everything up to now: $\sigma^2 I$

10. The Aitken Model

we relax the assumption of homogeneous variance, e.g.

different sample sizes when analyzing averages will give heterogeneous Var.

Orthogonal and Orthonormal Vectors

The $m \times 1$ vectors $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n$ are said to be *orthogonal* if and only if $\boldsymbol{p}_i^\top \boldsymbol{p}_j = 0$ for all $i \neq j$.

The $m \times 1$ vectors $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n$ are said to be *orthonormal* if and only if

$$\boldsymbol{p}_i^\top \boldsymbol{p}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Orthogonal Matrices

A square matrix $\boldsymbol{P}$ is said to be *orthogonal* if and only if $\boldsymbol{P}^\top \boldsymbol{P} = \boldsymbol{I}$.

Note that because $\boldsymbol{P}$ is square, $\boldsymbol{P}^\top \boldsymbol{P} = \boldsymbol{I}$ implies that $(\boldsymbol{P}^\top)^{-1} = \boldsymbol{P}$ and $\boldsymbol{P}^{-1} = \boldsymbol{P}^\top$. Thus, $\boldsymbol{P}^\top \boldsymbol{P} = \boldsymbol{P}\boldsymbol{P}^\top = \boldsymbol{I}$.

It follows that a square matrix $\boldsymbol{P}$ is orthogonal if and only if the rows of $\boldsymbol{P}$ are orthonormal vectors and the columns of $\boldsymbol{P}$ are orthonormal vectors.

The Spectral Decomposition Theorem

An $n \times n$ symmetric matrix $\boldsymbol{H}$ may be decomposed as

$$\boldsymbol{H} = \boldsymbol{P}\boxed{\boldsymbol{\Lambda}}\boldsymbol{P}^\top = \sum_{i=1}^{n} \lambda_i \, \boldsymbol{p}_i \, \boldsymbol{p}_i^\top,$$

where

- $\boldsymbol{P} = [\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n]$ is an $n \times n$ orthogonal matrix whose columns $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n$ are the orthonormal eigenvectors of $\boldsymbol{H}$, and
- $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is a diagonal matrix whose diagonal entries $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ are the eigenvalues of $\boldsymbol{H}$ (with λ_i corresponding to $\boldsymbol{p}_i$ for $i = 1, \ldots, n$).

Homework Problem

Suppose H is a symmetric matrix.

a) Prove that H is non-negative definite (NND) if and only if all its eigenvalues are non-negative.

b) Prove that H is positive definite (PD) if and only if all its eigenvalues are positive.

Let $\boxed{\boldsymbol{\Lambda}^{1/2}} = \text{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})$, and let $\boldsymbol{B} = \boldsymbol{P}\boldsymbol{\Lambda}^{1/2}\boldsymbol{P}^\top$. Then $\boldsymbol{B}$ is NND because all its eigenvalues $(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})$ are non-negative. Furthermore,

$$\boldsymbol{B}\boldsymbol{B} = \boldsymbol{P}\boldsymbol{\Lambda}^{1/2}\boldsymbol{P}^\top\boldsymbol{P}\boldsymbol{\Lambda}^{1/2}\boldsymbol{P}^\top = \boldsymbol{P}\boldsymbol{\Lambda}^{1/2}\boldsymbol{\Lambda}^{1/2}\boldsymbol{P}^\top = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^\top = \boldsymbol{H}. \quad \square$$

The Aitken Model

previously : $\sigma^2 I$

an alternative decomposition is based on the Cholesky decomposition

// end lecture 19 03-07-25

- $\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}, \; E(\boldsymbol{\epsilon}) = \boldsymbol{0}, \; \mathrm{Var}(\boldsymbol{\epsilon}) = \boxed{\sigma^2 \boldsymbol{V}}$

- Identical to the Gauss-Markov linear model except that

$$\mathrm{Var}(\boldsymbol{\epsilon}) = \sigma^2 \boldsymbol{V} \text{ instead of } \sigma^2 \boldsymbol{I}.$$

- $\boldsymbol{V}$ is assumed to be a known positive definite variance matrix.

- σ^2 is an unknown positive variance parameter.

We need a transformation of our model that results in a new model fulfilling GMM NE assumpt.