

Moment generating functions

More applications (cont'd)

3. Convergence (more on this later)

- Suppose a r.v. X has mgf $M_X(t)$ and suppose X_1, X_2, \dots are a sequence of random variables, where X_n has mgf $M_{X_n}(t)$ for each $n \geq 1$.

If

$$\lim_{n \rightarrow \infty} M_{X_n}(t) = M_X(t)$$

holds for all $t \in (-h, h)$ for some $h > 0$, then the sequence X_1, X_2, \dots converges (in distribution) to X .

$$\lim_{x \rightarrow \infty} (f(x))^g(x) = 1^\infty \quad \Rightarrow \quad \lim_{x \rightarrow \infty} (f(x))^g(x) = e^{\lim_{x \rightarrow \infty} (f(x)-1)g(x)}$$

Example: Suppose X_n pmf $f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$, $x = 0, 1, \dots, n$.

Then X_n is a Binomial(n, p) r.v., counting the number of "successes" in n separate trials (more later).

Suppose that $n \rightarrow \infty$ and $\lambda = np$ remains constant.

Recall: If $X \sim \text{Bin}(n, p) \Rightarrow M_X(t) = (pe^t + 1 - p)^n$

$$\text{bign } n \uparrow \quad p_{\text{small}} = \frac{\lambda}{n}, p \rightarrow 0$$

$$\lim_{n \rightarrow \infty} M_{X_n}(t) = \lim_{n \rightarrow \infty} (pe^t + 1 - p)^n = \lim_{n \rightarrow \infty} (1 + p(e^t - 1))^n$$

$$= \lim_{n \rightarrow \infty} \left(1 + \frac{p(e^t - 1)}{n}\right)^n$$

$$= e^{\lambda(e^t - 1)}$$

Recall: $\lim_{n \rightarrow \infty} (1 + \frac{a}{n})^n = e^a$

Note: Poisson distribution is often used to model counts of rare events. A random variable X having Poisson distribution has

$$P(X=x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0, 1, 2, \dots$$

$$\text{pmf } f(x) = e^{-\lambda} \frac{\lambda^x}{x!}, \quad x = 0, 1, 2, 3, \dots \quad \& \quad \text{mgf } M_X(t) = Ee^{tX} = e^{\lambda(e^t - 1)}, \quad t \in \mathbb{R}$$

$$\lim_{n \rightarrow \infty} M_{X_n}(t) = e^{\lambda(e^t - 1)} \quad \rightarrow \quad \text{MGF of a Poisson}(\lambda)$$

(ASIDE)

$$\rightarrow M_X(t) = \mathbb{E}[e^{tX}] \rightarrow \text{MGF}$$

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Other generating functions

There is nothing particularly illuminating about mgf's: these are just a technical device that are sometimes useful for proving theorems. In fact, they are not the only transforms for technical purposes or even the most useful (e.g., Fourier transforms/characteristic functions are often more useful in STAT 642).

- ✓ 1. $\log M_X(t)$ is the cumulant generating function

$$\frac{d^n \log M_X(t)}{dt^n} \Big|_{t=0} = \text{nth cumulant}$$

$$\left. \frac{d^n M_X(t)}{dt^n} \right|_{t=0} = \mathbb{E}(X^n)$$

where 1st cumulant is $\underline{\underline{\mathbb{E}X}}$ & 2nd cumulant is $\underline{\underline{\text{Var}(X)}}$

$$M_X(t) = \mathbb{E}[e^{tX}]$$

2. $E t^X$ is the factorial moment generating function (fmgf)

$$\frac{d^n E t^X}{dt^n} \Big|_{t=1} = EX(X-1)\cdots(X-n+1)$$

For a discrete r.v. the fmfgf is also called the probability generating function

$$n=2 \Rightarrow d^2 E$$

3. $\phi_X(t) = \mathbb{E} e^{itX} = \mathbb{E} \cos(tx) + i \mathbb{E} \sin(tx)$ is the characteristic function

- above $e^{itx} = \cos(tx) + i \sin(tx)$, $i = \sqrt{-1}$ (i.e., requires complex numbers)

- always exists for all $t \in \mathbb{R}$

- uniquely determines the distribution of X

- derivatives can be used to obtain moments

$$e^{i\theta} = \cos \theta + i \sin \theta$$

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One last thing

We have several times in lecture up to this point interchanged “orders of limits” (i.e., switch the order of derivatives and expectation or derivatives and summations or derivatives and integrals). This interchange is also implicit in using mgf’s to compute moments:

$$\begin{aligned} \frac{d^n M_X(t)}{dt^n} \Big|_{t=0} & \stackrel{\textcircled{N}}{=} \frac{d^n E e^{tX}}{dt^n} \Big|_{t=0} \\ & \stackrel{\textcircled{P}}{=} E \left(\frac{d^n e^{tX}}{dt^n} \Big|_{t=0} \right) \\ & = E \left(X^n e^{tX} \Big|_{t=0} \right) \\ & = \underline{\underline{E X^n}} \end{aligned}$$

We don't need to worry about the validity of these interchanges here (it's covered in STAT 642 where it makes more sense with the right technical material in hand).

Section 2.4 of Casella & Berger addresses the issue of conditions under which such interchanges are permissible; this is for mathematical completeness.

Inequalities

Markov inequality

- So far, we have discussed computing exact probabilities
- Inequalities can provide insight with less calculation and have theoretical importance/uses

This material can also be found in Section 3.6, Casella & Berger (though this has nothing to do with two or more random variables at a time)

- **Markov inequality:** Suppose X is a r.v. and $g(x) \geq 0$. Then, for any $r > 0$,

$$P(g(X) \geq r) \leq \frac{Eg(X)}{r}.$$

$$\begin{aligned} g(x) &\geq r \\ \frac{g(x)}{r} &\geq 1 \end{aligned}$$

Proof.

$$\begin{aligned} P(g(x) \geq r) &= \int_{\{x: g(x) \geq r\}} f_X(x) dx \leq \int_{\{x: g(x) \geq r\}} \frac{g(x)}{r} f_X(x) dx \\ &\leq \frac{g(x)}{r} \int_{\mathbb{R}} f_X(x) dx \\ &= \frac{1}{r} \int_{\mathbb{R}} g(x) f_X(x) dx = \frac{1}{r} E[g(X)]. \end{aligned}$$

- Examples

– If $g(x) = |x|$ and $r > 0$, then

$$P(|X| \geq r) \leq \frac{E|X|}{r}$$

– If $g(x) = |x|^n$ and $r > 0$, then

$$P(|X| \geq r) = P(|X|^n \geq r^n) \leq \frac{E|X|^n}{r^n}$$

– If $g(x) = e^{tx}$ for $t > 0$ and $a \in \mathbb{R}$,

$$P(X \geq a) = P(e^{tX} \geq e^{at}) \leq \frac{Ee^{tX}}{e^{at}} = \frac{M_X(t)}{e^{at}}$$