Common univariate distributions

Discrete distributions: Poisson

$X \sim$ Poisson(λ), $\lambda > 0$

- pmf given by

$$\mathbb{P}(X=x) = f_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, \ldots,$$

- Motivation: widely used to model "rare event" count data

  (e.g., number of car accidents in a county)

- Note that: $\overset{(*)}{\boxed{e^a = \sum_{k=0}^{\infty} \frac{a^k}{k!}}}$ for any real a

$$\sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} \overset{(*)}{=} e^{-\lambda} e^{\lambda} = 1 \checkmark$$

Recall: $f(x)$ is pmf $<$ $f(x) \geq 0$ ✓, $\sum f(x) = 1$

$$\boxed{f_X(x) = \frac{e^{-\lambda}\lambda^x}{x!} > 0} \checkmark$$

- Mean: $\boxed{EX = \lambda,}$ follows from $EX = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!}$

$$\mathbb{E}(X) \overset{def}{=} \sum_{x=0}^{\infty} x \mathbb{P}(X=x) = \sum_{x=0}^{\infty} \frac{x e^{-\lambda}\lambda^x}{x!} = \sum_{x=1}^{\infty} \frac{\cancel{x} e^{-\lambda}\lambda^{x-1+1}}{\cancel{x}(x-1)!}$$

$$= \lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1} e^{-\lambda}}{(x-1)!} \overset{y=x-1}{=} \lambda \underbrace{\sum_{y=0}^{\infty} \frac{\lambda^y e^{-\lambda}}{y!}} = \lambda$$

- Variance: $\mathrm{Var}(X) = \lambda$, can derive by showing $\boxed{EX(X-1) = \lambda^2}$

$$\mathrm{Var}(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \underbrace{\mathbb{E}[X(X-1)]} + \underbrace{\mathbb{E}(X)} - (\mathbb{E}X)^2 = \cancel{\lambda^2} + \lambda - \cancel{\lambda^2} = \lambda$$

$$\mathbb{E}[X(X-1)] = \sum_{x=0}^{\infty} \frac{x(x-1) e^{-\lambda}\lambda^x}{x!} = \sum_{x=2}^{\infty} \frac{\cancel{x(x-1)} e^{-\lambda}\lambda^{x-2+2}}{\cancel{x(x-1)}(x-2)!}$$

$$= \lambda^2 \sum_{x=2}^{\infty} \frac{e^{-\lambda}\lambda^{x-2}}{(x-2)!} \overset{y=x-2}{=} \lambda^2 \overbrace{\sum_{y=0}^{\infty} \frac{e^{-\lambda}\lambda^y}{y!}}^{=1} = \lambda^2$$

- mgf: $M_X(t) = Ee^{tX} = \boxed{e^{\lambda(e^t-1)}, t \in \mathbb{R}}$

$$M_X(t) \overset{def}{=} \mathbb{E}\left[e^{tX}\right] = \sum_{x=0}^{\infty} \frac{e^{tx} e^{-\lambda}\lambda^x}{x!} = \sum_{x=0}^{\infty} \frac{[e^t\lambda]^x e^{-\lambda}}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^t\lambda)^x}{x!}$$

$$= e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t-1)}$$

Note: $e^k = \sum_{x=0}^{\infty} \frac{k^x}{x!}$

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Done!

Discrete distributions: Poisson

A way to justify the Poisson as a sensible pmf is the following:

think of a specified interval of time as represented by n subintervals, and the occurrence of an event in each subinterval is specified by an independent Bernoulli(p) trial such that $np = \lambda$. Let X = total number of occurrences.

Example: Roughly, 1 e-mail per second goes through a certain network router. Using a Poisson model for X =# of e-mails handled in the next 5 seconds, find the probability that at least one e-mail is handled in the next 5 seconds.

X: # of emails in next 5 sec. Poisson($\lambda=5$)

$$\mathbb{P}(X \geq 1) = \sum_{x=1}^{\infty} \frac{e^{-5}\,5^{x}}{x!}$$

$P(A^c) = 1 - \mathbb{P}(A)$

$$P(X \geq 1) = 1 - \mathbb{P}(X < 1) = 1 - \mathbb{P}(X=0) = 1 - \frac{e^{-5}\,5^{0}}{0!} = 1 - e^{-5}$$

Common univariate distributions

Continuous distributions: Uniform

$X \sim$Uniform(a, b) $\quad -\infty < a < b < \infty$

- pdf given by

$$f_X(x) = \frac{1}{b-a}, \quad a < x < b$$

- Motivation: equally likely outcome model over a finite range (a, b)

- a is lower endpoint of the range; b is the upper endpoint

- $\mathrm{E}X^r = \dfrac{1}{b-a} \displaystyle\int_a^b x^r \, dx = \dfrac{b^{r+1} - a^{r+1}}{(r+1)(b-a)}$ for $r > 0$

$$\mathbb{E}(X^r) = \int_a^b x^r \frac{1}{b-a} dx = \frac{1}{b-a}\int_a^b x^r = \frac{1}{b-a}\,\frac{1}{r+1}\,x^{r+1}\Big]_a^b$$

- Mean: $\mathrm{E}X = (a+b)/2$

$$\int_a^b \frac{x}{b-a} dx = \frac{1}{b-a}\,\frac{x^2}{2}\Big]_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{\cancel{(b-a)}(b+a)}{2\cancel{(b-a)}}$$

- Variance: $\mathrm{Var}(X) = \mathrm{E}X^2 - [\underbrace{\mathrm{E}X}_{\frac{a+b}{2}}]^2 = \boxed{(b-a)^2/12}$ (Check!)

- Important case: $U \sim$Uniform$(a = 0, b = 1)$

1. $f_U(u) = 1$, $0 < u < 1$; $\quad \mathrm{E}(U) = 1/2$; $\quad \mathrm{Var}(U) = 1/12$

(*) 2. If Y has a *continuous* cdf $F_Y(y)$ then the r.v. $U = F_Y(Y) \sim$Uniform$(0, 1)$

(again called the "probability integral transform (PIT)")

3. If $U \sim$Uniform$(0, 1)$ and $F_Y(y)$ is a *continuous* cdf, then the r.v. $Y = F_Y^{-1}(U)$ has distribution F_Y

(useful for simulating r.v.s; more later)

Parameters

Common univariate distributions

Continuous distributions: Gamma

$X \sim$Gamma(α, β) $\quad \alpha > 0, \beta > 0$

- pdf given by

$$\underbrace{f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta},} \quad 0 < x < \infty$$

$$f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$$

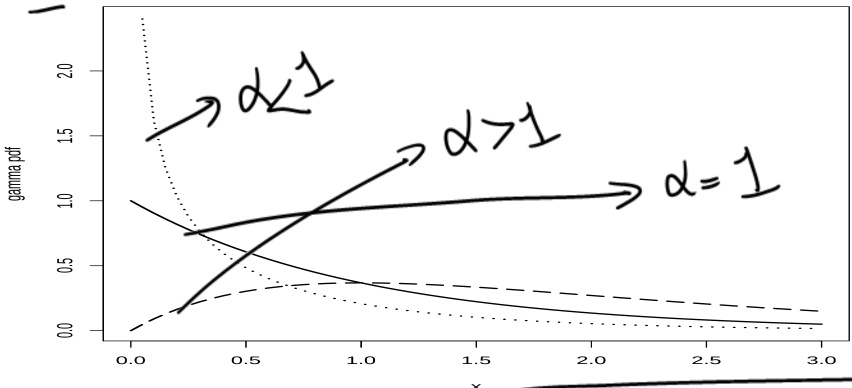
$$f_X(0) = \frac{1}{\Gamma(\alpha)\beta^\alpha} (0)^{\boxed{\alpha-1}} e^{-\frac{0}{\beta}} = \infty$$

If $\alpha < 1$

- Motivation: flexible family for positive quantities

- $\alpha > 0$ is shape parameter.

  ($\alpha < 1$ density unbounded near $x = 0$, $\alpha > 1$ density is zero at $x = 0$)



- $\beta > 0$ is scale parameter. i.e., if $X \sim$Gamma(α, β) then $Z = \frac{X}{\beta} \sim$Gamma$(\alpha, 1)$

$$F_Z(z) \overset{def}{=} \mathbb{P}(Z \le z) = \mathbb{P}\left(\frac{X}{\beta} \le z\right) = \underbrace{\mathbb{P}(X \le \beta z)} = F_X(\beta z)$$

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{d}{dz} F_X(\beta z) = \beta f_X(\beta z) = \cancel{\beta} \frac{1}{\Gamma(\alpha)\cancel{\beta^\alpha}} \cancel{(\beta} z)^{\alpha-1} e^{-\frac{\cancel{\beta} z}{\cancel{\beta}}}$$

$$= \frac{1}{\Gamma(\alpha)} e^{-z/1} z^{\alpha-1}$$

$\boxed{Z \sim \text{Gamma}(\alpha, 1)}$

- $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx, \ \alpha > 0$, is the gamma function, which ensures that $f_X(x)$ is a density

  Some properties

  1. $\Gamma(1 + \alpha) = \alpha\Gamma(\alpha)$ for $\alpha > 0$
  2. $\Gamma(\alpha) = (\alpha - 1)!$ for integer $\alpha \ge 1$
  3. $\Gamma(1/2) = \sqrt{\pi}$

Note: $\int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx$

$x/\beta = y$, $dx = \beta dy$

$$\frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty (\beta y)^{\alpha-1} e^{-y} \beta \, dy$$

$$= \frac{\cancel{\beta^\alpha}}{\Gamma(\alpha)\cancel{\beta^\alpha}} \underbrace{\int_0^\infty y^{\alpha-1} e^{-y} dy} = \frac{\cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}} = 1$$

Common univariate distributions

Continuous distributions: Gamma (cont'd)

$X \sim$Gamma(α, β) $\quad \alpha > 0, \beta > 0$

- $\mathrm{E}X^r = \underbrace{\beta^r \Gamma(\alpha + r)/\Gamma(\alpha)}$ for $r > 0$

Proof: $\quad \mathrm{E}X^r = \dfrac{1}{\Gamma(\alpha)\beta^\alpha} \displaystyle\int_0^\infty x^{r+\alpha-1} e^{-x/\beta} dx$

$$\xrightarrow[dx = \beta dy]{x/\beta = y} \frac{1}{\Gamma(\alpha)\beta^\alpha} \int^\infty (\beta y)^{r+\alpha-1} e^{-y} \beta dy$$

$$= \frac{\beta^{\cancel{r+\alpha-1+1}}}{\Gamma(\alpha)\cancel{\beta^\alpha}} \int_0^\infty \underbrace{y^{\boxed{r+\alpha}-1} e^{-y}} = \frac{\beta^r \Gamma(\alpha+r)}{\Gamma(\alpha)}$$

$$\mathrm{E}X^r = \frac{\beta^r \Gamma(\alpha+r)}{\Gamma(\alpha)} \quad (*)$$

- Mean: $\mathrm{E}X = \alpha\beta$

$$(*) \xRightarrow[r=1]{} \mathrm{E}X = \frac{\beta\,\Gamma(\alpha+1)}{\Gamma(\alpha)} = \frac{\beta\alpha\cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}} = \alpha\beta$$

- Variance: $\mathrm{Var}(X) = \mathrm{E}X^2 - [\mathrm{E}X]^2 = \boxed{\alpha\beta^2}$

$$(*) \xRightarrow[r=2]{} \mathrm{E}X^2 = \frac{\beta^2\Gamma(\alpha+2)}{\Gamma(\alpha)} = \frac{\beta^2(\alpha+1)\Gamma(\alpha+1)}{\Gamma(\alpha)} = \frac{\beta^2(\alpha+1)\alpha\cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}} = \boxed{\alpha(\alpha+1)\beta^2}$$

$$\mathrm{Var}(X) = \alpha(\alpha+1)\beta^2 - \alpha^2\beta^2 = \cancel{\alpha^2\beta^2} + \alpha\beta^2 - \cancel{\alpha^2\beta^2} = \alpha\beta^2$$

- mgf: $M_X(t) = \mathrm{E}e^{tX} = (1 - \beta t)^{-\alpha}, \; t < 1/\beta$

- Relationship of gamma and Poisson cdfs for integer α:

$$F_X(x|\alpha, \beta) = P(Y \geq \alpha) \quad \text{where } Y \sim \text{Poisson}(x/\beta)$$