Common univariate distributions

Discrete distributions: Binomial distribution

Binomial distribution, $X \sim$Binom(n,p), $0<p<1$

- pmf given by
$$f_X(x) = f_X(x|n,p) = P(X=x|n,p) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0,1,\dots,n$$

- Motivation: distribution for the number of successes in n independent Bernoulli(p) trials, i.e., if $Y_1,\dots,Y_n$ are independent Bern(p), where Y_i is the outcome of the ith trial ($Y_i = 1$ if the trial is "success" and 0 if "failure"), then $X = \sum_{i=1}^n Y_i$ is Binom(n,p) distributed

$$\sum_{x=0}^{n} \mathbb{P}(X=x) = 1$$

For a given $x = 0,1,\dots,n$, $P(X=x) = P\left(\sum_{i=1}^n Y_i = x\right)$

$$= \sum_{\substack{(y_1,\dots,y_n)\in\{0,1\}^n,\\ \sum_{i=1}^n y_i = x}} P(Y_1 = y_1,\dots,Y_n = y_n) = \sum_{\substack{(y_1,\dots,y_n)\in\{0,1\}^n,\\ \sum_{i=1}^n y_i = x}} \prod_{i=1}^n P(Y_i = y_i)$$

$$= \sum_{\substack{(y_1,\dots,y_n)\in\{0,1\}^n,\\ \sum_{i=1}^n y_i = x}} \prod_{i=1}^n p^{y_i}(1-p)^{1-y_i} = \sum_{\substack{(y_1,\dots,y_n)\in\{0,1\}^n,\\ \sum_{i=1}^n y_i = x}} p^x(1-p)^{n-x}$$

$= p^x(1-p)^{n-x} \times$ *"# of ways to choose exactly x components of $(y_1,\dots,y_n)$ to be 1"*

- mean: $\mathrm{E}X = np$ (proof next slide)
- variance: $\mathrm{Var}(X) = np(1-p)$
- Moment generating function: $M_X(t) = \mathrm{E}e^{tX} = [pe^t + (1-p)]^n$ for any $t \in \mathbb{R}$

  We've actually proven this already in the mgf section; the result follows from $M_X(t) = \sum_{x=0}^n \binom{n}{x}(pe^t)^x(1-p)^{n-x}$ and the binomial formula for $(a+b)^n$

Common univariate distributions

Discrete distributions: Binomial distribution (cont'd)

Derive mean

1. Using mgf $M_X(t) = \mathrm{E}e^{tX} = [pe^t + (1-p)]^n$,

$$\left.\frac{d[pe^t + (1-p)]^n}{dt}\right|_{t=0} = n[pe^t + (1-p)]^{n-1}pe^t\Big|_{t=0} = np$$

2. Or let $X = \sum_{i=1}^n Y_i$ where $Y_1, \ldots, Y_n$ are independent Bern(p), so that

Bin(n,p)

$$\mathrm{E}X = \mathrm{E}\left(\sum_{i=1}^n Y_i\right) = \sum_{i=1}^n \mathrm{E}Y_i = \sum_{i=1}^n p = np$$

3. Or using the direct definition

$$\begin{aligned}
\mathrm{E}X = \sum_{x=0}^n x\binom{n}{x}p^x(1-p)^{n-x} &= \sum_{x=1}^n x\frac{n!}{x!(n-x)!}p^x(1-p)^{n-x} \\
&= \sum_{x=1}^n x\frac{n}{x}\frac{(n-1)!}{(x-1)!(n-x)!}p^x(1-p)^{n-x} \\
&= n\sum_{x=1}^n \frac{(n-1)!}{(x-1)!(n-x)!}p^x(1-p)^{n-x} \\
&= n\sum_{x=1}^n \binom{n-1}{x-1}p^x(1-p)^{n-x} \\
&= n\sum_{z=0}^{n-1} \binom{n-1}{z}p^{z+1}(1-p)^{n-(z+1)} \quad (z = x-1) \\
&= np\sum_{z=0}^{n-1} \binom{n-1}{z}p^z(1-p)^{n-1-z}
\end{aligned}$$

Variance derivation is similar but messy: compute

$$\mathrm{E}X(X-1) = \sum_{x=0}^n x(x-1)\binom{n}{x}p^x(1-p)^{n-x} = \sum_{x=2}^n \frac{n!}{(x-2)!(n-2)!}p^x(1-p)^{n-x} = n(n-1)p^2$$

Then, $\mathrm{Var}(X) = \mathrm{E}X(X-1) + \mathrm{E}X - (\mathrm{E}X)^2 = n(n-1)p^2 + np - n^2p^2 = np(1-p)$

Common univariate distributions

Discrete distributions: Binomial distribution (cont'd)

Example: Test newly manufactured widgets - suppose that an inspection (a functionality test) is independently performed on each of $n = 500$ manufactured widgets. Suppose that, for high quality widgets, the probability that a given widget fails the test is 0.01. In $n = 500$ tests, what is the probability of no failures?

$X = \#$ of failures in $n = 500$ testes $\quad X \sim \text{Bin}(\underset{n}{500}, \underset{p}{.01})$

$$\mathbb{P}(X=0) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \binom{500}{0} p^0 (1-.01)^{500-0} \approx .0066$$

$$\mathbb{E}(X) = np = (500)(.01) = 5$$

Recall: Binomial dist. $\mathbb{P}(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$

$(a+b)^n = \sum_{x=0}^{n} \binom{n}{x} b^x a^{n-x}$

Common univariate distributions

Discrete distributions: Negative Binominal

$X \sim$Neg-Binom(r, p), $0 < p < 1$

r is integer $r \geq 1$

- pmf given by

pmf $\sum_{x=r}^{\infty} \mathbb{P}(X=x) = 1$

$$\mathbb{P}(X=x) = f_X(x) = f_X(x|r,p) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, \quad x = r, r+1, \ldots,$$

fix

Let's consider a sequence of S's + F's of length x where rth success is on the x^{th} trial

$S_s + F_s$ length of x rth

- Motivation: distribution for the number of independent Bernoulli(p) trials needed to obtain r successes

- $Y = X - r$ (number of failures prior to the rth success) also common

$$\mathbb{P}(Y=y) = f_Y(y|r,p) = \binom{y+r-1}{y} p^r (1-p)^y, \quad y = 0, 1, \ldots,$$

$$\mathbb{P}(Y=y) = \binom{y+r-1}{r-1} p^r (1-p)^y$$

$$\binom{y+r-1}{y} = \binom{y+r-1}{r-1}$$

- Showing that these probabilities sum to 1 is not easy (next slide)

- Be careful: both r.v.s X and Y (different) are called "negative binomial"

$Y = X - r \Rightarrow \mathbb{E}(Y) = \mathbb{E}(X-r) = \mathbb{E}(X) - r$

- Mean: $\mathrm{E}Y = \dfrac{r(1-p)}{p}$ and hence $\mathrm{E}X = \mathrm{E}Y + r = \dfrac{r}{p}$

- Variance: $\mathrm{Var}(Y) = \dfrac{r(1-p)}{p^2} = \mathrm{Var}(X)$

- $M_Y(t) = \mathrm{E}e^{tY} = \left[\dfrac{p}{1-(1-p)e^t}\right]^r$, $\boxed{t < -\log(1-p)}$,

$M_X(t) = \mathrm{E}e^{t(Y+r)} = \mathrm{E}e^{rt}e^{tY} = e^{rt}M_Y(t)$

$\underbrace{M_{Y+r}(t)}$

Common univariate distributions

Discrete distributions: Negative Binomial (cont'd)

To show probabilities sum to 1:

1. Newton's negative binomial formula : if $\alpha < 0$ and $|x| < 1$,

$\alpha = -r$, ~~k=y~~ $\boxed{x = p-1}$

$$\sum \binom{-r}{y}(p-1)^y = (1+(p-1))^{-r} = p^{-r}$$

$$**\quad (1+x)^{\alpha} = \sum_{k=0}^{\infty} g^{(k)}(0)\frac{(x-0)^k}{k!} = \sum_{k=0}^{\infty}\binom{\alpha}{k}^* x^k \qquad \binom{\alpha}{k}^* \equiv \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$$

Taylor expanding $g(x) = (1+x)^{\alpha}$ around 0: $g^{(0)}(0) = g(0) = 1$, $g^{(1)}(0) = g'(0) = \alpha$

2. for integers $r \geq 1$ and $k \geq 0$, note that

$$\binom{-r}{k}^*(-1)^k = (-1)^k\frac{(-r)(-r-1)\cdots(-r-k+1)}{k!} = \frac{(r)(r+1)\cdots(r+k-1)}{k!} \overset{*}{=} \binom{r+k-1}{k}$$

$$1 \overset{?}{=} \sum_{y=0}^{\infty} f_Y(y) = \underbrace{\sum_{y=0}^{\infty}\binom{y+r-1}{y}p^r(1-p)^y}_{\text{pmf}} \overset{*}{=} \sum_{y=0}^{\infty}\binom{-r}{y}^*(-1)^y p^r(1-p)^y$$

$$= p^r\sum_{y=0}^{\infty}\binom{-r}{y}^*\boxed{(p-1)^y} \qquad (-1)^y(1-p)^y = (p-1)^y$$

$$= p^r\underbrace{[1+(p-1)]^{-r}}_{p^{-r}} = 1 \qquad \sum\binom{-r}{y}(p-1)^y \overset{**}{=} (1+(p-1))^y$$

Show $M_Y(t) = Ee^{tY} = \left[\frac{p}{1-(1-p)e^t}\right]^r$ for $t < -\log(1-p)$

$$M_Y(t) \overset{\text{def}}{=} \mathbb{E}\left[e^{tY}\right] = \sum_{y=0}^{\infty} e^{ty}\binom{y+r-1}{y}p^r(1-p)^y$$

$$= p^r\sum_{y=0}^{\infty}\binom{y+r-1}{y}\left[e^t(1-p)\right]^y$$

$$\cancel{= p^r\left[1+e^t(1-p)\right]^{y+r-1}} \quad \text{wrong}$$

$$= p^r\sum_{y=0}^{\infty}\underbrace{\binom{-r}{y}^*(-1)^y}_{\text{step 2}}\boxed{e^{ty}}(1-p)^y \qquad \boxed{(p-1)^y}$$

$$= p^r \sum_{y=0}^{\infty} \binom{-r}{y}^* \left[e^t(p-1)\right]^y$$ step1

$M_X(t)$

Common univariate distributions

$$p^r \left[1+e^t(p-1)\right]^{-r}$$

Discrete distributions: Geometric

$$= \left[\frac{p}{1+e^t(p-1)}\right]^r$$

$$= \left[\frac{p}{1+\cancel{(-1)}(1-p)e^t}\right]$$

Note: If $|e^t(p-1)|<1$ or $t < -\log(1-p)$

$X \sim \text{Geom}(p)$, $0 < p < 1$

- special case of Negative Binomial$(r = 1, p)$

- Motivation: distribution for the number of independent Bernoulli(p) trials needed to obtain 1st success

- pmf given by

$$f_X(x) = f_X(x|p) = p(1-p)^{x-1}, \quad x = 1, 2, 3, \ldots,$$

- Mean: $\mathrm{E}X = \dfrac{1}{p}$

- Variance: $\mathrm{Var}(X) = \dfrac{1-p}{p^2}$

- $M_X(t) = Ee^{tX} = \dfrac{pe^t}{1-(1-p)e^t}$ for $t < -\log(1-p)$