

## HW 2

NAME: SAM OLSON

COLLABORATORS: KYU-TAE, BEN, SARAH, SABRINA

### 1.

Q: Suppose a random variable  $X$  has the following cdf from class (which is neither a step function nor continuous):

$$F(x) = \begin{cases} 0 & x < 0 \\ (1+x)/2 & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

(a): Find the following probabilities:  $P(X > \frac{1}{2})$   $P(X \geq \frac{1}{2})$   $P(0 < X \leq \frac{1}{2})$   $P(0 \leq X \leq \frac{1}{2})$

(b): Conditional on the event “ $X > 0$ ”, the corresponding conditional pdf of  $X$  (i.e. given  $X > 0$ ) is as follows at  $x \in \mathbb{R}$ :

$$P(X \leq x | X > 0) = \frac{P(X \leq x, X > 0)}{P(X > 0)} = \frac{P(0 < X \leq x)}{P(X > 0)} = \frac{F(x) - F(0)}{1 - F(0)}$$

Giving:

$$P(X \leq x | X > 0) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x \leq 1 \\ 1 & x > 1 \end{cases}$$

Based on the conditional cdf above, show that the distribution of  $X$ , conditional on “ $X > 0$ ”, is the same (i.e. has the same cdf) as that of a random variable  $Y$  which is “uniform” on the interval  $(0, 1)$ , having constant pdf  $f_Y(y) = 1$  for  $0 < y < 1$  (with  $f_Y(y) = 0$  for all other  $y \in \mathbb{R}$ )

A:

(a):

Note: The random variable  $X$  is continuous for  $0 \leq x \leq 1$

$$F(x) = \begin{cases} 0 & x < 0 \\ (1+x)/2 & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

$$f(x) = \begin{cases} 0 & x < 0 \\ 1/2 & 0 \leq x \leq 1 \\ 0 & x > 1 \end{cases}$$

Given the above cdf of  $X$ , we may write the pdf of  $X$  for  $0 \leq x \leq 1$  as:

$$(F(x))/\curvearrowright = ((1+x)/2)/\curvearrowright = \frac{1}{2}$$

We then solve the following relations:

$$P(X > \frac{1}{2}) = \int_{1/2}^1 f(x)dx = \int_{1/2}^1 \frac{1}{2}dx = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$P(X \geq \frac{1}{2}) = \int_{1/2}^1 f(x)dx = \int_{1/2}^1 \frac{1}{2}dx = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$P(0 < X \leq \frac{1}{2}) = \int_0^{1/2} f(x)dx = \int_0^{1/2} \frac{1}{2}dx = \frac{1}{4} - 0 = \frac{1}{4}$$

$$P(0 \leq X \leq \frac{1}{2}) = \int_0^{1/2} f(x)dx = \int_0^{1/2} \frac{1}{2}dx = \frac{1}{4} - 0 = \frac{1}{4}$$

(b):

We are given the following relation to hold (given the definition of conditional probability):

$$P(X \leq x|X > 0) = \frac{P(X \leq x, X > 0)}{P(X > 0)} = \frac{P(0 < X \leq x)}{P(X > 0)} = \frac{F(x) - F(0)}{1 - F(0)}$$

For  $x > 1$ ,

$$P(X \leq x|X > 0) =$$

And for  $x < 0$ ,

$$P(X \leq x|X > 0) =$$

Then for  $0 < x \leq 1$  we have:

$$P(X \leq x|X > 0) = \frac{F(x) - F(0)}{1 - F(0)} = \frac{\frac{(x+1)}{2} - \frac{1}{2}}{1 - \frac{1}{2}} = \frac{\frac{(x)}{2} + \frac{1}{2} - \frac{1}{2}}{\frac{1}{2}} = \frac{x}{2} / \frac{1}{2} = x$$

We may then conclude:

$$P(X \leq x|X > 0) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x \leq 1 \\ 1 & x > 1 \end{cases}$$

The above may be considered the cdf of the distribution of  $X$ , conditional on “ $X > 0$ ”.

Consider then a random variable  $Y$  which is “uniform” on the interval  $(0, 1)$ , having constant pdf  $f_Y(y) = 1$  for  $0 < y < 1$  (with  $f_Y(y) = 0$  for all other  $y \in \mathbb{R}$ ), as for a random variable  $Y \sim \text{Uniform}(0, 1)$ . We may write its pdf as:

$$f_Y(y) = \begin{cases} 1 & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Taking this, we may find its cdf as:

$$F_Y(y) = \begin{cases} 0 & y \leq 0 \\ y & 0 < y \leq 1 \\ 1 & y > 1 \end{cases}$$

Note: I am unsure what can be taken for granted in this instance of “what we know” about Y, i.e. “we know its cdf”. In that vein, I want to emphasize that as  $\int 1dy = y$  and  $\int_0^1 1dy = y|_0^1 = 1 - 0 = 1$ , we may write the above cdf of Y,  $F_Y(y)$  as given.

And conclude that: Based on the conditional cdf above, that the distribution of X, conditional on “ $X > 0$ ”, is the same (has the same cdf) as that of a random variable Y which is “uniform” on the interval (0, 1).

## 2.

Q: Statistical reliability involves studying the time to failure of manufactured units. In many reliability textbooks, one can find the exponential distribution:

$$f(x) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

where  $\theta > 0$  is a fixed value, for modeling the time  $X$  that a random unit runs until failure (i.e.  $X$  is a survival time). Show that if  $X$  has an exponential distribution as above, then:

$$P(X > s + t | X > t) = P(X > s)$$

for any values  $t, s > 0$ ; this feature is called the “memoryless” property of the exponential distribution.

A:

Let  $X$  be a random variable with Exponential distribution as given above, with parameter  $\theta > 0$ . Let  $t, s > 0$ .

For  $x > 0$ , the pdf given is  $\frac{1}{\theta} e^{-\frac{x}{\theta}}$ , thus, for the same  $x > 0$  the cdf is:

$$F_X(x) = \int_{x>0} f(x) dx = \int \frac{1}{\theta} e^{-\frac{x}{\theta}} dx = 1 - e^{-\frac{x}{\theta}}$$

Thus:

$$P(X > s + t | X > t) = P(X > s) = \frac{P(X > s+t, X > t)}{P(X > t)}$$

$$P(X > s + t | X > t) = P(X > s) = \frac{P(X > s+t)}{P(X > t)}$$

$$P(X > s + t | X > t) = \frac{1 - F_X(s+t)}{1 - F_X(t)} = \frac{1 - P(X \leq s+t)}{1 - P(X \leq t)}$$

With note of the following relation:

$$F_X(s) = \int_0^s \frac{1}{\theta} e^{-\frac{x}{\theta}} dx = (-e^{-\frac{s}{\theta}}) - (-e^{-\frac{0}{\theta}}) = (-e^{-\frac{s}{\theta}}) - (-1) = 1 - e^{-\frac{s}{\theta}}$$

We then have:

$$P(X > s + t | X > t) = \frac{1 - (1 - \frac{1}{\theta} e^{-\frac{s+t}{\theta}})}{1 - (1 - \frac{1}{\theta} e^{-\frac{t}{\theta}})}$$

Cancelling out (most) like terms gives us:

$$P(X > s + t | X > t) = \frac{1 - F(s+t)}{1 - F(t)} = \frac{e^{-\frac{s+t}{\theta}}}{e^{-\frac{t}{\theta}}} = e^{\frac{-(s+t) - (-t)}{\theta}} = e^{-\frac{s}{\theta}}$$

However, we know that this is exactly  $P(X > s) = 1 - P(X \leq s) = 1 - (1 - e^{-\frac{s}{\theta}}) = e^{-\frac{s}{\theta}}$ !, giving us:

$$P(X > s + t | X > t) = P(X > s)$$

### 3. 2.3:

Q: Suppose X has the Geometric pmf:

$f_X(x) = \frac{1}{3}(\frac{2}{3})^x$ ,  $x = 0, 1, 2, \dots$  Determine the probability distribution of  $Y = \frac{X}{X+1}$ . Note that here X and Y are discrete random variables. To specify the probability distribution of Y, specify its pmf.

A:

$$f_Y(y) = P(Y = y) = P(\frac{X}{X+1} = y)$$

Using this relation we have:  $y(X + 1) = X \rightarrow yX + y = X \rightarrow y = X - yX \rightarrow y = X(1 - y)$

Thus we have:  $X = \frac{y}{1-y}$

Returning then to the original function for the pmf, we have:

$$f_Y(y) = P(X = \frac{y}{1-y}) = \frac{1}{3}(\frac{2}{3})^{\frac{y}{1-y}}$$

We must then identify the support of Y given  $x = 0, 1, 2, \dots$

For the support of X as given,  $x = 0, 1, 2, \dots \rightarrow y = \frac{X}{X+1} = \frac{0}{1}, \frac{1}{2}, \frac{2}{3}, \dots$

Thus we define the discrete random variable Y by its pmf and support respectively as:

$$f_Y(y) = \frac{1}{3}(\frac{2}{3})^{\frac{y}{1-y}} \text{ for } y = \frac{0}{1}, \frac{1}{2}, \frac{2}{3}, \dots$$

## 4. 2.4:

Q:

Let  $\lambda$  be a fixed positive constant, and define the function  $f(x)$  by:

$$f(x) = \frac{1}{2}\lambda e^{-\lambda x} \text{ if } x \geq 0 \text{ and } f(x) = \frac{1}{2}\lambda e^{\lambda x} \text{ if } x < 0$$

(a): Verify that  $f(x)$  is a pdf.

(b): If  $X$  is a random variable with pdf given by  $f(X)$ , find  $P(X < t) \forall t$ . Evaluate all integrals.

(c): Find  $P(|X| < t) \forall t$ . Evaluate all integrals.

A:

(a): (1):  $f(x)$  is a pdf so long as it is well defined, i.e.  $f(x) \geq 0 \forall x \in \mathbb{X}$  (2): and so long as  $\int_{x \in \mathbb{X}} f(x) dx = 1$

Then  $f(x)$  is a (proper) pdf

(1):  $f(x)$  is well-defined, i.e. ever negative.

For  $x \geq 0$ ,  $e^{-x} \geq 0$ , so by including additional, fixed (positive!) constants such as  $\lambda$ ,  $f(x) \geq 0$  for  $x \geq 0$ .

For  $x < 0$ ,  $f(x) = e^{\lambda x} \geq 0$ , so by including additional, fixed positive constants such as  $\lambda$ ,  $f(x) \geq 0$  for  $x < 0$

Taken collectively,  $f(x) \geq 0$  for all  $x \in \mathbb{X}$

(2):

$$\int_{x \in \mathbb{X}} f(x) dx = \int_{x < 0} \frac{1}{2}\lambda e^{\lambda x} + \int_{x \geq 0} \frac{1}{2}\lambda e^{-\lambda x}$$

$$\int_{x \in \mathbb{X}} f(x) dx = \int_{-\infty}^0 \frac{1}{2}\lambda e^{\lambda x} + \int_0^{\infty} \frac{1}{2}\lambda e^{-\lambda x}$$

Note, we can factor out a constant term from both integrals, giving us:

$$\int_{x \in \mathbb{X}} f(x) dx = \frac{1}{2}\lambda \left( \int_{-\infty}^0 e^{\lambda x} + \int_0^{\infty} e^{-\lambda x} \right) = \frac{1}{2}\lambda \left[ \frac{e^{\lambda x}}{\lambda} \Big|_{-\infty}^0 + \left( -\frac{e^{-\lambda x}}{\lambda} \Big|_0^{\infty} \right) \right]$$

$$\int_{x \in \mathbb{X}} f(x) dx = \frac{1}{2}\lambda \left( \frac{1}{\lambda} - \left( -\frac{1}{\lambda} \right) \right) = \frac{1}{2}\lambda \left( \frac{2}{\lambda} \right) = 1$$

We may then conclude that  $f(x)$  is a (proper) pdf.

(b):

If  $X$  is a random variable with pdf given by  $f(X)$ , find  $P(X < t) \forall t$ .

$$P(X < t) = \begin{cases} \int_{-\infty}^t \frac{1}{2}\lambda e^{\lambda x} dx & t > 0 \\ \int_{-\infty}^0 \frac{1}{2}\lambda e^{\lambda x} dx + \int_0^t \frac{1}{2}\lambda e^{-\lambda x} dx & t \geq 0 \end{cases}$$

We then evaluate the integrals of each, giving:

(1):

$$\int_{-\infty}^t \frac{1}{2} \lambda e^{\lambda x} dx = \frac{1}{2} \lambda e^{\lambda t} \Big|_{-\infty}^t = \frac{1}{2} e^{\lambda t} - 0 = \frac{1}{2} e^{\lambda t}$$

(2)

$$\int_0^t \frac{1}{2} \lambda e^{-\lambda x} dx = -\frac{1}{2} e^{-\lambda x} \Big|_0^t = \frac{1}{2} - \frac{1}{2} e^{-\lambda t}$$

(3):

$$\int_{-\infty}^0 \frac{1}{2} \lambda e^{\lambda x} dx = \frac{1}{2} e^{\lambda x} \Big|_{-\infty}^0 = \frac{1}{2} - 0$$

(4): For the case of (2) + (3),

$$\frac{1}{2} + \frac{1}{2} - \frac{1}{2} e^{-\lambda t} = 1 - \frac{1}{2} e^{-\lambda t}$$

Thus we're left with:

$$P(X < t) = \begin{cases} \frac{1}{2} e^{\lambda t} & t > 0 \\ 1 - \frac{1}{2} e^{-\lambda t} & t \geq 0 \end{cases}$$

(c):

$$P(|X| < t) \quad \forall t,$$

$$P(|X| < t) = P(-t < X < t) = \int_{-t}^0 \frac{1}{2} \lambda e^{\lambda x} + \int_0^t \frac{1}{2} \lambda e^{-\lambda x}$$

$$P(|X| < t) = \frac{1}{2} \left[ \frac{e^{\lambda x}}{\lambda} \Big|_{-t}^0 + \left( -\frac{e^{-\lambda x}}{\lambda} \Big|_0^t \right) \right] = \frac{1}{2} [(1 - e^{-\lambda t}) + (-e^{-\lambda t} + 1)] = \frac{1}{2} (2)(1 - e^{-\lambda t}) = 1 - e^{-\lambda t}$$

## 5. 2.6 (b, c):

Q: In each of the following find the pdf of  $Y$ . (Do not need to verify the pdf/evaluate the integration, per Instructions).

(b):  $f_X(x) = \frac{3}{8}(x+1)^2$ ,  $-1 < x < 1$ ;  $Y = 1 - X^2$

(c):

$$f_X(x) = \frac{3}{8}(x+1)^2$$

,  $-1 < x < 1$ ;  $Y = 1 - X^2$  if  $X \leq 0$  and  $Y = 1 - X$  if  $X > 0$

A:

(b):  $f_X(x) = \frac{3}{8}(x+1)^2$ ,  $-1 < x < 1$ ;  $Y = 1 - X^2$

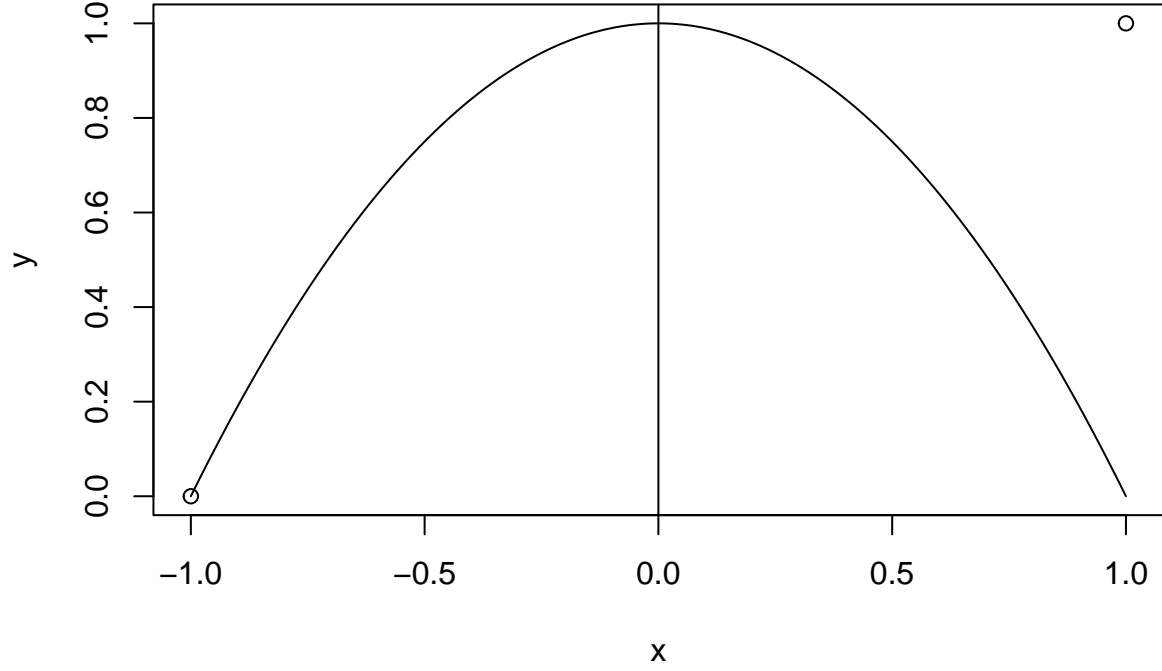
Then for the pdf of  $Y$ , we have:

$$f_Y(y) = \begin{cases} \sum_{i=1}^k f_X(g_i^{-1}(y)) \left| \frac{d}{dy} g_i^{-1}(y) \right| & y \in \mathbb{Y} \\ 0 & \text{otherwise} \end{cases}$$

Let us consider the following to motivate our partitions of the sample space:

```
x <- seq(from = -1, to = 1, by = 0.01)
y <- (1 - x^2)
plot(x = c(-1, 1), y = c(0, 1), xlab = "x", ylab = "y")
lines(x, y)
abline(v = 0)
```





We see three distinct partitions to ensure monotone functions:

$$A_1 = (-1, 0) \quad A_2 = \{0\} \quad A_3 = (0, 1)$$

We then have their respective functions,  $g_i(x)$  as follows:

$$g_1 = (1 - x^2) \quad g_2 = 0 \quad g_3 = (1 - x^2)$$

Then, with note from the results of the following theorem (2.1.8):

$$f_Y(y) = \begin{cases} \sum_{i=1}^k f_X(g_i^{-1}(y)) \left| \frac{d}{dy} g_i^{-1}(y) \right| & y \in \mathbb{Y} \\ 0 & \text{otherwise} \end{cases}$$

We have, for  $0 < y < 1$ ,

$$g_1(x) = g_3(x) = 1 - x^2 \rightarrow g^{-1}(y) = -(1 - y)^{1/2}$$

$$\therefore \left| \frac{d}{dy} g_1^{-1}(y) \right| = \frac{1}{2(1-y)^{1/2}} = \left| \frac{d}{dy} g_3^{-1}(y) \right|$$

Note however that we are dealing with two distinct functions, one positive and the other negative:

(1):

$$f_X(g_1^{-1}(y)) \left| \frac{d}{dy} g_1^{-1}(y) \right| = \frac{3}{8} (1 - (1 - y)^{1/2})^2 \left( \frac{1}{2(1 - y)^{1/2}} \right)$$

(2):

$$f_X(g_3^{-1}(y)) \left| \frac{d}{dy} g_3^{-1}(y) \right| = \frac{3}{8} (1 + (1 - y)^{1/2})^2 \left( \frac{1}{2(1 - y)^{1/2}} \right)$$

Such that we combine (1) and (2) together to get, for  $0 < y < 1$ :

$$f_Y(y) = \frac{3}{8}(1-(1-y)^{1/2})^2(\frac{1}{2(1-y)^{1/2}}) + \frac{3}{8}(1+(1-y)^{1/2})^2(\frac{1}{2(1-y)^{1/2}}) = \frac{3}{8} \frac{1}{2}(1-y)^{-1/2}[(1-(1-y)^{1/2})^2 + (1+(1-y)^{1/2})^2]$$

Notice the second term of the expansion between the two values will cancel each other out, leaving us (after much algebra and simplification):

$$f_Y(y) = \begin{cases} f_Y(y) = \frac{3}{8}(1-y)^{-1/2} + \frac{3}{8}(1-y)^{1/2} & 0 < y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(c):

$$f_X(x) = \frac{3}{8}(x+1)^2$$

,  $-1 < x < 1$ ;  $Y = 1 - X^2$  if  $X \leq 0$  and  $Y = 1 - X$  if  $X > 0$

Similar to part (b), we see three distinct partitions to ensure monotone functions:

$$A_1 = (-1, 0) \quad A_2 = \{0\} \quad A_3 = (0, 1)$$

We then have their respective functions,  $g_i(x)$  as follows:

$$g_1 = (1 - x^2) \quad g_2 = 0 \quad g_3 = (1 - x^2)$$

Thus, with note of the relevant theorem:

$$f_Y(y) = \begin{cases} \sum_{i=1}^k f_X(g_i^{-1}(y)) \left| \frac{d}{dy} g_i^{-1}(y) \right| & y \in \mathbb{Y} \\ 0 & \text{otherwise} \end{cases}$$

Such that for  $0 < y < 1$ :

$$g_1(x) = 1 - x^2 \rightarrow g^{-1}(y) = (1 - y)^{1/2}$$

$$\therefore \left| \frac{d}{dy} g_1^{-1}(y) \right| = \frac{1}{2(1-y)^{1/2}}$$

$$g_3(x) = 1 - x \rightarrow g^{-1}(y) = 1 - y$$

$$\therefore \left| \frac{d}{dy} g_i^{-1}(y) \right| = |-1| = 1$$

There are two relevant summations:

(1):

$$\frac{3}{8}(1 - (1 - y)^{1/2})^2 \frac{1}{2}(1 - y)^{-1/2} = \frac{3}{16}(1 - (1 - y)^{1/2})^2(1 - y)^{-1/2}$$

(2):

$$\frac{3}{8}((1 - y) + 1)^2 = \frac{3}{8}(2 - Y)^2$$

Taken together, we have the sum of (1) and (2), written:

$$f_Y(y) = \begin{cases} f_Y(y) = \frac{3}{16}(1 - (1 - y)^{1/2})^2(1 - y)^{-1/2} + \frac{3}{8}(2 - y)^2 & 0 < y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

## 6. 2.9:

Q: If the random variable  $X$  has pdf:

$$f(x) = \begin{cases} \frac{x-1}{2} & 1 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

find a monotone function  $u(x)$  such that the random variable  $Y = u(X)$  has a Uniform(0,1) distribution.

A:

We may take advantage of Thm 2.1.10, and let the random variable  $Y$  be defined as  $Y = u(X) = F_x(x)$

Taking advantage of the fact that  $u(x) = F_x(x) \rightarrow F_x(X) \sim \text{Uniform}(0,1)$

That is to say define the random variable  $Y$  as the cdf of the random variable  $X$ .

$$F_x(x) = P(X \leq x) = \int_{-\infty}^x f(t)dt = \int_{-\infty}^x \frac{t-1}{2} = \int_{-\infty}^1 \frac{t-1}{2} + \int_1^x \frac{t-1}{2} = 0 + \int_1^x \frac{t-1}{2}$$

$$F_x(x) = \int_1^x \frac{t-1}{2} = \frac{(t-1)^2}{4} \Big|_1^x = \frac{(x-1)^2}{4} - 0 = \frac{(x-1)^2}{4}$$

Such that we may define the monotone function  $u(x)$  by:

$$u(x) = \begin{cases} 0 & x \leq 1 \\ \frac{(x-1)^2}{4} & 1 < x < 3 \\ 1 & x \geq 3 \end{cases}$$

## 7. 2.22 (a, b):

Q: Let  $X$  have the pdf:

$$f(x) = \frac{4}{\beta^3 \sqrt{\pi}} x^2 e^{-\frac{x^2}{\beta^2}}$$

,  $0 < x < \infty$ ,  $\beta > 0$

(a): Verify that  $f(x)$  is a pdf.

(b): Find  $E(X)$

A:

(a):

There are two conditions to verify that  $f(x)$  is a pdf, the first is: (1):  $f(x) \geq 0$ ,  $\forall x$ . This one is apparent under the conditions  $0 < x < \infty$ ,  $\beta > 0$ . We must then establish condition (2):

(2):  $\int_{\Omega} f(x) dx = 1$ , or, the sum of the pdf over the sample space is 1 (note: this is for the continuous case, which we have).

We thus have:

$$\int_{\Omega} f(x) dx = \int_0^{\infty} \frac{4}{\beta^3 \sqrt{\pi}} x^2 e^{-\frac{x^2}{\beta^2}} dx$$

Set  $\frac{y}{\sqrt{2}} = \frac{x}{\beta} \rightarrow dx = \frac{\beta}{\sqrt{2}} dy$

And  $x^2 = \frac{\beta^2 y^2}{2}$

Such that we may write:

$$\int_{\Omega} f(x) dx = \int_0^{\infty} \frac{4}{\beta^3 \sqrt{\pi}} \frac{\beta^2 y^2}{2} e^{-\frac{y^2}{2}} \frac{\beta}{\sqrt{2}} dy = \int_0^{\infty} \frac{2}{\sqrt{2\pi}} y^2 e^{-y^2/2} dy$$

We may then make use of our assumption/hint, namely:

$$1 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x^2 e^{-\frac{x^2}{2}} dx = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} x^2 e^{-\frac{x^2}{2}} dx \rightarrow \sqrt{2\pi}/2 = \int_0^{\infty} x^2 e^{-\frac{x^2}{2}} dx$$

Incorporating this into the above relation on  $f(x)$  gives us (taking out the constant term from the integral):

$$\int_{\Omega} f(x) dx = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} y^2 e^{-y^2/2} dy = \frac{2}{\sqrt{2\pi}} \frac{\sqrt{2\pi}}{2} = 1$$

We have shown then that conditions (1) and (2) hold, and as such,  $f(x)$  is a pdf!

(b):

Q: Find  $\mathbb{E}(X)$

Note: For the random variable  $X$  given from the prior  $f(x)$ , we have  $\mathbb{E}(X) = \int_{\Omega} x f(x) dx$

We may calculate this as follows:

$$\mathbb{E}(X) = \int_0^{\infty} x \frac{4}{\beta^3 \sqrt{\pi}} x^2 e^{-\frac{x^2}{\beta^2}} dx$$

Let us take note of Integration by parts, that is:

$$\int u dv = uv - \int v du$$

For the above relation, let

$$u = \frac{4x^2}{\beta^3 \sqrt{\pi}} \rightarrow du = \frac{8x}{\beta^3 \sqrt{\pi}}$$

and

$$dv = x e^{-\frac{x^2}{\beta^2}} \rightarrow v = \int_0^{\infty} x e^{-\frac{x^2}{\beta^2}}$$

Of interest is  $uv$ , which may be written:

$$uv = \left[ \frac{4}{\beta^3 \sqrt{\pi}} x^2 \left( -\frac{\beta^2}{2} e^{-\frac{x^2}{\beta^2}} \right) \right] \Big|_0^{\infty}$$

Note: We have a number of constants, such that the above simplifies to:

$$uv = \frac{4}{\beta^3 \sqrt{\pi}} \left( -\frac{\beta^2}{2} \right) [x^2 e^{-\frac{x^2}{\beta^2}}] \Big|_0^{\infty}$$

And we note the following:

$$[x^2 e^{-\frac{x^2}{\beta^2}}] \Big|_0^{\infty} = 0 - 0 = 0$$

Such that our term  $uv$  is equal to zero, leaving us with:

$$\mathbb{E}(X) = 0 + \int_0^{\infty} x \frac{4}{\beta^3 \sqrt{\pi}} x^2 e^{-\frac{x^2}{\beta^2}} dx = 0 + \frac{4}{\beta \sqrt{\pi}} \int_0^{\infty} x e^{-\frac{x^2}{\beta^2}} dx$$

$$\mathbb{E}(X) = 0 + \frac{4}{\beta \sqrt{\pi}} \left( -\frac{1}{2} \beta^2 e^{-\frac{x^2}{\beta^2}} \Big|_0^{\infty} \right) = \frac{4}{\beta \sqrt{\pi}} \frac{\beta^2}{2} = \frac{2\beta}{\sqrt{\pi}}$$

We conclude then:

$$\mathbb{E}(X) = \frac{2\beta}{\sqrt{\pi}}$$

## 8.

Q: Suppose that a random variable  $U$  has a  $\text{Uniform}(0,1)$  distribution

(i.e. pdf  $f_U(u) = 1$  for  $0 < u < 1$ )

(a): Suppose a random variable  $X$  has a cdf  $F(x)$  which is strictly increasing and continuous on  $x \in \mathbb{R}$ ; this implies that, for any real value of  $0 < u < 1$ , there is an inverse  $F^{-1}(u) = x \in \mathbb{R}$  so that  $F(x) = F(F^{-1}(u)) = u$ . Define a random variable  $Y = F^{-1}(U)$  based on the random variable  $U$ . Show that  $X$  and  $Y$  have the same cdf (i.e. the same distributions).

Hint: Use that, because  $F$  is strictly increasing,  $P(Y \leq y) = P(F(Y) \leq F(y))$  holds for any  $y \in \mathbb{R}$ , i.e.,  $Y$  can be less than or equal to  $y$  if and only if  $F(Y)$  is less than or equal to  $F(y)$ . Note that  $F(y) \in (0,1)$  for any real  $y$ .

(b): If there is a computer program (i.e. random number generator) that produces numbers uniformly distributed between zero and one (i.e., according to the pdf  $F_U(u)$ ), explain how these numbers could be used to generate values distributed according to the pdf  $f_Z(z) = \frac{e^{-|z|}}{2}$ ,  $-\infty < z < \infty$ .

Hint: Use (a) where  $F$  now becomes the cdf of  $Z$ ; you need to find  $F^{-1}(u)$  for a given  $0 < u < 1$  by solving the expression  $F(z) = u$  for  $z \in \mathbb{R}$

A:

(a):

Let  $U$  and  $X$  be random variables.

Define the following relations to hold:

For any real value of  $0 < u < 1$ , there is an inverse  $F^{-1}(u) = x \in \mathbb{R}$  so that  $F(x) = F(F^{-1}(u)) = u$ .

Let us then define a random variable  $Y$  as follows:  $Y = F^{-1}(U)$

Note:  $F$  is strictly increasing, and  $F^{-1}$  is also strictly increasing.

Thus if we define  $Y \leq y \rightarrow F(Y) \leq F(y)$ . Similarly, if we define  $F(Y) \leq F(y) \rightarrow F^{-1}(F(Y)) \leq F^{-1}(F(y)) \rightarrow Y \leq y$

Such that we have shown:

$$Y \leq y \iff F(Y) \leq F(y)$$

for a strictly increasing function  $F$ .

Then consider the cdf of the random variable  $X$ , and the following consequence of  $F$  being strictly increasing:

$$F(x) = F(F^{-1}(u)) = u \rightarrow F^{-1}(u) = x$$

Given Our relations of the random variables  $Y$  and  $U$ , namely that  $F$  is strictly increasing, then the values the random variables take,  $y$  and  $u$  respectively may be written:

$$Y = F^{-1}(U) \rightarrow y = F^{-1}(u) \rightarrow F(y) = u$$

such that the above relations give us:

(1):

$$F(Y) = P(Y \leq y) = P(F^{-1}(U) \leq F^{-1}(u)) = P(F(F^{-1}(U)) \leq F(F^{-1}(u))) = P(U \leq u)$$

(2):

$$F(X) = P(F(X) \leq u) = P(U \leq u)$$

And taking (1) and (2) together, we may conclude that the random variable X and Y have the same cdfs.

(b):

We are given the pdf of Z, so we derive its cdf as follows:

$$F_Z(z) = \begin{cases} \int_{-\infty}^z \frac{e^t}{2} dt & z < 0 \\ \int_{-\infty}^0 \frac{e^t}{2} dt + \int_0^z \frac{e^{-t}}{2} dt & z > 0 \end{cases}$$

We then evaluate the following integrals such that we have:

$$\int_{-\infty}^z \frac{e^t}{2} dt = \frac{e^z}{2}$$

$$\int_{-\infty}^0 \frac{e^t}{2} dt = \frac{1}{2}$$

$$\int_0^z \frac{e^{-t}}{2} dt = \frac{1}{2} - \frac{1}{2}e^{-z}$$

Such that we have, for  $z > 0$ :

$$F_Z(z) = \frac{1}{2} + \frac{1}{2} - \frac{1}{2}e^{-z} = 1 - \frac{1}{2}e^{-z}$$

Thus we have the cdf of Z as:

$$F_Z(z) = \begin{cases} \frac{e^z}{2} & z < 0 \\ 1 - \frac{1}{2}e^{-z} & z > 0 \end{cases}$$

We then take the inverse  $F^{-1}$  to transform this from the random variable Z to the random variable U:

$$F^{-1}\left(\frac{e^z}{2}\right) = \ln(2u)$$

$$F^{-1}\left(1 - \frac{1}{2}e^{-z}\right) = \frac{1}{\ln(2-2u)}$$

We may then write  $F^{-1}(u)$

$$F_Z^{-1}(u) = \begin{cases} \ln(2) - \ln(u) & 0 < u < 1/2 \\ \frac{1}{\ln(2-2u)} & 1/2 < u < 1 \end{cases}$$