Convergence concepts

Introduction (cont'd)

We will consider two types of ways in which sequences of r.v.s can "converge"

$X_1, X_2, \dots, X_n$ i.i.d $\quad X_n \xrightarrow{d} X \to$ random variable

1. Convergence in distribution
2. Convergence in probability $\longrightarrow$ $X_n \xrightarrow{P} X \equiv c \to$ Constant
   (a) mostly, convergence in probability to a constant (a degenerate r.v.)

In our text, there are other types of convergence ("convergence almost surely") **which we will not discuss**; these are topics discussed in STAT 6420 in more detail (the textbook attempts these but this only muddies the water).

Arguably, in statistics, convergence in distribution is most common

- conveniently, this type of convergence only requires working with probability distributions directly
- convergence in probability to a constant is the same as convergence in distribution to a constant (as we will see)

Set up: Let $Y_1, Y_2, \dots, Y_n, \dots$ be a sequence of r.v.s

- the distribution of Y_n can change with n
- One common situation
  1. $X_1, X_2, \dots$ is an iid sequence
  2. $Y_n = g(X_1, \dots, X_n)$ for each n
  3. functions $g()$ include: mean, sample variance, minimum, maximum

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Convergence in probability

Definition: Y_n **converges in probability** to Y, denoted as $Y_n \xrightarrow{p} Y$ as $n \to \infty$, if

$$\text{for any given/fixed } \epsilon > 0, \quad \lim_{n\to\infty} P(\underbrace{|Y_n - Y| \geq \epsilon}_{A}) = 0.$$

Note: Often the limit Y is a constant (e.g., $Y = a$ or $P(Y = a) = 1$)

$$\overbrace{\{\omega : |Y_n(\omega) - Y(\omega)| \geq \epsilon\}}^{A} = 1$$

Interpretation: $\equiv$

- $\{|Y_n - Y| \geq \epsilon\}$ is an event & the definition requires that the probability of this event (for a fixed $\epsilon > 0$) tends to zero as n gets large

$$\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$$

- equivalently, $\{|Y_n - Y| < \epsilon\}$ is an event & the definition requires that the probability of this event (for any given $\epsilon > 0$) tends to one as n gets large

$$\lim_{n\to\infty} \mathbb{P}(|Y_n - Y| < \varepsilon) = 1$$

- with arbitrarily high probability, Y_n will be close to Y (within ϵ for any given ϵ) for large n

$A \subseteq B$ If $\forall \omega \in A \Longrightarrow \omega \in B$

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Convergence in probability: examples

Example 1: Suppose $Z \sim N(0,1)$, $X \sim N(1,1)$ and, independent of Z, X, let $B_n \sim$Bernoulli$(\overset{p}{1-1/n})$ for each $n \geq 1$. Define

$$B_n = \begin{cases} 1 & \text{with prob } 1-\frac{1}{n} \quad p \\ 0 & \text{with prob } 1-\left[1-\frac{1}{n}\right] = \frac{1}{n} \quad 1-p \end{cases}$$

$$Y_n = B_n Z + (1-B_n)X.$$

Show that $\boxed{Y_n \xrightarrow{p} Z.}$

Fix/pick $\varepsilon > 0$

$$\longrightarrow \mathbb{P}(|Y_n - Z| > \varepsilon) \overset{\text{def of } Y_n}{=} \mathbb{P}(|B_n Z + (1-B_n)X - Z| > \varepsilon)$$

$$= \mathbb{P}(|(B_n - 1)Z + (1-B_n)X| > \varepsilon)$$

$$= \mathbb{P}(\underbrace{|(1-B_n)(X-Z)| > \varepsilon}_{\text{this event implies } B_n \neq 1})$$

Question:

If $\boxed{|(1-B_n)(X-Z)| > \varepsilon}$ A

$\Longrightarrow B_n \neq 1$ B

$A \subseteq B$

$\Rightarrow \boxed{\mathbb{P}(A) \leq \mathbb{P}(B)}$

$$\leq \mathbb{P}(B_n \neq 1) = \mathbb{P}(B_n = 0) = \frac{1}{n}$$

$$\lim_{n\to\infty} \mathbb{P}(|Y_n - Z| > \varepsilon) \leq \lim_{n\to\infty} \frac{1}{n} = 0 \longrightarrow$$

$$\Rightarrow Y_n \xrightarrow{P} Z$$

Example 2: Suppose $U_n \sim$Uniform$(0, 1/n)$. Show $U_n \xrightarrow{p} 0$.

Fix/pick $\varepsilon > 0$,

$$\mathbb{P}(|U_n - 0| \geq \varepsilon) = \mathbb{P}(|U_n| \geq \varepsilon) = \mathbb{P}(U_n \geq \varepsilon) \overset{U_n \sim Uni(0,\frac{1}{n})}{=\!=}$$

$$U_n \sim Uni(0, \tfrac{1}{n}) \Rightarrow f_{U_n}(u) = \frac{1}{\frac{1}{n} - 0} = n$$

$$= \int_{\varepsilon}^{\frac{1}{n}} n \, du$$

$N \equiv N(\varepsilon)$

$$= n\left(\frac{1}{n} - \varepsilon\right) = \begin{cases} 0 & \varepsilon \geq \frac{1}{n} \\ n\left(\frac{1}{n} - \varepsilon\right) & \varepsilon < \frac{1}{n} \end{cases}$$

$$\mathbb{P}(X > a) = \int_a^{\infty} f_X(x)\,dx$$

As $n \to \infty$, $\frac{1}{n} \to 0$, $N \equiv N(\varepsilon)$

$$\varepsilon = \frac{1}{n} \Rightarrow \int_{\frac{1}{n}}^{\frac{1}{n}} n\,du = 0 \checkmark$$

$$\forall n \geq N \lim_{n\to\infty} \mathbb{P}(|U_n - 0| \geq \varepsilon) = 0$$

$\frac{1}{n}$ ε

$$\varepsilon > \frac{1}{n} \Rightarrow \int_{\varepsilon}^{1/n} f_{U_n}(u)du = -\int_{\frac{1}{n}}^{\varepsilon} f_{U_n}(u)du = -\underbrace{\int_{1/n}^{1/n} n\,du}_{=0} - \int_{1/n}^{\varepsilon} \cancel{f}\overset{0}{} du$$

$\lim_{n\to\infty} \mathbb{P}(|U_n| > \varepsilon) = 0$ for $\varepsilon \geq 1/n$ $\quad U_n \sim \mathrm{Uni}(0, \frac{1}{n})$

$\mathbb{E}(U_n) = \frac{\frac{1}{n} - 0}{2} = \frac{1}{2n}$

$\mathbb{P}(|U_n| \geq \varepsilon) \leq \frac{\mathbb{E}|U_n|}{\varepsilon}$ $\quad \frac{\mathbb{E}(U_n)}{\varepsilon} = \frac{1}{2n}\frac{1}{\varepsilon} \to 0$ as $n \to \infty$

$\Rightarrow U_n \xrightarrow{p} 0$.

Convergence concepts

Convergence in probability: weak law of large numbers

Theorem: Weak Law of Large Numbers (WLLN). Suppose $X_1, X_2, \ldots$ are iid having $\mathrm{E}X_1 = \mu$ and $\mathrm{Var}(X_1) = \sigma^2 < \infty$. Let $Y_n = \bar{X}_n = \sum_{i=1}^n X_i/n$. Then

$$Y_n = \bar{X}_n \xrightarrow{p} \mu \quad \text{as } n \to \infty.$$

Proof: Pick/fix $\epsilon > 0$. Then,

$$P(|Y_n - \mu| \geq \epsilon) = P(|\bar{X}_n - \mu| \geq \epsilon)$$

Examples:

1. $X_1, X_2, \ldots$ iid Bernoulli(p):

2. Let $X_1, X_2, \ldots$ iid with $\mathrm{E}X_1^4 < \infty$. Define $W_i = X_i^2$, $i \geq 1$