STAT 5000

Statistical Methods I

Week 8

Fall 2024

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Unit 2

Blocking: Efficiency & Diagnostics

- Is RCBD better than CRD?
  - If the experiment was repeated on similar e.u.'s, should you block?
  - Not a question about how to analyze the observed data. Analysis should match the design.
- How to measure "better"?
  - Consider the **error variance** for each design: σ^2_{CRD} versus σ^2_{RCBD}
  - *Efficiency* of RCBD relative to CRD is $\sigma^2_{CRD}/\sigma^2_{RCBD}$.
  - Efficiency $> 1 \Rightarrow$ RCBD provides more precise estimates of treatment mean contrasts.

- Can also express Efficiency in terms of sample sizes

$$Var(\bar{Y}_{\cdot j} - \bar{Y}_{\cdot k}) = \sigma_e^2 \left(\frac{2}{n}\right)$$

To have $Var(\bar{Y}_{\cdot j} - \bar{Y}_{\cdot k})$ the same for both designs, we need

$$\sigma^2_{CRD} \left(\frac{2}{n_{CRD}}\right) = \sigma^2_{RCBD} \left(\frac{2}{n_{RCBD}}\right)$$

$$\Rightarrow \text{Efficiency} \;= \frac{\sigma^2_{CRD}}{\sigma^2_{RCBD}} = \frac{n_{CRD}}{n_{RCBD}}$$

- i.e. Efficiency = 1.5 $\Rightarrow$ CRD requires 50% more units per treatment than the RCBD

- In the randomized block design that provided the data $\hat{\sigma}^2_{RCBD} = MS_{error}$
- Snedecor and Cochran give

$$\hat{\sigma}^2_{CRD} = \frac{(n-1)MS_{blocks} + n(J-1)MS_{error}}{nJ-1}$$

  as an unbiased estimate of the error variance if a completely randomized design had been used instead (proof in Cochran and Cox, 1957)
- One complication is that $\hat{\sigma}^2_{CRD}$ and $\hat{\sigma}^2_{RCBD}$ have different degrees of freedom.

- Fisher used “relative amount of information”, an estimated efficiency,
$$\frac{(\mathrm{df}_{RCBD}+1)(\mathrm{df}_{CRD}+3)\hat{\sigma}^2_{CRD}}{(\mathrm{df}_{RCBD}+3)(\mathrm{df}_{CRD}+1)\hat{\sigma}^2_{RCBD}}$$
to adjust for differing d.f.
- Typical values of efficiency depend on the subject matter
- Values of 1.10 to 1.30 are common (e.g., blocking often reduces the number of units by 10 to 30 percent)

Example: Penicillin Experiment

ANOVA Table

Source	d.f.	SS	MS	F	p-value
Blocks	4	264	66.000	3.50	0.0407
Processes	3	70	23.333	1.24	0.3387
Error	12	226	18.833		
Total	19	560			

Example: Penicillin Experiment

- Randomized Complete Block Design: $\hat{\sigma}^2_{RCBD} = MS_{error} = 18.833$
- Completely Randomized Design:

$$\begin{aligned}\hat{\sigma}^2_{CRD} &= \frac{(n-1)MS_{blocks} + n(J-1)MS_{error}}{nJ-1} \\ &= \frac{(4)(66.0) + (5)(3)(18.833)}{19} = 28.76289\end{aligned}$$

- Estimated efficiency:

$$= \frac{(12+1)(16+3)(28.76289)}{(12+3)(16+1)(18.833)} = 1.48$$

- To have the same efficiency, $n_{\text{CRD}} = 1.48 n_{\text{RCBD}}$

Assumptions (treatments used equally often in each block)

- Independence of errors
- Homogeneous error variance
- Normality of errors
- Block and treatment effects are additive (no interaction)

- Relative importance of first three assumptions and diagnoses are similar to before
- Non-parametric test: Friedman test performs an ANOVA with observed data replaced by ranks within blocks

Diagnose Assumption: Additive Model

- *Additivity*: treatment effect is the same within each block.

$$\text{Additive Model: } Y_{ij} = \mu + \beta_i + \tau_j + \epsilon_{ij}$$

- *Non-additivity*: treatment effect varies depending on block.

$$\text{Non-Additive Model: } Y_{ij} = \mu + \beta_i + \tau_j + (\beta\tau)_{ij} + \epsilon_{ij}$$

- Unless replicates of treatments within blocks, we cannot test for significance of the interaction $(\beta\tau)_{ij}$

Diagnose Assumption: Tukey's Test for Non-Additivity

- Used when no replicates of treatments within blocks
- Detects one specific pattern of non-additivity: multiplicative interaction between block and treatment effects.

$$\text{Tukey Model: } Y_{ij} = \mu + \beta_i + \tau_j + \kappa\beta_i\tau_j + \epsilon_{ij}$$

- Tukey constructed an F-test for $H_0 : \kappa = 0$ vs. $H_a : \kappa \neq 0$

Unit 2

Blocking: Latin Squares

More than 1 Blocking Factor

- Can use a broader definition of blocks
- Example: if gender and age are both blocking factors, then one could use as blocks: males 20-29, males 30-39, females 20-29, etc.
- Problem: have many blocks = need many experimental units
- Special case: Latin Square Designs

Latin Squares Design

- Two blocking variables
- Number of levels for each blocking factor = number of treatments (or its multiple)
  - 3 treatments: each block has three levels (or 6, 9, 12, etc.)
  - 4 treatments: each block has four levels (or 8, 12, 16, etc.)
- Each block contains only one unit for each treatment
- Each level of each blocking variable gets all treatments

Example: **Fuel Efficiency Study**

- Block 1 (row blocks) = Drivers (1 through 4)
- Block 2 (column blocks) = Cars (1 through 4)
- Treatments = Fuel Additives (A, B, C, D)
- Each treatment occurs once in each row and once in each column

Example: Fuel Efficiency Study

	Cars			
Drivers	1	2	3	4
1	B	A	C	D
2	A	C	D	B
3	C	D	B	A
4	D	B	A	C

- Advantages
  - Can estimate treatment effects in a small study
  - Can use two blocking factors to reduce variability
- Limitations
  - Levels of each blocking variable must equal (or be a multiple of) the number of treatments
  - Analysis assumes no interactions between blocking factors and treatments: critical, because each block contains only one unit for each treatment
  - Few degrees of freedom for error, can increase by using multiple Latin squares

Model

$$Y_{ijk} = \mu + \beta_i + \gamma_j + \tau_k + \epsilon_{ijk}$$

where $i, j, k = 1, 2, \ldots, r$

- β_i first blocking factor effect
- γ_j second blocking factor effect
- τ_k is a fixed treatment effect
- k is the treatment and is determined by (i, j)
- $\epsilon_{ijk} \sim N(0, \sigma^2)$

ANOVA

source	d.f.	SS
Block 1	$r-1$	$r\sum_i(\bar{Y}_{i..}-\bar{Y}_{...})^2$
Block 2	$r-1$	$r\sum_j(\bar{Y}_{.j.}-\bar{Y}_{...})^2$
Treatment	$r-1$	$r\sum_k(\bar{Y}_{..k}-\bar{Y}_{...})^2$
Error	$(r-1)(r-2)$	SS_{error}
Total	r^2-1	$\sum_i\sum_j(Y_{ij.}-\bar{Y}_{...})^2$

where

- $\bar{Y}_{...} = r^{-2}\sum_i\sum_j Y_{ij.}$
- $SS_{\text{error}} = SS_{\text{total}} - SS_{\text{block 1}} - SS_{\text{block2}} - SS_{\text{trt}}$

Inference

- Tests for treatment based on usual F-test
- CIs for means, pairwise comparisons, contrasts as in one-way ANOVA

Unit 2

Multi-Factor Designs

Factor & Levels

- A **factor** is an explanatory variable studied in an investigation.
- The different values of a factor are called **levels**.
- Often correspond to treatments in an experiment:

Consider the fuel efficiency study:

- A treatment factor is the fuel additive
- The levels correspond to the different additive types

- Examine effects of two or more factors within a single experiment/study
- Examples:
  - Vary price (3 levels) and type of advertising media (2 levels) to explore effect on sales
  - Examine the effects of varieties (4 levels) and soil type (3 levels) on corn yield
- Can learn about <u>interactions</u>:

  The effects of changing the levels of one factor are not the same across all levels of another factor

Factorial Experimental Design

- **Factorial designs** use combinations of levels of two or more factors as treatments
- Example
  - Factor A - 3 levels (a_1, a_2, a_3)
  - Factor B - 2 levels (b_1, b_2)
  - Combinations of A and B $\Rightarrow$ 6 Treatments

$$(a_1b_1,\ a_1b_2,\ a_2b_1,\ a_2b_2,\ a_3b_1,\ a_3b_2)$$

Terminology

- **Complete (full) factorial**: all possible combinations of factor levels are used
- Fractional factorial: only a subset used are used

- Complete designs are commonly used.
- Fractional designs are important in industrial applications.

Different Types of Factorial Designs:

- Factorial designs with two treatment factors
- Factorial designs with blocking
- Factorial designs with more than two factors
- Factorial designs with no replication
- Unbalanced factorial designs – combinations of factor levels are not all used the same number of times
(ANOVA considered in Stat 5100)

Larvae Example: Examine the effects of different concentrations of copper and zinc in water on the ability of minnow larvae to produce protein

- Factor A: Concentration of copper (0 or 150 ppm)
- Factor B: Concentration of zinc (0, 750 or 1500 ppm)
- Treatments: All 6 combination of 2 levels of copper and 3 levels of zinc (complete/full factorial treatment design)

Copper Conc.	Zinc Concentration		
	0 ppm	750 ppm	1500 ppm
0 ppm			
150 ppm			

Larvae Example: Examine the effects of different concentrations of copper and zinc in water on the ability of minnow larvae to produce protein

- Experimental units: Twelve water tanks containing minnow larvae
- Experimental design: CRD, Experimental units are randomly assigned to the 6 treatments with 2 units per treatment.
- Response Variable: protein content (μg/larva)

Larvae Example: Data

Copper Conc.	Zinc Concentration 0 ppm	750 ppm	1500 ppm
0 ppm	$Y_{111} = 201$	$Y_{121} = 173$	$Y_{131} = 115$
	$Y_{112} = 186$	$Y_{122} = 162$	$Y_{132} = 124$
150 ppm	$Y_{211} = 163$	$Y_{221} = 184$	$Y_{231} = 114$
	$Y_{212} = 182$	$Y_{222} = 157$	$Y_{232} = 108$

Notation:

- Factor A indexed $i = 1, \ldots, a$
- Factor B indexed $j = 1, \ldots, b$
- Replications indexed $k = 1, \ldots, n$
- Total number of observations $= nab$

Notation: Data

- Y_{ijk} = response of the k^{th} repetition of level i of factor A and level j of factor B
- $\bar{Y}_{ij\cdot} = \frac{1}{n}\sum_{k=1}^{n} Y_{ijk}$ = mean response of observations in level i of factor A and level j of factor B
- $\bar{Y}_{i\cdot\cdot} = \frac{1}{nb}\sum_{k=1}^{n}\sum_{j=1}^{b} Y_{ijk}$ = mean response of observations in level i of factor A
- $\bar{Y}_{\cdot j\cdot} = \frac{1}{na}\sum_{k=1}^{n}\sum_{i=1}^{a} Y_{ijk}$ = mean response of observations in level j of factor B
- $\bar{Y}_{\cdots} = \frac{1}{nab}\sum_{k=1}^{n}\sum_{i=1}^{a}\sum_{j=1}^{b} Y_{ijk}$ = overall mean response

Notation: Cell Means Model

$$Y_{ijk} = \mu_{ij} + \epsilon_{ijk} \qquad \epsilon_{ijk} \text{ are i.i.d. } N(0, \sigma^2)$$

- μ_{ij} = mean response to level *i* of factor A and level *j* of factor B
- $\bar{\mu}_{i\cdot} = \frac{1}{b}\sum_j \mu_{ij}$ = mean response of factor A at level *i*, averaging across the levels of factor B
- $\bar{\mu}_{\cdot j} = \frac{1}{a}\sum_i \mu_{ij}$ = mean response of factor B at level *j*, averaging across the levels of factor A
- $\bar{\mu}_{\cdot\cdot} = \frac{1}{ab}\sum_i \sum_j \mu_{ij}$ = mean response, averaging across the levels of both factors
- σ^2 = variance of responses in level *i* of factor A and level *j* of factor B

Research Questions:

- Are the 6 response means (μ_{ij}) the same?
- Are mean responses to copper levels the same, averaging over zinc levels? $\bar{\mu}_{1\cdot} = \bar{\mu}_{2\cdot}$?
- Are mean responses to zinc levels the same, averaging over copper levels? $\bar{\mu}_{\cdot 1} = \bar{\mu}_{\cdot 2} = \bar{\mu}_{\cdot 3}$?
- Are differences in mean responses between copper levels the same across zinc levels? $(\mu_{11} - \mu_{21}) = (\mu_{12} - \mu_{22}) = (\mu_{13} - \mu_{23})$?

Larvae Example: Model

As in the minnow larvae experiment, suppose there are 2 levels of factor A (i=1,2), 3 levels of factor B (j=1,2,3) and 2 units assigned to each of the combinations of the two factors (k=1,2)

$$\begin{bmatrix} Y_{111} \\ Y_{112} \\ Y_{121} \\ Y_{122} \\ Y_{131} \\ Y_{132} \\ Y_{211} \\ Y_{212} \\ Y_{221} \\ Y_{222} \\ Y_{231} \\ Y_{232} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_{11} \\ \mu_{12} \\ \mu_{13} \\ \mu_{21} \\ \mu_{22} \\ \mu_{23} \end{bmatrix} + \begin{bmatrix} \epsilon_{111} \\ \epsilon_{112} \\ \epsilon_{121} \\ \epsilon_{122} \\ \epsilon_{131} \\ \epsilon_{132} \\ \epsilon_{211} \\ \epsilon_{212} \\ \epsilon_{221} \\ \epsilon_{222} \\ \epsilon_{231} \\ \epsilon_{232} \end{bmatrix}$$

Larvae Example: Least Squares Estimate

The model has the form of a linear model: $\mathbf{Y} = \mathbf{X}\beta + \epsilon$

The least squares estimates of the mean responses for the six combinations of the levels of the two factors are

$$\hat{\beta} = \begin{bmatrix} \hat{\mu}_{11} \\ \hat{\mu}_{12} \\ \hat{\mu}_{13} \\ \hat{\mu}_{21} \\ \hat{\mu}_{22} \\ \hat{\mu}_{23} \end{bmatrix} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y} = \begin{bmatrix} \bar{Y}_{11\cdot} \\ \bar{Y}_{12\cdot} \\ \bar{Y}_{13\cdot} \\ \bar{Y}_{21\cdot} \\ \bar{Y}_{22\cdot} \\ \bar{Y}_{23\cdot} \end{bmatrix} = \begin{bmatrix} 193.5 \\ 167.5 \\ 119.5 \\ 172.5 \\ 170.5 \\ 111.0 \end{bmatrix}$$

ANOVA Table

- The following formulas assume equal sample sizes $n_{ij} = n$. (Note that the total sample size is abn)
- The ANOVA Table for the $a \times b$ treatments is

Source	d.f.	Sum of Squares
Model	$ab - 1$	$SS_{\text{model}} = n \sum_i \sum_j (\bar{Y}_{ij\cdot} - \bar{Y}_{\dots})^2$
Error	$ab(n-1)$	$SS_{\text{error}} = \sum_i \sum_j \sum_k (Y_{ijk} - \bar{Y}_{ij\cdot})^2$
Total	$abn - 1$	$SS_{\text{total}} = \sum_i \sum_j \sum_k (Y_{ijk} - \bar{Y}_{\dots})^2$

Larvae Example: ANOVA Table

Source	d.f.	SS	MS	F-test	p-value
Model	5	10755.75	2151.15	16.62	0.0019
Error	6	776.50	129.42		
Total	11	11532.25			

F-test for Treatment Effects

- H_0 : μ_{ij} are equal for all $i = 1, \ldots, a$ and $j = 1, \ldots, b$
- H_a : at least one μ_{ij} is different
- Reject H_0 if

$$F = \frac{MS_{\text{model}}}{MS_{\text{error}}} > F_{ab-1, ab(n-1), 1-\alpha}$$

Example:

- F = 16.62 with p-value 0.0019 $\Rightarrow$ Reject H_0
- There is substantial evidence that at least one of the mean responses for the six treatments is different.

Additional Research Questions:

- Question 2: Are mean responses to copper levels the same, averaging over zinc levels? ($\bar{\mu}_{1\cdot} = \bar{\mu}_{2\cdot}$)
- Question 3: Are mean responses to zinc levels the same, averaging over copper levels? ($\bar{\mu}_{\cdot 1} = \bar{\mu}_{\cdot 2} = \bar{\mu}_{\cdot 3}$)
- Question 4: Are differences in mean responses between copper levels consistent across zinc levels? ($\mu_{11} - \mu_{21} = \mu_{12} - \mu_{22} = \mu_{13} - \mu_{23}$)

Questions 2, 3 and 4 can be addressed with contrasts on the cell means μ_{ij}

Contrasts and Factor Effects: Question 2

Are the means equal for copper levels?

- H_0: $\bar{\mu}_{1\cdot} - \bar{\mu}_{2\cdot} = 0$
- Contrast:

$$\bar{\mu}_{1\cdot} - \bar{\mu}_{2\cdot} = \frac{\mu_{11} + \mu_{12} + \mu_{13}}{3} - \frac{\mu_{21} + \mu_{22} + \mu_{23}}{3}$$

Contrasts and Factor Effects: Question 3

Are the means equal for zinc levels?

- H_0: $\bar{\mu}_{\cdot 1} = \bar{\mu}_{\cdot 2} = \bar{\mu}_{\cdot 3}$
- Contrasts:

$$\bar{\mu}_{\cdot 1} - \bar{\mu}_{\cdot 2} = \frac{\mu_{11} + \mu_{21}}{2} - \frac{\mu_{12} + \mu_{22}}{2}$$

$$\frac{\bar{\mu}_{\cdot 1} + \bar{\mu}_{\cdot 2}}{2} - \bar{\mu}_{\cdot 3} = \frac{\mu_{11} + \mu_{21} + \mu_{12} + \mu_{22}}{4} - \frac{\mu_{13} + \mu_{23}}{2}$$

Contrasts and Factor Effects: **Question 4**

Are difference in mean response for copper levels consistent across zinc levels?

- $H_0 : \mu_{11} - \mu_{21} = \mu_{12} - \mu_{22} = \mu_{13} - \mu_{23}$
- Contrasts:

$$(\mu_{11} - \mu_{21}) - (\mu_{12} - \mu_{22})$$

$$\frac{(\mu_{11} - \mu_{21}) + (\mu_{12} - \mu_{22})}{2} - (\mu_{13} - \mu_{23})$$

Contrasts and Factor Effects:

- For convenience, orthogonal contrasts are used. This will enable us to add contrast sums of squares to develop an F-test for the null hypothesis
- Five contrasts are orthogonal among the cell means μ_{ij}

	Cell Means					
	μ_{11}	μ_{12}	μ_{13}	μ_{21}	μ_{22}	μ_{23}
γ_1: Copper effect	1/3	1/3	1/3	-1/3	-1/3	-1/3
γ_2: Zinc effect 1	1/2	-1/2	0	1/2	-1/2	0
γ_3: Zinc effect 2	1/4	1/4	-1/2	1/4	1/4	-1/2
γ_4: Interaction 1	1	-1	0	-1	1	0
γ_5: Interaction 2	-1/2	-1/2	1	1/2	1/2	-1

Contrasts and Factor Effects:

- Obtain sums of squares for each of these orthogonal contrasts.

Contrast	SS
γ_1: Copper effect	234.08
γ_2: Zinc effect 1	392.00
γ_3: Zinc effect 2	9841.50
γ_4: Interaction 1	288.00
γ_5: Interaction 2	0.16667

- Because these 5 (=6-1) contrasts are orthogonal,

$$SS_{\text{model}} = SS_{\gamma_1} + SS_{\gamma_2} + SS_{\gamma_3} + SS_{\gamma_4} + SS_{\gamma_5}$$

Factor Effects

- **Main effect**: difference (or contrast) between levels of one factor averaged over all levels of the other factor(s).
- **Simple effect**: difference (or contrast) between levels of one factor at one specific level of the other. e.g., difference between different copper concentration levels when zinc concentration = 0ppm .
- **Interaction** exists when simple effects are not the same.
  - Interaction measures the differences between the simple effects of one factor at different levels of the other factor.
  - Equivalent to non-parallel lines in a plot of means.
  - Could be difference in the magnitude or in direction of responses.

Contrasts and Factor Effects: Larvae Example

- Divide model sums of squares into main effects plus interaction

Source	d.f.	Sum of Squares
Copper	1	$SS_{\gamma_1} = 234.08$
Zinc	2	$SS_{\gamma_2} + SS_{\gamma_3} = 10233.50$
Interaction	2	$SS_{\gamma_4} + SS_{\gamma_5} = 288.17$
Error	6	$SS_{\text{error}} = 776.50$
Total	11	$SS_{\text{total}} = 11532.25$

Larvae Example: ANOVA Table

- Including main effects and interactions:

Source	d.f.	SS	MS	F-test	p-value
Copper levels	1	234.08	234.08	1.809	0.2272
Zinc levels	2	10233.50	5116.75	39.536	0.0004
Interaction	2	288.17	144.085	1.113	0.3881
Error	6	776.50	129.42		
Total	11	11532.25			

Two-Way ANOVA Table

- In general, partition SS_{model} to assess main effects for each factor and interaction

source of variation	degrees of freedom	sums of squares
Factor A	$a-1$	$nb\sum_i(\bar{Y}_{i..}-\bar{Y}_{...})^2$
Factor B	$b-1$	$na\sum_j(\bar{Y}_{.j.}-\bar{Y}_{...})^2$
Interaction AB	$(a-1)(b-1)$	$n\sum_i\sum_j(\bar{Y}_{ij.}-\bar{Y}_{i..}-\bar{Y}_{.j.}+\bar{Y}_{...})^2$
Error	$ab(n-1)$	$\sum_i\sum_j\sum_k(Y_{ijk}-\bar{Y}_{ij.})^2$
Total	$abn-1$	$\sum_i\sum_j\sum_k(Y_{ijk}-\bar{Y}_{...})^2$

Expected Mean Squares

$E(MS_{error}) = \sigma^2$

$E(MS_A) = \sigma^2 + nb\sum_i(\bar{\mu}_{i.} - \bar{\mu}_{..})^2/(a-1)$

$E(MS_B) = \sigma^2 + na\sum_j(\bar{\mu}_{.j} - \bar{\mu}_{..})^2/(b-1)$

$E(MS_{AB}) = \sigma^2 + n\frac{\sum_i\sum_j(\mu_{ij}-\bar{\mu}_{i.}-\bar{\mu}_{.j}+\bar{\mu}_{..})^2}{(a-1)(b-1)}$

- All sums of squares (or mean squares) are independent
- Test hypotheses about marginal means and interaction effects using F-tests

F-test for Factor A Main Effect

- $H_0 : \bar{\mu}_{1.} = \bar{\mu}_{2.} = \cdots = \bar{\mu}_{a.}$
- H_a : at least one $\bar{\mu}_{i.}$ is different, $i = 1, \ldots, a$
- Reject H_0 if

$$F = \frac{MS_A}{MS_{\text{error}}} \geq F_{a-1, ab(n-1), 1-\alpha}$$

Larvae Example: **F-test for Copper Main Effect**

- $H_0 : \bar{\mu}_{1.} = \bar{\mu}_{2.}$
- $H_a : \bar{\mu}_{1.} \neq \bar{\mu}_{2.}$
- $F = 1.809$ with p-value $= 0.2272 \implies$ Fail to reject H_0
- There is no evidence of a main effect of copper.

F-test for Factor B Main Effect

- $H_0 : \bar{\mu}_{.1} = \bar{\mu}_{.2} = \cdots = \bar{\mu}_{.b}$
- H_a : at least one $\bar{\mu}_{.j}$ is different, $j = 1, \ldots, b$
- Reject H_0 if

$$F = \frac{MS_B}{MS_{\text{error}}} \geq F_{b-1, ab(n-1), 1-\alpha}$$

Larvae Example: F-test for Zinc Main Effect

- $H_0 : \bar{\mu}_{.1} = \bar{\mu}_{.2} = \bar{\mu}_{.3}$
- H_a : at least one $\bar{\mu}_{.j}$ is different, $j = 1, 2, 3$
- $F = 39.536$ with p-value $= 0.0004 \implies$ Reject H_0
- There is overwhelming evidence of a main effect of zinc.

F-test for Interaction

- $H_0 : (\mu_{ij} - \mu_{kj}) = (\mu_{ir} - \mu_{kr})$ for all $i \neq k$ and $j \neq r$
- H_a : at least one $(\mu_{ij} - \mu_{kj}) \neq (\mu_{ir} - \mu_{kr})$ for some $i \neq k$ and $j \neq r$
- Reject H_0 if

$$F = \frac{MS_{AB}}{MS_{\text{error}}} \geq F_{(a-1)(b-1),ab(n-1),1-\alpha}$$

Larvae Example: F-test for Copper-Zinc Interaction

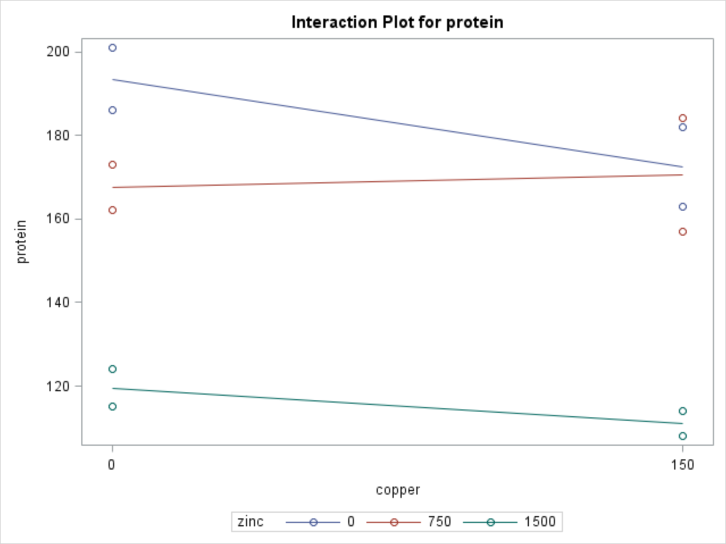
- $H_0 : (\mu_{ij} - \mu_{kj}) = (\mu_{ir} - \mu_{kr})$ for all $i \neq k$ and $j \neq r$
- H_a : at least one $(\mu_{ij} - \mu_{kj}) \neq (\mu_{ir} - \mu_{kr})$ for some $i \neq k$ and $j \neq r$
- $F = 1.113$ with p-value $= 0.3881 \implies$ Fail to reject H_0
- There is no evidence of any interaction effect between copper and zinc.
  - The effect of copper is the same for all levels of zinc and the effect of zinc is the same for all levels of copper.
  - No interaction $\Rightarrow$ homogeneous simple effects

Interpretation of Results When There Is NO Interaction

- Interpretation of marginal means is straightforward
- *F*-test for each factor: Are there differences (effects) in response means for different levels of the factor, averaging across all levels of other factors?
- Contrasts in marginal means: estimate contrast of mean responses across levels of one factor averaging across all levels of any other factors

Two Factor Study–Interactions

- When interactions are present:
  - The effect of factor A is not the same at every level of factor B
  - The effect of factor B is not the same at every level of factor A
- Can see interactions by plotting the sample response means versus levels of factor A and connect points within each level of factor B
- Can also see interactions in tables of sample means. Look at differences between two levels of one factor at each level of the other factor.


Interaction Plot for protein
protein
200
180
160
140
120
0
150
copper
zinc
0
750
1500

What if there is an interaction?

- Main effect, $\bar{\mu}_{\cdot 1} - \bar{\mu}_{\cdot 2}$, is not the same as some simple effect, $\mu_{i1} - \mu_{i2}$
- Each simple effect for the first factor ($\mu_{ij} - \mu_{rj}$) is conditional on a level of the other factor
- Why is there an interaction? Are effects additive on some other scale?
- Is the interaction effect practically significant?
- Should I report main effect or simple effect? Do differences in marginal means have any practical importance?

Questions?

Contact me:

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Student Office Hours: Thursdays @ 10-11 am