Convergence concepts

Slutsky's theorem & Delta method

Slutsky's theorem: If $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{p} c$, then

$\sigma = \sigma_P$

$\hat{\sigma} \xrightarrow{P} \sigma$

$\lim_{n\to\infty} P(|\hat{\sigma}-\sigma| > \varepsilon) = 0$

1. it holds that

$$\begin{pmatrix} X_n \\ Y_n \end{pmatrix} \xrightarrow{d} \begin{pmatrix} X \\ c \end{pmatrix}$$

2. if $g : \mathbb{R}^2 \to \mathbb{R}$ is continuous at (x, c) for any $x \in \mathbb{R}$, then

$$g(X_n, Y_n) \xrightarrow{d} g(X, c)$$

3. In particular,

$X_n \xrightarrow{d} X, Y_n \xrightarrow{P} c$ $\quad g(x,y) := xy$ $\quad g(x,y) = \frac{x}{y}$ $(y \neq 0)$

$$X_n + Y_n \xrightarrow{d} X + c, \qquad X_n Y_n \xrightarrow{d} Yc, \qquad \& \text{ if } c \neq 0,\ X_n/Y_n \xrightarrow{d} X/c.$$

$g(x,y) = x+y$

$g(X_n, Y_n) := X_n + Y_n \to g(X,c) = X + c$

Examples: Let $X_1, X_2, \ldots$ be iid with mean μ and variance σ^2

① $\begin{cases} \sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} Z \sim N(0,\sigma^2) & \text{by CLT} \\ \hat{\sigma} = \sqrt{\hat{\sigma}^2} \xrightarrow{P} \sigma \text{ as } n \to \infty \end{cases}$

$$\Longrightarrow \frac{\sqrt{n}(\bar{X}_n - \mu)}{\hat{\sigma}} \xrightarrow{d} \frac{Z}{\sigma} \sim \frac{N(0,\sigma^2)}{\sigma} \sim N(0,1) \quad \blacksquare$$

② $\dfrac{\sqrt{n}(\bar{X}_n - \mu)}{S_n}$

$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} Z \sim N(0,\sigma^2)$ (I)

Recall:

$S_n^2 \xrightarrow{P} \sigma^2 \Rightarrow \sqrt{S_n^2} \xrightarrow{P} \sqrt{\sigma^2}$ by Continuous mapping Theorem (II)

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{S_n} \xrightarrow{d} \frac{Z}{\sigma} \sim \frac{N(0,\sigma^2)}{\sigma} \sim N(0,1)$$

③ $$\underbrace{\frac{\sqrt{n}(\bar{X}_n-\mu)}{S_n}}_{\substack{\text{from ②, we know} \\ \frac{\sqrt{n}(\bar{X}_n-\mu)}{S_n} \xrightarrow{d} N(0,1)}} + \underbrace{\bar{X}_n}_{\substack{\bar{X}_n \xrightarrow{P} \mu \\ \text{by WLLN}}} \xrightarrow{d} \underset{\substack{\downarrow \\ N(0,1)}}{Z} + \underbrace{\mu} \sim N(\mu, 1)$$

④ $$\sqrt{n}(\bar{X}_n-\mu)^2 = \underbrace{\sqrt{n}(\bar{X}_n-\mu)}_{\xrightarrow{d} N(0,\sigma^2)} \underbrace{(\bar{X}_n-\mu)}_{\xrightarrow{P} 0} \xrightarrow{d} 0 \; N(0,\sigma^2) = 0$$

$$\left(\sqrt{n}(\bar{X}_n-\mu)^2 \xrightarrow{P} 0 \text{ as } n \to \infty\right)$$

⑤ $$\left[1 \cdot \frac{\sqrt{n}(\bar{X}_n-\mu)}{\underbrace{S_n}}\right]^2 \xrightarrow{d} \left[1 \cdot N(0,1)\right]^2 \sim \chi^2_1$$

$$\begin{cases} \dfrac{\sqrt{n}(\bar{X}_n-\mu)}{S_n} \xrightarrow{d} N(0,1) \\ 1 \xrightarrow{P} 1 \text{ as } n \to \infty \end{cases}$$

Convergence concepts

Slutsky's theorem & Delta method

The Delta Method finds the limiting distribution of a function $g(Y_n)$ of a sequence of r.v.s Y_n, which have a normal limit distribution (after centering/scaling)

Theorem (Delta Method): If $\sqrt{n}(Y_n - m) \xrightarrow{d} N(0, c^2)$ and $g'(m) \neq 0$, then

$$\sqrt{n}[g(Y_n) - g(m)] \xrightarrow{d} N(0, [g'(m)]^2 c^2).$$

Interpretation: If $Y_n \overset{a}{\sim} N(m, c^2/n)$ then $g(Y_n) \overset{a}{\sim} N(g(m), [g'(m)]^2 c^2/n)$.

$$\sqrt{n}(Y_n - m) \longrightarrow N(0, c^2)$$

$$\equiv$$

$$Y_n \overset{a}{\sim} N(m, c^2/n) \qquad g(Y_n) \overset{a}{\sim} N\left(g(m), \frac{g'(m)c^2}{n}\right)$$

Example: Suppose $X_1, X_2, \ldots,$ are iid Exponential(θ).

By CLT, $\sqrt{n}(\bar{X}_n - \theta) \xrightarrow{d} N(0, \theta^2)$. Consider $g(\bar{X}_n) = \log(\bar{X}_n)$.

$$\sqrt{n}\left(g(\bar{X}_n) - g(\theta)\right) \xrightarrow{d} ?$$

$$g(x) = \log x \Rightarrow g'(x) = \frac{1}{x}$$

$$\sqrt{n}(\log \bar{X}_n - \log \theta) \xrightarrow{d} N\left(0, (g'(\theta))^2 \theta^2\right), \quad g'(x) = \frac{1}{x},\ g'(\theta) = \frac{1}{\theta}$$

$$= N(0, 1) \quad \blacksquare$$

Note: $\left[\log \bar{X}_n \overset{a}{\sim} N\left(\log\theta, \frac{\left(\frac{1}{\theta^2}\right)\cancel{\theta^2}}{n}\right) = N\left(\log\theta, \frac{1}{n}\right) \text{ as } n \to \infty \right]$

ex: Let $X_1, X_2, \dots$ be i.i.d Poisson(λ), $\lambda > 0$.

Find the limiting distribution of $\sqrt{n}\left(2\sqrt{\bar{X}_n} - 2\sqrt{\lambda}\right)$.

$$\equiv \qquad \underbrace{\sqrt{n}\left(2\sqrt{\bar{X}_n} - 2\sqrt{\lambda}\right)}\xrightarrow{d} ?$$

$$\underbrace{\sqrt{n}(\bar{X}_n - \lambda)} \xrightarrow{d} \underbrace{N(0, \lambda)}$$

$$g(x) = 2\sqrt{x} \longrightarrow g'(x) = \frac{1}{\sqrt{x}}, \quad g'(\lambda) = \frac{1}{\sqrt{\lambda}}, \quad (g'(\lambda))^2 = \frac{1}{\lambda}$$

$$\Longrightarrow \begin{cases} \sqrt{n}\left(g(\bar{X}_n) - g(\lambda)\right) \xrightarrow{d} N\left(0, \frac{1}{\cancel{\lambda}}\cancel{\lambda}\right) = N(0,1) \\ \equiv \\ \sqrt{n}\left(2\sqrt{\bar{X}_n} - 2\sqrt{\lambda}\right) \xrightarrow{d} N(0,1) \end{cases}$$

How about $\sqrt{n}(\log \bar{X}_n - \log \lambda) \xrightarrow{d} N\left(0, \left(\frac{1}{\lambda}\right)^2 \lambda\right)$

$$g(x) = \log x \quad g'(x) = \frac{1}{x} \quad g'(\lambda) = \frac{1}{\lambda} \qquad = N\left(0, \frac{1}{\lambda}\right)$$

$$\Longrightarrow \sqrt{n}(\log \bar{X}_n - \log \lambda) \xrightarrow{d} N\left(0, \frac{1}{\lambda}\right)$$

END EXAM MATERIAL :)