HW12

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STAT 5000 HOMEWORK #11
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Q1

For each of the following models Y_i are the responses, β_i are parameters, X_i are fixed values, and ϵ_i denotes random errors with variance σ^2. Indicate if it is a linear model, a nonlinear model, or an intrinsically linear model (a nonlinear model that can be transformed into a linear model).

(a)

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 \log(X_{2i}) + \beta_3 X_{3i} + \epsilon_i \text{ with } \mathbb{E}(\epsilon_i) = 0$$

Linear Model: This is a Linear Model because the response Y_i depends linearly on the parameters $\beta_0, \beta_1, \beta_2, \beta_3$, and the predictor transformations (e.g., $\log(X_{2i})$) are fixed functions of the independent variables.

(b)

$$Y_i = \beta_0 \exp(\beta_1 X_{1i}) + \epsilon_i \text{ with } \mathbb{E}(\epsilon_i) = 0$$

This is a Nonlinear Model because the parameter β_1 appears in the exponent, making the model nonlinearly dependent on β_1, and it cannot be transformed into a linear model.

(c)

$$Y_i = [1 + \exp(\beta_0 + \beta_1 X_{1i} + \epsilon_i)]^{-1} \text{ with } \mathbb{E}(\epsilon_i) = 0$$

This is a Nonlinear Model because the parameters β_0 and β_1 appear inside a nonlinear transformation (exp) and within a complex function $[1 + \exp(\cdot)]^{-1}$, making the model nonlinearly dependent on the parameters. It cannot be transformed into a linear model.

(d)

$$Y_i = (\beta_0 + \beta_1 X_{1i})\epsilon_i \text{ with } \mathbb{E}(\epsilon_i) = 1$$

This is a Nonlinear Model because the error term ϵ_i is multiplied by the parameter-dependent expression $(\beta_0 + \beta_1 X_{1i})$, making the model nonlinearly dependent on the parameters. It cannot be transformed into a linear model.$

(e)

$$Y_i = \epsilon_i \exp(\beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i}) \text{ with } \mathbb{E}(\epsilon_i) = 1$$

This is an Intrinsically Linear Model because the multiplicative structure can be transformed into a linear model by taking the natural logarithm of both sides, resulting in a model linear in the parameters β_0, β_1, and β_2.

Q2

Only square, nonsingular matrices have inverses, but every matrix has a generalized inverse. For example, let

$$A = \begin{bmatrix} 1 \\ 2 \\ 5 \\ -2 \end{bmatrix}$$

Show that $B = [1\ 0\ 0\ 0]$ satisfies the definition of a generalized inverse for A.

To show that $B = [1\ 0\ 0\ 0]$ satisfies the definition of a generalized inverse for A, we need to confirm that:

$$ABA = A$$

Given:

$$A = \begin{bmatrix} 1 \\ 2 \\ 5 \\ -2 \end{bmatrix}, \quad B = [1\ 0\ 0\ 0]$$

$$AB = \begin{bmatrix} 1 \\ 2 \\ 5 \\ -2 \end{bmatrix} [1\ 0\ 0\ 0] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix}$$

$$ABA = AB \cdot A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 5 \\ -2 \end{bmatrix}$$

$$ABA = \begin{bmatrix} 1(1) + 0(2) + 0(5) + 0(-2) \\ 2(1) + 0(2) + 0(5) + 0(-2) \\ 5(1) + 0(2) + 0(5) + 0(-2) \\ -2(1) + 0(2) + 0(5) + 0(-2) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \\ -2 \end{bmatrix} = A$$

Our goal was to verify ABA = A. Since we confirmed this equation holds, we have confirmed B is a generalized inverse of A.

Q3

Consider the linear model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ where

$$\mathbf{Y} = \begin{bmatrix} Y_{11} \\ Y_{12} \\ Y_{21} \\ Y_{22} \\ Y_{23} \\ Y_{24} \\ Y_{31} \\ Y_{32} \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}, \quad \boldsymbol{\epsilon} \sim N(0, \sigma^2 \mathbf{I}).$$

Determine which of the following linear functions, $c_i^T \boldsymbol{\beta}$, of the model parameters are estimable. Briefly justify your answer. For estimable functions only, find a constant matrix $\mathbf{A}_i$ such that $\mathbf{A}_i \mathbb{E}(\mathbf{Y}) = c_i^T \boldsymbol{\beta}$.

(a)

$c_i^T \boldsymbol{\beta} = \alpha_1 - \frac{1}{2}(\alpha_2 + \alpha_3)$

Estimable: The function $c_i^T \boldsymbol{\beta} = \alpha_1 - \frac{1}{2}(\alpha_2 + \alpha_3)$ can be expressed as a linear combination of the rows of the design matrix $\mathbf{X}$, specifically using weights $a_1 = 1, a_2 = -\frac{1}{2}, a_3 = -\frac{1}{2}$, which satisfy the row space condition.

Constant matrix:

$$\mathbf{A}_i = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

(b)

$c_i^T \boldsymbol{\beta} = 3\mu + \alpha_1 + 2\alpha_2$

Not Estimable: The function $c_i^T \boldsymbol{\beta} = 3\mu + \alpha_1 + 2\alpha_2$ is not estimable because μ is confounded with the group means $(\alpha_1, \alpha_2, \alpha_3)$ due to the rank deficiency of the design matrix.

(c)

$c_i^T \boldsymbol{\beta} = \alpha_2 + \alpha_3$

Estimable: The function $c_i^T \boldsymbol{\beta} = \alpha_2 + \alpha_3$ can be expressed as a linear combination of the rows of the design matrix $\mathbf{X}$, specifically using weights $a_1 = 0, a_2 = 0, a_3 = 1, a_4 = 1$, which satisfy the row space condition.

Constant matrix:

$$\mathbf{A}_i = \begin{bmatrix} 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

(d)

$c_i^T \boldsymbol{\beta} = 3\mu - \alpha_1 - \alpha_2 - \alpha_3$

Not Estimable: The function $c_i^T \boldsymbol{\beta} = 3\mu - \alpha_1 - \alpha_2 - \alpha_3$ simplifies to 3μ, which is not estimable because μ is confounded with the group means $(\alpha_1, \alpha_2, \alpha_3)$ due to the rank deficiency of the design matrix.

Q4

A food scientist performed an experiment to study the effects of combining two different fats and three different surfactants on the specific volume of bread loaves. Two batches of dough were made for each of the six combinations of fat and surfactant. Ten loaves of bread were made from each batch of dough and the average volume of the ten loaves was recorded for each batch. In total, there are 12 observations. Consider the two-way ANOVA model

$$Y_{ijk} = \mu + \alpha_i + \tau_j + (\alpha\tau)_{ij} + \epsilon_{ijk}$$

where $\epsilon_{ijk} \sim N(0, \sigma^2)$,

and Y_{ijk} denotes the average of the volumes of ten loaves of bread made from the k-th batch of dough using the i-th fat and the j-th surfactant.

Determine which of the following linear functions of the model parameters are estimable. Briefly justify your answer.

(a)

μ

Not Estimable: The parameter μ is not estimable because it is confounded with the main effects α_i (fat) and τ_j (surfactant), as well as the interaction effects $(\alpha\tau)_{ij}$. The design matrix has rank deficiency due to the identifiability constraints (e.g., sum-to-zero) imposed on these effects, preventing μ from being uniquely determined.

(b)

$\alpha_1 - \alpha_2$

Estimable: The function $\alpha_1 - \alpha_2$ is estimable because it is a contrast of the main effects α_i (fat). Contrasts are not affected by the rank deficiency of the design matrix imposed by identifiability constraints (e.g., sum-to-zero).

(c)

$(\alpha\tau)_{12}$

Not Estimable: The interaction effect $(\alpha\tau)_{12}$ is not estimable because it cannot be uniquely separated from the other interaction effects or main effects due to the rank deficiency of the design matrix and the sum-to-zero constraints imposed on the interaction terms.

(d)

$(\alpha\tau)_{11} - (\alpha\tau)_{12}$

Estimable: The function $(\alpha\tau)_{11} - (\alpha\tau)_{12}$ is estimable because it is a contrast of interaction effects. Contrasts of interaction effects are estimable as they are not affected by the rank deficiency imposed by the sum-to-zero constraints on the interaction terms.

(e)

$(\alpha\tau)_{11} - (\alpha\tau)_{12} - (\alpha\tau)_{21} + (\alpha\tau)_{22}$

Estimable: The function $(\alpha\tau)_{11} - (\alpha\tau)_{12} - (\alpha\tau)_{21} + (\alpha\tau)_{22}$ is estimable because it is a contrast of interaction effects, which are not affected by the rank deficiency imposed by the sum-to-zero constraints on the interaction terms.