Statistics 520 - Assignment 3

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Assignment 3

1. **(10 pt.)** Suppose that a random variable Y has a beta distribution with parameters α and β. A standard form for the probability density function of Y is, for $\alpha > 0$ and $\beta > 0$,

$$f(y \mid \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1}, \quad 0 < y < 1.$$

Put this density in canonical exponential family form.

Using properties of exponential families, find $E\{\log(Y)\}$ and $E\{\log(1-Y)\}$ expressed in terms of the original α and β parameters.

Note: Use $\Gamma'(x)$ to denote the derivative of the gamma function,

$$\frac{d}{dx}\Gamma(x).$$

Answer

The canonical exponential family form of the density is:

$$f(y \mid \alpha, \beta) = \exp\{(\alpha-1)\log(y) + (\beta-1)\log(1-y) + \log\left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right)\} \ \mathbb{I}[y \in (0,1)]$$

Where:

$\theta_1 = \alpha - 1, \quad \theta_2 = \beta - 1, \quad \theta_1 > -1, \quad \theta_2 > -1.$

$T = (T_1, T_2)$ for $T_1(y) = \log(y), \ T_2(y) = \log(1-y)$.

$B(\theta) = -\log\left(\frac{\Gamma(\theta_1+\theta_2+2)}{\Gamma(\theta_1+1)\Gamma(\theta_2+1)}\right) = \log\Gamma(\theta_1+1) + \log\Gamma(\theta_2+1) - \log\Gamma(\theta_1+\theta_2+2)$.

$c(y) = \mathbb{I}[y \in (0,1)]$, where $\mathbb{I}$ denotes the indicator function

Note: Though $B(\theta)$ is as given above, a simplified version which makes taking partial derivatives easier is the equivalent form:

Using properties of exponential families, and noting that the natural parameters (θ_1, θ_2) are linearly related to the parameters (α, β):

$$\begin{aligned} E\{\log(Y)\} = E\{T_1(Y)\} = \frac{\partial}{\partial\theta_1}B(\theta) = \frac{\partial}{\partial\alpha}\left(\log\Gamma(\alpha) + \log\Gamma(\beta) - \log\Gamma(\alpha+\beta)\right) \\ = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} - \frac{\Gamma'(\alpha+\beta)}{\Gamma(\alpha+\beta)} \end{aligned}$$

And

$$
\begin{aligned}
E\{\log(1-Y)\} = E\{T_2(Y)\} = \frac{\partial}{\partial \theta_2} B(\theta) = \frac{\partial}{\partial \beta} \left(\log \Gamma(\alpha) + \log \Gamma(\beta) - \log \Gamma(\alpha+\beta)\right) \\
= \frac{\Gamma'(\beta)}{\Gamma(\beta)} - \frac{\Gamma'(\alpha+\beta)}{\Gamma(\alpha+\beta)}
\end{aligned}
$$

2. **(5 pt.)** Suppose that a random variable Y has a Poisson distribution with parameter λ. A standard form for the probability mass function of Y is, for $\lambda > 0$,

$$f(y \mid \lambda) = \frac{1}{y!}\lambda^y \exp(-\lambda), \quad y = 0, 1, 2, \ldots$$

Put this probability mass function in canonical exponential family form.

Using properties of exponential families, verify that $E(Y) = \lambda$.

Answer

The canonical exponential family form of the density is:

$$f(y \mid \lambda) = \exp\{y \log(\lambda) - \lambda - \log(y!)\}$$

$\theta_1 = \log(\lambda), \quad \theta_1 \in \mathbb{R}.$

$T_1(y) = y.$

$B(\theta) = \exp(\theta_1).$

$c(y) = -\log(y!)$

Using properties of exponential families, and noting the natural parameter θ_1 is non-linearly related to the parameter λ:

$$E\{T_1(Y)\} = \frac{\partial}{\partial \theta_1} B(\theta) = \frac{\partial}{\partial \theta_1} e^{\theta_1} = e^{\log(\lambda)} = \lambda$$

3. Suppose that a random variable Y has a gamma distribution with parameters α and β. A standard form for the probability density function of Y is, for $\alpha > 0$ and $\beta > 0$,

$$f(y \mid \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} \exp(-\beta y), \quad y > 0.$$

Note: You may have seen a gamma density written with a parameter that is equal to $1/\beta$ in the above expression. Use the parameterization given above to answer this question (I think it will be easier).

(a) (5 pts.) Write the gamma density in the form of a two-parameter exponential family. Using properties of exponential families, derive the expected values of Y and $\log(Y)$.

(b) (5 pts.) Write the gamma density in the form of an exponential dispersion family with parameters θ and ϕ. Derive the expected value of Y.

Answer

(a)

The gamma density in the form of a (canonical) two-parameter exponential family is of the form:

$$f(y \mid \alpha, \beta) = \exp\{(\alpha - 1)\log(y) - \beta y + \alpha \log(\beta) - \log\Gamma(\alpha)\}\mathbb{I}[y > 0]$$

Where:

$\theta_1 = \alpha - 1 \quad \theta_2 = -\beta, \quad \theta_1 > -1, \quad \theta_2 < 0$

$T = (T_1(y), T_2(y)), \quad T_1(y) = \log(y), \quad T_2(y) = y,$

$B(\theta) = \log\Gamma(\theta_1 + 1) - (\theta_1 + 1)\log(-\theta_2)$

And

$c(y) = \mathbb{I}[y > 0]$, where $\mathbb{I}$ denotes the indicator function

Using properties of exponential families, and noting the natural parameters (θ_1, θ_2) are linear functions of the parameters (α, β), allowing substitution during evaluation (instead of at the end), then:

$$E(\log(Y)) = E\{T_1(Y)\} = \frac{\partial}{\partial\theta_1}B(\theta) = \frac{\partial}{\partial\alpha}(\log\Gamma(\alpha) - \alpha\log(\beta)) = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} - \log(\beta)$$

(Another Digamma function, in the flesh!)

Also:

$$E(Y) = E\{T_2(Y)\} = \frac{\partial}{\partial\theta_2}B(\theta) = \frac{\partial}{\partial(-\beta)}(\log(\Gamma(\alpha) - \alpha\log(\beta)) = \frac{\alpha}{\beta}$$

(b)

Now, taking the canonical form, we may then write the exponential dispersion family form as:

$$\begin{aligned}
f(y \mid \alpha, \beta) &= \exp\Big((\alpha - 1)\log y - \beta y + \alpha\log\beta - \log\Gamma(\alpha)\Big) \\
&= \exp\Big\{\alpha\Big(\log(\frac{\beta}{\alpha}) - y\frac{\beta}{\alpha}\Big) + \big((\alpha - 1)\log y + \alpha\log\alpha - \log\Gamma(\alpha)\big)\Big\} \\
&= \exp\{\,\phi\,(y\theta - b(\theta)) + c(y, \phi)\}\,, \qquad y > 0,
\end{aligned}$$

where

$$\phi = \alpha, \quad \theta = -\frac{\beta}{\alpha} \quad \phi > 0, \quad \theta < 0$$

And

$$b(\theta) = -\log(-\theta), \quad \text{and } c(y, \phi) = (\phi - 1)\log y + \phi \log \phi - \log \Gamma(\phi)$$

Using the properties of an exponential dispersion family, we may calculate expectation via:

$$E(Y) = \frac{d}{d\theta} b(\theta) = \frac{d}{d\theta}\left(-\log(-\theta)\right) = -\frac{1}{\theta} = \frac{\alpha}{\beta}$$