STAT 543O

Lec 31, F, Apr 11

Office Hours:
11:30-12
1-2PM

- No homework this week! Homework 7 solutions posted
- Exam 2 is coming up (≈ 1 week away)
  on W, April 16, 6:15-8:15 PM, 3rd floor seminar room
- No class on that W.
- I'll post: study guide (sufficiency/completeness/tests)
  No Bayes tests on exam
  - practice exams
  - bring new 1 page (front/back) formula sheet on exam 2 material
    (I'll post one to use if you'd like)
  - can bring calculator & previous formula sheet for exam 1
- I'll provide table of distributions / STAT 542 facts on test as before

Interval Estimation I

"Inverting a Test"

Theorem: Let $X_1, \ldots, X_n$ have joint pdf/pmf $f(\underset{\sim}{x}|\underset{\sim}{\theta})$, $\underset{\sim}{\theta} \in \Theta \subset \mathbb{R}^p$ and let $A(\underset{\sim}{\theta_0})$ denote the acceptance region of a *simple test of size* α for testing $H_0 : \underset{\sim}{\theta} = \underset{\sim}{\theta_0}$ vs. $H_1 : \underset{\sim}{\theta} \neq \underset{\sim}{\theta_0}$ (for $p = 1$, $H_1 : \theta < \theta_0$ or $\theta > \theta_0$ is allowed too). Define sets $C_{\underset{\sim}{x}} \subset \Theta$, $\underset{\sim}{x} \in \mathbb{R}^n$ as

test is 0 or 1

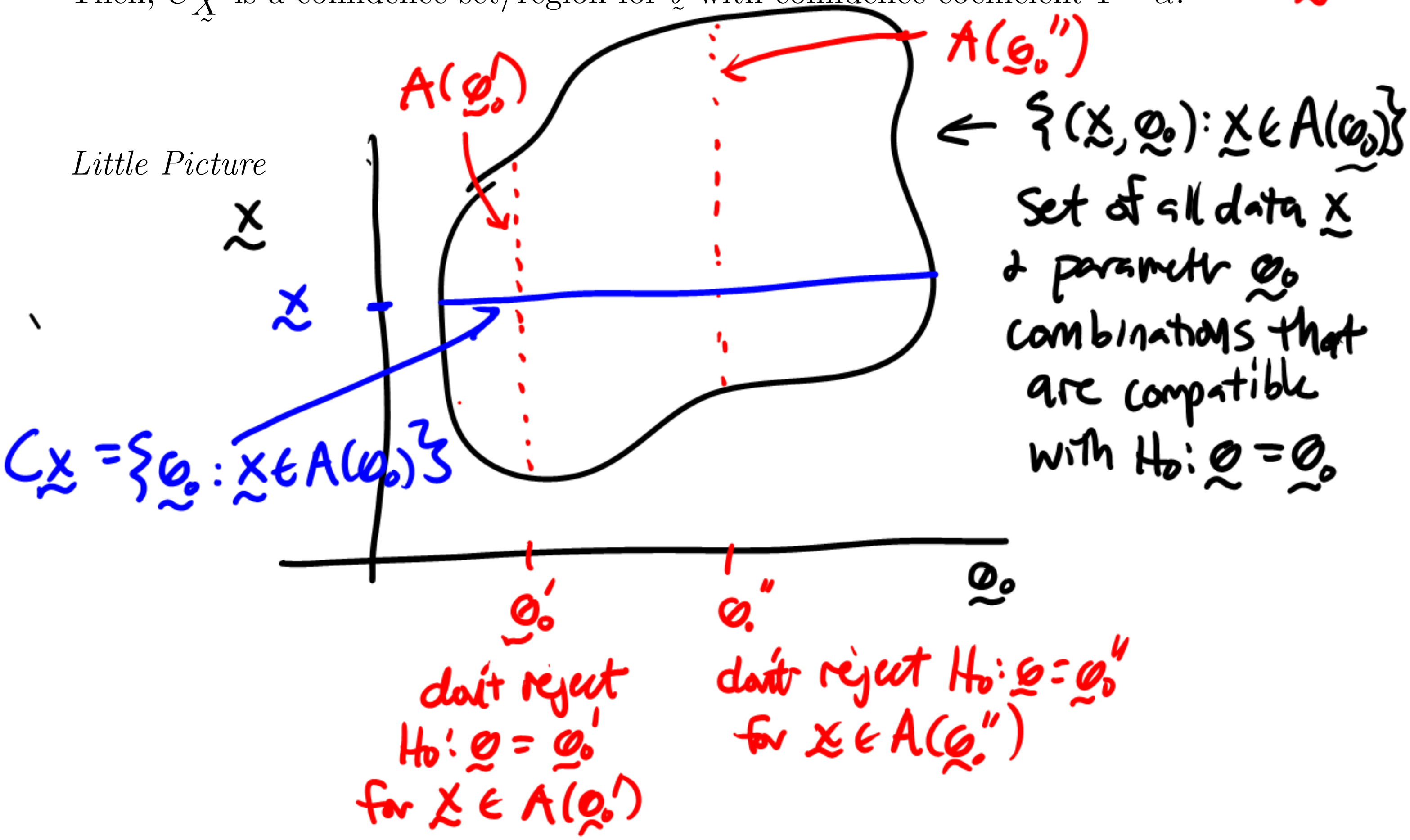
$$C_{\underset{\sim}{x}} = \{\underset{\sim}{\theta_0} : \underset{\sim}{x} \in A(\underset{\sim}{\theta_0})\}$$

↑ given data $\underset{\sim}{x}$

recall $\underset{\sim}{x} \in A(\underset{\sim}{\theta_0})$ means "don't reject $H_0 : \underset{\sim}{\theta} = \underset{\sim}{\theta_0}$"

Then, $C_{\underset{\sim}{X}}$ is a confidence set/region for $\underset{\sim}{\theta}$ with confidence coefficient $1 - \alpha$.

Little Picture



Interval Estimation I

Inverting a Test, cont'd

Proof of Theorem: Note that

1.

$$\begin{aligned}
\min_{\underset{\sim}{\theta}_0 \in \Theta} P_{\underset{\sim}{\theta}_0}\big(\underset{\sim}{X} \in A(\underset{\sim}{\theta}_0)\big) &= \min_{\underset{\sim}{\theta}_0 \in \Theta} P_{\underset{\sim}{\theta}_0}(\text{“do not reject } H_0 : \underset{\sim}{\theta} = \underset{\sim}{\theta}_0\text{”}) \\
&= \min_{\underset{\sim}{\theta}_0 \in \Theta} [1 - P_{\underset{\sim}{\theta}_0}(\text{“reject } H_0 : \underset{\sim}{\theta} = \underset{\sim}{\theta}_0\text{”})] \\
&= \min_{\underset{\sim}{\theta}_0 \in \Theta} [1 - \alpha] \\
&= 1 - \alpha,
\end{aligned}$$

and

2. for any $\underset{\sim}{\theta}_0 \in \Theta$, any $\underset{\sim}{x} \in \mathbb{R}^n$, it holds that

$$\underset{\sim}{x} \in A(\underset{\sim}{\theta}_0) \;\Leftrightarrow\; \underset{\sim}{\theta}_0 \in C_{\underset{\sim}{x}}.$$

Hence,

$$\begin{aligned}
\min_{\underset{\sim}{\theta}_0 \in \Theta} P_{\underset{\sim}{\theta}_0}(\underset{\sim}{\theta}_0 \in C_{\underset{\sim}{X}}) &= \min_{\underset{\sim}{\theta}_0 \in \Theta} P_{\underset{\sim}{\theta}_0}(\underset{\sim}{X} \in A(\underset{\sim}{\theta}_0)) \\
&= 1 - \alpha.
\end{aligned}$$

Interval Estimation I

Inverting a Test: Illustration

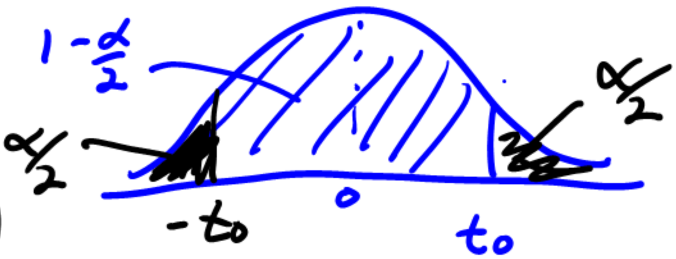
Example: Let $X_1, \ldots, X_n$ be iid $N(\mu, \sigma^2)$, $-\infty < \mu < \infty$, $\sigma > 0$. Find

1. a C.I. (confidence interval) for μ with C.C. $1-\alpha$ (two-sided)
2. a 1-sided lower confidence bound for μ with C.C. $1-\alpha$, i.e., $(L(\underset{\sim}{X}), \infty)$

Solution for 1. Consider a test function

$$\phi_{\mu_0}(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \dfrac{|\bar{X}_n - \mu_0|}{S/\sqrt{n}} > t_0 \\ 0 & \text{o.w.} \end{cases}$$

for testing $H_0: \mu = \mu_0$ vs $H_1: \mu \neq \mu_0$ (given some $\mu_0 \in \mathbb{R}$)
where $t_0 \equiv (1-\frac{\alpha}{2})$ percentile of T_{n-1} distribution.



$E_{\mu_0}\phi_{\mu_0}(\underset{\sim}{X})$
$= P_{\mu_0}\left(|\bar{X}_n - \mu_0| / S/\sqrt{n} > t_0 \right)$
$= \alpha$, i.e. $\phi_{\mu_0}(\underset{\sim}{X})$ is a simple test (0 or 1) of size α for $H_0: \mu = \mu_0$ vs $H_1: \mu \neq \mu_0$

Note: acceptance region of $\phi_{\mu_0}(\underset{\sim}{x})$ is $A(\mu_0) = \left\{ \underset{\sim}{x} : \dfrac{|\bar{X}_n - \mu_0|}{S/\sqrt{n}} \le t_0 \right\}$
$= \{ \underset{\sim}{x} : \phi_{\mu_0}(\underset{\sim}{x}) = 0 \}$

Hence, $C_{\underset{\sim}{x}} = \{ \mu_0 : \underset{\sim}{x} \in A(\mu_0) \}$

depends on $\underset{\sim}{x}$

$= \left\{ \mu_0 : \dfrac{|\bar{X}_n - \mu_0|}{S/\sqrt{n}} \le t_0 \right\}$

$= \left\{ \mu_0 : -t_0 \le \dfrac{\bar{X}_n - \mu_0}{S/\sqrt{n}} \le t_0 \right\}$

and $C_{\underset{\sim}{x}}$ is a CI with C.C. $1-\alpha$

$= \left\{ \mu_0 : -t_0 \frac{S}{\sqrt{n}} \le \bar{X}_n - \mu_0 \le t_0 S/\sqrt{n} \right\}$

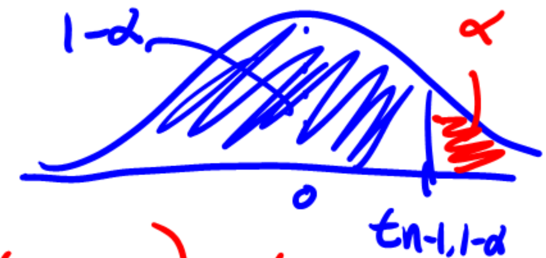
$= \left\{ \mu_0 : \bar{X}_n - t_0 S/\sqrt{n} \le \mu_0 \le \bar{X}_n + t_0 S/\sqrt{n} \right\}$

$= \left[\bar{X}_n - t_0 S/\sqrt{n}, \bar{X}_n + t_0 S/\sqrt{n} \right]$

Solution to 2: Consider the following test for $H_0: \mu = \mu_0$ vs $H_1: \mu > \mu_0$ given by

$$\phi_{\mu_0}(\underset{\sim}{X}) = \begin{cases} 1 & (\bar{X}_n - \mu_0)/s/\sqrt{n} > t_{n-1,1-\alpha} \\ 0 & \text{o.w.} \end{cases}$$

where $t_{n-1,1-\alpha} \equiv 1-\alpha$ percentile



Note: $E_{\mu_0} \phi_{\mu_0}(\underset{\sim}{X}) = P_{\mu_0}\left(\frac{\bar{X}_n - \mu_0}{s/\sqrt{n}} > t_{n-1,1-\alpha} \right) = \alpha$

↑ size