PS1

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2025-01-27

Problem 1

Find the method of moment estimators (MMEs) of the unknown parameters based on a random sample $X_1, X_2, \ldots, X_n$ of size n from the following distributions:

1. Negative Binomial $(3, p)$, unknown p:
2. Double Exponential (μ, σ), unknown μ and σ:

See "Table of Common Distributions" in Casella & Berger (pages 623–623) for the definitions/properties of the above distributions.

Problem 2

Problem 7.1, Casella & Berger:

Hint: For context, there is only one (discrete) data observation X which has possible outcomes as $0, 1, 2, 3, 4$. For a given outcome x of X, the likelihood $(L(\theta) \equiv f(x|\theta)$ is given by the pmf as a function of $\theta \in \Theta \equiv \{1, 2, 3\}$.

One observation is taken on a discrete random variable X with pmf $f(x|\theta)$, where $\theta \in \{1, 2, 3\}$. Find the MLE of θ.

x	$f(x\|1)$	$f(x\|2)$	$f(x\|3)$
0	$\frac{1}{3}$	$\frac{1}{4}$	0
1	$\frac{1}{3}$	$\frac{1}{4}$	0
2	0	$\frac{1}{4}$	$\frac{1}{4}$
3	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{2}$
4	$\frac{1}{6}$	0	$\frac{1}{4}$

Problem 3

An indicator function $I(A)$ of an event A has the form:

$$I(A) = \begin{cases} 1, & \text{if event } A \text{ holds true,} \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that $A_1, \ldots, A_n$ are n separate events. Show that:

$$\prod_{i=1}^{n} I(A_i) = I(B),$$

where B is the event that $B = \bigcap_{i=1}^{n} A_i$.

Problem 4

Maximum-Likelihood & Indicator Functions

Given a random sample $X_1, \ldots, X_n$ from a pdf/pmf $f(x|\theta)$, $\theta \in \Theta \subset \mathbb{R}$, we know that the likelihood function will generically be

$$L(\theta) = \prod_{i=1}^{n} f(x_i|\theta), \quad \theta \in \Theta,$$

but there's one subtle point to again highlight about how to exactly write the likelihood expression depending on the support of $f(x|\theta) > 0$.

- Recall the support or range of $f(x|\theta)$ is a set

$$S_\theta = \{x \in \mathbb{R} : f(x|\theta) > 0\},$$

which could possibly depend on $\theta \in \Theta$. For example, an exponential distribution has a pdf

$$f(x|\theta) = \begin{cases} \frac{1}{\theta} e^{-x/\theta}, & x > 0, \\ 0, & \text{otherwise}, \end{cases}$$

with a parameter $\theta > 0$, and in this case the support $S_\theta = (0, \infty)$ doesn't depend on $\theta \in \Theta = (0, \infty)$.

On the other hand, the pdf (1):

(1)

$$f(x|\theta) = \begin{cases} \frac{2x}{\theta^2}, & 0 < x \leq \theta, \\ 0, & \text{otherwise}, \end{cases}$$

with parameter $\theta > 0$, does have a support $S_\theta = (0, \theta]$ depending on $\theta \in \Theta = (0, \infty)$.

- It's always true that $f(x|\theta) = f(x|\theta) I(x \in S_\theta)$ for all $x \in \mathbb{R}$ and so always true that (2):

(2)

$$L(\theta) = \prod_{i=1}^{n} \left[f(x_i|\theta) I(x_i \in S_\theta) \right] = \left(\prod_{i=1}^{n} f(x_i|\theta) \right) I(x_1, \ldots, x_n \text{ are all in } S_\theta).$$

Questions

(a) If $X_1, \ldots, X_n$ are a random sample from an exponential pdf $f(x|\theta)$, $\theta > 0$ (and so $X_1, \ldots, X_n$ are positive values), show that the likelihood function (2) can be written as

$$L(\theta) = \frac{1}{\theta^n} e^{-\sum_{i=1}^n x_i/\theta},$$

and that the MLE of θ is $\bar{X}_n$. (Message here: The support of an exponential doesn't depend on θ, so we don't have to worry about indicating the support.)

(b) If $X_1, \ldots, X_n$ are a random sample from the pdf

$$f(x|\theta) = \begin{cases} \frac{2x}{\theta^2}, & 0 < x \leq \theta, \\ 0, & \text{otherwise}, \end{cases}$$

(and so $X_1, \ldots, X_n > 0$ are less than or equal to θ), show that the likelihood function (2) can be written as

$$L(\theta) = \frac{2^n \prod_{i=1}^n x_i}{\theta^{2n}} I\left(\max_{1 \leq i \leq n} x_i \leq \theta\right),$$

and that the MLE of θ is $\max_{1 \leq i \leq n} X_i$. (Message here: The support in this case depends on θ, so we should think about indicator functions in writing the likelihood.)

Problem 5

Problem 7.6(b)-(c), Casella & Berger (Skip part (a).)

Let $X_1, \dots, X_n$ be a random sample from the pdf

$$f(x|\theta) = \theta x^{-2}, \quad 0 < \theta \leq x < \infty.$$

(b) Find the MLE of θ.

(c) Find the method of moments estimator of θ.