PS2

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Problem 1

Suppose $\mathbf{y} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu}^T = [1 \quad 2 \quad 3]$

$$\boldsymbol{\mu}^T = [1 \quad 2 \quad 3] \quad \text{and} \quad \boldsymbol{\Sigma} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Further, define a 3×3 matrix A and a 2×3 matrix B as follows

$$\mathbf{A} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

a) Determine the distribution of $u = \mathbf{1}_3^T\mathbf{y}$.

b) Determine the distribution of $\mathbf{v} = \mathbf{A}\mathbf{y}$.

c) Determine the distribution of $\mathbf{w}$, where $\mathbf{w}^T = [\mathbf{A}\mathbf{y} \quad \mathbf{B}\mathbf{y}]$.

d) Which of the distributions obtained in (a)-(c) are singular distributions? Recall that a distribution is singular if $\boldsymbol{\Sigma}$ is positive definite. Note that there are many algebraic properties of $\boldsymbol{\Sigma}$ that can be used to show that $\boldsymbol{\Sigma}$ is singular/nonsingular.

Problem 2

Suppose $\mathbf{X}$ and $\mathbf{W}$ are any two matrices with n rows for which $\mathcal{C}(\mathbf{X}) = \mathcal{C}(\mathbf{W})$. Show that $\mathbf{P_X} = \mathbf{P_W}$.

Problem 3

Consider a competition among 5 table tennis players labeled 1 through 5. For $1 \leq i < j \leq 5$, define y_{ij} to be the score for player i minus the score for player j when player i plays a game against player j. Suppose for $1 \leq i < j \leq 5$,

$$y_{ij} = \beta_i - \beta_j + \epsilon_{ij},$$

where $\beta_1, \ldots, \beta_5$ are unknown parameters and the ϵ_{ij} terms are random errors with mean 0. Suppose four games will be played that will allow us to observe y_{12}, y_{34}, y_{25}, and y_{15}. Let

$$\mathbf{y} = \begin{bmatrix} y_{12} \\ y_{34} \\ y_{25} \\ y_{15} \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{12} \\ \epsilon_{34} \\ \epsilon_{25} \\ \epsilon_{15} \end{bmatrix}.$$

a) Define a model matrix $\mathbf{X}$ so that model (1) may be written as $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$.

b) Is $\beta_1 - \beta_2$ estimable? Prove that your answer is correct.

c) Is $\beta_1 - \beta_3$ estimable? Prove that your answer is correct.

d) Find a generalized inverse of $\mathbf{X}^\top \mathbf{X}$.

e) Find a solution to the normal equations in this particular problem involving table tennis players.

f) Find the Ordinary Least Squares (OLS) estimator of $\beta_1 - \beta_5$.

g) Give a linear unbiased estimator of $\beta_1 - \beta_5$ that is not the OLS estimator.

Problem 4

Consider a linear model for which

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix}, \quad \text{and} \quad \mathbf{X} = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}.$$

a) Obtain the normal equations for this model and solve them.

b) Are all functions $\mathbf{c}^\top \boldsymbol{\beta}$ estimable? Justify your answer.

c) Obtain the least squares estimator of $\beta_1 + \beta_2 + \beta_3 + \beta_4$.

Problem 5

Suppose the Gauss-Markov model with normal errors (GMMNE) holds.

a) Suppose $\mathbf{C}\boldsymbol{\beta}$ is estimable. Derive the distribution of $\mathbf{C}\hat{\boldsymbol{\beta}}$, the OLSE of $\mathbf{C}\beta$.

b) Now suppose $\mathbf{C}\boldsymbol{\beta}$ is NOT estimable. Provide a fully simplified expression for $\text{Var}\left(\mathbf{C}(\mathbf{X}^\top\mathbf{X})^\top\mathbf{X}^\top\mathbf{y}\right)$.

c) Now suppose $H_0 : \mathbf{C}\boldsymbol{\beta} = \mathbf{d}$ is testable and that $\mathbf{C}$ has only one row and $\mathbf{d}$ has only one element so that they may be written as $\mathbf{c}^\top$ and $\mathbf{d}$, respectively. Prove the result on slide 29 of slide set 2 of Key Linear Model Results.

Problem 6

Provide an example that shows that a generalized inverse of a symmetric matrix need not be symmetric. (Comment: For this reason, we cannot assume that $(\mathbf{X}^\top\mathbf{X})^- = [(\mathbf{X}^\top\mathbf{X})^-]^\top$.)