Ph.D. PRELIMINARY EXAMINATION

March 18, 2003

PART I: Theory
(Majors)

Let $(X_i, Y_i)'$, $i = 1, 2, \ldots$ be a collection of independent and identically distributed (iid) bivariate random vectors on a probability-space $(\Omega, \mathcal{F}, P)$. Define the random variables

$$Z_i = X_i + \frac{1}{\sqrt{2}} Y_{i+2}, \ i \geq 1 \ .$$

Let $S_n = X_1 + \cdots + X_n$ and $\bar{Z}_n = n^{-1}(Z_1 + \cdots + Z_n)$, $n \geq 1$.

1. Suppose that the probability density function (pdf) of X_1 is

$$f(x) = \begin{cases} C|x|^{-2}(1+|x|)^{-1}(\log(1+|x|))^{-\alpha} & \text{if } |x| > 1 \\ 0 & \text{otherwise} \end{cases}$$

for $\alpha \in (0, \infty)$, $\alpha \neq 1$ where $C \in (0, \infty)$ is a constant depending on α that makes $f(x)$ a pdf. Find constants $a_n > 0$, $b_n \in I\!R$, $n \geq 1$ such that

$$a_n^{-1}(S_n - b_n) \longrightarrow^d N(0, 1) \ .$$

NOTE:

(i) $EX_1^2 < \infty$ if and only if $\alpha > 1$.

(ii) The constants of a_n, b_n depend on α. You need to specify a_n, b_n for all $\alpha \in (0, \infty)$, $\alpha \neq 1$.

(iii) You must identify/state any standard result(s) that you are using.

2. Next suppose that $EX_1^2 < \infty$, $EY_1^2 < \infty$, and $\sigma^2 \equiv \text{Var}(X_1 + \frac{1}{\sqrt{2}} Y_1) > 0$.

   (i) Show that for any positive integers k, r, Z_k and Z_{k+r} are independent whenever $r \geq 3$.

   (ii) We want to show that $T_n = \sqrt{n}(\bar{Z}_n - EZ_1)$ is asymptotically normal. Because the Z_i's are not independent, the classical Central Limit Theorem (CLT) does not apply. However, it is possible to show that $T_n \longrightarrow^d N(0, \sigma^2)$ as follows. Write $W_i = X_i + \frac{1}{\sqrt{2}} Y_i$, $i \geq 1$.

      (a) Let $T_{1n} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (W_i - EW_i)$. Show that

$$T_{1n} \longrightarrow^d N(0, \sigma^2)$$

      where $\sigma^2 = \text{Var}(W_1) = \text{Var}(X_1 + \frac{1}{\sqrt{2}} Y_1)$ is as defined above.

      (b) Show that $T_n - T_{1n} = (Y_{n+2} + Y_{n+1} - Y_1 - Y_2)/\sqrt{2n}$ and that $E(T_n - T_{1n})^2 \to 0$ as $n \to \infty$.

      (c) Conclude that $T_n \longrightarrow^d N(0, \sigma^2)$.

   (iii) (a) Using part 2(ii) or otherwise, show that $ET_n^2 \to \sigma^2$ as $n \to \infty$.

      (b) Let $T_\infty \sim N(0, \sigma^2)$. Show that for any $\beta \in (0, 2)$,

$$E|T_n|^\beta \to E|T_\infty|^\beta \quad \text{as} \quad n \to \infty.$$

SOLUTION — Ph.D. Prelim 2003/Theory (I)

1. For $\alpha > 1$,

$$E X_1^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx = 2c \int_1^{\infty} (1+|x|)^{-1} \left(\log(1+|x|)\right)^{-\alpha} dx$$

$< \infty$.

Hence, by the classical CLT for iid r.v.s,

$$\frac{1}{\sigma_X \sqrt{n}} (S_n - nEX_1) \xrightarrow{d} N(0,1)$$

where $\sigma_X^2 = Var(X_1)$ (here $EX_1 = 0$, by the symmetry of $f(x)$).

Next suppose $0 < \alpha < 1$. Consider the function

$$\mu(x) \equiv \int_{-x}^{x} y^2 f(y)\,dy, \quad x > 0.$$

By substitution ($t = \log(1+y)$), we have, for $x > 1$,

$$\mu(x) = 2c \int_1^x (1+y)^{-1} \left[\log(1+y)\right]^{-\alpha} dy$$

$$= 2c \int_{\log 2}^{\log(x+1)} (t)^{-\alpha}\, dt =$$

$$= \frac{2c}{1-\alpha} \cdot \left\{ \left[\log(x+1)\right]^{1-\alpha} - (\log 2)^{1-\alpha} \right\},$$

Check that

$$\lim_{x\to\infty} \frac{\mu(ax)}{\mu(x)} = \lim_{x\to\infty} \frac{\left[\log(ax+1)\right]^{1-\alpha} - (\log 2)^{1-\alpha}}{\left[\log(x+1)\right]^{1-\alpha} - (\log 2)^{1-\alpha}}$$

$$= 1 \quad \text{for any } a > 0.$$

Hence, X_1 belongs to the domain of attraction of the normal law.

The scaling sequence a_n is determined by

$$\frac{n\,\mu(a_n)}{a_n^2} \longrightarrow 1.$$

$\Longrightarrow$ $a_n = \left\{\frac{2c}{1-\alpha}\cdot n\right\}^{1/2}\left(\log\sqrt{n}\right)^{\frac{1-\alpha}{2}}$, $n>1$ is a possible choice.

$\hookrightarrow$ (**)

Hence, for $0<\alpha<1$,

$$a_n^{-1}\left(S_n - 0\right) \longrightarrow^{d} N(0,1)$$

where a_n is as in (**)

2. (i) Easy

$Z_k \in \sigma\left\langle\left\{\binom{X_k}{Y_k}, \binom{X_{k+2}}{Y_{k+2}}\right\}\right\rangle$, which is indep.

of $\sigma\left\langle\left\{\binom{X_{k+3}}{Y_{k+3}}, \cdots\right\}\right\rangle$.

(ii) (a) Follows from the CLT for iid random variables.

(b) By ~~Cauchy-Schwarz~~ the inequality, $(a+b)^2 \le 2(a^2+b^2)$,

$$E\left(T_n - T_{1n}\right)^2 = \frac{1}{4n} E\left[\left(Y_{n+2}+Y_{n+1}\right) - \left(Y_1+Y_2\right)\right]^2$$

$$\le \frac{2}{4n}\left[E\left(Y_{n+2}+Y_{n+1}\right)^2 + E\left(Y_1+Y_2\right)^2\right]$$

$$= \frac{4}{4n} E\left(Y_1+Y_2\right)^2 \quad \text{(as } Y_i\text{'s are iid).}$$

$$= \frac{E\left(Y_1+Y_2\right)^2}{n} \longrightarrow 0 \quad \text{as } n\to\infty.$$

(ii) (c) Follows from (ii)(a) and (ii)(b), and Slutsky's Theorem.

(iii) (a). Clearly $E T_{1n}^2 = E W_1^2 = \sigma^2$. Hence, by (iii)(b) and Cauchy-Schwarz inequality,

$$\left|E T_n^2 - \sigma^2\right|$$

$$= \left|E\left[T_{1n} + (T_n - T_{1n})\right]^2 - \sigma^2\right|$$

$$= \left|\underbrace{E T_{1n}^2}_{=\sigma^2} + 2 E T_{1n}(T_n - T_{1n}) + E(T_n - T_{1n})^2 - \sigma^2\right|$$

$$\le 2\left|E T_{1n}(T_n - T_{1n})\right| + E(T_n - T_{1n})^2$$

C-S inequality

$$\le 2 \left(\underbrace{E T_{1n}^2}_{=\sigma^2}\right)^{\frac{1}{2}} \left(\underbrace{E(T_n - T_{1n})^2}_{\to 0}\right)^{\frac{1}{2}} + \underbrace{E(T_n - T_{1n})^2}_{\to 0}$$

$$\longrightarrow 0 \quad \text{as } n \to \infty$$

Hence, $E T_n^2 \to \sigma^2$ as $n \to \infty$.

(iii) (b) Note that by (iii)(a),

$\sup\{E T_n^2 : n \ge 1\} < \infty$. Hence, $\{|T_n|^\beta\}_{n \ge 1}$ is uniformly integrable for all $\beta < 2$. Also, by (ii)(c), and the continuous mapping Theorem, $|T_n|^\beta \xrightarrow{d} |T_\infty|^\beta$. Hence, it follows that

$$E|T_n|^\beta \longrightarrow E|T_\infty|^\beta.$$

This problem consists of 3 unrelated parts, labeled A, B, and C.
For a subset A of a nonempty set Ω, let $\mathbb{1}_A$ denote the indicator function of a set A, i.e., $\mathbb{1}_A(\omega) = 1$ if $\omega \in A$ and $\mathbb{1}_A(\omega) = 0$ otherwise.

A. 1. State Kolmogorov's 3-series theorem.

2. Show that for a random variable X,

$$EX^2 \mathbb{1}_{[0,n]}(|X|) \leq \sqrt{n} E|X| \mathbb{1}_{[0,\sqrt{n}]}(|X|) + nE|X| \mathbb{1}_{(\sqrt{n},\infty)}(|X|) .$$

3. Let $\{X_n\}_{n\geq 1}$ be a sequence of iid random variables with $EX_1 = 0$ and $E|X_1|[\log(1+|X_1|)]^2 < \infty$. Show that $\sum_{n=1}^{\infty} X_n/n$ converges almost surely.

B. 1. Define the conditional expectation of a random variable X given a σ-field $\mathcal{G}$.

2. Consider the probability space $(\Omega, \mathcal{F}, P)$ where $\Omega = (0,1]$, $\mathcal{F} = \mathcal{B}((0,1])$, the Borel σ-field on $(0,1]$ and $P =$ the Lebesgue measure on $((0,1], \mathcal{B}((0,1]))$. Let

$$\mathcal{G}_n = \sigma\Big\langle \Big\{ \Big(\frac{i-1}{n}, \frac{i}{n}\Big] : 1 \leq i \leq n \Big\} \Big\rangle, \ n > 1$$

and let X be an integrable random variable on $(\Omega, \mathcal{F}, P)$.

(a) Verify that (a version of) the conditional expectation of X given $\mathcal{G}_n$, denoted by $E(X|\mathcal{G}_n)$, is

$$E(X|\mathcal{G}_n) = n \sum_{i=1}^{n} \int_{(\frac{i-1}{n}, \frac{i}{n}]} X dP \cdot \mathbb{1}_{(\frac{i-1}{n}, \frac{i}{n}]} .$$

(b) Find $E\{E(X|\mathcal{G}_n)|\mathcal{G}_{n+1}\}$ for $n = 2$.

C. Let $\{X_n\}_{n\geq 1}$ be a sequence of iid random variables on a probability space $(\Omega, \mathcal{F}, P)$ such that the distribution F_X of X_1 is absolutely continuous with respect to a probability distribution λ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, where $\mathcal{B}(\mathbb{R})$ denotes the Borel σ-field on $\mathbb{R}$. Let $f = \frac{dF_X}{d\lambda}$ denote the density of F_X with respect to λ. Suppose that f is everywhere positive. Let $\mathcal{F}_n = \sigma\langle X_1, \ldots, X_n \rangle$ and $Z_n = \prod_{i=1}^{n} [f(X_i)]^{-1}$, $n \geq 1$,

1. Show that $\{Z_n, \mathcal{F}_n\}_{n\geq 1}$ is a martingale.

2. Let $L_n = -\sum_{i=1}^{n} \log f(X_i)$, $n \geq 1$. Is $\{L_n, \mathcal{F}_n\}_{n\geq 1}$ a supermartingale? Justify your answer!

A. 1. ___

2.
$$E\, X^2 \mathbb{1}(|X| \le n)$$
$$= E X^2 \mathbb{1}(\sqrt{n} < |X| \le n) + E X^2 \mathbb{1}(|X| \le \sqrt{n})$$
$$\le n\, E|X|\, \mathbb{1}(\sqrt{n} < |X| \le n) + \sqrt{n}\, E|X|\, \mathbb{1}(|X| \le \sqrt{n})$$
$$\le n\, E|X|\, \mathbb{1}(|X| > \sqrt{n}) + \sqrt{n}\, E|X|\, \mathbb{1}(|X| \le \sqrt{n})$$

3. By Kolmogorov's 3-series Theorem, it is enough to show ~~that~~

(i) $\sum_{n=1}^\infty P(|\frac{X_n}{n}| > 1) < \infty$, (ii) $\sum_{n=1}^\infty E(\frac{X_n}{n}) \mathbb{1}(|\frac{X_n}{n}| \le 1)$ converges and (iii) $\sum_{n=1}^\infty Var\left((\frac{X_n}{n}) \cdot \mathbb{1}(|\frac{X_n}{n}| \le 1) \right) < \infty$.

Pf of (i): $\sum_{n=1}^\infty P\left(\frac{|X_n|}{n} > 1\right) = \sum_{n=1}^\infty P(|X_1| > n) \le E|X_1| < \infty$.

Pf of (ii) $\sum_{n=1}^\infty n^{-1} |E X_n \mathbb{1}(|X_n| \le n)| = \sum_{n=1}^\infty n^{-1} |E X_n \mathbb{1}(|X_n| > n)|$ (as $EX_n = 0$)

$$0 \le \sum_{n=1}^\infty n^{-1} E|X_1| \mathbb{1}(|X_1| > n) \le \sum_{n=1}^\infty n^{-1} \frac{E|X_1| (\log(1+|X_1|))^2}{[\log(1+n)]^2}$$
$$= \rho \cdot \sum_{n=1}^\infty n^{-1} [\log(1+n)]^{-2} < \infty,$$
where $\rho \equiv E|X_1| [\log(1+|X_1|)]^{+2}$.

Pf of (iii): By part A.2,

$$\sum_{n=1}^\infty Var\left(\frac{X_n}{n} \cdot \mathbb{1}(|\frac{X_n}{n}| \le 1)\right) \le \sum_{n=1}^\infty n^{-2} E X_1^2 \mathbb{1}(|X_1| \le n)$$
$$\le \sum_{n=1}^\infty n^{-2} [n E|X_1| \mathbb{1}(|X_1| > \sqrt{n})] + \sum_{n=1}^\infty n^{-3/2} E|X_1| \cdot \mathbb{1}(|X_1| \le \sqrt{n})$$
$$\le \sum_{n=1}^\infty n^{-1} \cdot \frac{E|X_1| \log(1+|X_1|)}{[\log(\sqrt{n}+1)]^2} + \sum_{n=1}^\infty n^{-3/2} \cdot E|X_1| < \infty.$$

B. (1) —

2. (a). Check that ~~EX~~ the right side is $\mathcal{G}_n$-measurable as each set $\left(\frac{i-1}{n}, \frac{i}{n}\right] \in \mathcal{G}_n$, $1 \le i \le n$. Hence, it is enough to show that for any $A \in \mathcal{G}_n$,

$$\int_A [Y h_n] \, dP = \int_A X \, dP. \longrightarrow (*)$$

~~Since~~ For any $1 \le j \le n$,

$$\int_{\left(\frac{j-1}{n}, \frac{j}{n}\right]} \left\{ n \sum_{i=1}^{n} \underbrace{\int_{\left(\frac{i-1}{n}, \frac{i}{n}\right]} X \, dP}_{\text{a constant}} \cdot \mathbb{1}_{\left(\frac{i-1}{n}, \frac{i}{n}\right]} \right\} dP$$

$$= \sum_{i=1}^{n} \left(n \cdot \int_{\left(\frac{i-1}{n}, \frac{i}{n}\right]} X \, dP \right) \cdot \int_{\left(\frac{j-1}{n}, \frac{j}{n}\right]} \mathbb{1}_{\left(\frac{i-1}{n}, \frac{i}{n}\right]} \, dP$$

disjoint, unless $i = j$

$$= \underbrace{\left(n \int_{\left(\frac{j-1}{n}, \frac{j}{n}\right]} X \, dP \right) \int_{\left(\frac{j-1}{n}, \frac{j}{n}\right]} \mathbb{1}_{\left(\frac{j-1}{n}, \frac{j}{n}\right]} \, dP}_{i=j \text{ term of the sum}} + \underbrace{0}_{i \ne j \text{ terms.}}$$

$$= n \int_{\left(\frac{j-1}{n}, \frac{j}{n}\right]} X \, dP \cdot \underbrace{P\left(\left(\frac{j-1}{n}, \frac{j}{n}\right]\right)}_{= \frac{1}{n}, \text{ as } P \text{ is the Lebesgue measure on } (0,1]}$$

$$= \int_{\left(\frac{j-1}{n}, \frac{j}{n}\right]} X \, dP. \longrightarrow (**)$$

Thus, $(*)$ holds for $A = \left(\frac{j-1}{n}, \frac{j}{n}\right]$. Note that any set in $\mathcal{G}_n$

can be written as $\bigcup_{i \in I} \left(\frac{i-1}{n}, \frac{i}{n}\right]$, $I \subset \{1, \cdots, n\}$.

Hence, $(*)$ now follows from $(**)$ and the linearity of integrals.

2(b). By part B.2(a), for any $m \geq 2$,

$$E(X|\mathcal{G}_m) = m \sum_{i=1}^{m} \left(\int_{\left(\frac{i-1}{m}, \frac{i}{m}\right]} x \, dP\right) \cdot \mathbb{1}_{\left(\frac{i-1}{m}, \frac{i}{m}\right]}$$

$$\Rightarrow E\left\{E(X|\mathcal{G}_n) \middle| \mathcal{G}_{n+1}\right\} = E\left\{\left(2 \sum_{i=1}^{2} \int_{\left(\frac{i-1}{2}, \frac{i}{2}\right]} x \, dP \cdot \mathbb{1}_{\left(\frac{i-1}{2}, \frac{i}{2}\right]}\right) \middle| \mathcal{G}_{n+1}\right\}$$

$$= 2 \cdot E\left\{\left(\int_0^{1/2} x \, dP\right) \cdot \mathbb{1}_{(0, 1/2]} \middle| \mathcal{G}_{n+1}\right\}$$

$$+ 2 \, E\left\{\left(\int_{(1/2, 1]} x \, dP\right) \cdot \mathbb{1}_{(1/2, 1]} \middle| \mathcal{G}_{n+1}\right\}$$

$$= \underbrace{2\left(\int_0^{1/2} x \, dP\right)}_{= a_1} \cdot E\left(\mathbb{1}_{(0, 1/2]} \middle| \mathcal{G}_3\right)$$

$$+ \underbrace{2\left(\int_{1/2}^{1} x \, dP\right)}_{= a_2, \text{ say}} \cdot E\left(\mathbb{1}_{(1/2, 1]} \middle| \mathcal{G}_3\right)$$

$$= 2a_1 \left[3 \sum_{i=1}^{3} \int_{\left(\frac{i-1}{3}, \frac{i}{3}\right]} \mathbb{1}_{(0, 1/2]} \, dP \cdot \mathbb{1}_{\left(\frac{i-1}{3}, \frac{i}{3}\right]}\right]$$

$$+ 2a_2 \left[3 \sum_{i=1}^{3} \int_{\left(\frac{i-1}{3}, \frac{i}{3}\right]} \mathbb{1}_{(1/2, 1]} \, dP \cdot \mathbb{1}_{\left(\frac{i-1}{3}, \frac{i}{3}\right]}\right]$$

$$= 2a_1\Big[3\Big\{\underbrace{\tfrac{1}{3}\cdot \mathbb{1}_{(0,\frac{1}{3}]}}_{i=1} + \underbrace{\int_{1/3}^{1/2} dx}_{i=2}\cdot \mathbb{1}_{(\frac{1}{3},\frac{2}{3}]} + 0\Big\}\Big]$$

$$+2a_2\Big[3\Big\{\underbrace{0}_{i=1} + \underbrace{\int_{1/2}^{2/3} dx}_{i=2}\cdot \mathbb{1}_{(\frac{1}{3},\frac{2}{3}]} + \int_{2/3}^{1} dx\cdot \mathbb{1}_{(\frac{2}{3},1]}\Big\}\Big]$$

$$= a_1\Big[2\cdot \mathbb{1}_{(0,\frac{1}{3}]} + \mathbb{1}_{(\frac{1}{3},\frac{2}{3}]}\Big] + a_2\Big[\mathbb{1}_{(\frac{1}{3},\frac{2}{3}]} + 2\,\mathbb{1}_{(\frac{2}{3},1]}\Big]$$

where $a_1 = \int_0^{1/2} x\,dP$, $a_2 = \int_{1/2}^{1} x\,dP$. (The integrals $\int_a^b (\cdot)\,dP$ are unambiguously defined as $P(\{a\}) = 0 = P(\{b\})$).

C. (i) Check that Z_n is $\mathcal{G}_n$-measurable for each $n \geq 1$. And $Z_n \geq 0$ a.s. $\Rightarrow$

$$E(Z_{n+1}\mid \mathcal{G}_n) = E\Big(\prod_{i=1}^{n+1}\{f(X_i)\}^{-1}\,\Big|\, X_1,\dots,X_n\Big)$$

$= \prod_{i=1}^{n}\{f(X_i)\}^{-1}\cdot E\{f(X_{n+1})\}^{-1}$, by the independence of X_{n+1} and $\{X_1,\dots,X_n\}$.

$\overset{(a)}{=} Z_n \cdot E\{f(X_1)\}^{-1}$

$= Z_n \int \{f(x)\}^{-1}\, dF_X(x) = Z_n \cdot \int_{\{f(x)>0\}} \{f(x)\}^{-1}\cdot f(x)\, d\lambda(x)$

$= Z_n \cdot 1$

$= Z_n$.

Hence, $\{Z_n, \mathcal{G}_n\}_{n\geq 1}$ is a martingale.

C. 2. Clearly $L_n = \log Z_n$ and the function $g(x) = \log x$, $x > 0$ is a concave function.

Hence, by conditional Jensen's inequality,

$$
\begin{aligned}
E(L_{n+1} \mid \mathcal{F}_n) &= E(\log Z_{n+1} \mid \mathcal{F}_n) \\
&\leq \log E(Z_{n+1} \mid \mathcal{F}_n) \\
&= \log Z_n, \\
&= L_n
\end{aligned}
$$

as $\{Z_n, \mathcal{F}_n\}_{n \geq 1}$ is a martingale.

Hence, $\{L_n, \mathcal{F}_n\}_{n \geq 1}$ is a supermartingale!

Let $X_1, \ldots, X_n$ be iid observations from $Unif[\theta - \frac{1}{2}, \theta + \frac{1}{2}]$ with probability density function (with respect to Lebesgue measure)

$$f(x;\theta) = \begin{cases} 1 & \text{if } \theta - \frac{1}{2} \le x \le \theta + \frac{1}{2} \\ 0 & \text{otherwise,} \end{cases}$$

where $-\infty < \theta < \infty$ is the unknown parameter. Consider the square error loss for estimating θ, i.e., $L(\theta, d) = (d - \theta)^2$. Let $\widehat{\theta}_{1,n} = \frac{X_{(1)}+X_{(n)}}{2}$ and $\widehat{\theta}_{2,n} = X_{(n)} - \frac{1}{2}$, where $X_{(1)}$ and $X_{(n)}$ denote the first and last order statistics, respectively.

1. Identify a minimal sufficient statistic for θ and show that it is indeed minimal.
2. Is the minimal sufficient statistic complete? Prove your answer.
3. Show that both $\widehat{\theta}_{1,n}$ and $\widehat{\theta}_{2,n}$ are maximum likelihood estimators of θ.
4. Are they unbiased estimators? Explain your answer (no detailed calculation of the expectations is required).
5. Consider a prior distribution $Unif[-A, A]$ on θ with $A > 0$. Find a Bayes estimator of θ.
6. Prove that $\widehat{\theta}_{1,n}$ is an equalizer.
7. Show that $\widehat{\theta}_{1,n}$ is a minimax estimator of θ.
8. Find a method of moment estimator of θ, say $\widehat{\theta}_{3,n}$ and identify its appropriate limiting distribution.
9. Identify an appropriate limiting distribution of $\widehat{\theta}_{2,n}$.
10. Show that $\widehat{\theta}_{1,n} - \theta = O_p(\frac{1}{n})$ (i.e., $\{n(\widehat{\theta}_{1,n} - \theta), n \ge 1\}$ is bounded in probability).
11. Which of the two estimators do you prefer, $\widehat{\theta}_{1,n}$ or $\widehat{\theta}_{3,n}$? Explain.

A possibly useful fact: For $Unif[0,1]$, the variance is $\frac{1}{12}$.

Solution — Theory III

1. $f(\underset{\sim}{x};\theta) = I\{\theta-\frac{1}{2} \le x_{(1)} \le x_{(n)} \le \theta+\frac{1}{2}\}$

$$\frac{f(\underset{\sim}{x};\theta)}{f(\underset{\sim}{y};\theta)} = \frac{I\{\theta-\frac{1}{2} \le x_{(1)} \le x_{(n)} \le \theta+\frac{1}{2}\}}{I\{\theta-\frac{1}{2} \le y_{(1)} \le y_{(n)} \le \theta+\frac{1}{2}\}}$$ is constant as a function of θ iff $x_{(1)} = y_{(1)}$ and $x_{(n)} = y_{(n)}$ (check).

Thus a minimal sufficient statistic for θ is $(X_{(1)}, X_{(n)})$.

2. No. Since θ is a location parameter, $X_{(n)} - X_{(1)}$ is an ancillary statistic. Thus $E_\theta(X_{(n)}-X_{(1)}) = c$ for some constant $c>0$. It follows that $E_\theta(X_{(n)}-X_{(1)}-c)=0 \;\; \forall\, \theta \in \mathbb{R}$. But clearly $P_\theta(X_{(n)}-X_{(1)}-c=0)=0 \ne 1$, therefore $(X_{(1)}, X_{(n)})$ is not complete.

3. Note that $L(\theta) = \begin{cases} 1 & \text{when } X_{(n)}-\frac{1}{2} \le \theta \le X_{(1)}+\frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$

Thus $\hat{\theta}_1$ and $\hat{\theta}_2$ both yield $L(\theta)=1$ (when $X_{(n)}-X_{(1)} \le 1$, which happens w.p. 1 under P_θ, $\theta \in \mathbb{R}$).

4. Consider $Y_i = X_i - \theta$. Then $\{Y_i\}$ iid $\sim$ Unif$[-\frac{1}{2}, \frac{1}{2}]$.
Let $Z_i = -Y_i$. Then $\{Z_i\}$ iid $\sim$ Unif$[-\frac{1}{2}, \frac{1}{2}]$.
It follows that $Z_{(1)}$ and $Y_{(1)}$ have the same distribution.
But $Z_{(1)} = -Y_{(n)}$, so $Y_{(1)}$ and $-Y_{(n)}$ have the same distribution.
It follows that $E_\theta\left[\frac{X_{(1)}+X_{(n)}}{2} - \theta\right] = E_\theta \frac{X_{(1)}-\theta}{2} + E_\theta \frac{X_{(n)}-\theta}{2}$

$$= \frac{1}{2} E\, Y_{(1)} + \frac{1}{2} E\, Y_{(n)} = 0.$$

So $\hat{\theta}_1$ is unbiased.
Clearly $\hat{\theta}_2$ is biased since $X_{(n)} - \frac{1}{2} < \theta$ a.s. P_θ.

5. The Bayes estimator θ_B is the posterior mean, $\hat{\theta}_B = E_{\theta|\underset{\sim}{x}}\, \theta$.

Note $\quad \pi(\theta|\underset{\sim}{x}) \propto I_{\{|\theta| \le A\}}\, I_{\{X_{(n)}-\frac{1}{2} \le \theta \le X_{(1)}+\frac{1}{2}\}}$

$$\propto I_{\{\max(X_{(n)}-\frac{1}{2}, -A) \le \theta \le \min(X_{(1)}+\frac{1}{2}, A)\}}.$$

Thus $\quad \hat{\theta}_B = \frac{1}{2}\left(\max\left(X_{(n)}-\frac{1}{2}, -A\right) + \min\left(X_{(1)}+\frac{1}{2}, A\right)\right).$

6. $R(\hat{\theta}_1, \theta) = E_\theta \left(\frac{X_{(1)}-\theta + X_{(n)}-\theta}{2}\right)^2 = E\left(\frac{Y_{(1)}+Y_{(n)}}{2}\right)^2.$

Since the distribution of $\underset{\sim}{Y}$ does not depend on θ, we know $\hat{\theta}_1$ is an equalizer.

7. Note that for $-A+\frac{1}{2} \le \theta \le A-\frac{1}{2}$, $(A>\frac{1}{2})$ w.p. 1 P_θ,

$$\min\left(X_{(1)}+\tfrac{1}{2}, A\right) = X_{(1)}+\tfrac{1}{2}$$
$$\max\left(X_{(n)}-\tfrac{1}{2}, -A\right) = X_{(n)}-\tfrac{1}{2}.$$

It follows that $\quad R(\hat{\theta}_B; \theta) = R(\hat{\theta}_1, \theta)$ for $-A+\frac{1}{2} \le \theta \le A-\frac{1}{2}$.

Thus the Bayes risk of $\hat{\theta}_B$ satisfies

$$\frac{1}{2A}\int_{-A}^{A} R(\hat{\theta}_B, \theta)\, d\theta \ge \frac{1}{2A}\int_{-A+\frac{1}{2}}^{A-\frac{1}{2}} R(\hat{\theta}_1, \theta)\, d\theta$$

$$= \frac{2A-1}{2A} R(\hat{\theta}_1, \theta) \qquad \left(R(\hat{\theta}_1, \theta) \text{ is constant}\right).$$

So ~~it~~ the Bayes risk converges to $R(\hat{\theta}_1, \theta)$. By a theorem, we know $\hat{\theta}_1$ is a minimax estimator.

8. $EX_1 = \theta$, So a MME of θ is $\hat{\theta}_3 = \overline{X}_n$.

By CLT, $\sqrt{n}(\overline{X}_n - \theta) \xrightarrow{\mathcal{D}} N(0, \frac{1}{12})$.

9. For $x \geq 0$,

$$P_\theta\left(-\left(X_{(n)} - \tfrac{1}{2} - \theta\right) \leq x\right)$$

$$= P\left(-\left(Y_{(n)} - \tfrac{1}{2}\right) \leq x\right)$$

$$= 1 - P\left(Y_{(n)} \leq \tfrac{1}{2} - x\right)$$

$$= \begin{cases} 1 - \left(P\left(Y_1 \leq \tfrac{1}{2} - x\right)\right)^n & 0 < x \leq 1 \\ 1 & x > 1. \end{cases}$$

It follows that for any $x > 0$, when n is large enough,

$$P_\theta\left(-\left(\hat{\theta}_2 - \theta\right) \leq \frac{x}{n}\right) = 1 - \left(1 - \frac{x}{n}\right)^n \to 1 - e^{-x}.$$

Therefore, $-n(\hat{\theta}_2 - \theta) \xrightarrow{\mathcal{D}} Exp(1)$.

10. From 9), $\hat{\theta}_2 - \theta = O_p(\frac{1}{n})$. Due to the symmetry as seen in 4), $X_{(1)} + \frac{1}{2} - \theta = O_p(\frac{1}{n})$. It follows

$$\hat{\theta}_1 - \theta = \frac{1}{2}\left(X_{(1)} + \tfrac{1}{2} - \theta\right) + \frac{1}{2}\left(X_{(n)} - \tfrac{1}{2} - \theta\right) = O_p\left(\tfrac{1}{n}\right).$$

11. We prefer $\hat{\theta}_1$. From 8) and 10), we know $\hat{\theta}_3$ converges much more slowly to θ than $\hat{\theta}_1$. Also $\hat{\theta}_1$ is minimax.

Ph.D. Prelim

Theory IV **March 2003**

a) Define the admissibility of a (nonrandomized) decision function δ in terms of its risk function $R(\theta, \delta)$ mapping from a parameter space Θ to $\Re$ (the set of all real numbers).

b) The statement below is false. Use a simple 2-dimensional sketch of a risk set $\mathcal{S}$ to provide a counterexample. (Explain how your picture provides this counterexample.)

$$\text{If } \exists\, \theta_0 \in \Theta \text{ such that } R(\theta_0, \phi) = \inf_{\phi'} R(\theta_0, \phi'), \text{ then } \phi \text{ is admissible.}$$

In all that follows, consider $X \sim \mathrm{N}(\theta, 1)$ for $\theta \in [-k, k]$ and the estimation of θ under the squared error loss function. k is some known positive constant and the action space is $\mathcal{A} = \Re$.

c) In this context, we may restrict attention to nonrandomized decision rules. Why?

d) In this context $\delta(x) \equiv x$ is inadmissible. Propose a modification of $\delta(x)$ (say $\delta^*(x)$) that is better than $\delta(x)$ and prove that it is better. (It may be helpful to note that for $\theta \le k \le x$, $|\theta - x| = |\theta - k| + |x - k|$ and that for $x \le -k \le \theta$, $|\theta - x| = |\theta - (-k)| + |x - (-k)|$.)

e) Let G_1 be a prior distribution that puts mass $\frac{1}{2}$ on each of $-k$ and k. Find an explicit formula for the Bayes estimator of θ against the prior G_1, say $\delta_{1,k}$.

f) What happens to $R(0, \delta_{1,k})$ as $k \longrightarrow \infty$? Justify your answer carefully. (You may use without proof the facts that $h(y) = (e^y - e^{-y})/(e^y + e^{-y})$ is an increasing function with $h(-y) = -h(y)$ and $|h(y)| < 1$. And, for example, $h(1) \approx .7616$.)

g) Let G_2 be the uniform distribution on the interval $[-k, k]$. Find an explicit formula for the Bayes estimator of θ against the prior G_2, say $\delta_{2,k}$. You may write this in terms of the standard normal probability density function ϕ and the standard normal cumulative probability function Φ. You may find it helpful to know that

$$\int \gamma \phi(\gamma - \mu) d\gamma = -\phi(\gamma - \mu) + \mu \Phi(\gamma - \mu)$$

h) Argue carefully that for any choice of k, the corresponding estimator $\delta_{2,k}$ is admissible. (If you leave any technical gaps in your argument, point out where they are.)

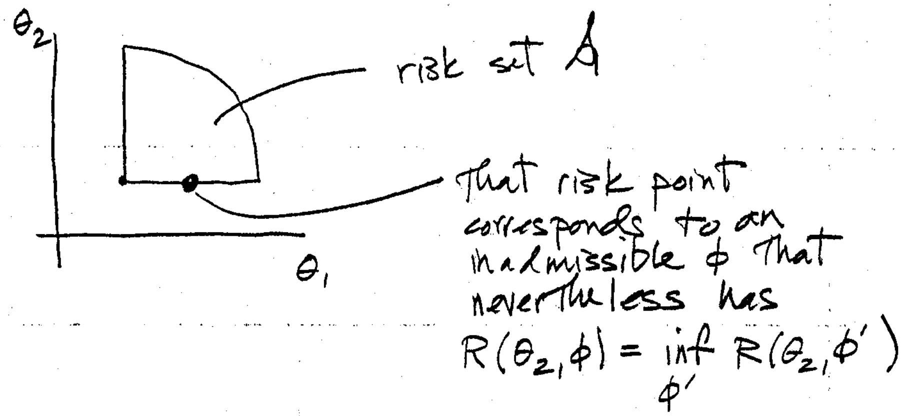
i) If $X_1, X_2, \ldots, X_n$ are iid $\mathrm{N}(\theta, 1)$ for $\theta \in [-k, k]$ and one considers the squared error loss estimation of θ, it suffices to consider nonrandomized estimators that are functions of $\overline{X}$. Why?

a) δ admissible means that there is no ϕ that is better than δ, i.e. there is no ϕ with

$$R(\theta,\phi) \le R(\theta,\delta) \quad \forall \theta \in \Theta$$

$$\text{and} \quad R(\theta_0,\phi) < R(\theta_0,\delta) \quad \text{for some } \theta_0 \in \Theta$$

b) This is not true. Consider the picture



c) The action space is convex and $L(\theta,a) = (\theta-a)^2$ is convex in θ.

d) $\delta(x) = x$ is inadmissible. It can take values outside the interval $[-k,k]$ where θ is assumed to reside. Define

$$\delta^*(x) = \begin{cases} x & \text{if } |x| \le k \\ -k & \text{if } x < -k \\ k & \text{if } x > k \end{cases}$$

I claim that $\delta^*(x)$ is better than $\delta(x)$. Consider

$$L(\theta,\delta(x)) - L(\theta,\delta^*(x)) = (\theta-x)^2 - (\theta-\delta^*(x))$$

If $x \in [-k,k]$ this is 0. If $x > k$ we have

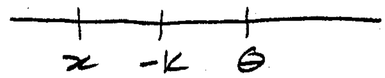
$$\theta \quad k \quad x$$

so that

$$|\theta - x| = |\theta - k| + |x - k|$$

So
$$(\theta - x)^2 = (\theta - k)^2 + 2|\theta - k||x - k| + (x-k)^2$$
and
$$(\theta - x)^2 - (\theta - k)^2 \geq (x-k)^2$$

Similarly, if $x < -k$ we have



so that
$$|\theta - x| = |\theta - (-k)| + |x - (-k)|$$
and
$$(\theta - x)^2 - (\theta - (-k))^2 \geq (x - (-k))^2$$

So (since both δ and δ^* clearly have finite second moments and thus finite risk functions), for $\theta \in \Theta = [-k, k]$
$$R(\theta, \delta) - R(\theta, \delta^*) = \int (\theta - x)^2 - (\theta - \delta^*(x))^2 f(x|\theta)\, dx$$
$$\geq \int_{-\infty}^{-k} (x - (-k))^2 f(x|\theta)\, dx$$
$$+ \int_{k}^{\infty} (x-k)^2 f(x|\theta)\, dx > 0$$

and δ^* is actually uniformly better than $\delta(x) = x$

e) The posterior like the prior has support $\{-k, k\}$. With ϕ the standard normal pdf, the posterior probability that $\theta = k$ is

$$\frac{\phi(x-k)}{\phi(x-k) + \phi(x-(-k))} = \frac{\phi(x-k)}{\phi(x-k) + \phi(x+k)}$$

so the posterior mean will be

$$\delta_{1,k}(x) = \frac{k\phi(x-k) + (-k)\phi(x+k)}{\phi(x-k)+\phi(x+k)}$$

$$= k\left(\frac{e^{kx} - e^{-kx}}{e^{kx}+e^{-kx}}\right)$$

f) The risk of $\delta_{1,k}$ at 0 explodes as $k \to \infty$.
For any $c>0$, if $k > \frac{1}{c}$

$$|\delta_{1,k}(x)| \geq k(.7616) \quad \forall\ x \text{ with } |x|>c$$

So for $k > \frac{1}{c}$

$$R(0,\delta_{1,k}) \geq k^2(.7616)^2\left(1 - \left(\Phi(c) - \Phi(-c)\right)\right)$$

which clearly goes to infinity as $k \to \infty$

g) The posterior mean here is

$$\delta_{2,k} = \frac{\int_{-k}^{k} \theta\,\phi(x-\theta)\,d\theta}{\int_{-k}^{k}\phi(x-\theta)\,d\theta} = \frac{\phi(-k-x) - \phi(k-x) + x\left[\Phi(k-x) - \Phi(-k-x)\right]}{\Phi(k-x)-\Phi(-k-x)}$$

$$= x + \frac{\phi(x+k)-\phi(x-k)}{\Phi(k-x)-\Phi(-k-x)}$$

h) We may restrict attention to nonrandomized estimators (see i)). The risk function of a nonrandomized estimator is

$$\int(\theta-\delta(x))^2\phi(x-\theta)\,dx = \theta^2 - 2\theta\int\delta(x)f(x-\theta)\,dx + \int\delta^2(x)\phi(x-\theta)\,dx$$

which is cont$\underline{s}$ in θ provided $\int\delta(x)\phi(x-\theta)\,dx$ and $\int\delta^2(x)\phi(x-\theta)\,dx$ are cont$\underline{s}$ in θ which follows from properties of exponential families. Then since every neighborhood of a point in Θ has nonempty intersection with the interior of Θ and every open interval in Θ gets positive probability under G_2, the Bayes estimator is guaranteed to be admissible

i) The Factorization Theorem shows $\sum X_i$ to be sufficient for θ and decision rules that are functions of sufficient statistics form an essentially complete class.