Calculation of Type I, II, and III Sums of Squares

SS (time | 1, temp, time x temp)

Every Type I, II, or III sum of squares is the error sums of squares for a reduced model minus the error sum of squares for a model that adds one term to the reduced model:

$$\boldsymbol{y}^\top(\boldsymbol{I}-\boldsymbol{P}_{\boldsymbol{X}_{\text{reduced}}})\boldsymbol{y}-\boldsymbol{y}^\top(\boldsymbol{I}-\boldsymbol{P}_{\boldsymbol{X}_{\text{reduced+term}}})\boldsymbol{y}=\boldsymbol{y}^\top(\boldsymbol{P}_{\boldsymbol{X}_{\text{reduced+term}}}-\boldsymbol{P}_{\boldsymbol{X}_{\text{reduced}}})\boldsymbol{y},$$

where $\mathcal{C}(\boldsymbol{X}_{\text{reduced}}) \subset \mathcal{C}(\boldsymbol{X}_{\text{reduced+term}}) \subseteq \mathcal{C}(\boldsymbol{X})$.

As usual, $\boldsymbol{X}$ represents the model matrix for the most complex model under consideration (a.k.a., the full model).

For all Type III sums of squares, the reduced+term model is the full model.

For example, $SS(\text{temp}|1, \text{time})$ is

$$\boldsymbol{y}^\top(\boldsymbol{I}-\boldsymbol{P}_{\boldsymbol{X}_{\text{reduced}}})\boldsymbol{y}-\boldsymbol{y}^\top(\boldsymbol{I}-\boldsymbol{P}_{\boldsymbol{X}_{\text{reduced+term}}})\boldsymbol{y} = \boldsymbol{y}^\top(\boldsymbol{P}_{\boldsymbol{X}_{\text{reduced+term}}}-\boldsymbol{P}_{\boldsymbol{X}_{\text{reduced}}})\boldsymbol{y},$$

where we can choose

$$\boldsymbol{X}_{\text{reduced}} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{X}_{\text{reduced+term}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

How does this apply to a Type III Sum of Squares like $SS(\mathrm{time}|1, \mathrm{temp}, \mathrm{time} \times \mathrm{temp})$?

Here the reduced+term model is actually the full cell means model that includes an intercept, time and temperature main effects, and time $\times$ temperature interaction.

The reduced model has an intercept, temperature main effect, and time $\times$ temperature interaction but no time main effect.

How do we fit such a reduced model?

Code like `lm(y~b+a:b)` or `model y=b a*b;` does not fit the reduced model with no `a` main effects.

```
> model.matrix(~temp+time:temp)
   (Intercept) temp30 temp20:time6 temp30:time6
1            1      0            0            0
2            1      0            0            0
3            1      1            0            0
4            1      1            0            0
5            1      1            0            0
6            1      0            1            0
7            1      0            1            0
8            1      0            1            0
9            1      0            1            0
10           1      1            0            1
```

add

This model matrix has rank 4 and the same column space as $\boldsymbol{X}$.

Removing Time Main Effect from Cell Means Model

Time	Temp 20°	Temp 30°	Marginal Mean
3 months	μ_{11}	μ_{12}	$\bar{\mu}_{1\cdot} = \boxed{(\underline{\mu_{11} + \mu_{12}})/2}$
6 months	μ_{21}	μ_{22}	$\bar{\mu}_{2\cdot} = \boxed{(\underline{\mu_{21} + \mu_{22}})/2}$

$\bar{\mu}_{1\cdot} - \bar{\mu}_{2\cdot}$

$\mu_{11} + \mu_{12} = \mu_{21} + \mu_{22}$

The parameters μ_{11}, μ_{12}, μ_{21}, and μ_{22} can be anything as long as

$$\underline{\bar{\mu}_{1\cdot} = \bar{\mu}_{2\cdot}} \iff \mu_{22} = \mu_{11} + \mu_{12} - \mu_{21}.$$

A Model Matrix for Model with $1, \text{temp}, \text{time} \times \text{temp}$

$$\boldsymbol{X}_{\text{reduced}}\boldsymbol{\beta}_{\text{reduced}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \mu_{11} \\ \mu_{12} \\ \mu_{21} \end{bmatrix} = \begin{bmatrix} \mu_{11} \\ \mu_{11} \\ \mu_{12} \\ \mu_{12} \\ \mu_{12} \\ \mu_{21} \\ \mu_{21} \\ \mu_{21} \\ \mu_{21} \\ \mu_{11} + \mu_{12} - \mu_{21} \end{bmatrix}$$

$\mu_{22} = \mu_{11} + \mu_{12} - \mu_{21}$

```
#Type III Sum of Squares for SS(time|1,temp,time x temp)
> x0=x[,1:3]
> x0[10,]=c(1,1,-1)
> x0
1          1        0        0
2          1        0        0
3          0        1        0
4          0        1        0
5          0        1        0
6          0        0        1
7          0        0        1
8          0        0        1
9          0        0        1
10         1        1       -1
> anova(lm(y~0+x0),lm(y~0+x))
Analysis of Variance Table
Model 1: y ~ 0 + x0
Model 2: y ~ 0 + x
  Res.Df RSS Df Sum of Sq  F  Pr(>F)
1      7  24
2      6  12  1        12  6 0.04983 *
```

this model matrix no longer accounts for a time main effect.

$$F = \frac{y^T (P_x - P_{x_0}) y / 1}{y^T (I - P_x) y / 6}$$

$$F = \frac{(24-12)/1}{12/6} = \frac{12}{2} = 6$$

Alternative Computation of Sums of Squares

Let $SS = \boldsymbol{y}^\top(\boldsymbol{P}_{\boldsymbol{X}_{\text{reduced+term}}} - \boldsymbol{P}_{\boldsymbol{X}_{\text{reduced}}})\boldsymbol{y}$ represent any Type I, II, or III sum of squares.

Let $q = \text{rank}(\boldsymbol{X}_{\text{reduced+term}}) - \text{rank}(\boldsymbol{X}_{\text{reduced}})$ be the degrees of freedom associated with SS.

Let $\boldsymbol{C}$ be any $q \times p$ matrix whose rows are a basis for the row space of $(\boldsymbol{P}_{\boldsymbol{X}_{\text{reduced+term}}} - \boldsymbol{P}_{\boldsymbol{X}_{\text{reduced}}})\boldsymbol{X}$.

Alternative Computation of Sums of Squares

$$y^\top (P_{j+1} - P_j) y = (C\hat{\beta})^\top \{C(X^\top X)^- C^\top\}^{-1} C\hat{\beta}$$

Then the ANOVA F statistic

$$\frac{SS/q}{MSE} = \frac{\widehat{\boldsymbol{\beta}}^\top \boldsymbol{C}^\top [\boldsymbol{C}(\boldsymbol{X}^\top \boldsymbol{X})^- \boldsymbol{C}^\top]^{-1} \boldsymbol{C}\widehat{\boldsymbol{\beta}}/q}{\widehat{\sigma}^2}.$$

Thus, any SS can be computed as $\widehat{\boldsymbol{\beta}}^\top \boldsymbol{C}^\top [\boldsymbol{C}(\boldsymbol{X}^\top \boldsymbol{X})^- \boldsymbol{C}^\top]^{-1} \boldsymbol{C}\widehat{\boldsymbol{\beta}}$ for an appropriate matrix $\boldsymbol{C}$.