HW4

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Problem 1

Problem 6.2, Casella and Berger (2nd Edition)

6.2 Let $X_1, \ldots, X_n$ be independent random variables with densities

$$f_{X_i}(x|\theta) = \begin{cases} e^{\theta - x} & x \geq i\theta \\ 0 & x < i\theta. \end{cases}$$

Prove that $T = \min_i(X_i/i)$ is a sufficient statistic for θ.

Problem 2

Example of Rao-Blackwell theorem, which is largely a STAT 5420 problem in computation.

Let X_1 and X_2 be iid Bernoulli(p), $0 < p < 1$.

a)

Show $S = X_1 + X_2$ is sufficient for p.

b)

Identify the conditional probability $P(X_1 = x|S = s)$; you should know which values of x, s to consider.

c)

Find the conditional expectation $T \equiv E(X_1|S)$, i.e., as a function of the possibilities of S. Note that T is a statistic.

d)

Show X_1 and T are both unbiased for p.

e)

Show $\mathrm{Var}_p(T) \leq \mathrm{Var}_p(X_1)$, for any p.

Problem 3

Problem 6.21 a)-b), Casella and Berger (2nd Edition)

6.21 Let X be one observation from the pdf

$$f(x|\theta) = \left(\frac{\theta}{2}\right)^{|x|} (1-\theta)^{1-|x|}, \quad x = -1, 0, 1, \quad 0 \leq \theta \leq 1.$$

a)

Is X a complete sufficient statistic?

b)

Is $|X|$ a complete sufficient statistic?

Problem 4

Problem 6.24, Casella and Berger (2nd Edition)

6.24 Consider the following family of distributions:

$$\mathcal{P} = \{P_\lambda(X = x) : P_\lambda(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}; x = 0, 1, 2, \ldots; \lambda = 0 \text{ or } 1\}.$$

This is a Poisson family with λ restricted to be 0 or 1. Show that the family $\mathcal{P}$ is *not complete*, demonstrating that completeness can be dependent on the range of the parameter. (See Exercises 6.15 and 6.18.)

Problem 5

Problem 7.57, Casella and Berger (2nd Edition) You may assume $n \geq 3$.

One has to Rao-Blackwellize on the complete/sufficient statistic here

$$\sum_{i=1}^{n+1} X_i.$$

7.57 Let $X_1, \ldots, X_{n+1}$ be iid Bernoulli(p), and define the function $h(p)$ by

$$h(p) = P\left(\sum_{i=1}^{n} X_i > X_{n+1} \middle| p\right),$$

the probability that the first n observations exceed the $(n+1)$st.

a)

Show that

$$T(X_1, \ldots, X_{n+1}) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} X_i > X_{n+1} \\ 0 & \text{otherwise} \end{cases}$$

is an unbiased estimator of $h(p)$.

b)

Find the best unbiased estimator of $h(p)$.