fixed effects: seen previously

random effect: new concept
with goal of broadening

11. Linear Mixed-Effects Models

Scope of inference

by accounting for random effects, we increase Var(y)
implying that we will need stronger signals with respect
to trt differences before concluding there are
indeed treatment differences.

Motivation

In a linear model we distinguish between two types of effects:

fixed effects vs. **random** effects

Which effect to chose depends on

1. the context of the data,
2. the research questions of interest, and
3. how the data are collected.

Motivation

Fixed Effects

Data have been gathered from all the levels of the factor that are of interest. Number of levels is typically small.

↳ typically only these levels are of interest

1. drug with four different levels of concentration of an active ingredient.
2. sex
3. age grouped into different categories
4. any character variables

Motivation

Random Effects

Factor variable has many possible levels and levels typically represent a larger population of interest. However, observing all levels is not feasible and we only have a random sample of levels in the data.

1. Assessing effectiveness of a new curriculum after statewide implementation: school effect
2. operator or machine effect in Gauge R&R studies
3. hospital effect

We are interested in whether the factor has a significant effect in explaining the response, but only in a general way.

Motivation

treatment – fixed effect

Example

Two surgical procedures are being compared. Patients are randomized to treatment. Five different surgical teams are used. To prevent possible confounding of treatment and surgical team, each team is trained in both procedures, and each team performs equal numbers of surgery of each of the two types. The purpose of the experiment is to compare the procedures with the intent to generalize to other surgical teams.

Thus surgical team should be considered as a random factor, not a fixed factor.

Motivation

Some general remarks:

- Data analysis differs depending on type of effect (fixed or random); hence misspecification of the type of effect can lead to incorrect conclusions.
- Random factor analysis is usually used if there is reason to believe that the levels observed in the experiment could reasonably be a random sample of all levels.
- An interaction term involving both a fixed and a random factor should be considered a random factor.
- A factor that is nested in a random factor should be considered random.

If a model contains both fixed and random effects, we call it a mixed effects model.

The Linear Mixed-Effects Model

- $\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boxed{\boldsymbol{Z}\boldsymbol{u}} + \boldsymbol{e}$
- $\boldsymbol{X}$ is an $n \times p$ matrix of known constants
- $\boldsymbol{\beta} \in \mathbb{R}^p$ is an unknown parameter vector
- $\boldsymbol{Z}$ is an $n \times q$ matrix of known constants
- $\boldsymbol{u}$ is a $q \times 1$ random vector — We model its Variance
- $\boldsymbol{e}$ is an $n \times 1$ vector of random errors

The Linear Mixed-Effects Model

- $\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{Zu} + \boldsymbol{e}$

- The elements of $\boldsymbol{\beta}$ are considered to be non-random and are called *fixed effects*.

- The elements of $\boldsymbol{u}$ are random variables and are called *random effects*.

- The elements of the error vector $\boldsymbol{e}$ are always considered to be random variables.

- Because the model includes both fixed and random effects (in addition to the random errors), it is called a *mixed-effects* model or, more simply, a *mixed* model.

- The model is called a *linear* mixed-effects model because (as we will soon see)

no longer considered random

$$\mathrm{E}(\boldsymbol{y}|\boldsymbol{u}) = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{Z}\boldsymbol{u},$$

a linear function of fixed and random effects.

We assume that

$$E(\boldsymbol{e}) = \mathbf{0} \qquad \mathrm{Var}(\boldsymbol{e}) = \underline{\boldsymbol{R}}$$

$$E(\boldsymbol{u}) = \mathbf{0} \qquad \mathrm{Var}(\boldsymbol{u}) = \underline{\boldsymbol{G}}$$

random effect does not affect the mean structure

$$\mathrm{Cov}(\boldsymbol{e},\ \boldsymbol{u}) = \mathbf{0}.$$

It follows that

$$
\begin{aligned}
\mathrm{E}(\boldsymbol{y}) &= \mathrm{E}(\boldsymbol{X\beta} + \boldsymbol{Zu} + \boldsymbol{e}) \\
&= \boldsymbol{X\beta} + \boldsymbol{Z}\underbrace{\mathrm{E}(\boldsymbol{u})}_{=0} + \underbrace{\mathrm{E}(\boldsymbol{e})}_{=0} \\
&= \underline{\underline{\boldsymbol{X\beta}}}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}(\boldsymbol{y}) &= \mathrm{Var}(\cancel{\boldsymbol{X\beta}}^{\text{fixed}} + \boldsymbol{Zu} + \boldsymbol{e}) \\
&= \mathrm{Var}(\underline{\boldsymbol{Zu}} + \underline{\boldsymbol{e}}) \\
&\overset{\text{indep.}}{=} \mathrm{Var}(\boldsymbol{Zu}) + \mathrm{Var}(\boldsymbol{e}) \\
&= \boldsymbol{Z}\mathrm{Var}(\boldsymbol{u})\boldsymbol{Z}^\top + \boldsymbol{R} \\
&= \boldsymbol{ZGZ}^\top + \boldsymbol{R} \equiv \boldsymbol{\Sigma}.
\end{aligned}
$$

We usually consider the special case in which

$$\begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{e} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}, \begin{bmatrix} \boldsymbol{G} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{R} \end{bmatrix} \right).$$

This implies

$$\boldsymbol{y} \sim \mathcal{N}(\boldsymbol{X\beta}, \underbrace{\boldsymbol{ZGZ}^\top + \boldsymbol{R}}_{\text{Var}(y)}).$$

The conditional mean and variance, given the random effects, are

$$\mathrm{E}(\boldsymbol{y}|\boldsymbol{u}) = \boldsymbol{X\beta} + \boldsymbol{Zu} \quad \text{and} \quad \mathrm{Var}(\boldsymbol{y}|\boldsymbol{u}) = \boldsymbol{R}.$$

Example 1

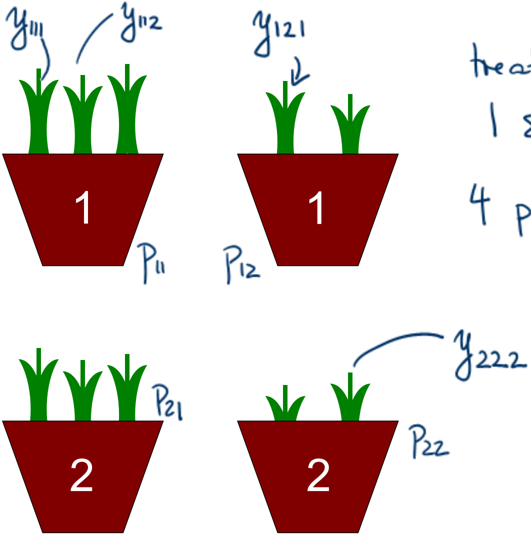
Suppose an experiment was conducted to compare the height of plants grown at two soil moisture levels (labeled 1 and 2). The soil moisture levels were randomly assigned to 4 pots with 2 pots per moisture level. For each moisture level, 3 seeds were planted in one pot and 2 seeds were planted in the other. After a four-week growing period, the height of each seedling was measured. Let $\boxed{y_{ijk}}$ denote the height for soil moisture level i, pot j, seedling k.

3 plants in 2 pots

2 plants in 2 pots

treatment 1 & 2

4 pots



P_{11}, P_{12}, P_{21}, P_{22} - random pot effect

Consider the model

$i = 1,2$ treatment effect

$$y_{ijk} = \mu + \alpha_i + p_{ij} + e_{ijk}.$$

random pot effect

We assume $p_{11}, p_{12}, p_{21}, p_{22} \stackrel{i.i.d.}{\sim} \mathcal{N}(0, \sigma_p^2)$ and independent of the e_{ijk} terms.

Further $e_{ijk} \stackrel{i.i.d.}{\sim} \mathcal{N}(0, \sigma_e^2)$.

This model can be written in the form

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{Zu} + \boldsymbol{e},$$

where

$$\boldsymbol{y} = \left[\begin{array}{c} y_{111}\\ y_{112}\\ y_{113}\\ y_{121}\\ y_{122}\\ \hline y_{211}\\ y_{212}\\ y_{213}\\ y_{221}\\ y_{222} \end{array}\right] \begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right\} \text{trt } 1 \\ \left.\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right\} \text{trt } 2 \end{array}, \quad \boldsymbol{X} = \begin{array}{c} \begin{array}{ccc} \mu & & \end{array} \\ \begin{bmatrix} 1 & 1 & 0\\ 1 & 1 & 0\\ 1 & 1 & 0\\ 1 & 1 & 0\\ 1 & 1 & 0\\ 1 & 0 & 1\\ 1 & 0 & 1\\ 1 & 0 & 1\\ 1 & 0 & 1\\ 1 & 0 & 1 \end{bmatrix} \\ \begin{array}{ccc} & \alpha_1 & \alpha_2 \end{array} \end{array}, \quad \boldsymbol{\beta} = \left[\begin{array}{c} \mu\\ \hline \alpha_1\\ \hline \alpha_2 \end{array}\right],$$

$$
\boldsymbol{Z} = \begin{array}{c}
\begin{array}{cccc} P_{11} & P_{12} & & \end{array} \\
\begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix} \\
\begin{array}{cccc} & & P_{21} & P_{22} \end{array}
\end{array},
\quad \boldsymbol{u} = \begin{bmatrix} p_{11} \\ p_{12} \\ p_{21} \\ p_{22} \end{bmatrix},
\quad \boldsymbol{e} = \begin{bmatrix} e_{111} \\ e_{112} \\ e_{113} \\ e_{121} \\ e_{122} \\ e_{211} \\ e_{212} \\ e_{213} \\ e_{221} \\ e_{222} \end{bmatrix}.
$$

trt 1 trt 2

$$
\begin{bmatrix} y_{111} \\ y_{112} \\ y_{113} \\ y_{121} \\ y_{122} \\ y_{211} \\ y_{212} \\ y_{213} \\ y_{221} \\ y_{222} \end{bmatrix} = \begin{bmatrix} \mu + \alpha_1 \\ \mu + \alpha_1 \\ \mu + \alpha_1 \\ \mu + \alpha_1 \\ \mu + \alpha_1 \\ \mu + \alpha_2 \\ \mu + \alpha_2 \\ \mu + \alpha_2 \\ \mu + \alpha_2 \\ \mu + \alpha_2 \end{bmatrix} + \begin{bmatrix} p_{11} \\ p_{11} \\ p_{11} \\ p_{12} \\ p_{12} \\ p_{21} \\ p_{21} \\ p_{21} \\ p_{22} \\ p_{22} \end{bmatrix} + \begin{bmatrix} e_{111} \\ e_{112} \\ e_{113} \\ e_{121} \\ e_{122} \\ e_{211} \\ e_{212} \\ e_{213} \\ e_{221} \\ e_{222} \end{bmatrix}
$$

$$
\begin{aligned}
y_{111} &= \mu + \alpha_1 + p_{11} + e_{111} \\
y_{112} &= \mu + \alpha_1 + p_{11} + e_{112} \\
y_{113} &= \mu + \alpha_1 + p_{11} + e_{113} \\
y_{121} &= \mu + \alpha_1 + p_{12} + e_{121} \\
y_{122} &= \mu + \alpha_1 + p_{12} + e_{122} \\
y_{211} &= \mu + \alpha_2 + p_{21} + e_{211} \\
y_{212} &= \mu + \alpha_2 + p_{21} + e_{212} \\
y_{213} &= \mu + \alpha_2 + p_{21} + e_{213} \\
y_{221} &= \mu + \alpha_2 + p_{22} + e_{221} \\
y_{222} &= \mu + \alpha_2 + p_{22} + e_{222}
\end{aligned}
$$

- variance of random effect $\boldsymbol{u}$:

$$\begin{aligned} \boldsymbol{G} = \mathrm{Var}(\boldsymbol{u}) &= \mathrm{Var}\left([p_{11}, p_{12}, p_{21}, p_{22}]^\top \right) \\ &= \sigma_p^2 \boldsymbol{I}_{4\times 4} \end{aligned}$$

$$\begin{pmatrix} \sigma_p^2 & & & \\ & \sigma_p^2 & & \\ & & \sigma_p^2 & \\ & & & \sigma_p^2 \end{pmatrix}$$

- variance of random error $\boldsymbol{e}$:

$$\boldsymbol{R} = \mathrm{Var}(\boldsymbol{e}) = \sigma_e^2 \boldsymbol{I}_{10\times 10}$$

$$\begin{pmatrix} \sigma_e^2 & & \\ & \ddots & \\ & & \sigma_e^2 \end{pmatrix}$$

- variance of $\boldsymbol{y}$:

$$\begin{aligned} \mathrm{Var}(\boldsymbol{y}) &= \boldsymbol{Z}\boldsymbol{G}\boldsymbol{Z}^\top + \boldsymbol{R} \\ &= \boldsymbol{Z}\sigma_p^2 \boldsymbol{I}\boldsymbol{Z}^\top + \sigma_e^2 \boldsymbol{I} \\ &= \sigma_p^2 \boldsymbol{Z}\boldsymbol{Z}^\top + \sigma_e^2 \boldsymbol{I}. \end{aligned}$$

block diagonal matrix

3 plants within pot 1 : P_{11}

plants from different pots are independ.

$$
ZZ^\top = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}
$$

to account for correlation between plants within the same pot $\hat{=}$ pot effect of the block diagonal matrix on slide 21

end lecture 21

03-12-25

Thus, $\mathrm{Var}(\boldsymbol{y}) = \sigma_p^2 \boldsymbol{Z}\boldsymbol{Z}^\top + \sigma_e^2 \boldsymbol{I}$ is a block diagonal matrix.

The first block is

pot 1

Var (y_{111}) Cov (y_{111}, y_{112})

$$\mathrm{Var}\begin{bmatrix} y_{111} \\ y_{112} \\ y_{113} \end{bmatrix} = \begin{bmatrix} \sigma_p^2 + \sigma_e^2 & \sigma_p^2 & \sigma_p^2 \\ \sigma_p^2 & \sigma_p^2 + \sigma_e^2 & \sigma_p^2 \\ \sigma_p^2 & \sigma_p^2 & \sigma_p^2 + \sigma_e^2 \end{bmatrix}.$$

Structure of the first matrix on the diagonal of $ZZ^\top$

$$(\sigma_p^2 + \sigma_e^2) = \sigma_e^2\left(\frac{\sigma_p^2}{\sigma_e^2} + 1\right)$$