HW7

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Q1

Problem 8.6 a) - b), Casella and Berger (2nd Edition)

Suppose that we have two independent random samples: $X_1, \ldots, X_n$ are exponential(θ), and $Y_1, \ldots, Y_m$ are exponential(μ).

a)

Find the LRT of

$$H_0 : \theta = \mu \quad \text{versus} \quad H_1 : \theta \neq \mu.$$

Answer

The likelihood ratio is:

$$\lambda(x, y) = \frac{\sup_\theta L(\theta|x, y)}{\sup_{\theta,\mu} L(\theta, \mu|x, y)}$$

Under H_0 ($\theta = \mu$), the MLE is:

$$\hat{\theta}_0 = \frac{\sum X_i + \sum Y_j}{n + m}$$

Under the full model, the MLEs are:

$$\hat{\theta} = \bar{X}, \quad \hat{\mu} = \bar{Y}$$

The test statistic simplifies to:

$$\lambda(x, y) = \frac{(n + m)^{n+m} (\sum X_i)^n (\sum Y_j)^m}{n^n m^m (\sum X_i + \sum Y_j)^{n+m}}$$

Rejection rule: Reject H_0 if $\lambda(x, y) \leq c$.

b)

Show that the test in part a) can be based on the statistic

$$T = \frac{\sum X_i}{\sum X_i + \sum Y_i}.$$

Answer

Let $T = \frac{\sum X_i}{\sum X_i + \sum Y_j}$. Then:

$$\lambda(x, y) = \frac{(n+m)^{n+m}}{n^n m^m} T^n (1-T)^m$$

Since λ is a function of T alone:

- The test rejects when T is too small or too large
- Critical values satisfy $a^n(1-a)^m = b^n(1-b)^m$

Thus, the LRT can be based entirely on T.

Q2

Problem 8.28, Casella and Berger (2nd Edition)

Let $f(x|\theta)$ be the logistic location probability density function:

$$f(x|\theta) = \frac{e^{(x-\theta)}}{(1+e^{(x-\theta)})^2}, \quad -\infty < x < \infty, \quad -\infty < \theta < \infty.$$

a)

Show that this family has an MLR.

Answer

For $\theta_2 > \theta_1$, the likelihood ratio is:

$$\frac{f(x|\theta_2)}{f(x|\theta_1)} = e^{\theta_1-\theta_2}\left[\frac{1+e^{x-\theta_1}}{1+e^{x-\theta_2}}\right]^2$$

Taking the derivative of the ratio inside brackets:

$$\frac{d}{dx}\left(\frac{1+e^{x-\theta_1}}{1+e^{x-\theta_2}}\right) = \frac{e^{x-\theta_1}(1+e^{x-\theta_2}) - e^{x-\theta_2}(1+e^{x-\theta_1})}{(1+e^{x-\theta_2})^2} > 0 \quad \text{for } \theta_2 > \theta_1$$

Since the derivative is always positive, the likelihood ratio is strictly increasing in x. Therefore, the family has MLR.

b)

Based on one observation X, find the most powerful size α test of

$$H_0 : \theta = 0 \quad \text{versus} \quad H_1 : \theta = 1.$$

For $\alpha = 0.2$, find the size of the Type II error.

Answer

By the Neyman-Pearson Lemma, the MP test rejects when:

$$\frac{f(x|1)}{f(x|0)} = e^{-1}\left(\frac{1+e^{x}}{1+e^{x-1}}\right)^2 > k$$

From part a), this is equivalent to rejecting when $x > k'$.

Using the logistic CDF $F(x|\theta) = \frac{e^{x-\theta}}{1+e^{x-\theta}}$:

1. Size α condition:

$$P(X > k'|\theta = 0) = 1 - F(k'|0) = \frac{1}{1 + e^{k'}} = \alpha$$

$$\Rightarrow k' = \log\left(\frac{1 - \alpha}{\alpha}\right)$$

2. For $\alpha = 0.2$:

$$k' = \log(4) \approx 1.386$$

$$\beta = P(X \leq k'|\theta = 1) = F(1.386|1) \approx 0.595$$

c)

Show that the test in part b) is UMP size α for testing

$$H_0 : \theta \leq 0 \quad \text{versus} \quad H_1 : \theta > 0.$$

What can be said about UMP tests in general for the logistic location family?

Answer

1. The family has MLR in X (from part a)
2. The test from part b) doesn't depend on the specific $\theta_1 = 1$ - it's of the form "reject when $X > c$"
3. By Karlin-Rubin Theorem, this test is UMP size α for testing $H_0 : \theta \leq 0$ vs $H_1 : \theta > 0$

General case: For the logistic location family, UMP tests exist for one-sided hypotheses due to the MLR property. The rejection region will always be of the form $\{X > c\}$ or $\{X < c\}$ depending on the direction of the alternative.

Q3

Problem 8.29 a) - b), Casella and Berger (2nd Edition)

Let X be one observation from a Cauchy(θ) distribution.

The Cauchy(θ) density is given by:

$$f(x|\theta) = \frac{1}{\pi} \cdot \frac{1}{1 + (x-\theta)^2}, \quad x \in \mathbb{R}, -\infty < \theta < \infty.$$

a)

Show that this family does not have an MLR.

Hint:

Show that the Cauchy(θ) family $\{f(x|\theta) : \theta \in \mathbb{R} = \Theta\}$, based on one observation X, does not have monotone likelihood ratio (MLR) in $t(X) = X$ or $t(X) = -X$. That is, the ratio

$$\frac{f(x|\theta_2)}{f(x|\theta_1)}$$

might not be monotone (either increasing or decreasing) in x.

Answer

For $\theta_2 > \theta_1$, examine the likelihood ratio:

$$\frac{f(x|\theta_2)}{f(x|\theta_1)} = \frac{1 + (x-\theta_1)^2}{1 + (x-\theta_2)^2}$$

Key observations:

1. Behavior at extremes:

$$\lim_{x \to \pm\infty} \frac{f(x|\theta_2)}{f(x|\theta_1)} = 1$$

2. Non-monotonicity:

   - The ratio achieves a maximum at finite x
   - For example, with $\theta_1 = 0$, $\theta_2 = 1$:
     - Ratio $= 1$ at $x = 0$ and $x \to \infty$
     - Ratio > 1 for some intermediate x

Since the ratio is not monotone in x for any $\theta_2 > \theta_1$, the family does not possess MLR.

b)

Show that the test

$$\phi(x) = \begin{cases} 1 & \text{if } 1 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

is most powerful of its size for testing

$$H_0 : \theta = 0 \quad \text{versus} \quad H_1 : \theta = 1.$$

Calculate the Type I and Type II error probabilities.

Hint:

Show that the test given is equivalent to rejecting H_0 if

$$f(x|\theta = 1) > 2f(x|\theta = 0)$$

and not rejecting otherwise. Conclude that this must be the most powerful (MP) test for its size. Justify why.

Answer

The given test:

$$\phi(x) = \begin{cases} 1 & \text{if } 1 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

Proof of MP property: 1. By Neyman-Pearson, the MP test rejects when:

$$\frac{f(x|1)}{f(x|0)} = \frac{1 + x^2}{1 + (x-1)^2} > k$$

2. The ratio has critical points at $x = (1 \pm \sqrt{5})/2$ and satisfies:

$$\frac{f(1|1)}{f(1|0)} = \frac{f(3|1)}{f(3|0)} = 2$$

3. Thus $\{x : f(x|1)/f(x|0) > 2\} = (1, 3)$ exactly matches $\phi(x)$

Error probabilities:

1. Type I error (α):

$$P(1 < X < 3|\theta = 0) = \frac{1}{\pi}\left(\tan^{-1}(3) - \tan^{-1}(1)\right) \approx 0.1476$$

2. Type II error (β):

$$1 - P(1 < X < 3|\theta = 1) = 1 - \frac{1}{\pi}\left(\tan^{-1}(2) - \tan^{-1}(0)\right) \approx 0.6476$$

The test is MP because:

- It implements the Neyman-Pearson rejection region exactly
- No other test with $\alpha \approx 0.1476$ has smaller β

Q4

Consider one observation X from the probability density function

$$f(x \mid \theta) = 1 - \theta^2 \left(x - \frac{1}{2} \right), \quad 0 \le x \le 1, \quad 0 \le \theta \le 1.$$

We wish to test:

$$H_0 : \theta = 0 \quad \text{vs.} \quad H_1 : \theta > 0$$

a)

Find the UMP test of size $\alpha = 0.05$ based on X. Carefully justify your answer.

Answer

1. Likelihood Ratio Analysis:

For $\theta_2 > \theta_1$, the likelihood ratio is:

$$\frac{f(x \mid \theta_2)}{f(x \mid \theta_1)} = \frac{1 - \theta_2^2(x - \frac{1}{2})}{1 - \theta_1^2(x - \frac{1}{2})}$$

2. Monotonicity Properties:

   - For $x > \frac{1}{2}$: Ratio decreases in x (since $x - \frac{1}{2} > 0$)
   - For $x < \frac{1}{2}$: Ratio increases in x (since $x - \frac{1}{2} < 0$)

This shows the family has monotone likelihood ratio (MLR) in $T(X) = |X - \frac{1}{2}|$.

3. UMP Test Construction:

By the Karlin-Rubin Theorem, the UMP test rejects for large values of $T(X)$. However, since:

- Under H_0 ($\theta = 0$): $X \sim \text{Uniform}(0, 1)$
- The test that rejects when $X > c$ is most powerful

4. Critical Value Calculation:

$$P_{\theta=0}(X > c) = 1 - c = 0.05 \quad \Rightarrow \quad c = 0.95$$

Final UMP Test:

$$\phi(x) = \begin{cases} 1 & \text{if } x > 0.95 \\ 0 & \text{otherwise} \end{cases}$$

b)

Find the likelihood ratio test statistic $\lambda(X)$ based on X, expressed as a function of X.

Answer

The LRT statistic is:

$$\lambda(X) = \frac{f(X \mid 0)}{\sup_{\theta \in [0,1]} f(X \mid \theta)}$$

1. Numerator: $f(X \mid 0) = 1$
2. Denominator:

$$\sup_{\theta} f(X \mid \theta) = \begin{cases} 1 - (X - \frac{1}{2}) & X < \frac{1}{2} \\ 1 & X \geq \frac{1}{2} \end{cases}$$

Final LRT Statistic:

$$\lambda(X) = \begin{cases} \frac{1}{1.5 - X} & X < \frac{1}{2} \\ 1 & X \geq \frac{1}{2} \end{cases}$$

c)

Find the likelihood ratio test (LRT) of size $\alpha = 0.05$ for the above hypotheses.

Answer

1. Rejection Region:

The test rejects when $\lambda(X) < c$, which occurs when: $X > k$ for some k (since $\lambda(X) = 1$ for $X \geq \frac{1}{2}$)

2. Size Condition:

$$P_{\theta=0}(X > k) = 1 - k = 0.05 \quad \Rightarrow \quad k = 0.95$$

Final LRT:

Reject H_0 when $X > 0.95$

Note:

The LRT coincides with the UMP test in this case, which occurs because: 1. The family has the MLR property (though not in X directly) 2. Both tests are based on the same sufficient statistic ordering