p$-measure on $(\Omega, \mathcal{F})$ that is uniquely determined by its values on $\mathcal{F}_0$ by

for $A$ in $\mathcal{F}_0$, $A = \{\omega : (\delta_1, \ldots, \delta_n) \in A_n\}$, $n < \infty$

$$\tilde{P}_0(A) = P_n(A_n) = \sum_{\omega_n \in A_n} P\{\omega_n\}$$

where $\omega_n = (\delta_1, \delta_2, \ldots, \delta_n)$.

with $P\{\omega_n\}$ as in (a).

It can be verified that $\tilde{P}$ is c.a. on $\mathcal{F}_0$ and $\mathcal{F}_0$ is a semi algebra. So by the extn theorem there is a measure $P$ on $\mathcal{F} = \sigma(\mathcal{F}_0)$ $\ni$ $P = \tilde{P}$ on $\mathcal{F}_0$.

Let $A \equiv \{\omega = \lim_n \frac{1}{n}\sum_1^n \delta_i \text{ exists}\}$. This event $A \in \mathcal{F}$ but is not decidable by knowing $\omega_n$ for any $n$ where $\omega_n = (\delta_1, \ldots, \delta_n)$ if $\omega = (\delta_1, \delta_2, \ldots\ldots)$

c) For any $\epsilon > 0$,

$$P\left(\left|Y_n - \frac{1}{n}\sum_1^n p_i\right| > \epsilon\right)$$

$$\leq \frac{E|\quad|^4}{\epsilon^4} = \frac{1}{n^4\epsilon^4} E\left(\sum_{i=1}^n (\delta_i - p_i)\right)^4$$

$$\leq \frac{1}{n^4\epsilon^4}\left(\sum_1^n E(\delta_i - p_i)^4 + \sum_{\substack{i \neq j \\ i,j=1}}^n E(\delta_i - p_i)^2 E(\delta_j - p_j)^2\right)$$

$$\leq \frac{1}{n^4\epsilon^4}\left(n + n(n-1)\right) \leq \frac{C_\epsilon}{n^2}$$

for some $C_\epsilon < \infty$

$$\Rightarrow \sum_1^\infty P\left(|Y_n - \frac{1}{n}\sum_1^n p_i| > \epsilon\right) < \infty \quad \forall \epsilon > 0$$

By Borel Cantelli, $\forall \epsilon > 0$, $\exists A_\epsilon \ni P(A_\epsilon) = 1$

$\omega \in A_\epsilon \Rightarrow \left| Y_n - \frac{1}{n}\sum_1^n p_i \right| \cancel{\to 0} \le \epsilon$ for all large $n_\epsilon(\omega)$

Let $A = \bigcap_{r=1}^{\infty} A_{\frac{1}{r}}$. Then $P(A) = 1$

$\omega \in A \Rightarrow \omega \in A_{\frac{1}{r}} \Rightarrow \forall r$

$\Rightarrow \forall r, \exists n_r(\omega) \ni |Y_n - \frac{1}{n}\sum_1^n p_i| < \frac{1}{r}$ for $n \geq n_r$.

$\Rightarrow \lim_{n\to\infty} |Y_n - \frac{1}{n}\sum_1^n p_i| = 0$

Since $\frac{1}{n}\sum_1^n p_i \longrightarrow p$ by hyp

$Y_n \longrightarrow p$ w.p-1.

II (a) Thm If $\tilde{P}$ is a set fn on a semi algebra $\mathcal{C}$ that satisfies

$\tilde{P}(\phi) = 0$ + $\tilde{P}(\cdot)$ is c.a. on $\mathcal{C}$

then $\tilde{P}$ can be extended to be a measure $P$ on $\sigma(\mathcal{C}) \ni \bar{P} = \tilde{P}$ on $\mathcal{C}$.

(b) i) $\nu(\cdot) = P(X \in \cdot) = P(\theta \in S_1, X \in \cdot)$

$$= \int_{S_1} Q(\theta, \cdot)\, \lambda(d\theta)$$

ii) By MCT $\mu_{A_1}(\cdot)$ is a measure on $\mathcal{F}_2$

Next $\mu_{A_1}(\cdot) = P(\theta \in A_1, X \in \cdot) \leq P(X \in \cdot) = \nu(\cdot)$

So $\mu_{A_1}(\cdot) << \nu(\cdot)$.

iii) ~~Also by~~ For fixed $A_1$, by the Radon-Nikodym Thm

$\exists$ a fn $Q(x, A_1)$ : $S_2 \to [0,1]$, $\mathcal{F}_2$ mble $\ni$

$$\mu_{A_1}(A_2) = \int_{A_2} Q(x, A_1)\, \nu(dx)$$

iv)

$$P(\theta \in A_1, X \in A_2)$$

$$= \int_{A_1} Q(\theta, A_2)\, \lambda(d\theta)$$

$$= \int_{A_1} \left( \frac{1}{\sqrt{2\pi}} \int_{A_2} e^{-\frac{(x-\theta)^2}{2}} dx \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{\theta^2}{2}} d\theta$$

$$= \frac{1}{\sqrt{2\pi}} \left( \int_{A_2} \left( \int_{A_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\theta)^2}{2} - \frac{\theta^2}{2}} d\theta \right) dx \right.$$

$$= \cancel{d} \int_{A_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \left( \frac{1}{\sqrt{2\pi}} \int_{A_1} e^{-\theta^2 + x\theta} d\theta \right) dx$$

$$= \int_{A_2} \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-\frac{x^2}{4}} \left( \int_{A_1} \frac{1}{\sqrt{2\pi}\sqrt{\frac{1}{2}}} e^{-\frac{(\theta - \frac{x}{2})^2}{2(\frac{1}{2})}} d\theta \right) dx$$

$\Rightarrow$ $\nu(\cdot) \sim N(0, 2)$, $Q(x, \cdot) \sim N\left(\frac{x}{2}, \frac{1}{2}\right)$

III (a) ① (DCT) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space

Let $\{f_n\}_{n\geq 1}$ ~~$\in L^1(\Omega, \mathcal{F}, \mu)$~~ be $\ni$ $\forall n$
$f_n : \Omega \to \mathbb{R}$ is $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$ mble.

i) $f_n \to f$ a-e. $(\mu)$

ii) $|f_n| \leq g$ a-e-$(\mu)$

iii) $g \in L^1(\Omega, \mathcal{F}, \mu)$

Then i) ~~$f \in L^1(\Omega,\mathcal{F},\mu)$ $\int f_n d\mu \to \int f d\mu$~~ $f_n, f \in L^1(\Omega, \mathcal{F}, \mu)$

ii) $\int |f_n - f| d\mu \to 0$

iii) $\int f_n d\mu \to \int f d\mu$

(b) ~~Fatou~~ Scheffe's Thm.

Let $f_n, f : \Omega \to \mathbb{R}^+$ be $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$ mble.

Suppose $f_n \to f$ a-e. $\mu$

& $\int f_n d\mu \to \int f d\mu < \infty$.

Then $\int |f_n - f| d\mu \to 0$.

Pf Let ~~$g$~~ $h_n = (f - f_n)^+$ Then $0 \leq h_n \leq f$

Since $f_n \to f$ a.e $\mu$, $h_n \to 0$ a-e-$(\mu)$

and $|h_n| \leq f \in L^1(\mu)$. So by DCT $\int h_n d\mu \to 0$.

Also since $\int (f - f_n) d\mu = \int f ~~d\mu~~ - \int f_n d\mu \to 0$

$\int (f - f_n)^- d\mu = \int (f - f_n) d\mu - \int (f - f_n)^+ d\mu \to 0$.

$\Rightarrow \int |f - f_n| d\mu = \int (f - f_n)^+ d\mu + \int (f - f_n)^- d\mu \to 0$.

(c) ~~Let~~ $X_n$ has pdf $f_n(x) = \frac{1}{\sqrt{2\pi}\,\sigma_n} e^{-\frac{(x-\mu_n)^2}{2\sigma_n^2}}$

$X$ has pdf $f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

Then since $\mu_n \to \mu$, $\sigma_n \to \sigma$

$$f_n(x) \longrightarrow f(x) \quad \forall\, x \in \mathbb{R}.$$

Applying Scheffe to $(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$, $m$-Lebesgue measure

$$\int |f_n - f|\, dm \longrightarrow 0.$$

If $h : \mathbb{R} \to \mathbb{R}$ is bdd measurable

$$\begin{aligned}
&\left| E\,h(X_n) - E\,h(X) \right| \\
&= \left| \int h(x)\, f_n(x)\, dx - \int h(x) f(x)\, dx \right| \\
&\le \int |h(x)|\, |f_n(x) - f(x)|\, dx \\
&\le M \int |f_n(x) - f(x)|\, dx
\end{aligned}$$

where $M = \sup_{x \in \mathbb{R}} |h(x)|$.

& hence $\to 0$ as $n \to \infty$.

IV a) Let $\{X_i\}_{i\geq 1}$ be pairwise indep + id-d. r.v.

with cdf $F(\cdot)$. ~~B~~ Let

$$F_n(x) \equiv \frac{1}{n} \sum_{j=1}^{n} I(X_j \leq x) \quad , x \in \mathbb{R}, \; n \geq 1.$$

be the empirical cdf.

Glivenko–Cantelli Thm

$$\sup_x |F_n(x) - F(x)| \longrightarrow 0 \quad w/p\ 1$$
as $n \to \infty$.

b) The given integral is

$$E\left(\sin\left(\frac{\sum_{i=1}^{n} \sum_{j=1}^{n} X_i^2 X_j}{\left(\sum_1^n X_i^4\right)^2}\right)\right).$$

where $\{X_i\}_{i\geq 1}$ are iid exp(1) r.v.

By SLLN $\quad \frac{1}{n} \sum_1^n X_i^p \to EX_1^p$ for all $p < \infty$.

So by SLLN

$$\frac{\sum_{i=1}^{n} \sum_{j=1}^{n} X_i^2 X_j}{\left(\sum_1^n X_i^4\right)^2} = \frac{\left(\frac{1}{n} \sum_1^n X_i^2\right)\left(\frac{1}{n} \sum_1^n X_j\right)}{\left(\frac{1}{n} \sum_1^n X_i^4\right)^2}$$

$$\longrightarrow \frac{EX_1^2 \; EX_1}{EX_1^4}.$$

Since $\sin(\cdot)$ is a bounded cont. fn.

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the given integral converges to

$$\ln\left(\frac{EX_1^2\, EX_1}{EX_1^4}\right).$$

$$EX_1^p = \int_0^\infty e^{-x} x^{p+1}\, dx = \Gamma(p+1).$$

By SLLN
$$\frac{\sum_{j=1}^n \sum_{i=1}^n X_i^2 X_i}{\left(\sum_{i=1}^n X_i^4\right)^2} \rightarrow \frac{(EX_1^2)(EX_1)}{(EX_1^4)}$$

c) (CLT) Let $\{X_i\}_{i\geq 1}$ be iid, $EX_i = 0$, $EX_i^2 = 1$.

Then $\frac{1}{\sqrt{n}}\sum_1^n X_i \xrightarrow{d} N(0,1)$

$$Y_i = X_i + \epsilon_i \quad \forall i$$

$$\Rightarrow \frac{1}{\sqrt{n}}\sum_1^n Y_i = \frac{1}{\sqrt{n}}\sum_1^n X_i + \frac{1}{\sqrt{n}}\sum_1^n \epsilon_i$$

By CLT, $\frac{1}{\sqrt{n}}\sum_1^n X_i \xrightarrow{d} N(0,1)$

By hyp $\frac{1}{\sqrt{n}}\sum_1^n \epsilon_i \xrightarrow{P} 0$

By Slutzky, $\frac{1}{\sqrt{n}}\sum_1^n Y_i \longrightarrow N(0,1)$

Also by Polya's thm, the cdf of $\frac{1}{\sqrt{n}}\sum_1^n Y_i \longrightarrow \Phi(\cdot)$ uniformly

ie $\sup_x \left| P\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n Y_i \leq x\right) - \Phi(x)\right| \longrightarrow 0.$

Thus

$$\left| P\left( \frac{1}{\sqrt{n}} \sum_{1}^{n} Y_i \le x + \frac{\log n}{\sqrt{n}} \right) - \Phi\left( x + \frac{\log n}{\sqrt{n}} \right) \right| \longrightarrow 0 .$$

Since $\Phi(\cdot)$ is conts

$$\Phi\left( x + \frac{\log n}{\sqrt{n}} \right) \longrightarrow \Phi(x).$$

Thus $\quad P\left( \sum_{i}^{n} Y_i \le x\sqrt{n} + \log n \right) \longrightarrow \Phi(x).$

Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with the probability density function (pdf)

$$f(x;\theta) = \begin{cases} \frac{2}{\pi\theta}\exp\{-\frac{x^2}{\pi\theta^2}\} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

where $\theta > 0$ is an unknown parameter. Note that

$$E(X_1) = \theta,\ E(X_1^2) = \left(\frac{\pi}{2}\right)\theta^2,\ E(X_1^4) = \left(\frac{3\pi^2}{4}\right)\theta^4.$$

(a) Find the Cramér-Rao lower bound for the variance of an unbiased estimator of $\theta^2$. Is there a uniformly minimum variance unbiased estimator (UMVUE) of $\theta^2$ whose variance is the same as the Cramér-Rao lower bound? If so, what is the UNVUE?

(b) Prove that the random variable $Y = \frac{2}{\pi\theta^2}\sum_{i=1}^n X_i^2$ has a $\chi_n^2$ distribution with pdf

$$f(y) = \frac{1}{2^{n/2}\Gamma(n/2)} e^{(-y/2)} y^{(n/2-1)} \quad (y \geq 0),$$

where $\Gamma(1/2) = \sqrt{\pi}, \Gamma(1) = 1$, and $\Gamma(n/2) = (n/2-1)\Gamma(n/2-1)$ for $n \geq 3$.

(c) Use the result in part (b) to find a multiple of $(\sum_{i=1}^n X_i^2)^{1/2}$ that is an unbiased estimator of $\theta$. Is this estimator a UMVUE of $\theta$? Is the sample mean, $\overline{X}$, a UMVUE of $\theta$? Explain your answers.

(d) Find the maximum likelihood estimators (MLE's) for $\theta$ and $\theta^2$, and determine their asymptotic distributions as $n \to \infty$.

(e) Construct an exact 95% confidence interval for $\theta$ using the result from part (b). Then construct an approximate 95% confidence interval for $\theta$ using the result from part (d). What are the differences between the two intervals?

(f) Derive the most powerful size $\alpha$ test of the null hypothesis $H_0 : \theta = \theta_0$ against the alternative hypothesis $H_1 : \theta = \theta_1$. Here, $\theta_0$ and $\theta_1 > \theta_0$ are given positive numbers. Express the test in terms of $\sum_{i=1}^n X_i^2$ and a quantile of the $\chi^2$ distribution.

(g) Is the test in part (f) a uniformly most powerful (UMP) size $\alpha$ test of $H_0 : \theta \leq \theta_0$ against $H_1 : \theta > \theta_0$? Explain your answer. Hint: The exact form of the test in part (f) is not needed to answer this question.

(h) Derive the likelihood ratio size $\alpha$ test of $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$. Is this test a UMP test? Explain your answer.

Theory III Solutions Ph.D. Prelim Exam, 2005

(a) To obtain the Cramér–Rao lower bound, note that

$$\frac{\partial}{\partial\theta}\log f(x;\theta) = \frac{\partial}{\partial\theta}\left[\log\left(\frac{2}{\pi}\right) - \log\theta - \frac{x^2}{\pi\theta^2}\right] = -\frac{1}{\theta} + \frac{2}{\pi\theta^3}x^2.$$

Thus, $I_1(\theta) = Var\left[\frac{\partial}{\partial\theta}\log f(X_1;\theta)\right] = \left(\frac{2}{\pi\theta^3}\right)^2 Var(X_1^2) = \frac{2}{\theta^2}$.

The C–R bound is then $\frac{(2\theta)^2}{n\cdot I_1(\theta)} = \frac{2\theta^4}{n}$.

To obtain a UMVUE of $\theta^2$, note that $\sum_{i=1}^n X_i^2$ is a complete and sufficient statistic and that $\frac{2}{n\pi}\sum_{i=1}^n X_i^2$ is an unbiased estimate of $\theta^2$. Thus, $\frac{2}{n\pi}\sum_{i=1}^n X_i^2$ is a UMVUE of $\theta^2$, and its variance is $\frac{2\theta^4}{n}$. This UMVUE indeed has the variance equal to the C–R bound.

(b) The pdf of $Y_1 = \frac{2}{\pi\theta^2}X_1^2$ is $f_{Y_1}(y) = f\left(\sqrt{\frac{\pi\theta^2 y}{2}};\theta\right)\cdot$

$$\left(\sqrt{\frac{\pi\theta^2 y}{2}}\right)'_y = \frac{2}{\pi\theta}\exp\left\{-\frac{1}{\pi\theta^2}\cdot\frac{\pi\theta^2 y}{2}\right\}\cdot\frac{1}{2}\sqrt{\frac{\pi\theta^2}{2y}}$$

$$= \sqrt{\frac{1}{2\pi y}}\exp\left\{-\frac{y}{2}\right\} = \frac{1}{2^{1/2}\cdot\Gamma(1/2)}e^{-y/2}\cdot y^{(1/2-1)} \quad (y \ge 0),$$

which is the pdf of $\chi_1^2$ distribution. Thus, $Y$ has a $\chi_n^2$ distribution because $Y$ is the sum of $n$ independent $\chi_1^2$-distributed random variables.

(c) Note that $E(Y^{1/2}) = \int_0^\infty y^{\frac{1}{2}} f(y)\,dy = \int_0^\infty \frac{1}{2^{n/2}\Gamma(n/2)} e^{-y/2} \cdot y^{(n+1)/2-1} dy$

$$= \int_0^\infty \frac{1}{2^{(n+1)/2} \cdot \Gamma((n+1)/2)} e^{-y/2} \cdot y^{(n+1)/2-1} dy \cdot \frac{2^{1/2} \cdot \Gamma((n+1)/2)}{\Gamma(n/2)} = \frac{2^{1/2} \cdot \Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})}$$

Thus $E\left(\sum_{i=1}^n X_i^2\right)^{1/2} = E\left(\frac{\pi\theta^2}{2} Y\right)^{1/2} = \sqrt{\frac{\pi\theta^2}{2}} \cdot \frac{2^{1/2} \cdot \Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} = \frac{\sqrt{\pi} \cdot \Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} \cdot \theta.$

Hence $\frac{\Gamma(\frac{n}{2})}{\sqrt{\pi} \cdot \Gamma(\frac{n+1}{2})} \left(\sum_{i=1}^n X_i^2\right)^{1/2}$ is an unbiased estimator of $\theta$. This estimator is a UMVUE of $\theta$ because it is a function of the complete and sufficient statistic $T = \sum_{i=1}^n X_i^2$ and is unbiased. By the Rao-Blackwell Theorem, the sample mean $\bar{X}$ is not a UMVUE of $\theta$.

(d) From part (a), we have

$$\ell'(\theta) = \frac{\partial}{\partial\theta} \sum_{i=1}^n \log f(X_i;\theta) = -\frac{n}{\theta} + \frac{2}{\pi\theta^3} \sum_{i=1}^n X_i^2.$$

Note that for $\hat{\theta} = \sqrt{\frac{2}{\pi} \cdot \frac{1}{n} \sum_{i=1}^n X_i^2}$, $\ell'(\hat{\theta}) = 0$ and $\ell'(\theta) > 0$ for $\theta < \hat{\theta}$, $\ell'(\theta) < 0$ for $\theta > \hat{\theta}$. Thus, $\hat{\theta}$ is the unique $\theta$ at which $\ell(\theta)$ is maximized. Hence $\hat{\theta}$ is the MLE of $\theta$, and $\hat{\theta}^2 = \frac{2}{\pi} \cdot \frac{1}{n} \sum_{i=1}^n X_i^2$ is the MLE of $\theta^2$.

By part (a), $I_1(\theta) = \frac{2}{\theta^2}$. Thus the asymptotic distributions of $\hat{\theta}$ and $\hat{\theta}^2$ are $\sqrt{n}(\hat{\theta} - \theta) \to N(0, \frac{\theta^2}{2})$ and $\sqrt{n}(\hat{\theta}^2 - \theta^2) \to N(0, 2\theta^4)$.

(e) Let $\chi^2_n(0.025)$ and $\chi^2_n(0.975)$ denote the 0.025th and 0.975th quantiles of the $\chi^2_n$ distribution. Then by part (b), $P(\chi^2_n(0.025) < \frac{2}{\pi\theta^2}\sum_{i=1}^n X_i^2 < \chi^2_n(0.975)) = 0.95$.

That is, $P\left(\sqrt{\frac{2}{\pi}\sum_{i=1}^n X_i^2 / \chi^2_n(0.975)} < \theta < \sqrt{\frac{2}{\pi}\sum_{i=1}^n X_i^2 / \chi^2_n(0.025)}\right) = 0.95$.

Thus, a 95% confidence interval for $\theta$ is

$$\left(\sqrt{\frac{\frac{2}{\pi}\sum_{i=1}^n X_i^2}{\chi^2_n(0.975)}}, \sqrt{\frac{\frac{2}{\pi}\sum_{i=1}^n X_i^2}{\chi^2_n(0.025)}}\right).$$

By part (d), an approximate 95% confidence interval for $\theta$ is $(\hat{\theta} - 1.96 \cdot \frac{\hat{\theta}}{\sqrt{2n}}, \hat{\theta} + 1.96 \cdot \frac{\hat{\theta}}{\sqrt{2n}})$; that is,

$$\left(\sqrt{\frac{2}{\pi \cdot n}\sum_{i=1}^n X_i^2} \cdot \left(1 - \frac{1.96}{\sqrt{2n}}\right), \sqrt{\frac{2}{\pi \cdot n}\sum_{i=1}^n X_i^2}\left(1 + \frac{1.96}{\sqrt{2n}}\right)\right).$$

The first interval is an exact 95% CI for $\theta$ for any $n$ while the second is valid only for large $n$. The second interval is symmetric and centered at $\sqrt{\frac{2}{\pi}\sum_{i=1}^n X_i^2 / n}$ while the first is not centered at $\sqrt{\frac{2}{\pi}\sum_{i=1}^n X_i^2 / n}$.

(f) Note that $\frac{f(x;\theta_1)}{f(x;\theta_0)} = \frac{\theta_0}{\theta_1} \exp\{\frac{1}{\pi}\cdot(\frac{1}{\theta_0^2} - \frac{1}{\theta_1^2}) x^2\}$ is increasing in $x^2$. Thus, by the Neyman-Pearson Lemma, the most powerful test is given by

$$\phi(x_1, \cdots, x_n) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} X_i^2 \geq c \\ 0 & \text{if } \sum_{i=1}^{n} X_i^2 < c \end{cases}$$

where $c$ satisfies that $P_{\theta_0}[\sum_{i=1}^{n} X_i^2 \geq c] = \alpha$. From part (b) we have $P_{\theta_0}[\frac{2}{\pi\theta_0^2} \sum_{i=1}^{n} X_i^2 \geq \chi_n^2(1-\alpha)] = \alpha$, where $\chi_n^2(1-\alpha)$ is the $(1-\alpha)$th quantile of the $\chi_n^2$ distribution.

Thus, $c = \frac{\pi\theta_0^2}{2} \cdot \chi_n^2(1-\alpha)$.

(g) Yes, because $f(x;\theta)$ has monotone likelihood ratio in $X^2$ and the test is UMP for testing $H_0: \theta \leq \theta_0$ against $H_1: \theta > \theta_0$.

(h) From part (d), we have $\hat{\theta} = \sqrt{\frac{2}{\pi} \cdot \frac{1}{n} \sum_{i=1}^{n} X_i^2}$ as the MLE of $\theta$. Thus, $\log \lambda(X_1, \cdots, X_n) = \sum_{i=1}^{n} [\log f(X_i; \hat{\theta}) - \log f(X_i; \theta_0)]$

$$= n \log \theta_0 + \frac{1}{\pi \theta_0^2} \sum_{i=1}^{n} X_i^2 - \frac{n}{2} \log \left( \frac{2}{\pi} \cdot \frac{1}{n} \sum_{i=1}^{n} X_i^2 \right) - \frac{n}{2}$$

Let $U = \frac{2}{\pi \theta_0^2} \sum_{i=1}^{n} X_i^2$. Then by part (b), $U \sim \chi_n^2$ under $H_0$.

We have $\log \lambda(X_1, \cdots, X_n) = \left(\frac{1}{2} U - \frac{n}{2} \log U\right) - \left(\frac{n}{2} - \frac{n}{2} \log n\right)$

Thus, the likelihood ratio size $\alpha$ test is given by

$$\phi_1(X_1, \cdots, X_n) = \begin{cases} 1 & \text{if } U - n \log U \geq c_1 \\ 0 & \text{if } U - n \log U < c_1 \end{cases}$$

where $c_1$ is such that $P_{\theta_0}[U - n \log U \geq c_1] = \alpha$.

That is, $c_1$ is the $(1-\alpha)$th quantile of the distribution for the random variable $U - n \log U$, with $U \sim \chi_n^2$.

This test is not a UMP test because there does not exist a UMP test for testing $H_0: \theta = \theta_0$ against $H_1: \theta \neq \theta_0$.

Let $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically distributed (iid) random variables with common cumulative distribution function (cdf)

$$F(x) = \begin{cases} 0 & \text{for} \quad x \in (-\infty, 0) \\ 0.1 & \text{for} \quad x \in [0, 1) \\ 0.3x - 0.2 & \text{for} \quad x \in [1, 3) \\ 1.0 & \text{otherwise.} \end{cases}$$

Note that the $X_n$-s are neither discrete nor continuous random variables. You may assume the fact that $EX_1^r = (0.3)\left[\frac{3^{r+1}-1}{r+1} + 3^r\right]$ for any $r \in (0, \infty)$; In particular, $EX_1 = 2.1$, $EX_1^2 = 5.3$ and $EX_1^4 = 38.82$.

1. Find the quantile function $F^{-1}(u)$ of $F$.
   (**Note:** Recall that $F^{-1}(u) = \inf\{x : F(x) \geq u\},\ u \in (0, 1)$.)

2. Define the random variables $Y_i, i \geq 1$ by

$$Y_i = \begin{cases} 0 & \text{for} \quad X_i \in (-\infty, 2) \\ 1 & \text{otherwise.} \end{cases}$$

   Find the distribution of $S_n = \sum_{i=1}^n Y_i$. You need to explicitly write down parameters of the distribution of $S_n$ and state any standard fact(s) that you are using.
   (**Hint:** First find the probability distribution of $Y_1$.)

3. Find $E(X_1 + Y_2)^2$.

4. Show that $S_n / \sum_{i=1}^n X_i \to_p c$ for some constant $c$. Find $c$.

5. Find the limit distributions of

   (i) $\left[\sum_{i=1}^n X_i^2 - nE(X_1^2)\right]/\sqrt{n}$,

   (ii) $\left\{\left[\sum_{i=1}^n X_i - nE(X_1)\right]/\sqrt{n}\right\}^2$, and

   (iii) $\left[\left(n^{-1}\sum_{i=1}^n X_i\right)^2 - \left(EX_1\right)^2\right]\sqrt{n}$.

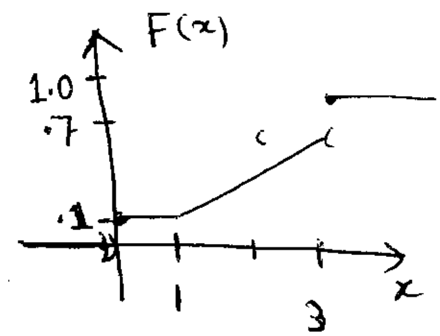
NOTE: $$E X_1^r = 0^r \cdot (.1) + \int_1^3 (x^r)(0.3)\,dx + 3^r \cdot (.3)$$

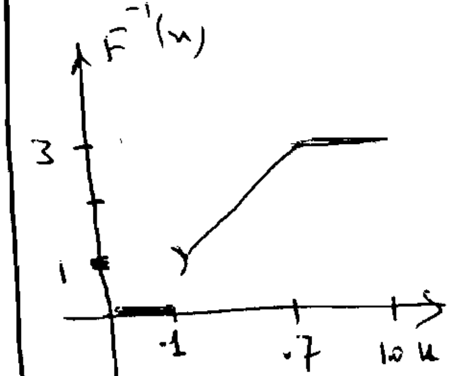
$$= (0.3)\left[\frac{3^{r+1}-1}{r+1} + 3^r\right]$$

1.

$$F^{-1}(u) = \inf\{x : F(x) \geq u\}$$

$$= \begin{cases} 0 & \text{if } 0 < u \leq .1 \\ \frac{(u+.2)}{.3} & \text{if } .1 < u < .7 \\ 3 & \text{if } u \in [.7, 1). \end{cases}$$



2. $P(Y_1 = 0) = P(X_1 < 2) = F(2-) = (.3)(2) - .2$

$$= 0.4 \quad = 1 - P(Y_1 = 1).$$

Thus, $\{Y_i\}_{i \geq 1}$ is a sequence of $\overset{\text{iid}}{\text{Bernoulli}}(p)$ random variables with $p = 0.6$. Hence,

$$S_n = \sum_{i=1}^{n} Y_i \sim \text{Binomial}(n, 0.6).$$

3.
$$E(X_1 + Y_2)^2 = EX_1^2 + EY_2^2 + 2EX_1Y_2$$

$$= (0.3)\left[\frac{3^3 - 1}{3} + 3^2\right] + \left[Var(Y_2) + (EY_2)^2\right]$$

$$+ 2 \cdot (EX_1)(EY_2), \text{ by indep.}$$

$$= (0.3)\left[\frac{26 + 27}{3}\right] + \left[(0.4)(0.6) + (0.6)^2\right]$$

$$+ 2\left((0.3)\left[\frac{3^2 - 1}{2} + 3\right]\right)(0.6), \text{ etc.}$$

$$= \ldots \quad (\text{simplify!})$$

4. By the WLLN,

$$n^{-1} S_n \xrightarrow{p} EY_1 = 0.6, \quad \text{and}$$

$$n^{-1} \sum_{i=1}^n X_i \xrightarrow{p} EX_1 = (0.3)\left[\frac{8+6}{2}\right] = 2.1$$

Hence, by Slutsky's theorem,

$$\frac{S_n}{\sum_{i=1}^n X_i} = \frac{n^{-1} S_n}{n^{-1} \sum_{i=1}^n X_i} \xrightarrow{p} \frac{0.6}{2.1} = ?$$

5. (i) Since $\{X_i\}_{i\geq 1}$ is a sequence of iid ~~r.v.s~~, random variables, $\{X_i^2\}_{i\geq 1}$ is also iid. Hence, by the CLT,

$$\frac{1}{\sqrt{n}}\left(\sum_{i=1}^n X_i^2 - n EX_1^2\right) \xrightarrow{d} N(0, Var(X_1^2)).$$

Here, $Var(X_1) = EX_1^4 - (EX_1^2)^2$, which can be found using the formula for $EX_1^r$.

5 (ii)

By the CLT,

$$T_n \equiv \frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n} X_i - n\,EX_1\right) \xrightarrow{d} N(0,\ Var(X_1))$$

Here, $Var(X_1) = EX_1^2 - (EX_1)^2 = (0.3)\left(\frac{26+27}{3}\right) - (2.1)^2$
$= 5.3 - 4.41 = 0.89$

Hence, by the continuous mapping theorem, with $g(x) = x^2$, $x \in \mathbb{R}$, we have

$$\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i - n\,EX_1\right]^2 = g(T_n) \xrightarrow{d} g(T_\infty)$$

where $T_\infty \sim N(0,\ var(X_1))$.

5 (iii) By the CLT, $\sqrt{n}(\bar{X}_n - \underbrace{EX_1}_{\mu}) \xrightarrow{d} N(0, \underbrace{Var(X_1)}_{\sigma^2})$

By the Delta method,

$$\sqrt{n}\left((\bar{X}_n)^2 - \mu^2\right) = \sqrt{n}\left(g(\bar{X}_n) - g(\mu)\right)$$

$$\xrightarrow{d} N(0, \sigma^2) \cdot g'(\mu)$$ . Here $g'(\mu) = 2\cdot(2.1) \neq 0$

$$\Rightarrow \sqrt{n}\left(\bar{X}_n^2 - \mu^2\right) \xrightarrow{d} N(0,\ (0.89)(4.2)^2).$$

1. Let $X$ have a Beta$(\alpha, \beta)$ distribution with probability density function (pdf) given by

$$f_X(x|\alpha,\beta) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha$ and $\beta$ are positive. Derive the expected value of $X^2$ directly from the pdf.

2. Let $X$ have a Beta$(\alpha, \beta)$ distribution as in question 1, and let $Y = \delta + \sigma X$, $\sigma > 0$.

   (a) What is the pdf of $Y$?

   (b) Does the distribution of $Y$ define an exponential family of distributions? Why or why not?

3. For positive values of $\sigma$ and $\delta$, let the random variable $W$ have the following pdf:

$$f_W(w|\sigma,\delta) = \begin{cases} \frac{3}{\sigma}\left(\frac{w-\delta}{\sigma}\right)^2 & \delta < w < \delta + \sigma \\ 0 & \text{otherwise} \end{cases}$$

Assume $W_1, W_2, \ldots, W_n$ are independent and identically distributed with probability density function (pdf) $f_W$.

   (a) Find the maximum likelihood estimator of $\sigma$ for a given value (a fixed or known value) of $\delta$.

   (b) Find the maximum likelihood estimator of $\delta$ for a given value (a fixed or known value) of $\sigma$.

4. For positive values of $\sigma$, let the random variable $V$ have the following pdf:

$$f_V(v|\sigma) = \begin{cases} \frac{3}{\sigma}(\frac{v}{\sigma})^2 & 0 < v < \sigma \\ \\ 0 & \text{otherwise} \end{cases}$$

Let $V_1, V_2, \ldots, V_n$ be independent and identically distributed random varibles with the same distribution as $V$.

(a) For what quantity is $\bar{V} = \frac{1}{n}\sum_{i=1}^{n} V_i$ a consistent estimator? Outline an argument that would show that $\bar{V}$ is consistent.

(b) What is the large sample distribution of $\sqrt{n}(\frac{4}{3}\bar{V} - \sigma)$?

(c) What is the large sample distribution of $\sqrt{n}(\frac{16}{9}\bar{V}^2 - \sigma^2)$?

5. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed Beta distributions with positive-valued parameters $\alpha$ and $\beta$.

(a) Suppose that $\beta = 1$. What is the maximum likeihood estimator for $\alpha$?

(b) Consider the following hypotheses:

$$H_0 : \alpha = 3, \beta = 1$$

$$H_A : \beta = 1.$$

Describe a uniformly most powerful (UMP) test of these hypotheses.

(c) Suppose that neither $\alpha$ nor $\beta$ are known. A method of moments estimator for $\tau = \alpha + \beta$ is

$$\hat{\tau} = \frac{\bar{x}(1-\bar{x})}{(1/n)\sum x_i^2 - \bar{x}^2} - 1.$$

Is $\hat{\tau}$ a uniform minimum variance unbiased estimator (UMVUE) for $\tau$? Why or why not?

Solutions G-Major Theory II Summer 2005

① $EX^2 = \int_0^1 x^2 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx$

$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{(\alpha+2)-1}(1-x)^{\beta-1}\,dx$ ← kernel of Beta$(\alpha+2, \beta)$

$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)}$

$= \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)}$ .

② $Y = \delta + \sigma X \qquad X = \frac{Y-\delta}{\sigma} \qquad \frac{dX}{dY} = \frac{1}{\sigma}$

a) $f_Y(y) = f_X\left(\frac{y-\delta}{\sigma}\right)\frac{1}{\sigma} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{1}{\sigma} \left(\frac{y-\delta}{\sigma}\right)^{\alpha-1}\left(1-\frac{y-\delta}{\sigma}\right)^{\beta-1}$

for $\delta < y < \delta\sigma$

$= 0$ for other $y$.

b) No, the density cannot be written in exp. fam. form.

③ a) MLE of $\sigma$ is the maximum of $w_1-\delta, w_2-\delta, \ldots, w_n-\delta$.

b) MLE of $\delta$ is any value in the range

$\max_i w_i - \sigma \le \delta \le \min_i w_i$

(4) $EV = \int_0^\sigma v \frac{3}{\sigma^3} v^2 dv = \frac{3}{\sigma^3}\int_0^\sigma v^3 dv = \frac{3v^4}{\sigma^3 4}\Big|_0^\sigma = \frac{3\sigma}{4}$

$EV^2 = \int_0^\sigma v^2 \frac{3}{\sigma^3} v^2 dv = \frac{3}{\sigma^3}\int_0^\sigma v^4 dv = \frac{3v^5}{\sigma^3 5}\Big|_0^\sigma = \frac{3\sigma^2}{5}$

$Var(V) = EV^2 - (EV)^2 = \frac{3}{5}\sigma^2 - \left(\frac{9}{16}\right)\sigma^2 = \frac{3}{80}\sigma^2$

a) $\bar{V}$ is consistent for $\frac{3}{4}\sigma$ by WLLN.

b) $\sqrt{n}\left(\frac{4}{3}\bar{V} - \sigma\right) \sim N\left(0,\ \frac{3}{80}\sigma^2\left(\frac{16}{9}\right) = \frac{1}{15}\sigma^2\right)$

by CLT

c) $g(x) = x^2 \quad g'(x) = 2x$ replace $x$ with $\sigma$

$\sqrt{n}\left(\frac{16}{9}\bar{V}^2 - \sigma^2\right) \sim N\left(0,\ \frac{1}{15}\sigma^2 (4\sigma^2)\right)$

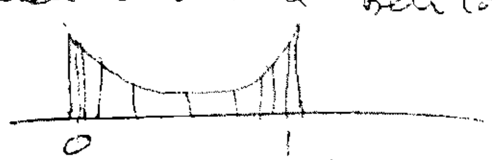
$= N\left(0,\ \frac{4}{15}\sigma^4\right)$.

by Delta method.

(6)

a) Goodness of fit.

Divide the area under a Beta$(\alpha, \beta)$ density into deciles



Count the # of points in each interval.

$$X^2 = \sum_{i=1}^{10} \frac{(O_i - n\frac{1}{10})^2}{n\frac{1}{10}} \sim \chi^2_9$$

b) If a Beta distribution is rejected consider a mixture of Betas or a mixture of a point mass and a Beta.

(5) $f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$

a) $\beta = 1$ $\quad f(x) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)\Gamma(1)} x^{\alpha-1} = \alpha x^{\alpha-1}$

$$f(\underset{\sim}{x}) = \alpha^n \left(\prod x_i\right)^{\alpha-1} = L$$

$$\log L = n \log \alpha + (\alpha-1) \sum \log x_i$$

$$\frac{d \log L}{d\alpha} = \frac{n}{\alpha} + \sum \log x_i = 0$$

$$\hat{\alpha} = \frac{n}{\sum \log x_i}$$

b) $H_0: \alpha = 3, \beta = 1$ $\quad$ These are simple hypotheses
$H_a: \beta = 1$ $\quad$ so use a likelihood ratio test

$$LR = \frac{f_0}{f_a} = \frac{3^n \left(\prod x_i\right)^2}{\hat{\alpha}^n \left(\prod x_i\right)^{\hat{\alpha}-1}}$$

$-2 \log LR \sim \chi^2_1$ (asymptotic)

or

simulation of many samples of size $n$ under the null to produce a reference distribution

c) $\hat{\tau}$ is not UMVUE for $\tau$ because
1) its not unbiased and
2) its not based on a sufficient statistic

(6) see previous page