Homework 3 – STAT 5430

Due Monday, Feb 17 by midnight in gradescope;

1. Suppose $X_1, \ldots, X_n$ are iid Bernoulli(p), $0 < p < 1$.
   (a) Find the information number $I_n(p)$ and make a rough sketch of $I_n(p)$ as a function of $p \in (0, 1)$.
   (b) Find the value of $p \in (0, 1)$ for which $I_n(p)$ is minimal. (This value of p corresponds to the "hardest" case for estimating p. That is, when data are generated under this value of p from the model, the variance of an UE of p is potentially largest.)
   (c) Show that $\bar{X}_n = \sum_{i=1}^n X_i/n$ is the UMVUE of p.

2. Suppose that the random variables $Y_1, \ldots, Y_n$ satisfy

$$Y_i = \beta x_i + \varepsilon_i, \quad i = 1, \ldots, n$$

   where $x_1, \ldots, x_n$ are fixed constants and $\varepsilon_1, \ldots, \varepsilon_n$ are iid $N(0, \sigma^2)$; here we assume $\sigma^2 > 0$ is known.
   (a) Find the MLE of β.
   (b) Find the distribution of the MLE.
   (c) Find the CRLB for estimating β. (Hint: you'll have to work with the joint distribution $f(y_1, \ldots, y_n|\beta)$ directly, since $Y_1, \ldots, Y_n$ are not iid.)
   (d) Show the MLE is the UMVUE of β.

3. Suppose $X_1, \ldots, X_n$ are iid normal $N(\theta, 1)$, where $\theta \in \mathbb{R}$. It turns out that $T = (\bar{X}_n)^2 - n^{-1}$ is the UMVUE of $\gamma(\theta) = \theta^2$. (We can show this later in the course; our goal here is to show that the UMVUE can exist without obtaining the CRLB.)
   (a) Show T is an UE of $\gamma(\theta) = \theta^2$ and find the variance $\mathrm{Var}_\theta(T)$ of T. (Note $Z \equiv \sqrt{n}(\bar{X}_n - \theta) \sim N(0, 1)$ and one can write $T = (Z^2/n) + (2\theta Z/\sqrt{n}) + \theta^2 - n^{-1}$, where $Z^2 \sim \chi_1^2$, $E_\theta Z^2 = 1$, $\mathrm{Var}_\theta(Z^2) = 2$.)
   (b) Find the CRLB for an UE of $\gamma(\theta) = \theta^2$.
   (c) Show that $\mathrm{Var}_\theta(T) >$CRLB for all values of $\theta \in \mathbb{R}$.

4. Problem 7.58, Casella & Berger. ("better" here refers to MSE as a criterion)

5. Let $X_1, \ldots, X_n$ be iid Bernoulli(θ), $\theta \in (0, 1)$. Find the Bayes estimator of θ with respect to the uniform(0,1) prior under the loss function

$$L(t, \theta) = \frac{(t - \theta)^2}{\theta(1 - \theta)}$$