HW7

Sam Olson

Outline

- Q1: Edits
- Q2: Edits
- Q3: Edits
- Q4: Edits

Q1

Problem 8.6 a) - b), Casella and Berger (2nd Edition)

Suppose that we have two independent random samples: $X_1, \ldots, X_n$ are exponential(θ), and $Y_1, \ldots, Y_m$ are exponential(μ).

a)

Find the LRT of

$$H_0 : \theta = \mu \quad \text{versus} \quad H_1 : \theta \neq \mu.$$

Answer

The likelihood ratio test (LRT) statistic is:

$$\lambda(x, y) = \frac{\max_\theta L(\theta \mid x, y)}{\max_{\theta,\mu} L(\theta, \mu \mid x, y)}$$

Under H_0 ($\theta = \mu$):

The MLE for θ is obtained from the combined sample:

$$\hat{\theta}_0 = \frac{\sum_{i=1}^n X_i + \sum_{j=1}^m Y_j}{n+m}$$

Under the full model:

The MLEs are the sample means:

$$\hat{\theta} = \bar{X} = \frac{\sum X_i}{n}, \quad \hat{\mu} = \bar{Y} = \frac{\sum Y_j}{m}$$

Substituting the MLEs, we get:

$$\lambda(x,y) = \frac{(\hat{\theta}_0)^{-(n+m)}e^{-(n+m)}}{(\hat{\theta})^{-n}e^{-n}(\hat{\mu})^{-m}e^{-m}} = \frac{(\bar{X})^n(\bar{Y})^m}{\left(\frac{\sum X_i+\sum Y_j}{n+m}\right)^{n+m}}$$

Simplifying, this becomes:

$$\lambda(x,y) = \frac{(n+m)^{n+m}(\sum X_i)^n(\sum Y_j)^m}{n^n m^m(\sum X_i+\sum Y_j)^{n+m}}$$

Rejection Rule: Reject H_0 if $\lambda(x,y) \leq c$, where c is chosen for significance level α.

$$\varphi(x,y) = \begin{cases} 1 & \text{if } \lambda(x,y) \leq c, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\lambda(x,y) = \frac{(n+m)^{n+m}(\sum X_i)^n(\sum Y_j)^m}{n^n m^m(\sum X_i+\sum Y_j)^{n+m}}$$

And c is chosen such that $\mathbb{P}(\varphi(X,Y)=1 \mid H_0) = \alpha$.

b)

Show that the test in part a) can be based on the statistic

$$T = \frac{\sum X_i}{\sum X_i + \sum Y_i}.$$

Answer

Let $T = \frac{\sum X_i}{\sum X_i+\sum Y_j}$.

Rewriting $\lambda(x,y)$ in terms of T:

$$\lambda(x,y) = \frac{(n+m)^{n+m}}{n^n m^m}\left(\frac{\sum X_i}{\sum X_i+\sum Y_j}\right)^n\left(\frac{\sum Y_j}{\sum X_i+\sum Y_j}\right)^m = \frac{(n+m)^{n+m}}{n^n m^m}T^n(1-T)^m$$

Since $\lambda(x,y)$ depends on the data only through T, the LRT can be based entirely on T.

Rejection Region:

The test rejects H_0 when T is too small or too large, i.e.,

$$T \leq a \quad \text{or} \quad T \geq b$$

where a and b are critical values satisfying:

$$P(T \leq a \mid H_0) + P(T \geq b \mid H_0) = \alpha$$

Distribution of T under H_0:

Under H_0 ($\theta = \mu$):

- $\sum X_i \sim \text{Gamma}(n, \theta)$
- $\sum Y_j \sim \text{Gamma}(m, \theta)$

Thus,

$$T = \frac{\sum X_i}{\sum X_i + \sum Y_j} \sim \text{Beta}(n, m)$$

Alternatively, $\frac{T}{1-T} = \frac{\sum X_i/n}{\sum Y_j/m} \sim F_{2n,2m}$, which can be used to compute critical values.

Q2

Problem 8.28, Casella and Berger (2nd Edition)

Let $f(x|\theta)$ be the logistic location probability density function:

$$f(x|\theta) = \frac{e^{(x-\theta)}}{(1+e^{(x-\theta)})^2}, \quad -\infty < x < \infty, \quad -\infty < \theta < \infty.$$

a)

Show that this family has an MLR.

Answer

To show the family has a monotone likelihood ratio (MLR) in x, consider the likelihood ratio for $\theta_2 > \theta_1$:

$$\frac{f(x|\theta_2)}{f(x|\theta_1)} = e^{\theta_1-\theta_2}\left[\frac{1+e^{x-\theta_1}}{1+e^{x-\theta_2}}\right]^2.$$

Define $g(x) = \frac{1+e^{x-\theta_1}}{1+e^{x-\theta_2}}$. Its derivative is:

$$g'(x) = \frac{e^{x-\theta_1}(1+e^{x-\theta_2}) - e^{x-\theta_2}(1+e^{x-\theta_1})}{(1+e^{x-\theta_2})^2} = \frac{e^{x-\theta_1} - e^{x-\theta_2}}{(1+e^{x-\theta_2})^2} > 0,$$

where the inequality holds because $\theta_2 > \theta_1$. Thus, $g(x)$ is strictly increasing in x, and so is the likelihood ratio.

Conclusion: The family $\{f(x|\theta)\}$ has MLR in x.

b)

Based on one observation X, find the most powerful size α test of

$$H_0 : \theta = 0 \quad \text{versus} \quad H_1 : \theta = 1.$$

For $\alpha = 0.2$, find the size of the Type II error.

Answer

By the Neyman-Pearson Lemma, the most powerful (MP) test rejects H_0 when:

$$\frac{f(x|1)}{f(x|0)} = e^{-1}\left(\frac{1+e^x}{1+e^{x-1}}\right)^2 > k.$$

Since the likelihood ratio is increasing in x (from part (a)), the MP test rejects if $X > k'$, where k' is determined by the size α.

The CDF of the logistic distribution is:

$$F(x|\theta) = \frac{e^{x-\theta}}{1+e^{x-\theta}}.$$

Under H_0, the size condition is:

$$\mathbb{P}(X > k' \mid \theta = 0) = 1 - F(k'|0) = \frac{1}{1+e^{k'}} = \alpha.$$

Solving for k':

$$k' = \log\left(\frac{1-\alpha}{\alpha}\right).$$

For $\alpha = 0.2$:

$$k' = \log(4) \approx 1.386.$$

Under H_1:

$$\beta = \mathbb{P}(X \leq k' \mid \theta = 1) = F(k'|1) = \frac{e^{k'-1}}{1+e^{k'-1}} \approx \frac{e^{0.386}}{1+e^{0.386}} \approx 0.595.$$

Conclusion: The MP level-0.2 test rejects when $X > 1.386$, with a Type II error rate of approximately 0.595.

c)

Show that the test in part b) is UMP size α for testing

$$H_0 : \theta \leq 0 \quad \text{versus} \quad H_1 : \theta > 0.$$

What can be said about UMP tests in general for the logistic location family?

Answer

1. MLR Property: From part (a), the family has MLR in X.
2. Karlin-Rubin Theorem: Since the MP test for $\theta = 0$ vs $\theta = 1$ rejects for large X and does not depend on the specific $\theta_1 = 1$, it is uniformly most powerful (UMP) for $H_0 : \theta \leq 0$ vs $H_1 : \theta > 0$.

General Conclusion:
For the logistic location family, UMP tests exist for one-sided hypotheses and take the form "Reject H_0 if $X > c$."

Q3

Problem 8.29 a) - b), Casella and Berger (2nd Edition)

Let X be one observation from a Cauchy(θ) distribution.

The Cauchy(θ) density is given by:

$$f(x|\theta) = \frac{1}{\pi} \cdot \frac{1}{1+(x-\theta)^2}, \quad x \in \mathbb{R}, -\infty < \theta < \infty.$$

a)

Show that this family does not have an MLR.

Hint:

Show that the Cauchy(θ) family $\{f(x|\theta) : \theta \in \mathbb{R} = \Theta\}$, based on one observation X, does not have monotone likelihood ratio (MLR) in $t(X) = X$ or $t(X) = -X$. That is, the ratio

$$\frac{f(x|\theta_2)}{f(x|\theta_1)}$$

might not be monotone (either increasing or decreasing) in x.

Answer

For $\theta_2 > \theta_1$, the likelihood ratio is:

$$\frac{f(x|\theta_2)}{f(x|\theta_1)} = \frac{1+(x-\theta_1)^2}{1+(x-\theta_2)^2}.$$

1. Limiting Behavior:

$$\lim_{x\to\pm\infty} \frac{f(x|\theta_2)}{f(x|\theta_1)} = 1.$$

2. Non-Monotonicity:
   - The ratio achieves a maximum at finite x. For example, let $\theta_1 = 0$, $\theta_2 = 1$:

     $$\frac{f(x|1)}{f(x|0)} = \frac{1+x^2}{1+(x-1)^2}.$$

     - At $x = 0$: Ratio $= 1$.
     - At $x = 1$: Ratio $= 2$.
     - As $x \to \infty$: Ratio $\to 1$.
   - The ratio increases from $x = 0$ to $x = 1$ and then decreases, proving non-monotonicity.

Conclusion: The Cauchy(θ) family lacks MLR in X or $-X$.

b)

Show that the test

$$\phi(x) = \begin{cases} 1 & \text{if } 1 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

is most powerful of its size for testing

$$H_0 : \theta = 0 \quad \text{versus} \quad H_1 : \theta = 1.$$

Calculate the Type I and Type II error probabilities.

Hint:

Show that the test given is equivalent to rejecting H_0 if

$$f(x|\theta = 1) > 2f(x|\theta = 0)$$

and not rejecting otherwise. Conclude that this must be the most powerful (MP) test for its size. Justify why.

Answer

Consider the test:

$$\phi(x) = \begin{cases} 1 & \text{if } 1 < x < 3, \\ 0 & \text{otherwise.} \end{cases}$$

By the Neyman-Pearson Lemma, the MP test rejects H_0 when:

$$\frac{f(x|1)}{f(x|0)} = \frac{1 + x^2}{1 + (x-1)^2} > k.$$

- The ratio $\frac{f(x|1)}{f(x|0)}$ has critical points at $x = \frac{1 \pm \sqrt{5}}{2}$.
- At $x = 1$ and $x = 3$:

$$\frac{f(1|1)}{f(1|0)} = \frac{f(3|1)}{f(3|0)} = 2.$$

- The set $\{x : \frac{f(x|1)}{f(x|0)} > 2\} = (1, 3)$ exactly matches $\phi(x)$.

Thus, $\phi(x)$ is the most powerful test for its size.

Under H_0:

$$\alpha = \mathbb{P}(1 < X < 3 \mid \theta = 0) = \frac{1}{\pi}\left(\tan^{-1}(3) - \tan^{-1}(1)\right) \approx 0.1476.$$

Under H_1:

$$\beta = 1 - \mathbb{P}(1 < X < 3 \mid \theta = 1) = 1 - \frac{1}{\pi}\left(\tan^{-1}(2) - \tan^{-1}(0)\right) \approx 0.6476.$$

Conclusion:
- $\phi(x)$ is MP with $\alpha \approx 0.1476$ and $\beta \approx 0.6476$.
- The rejection region $(1, 3)$ is unique for this α, as guaranteed by Neyman-Pearson.

Q4

Consider one observation X from the probability density function

$$f(x \mid \theta) = 1 - \theta^2 \left(x - \frac{1}{2}\right), \quad 0 \le x \le 1, \quad 0 \le \theta \le 1.$$

We wish to test:

$$H_0 : \theta = 0 \quad \text{vs.} \quad H_1 : \theta > 0$$

a)

Find the UMP test of size $\alpha = 0.05$ based on X. Carefully justify your answer.

Answer

1. Likelihood Ratio Analysis:

For $\theta_2 > \theta_1 \ge 0$, the likelihood ratio is:

$$\frac{f(x \mid \theta_2)}{f(x \mid \theta_1)} = \frac{1 - \theta_2^2(x - \frac{1}{2})}{1 - \theta_1^2(x - \frac{1}{2})}$$

2. Monotonicity Properties:

- When $x > \frac{1}{2}$: the ratio is decreasing in x
- When $x < \frac{1}{2}$: the ratio is increasing in x

So, the family does not have global monotone likelihood ratio in X, but the likelihood function tilts rightward under H_1, which suggests rejecting for large X values is most powerful.

3. UMP Test Construction:

Under $H_0 : \theta = 0$, $X \sim \text{Uniform}(0, 1)$. So we define the test:

$$\phi(x) = \begin{cases} 1 & \text{if } x > c \\ 0 & \text{otherwise} \end{cases}$$

4. Critical Value Calculation:

$$P_{\theta=0}(X > c) = 1 - c = 0.05 \quad \Rightarrow \quad c = 0.95$$

Final UMP Test:

$$\phi(x) = \begin{cases} 1 & \text{if } x > 0.95 \\ 0 & \text{otherwise} \end{cases}$$

b)

Find the likelihood ratio test statistic $\lambda(X)$ based on X, expressed as a function of X.

Answer

The likelihood ratio test statistic is:

$$\lambda(X) = \frac{f(X \mid 0)}{\max_{\theta \in [0,1]} f(X \mid \theta)} = \frac{1}{\max_{\theta} \left[1 - \theta^2(X - \frac{1}{2})\right]}$$

Case 1: $X \geq \frac{1}{2}$
Maximum occurs at $\theta = 0$:

$$\max_{\theta} f(X \mid \theta) = 1$$

Case 2: $X < \frac{1}{2}$
Maximum occurs at $\theta = 1$:

$$\max_{\theta} f(X \mid \theta) = 1 + \left(\frac{1}{2} - X\right) = 1.5 - X$$

Final LRT Statistic:

$$\lambda(X) = \begin{cases} \frac{1}{1.5 - X} & \text{if } X < \frac{1}{2} \\ 1 & \text{if } X \geq \frac{1}{2} \end{cases}$$

c)

Find the likelihood ratio test (LRT) of size $\alpha = 0.05$ for the above hypotheses.

Answer

1. Rejection Region:

From part (b), $\lambda(X) = 1$ for $X \geq \frac{1}{2}$, and is increasing for $X < \frac{1}{2}$. So to make the test most powerful while maintaining the correct size, we reject for large values of X.

2. Size Condition:

$$P_{\theta=0}(X > k) = 1 - k = 0.05 \quad \Rightarrow \quad k = 0.95$$

Final LRT:

Reject H_0 when $X > 0.95$

Note: The LRT coincides with the UMP test derived in part (a) because:

1. Although the MLR is not strictly monotone in X, the density under H_1 favors larger values of X
2. The test based on rejecting for large X values maximizes power subject to size, satisfying both Neyman-Pearson and UMP conditions