

Functions of a random variable

Introduction

- Consider a random variable $X \sim F_X(\cdot)$ and a function $g: \mathbb{R} \rightarrow \mathbb{R}$

\rightarrow is a R.V.

any function

- Then, $Y = g(X)$ is also a r.v., having its own cdf $F_Y(\cdot)$

Y is a function of $X \Rightarrow$ we can describe the probabilistic behavior of Y in terms of that X .

- Formally, there is also an inverse mapping g^{-1} defined by

$$g^{-1}(A) = \{x \in \mathbb{R} : g(x) \in A\} \quad \text{for any } A \subset \mathbb{R}.$$

$y = g(x) \Rightarrow g(x): \mathcal{X} \rightarrow \mathcal{Y}$, the sample space of Y
 the sample space of X \leftarrow $g^{-1}(A) = \{x \in \mathcal{X} : g(x) \in A\}$

- Distribution of Y is determined by the distribution of X and the function g

$$P_Y(Y \in A) = P_X(g(X) \in A) = P_X(X \in g^{-1}(A)) \quad \text{for } A \subset \mathbb{R}$$

\hookrightarrow This means the distribution of Y depends on the functions F_X and g .

- If X has pdf/pmf $f_X(x)$, then the **range** or **support** of X is

$$\mathcal{X} = \{x \in \mathbb{R} : f_X(x) > 0\}.$$

If Y has pdf/pmf $f_Y(y)$, then the range or support of Y will be

$$\mathcal{Y} = \{y \in \mathbb{R} : f_Y(y) > 0\} = \{g(x) : x \in \mathcal{X}\}.$$

Note: The mapping g^{-1} takes sets into sets;

$g^{-1}(A)$ is the set of points in \mathcal{X} that $g(x)$ takes into the set A .

Note: $A = \{y\}$ (A is a point set) $g^{-1}(\{y\}) = \{x \in \mathcal{X} : g(x) = y\}$

Now, $Y = g(X)$, then for any $A \subset \mathcal{Y}$,
 \mathcal{Y}
Sample Space
of Y

$$\begin{aligned}\mathbb{P}_Y(Y \in A) &= \mathbb{P}(g(X) \in A) \\ &= \mathbb{P}(\{x \in \mathcal{X} : g(x) \in A\}) \\ &= \mathbb{P}_X(X \in g^{-1}(A)).\end{aligned}$$

Functions of a random variable

Determining distributions: the discrete case

We'll separately treat two cases:

1. functions of discrete random variables (not much to this)
2. functions of continuous random variables (much more to this)

Result: If X is a discrete r.v. with pmf $f_X(x)$ (i.e., X has range $\mathcal{X} = \{x \in \mathbb{R} : f_X(x) > 0\}$ either finite or countably infinite), then $Y = g(X)$ is a discrete r.v. with pmf

$$f_Y(y) = P(Y = y) = \begin{cases} \sum_{x \in g^{-1}\{y\}} f_X(x) = \sum_{x \in \mathcal{X}, g(x)=y} f_X(x) & \text{for } y \in \mathcal{Y} \\ 0 & \text{for } y \notin \mathcal{Y} \end{cases}$$

where the range of Y is $\mathcal{Y} = \{g(x) : x \in \mathcal{X}\} = \{y \in \mathbb{R} : f_Y(y) > 0\}$.

Example: $X = \#$ of heads in 3 independent tosses of a fair coin

$$P(X=0) = \frac{1}{8} \quad P(X=1) = \frac{3}{8} \quad P(X=2) = \frac{3}{8} \quad P(X=3) = \frac{1}{8}$$

Let $Y = g(X) = (X-1)^2$.

x	$f_X(x)$	$Y = g(x)$
0	$1/8$	$(0-1)^2 = 1$
1	$3/8$	0
2	$3/8$	1
3	$1/8$	4

Range of $Y = \{0, 1, 4\}$

pmf of Y

$$f_Y(y) = P(Y=y) = \begin{cases} 3/8 & \text{If } Y=0 \\ 4/8 & \text{If } Y=1 \\ 1/8 & \text{If } Y=4 \\ 0 & \text{ow} \end{cases}$$

ex: We say $X \sim \text{Bin}(n, p)$ If $0 \leq p \leq 1$

$$f_X(x) = \mathbb{P}(X=x) = \binom{n}{x} p^x (1-p)^{n-x} \text{ for } x=0,1,2,\dots,n.$$

Define $g(x) = n-x$, $Y = n-X$.

$$X = \{0, 1, 2, \dots, n\}$$

$$Y = \{0, 1, 2, \dots, n\}$$

Now, $\forall y \in Y$, $n-x = g(x) = y \Leftrightarrow x = n-y \Rightarrow$

$\bar{g}^{-1}(\{y\})$ is the single point $x = n-y$ and

$$f_Y(y) = \sum_{x \in \bar{g}^{-1}(y)} f_X(x) = f_X(n-y)$$

$$\begin{aligned} \binom{n}{n-y} &= \binom{n}{y} \\ &= \binom{n}{y} p^{n-y} (1-p)^{n-(n-y)} \\ &= \binom{n}{y} (1-p)^y p^{n-y} \end{aligned}$$

$\Rightarrow Y$ is binomial dist. but with parameters n and $(1-p) \equiv$

$$Y \sim \text{Bin}(n, 1-p)$$

Functions of a random variable

Determining distributions: continuous case

For a continuous r.v. X , the r.v. $Y = g(X)$ will typically (but not always) be continuous.

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = P(\{x \in \mathcal{X} : g(x) \leq y\})$$

To determine the distribution of Y , one can try either of two approaches:

1. compute the cdf $F_Y(\cdot)$ of Y as

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = \int_{\{x \in \mathbb{R} : g(x) \leq y\}} f_X(x) dx$$

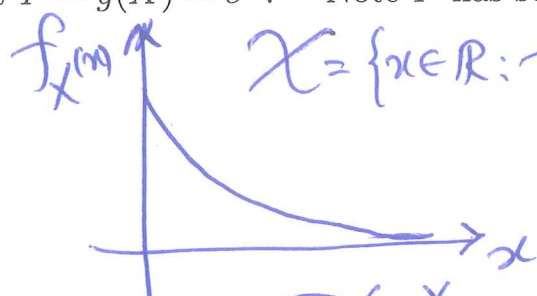
This is a general approach, but its success depends on computing the integral.

2. compute the pdf $f_Y(\cdot)$ directly through a transformation technique which is only valid if the function g is monotone or "piecewise monotone."

Example 1. Let X have pdf $f_X(x) = e^{-x}$, $x > 0$.

Let $Y = g(X) = e^X$. Note Y has support $\mathcal{Y} = \{e^x : x \in \mathcal{X}\} = \{e^x : x > 0\} = (1, \infty)$

$$\mathcal{X} = \{x \in \mathbb{R} : f_X(x) > 0\} = (0, \infty)$$



$$P(Y \leq y) = P(e^X \leq y) = 0 \text{ for } y \leq 1$$

$$P(Y \leq y) = P(e^X \leq y) = P(X \leq \log y) = \int_{-\infty}^{\log y} e^{-x} dx$$

$$= -e^{-x} \Big|_{-\infty}^{\log y} = 1 - \frac{1}{y} \text{ for } y > 1 \Rightarrow F_Y(y) = \begin{cases} 0 & \text{If } y \leq 1 \\ 1 - \frac{1}{y} & \text{If } y > 1 \end{cases}$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \begin{cases} 0 & \text{If } y \leq 1 \\ \frac{1}{y^2} & \text{If } y > 1 \end{cases}$$

Support of Y is $\mathcal{Y} = \{y : f_Y(y) > 0\} = (1, \infty)$

