Shortcut for Obtaining SS from DF

y_{ijk}

trt: $i = 1, \ldots, t$ Obs. units: $k = 1, \ldots, m$

exp. units: $j = 1, \ldots, n$

$\bar{y}_{\cdots}$ = overall mean

represents overall mean $\bar{y}_{\cdots}$

Source	DF	Sum of Squares
trt	$t - 1$	$\sum_{i=1}^{t}\sum_{j=1}^{n}\sum_{k=1}^{m}(\bar{y}_{i\cdot\cdot} - \bar{y}_{\cdots})^2$
$xu(trt)$	$t(n-1)$	$\sum_{i=1}^{t}\sum_{j=1}^{n}\sum_{k=1}^{m}(\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot})^2$
$ou(xu, trt)$	$tn(m-1)$	$\sum_{i=1}^{t}\sum_{j=1}^{n}\sum_{k=1}^{m}(y_{ijk} - \bar{y}_{ij\cdot})^2$
$c.total$	$tnm - 1$	$\sum_{i=1}^{t}\sum_{j=1}^{n}\sum_{k=1}^{m}(y_{ijk} - \bar{y}_{\cdots})^2$

$(\bar{y}_{i\cdot\cdot} - \bar{y}_{\cdots})$ i

$(\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot})^2$ $ij - i$, $tn - t$

$\hat{=}$ Variation of exp. units within each trt-group

$ijk - ij$, $tnm - tn$

$(y_{ijk} - \bar{y}_{ij\cdot})^2$

$\hat{=}$ Variation of repeat measurements within each exp. unit

Source	DF	Sum of Squares	Mean Square
trt	$t-1$	$nm\sum_{i=1}^{t}(\overline{y}_{i\cdot\cdot} - \overline{y}_{\cdot\cdot\cdot})^2$	$E\left(SS(trt) / t-1 \right)$
$xu(trt)$	$tn-t$	$m\sum_{i=1}^{t}\sum_{j=1}^{n}(\overline{y}_{ij\cdot} - \overline{y}_{i\cdot\cdot})^2$	SS/DF
$ou(xu,trt)$	$tnm-tn$	$\sum_{i=1}^{t}\sum_{j=1}^{n}\sum_{k=1}^{m}(y_{ijk} - \overline{y}_{ij\cdot})^2$	
$c.total$	$tnm-1$	$\sum_{i=1}^{t}\sum_{j=1}^{n}\sum_{k=1}^{m}(y_{ijk} - \overline{y}_{\cdot\cdot\cdot})^2$	

Expected Mean Squares

- Based on our linear mixed-effects model ($\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{Zu} + \boldsymbol{e}$ and associated assumptions), we can find the expected value of each mean square in the ANOVA table.

- Examining these expected values helps us see ways to

  test for trt effects or e.g. $\sigma_u^2 = 0$

  1) test hypotheses of interest by computing ratios of mean squares, and

  σ_u^2 & σ_e^2

  2) estimate variance components by computing linear combinations of mean squares.

Expected Mean Squares

- For balanced designs, there are shortcuts (not presented here) for writing down expected mean squares.
- Rather than memorizing shortcuts, I think it is better to know how to derive expected mean squares.
- Before going through one example derivation, we will prove a useful result that you may already be familiar with.

Expectation of Sample Variance Numerator

Suppose $w_1, \ldots, w_k \overset{ind}{\sim} (\mu_w, \sigma_w^2)$. Then

$$
\begin{aligned}
k\sigma_w^2 &= \sum_{i=1}^{k} \sigma_w^2 = \sum_{i=1}^{k} \underbrace{E(w_i - \mu_w)^2}_{\sigma_w^2} = \sum_{i=1}^{k} E(w_i - \bar{w}_\cdot + \bar{w}_\cdot - \mu_w)^2 \\
&= E\left\{ \sum_{i=1}^{k} (w_i - \bar{w}_\cdot + \bar{w}_\cdot - \mu_w)^2 \right\} \\
&= E\left\{ \sum_{i=1}^{k} \left[(w_i - \bar{w}_\cdot)^2 + (\bar{w}_\cdot - \mu_w)^2 + \underbrace{2(\bar{w}_\cdot - \mu_w)}_{\text{constant}}(w_i - \bar{w}_\cdot) \right] \right\} \\
&= E\left\{ \sum_{i=1}^{k} (w_i - \bar{w}_\cdot)^2 + \sum_{i=1}^{k} (\bar{w}_\cdot - \mu_w)^2 + \cancel{2(\bar{w}_\cdot - \mu_w) \underbrace{\sum_{i=1}^{k} (w_i - \bar{w}_\cdot)}_{=0}} \right\}
\end{aligned}
$$

k

no change

$(a+b)^2 = a^2 + b^2 + 2ab$

move sum inside

Expectation of Sample Variance Numerator (ctd.)

constant

$$
\begin{aligned}
k\sigma_w^2 &= \mathrm{E}\left\{\sum_{i=1}^{k}(w_i-\bar{w}.)^2+\sum_{i=1}^{k}(\bar{w}.-\mu_w)^2\right\}\\
&= \mathrm{E}\left\{\sum_{i=1}^{k}(w_i-\bar{w}.)^2+k(\bar{w}.-\mu_w)^2\right\}\\
&= \mathrm{E}\left\{\sum_{i=1}^{k}(w_i-\bar{w}.)^2\right\}+kE(\bar{w}.-\mu_w)^2\\
&= \mathrm{E}\left\{\sum_{i=1}^{k}(w_i-\bar{w}.)^2\right\}+k\mathrm{Var}(\bar{w}.)\\
&= \mathrm{E}\left\{\sum_{i=1}^{k}(w_i-\bar{w}.)^2\right\}+\cancel{k}\sigma_w^2/\cancel{k}=\mathrm{E}\left\{\sum_{i=1}^{k}(w_i-\bar{w}.)^2\right\}+\sigma_w^2
\end{aligned}
$$

σ_w^2 / k

We have shown

$$k\sigma_w^2 = \mathrm{E}\left\{\sum_{i=1}^{k}(w_i - \bar{w}_.)^2\right\} + \sigma_w^2.$$

Therefore,

$$\mathrm{E}\left\{\sum_{i=1}^{k}(w_i - \bar{w}_.)^2\right\} = (k-1)\sigma_w^2.$$

This is just a special case of the Gauss-Markov model result
$\mathrm{E}(\widehat{\sigma}^2) = \sigma^2$. $\quad (\boldsymbol{y} = [w_1, \dots, w_k]^\top,\ \boldsymbol{X} = \mathbf{1},\ \boldsymbol{\beta} = [\mu_w],\ \sigma^2 = \sigma_w^2)$

Expected Value of MS_{trt}

from slide 9: $\frac{SS(trt)}{df}$ & then take expectation

write out $\overline{y}_{i\cdot\cdot}$ and $\overline{y}_{\cdot\cdot\cdot}$ * in terms of our model

$$\begin{aligned}
E(MS_{trt}) &= \frac{nm}{t-1}\sum_{i=1}^t E(\overline{y}_{i\cdot\cdot} - \overline{y}_{\cdot\cdot\cdot})^2 \\
&= \frac{nm}{t-1}\sum_{i=1}^t E(\mu + \tau_i + \overline{u}_{i\cdot} + \overline{e}_{i\cdot\cdot} - \mu - \overline{\tau}_{\cdot} - \overline{u}_{\cdot\cdot} - \overline{e}_{\cdot\cdot\cdot})^2 \\
&= \frac{nm}{t-1}\sum_{i=1}^t E(\tau_i - \overline{\tau}_{\cdot} + \overline{u}_{i\cdot} - \overline{u}_{\cdot\cdot} + \overline{e}_{i\cdot\cdot} - \overline{e}_{\cdot\cdot\cdot})^2 \\
&= \frac{nm}{t-1}\sum_{i=1}^t [(\tau_i - \overline{\tau}_{\cdot})^2 + E(\overline{u}_{i\cdot} - \overline{u}_{\cdot\cdot})^2 + E(\overline{e}_{i\cdot\cdot} - \overline{e}_{\cdot\cdot\cdot})^2] \\
&= \frac{nm}{t-1}[\sum_{i=1}^t (\tau_i - \overline{\tau}_{\cdot})^2 + E\{\sum_{i=1}^t (\overline{u}_{i\cdot} - \overline{u}_{\cdot\cdot})^2\} \\
&\qquad + E\{\sum_{i=1}^t (\overline{e}_{i\cdot\cdot} - \overline{e}_{\cdot\cdot\cdot})^2\}]
\end{aligned}$$

expand $(\ )^2$, cross products $= 0$ bc u_{ij} & e_{ijk} have mean 0

Variance comp.

*
$y_{ijk} = \mu + \tau_i + u_{ij} + e_{ijk}$

$\overline{y}_{i\cdot\cdot} = \mu + \tau_i + \overline{u}_{i\cdot} + \overline{e}_{i\cdot\cdot}$ $\quad$ $\overline{y}_{\cdot\cdot\cdot} = \mu + \overline{\tau}_{\cdot} + \overline{u}_{\cdot\cdot} + \overline{e}_{\cdot\cdot\cdot}$

So, to simplify $\mathrm{E}(MS_{trt})$ further, note that

$$\underline{\overline{u}_{1\cdot}, \ldots, \overline{u}_{t\cdot}} \overset{iid}{\sim} \mathcal{N}\left(0, \frac{\sigma_u^2}{n}\right).$$

Thus,

$$\mathrm{E}\left\{\sum_{i=1}^{t}(\overline{u}_{i\cdot} - \overline{u}_{\cdot\cdot})^2\right\} = (t-1)\frac{\sigma_u^2}{n}.$$

See slide 14 for why * (t-1)

Similarly,

$$\overline{e}_{1\cdot\cdot}, \ldots, \overline{e}_{t\cdot\cdot} \overset{iid}{\sim} \mathcal{N}\left(0, \frac{\sigma_e^2}{nm}\right)$$

so that

$$\mathrm{E}\left\{\sum_{i=1}^{t}(\overline{e}_{i\cdot\cdot} - \overline{e}_{\cdot\cdot\cdot})^2\right\} = \boxed{(t-1)\frac{\sigma_e^2}{nm}}.$$

use on next slide

It follows that

$(t-1)\cdot \sigma_u^2/n$

$$
\begin{aligned}
E(MS_{trt}) &= \frac{nm}{t-1}\left[\sum_{i=1}^{t}(\tau_i-\bar{\tau}_.)^2 + E\left\{\sum_{i=1}^{t}(\bar{u}_{i.}-\bar{u}_{..})^2\right\}\right. \\
&\qquad \left. + E\left\{\sum_{i=1}^{t}(\bar{e}_{i..}-\bar{e}_{...})^2\right\}\right] \\
&= \frac{nm}{t-1}\left[\sum_{i=1}^{t}(\tau_i-\bar{\tau}_.)^2 + (t-1)\frac{\sigma_u^2}{n}\right. \\
&\qquad \left. + (t-1)\frac{\sigma_e^2}{nm}\right] \\
&= \frac{nm}{t-1}\sum_{i=1}^{t}(\tau_i-\bar{\tau}_.)^2 + m\sigma_u^2 + \sigma_e^2.
\end{aligned}
$$

\# of obs. units within exp. units

no office hour today: 3/28/25

Similar calculations allow us to add an Expected Mean Squares (EMS) column to our ANOVA table.

Source	EMS
trt	$\sigma_e^2 + m\sigma_u^2 + \frac{nm}{t-1}\sum_{i=1}^{t}(\tau_i - \bar{\tau}.)^2$
$xu(trt)$	$\sigma_e^2 + m\sigma_u^2$
$ou(xu, trt)$	σ_e^2

end lecture 24

Wednesday

03-26-25