Multivariate Normal Distribution

Conditional distribution result

Result 6: Suppose $\boldsymbol{X} \sim MVN_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and consider a partition of $\boldsymbol{X}, \boldsymbol{\mu}, \boldsymbol{\Sigma}$ as

$$\boldsymbol{X} = \begin{pmatrix} \boldsymbol{X}^{(1)} \\ \boldsymbol{X}^{(2)} \end{pmatrix} \qquad \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}^{(1)} \\ \boldsymbol{\mu}^{(2)} \end{pmatrix} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}$$

where $\boldsymbol{X}^{(1)}$ is $p \times 1$ and $\boldsymbol{X}^{(2)}$ is $(k-p) \times 1$. Then, the conditional distribution of $\boldsymbol{X}^{(1)}|\boldsymbol{X}^{(2)} = \boldsymbol{x}^{(2)}$ is MVN, i.e.,

$E[X^{(1)}|X^{(2)}]$ $\quad$ $Var[X^{(1)}|X^{(2)}]$

$$\boldsymbol{X}^{(1)}|\boldsymbol{X}^{(2)} = \boldsymbol{x}^{(2)} \sim MVN_p\left(\boldsymbol{\mu}^{(1)} + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\boldsymbol{x}^{(2)} - \boldsymbol{\mu}^{(2)}), \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}\right)$$

$$\boxed{E[X^{(1)} | X^{(2)} = x^{(2)}] = \mu^{(1)} + \Sigma_{12}\Sigma_{22}^{-1}(x^{(2)} - \mu^{(2)})}$$

The proof uses the traditional definition

$$f(\boldsymbol{x}^{(1)}|\boldsymbol{x}^{(2)}) = \frac{f(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)})}{f(\boldsymbol{x}^{(2)})}$$

joint pdf of $(X^{(1)}, X^{(2)})$

along with

$$\boldsymbol{X}^{(2)} \sim MVN_{k-p}(\boldsymbol{\mu}^{(2)}, \boldsymbol{\Sigma}_{22})$$

but requires special matrix results

- Set $\boldsymbol{V} \equiv \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}$.
- Then, $\det(\boldsymbol{\Sigma}) = \det(\boldsymbol{V})\det(\boldsymbol{\Sigma}_{22})$ and

$$\boldsymbol{\Sigma}^{-1} = \begin{pmatrix} \boldsymbol{V}^{-1} & \boldsymbol{V}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1} \\ -\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{V}^{-1} & \boldsymbol{\Sigma}_{22}^{-1} + \boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{V}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1} \end{pmatrix}$$

Multivariate Normal Distribution

XX Bivariate normal distribution

- MVN with $k = 2$ is the bivariate normal distribution

$$\boldsymbol{X} = (X_1, X_2)' \sim MVN_2(\boldsymbol{\mu} = (\mu_1, \mu_2)', \boldsymbol{\Sigma})$$

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & (\sigma_1\sigma_2)\rho \\ \sigma_1\sigma_2\rho & \sigma_2^2 \end{pmatrix}$$

$Var(X_1) = Cov(X_1, X_1)$

$Var(X_1)$

$Cov(X_1, X_2)$

$Var(X_2)$

$Cov(X_2, X_2) = Var(X_2)$

$\sigma_{12} = \sigma_{21} = Cov(X_1, X_2)$

$\rho = \dfrac{Cov(X_1,X_2)}{\sqrt{\sigma_{11}\,\sigma_{22}}} = \dfrac{Cov(X_1,X_2)}{\sigma_1\sigma_2}$

$\Rightarrow Cov(X_1,X_2) = \rho\sigma_1\sigma_2$

$\det\Sigma = |\sigma_1^2\sigma_2^2 - \sigma_1^2\sigma_2^2\rho^2|$
$= \sigma_1^2\sigma_2^2|1-\rho^2|$
$= \sigma_1^2\sigma_2^2(1-\rho^2)$

$\sqrt{\det\Sigma} = \boxed{\sigma_1\sigma_2\sqrt{1-\rho^2}}$

- Parameters: $\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho$ where for $i = 1, 2$

$$\mu_i = \mathrm{E}X_i \quad \sigma_i^2 = \mathrm{Var}(X_i) \qquad \rho = \mathrm{Corr}(X_1, X_2) = \frac{\mathrm{Cov}(X_1, X_2)}{\sigma_1\sigma_2}$$

- Results follow from MVN results

Σ^{-1}

$$f_{X_1,X_2}(x_1, x_2) =$$

$$\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \times \exp\left[-\frac{1}{2}\frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)}\begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{pmatrix}' \begin{pmatrix} \sigma_2^2 & -(\sigma_1\sigma_2)\rho \\ -\sigma_1\sigma_2\rho & \sigma_1^2 \end{pmatrix} \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{pmatrix}\right]$$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\dfrac{1}{ad-bc}$

$$f_{X_1|X_2=x_2}(x_1|x_2) \sim N\left(\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(x_2 - \mu_2), \sigma_1^2(1-\rho^2)\right)$$

$E[X_1|X_2=x_2] = \mu_1 + \rho\dfrac{\sigma_1}{\sigma_2}(x_2 - \mu_2)$

$Var[X_1|X_2=x_2] = \sigma_1^2(1-\rho^2)$

- Conditional mean $\mathrm{E}[X_1|X_2 = x_2] = \mu_1 + \rho\frac{\sigma_1}{\sigma_2}(x_2 - \mu_2)$ is linear in x_2 while the conditional variance is constant

  (i.e., usual simple linear regression assumptions)

Multivariate Normal Distribution

Quadratic forms and chi-square distribtuions

One last fact about MVN random variables has important uses in statistics (e.g., making confidence sets to estimate mean vectors) and this concerns that certain "quadratic forms" have chi-square distributions

Σ^{-1} exists

Result 7: Suppose $\boldsymbol{X} \sim MVN_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with non-singular $\boldsymbol{\Sigma}$. Then,

$$Q(\mathbf{X}) = (\boldsymbol{X} - \boldsymbol{\mu})' \, \boldsymbol{\Sigma}^{-1} \, (\boldsymbol{X} - \boldsymbol{\mu}) \sim \chi^2_k.$$

$Z \sim N(0,1)$

$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}}$

$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$

To show this, one needs the following result:

(*) *Lemma* If $Z \sim N(0, 1)$, then $Z^2 \sim \chi^2_1$.

$$M_{Z^2}(t) \stackrel{\text{def}}{=} \mathbb{E}\left[e^{tZ^2}\right] = \int_{-\infty}^{\infty} e^{tz^2} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2\left(\frac{1}{2}-t\right)} \, dz \qquad \tilde{\sigma}^2 = \frac{1}{1-2t}$$

$$= \frac{1 \, \tilde{\sigma}}{\sqrt{2\pi}\,\tilde{\sigma}} \int_{-\infty}^{\infty} e^{\frac{-(z-0)^2}{2 \cdot \frac{1}{2 \left(\frac{1}{2}-t\right)}}} \, dz \qquad \frac{1}{2\left(\frac{1}{2}-t\right)} \to \tilde{\sigma}^2$$

$$= \tilde{\sigma} \Bigg[\underbrace{\frac{1}{\sqrt{2\pi}\,\tilde{\sigma}} \int_{-\infty}^{\infty} e^{\frac{-(z-0)^2}{2\tilde{\sigma}^2}} \, dz}_{=1} \Bigg]$$

$$= \tilde{\sigma} = \left(\frac{1}{1-2t}\right)^{1/2} = (1-2t)^{-1/2}$$

$\chi^2_\nu \Rightarrow M_{\chi^2_\nu}(t) = (1-2t)^{-\nu/2}$

But, $(1-2t)^{-1/2}$ is MGF of gamma$(1/2, 2) \sim \chi^2_1$

Multivariate Normal Distribution

Quadratic forms and chi-square distribtuions (cont'd)

$X \sim MVN_k(\mu, \Sigma) \qquad Q(X) = (X-\mu)'\Sigma^{-1}(X-\mu) \sim \chi^2_k$

Proof of Result 7: Let $\boldsymbol{Z} \sim MVN_k(\boldsymbol{0}, \boldsymbol{I}_k)$ be a vector of k independent standard normals and let $k \times k$ $\boldsymbol{A}$ be such that

$$\boldsymbol{A}\boldsymbol{A}' = \boldsymbol{\Sigma}$$

Then,

$$\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{Z} + \boldsymbol{\mu} \Rightarrow Y \sim X \quad \left(\begin{array}{l} Y \text{ has the Same} \\ \text{distribution of } X \end{array}\right)$$

has the same distribution as $\boldsymbol{X}$ so that

$X \sim Y = AZ + \mu$

$$(\boldsymbol{A}\boldsymbol{Z})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{A}\boldsymbol{Z})$$

$(AB)^{-1} = B^{-1}A^{-1}$

$AA' = \Sigma$

$\Sigma^{-1} = (AA')^{-1}$

$= (A')^{-1}A^{-1}$

has the same distribution as

$$Q(\mathbf{X}) = \underbrace{(\boldsymbol{X}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{X}-\boldsymbol{\mu})}$$

$$= \underbrace{(AZ)'}\Sigma^{-1}(AZ)$$

$$= Z'\underbrace{A'(A')^{-1}}\underbrace{A^{-1}A}Z$$

$$= \underbrace{Z' I Z} = Z'Z = \sum_{i=1}^{k} Z_i^2 \sim \chi^2_k$$

$\left(\begin{array}{l} Z = \begin{pmatrix} Z_1 \\ \vdots \\ Z_k \end{pmatrix} \text{indep. } N(0,1) \\ Z_i^2 \sim \chi^2_1 \ \forall i = 1, \ldots, k \\ \sum_{i=1}^{k} \chi^2_1 \sim \chi^2_k \end{array} \right.$

$$\Rightarrow Q(X) = (X-\mu)'\Sigma^{-1}(X-\mu) \sim \chi^2_k$$

STAT 542: Summary to date

Where we have been

& where we are headed

- Completed
  - Probability and random variables (definition, cdf, pdf/pmf)
  - Univariate distributions
    (definitions, expectation, transformations, families)
  - Multivariate distributions
    (joint distribution, covariance, conditional distribution, marginal distribution, independence, transformations, multivariate normal distributions)
- Next
  - Random samples
  - Order statistics
  - Normal sample theory

Random samples and iid variables

Definitions

random sample

a + b

- *Definition:* If $X_1, \ldots, X_n$ are **independent identically distributed** (iid) with $X_i \sim f_X(x_i)$, then we call $X_1, \ldots, X_n$ a **random sample** from the population $f_X(x)$.

- $Y = T(X_1, \ldots, X_n)$ is called a statistic.

  i.e., $T(X_1, \ldots, X_n)$ is computable from the data

$$Y = T(X_1, \ldots, X_n)$$

- The distribution of a statistic Y is sometime called the **sampling distribution** of the statistic.

- Examples

  1. sample mean: $\bar{X}_n = \sum_{i=1}^n X_i/n = T(X_1, X_2, \ldots, X_n)$

  2. sample variance:

$$S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2 = \frac{1}{n-1}\left(\sum_{i=1}^n X_i^2 - n\bar{X}_n^2\right)$$

  3. $X_{(1)} =$ minimum: $\min\{X_1, \ldots, X_n\} = T(X_1, \ldots, X_n)$ } order statistics

  4. $X_{(n)} =$ maximum: $\max\{X_1, \ldots, X_n\} = T(X_1, \ldots, X_n)$

Random samples and iid variables

Distribution of $\bar{X}_n$

i.i.d

Let $X_1, \ldots, X_n$ be a random sample from $f_X(x)$ with $\mu = \mathrm{E}X_i$ and $\sigma^2 = \mathrm{Var}(X_i)$

Important Results for $\bar{X}_n$: If $X_1, \ldots, X_n$ is a sample random with $\mu = \mathrm{E}X_i$ and $\sigma^2 = \mathrm{Var}(X_i)$, then

1. $\mathrm{E}\bar{X}_n = \mu$,

$$\mathbb{E}[\bar{X}_n] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}X_i = \frac{1}{n}\sum_{i=1}^{n} \mu = \frac{\cancel{n}\mu}{\cancel{n}} = \mu$$

2. $\mathrm{Var}(\bar{X}_n) = \dfrac{\sigma^2}{n}$

$$\mathrm{Var}(\bar{X}_n) = \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right)$$

X_i's are ind

$$= \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{Var}(X_i) = \frac{1}{n^2}\sum_{i=1}^{n} \sigma^2 = \frac{\cancel{n}\sigma^2}{\cancel{n^2}_{\,n}} = \frac{\sigma^2}{n}$$

MGF approach can sometimes apply for determining the exact distribution of $\bar{X}_n$

$$M_{\bar{X}_n}(t) = \mathrm{E}e^{t\bar{X}_n} = \mathrm{E}e^{n^{-1}t(X_1+\cdots+X_n)} = \mathrm{E}\prod_{i=1}^{n} e^{n^{-1}tX_i} = \prod_{i=1}^{n} \mathrm{E}e^{n^{-1}tX_i} = [M_{X_1}(t/n)]^n$$

Examples

1. Suppose $X_1, \ldots, X_n$ are iid Gamma(α, β)

2. Suppose $X_1, \ldots, X_n$ are iid $N(\mu, \sigma^2)$