Let X be a continuous random variable with cdf F given by

$$F(x) = \begin{cases} 1 - \dfrac{c}{\log x} & \text{if } x > e, \\ 0 & \text{otherwise,} \end{cases}$$

for some $c > 0$, where $\log x$ denotes the natural logarithm of x and where e denotes the natural exponent (i.e., $\log e = 1$). **The cdf F remains the same in all questions to follow.**

Part I

1. Show that $c = 1$ in the cdf F.

2. For a given $u \in (0,1)$, determine the quantile function $F^{-1}(u) \equiv \inf\{x \in \mathbb{R} : F(x) \geq u\}$.

3. Based on the quantile function F^{-1}, explain how one could simulate an observation X with distribution F.

4. Give the cdf of a random variable W^2, where W is defined as

$$W = \begin{cases} 2 & \text{if } X \geq e^3, \\ -3 & \text{if } e^2 \leq X < e^3, \\ -2 & \text{otherwise.} \end{cases}$$

5. Argue that X has a density given by $f(x) = [\log x]^{-2} x^{-1}$ for $x > e$, and $f(x) = 0$ otherwise.

6. If X_1, X_2 are iid random variables with cdf F, determine $\mathrm{E}[\log X_1 | X_2 \leq e^2]$.

7. Determine $\mathrm{E}[\log X | X \leq e^2]$.

8. Find the pdf of the random variable $Y = (X - 2e)X$.
Note: the quadratic equation $x^2 + bx + c = 0$ has two roots $(-b \pm \sqrt{b^2 - 4c})/2$.

9. Based on the distribution of $1 - 1/\log X$ or otherwise, show that $R = \log\log(X)$ has an exponential distribution with mean 1 (i.e., R has pdf e^{-r} for $r > 0$).

10. If X_1, X_2 are iid random variables with cdf F, determine the probability $P(\log X_1 < e, \log(\log X_1 \cdot \log X_2) > 1)$.

Part II

Using $X_1, X_2, X_3, X_4, X_5, X_6$ as iid random variables with cdf F, define three random variables Y_1, Y_2, Y_3 as

$$
\begin{aligned}
Y_1 &= \#\{i : i \in \{1,2\} \text{ and } X_i > e^3\} \\
Y_2 &= \#\{i : i \in \{1,2,3,4\} \text{ and } X_i > e^3\} \\
Y_3 &= \#\{i : i \in \{1,2,3,4,5,6\} \text{ and } X_i > e^3\}
\end{aligned}
$$

whereby each Y_j denotes the number of X_i's with $X_i > e^3$ over indices $i \leq 2j$, for $j = 1, 2, 3$.

11. Argue that Y_1, $Y_2 - Y_1, Y_3 - Y_2$ are a random sample from a Binominal$(2, p)$ distribution with $p = 1/3$.

12. Find the covariance $Cov(Y_1 + Y_2, Y_3)$.

13. Find the joint pmf of (Y_1, Y_2, Y_3).

14. Find the conditional pmf of Y_1 given $Y_3 = 3$.

15. Argue, intuitively, why the conditional distribution in Question **14** should not depend on p.

Part III

Let $X_1, X_2, \ldots, X_n$ be a sequence of iid random variables with cdf F. Define $T_i \equiv (\log X_i)^{-1/2}$ for any $i \geq 1$, where $\mathrm{E}(T_1^m) = 2/(m+2)$ for $m \geq 0$. Note also that

$$
\lim_{n\to\infty} \left(1 + \frac{1}{an}\right)^{bn} = e^{b/a}
$$

holds for any real values $a \neq 0$ and b.

16. For $X_{(1)} \equiv \min\{X_1, \ldots, X_n\}$, show that $1/X_{(1)} \xrightarrow{p} e^{-1}$ as $n \to \infty$.

17. For $X_{(n)} \equiv \max\{X_1, \ldots, X_n\}$, use the definition of distributional convergence to show that $[\log X_{(n)}]/n \xrightarrow{d} Z$ as $n \to \infty$, where Z is a continuous positive random variable.

18. Based on Questions **16-17**, determine the limiting distribution of $[X_{(1)} + \log X_{(n)}]/\sum_{i=1}^n T_i$ as $n \to \infty$, stating any standard results used.

19. Determine the limiting distribution of $n^{1/2}[(\sum_{i=1}^n T_i/n)^2 - (\mathrm{E}T_1)^2]$ as $n \to \infty$, stating any standard results used.

20. Determine the limiting distribution of $n^{-1/2}[\sum_{i=1}^n T_i^2 - n(\mathrm{E}T_1^2)]$ as $n \to \infty$, stating any standard results used.

Part I

1. For X to be continuous, we need $F(e) = 0$ where $F(e) = 1 - c$. Hence, $c = 1$.

2. For a given $u \in (0,1)$, the quantile function $F^{-1}(u)$ solves $u = F(F^{-1}(u)) = 1 - 1/\log F^{-1}(u)$ using that $F(\cdot)$ is continuous and strictly increasing on the support (e, ∞) of X. Consequently, $F^{-1}(u) = e^{1/(1-u)}$ for $u \in (0,1)$.

3. If we generate U as a uniform$(0,1)$ random variable, then the random variable $F^{-1}(U) = e^{1/(1-U)}$ follow the target cdf F in distribution.

4. The random variable W^2 assumes two values (4 or 9) in its range with

$$P(W^2 = 9) = P(W = -3) = P(e^2 \le X < e^3) = F(e^3) - F(e^2) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

and so $P(W^2 = 4) = 1 - 1/6 = 5/6$. The cdf of W^2 is then given by

$$P(W^2 \le t) = \begin{cases} 0 & \text{if } t < 4, \\ 5/6 & \text{if } 4 \le t < 9, \\ 1 & \text{if } t \ge 9. \end{cases}$$

5. The derivative of the cdf $F(x)$ is $F'(x) = 0$ for $x < e$ (as $F(x)$=0 in this range) and $F'(x) = [1 - (\log x)^{-1}]' = (\log x)^{-2} x^{-1}$ for $x > e$.

6. Due to independence of random variables,

$$\begin{aligned} \mathrm{E}[\log X_1 | X_2 \le e^2] \;=\; \mathrm{E}[\log X_1] &= \int_e^\infty (\log x) f(x) dx \;=\; \int_e^\infty (\log x)(\log x)^{-2} x^{-1} dx \\ &= \int_1^\infty y^{-1} dy \quad (y = \log x,\, dy = x^{-1} dx) \\ &= \log y \Big|_1^\infty = \infty. \end{aligned}$$

7. Conditional on $X \le e^2$, the density of X becomes $f(x)/F(e^2) = 2(\log x)^{-2} x^{-1}$ on a support $(e, e^2]$, where $F(e^2) = 1 - (\log e^2)^{-1} = 1/2$. Hence,

$$\begin{aligned} \mathrm{E}[\log X | X \le e^2] \;=\; \int_e^{e^2} (\log x) 2 (\log x)^{-2} x^{-1} dx &= 2 \int_1^2 y^{-1} dy \quad (y = \log x,\, dy = x^{-1} dx) \\ &= 2 \log y \Big|_1^2 = 2 \log 2. \end{aligned}$$

8. X has support (e, ∞) so that the function $g(x) = (x - 2e)x$ is negative on $(e, 2e)$ and positive on $(2e, \infty)$; also $g' = 2(x - e) > 0$ for $x > e$ so that $g(x)$ is strictly increasing on the support (e, ∞) of X. The support of $Y = g(X)$ then becomes $(g(e), g(\infty)) = (-e, \infty)$. One can find the density $f_Y(y)$ of Y using the pdf transformation technique on the density f of X: namely,

$$f_Y(y) = f(g^{-1}(y)) \left| \frac{g^{-1}(y)}{dy} \right|, \quad y \in (-e, \infty).$$

For $y \in (-e, \infty)$, solving $y = g(x) = x^2 - 2ex$ or $x^2 - 2ex - y = 0$ gives roots

$$\frac{2e \pm \sqrt{(2e)^2 + 4y}}{2} = e \pm \sqrt{e^2 + y}$$

(i.e., the quadratic form $x^2 + bx + c = 0$ has two roots given by $x = (-b \pm \sqrt{b^2 - 4c})/2$). We need the root to be in the support (e, ∞) of X so that $g^{-1}(y) = e + \sqrt{e^2 + y}$.

The density of Y becomes

$$f_Y(y) = f(g^{-1}(y)) \left| \frac{g^{-1}(y)}{dy} \right| = \begin{cases} f(e + \sqrt{e^2 + y}) \frac{1}{2\sqrt{e^2+y}} & \text{if } -e < y, \\ 0 & \text{otherwise.} \end{cases}$$

9. Note that $1 - 1/\log X = F(X)$ is uniform$(0,1)$ by the probability integral transform. In turn, $U \equiv 1 - F(X) = 1/\log X$ is uniform$(0,1)$ so that $-\log U = \log\log X$ is exponential with mean 1.

10. Using Question 9, $Y_1 = \log\log X_1$ and $Y_2 = \log\log X_2$ are iid exponential(1) and we may write

$$\begin{aligned} P(\log X_1 < e, \log(\log X_1 \cdot \log X_2) > 1) &= P(Y_1 < 1, Y_1 + Y_2 > 1) \\ &= \int_0^1 \int_{1-y_1}^{\infty} e^{-y_1} e^{-y_2} dy_2 dy_1 \\ &= \int_0^1 e^{-y_1} \left[-e^{-y_2} \Big|_{1-y_2}^{\infty} \right] dy_1 \\ &= \int_0^1 e^{-1} dy_1 = e^{-1}. \end{aligned}$$

11. If we use indicator functions $I(X_i > e^3)$ (i.e., 1 if $X_i > e^3$ and 0, otherwise), then $I(X_i > e^3)$, $i = 1, \ldots, 6$, are iid Bernoulli random variables with $p \equiv P(X_i > e^3) = 1 - F(e^3) = 1/3$. It then holds, by construction, that

$$\begin{aligned} Y_1 &= I(X_1 > e^3) + I(X_2 > e^3), \\ Y_2 - Y_1 &= I(X_3 > e^3) + I(X_4 > e^3), \\ Y_3 - Y_2 &= I(X_5 > e^3) + I(X_6 > e^3), \end{aligned}$$

so that $Y_1, Y_2 - Y_1, Y_3 - Y_2$ are iid Binomial$(2, p)$ random variables.

12. By Question 12, $Z_1 \equiv Y_1$, $Z_2 \equiv Y_2 - Y_1, Z_3 \equiv Y_3 - Y_2$ are iid Binomial$(2, p)$ random variables so that

$$Cov(Y_1 + Y_2, Y_3) = Cov(2Z_1 + Z_2, Z_1 + Z_2 + Z_3) = 2Var(Z_1) + Var(Z_2) = 3Var(Z_1) = 4/3,$$

using $Var(Z_i) = 2p(1-p) = 2(1/3)(2/3) = 4/9$.

13. In the notation above, we may write $Y_1 = Z_1, Y_2 = Z_1 + Z_2, Y_3 = Z_1 + Z_2 + Z_3$ so that the joint support/range of (Y_1, Y_2, Y_3) are integer-valued triples (y_1, y_2, y_3) where $y_1, y_2 -$

$y_1, y_3 - y_2 \in \{0,1,2\}$, or equivalently, triples where $0 \le y_1 \le 2, 0 \le y_2 \le 4, 0 \le y_3 \le 6$ and $y_1 \le y_2 \le y_3$. For (y_1, y_2, y_3) in this range, we have

$$\begin{aligned}
& P((Y_1,Y_2,Y_3) = (y_1,y_2,y_3)) \\
= \;& P(Z_1 = y_1, Z_2 = y_2 - y_1, Z_3 = y_3 - y_2) \\
= \;& P(Z_1 = y_1)P(Z_2 = y_2 - y_1)P(Z_3 = y_3 - y_2) \\
= \;& \binom{2}{y_1} p^{y_1}(1-p)^{2-y_1} \binom{2}{y_2 - y_1} p^{y_2-y_1}(1-p)^{2-(y_2-y_1)} \binom{2}{y_3-y_2} p^{y_3-y_2}(1-p)^{2-(y_3-y_2)} \\
= \;& \binom{2}{y_1}\binom{2}{y_2-y_1}\binom{2}{y_3-y_2} p^{y_3}(1-p)^{6-y_3}.
\end{aligned}$$

The joint distribution is then

$$P((Y_1,Y_2,Y_3) = (y_1,y_2,y_3)) = \begin{cases} \binom{2}{y_1}\binom{2}{y_2-y_1}\binom{2}{y_3-y_2} p^{y_3}(1-p)^{6-y_3} & \text{if } y_1, y_2 - y_1, y_3 - y_2 \in \{0,1,2\}, \\ 0 & \text{otherwise,} \end{cases}$$

with $p = 1/3$.

14. With the above notation again that $Y_1 = Z_1, Y_2 = Z_1 + Z_2, Y_3 = Z_1 + Z_2 + Z_3$ for Z_1, Z_2, Z_3 as iid Binomial$(2,p)$ random variables, the conditional probability mass function of Y_1 given $Y_3 = 3$ can be written as

$$\begin{aligned}
P(Y_1 = y_1 | Y_3 = 3) = \frac{P(Y_1 = y_1, Y_3 = 3)}{P(Y_3 = 3)} &= \frac{P(Z_1 = y_1, Z_2 + Z_3 = 3 - y_1)}{P(Z_1 + Z_2 + Z_3 = 3)} \\
&= \frac{P(Z_1 = y_1)P(Z_2 + Z_3 = 3 - y_1)}{P(Z_1 + Z_2 + Z_3 = 3)} \\
&= \frac{\binom{2}{y_1} p^{y_1}(1-p)^{2-y_1}\binom{4}{3-y_1} p^{3-y_1}(1-p)^{4-(3-y_1)}}{\binom{6}{3} p^3 (1-p)^{6-3}} \\
&= \frac{\binom{2}{y_1}\binom{4}{3-y_1}}{\binom{6}{3}}
\end{aligned}$$

for $y_1 = 0, 1, 2$ (the marginal support of Y_1 too); above we use that Z_1 is Binomial$(2,p)$, $Z_2 + Z_3$ is Binomial$(4,p)$, and that $Z_1 + Z_2 + Z_3$ is Binomial$(6,p)$. For any other values of y_1, the conditional probability mass function is zero.

15. The joint distribution of (Y_1, Y_2, Y_3) is driven by the joint distribution of $Z_1 \equiv Y_1, Z_2 \equiv Y_2 - Y_1, Z_3 \equiv Y_3 - Y_2$ (iid Binomial$(2,p)$), for which $Y_3 = Z_1 + Z_2 + Z_3$ is sufficient by the factorization theorem (that is, the joint distribution $\prod_{i=1}^3 \binom{2}{z_i} p^{z_i}(1-p)^{2-z_i}$ of (Z_1, Z_2, Z_3) depends on p only through $Y_3 = Z_1 + Z_2 + Z_3$). Conditional on Y_3, the conditional probability of any event involving (Y_1, Y_2, Y_3) or (Z_1, Z_2, Z_3) cannot depend on p by sufficiency.

16. Note that $1/X_{(1)} \xrightarrow{p} e^{-1}$ as $n \to \infty$ is equivalent to $X_{(1)} \xrightarrow{p} e$. To show the latter by definition, pick $\epsilon > 0$ and write

$$\begin{aligned}
P(|X_{(1)} - e| \le \epsilon) = P(X_{(1)} \le e + \epsilon) = 1 - P(X_{(1)} > e + \epsilon) &= 1 - P(\text{all } X_i > e + \epsilon) \\
&= 1 - [1 - F(e + \epsilon)]^n \\
&= 1 - [\log(e + \epsilon)]^{-n} \to 0
\end{aligned}$$

as $n \to \infty$, using above that $X_{(1)} > e$ with probability 1 and $\log(e + \epsilon) > 1$. Hence, $X_{(1)} \xrightarrow{p} e$.

17. Note first that $P([\log X_{(n)}]/n \leq z) = 0$ holds for $z \leq 0$ and any n, as $X_{(n)} > e$ or $\log X_{(n)} > 0$ with probability 1. Consequently,

$$\lim_{n\to\infty} P(X_{(n)}/n \leq z) = 0$$

follows for any $z \leq 0$. For some given $z > 0$, we can write, as $n \to \infty$,

$$P([\log X_{(n)}]/n \leq z) = P(X_{(n)} \leq e^{zn}) = P(\text{all } X_i \leq e^{zn}) = [F(e^{zn})]^n = \left(1 - \frac{1}{nz}\right)^n \to e^{-1/z}.$$

Note that a function defined by $F_Z(z) \equiv e^{-1/z}$ for $z > 0$, and $F_Z(z) \equiv 0$ for $z \leq 0$, is a valid and continuous cdf, defining a positive random variable Z. Hence, for any real z,

$$\lim_{n\to\infty} P(X_{(n)}/n \leq z) = F_Z(z)$$

so that $n^{-1} \log X_{(n)} \xrightarrow{d} Z$ as $n \to \infty$ for Z with cdf F_Z.

18. We can write

$$[X_{(1)} + \log X_{(n)}]/\sum_{i=1}^n T_i = \left[\frac{1}{n}X_{(1)} + \frac{1}{n}\log X_{(n)}\right] / \left(\frac{1}{n}\sum_{i=1}^n T_i\right).$$

By Questions 16-17, as $n \to \infty$, we have

$$\frac{1}{n}X_{(1)} \xrightarrow{p} 0, \quad \frac{1}{n}\log X_{(n)} \xrightarrow{d} Z$$

while

$$\frac{1}{n}\sum_{i=1}^n T_i \xrightarrow{p} \mathrm{E}T_1 = \frac{2}{3}$$

by the WLLN. By Slutsky's theorem, it follows that

$$[X_{(1)} + \log X_{(n)}]/\sum_{i=1}^n T_i \xrightarrow{d} \frac{3}{2}Z$$

as $n \to \infty$.

19. By the CLT, as the T_i's are iid with mean $\mathrm{E}T_1 = 2/3$ and variance $Var(T_1) = \mathrm{E}T_1^2 - [\mathrm{E}T_1]^2 = 1/2 - (2/3)^2 = 1/18$, we have

$$\sqrt{n}(\bar{T}_n - \mathrm{E}T_1) \xrightarrow{d} N(0, 1/18)$$

as $n \to \infty$, where $\bar{T}_n \equiv \sum_{i=1}^n T_i/n$. Defining a function $g(x) = x^2$, the delta method then gives

$$n^{1/2}\left[\left(\sum_{i=1}^n T_i/n\right)^2 - (\mathrm{E}T_1)^2\right] = n^{1/2}[g(\bar{T}_n) - g(\mathrm{E}T_1)] \xrightarrow{d} N(0, \sigma^2)$$

as $n \to \infty$, with

$$\sigma^2 \equiv [g'(\mathrm{E}T_1)]^2 \cdot 1/18 = (4/3)^2 \cdot 1/18.$$

20. By the CLT, as the T_i^2's are iid with mean $\mathrm{E}T_1^2 = 1/2$ and variance $Var(T_1^2) = \mathrm{E}T_1^4 - [\mathrm{E}T_1^2]^2 = 1/3 - (1/2)^2 = 1/12$, we have

$$n^{-1/2}\left[\sum_{i=1}^n T_i^2 - n(\mathrm{E}T_1^2)\right] = n^{1/2}\left[\frac{1}{n}\sum_{i=1}^n T_i^2 - (\mathrm{E}T_1^2)\right] \xrightarrow{d} N(0, 1/12)$$

as $n \to \infty$.

Part I

An experiment is conducted to estimate a parameter θ. Independent measurements Y's of θ can be made with two instruments, both of which measure with normal errors. For $i = 1, 2$, instrument i produces independent errors with a $N(0, \sigma_i^2)$ distribution, and the two error variances σ_1^2 and σ_2^2 are known. Thus, $Y = \theta + \epsilon$ where the distribution of ϵ is either $N(0, \sigma_1^2)$ or $N(0, \sigma_2^2)$. When a measurement y is made, a record is kept of the instrument used so that after n measurements, the data are

$$(a_1, y_1), \ldots, (a_n, y_n),$$

where $a_m = i$ if y_m is obtained using instrument i, $m = 1, \ldots, n$, $i = 1, 2$.

The choice between instruments is made independently for each observation in such a way that

$$P(a_m = 1) = P(a_m = 2) = \frac{1}{2},$$

for $m = 1, \ldots, n$. Let $\boldsymbol{x}$ denote the data set available, that is $\boldsymbol{x} = (x_1, \ldots, x_n)$ with $x_m = (a_m, y_m), m = 1, \ldots, n$.

1. Show that the maximum likelihood estimate of θ is

$$\hat{\theta}_n = \frac{\sum_{i=1}^n \frac{y_i}{\sigma_{a_i}^2}}{\sum_{i=1}^n \frac{1}{\sigma_{a_i}^2}}.$$

2. Derive the expected Fisher information for θ in terms of $n, \sigma_i^2, i = 1, 2$.

3. Show $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N\left(0, \frac{2\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right)$ as $n \to \infty$.

4. Find the UMVUE of θ.

For **Problems 5-10**, let $\lambda = \theta^2$.

5. Find the MLE of λ, call it $\hat{\lambda}_n$.

6. Using the result of **Problem 3**, find the limiting distribution of $\sqrt{n}(\hat{\lambda}_n - \lambda)$ as $n \to \infty$.

7. Let λ_0 be a known positive value. Using the result from **Problem 6**, derive an asymptotic size α test for testing $H_0 : \lambda = \lambda_0$ against $H_1 : \lambda \neq \lambda_0$.

8. Using the result from **Problem 6** or otherwise, show that $\hat{\lambda}_n$ is a consistent estimator of λ.

9. Find an asymptotic pivotal quantity $Q((X_1, \ldots, X_n), \lambda)$.

10. Construct an asymptotic confidence interval for λ with confidence coefficient $(1 - \alpha)$ using the pivotal quantity from **Problem 9**.

For **Problems 11-12** assume that the prior distribution of θ is $N(\mu, \tau^2)$.

11. Find the Bayes estimator of θ under the squared error loss.

12. Let θ_0 be a fixed number. Find the Bayes test for $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$ based on the posterior distribution obtained in **Problem 11** using the rejection region

$$\left\{\boldsymbol{x} : P(\theta \notin \Theta_0 | \boldsymbol{x}) > \frac{1}{2}\right\},$$

where Θ_0 is the null space.

Part II

Suppose that $Z_1, Z_2, \ldots, Z_n$ are iid random variables with $Z_1 \sim$ Uniform (0, ξ), $\xi > 0$. Let $\hat{\xi}_n = \max_{1 \leq i \leq n} Z_i$.

13. Show that $\hat{\xi}_n$ is the MLE of ξ.

14. Is $\hat{\xi}_n$ an unbiased estimator of ξ?

15. Show that $\hat{\xi}_n$ converges to ξ more quickly than usual: precisely, prove that

$$\frac{n(\xi - \hat{\xi}_n)}{\xi} \xrightarrow{d} W,$$

as $n \to \infty$, where W has an exponential distribution with parameter 1.

16. Derive the likelihood ratio test of $H_0 : \xi = 1$ versus $H_1 : \xi \neq 1$ which has significance level $\alpha = 0.05$.

1. The joint pdf of $\boldsymbol{X}$ is

$$\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_{a_i}^2}} \exp\Big(- \frac{1}{2\sigma_{a_i}^2}(y_i - \theta)^2\Big).$$

The loglikelihood is

$$\ell(\theta|\boldsymbol{x}) = -\sum_{i=1}^{n} \frac{(y_i - \theta)^2}{2\sigma_{a_i}^2} - \sum_{i=1}^{n} \log(\sigma_{a_i}\sqrt{(2\pi)}).$$

Differentiating $\ell(\theta|\boldsymbol{x})$ with respect to θ we have

$$\ell'(\theta|\boldsymbol{x}) = \sum_{i=1}^{n} \frac{(y_i - \theta)}{\sigma_{a_i}^2}.$$

Solving $\ell'(\theta|\boldsymbol{x}) = 0$ leads to

$$\hat{\theta}_n = \frac{\sum_{i=1}^{n} \frac{y_i}{\sigma_{a_i}^2}}{\sum_{i=1}^{n} \frac{1}{\sigma_{a_i}^2}}.$$

Since $\ell''(\theta|\boldsymbol{x}) = -\sum_{i=1}^{n} 1/\sigma_{a_i}^2 < 0$ for all θ, $\hat{\theta}_n$ is the MLE of θ.

2. We have

$$\begin{aligned} I_n = -E(\ell''(\theta|\boldsymbol{X})) &= \sum_{i=1}^{n} E\Big(\frac{1}{\sigma_{a_i}^2}\Big) \\ &= \frac{1}{2}\sum_{i=1}^{n} \Big(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\Big) \\ &= \frac{n(\sigma_1^2 + \sigma_2^2)}{2\sigma_1^2\sigma_2^2}. \end{aligned}$$

3. By the asymptotic normality of MLE,

$$\sqrt{I_n}(\hat{\theta}_n - \theta) \xrightarrow{d} N(0, 1).$$

Thus,

$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N\Big(0, \frac{2\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2}\Big).$$

4. Since $E(\hat{\theta}_n) = \theta$ and it is complete and sufficient for θ, $\hat{\theta}_n$ is UMVUE of θ.

5. Since $\hat{\theta}_n$ is the MLE of θ, by the invariance property of MLE, the MLE of λ is $\hat{\lambda}_n = \hat{\theta}_n^2$.

6. Note that $\frac{d\lambda(\theta)}{d\theta} = 2\theta$. By Delta method,

$$\sqrt{n}(\hat{\lambda}_n - \lambda) \xrightarrow{d} N\Big(0, \frac{8\lambda\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2}\Big).$$

7. Under $H_0 : \lambda = \lambda_0$,

$$\frac{\sqrt{n(\sigma_1^2 + \sigma_2^2)}(\hat{\lambda}_n - \lambda_0)}{2\sqrt{2}\sigma_1\sigma_2\sqrt{\lambda_0}} \xrightarrow{d} N(0, 1).$$

Thus, an asymptotic size α test for testing $H_0 : \lambda = \lambda_0$ against $H_1 : \lambda \neq \lambda_0$ is

$$\text{Reject } H_0 \text{ if } \frac{\sqrt{n(\sigma_1^2 + \sigma_2^2)}(|\hat{\lambda}_n - \lambda_0)|}{2\sqrt{2}\sigma_1\sigma_2\sqrt{\lambda_0}} > z_{\alpha/2},$$

where $z_{\alpha/2}$ is the $(1 - \alpha/2)$th quantile of $N(0, 1)$.

8. Since

$$(\hat{\theta}_n - \theta) = \frac{1}{\sqrt{I_n}}\sqrt{I_n}(\hat{\theta}_n - \theta),$$

and $\frac{1}{\sqrt{I_n}} \to 0$, by Slutsky's theorem $(\hat{\theta}_n - \theta) \xrightarrow{d} 0$, which is equivalent to $(\hat{\theta}_n - \theta) \xrightarrow{P} 0$. Since x^2 is a continuous function of x, by the continuous mapping theorem

$$\hat{\lambda}_n \xrightarrow{P} \lambda.$$

9. Since

$$\frac{\sqrt{n}(\hat{\lambda}_n - \lambda)}{\sqrt{\lambda}} \xrightarrow{d} N\Big(0, \frac{8\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2}\Big),$$

and $\hat{\lambda}_n \xrightarrow{P} \lambda$, by Slutsky's theorem

$$\frac{\sqrt{n}(\hat{\lambda}_n - \lambda)}{\sqrt{\hat{\lambda}_n}} \xrightarrow{d} N\Big(0, \frac{8\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2}\Big),$$

$$Q((X_1, \ldots, X_n), \lambda) = \frac{\sqrt{n}(\hat{\lambda}_n - \lambda)}{\sqrt{\hat{\lambda}_n}}$$

is an asymptotic pivotal quantity. (THE ANSWER CAN BE DIFFERENT.)

10. From the above we have

$$\Big[\hat{\lambda}_n - z_{\alpha/2}\frac{2\sqrt{2}\sigma_1\sigma_2}{\sqrt{\sigma_1^2 + \sigma_2^2}}\sqrt{\frac{\hat{\lambda}_n}{n}}, \hat{\lambda}_n + z_{\alpha/2}\frac{2\sqrt{2}\sigma_1\sigma_2}{\sqrt{\sigma_1^2 + \sigma_2^2}}\sqrt{\frac{\hat{\lambda}_n}{n}}\Big]$$

as an asymptotic $(1 - \alpha)$ confidence interval for λ.

11. The posterior density of θ is

$$\begin{aligned}\pi(\theta|\mathbf{x}) &\propto \pi(\theta)f(\boldsymbol{x}|\theta)\\ &\propto \exp\Big(-\frac{(\theta-\mu)^2}{2\tau^2}\Big)\prod_{i=1}^n \exp\Big(-\frac{1}{2\sigma_{a_i}^2}(y_i-\theta)^2\Big)\\ &\propto \exp\Big(-\frac{\theta^2}{2}\Big\{\sum_{i=1}^n\frac{1}{\sigma_{a_i}^2}+\frac{1}{\tau^2}\Big\}+\theta\Big\{\frac{\mu}{\tau^2}+\sum_{i=1}^n\frac{y_i}{\sigma_{a_i}^2}\Big\}\Big),\end{aligned}$$

which is a normal density with mean

$$\frac{\frac{\mu}{\tau^2}+\sum_{i=1}^n\frac{y_i}{\sigma_{a_i}^2}}{\frac{1}{\tau^2}+\sum_{i=1}^n\frac{1}{\sigma_{a_i}^2}}$$

and variance

$$\frac{1}{\frac{1}{\tau^2}+\sum_{i=1}^n\frac{1}{\sigma_{a_i}^2}}.$$

Thus, the Bayes estimator of θ under the squared error loss is

$$E(\theta|\boldsymbol{x})=\frac{\frac{\mu}{\tau^2}+\sum_{i=1}^n\frac{y_i}{\sigma_{a_i}^2}}{\frac{1}{\tau^2}+\sum_{i=1}^n\frac{1}{\sigma_{a_i}^2}}.$$

12. Now

$$P(\theta\notin\Theta_0|\boldsymbol{x})\geq 1/2 \Leftrightarrow P(\theta>\theta_0|\boldsymbol{x})\geq 1/2 \Leftrightarrow \theta_0\leq E(\theta|\boldsymbol{x}).$$

Thus, a Bayes test for testing $H_0:\theta\leq\theta_0$ vs $H_1:\theta>\theta_0$ is given by

$$\varphi(\boldsymbol{x})=\begin{cases}1 & \text{if } \theta_0\leq\frac{\frac{\mu}{\tau^2}+\sum_{i=1}^n\frac{y_i}{\sigma_{a_i}^2}}{\frac{1}{\tau^2}+\sum_{i=1}^n\frac{1}{\sigma_{a_i}^2}}\\ 0 & \text{otherwise}\end{cases}$$

13. The likelihood function is

$$L(\theta|\boldsymbol{z})=\frac{1}{\xi^n}I_{[z_{(n)},\infty)}(\xi),$$

which is decreasing function. So $\hat{\xi}_n$ is the MLE of ξ.

14. The cdf of $\hat{\xi}_n$ is

$$P(\hat{\xi}_n\leq w)=\Big(\frac{w}{\xi}\Big)^n,$$

implying its pdf is

$$f(w)=n\Big(\frac{w^{n-1}}{\xi^n}\Big),\ 0<w<\xi.$$

Thus,

$$E(\hat{\xi}_n) = \int_0^{\xi} n\Big(\frac{w^n}{\xi^n}\Big)dw = \frac{n}{n+1}\eta.$$

So, $\hat{\xi}_n$ is a biased estimator.

15. Now, for $w \geq 0$,

$$\begin{aligned} & P\Big(\frac{n(\xi - \hat{\xi}_n)}{\eta} \leq w\Big) \\ &= P(\hat{\xi}_n \geq \xi(1 - w/n)) \\ &= 1 - P(\hat{\xi}_n < \xi(1 - w/n)) \\ &= 1 - \Big(1 - \frac{w}{n}\Big)^n \to 1 - \exp(-w), \end{aligned}$$

as $n \to \infty$. Trivially, $P(\hat{\xi}_n > \eta(1 - w/n)) = 0$ if $w < 0$. Thus, $\hat{\xi}_n \xrightarrow{d} W$.

16. The likelihood ratio

$$\begin{aligned} \Lambda &= \frac{L(1|\boldsymbol{z})}{L(\hat{\eta}_n|\boldsymbol{z})} \\ &= \begin{cases} \frac{1}{\left(\frac{1}{\hat{\xi}_n}\right)^n} & \text{if } \hat{\xi}_n \leq 1 \\ 0 & \text{if } \hat{\xi}_n > 1 \end{cases} \\ &= \begin{cases} \hat{\xi}_n^n & \text{if } \hat{\xi}_n \leq 1 \\ 0 & \text{if } \hat{\xi}_n > 1. \end{cases} \end{aligned}$$

So reject H_0 if $\Lambda < k$ for some $k \in (0, 1)$ is equivalent to reject H_0 if either $\hat{\xi}_n \leq k^{1/n}$ or > 1.

For $\alpha = 0.05$, we choose k so that

$$P_{H_0}(\hat{\xi}_n \leq k^{1/n}) = k = .05.$$

Part I

Suppose $\Omega = [-1, 1]$, $\mathcal{A}$ is the Borel $\sigma-$field of Ω. Call a set $A \subseteq \Omega$ *symmetric* if $-y \in A$ whenever $y \in A$. Let $\mathcal{P} = \{P_x : x \in [0, 1]\}$ denote a collection of probability measures on $(\Omega, \mathcal{A})$ where $P_0(\{0\}) = 1$ and for $x \in (0, 1]$

$$P_x(\{x\}) = P_x(\{-x\}) = \frac{1}{2}.$$

Let $\mathcal{B} = \{A \in \mathcal{A} \ : \ A \text{ is symmetric}\}$ be the collection of symmetric subsets of Ω.

1. Show that $\mathcal{B}$ is a $\sigma-$field.
2. For any $\mathcal{A}-$measurable function $f : \Omega \to \mathbb{R}$, find a function f^* that serves as a conditional expectation of f given $\mathcal{B}$ under P_x for any $x \in [0, 1]$.

Suppose D_0 is a non-Borel subset of $[0, 1]$ and let

$$D = \{x \in \Omega : x \in D_0 \text{ or } -x \in D_0\}$$

and

$$\mathcal{C} = \{A \in \mathcal{A} : A \cap D \text{ is symmetric}\}$$

3. Show that $D \notin \mathcal{A}$.
4. Show that $\mathcal{C}$ is a $\sigma-$field.
5. Show that $\mathcal{B} \subseteq \mathcal{C}$.
6. Show that $\mathcal{B} \neq \mathcal{C}$.
7. Fix $a > 0$ and $a \notin D$, find the conditional expectation of an $\mathcal{A}-$measurable function g given $\mathcal{C}$ under P_a.

Part II

8. State Kolmogorov's three-series theorem.
9. Find all $a > 0$ for which $\sum_{n=1}^{\infty} X_n$ converges almost surely where $\{X_n\}$ are independent and X_n follows $\mathrm{U}(-1/n^a, 1/n^a)$ for each $n \geq 1$.

Part III

10. State the Lindeberg condition for the central limit theorem.

Suppose $\{X_j\}_{j\geq 1}$ are independent random variables such that X_j follows $\mathrm{U}(-j, j)$. Let $X_{nj} = X_j/n^a$, for $1 \leq j \leq n$, and $n \geq 1$.

11. Show that the Lindeberg condition holds for $\{X_{nj}\}$ for all $a > 0$.

12. Suppose $a = 3/2$. Show that $\sum_{j=1}^{n} X_{nj} \to N(0, \sigma^2)$ in distribution for some $\sigma^2 > 0$ and find the value of σ^2.

Part I

1. Clearly, $\Omega = [-1,1] \in \mathcal{B}$ and B is symmetric if and only if B^c is symmetric, so $\mathcal{B}$ is closed under complement. If $B_n \in \mathcal{B}$ for all n, then $x \in \cup B_n \iff x \in B_j$ for some j $\iff -x \in B_j \iff -x \in \cup B_n$. Thus $\mathcal{B}$ is closed under countable union. Hence $\mathcal{B}$ is a $\sigma-$field.

2. Suppose $f^* : \Omega \to \mathbb{R}$ defined as

$$f^*(\omega) = \frac{f(\omega) + f(-\omega)}{2}, \quad \omega \in \Omega.$$

Then, note that $f^*(\omega) = f^*(-\omega)$, $\forall \omega \in \Omega$. for any Borel set $R \subseteq \mathbb{R}$, $\omega \in (f^*)^{-1}(R)$ iff $f^*(\omega) \in R$ iff $f^*(-\omega) \in R$ iff $-\omega \in (f^*)^{-1}(R)$. Thus $(f^*)^{-1}(R) \in \mathcal{B}$ whence f^* is $\mathcal{B}$ measurable.

Next, we need to verify that for any $B \in \mathcal{B}$ and any $x \in [0,1]$

$$\int_B f dP_x = \int_B f^* dP_x.$$

This is obvious, since both side equals to $(f(x) + f(-x))/2$ if $x \in B$ and zero if $x \notin B$.

Suppose D_0 is a non-Borel subset of $[0,1]$ and let

$$D = \{x \in \Omega : x \in D_0 \text{ or } -x \in D_0\}$$

and

$$\mathcal{C} = \{A \in \mathcal{A} : A \cap D \text{ is symmetric}\}$$

3. If D were in $\mathcal{A}$, then so would $D \cap [0,1]$ which is D_0, giving a contradiction.

4. Since D is symmetric by construction (albeit non Borel), $\Omega \in \mathcal{C}$. Note that D^c is also symmetric. If $A \in \mathcal{C}$, then $A^c \cap D = (A \cup D^c)^c$ is also symmetric. If $A_n \in \mathcal{C}$ for all n, then $x \in (\cup A_n) \cap D \equiv \cup(A_n \cap D)$ iff $x \in A_j \cap D$ for some j iff $-x \in A_j \cap D$ (since $A_j \in \mathcal{C}$), iff $-x \in \cup(A_n \cap D) \equiv (\cup A_n) \cap D$. Thus $(\cup A_n) \cap D$ is symmetric and hence $\mathcal{C}$ is closed under countable union.

5. If $B \in \mathcal{B}$, then B is symmetric. Hence, $B \cap D$ is symmetric. Hence $B \in \mathcal{C}$.

6. Pick $z \in [0,1] - D_0$. Such a z exists because $D \neq [0,1]$. Then $z \in \mathcal{C}$ since $\{z\} \cap D = \phi$ is trivially symmetric. But $\{z\}$ is not in $\mathcal{B}$.

7. Pick a real number c such that $c \neq g(a), g(-a)$. Let g^* be the function:

$$g^*(x) = \begin{cases} c & \text{if} x \neq a, -a \\ g(x) & \text{if} x = a, -a \end{cases}$$

(note that g^* is not necessarily symmetric hence not necessarily $\mathcal{B}$-measurable). However,

- $(g^*)^{-1}(\{c\}) = [0,1] - \{a, -a\}$ is symmetric, hence $\in \mathcal{B} \subset \mathcal{C}$.
- If $g(a) = g(-a)$, then $(g^*)^{-1}(\{\pm a\}) = \{-a, a\}$ is symmetric, and hence $\in \mathcal{B} \subset \mathcal{C}$.
- If $g(a) \neq g(-a)$, $(g^*)^{-1}(\{\pm a\}) = \{\pm a\} \in \mathcal{C}$ since $\{\pm a\} \cap D = \phi$ is vacuously symmetric.

Thus g^* is $\mathcal{C}$-measurable.

Next, for any $C \in \mathcal{C}$,:

- if neither a nor $-a \in \mathcal{C}$, then $\int_C g^* dP_a = 0 = \int_C g dP_a$,
- else if $\{-a, a\} \in \mathcal{C}$, then $\int_C g^* dP_a = \frac{1}{2} g^*(a) + \frac{1}{2} g^*(-a) = \frac{1}{2} g(a) + \frac{1}{2} g(-a) = \int_C g dP_a$,
- else if $a \in \mathcal{C}$ but $-a \notin \mathcal{C}$, then $\int_C g^* dP_a = \frac{1}{2} g^*(a) = \frac{1}{2} g(a) = \int_C g dP_a$,
- else $a \notin \mathcal{C}$ but $-a \in \mathcal{C}$, then $\int_C g^* dP_a = \frac{1}{2} g^*(-a) = \frac{1}{2} g(-a) = \int_C g dP_a$,

Thus g^* serves as a conditional expectation of g given $\mathcal{C}$. In fact, any functions g^* that is symmetric at every x except $\pm a$, and is equal to g at $\pm a$ will work.

Part II

8. See Athreya and Lahiri.

9. Using Kolmogorov's three series-theorem, $\sum_n X_n$ converges almost surely if and only if

a. $\sum_n P(|X_n| \geq 1)$ converges.
b. $\sum_n E|Y_n|$ converges where $Y_n = X_n I(|X_n| \leq 1)$.
c. $\sum_n \mathrm{Var}(Y_n)$ converges.

Clearly, a cannot be negative since otherwise $\sum_n P(|X_n| > 1) = \sum_n (1 - n^a) = \infty$ and condition (a) will not hold. Note that, for $a \geq 0$, $Y_n = X_n$ almost surely. Hence

$$E|Y_n| = E|X_n| = \frac{n^a}{2} \int_{-1/n^a}^{1/n^a} |x| dx = \frac{1}{2n^a}.$$

Thus condition (b) holds iff $a > 1$. Similarly,

$$\mathrm{Var}(Y_n) = \mathrm{Var}(X_n) = \frac{(1/n^a + 1/n^a)^2}{12} = \frac{1}{3n^{2a}},$$

and condition (c) holds iff $a > 1/2$.

Combining conditions (a), (b), and (c), $\sum_n X_n$ converges a.s. iff $a > 1$.

Part III

10. See Athreya and Lahiri Definition 11.1.2.

Suppose $\{X_j\}_{j\geq 1}$ are independent random variables such that X_j follows $\mathrm{U}(-j, j)$. Let $X_{nj} = X_j/n^a$, for $1 \leq j \leq n$, and $n \geq 1$.

11. Note that, $\mathrm{E}X_{nj} = 0$ for all $n \geq 1$ and $1 \leq j \leq n$. Next,

$$\sigma_{nj}^2 = \mathrm{Var}X_{nj} = \mathrm{E}X_j^2/n^{2a} = \frac{(2j)^2}{12n^{2a}} = \frac{j^2}{3n^{2a}}.$$

Thus,

$$s_n^2 = \sum_{j=1}^{n} \sigma_{nj}^2 = \frac{1}{3n^{2a}} \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{18n^{2a}}$$

Note that, $s_n \sim n^{3/2-a}$. To verify Lindebeg condition, let $\epsilon > 0$ and $N > 0$ such that $\epsilon s_n n^a > n$ for all $n \geq N$. Hence, for all $n \geq N$, and $1 \leq j \leq n$,

$$\mathrm{E}X_{nj}^2 I(|X_{nj} > \epsilon) = \frac{1}{n^{2a}} \mathrm{E}X_j^2 I(|X_j| > \epsilon s_n n^a) = 0$$

since X_j follows $\mathrm{U}(-j, j)$. Thus, for all $n \geq N$,

$$\frac{1}{s_n^2} \sum_{j=1}^{n} \mathrm{E}X_{nj}^2 I(|X_{nj} > \epsilon s_n) = 0$$

so that the Lindeberg condition holds.

12. Using the Lindeberg CLT and the previous result,

$$\sum_{j=1}^{n} X_{nj}/s_n^2 \to^d N(0, 1).$$

Now, as was seen in the previous problem, when $a = 3/2$, $s_n^2 = n(n+1)(2n+1)/(18n^{2a}) \to 2/18 = 1/9$. Hence, $\sum_{j=1}^{n} X_{nj} \to^d N(0, 1/9)$.