STAT 543

Lec 24, W, Mar 26

MP & UMP testing → Homework 6 posted, due, M, Mar 31

Homework 5 solutions to be posted

- Exam 2 is coming up (3 weeks away) on W, April 16, 6:15–8:15 PM, 3rd floor seminar room
- No class on that W.
- I'll post: study guide (sufficiency/completeness/tests)
  - practice exams
  - bring new 1 page (front/back) formula sheet on exam 2 material (I'll post one to use if you'd like)
  - can bring calculator & previous formula sheet for exam 1
  - I'll provide table of distributions / STAT 542 facts on test as before

Recap: so we have MP tests for simple H_0 & simple H_1

Hypothesis Testing I

Uniformly Most Powerful (UMP) Tests

Definition: Let $f(\underset{\sim}{x}|\theta)$, $\theta \in \Theta \subset \mathbb{R}^p$, be the joint pdf/pmf of $\underset{\sim}{X} = (X_1, \ldots, X_n)$ and let Θ_0 be a nonempty proper subset of Θ. Then, a test rule $\varphi(\underset{\sim}{x})$ for testing $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \notin \Theta_0$ is called a **uniformly most powerful** (UMP) test of size α if

"best test" in this case when either H_0 or H_1 is composite

1. $\max_{\theta \in \Theta_0} \mathrm{E}_\theta \varphi(\underset{\sim}{X}) = \alpha$ ← right size
2. it holds that $\mathrm{E}_\theta \varphi(\underset{\sim}{X}) \geq \mathrm{E}_\theta \tilde{\varphi}(\underset{\sim}{X})$ for all $\theta \notin \Theta_0$, given any other test rule $\tilde{\varphi}(\underset{\sim}{x})$ with $\max_{\theta \in \Theta_0} \mathrm{E}_\theta \tilde{\varphi}(\underset{\sim}{X}) \leq \alpha$.

↑ any other θ under H_1

(UMP test DON'T always exist... but often DO exist if H_1 is "one-sided", e.g. $H_1 : \mu > 1$ or $H_1 : \mu \leq 2$)

Two General Methods of Finding UMP Tests

1. **Method I:** Based on Neyman-Pearson Lemma ← start here
2. **Method II:** Using Monotone Likelihood Ratio (MLR) property

Method I (Neyman-Pearson Lemma-based)

To find a UMP size α test for $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \notin \Theta_0$,

← carefully pick/fix one parameter from Θ_0

1. first fix one $\theta_0 \in \Theta_0$ (*suitably*) and also $\theta_1 \notin \Theta_0$
2. then use the Neyman-Pearson lemma to find a MP size α test $\varphi(\underset{\sim}{x})$ for $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$, where

   ← would have gotten same test $\varphi(\underset{\sim}{x})$ for any chosen $\theta_1 \notin \Theta_0$ placed in $H_1 : \theta = \theta_1$

   (a) $\varphi(\underset{\sim}{x})$ does not depend on $\theta_1 \notin \Theta_0$ and

   (b) $\max_{\theta \in \Theta_0} \mathrm{E}_\theta \varphi(\underset{\sim}{X}) = \alpha$ ← right size

Then $\varphi(\underset{\sim}{x})$ is a UMP size α test for $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \notin \Theta_0$.

(Homework 6 has the "proof" of this.)

Hypothesis Testing I

Illustration of Finding UMP Test (Method I)

Example: Let $X_1, \ldots, X_n$ be a random sample from $\mathrm{N}(\mu, 1)$, $\mu \in \mathbb{R}$. Find a UMP test of size α for $H_0 : \mu \geq \mu_0$ vs $H_1 : \mu < \mu_0$ (where μ_0 is a fixed real number).

"border parameter value" e.g $\mu_0 = 0$

Solution: pick/fix μ_0 from $\Theta_0 = \{\mu : \mu \geq \mu_0\} = [\mu_0, \infty)$

Then fix $\mu_1 < \mu_0$. Then, the MP test of size α for $H_0 : \mu = \mu_0$ vs $H_1 : \mu = \mu_1$ is

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } f(\underset{\sim}{x}|\mu_1) > k f(\underset{\sim}{x}|\mu_0) \\ r=0 & \text{if } \quad " \quad = \quad " \\ 0 & \text{if } \quad " \quad < \quad " \end{cases}$$

where $E_{\mu_0}\phi(\underset{\sim}{x}) = \alpha$.

$$f(\underset{\sim}{x}|\mu_1) \gtreqless k f(\underset{\sim}{x}|\mu_0) \Leftrightarrow \cancel{\left(\frac{1}{\sqrt{2\pi}}\right)^n} e^{-\sum_{i=1}^n (x_i-\mu_1)^2/2} \gtreqless k \cancel{\left(\frac{1}{\sqrt{2\pi}}\right)^n} e^{-\frac{\sum_{i=1}^n (x_i-\mu_0)^2}{2}}$$

$$\Leftrightarrow \left(\sum_{i=1}^n x_i\right)\underbrace{(\mu_1-\mu_0)}_{<0} \gtreqless k_1$$

$$\Leftrightarrow \sum_{i=1}^n x_i \lesseqgtr k_2 \equiv k_1/(\mu_1-\mu_0)$$

Hence, the MP test (size α) is

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \sum_{i=1}^n x_i < k_2 \\ 0 & \text{o.w.} \end{cases}$$

under μ_0, $\sum_{i=1}^n x_i \sim N(n\mu_0, n)$

where $\alpha = E_{\mu_0}\phi(\underset{\sim}{x}) = P_{\mu_0}\left(\sum_{i=1}^n x_i < k_2\right)$

$$= P_{\mu_0}\left(\frac{\sum x_i - n\mu_0}{\sqrt{n}} < \frac{k_2 - n\mu_0}{\sqrt{n}}\right) = P\Big(\underset{\uparrow N(0,1)}{Z} < \overbrace{\frac{k_2-n\mu_0}{\sqrt{n}}}^{z_\alpha}\Big)$$

So, the MP test (size α) for $H_0: \mu = \mu_0$ vs
$H_1: \mu = \mu_1$ $(\mu_1 < \mu_0)$

is $\phi(\underset{\sim}{X}) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} X_i < n\mu_0 + z_\alpha \sqrt{n} \\ 0 & \text{o.w} \end{cases}$

check

(a) Form of the MP test $\phi(\underset{\sim}{X})$ does NOT depend on $\mu_1 < \mu_0$

(b) so, $\phi(\underset{\sim}{X})$ will be UMP test of size α
for $H_0: \mu \geq \mu_0$ vs $H_1: \mu < \mu_0$ as long as
$\max_{\mu \geq \mu_0} E_\mu \phi(\underset{\sim}{X}) = \alpha$ [right size]

check $E_\mu \phi(\underset{\sim}{X}) = P_\mu\left(\sum_{i=1}^{n} X_i < n\mu_0 + z_\alpha \sqrt{n} \right)$

$$\xrightarrow{\text{standardize}} = P_\mu\left(\frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}} < \frac{n\mu_0 + z_\alpha\sqrt{n} - n\mu}{\sqrt{n}} \right)$$

$Z \sim N(0,1)$ $\quad = P\left(Z < z_\alpha + \sqrt{n}(\mu_0 - \mu)\right)$

$$= \Phi\big(z_\alpha + \underbrace{\sqrt{n}(\mu_0-\mu)}_{\leq 0 \text{ for } \mu \geq \mu_0}\big)$$

decrease as $\mu \uparrow$ from μ_0

so $\max_{\mu \geq \mu_0} E_\mu \phi(\underset{\sim}{X}) = \alpha$. //