# HW 2

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## 1.

Q: Suppose a random variable X has the following cdf from class (which is neither a step function nor continuous):

$$F(x) = \begin{cases} 0 & x < 0 \\ (1+x)/2 & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

(a): Find the following probabilities: $P(X > \frac{1}{2})$ $P(X \geq \frac{1}{2})$ $P(0 < X \leq \frac{1}{2})$ $P(0 \leq X \leq \frac{1}{2})$

(b): Conditional on the event "X > 0", the corresponding conditional pdf of X (i.e. given X > 0) is as follows at $x \in \mathbb{R}$:

$$P(X \leq x | X > 0) = \frac{P(X \leq x, X > 0)}{P(X > 0)} = \frac{P(0 < X \leq x)}{P(X > 0)} = \frac{F(x) - F(0)}{1 - F(0)}$$

Giving:

$$P(X \leq x | X > 0) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x \leq 1 \\ 1 & x > 1 \end{cases}$$

Based on the conditional cdf above, show that the distribution of X, conditional on "X > 0", is the same (i.e. has the same cdf) as that of a random variable Y which is "uniform" on the interval (0, 1), having constant pdf $f_Y(y) = 1$ for $0 < y < 1$ (with $f_Y(y) = 0$ for all other $y \in \mathbb{R}$)

A:

(a):

Note: The random variable X is continous for $0 \leq x \leq 1$

$$F(x) = \begin{cases} 0 & x < 0 \\ (1+x)/2 & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

Given the above cdf of X, we may write the pdf of X for $0 \leq x \leq 1$ as:

$$\frac{d}{dx}(F(x)) = \frac{d}{dx}[\frac{(1+x)}{2}] = \frac{1}{2}$$

Such that we may write the pdf of X as:

$$f(x) = \begin{cases} 0 & x < 0 \\ 1/2 & 0 \leq x \leq 1 \\ 0 & x > 1 \end{cases}$$

We then solve the following relations:

$$P(X > \frac{1}{2}) = P(1 \geq X > \frac{1}{2}) = F(1) - F(1/2) = \int\limits_{1/2}^{1} f(x)dx = \int\limits_{1/2}^{1} \frac{1}{2}dx = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$P(X \geq \frac{1}{2}) = P(1 \geq X \geq \frac{1}{2}) = F(1) - F(1/2) = \int\limits_{1/2}^{1} f(x)dx = \int\limits_{1/2}^{1} \frac{1}{2}dx = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$P(0 < X \leq \frac{1}{2}) = F(1/2) - F(0) = \int\limits_{0}^{1/2} f(x)dx = \int\limits_{0}^{1/2} \frac{1}{2}dx = \frac{1}{4} - 0 = \frac{1}{4}$$

$$P(0 \leq X \leq \frac{1}{2}) = F(1/2) - F(0) = \int\limits_{0}^{1/2} f(x)dx = \int\limits_{0}^{1/2} \frac{1}{2}dx = \frac{1}{4} - 0 = \frac{1}{4}$$

(b):

We are given the following relation to hold (given the definition of conditional probability):

$$P(X \leq x | X > 0) = \frac{P(X \leq x, X > 0)}{P(X > 0)} = \frac{P(0 < X \leq x)}{P(X > 0)} = \frac{F(x) - F(0)}{1 - F(0)}$$

For $x > 1$,

$P(X \leq x | X > 0) =$

And for $x < 0$,

$P(X \leq x | X > 0) =$

Then for $0 < x \leq 1$ we have:

$$P(X \leq x | X > 0) = \frac{F(x) - F(0)}{1 - F(0)} = \frac{\frac{(x+1)}{2} - \frac{1}{2}}{1 - \frac{1}{2}} == \frac{\frac{(x)}{2} + \frac{1}{2} - \frac{1}{2}}{\frac{1}{2}} = \frac{x}{2} / \frac{1}{2} = x$$

We may then conclude:

$$P(X \leq x | X > 0) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x \leq 1 \\ 1 & x > 1 \end{cases}$$

The above may be considered the cdf of the distribution of X, conditional on "X > 0".

Consider then a random variable Y which is "uniform" on the interval (0, 1), having constant pdf $f_Y(y) = 1$ for $0 < y < 1$ (with $f_Y(y) = 0$ for all other $y \in \mathbb{R}$), as for a random variable $Y$ Uniform$(0, 1)$. We may write its pdf as:

$$f_Y(y) = \begin{cases} 1 & 0 < y < 1 \\ 0 & otherwise \end{cases}$$

Taking this, we may find its cdf as:

$$F_Y(y) = \begin{cases} 0 & y \leq 0 \\ y & 0 < y \leq 1 \\ 1 & y > 1 \end{cases}$$

Note: I am unsure what can be taken for granted in this instance of "what we know" about Y, i.e. "we know its cdf". In that vein, I want to emphasize that as $\int 1dy = y$ and $\int\limits_0^1 1dy = y\Big|_0^1 = 1 - 0 = 1$, we may write the above cdf of Y, $F_Y(y)$ as given.

And conclude that: Based on the conditional cdf above, that the distribution of X, conditional on "X > 0", is the same (has the same cdf) as that of a random variable Y which is "uniform" on the interval (0, 1).

## 2.

Q: Statistical reliability involves studying the time to failure of manufactured units. In many reliability textbooks, one can find the exponential distribution:

$$f(x) = \begin{cases} \frac{1}{\theta}e^{-\frac{x}{\theta}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

where $\theta > 0$ is a fixed value, for modeling the time X that a random unit runs until failure (i.e. X is a survival time). Show that if X has an exponential distribution as above, then:

$P(X > s + t|X > t) = P(X > s)$

for any values t, s > 0; this feature is called the "memoryless" property of te exponential distribution.

A:

Let X be a random variable with Exponential distribution as given above, with parameter $\theta > 0$. Let t, s > 0.

For x > 0, the pdf given is $\frac{1}{\theta}e^{-\frac{x}{\theta}}$, thus, for the same x > 0 the cdf is:

$F_X(x) = \int_{x>0} f(x)dx = \int \frac{1}{\theta}e^{-\frac{x}{\theta}} dx = 1 - e^{-\frac{x}{\theta}}$

Thus:

$P(X > s + t|X > t) = P(X > s) = \frac{P(X>s+t, X>t)}{P(X>t)}$

$P(X > s + t|X > t) = P(X > s) = \frac{P(X>s+t)}{P(X>t)}$

$P(X > s + t|X > t) = \frac{1-F_X(s+t)}{1-F_X(t)} == \frac{1-P(X\leq s+t)}{1-P(X\leq t)}$

With note of the following relation:

$$F_X(s) = \int_0^s \frac{1}{\theta}e^{\frac{-x}{\theta}} dx = (-e^{\frac{-s}{\theta}}) - (-e^{\frac{0}{\theta}}) = (-e^{\frac{-s}{\theta}}) - (-1) = 1 - e^{\frac{-s}{\theta}}$$

We then have:

$$P(X > s + t|X > t) = \frac{1 - (1 - \frac{1}{\theta}e^{-\frac{s+t}{\theta}})}{1 - (1 - \frac{1}{\theta}e^{-\frac{t}{\theta}})}$$

Cancelling out (most) like terms gives us:

$$P(X > s + t|X > t) = \frac{1 - F(s + t)}{1 - F(t)} = \frac{e^{-\frac{s+t}{\theta}}}{e^{-\frac{t}{\theta}}} = e^{\frac{-(s+t)-(-t)}{\theta}} = e^{-\frac{s}{\theta}}$$

However, we know that this is exactly $P(X > s) = 1 - P(X \leq s) = 1 - (1 - e^{-\frac{s}{\theta}}) = e^{-\frac{s}{\theta}}$!, giving us:

$P(X > s + t|X > t) = P(X > s)$

## 3. 2.3:

Q: Suppose X has the Geometrc pmf:

$f_X(x) = \frac{1}{3}(\frac{2}{3})^x$, $x = 0, 1, 2, ...$ Determine the probability distribution of $Y = \frac{X}{X+1}$ Note that here X and Y are discrete random variables. To specify the probability distribution of Y, specify its pmf.

A:

$f_Y(y) = P(Y = y) = P(\frac{X}{X+1} = y)$

Using this relation we have: $y(X+1) = X \to yX + y = X \to y = X - yX \to y = X(1-y)$

Thus we have: $X = \frac{y}{1-y}$

Returning then to the original function for the pmf, we have:

$f_Y(y) = P(X = \frac{y}{1-y}) = \frac{1}{3}(\frac{2}{3})^{\frac{y}{1-y}}$

We must then identify the support of Y given $x = 0, 1, 2, ...$

For the support of X as given, $x = 0, 1, 2, ... \to y = \frac{X}{X+1} = \frac{0}{1}, \frac{1}{2}, \frac{2}{3}, ...$

Thus we define the discrete random variable Y by its pmf and support respectively as:

$f_y(y) = \frac{1}{3}(\frac{2}{3})^{\frac{y}{1-y}}$ for $y = 0, \frac{1}{2}, \frac{2}{3}, ...$

## 4. 2.4:

Q:

Let $\lambda$ be a fixed positive constant, and define the function $f(x)$ by:

$f(x) = \frac{1}{2}\lambda e^{-\lambda x}$ if $x \geq 0$ and $f(x) = \frac{1}{2}\lambda e^{\lambda x}$ if $x < 0$

(a): Verify that $f(x)$ is a pdf.

(b): If X is a random variable with pdf given by $f(X)$, find $P(X < t)$ $\forall t$. Evaluate all integrals.

(c): Find $P(|X| < t)$ $\forall t$. Evaluate all integrals.

A:

(a): (1): $f(x)$ is a pdf so long as it is well defined, i.e. $f(x) \geq 0$ $\forall x \in \mathbb{X}$ (2): and so long as $\int\limits_{x \in \mathbb{X}} f(x)dx = 1$

Then $f(x)$ is a (proper) pdf

(1): $f(x)$ is well-defined, i.e. ever negative.

For $x \geq 0$, $e^{-x} \geq 0$, so by including additional, fixed (positive!) constants such as $\lambda$, $f(x) \geq 0$ for $x \geq 0$.

For $x < 0$, $f(x) = e^{\lambda x} \geq 0$, so by including additional, fixed positive constants such as $\lambda$, $f(x) \geq 0$ for $x < 0$

Taken collectively, $f(x) \geq 0$ for all $x \in \mathbb{X}$

(2):

$$\int\limits_{x \in \mathbb{X}} f(x)dx = \int\limits_{x<0} \frac{1}{2}\lambda e^{\lambda x} + \int\limits_{x \geq 0} \frac{1}{2}\lambda e^{-\lambda x}$$

$$\int\limits_{x \in \mathbb{X}} f(x)dx = \int\limits_{-\infty}^{0} \frac{1}{2}\lambda e^{\lambda x} + \int\limits_{0}^{\infty} \frac{1}{2}\lambda e^{-\lambda x}$$

Note, we can factor out a constant term from both integrals, giving us:

$$\int\limits_{x \in \mathbb{X}} f(x)dx = \frac{1}{2}\lambda(\int\limits_{-\infty}^{0} e^{\lambda x} + \int\limits_{0}^{\infty} e^{-\lambda x}) = \frac{1}{2}\lambda[\frac{e^{\lambda x}}{\lambda}|_{-\infty}^{0} + (-\frac{e^{-\lambda x}}{\lambda}|_{0}^{\infty})]$$

$$\int\limits_{x \in \mathbb{X}} f(x)dx = \frac{1}{2}\lambda(\frac{1}{\lambda} - (-\frac{1}{\lambda})) = \frac{1}{2}\lambda(\frac{2}{\lambda}) = 1$$

We may then conclude that $f(x)$ is a (proper) pdf.

(b):

If X is a random variable with pdf given by $f(X)$, find $P(X < t)$ $\forall t$.

$$P(X < t) = \begin{cases} \int\limits_{-\infty}^{t} \frac{1}{2}\lambda e^{\lambda x}dx & t > 0 \\ \int\limits_{-\infty}^{0} \frac{1}{2}\lambda e^{\lambda x}dx + \int\limits_{0}^{t} \frac{1}{2}\lambda e^{-\lambda x}dx & t \geq 0 \end{cases}$$

We then evaluate the integrals of each, giving:

(1):

$\int\limits_{-\infty}^{t} \frac{1}{2}\lambda e^{\lambda x} dx = \frac{1}{2}\lambda e^{\lambda t}\big|_{-\infty}^{t} = \frac{1}{2}e^{\lambda t} - 0 = \frac{1}{2}e^{\lambda t}$

(2)

$\int\limits_{0}^{t} \frac{1}{2}\lambda e^{-\lambda x} dx = -\frac{1}{2}e^{-\lambda x}\big|_{0}^{t} = \frac{1}{2} - \frac{1}{2}e^{-\lambda t}$

(3):

$\int\limits_{-\infty}^{0} \frac{1}{2}\lambda e^{\lambda x} dx = \frac{1}{2}e^{\lambda x}\big|_{-\infty}^{0} = \frac{1}{2} - 0$

(4): For the case of (2) + (3),

$$\frac{1}{2} + \frac{1}{2} - \frac{1}{2}e^{-\lambda t} = 1 - \frac{1}{2}e^{-\lambda t}$$

Thus we're left with:

$$P(X < t) = \begin{cases} \frac{1}{2}e^{\lambda t} & t > 0 \\ 1 - \frac{1}{2}e^{-\lambda t} & t \geq 0 \end{cases}$$

(c):

$P(|X| < t)\ \forall t,$

$$P(|X| < t) = P(-t < X < t) = \int\limits_{-t}^{0} \frac{1}{2}\lambda e^{\lambda x} + \int\limits_{0}^{t} \frac{1}{2}\lambda e^{-\lambda x}$$

$$P(|X| < t) = \frac{1}{2}[\frac{e^{\lambda x}}{\lambda}\big|_{-t}^{0} + (-\frac{e^{-\lambda x}}{\lambda}\big|_{0}^{t})] = \frac{1}{2}[(1 - e^{-\lambda t}) + (-e^{-\lambda t} + 1)] = \frac{1}{2}(2)(1 - e^{-\lambda t}) = 1 - e^{-\lambda t}$$

## 5. 2.6 (b, c):

Q: In each of the following find the pdf of Y. (Do not need to verify the pdf/evaluate the integration, per Instructions).

(b): $f_X(x) = \frac{3}{8}(x+1)^2$, $-1 < x < 1$; $Y = 1 - X^2$

(c):

$$f_X(x) = \frac{3}{8}(x+1)^2$$

, $-1 < x < 1$; $Y = 1 - X^2$ if $X \leq 0$ and $Y = 1 - X$ if $X > 0$

A:

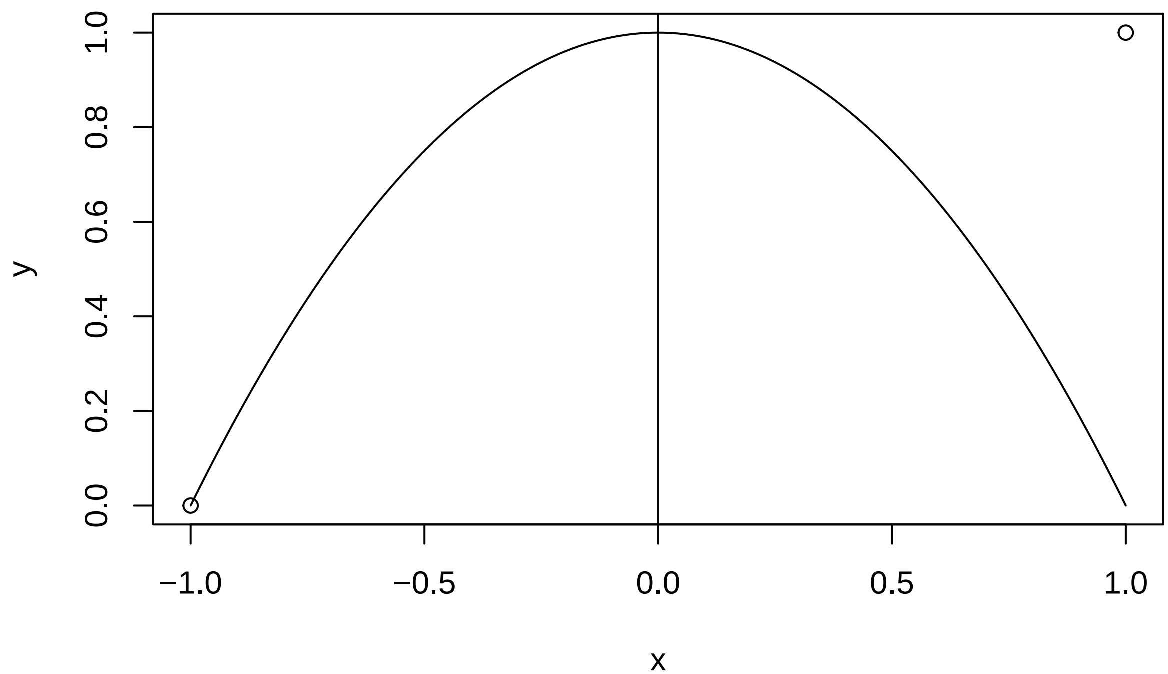
(b): $f_X(x) = \frac{3}{8}(x+1)^2$, $-1 < x < 1$; $Y = 1 - X^2$

Then for the pdf of Y, we have:

$$f_Y(y) = \begin{cases} \sum_{i=1}^{k} f_X(g_i^{-1}(y)) |\frac{d}{dy} g_i^{-1}(y)| & y \in \mathbb{Y} \\ 0 & otherwise \end{cases}$$

Let us consider the following to motivate our partitions of the sample space:

```
x <- seq(from = -1, to = 1, by = 0.01)
y <- (1 - x^2)
plot(x = c(-1, 1), y = c(0, 1), xlab = "x", ylab = "y")
lines(x, y)
abline(v = 0)
```




We see three distinct partitions to ensure monotone functions:

$A_1 = (-1, 0)$ $A_2 = \{0\}$ $A_3 = (0, 1)$

We then have their respective functions, $g_i(x)$ as follows:

$g_1 = (1 - x^2)$ $g_2 = 0$ $g_3 = (1 - x^2)$

Then, with note from the results of the following theorem (2.1.8):

$$f_Y(y) = \begin{cases} \sum_{i=1}^{k} f_X(g_i^{-1}(y))|\frac{d}{dy}g_i^{-1}(y)| & y \in \mathbb{Y} \\ 0 & otherwise \end{cases}$$

We have, for $0 < y < 1$,

$g_1(x) = g_3(x) = 1 - x^2 \to g^{-1}(y) = -(1 - y)^{1/2}$

$\therefore |\frac{d}{dy}g_1^{-1}(y)| = \frac{1}{2(1-y)^{1/2}} = |\frac{d}{dy}g_3^{-1}(y)|$

Note however that we are dealing with two distinct functions, one positive and the other negative:

(1):

$$f_X(g_1^{-1}(y))|\frac{d}{dy}g_1^{-1}(y)| = \frac{3}{8}(1 - (1 - y)^{1/2})^2(\frac{1}{2(1 - y)^{1/2}})$$

(2):

$$f_X(g_3^{-1}(y))|\frac{d}{dy}g_3^{-1}(y)| = \frac{3}{8}(1 + (1 - y)^{1/2})^2(\frac{1}{2(1 - y)^{1/2}})$$

Such that we combine (1) and (2) together to get, for $0 < y < 1$:

$$f_Y(y) = \frac{3}{8}(1-(1-y)^{1/2})^2(\frac{1}{2(1-y)^{1/2}}) + \frac{3}{8}(1+(1-y)^{1/2})^2(\frac{1}{2(1-y)^{1/2}}) = \frac{3}{8}\frac{1}{2}(1-y)^{-1/2}[(1-(1-y)^{1/2})^2+(1+(1-y)^{1/2})^2]$$

Notice the second term of the expansion between the two values will cancel each other out, leaving us (after much algebra and simplification):

$$f_Y(y) = \begin{cases} f_Y(y) = \frac{3}{8}(1-y)^{-1/2} + \frac{3}{8}(1-y)^{1/2} & 0 < y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(c):

$$f_X(x) = \frac{3}{8}(x+1)^2$$

, $-1 < x < 1$; $Y = 1 - X^2$ if $X \leq 0$ and $Y = 1 - X$ if $X > 0$

Similar to part (b), we see three distinct partitions to ensure monotone functions:

$A_1 = (-1, 0)$ $A_2 = \{0\}$ $A_3 = (0, 1)$

We then have their respective functions, $g_i(x)$ as follows:

$g_1 = (1 - x^2)$ $g_2 = 0$ $g_3 = (1 - x^2)$

Thus, with note of the relevant theorem:

$$f_Y(y) = \begin{cases} \sum_{i=1}^{k} f_X(g_i^{-1}(y))|\frac{d}{dy}g_i^{-1}(y)| & y \in \mathbb{Y} \\ 0 & \mathit{otherwise} \end{cases}$$

Such that for $0 < y < 1$:

$g_1(x) = 1 - x^2 \rightarrow g^{-1}(y) = (1-y)^{1/2}$

$\therefore |\frac{d}{dy}g_1^{-1}(y)| = \frac{1}{2(1-y)^{1/2}}$

$g_3(x) = 1 - x \rightarrow g^{-1}(y) = 1 - y$

$\therefore |\frac{d}{dy}g_i^{-1}(y)| = |-1 = 1$

There are two relevant summations:

(1):

$$\frac{3}{8}(1-(1-y)^{1/2})^2\frac{1}{2}(1-y)^{-1/2} = \frac{3}{16}(1-(1-y)^{1/2})^2(1-y)^{-1/2}$$

(2):

$$\frac{3}{8}((1-y)+1)^2 = \frac{3}{8}(2-Y)^2$$

Taken together, we have the sum of (1) and (2), written:

$$f_Y(y) = \begin{cases} f_Y(y) = \frac{3}{16}(1-(1-y)^{1/2})^2(1-y)^{-1/2} + \frac{3}{8}(2-y)^2 & 0 < y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

## 6. 2.9:

Q: If the random variable X has pdf:

$$f(x) = \begin{cases} \frac{x-1}{2} & 1 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

find a monotone function $u(x)$ such that the random variable $Y = u(X)$ has a Uniform(0,1) distribution.

A:

We may take advantage of Thm 2.1.10, and let the random variable Y be defined as $Y = u(X) = F_x(x)$

Taking advantage of the fact that $u(x) = F_x(x) \to F_x(X)$ ~Uniform $(0, 1)$

That is to say define the random variable Y as the cdf of the random variable X.

$F_x(x) = P(X \leq x) = \int\limits_{-\infty}^{x} f(t)dt = \int\limits_{-\infty}^{x} \frac{t-1}{2} = \int\limits_{-\infty}^{1} \frac{t-1}{2} + \int\limits_{1}^{x} \frac{t-1}{2} = 0 + \int\limits_{1}^{x} \frac{t-1}{2}$

$$F_x(x) = \int\limits_{1}^{x} \frac{t-1}{2} = \frac{(t-1)^2}{4}\Big|_1^x = \frac{(x-1)^2}{4} - 0 = \frac{(x-1)^2}{4}$$

Such that we may define the monotone function $u(x)$ by:

$$u(x) = \begin{cases} 0 & x \leq 1 \\ \frac{(x-1)^2}{4} & 1 < x < 3 \\ 1 & x \geq 3 \end{cases}$$

## 7. 2.22 (a, b):

Q: Let X have the pdf:

$$f(x) = \frac{4}{\beta^3\sqrt{\pi}}x^2e^{\frac{-x^2}{\beta^2}}$$

, $0 < x < \infty$, $\beta > 0$

(a): Verify that $f(x)$ is a pdf.

(b): Find $E(X)$

A:

(a):

There are two conditions to verify that $f(x)$ is a pdf, the first is: (1): $f(x) \geq 0$, $\forall x$. This one is apparent under the conditions $0 < x < \infty$, $\beta > 0$. We must then establish condition (2):

(2): $\int_\Omega f(x)dx = 1$, or, the sum of the pdf over the sample space is 1 (note: this is for the continuous case, which we have).

We thus have:

$$\int_\Omega f(x)dx = \int_0^\infty \frac{4}{\beta^3\sqrt{\pi}}x^2e^{\frac{-x^2}{\beta^2}}dx$$

Set $\frac{y}{\sqrt{2}} = \frac{x}{\beta} \to dx = \frac{\beta}{\sqrt{2}}dy$

And $x^2 = \frac{\beta^2y^2}{2}$

Such that we may write:

$$\int_\Omega f(x)dx = \int_0^\infty \frac{4}{\beta^3\sqrt{\pi}}\frac{\beta^2y^2}{2}e^{\frac{-y^2}{2}}\frac{\beta}{\sqrt{2}}dy = \int_0^\infty \frac{2}{\sqrt{2\pi}}y^2e^{-y^2/2}dy$$

We may then make use of our assumption/hint, namely:

$$1 = \int_\infty^\infty \frac{1}{\sqrt{2\pi}}x^2e^{\frac{-x^2}{2}}dx = \frac{2}{\sqrt{2\pi}}\int_0^\infty x^2e^{\frac{-x^2}{2}}dx \to \sqrt{2\pi}/2 = \int_0^\infty x^2e^{\frac{-x^2}{2}}dx$$

Incorporating this into the above relation on $f(x)$ gives us (taking out the constant term from the integral):

$$\int_\Omega f(x)dx = \frac{2}{\sqrt{2\pi}}\int_0^\infty y^2e^{-y^2/2}dy = \frac{2}{\sqrt{2\pi}}\frac{\sqrt{2\pi}}{2} = 1$$

We have shown then that conditions (1) and (2) hold, and as such, $f(x)$ is a pdf!

(b):

Q: Find $\mathbb{E}(X)$

Note: For the random variable X given from the prior $f(x)$, we have $\mathbb{E}(X) = \int_\Omega xf(x)dx$

We may calculate this as follows:

$$\mathbb{E}(X) = \int_0^\infty x \frac{4}{\beta^3\sqrt{\pi}} x^2 e^{\frac{-x^2}{\beta^2}} dx$$

Let us take note of Integration by parts, that is:

$$\int udv = uv - \int vdu$$

For the above relation, let

$$u = \frac{4x^2}{\beta^3\sqrt{\pi}} \rightarrow du = \frac{8x}{\beta^3\sqrt{\pi}}$$

and

$$dv = xe^{-\frac{x^2}{\beta^2}} \rightarrow v = \int_0^\infty xe^{-\frac{x^2}{\beta^2}}$$

Of interest is $uv$, which may be written:

$$uv = [\frac{4}{\beta^3\sqrt{\pi}} x^2(-\frac{\beta^2}{2} e^{-\frac{x^2}{\beta^2}})]\Big|_0^\infty$$

Note: We have a number of constants, such that the above simplifies to:

$$uv = \frac{4}{\beta^3\sqrt{\pi}}(-\frac{\beta^2}{2})[x^2 e^{-\frac{x^2}{\beta^2}}]\Big|_0^\infty$$

And we note the following:

$[x^2 e^{-\frac{x^2}{\beta^2}}]\Big|_0^\infty = 0 - 0 = 0$

Such that our term $uv$ is equal to zero, leaving us with:

$$\mathbb{E}(X) = 0 + \int_0^\infty x \frac{4}{\beta^3\sqrt{\pi}} x^2 e^{\frac{-x^2}{\beta^2}} dx = 0 + \frac{4}{\beta\sqrt{\pi}} \int_0^\infty xe^{-\frac{x^2}{\beta^2}} dx$$

$$\mathbb{E}(X) = 0 + \frac{4}{\beta\sqrt{\pi}}(-\frac{1}{2}\beta^2 e^{-\frac{x^2}{\beta^2}}\Big|_0^\infty) = \frac{4}{\beta\sqrt{\pi}}\frac{\beta^2}{2} = \frac{2\beta}{\sqrt{\pi}}$$

We conclude then:

$$\mathbb{E}(X) = \frac{2\beta}{\sqrt{\pi}}$$

## 8.

Q: Suppose that a random variable U has a Uniform(0,1) distribution

(i.e. pdf $f_U(u) = 1$ for 0 < u < 1)

(a): Suppose a random variable X has a cdf $F(x)$ which is strictly increasing and continuous on $x \in \mathbb{R}$; this implies that, for any real value of 0 < u < 1, there is an inverse $F^{-1}(u) = x \in \mathbb{R}$ so that $F(x) = F(F^{-1}(u)) = u$. Define a random variable $Y = F^{-1}(U)$ based on the random variable U. Show that X and Y have the same cdf (i.e. the same distributions).

Hint: Use that, because F is strictly increasing, $P(Y \leq y) = P(F(Y) \leq F(y))$ holds for any $y \in \mathbb{R}$, i.e., Y can be less than or equal to y if and only if F(Y) is less than or equal to $F(y)$. Noe that $F(y) \in (0,1)$ for any real y.

(b): If there is a computer program (i.e. random number generator) that produces numbers uniformly distributed between zero and one (i.e., according to the pdf $F_U(u)$), explain how these numbers could be used to generate values distributed according to the pdf $f_Z(z) = \frac{e^{-|z|}}{2}$, $-\infty < z < \infty$.

Hint: Use (a) where F now becomes the cdf of Z; you need to find $F^{-1}(u)$ for a given 0 < u < 1 by solving the expression $F(z) = u$ for $z \in \mathbb{R}$

A:

(a):

Let U and X be random variables.

Define the following relations to hold:

For any real value of 0 < u < 1, there is an inverse $F^{-1}(u) = x \in \mathbb{R}$ so that $F(x) = F(F^{-1}(u)) = u$.

Let us then define a random variable Y as follows: $Y = F^{-1}(U)$

Note: F is strictly increasing, and $F^{-1}$ is also strictly increasing.

Thus if we define $Y \leq y \rightarrow F(Y) \leq F(y)$. Similarly, if we define $F(Y) \leq F(y) \rightarrow F^{-1}(F(Y)) \leq F^{-1}(F(y)) \rightarrow Y \leq y$

Such that we have shown:

$$Y \leq y \iff F(Y) \leq F(y)$$

for a strictly increasing function $F$.

Then consider the cdf of the random variable X, and the following consequence of F being strictly increasing:

$$F(x) = F(F^{-1}(u)) = u \rightarrow F^{-1}(u) = x$$

Given Our relations of the random variables Y and U, namely that F is strictly increasing, then the values the random variables take, y and u respectively may be written:

$$Y = F^{-1}(U) \rightarrow y = F^{-1}(u) \rightarrow F(y) = u$$

such that the above relations give us:

(1):

$$F(Y) = P(Y \leq y) = P(F^{-1}(U) \leq F^{-1}(u)) = P(F(F^{-1}(U)) \leq F(F^{-1}(u))) = P(U \leq u)$$

(2):

$$F(X) = P(F(X) \leq u) = P(U \leq u)$$

And taking (1) and (2) together, we may conclude that the random variable X and Y have the same cdfs.

(b):

We are given the pdf of Z, so we derive its cdf as follows:

$$F_Z(z) = \begin{cases} \int\limits_{-\infty}^{z} \frac{e^t}{2} dt & z < 0 \\ \int\limits_{-\infty}^{0} \frac{e^t}{2} dt + \int\limits_{0}^{z} \frac{e^-t}{2} dt & z > 0 \end{cases}$$

We then evaluate the following integrals such that we have:

$$\int\limits_{-\infty}^{z} \frac{e^t}{2} dt = \frac{e^z}{2}$$

$$\int\limits_{-\infty}^{0} \frac{e^t}{2} dt = \frac{1}{2}$$

$$\int\limits_{0}^{z} \frac{e^{-t}}{2} = \frac{1}{2} - \frac{1}{2} e^{-z}$$

Such that we have, for $z > 0$:

$F_Z(z) = \frac{1}{2} + \frac{1}{2} - \frac{1}{2}e^{-z} = 1 - \frac{1}{2}e^{-z}$

Thus we have the cdf of Z as:

$$F_Z(z) = \begin{cases} \frac{e^z}{2} & z < 0 \\ 1 - \frac{1}{2}e^{-z} & z > 0 \end{cases}$$

We then take the inverse $F^{-1}$ to transform this from the random variable Z to the random variable U:

$F^{-1}(\frac{e^z}{2}) = ln(2u)$

$F^{-1}(1 - \frac{1}{2}e^{-z}) = \frac{1}{ln(2-2u)}$

We may then write $F^{-1}(u)$

As $F^{-1}(1 - \frac{1}{2}e^{-z} = -ln(-2(u-1)) = -ln(2-2u)$

$$F_Z^{-1}(u) = \begin{cases} ln(2u) & 0 < u \leq 1/2 \\ -ln(2-2u) & 1/2 < u < 1 \end{cases}$$

Misc:

In terms of directly answering, *"If there is a computer program (i.e. random number generator) that produces numbers uniformly distributed between zero and one (i.e., according to the pdf $F_U(u)$), explain how these numbers could be used to generate values distributed according to the pdf $f_Z(z)$."*

I will wave my hands energetically and point at the above derivation, saying something like, "Well, there you go. Try that out."

But realistically, the computer program defined above for the Uniform(0,1) distribution can be transformed using a particular function to convert the numbers to a new scale $(-\infty, \infty)$, where, for example $u = \frac{1}{2} \rightarrow z = 0$. This process can go both ways.

I am hesitant to say the above is a method for doing so, but honestly I just followed the Hint to what I believed to be its natural conclusion.