HW7

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Q1

Problem 8.6 a) - b), Casella and Berger (2nd Edition)

Suppose that we have two independent random samples: $X_1, \ldots, X_n$ are exponential(θ), and $Y_1, \ldots, Y_m$ are exponential(μ).

a)

Find the LRT of

$$H_0 : \theta = \mu \quad \text{versus} \quad H_1 : \theta \neq \mu.$$

Answer

The LRT statistic is of the form:

$$\lambda(x, y) = \frac{\max_\theta L(\theta \mid \underset{\sim}{x}, \underset{\sim}{y})}{\max_{\theta,\mu} L(\theta, \mu \mid \underset{\sim}{x}, \underset{\sim}{y})}$$

Where, under H_0 ($\theta = \mu$).

Generally, we know that, the MLE will be some weighted average of the observations, taking advantage of the one parameter exponential families known to be complete and their MLEs of a general form.

Under H_0 (to get the numerator of the LRT) the MLE for θ is of the form:

$$\hat{\theta}_{H_0} = \frac{\sum_{i=1}^n X_i + \sum_{j=1}^m Y_j}{n+m}$$

And, under the full model (the denominator of the LRT), the MLEs are the individual sample means, i.e.:

$$\hat{\theta}_{MLE} = \bar{X} = \frac{\sum X_i}{n}, \quad \hat{\mu}_{MLE} = \bar{Y} = \frac{\sum Y_j}{m}$$

Returning to the original expression, we then have:

$$\lambda(x, y) = \frac{(\hat{\theta}_0)^{-(n+m)} e^{-(n+m)}}{(\hat{\theta}_{MLE})^{-n} e^{-n} (\hat{\mu}_{MLE})^{-m} e^{-m}} = \frac{(\bar{X})^n (\bar{Y})^m}{\left(\frac{\sum X_i + \sum Y_j}{n+m}\right)^{n+m}} = \frac{(n+m)^{n+m} (\sum X_i)^n (\sum Y_j)^m}{n^n m^m (\sum X_i + \sum Y_j)^{n+m}}$$

We may then construct our test function, where our rejection rule is to "Reject H_0 if $\lambda(x,y) \leq c$", where c is calibrated based on the significance level α, i.e. our test function is of the form:

$$\varphi(x,y) = \begin{cases} 1 & \text{if } \lambda(x,y) \leq c, \\ 0 & \text{otherwise} \end{cases}$$

Where (to save space above):

$$\lambda(x,y) = \frac{(n+m)^{n+m} (\sum X_i)^n (\sum Y_j)^m}{n^n m^m (\sum X_i + \sum Y_j)^{n+m}}$$

And c is chosen such that $P(\varphi(X,Y) = 1 \mid H_0) = \alpha$.

b)

Show that the test in part a) can be based on the statistic

$$T = \frac{\sum X_i}{\sum X_i + \sum Y_i}$$

Answer

Let $T = \frac{\sum X_i}{\sum X_i + \sum Y_j}$.

Rewriting the LRT from part a) in terms of T:

$$\lambda(x,y) = \frac{(n+m)^{n+m}}{n^n m^m} \left(\frac{\sum X_i}{\sum X_i + \sum Y_j} \right)^n \left(\frac{\sum Y_j}{\sum X_i + \sum Y_j} \right)^m = \frac{(n+m)^{n+m}}{n^n m^m} T^n (1-T)^m$$

Since $\lambda(x,y)$ depends on the data only through T, the LRT can be based entirely on T.

Using the above, we may define the rejection region where the test rejects H_0 when T is "too small" or "too large" with constants a and b, where:

$$T \leq a \quad \text{or} \quad T \geq b$$

And where a and b are values satisfying:

$$P(T \leq a \mid H_0) + P(T \geq b \mid H_0) = \alpha$$

Under H_0 $(\theta = \mu)$, $\sum X_i \sim \text{Gamma}(n, \theta)$, $\sum Y_j \sim \text{Gamma}(m, \theta)$.

The above is taken as known because that the sum of iid Exponentials is Gamma, and a linear combination, specifically a ratio, of Gamma distributions with common rate parameter θ is a Beta.

Also, since both X and Y are independent of one another, their sums are also independent, and determining the parameters of the T Beta distribution becomes a matter of algebra (and the distribution of T does not involve θ in its parameters).

Specifically, we know:

$$T = \frac{\sum X_i}{\sum X_i + \sum Y_j} \sim \text{Beta}(n, m)$$

So the critical values being referenced above may be found via taking critical regions of the Beta distribution when n and m are known values (numbers of observations of X and Y respectively).

Q2

Problem 8.28, Casella and Berger (2nd Edition)

Let $f(x|\theta)$ be the logistic location probability density function:

$$f(x|\theta) = \frac{e^{(x-\theta)}}{(1+e^{(x-\theta)})^2}, \quad -\infty < x < \infty, \quad -\infty < \theta < \infty.$$

a)

Show that this family has an MLR.

Answer

Let $\theta_2 > \theta_1$.

We know the likelihood ratio statistic is given by:

$$\Lambda = \frac{f(x|\theta_2)}{f(x|\theta_1)} = e^{\theta_1-\theta_2}\left[\frac{1+e^{x-\theta_1}}{1+e^{x-\theta_2}}\right]^2$$

The derivative wrt X is of the form:

$$\Lambda' = \frac{e^{x-\theta_1}(1+e^{x-\theta_2}) - e^{x-\theta_2}(1+e^{x-\theta_1})}{(1+e^{x-\theta_2})^2} = \frac{e^{x-\theta_1} - e^{x-\theta_2}}{(1+e^{x-\theta_2})^2} > 0$$

And the inequality holds because of the assumption $\theta_2 > \theta_1$, which is allowed in the full parameter space.

Thus, our likelihood ratio is strictly increasing in x, meaning it is monotonic, i.e. that the family $f(x|\theta)$ from the logistic location probability density function has MLR in x.

b)

Based on one observation X, find the most powerful size α test of

$$H_0 : \theta = 0 \quad \text{versus} \quad H_1 : \theta = 1.$$

For $\alpha = 0.2$, find the size of the Type II error.

Answer

By the Neyman-Pearson Lemma, the MP test rejects H_0 when:

$$\Lambda = \frac{f(x|1)}{f(x|0)} = e^{-1}\left(\frac{1+e^x}{1+e^{x-1}}\right)^2 > k$$

From from part a), since the likelihood ratio is increasing in x, the MP test rejects if $X > k_1$, where k_1 is determined by the size α.

As we know the underlying distributions, let us consider the CDF of the logistic distribution:

$$F(x|\theta) = \frac{e^{x-\theta}}{1+e^{x-\theta}}$$

Under H_0, the size is given by the expression:

$$P(X > k_1 \mid \theta = 0) = 1 - F(k_1|0) = \frac{1}{1+e^{k_1}} = \alpha$$

Solving for k_1:

$$k_1 = \log\left(\frac{1-\alpha}{\alpha}\right) = \log(\alpha^{-1} - 1)$$

For $\alpha = 0.2$:

$$k_1 = \log(0.2^{-1} - 1) = \log(4) \approx 1.386$$

Under H_1, to calculate the Type II Error Rate:

$$\beta = P(X \leq k_1 \mid \theta = 1) = F(k_1|1) = \frac{e^{k_1-1}}{1+e^{k_1-1}} \approx \frac{e^{0.386}}{1+e^{0.386}} \approx 0.595$$

So, the MP level test of size $\alpha = 0.2$ rejects when our single observation $X > 1.386$, with a Type II error rate of 0.595.

c)

Show that the test in part b) is UMP size α for testing

$$H_0 : \theta \leq 0 \quad \text{versus} \quad H_1 : \theta > 0.$$

What can be said about UMP tests in general for the logistic location family?

Answer

Via MLR: From part a), the family has MLR in X.

Via Karlin-Rubin Thm. (Knew it would come up again!): Since the MP test for $\theta = 0$ vs $\theta = 1$ rejects for large X and does not depend on the specific parameter value, i.e., $\theta_1 = ...$ (alternative hypothesis parameter value in particular), the rejection region depends solely upon the observed value X, meaning the MP test is also the UMP test for $H_0 : \theta \leq 0$ vs $H_1 : \theta > 0$.

The above results extend to similar distributions within the the logistic location family, i.e., UMP tests for one-sided hypotheses both exist and take the form "Reject H_0 if $X > c$." I do not believe it would necessarily extend to rate parameter family of distributions however, as that tends to be a bit more complicated.

Q3

Problem 8.29 a) - b), Casella and Berger (2nd Edition)

Let X be one observation from a Cauchy(θ) distribution.

The Cauchy(θ) density is given by:

$$f(x|\theta) = \frac{1}{\pi} \cdot \frac{1}{1 + (x - \theta)^2}, \quad x \in \mathbb{R}, -\infty < \theta < \infty.$$

a)

Show that this family does not have an MLR.

Hint:

Show that the Cauchy(θ) family $\{f(x|\theta) : \theta \in \mathbb{R} = \Theta\}$, based on one observation X, does not have monotone likelihood ratio (MLR) in $t(X) = X$ or $t(X) = -X$. That is, the ratio

$$\frac{f(x|\theta_2)}{f(x|\theta_1)}$$

might not be monotone (either increasing or decreasing) in x.

Answer

Let $\theta_2 > \theta_1$ under the setup of the problem.

The likelihood ratio is of the form:

$$\Lambda = \frac{f(x|\theta_2)}{f(x|\theta_1)} = \frac{1 + (x - \theta_1)^2}{1 + (x - \theta_2)^2}$$

And it has limit(s):

$$\lim_{x \to \pm\infty} \frac{f(x|\theta_2)}{f(x|\theta_1)} = 1$$

As we seek to disprove that the ratio is not monotonic, we need only one example that displays non-monotonicity.

For example, let $\theta_1 = 0$, $\theta_2 = 1$ such that our base assumption that $\theta_2 > \theta_1$ holds.

Then:

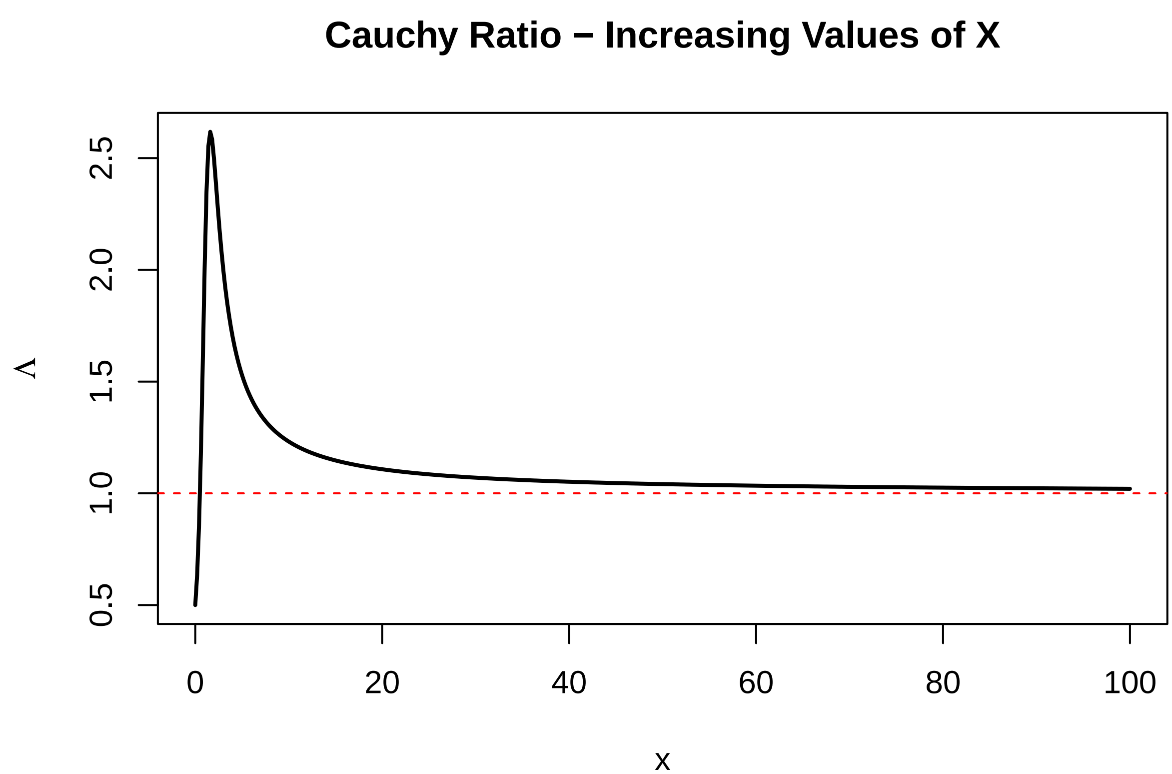
$$\Lambda = \frac{f(x|1)}{f(x|0)} = \frac{1 + x^2}{1 + (x - 1)^2}$$

```r
cauchy <- function(x) {
  numerator <- 1 + x^2
  denominator <- 1 + (x-1)^2
  numerator/denominator
}
```

```r
cauchy(0)
cauchy(1)
cauchy(2)
cauchy(3)
cauchy(1000)
```

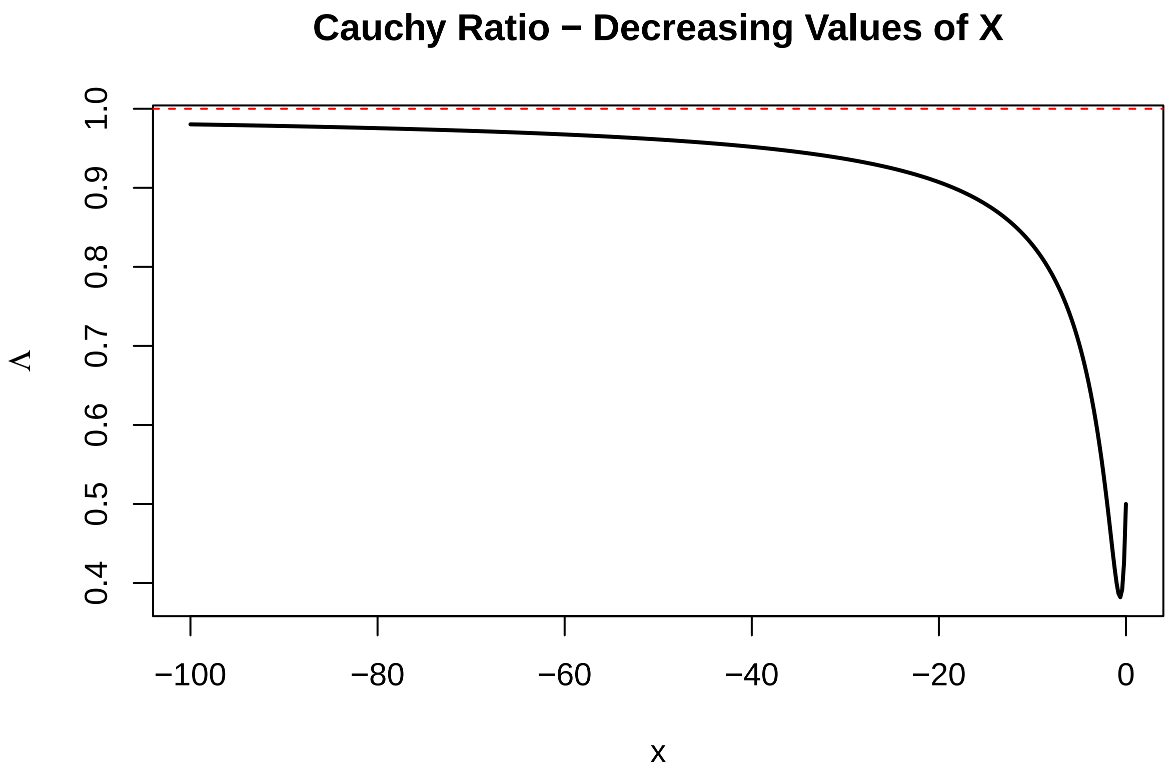
```r
x_vals <- seq(0, 100, length.out = 500)
y_vals <- cauchy(x_vals)

plot(x_vals, y_vals, type = "l", lwd = 2, col = "black",
     main = "Cauchy Ratio - Increasing Values of X",
     xlab = "x", ylab = expression(Lambda))
abline(h = 1, col = "red", lty = 2)
```




```r
x_vals <- seq(0, -100, length.out = 500)
y_vals <- cauchy(x_vals)

plot(x_vals, y_vals, type = "l", lwd = 2, col = "black",
     main = "Cauchy Ratio - Decreasing Values of X",
     xlab = "x", ylab = expression(Lambda))
abline(h = 1, col = "red", lty = 2)
```



At $x = 0$, $\Lambda = 0.5$. At $x = 1$, $\Lambda = 2$. At $x = 2$, $\Lambda = 2.5$. At $x = 1000$, $\Lambda = 1.002$ (as $x \to \infty$, $\Lambda \to 1$).

The ratio increases from 0 to around 2 and then decreases. So the ratio is not monotonic.

A similar argument can be made, and is shown above, for decreasing values of X also exhibiting non-monotonicity for this example.

Because the likelihood ratio is not monotonic, then the Cauchy(θ) family lacks MLR in X or $-X$.

b)

Show that the test

$$\phi(x) = \begin{cases} 1 & \text{if } 1 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

is most powerful of its size for testing

$$H_0 : \theta = 0 \quad \text{versus} \quad H_1 : \theta = 1.$$

Calculate the Type I and Type II error probabilities.

Hint:

Show that the test given is equivalent to rejecting H_0 if

$$f(x|\theta = 1) > 2f(x|\theta = 0)$$

and not rejecting otherwise. Conclude that this must be the most powerful (MP) test for its size. Justify why.

Answer

Consider the test provided in the hint:

$$\varphi(x) = \begin{cases} 1 & \text{if } 1 < x < 3, \\ 0 & \text{otherwise} \end{cases}$$

By the Neyman-Pearson Lemma, the MP test rejects H_0 when:

$$\frac{f(x|1)}{f(x|0)} = \frac{1 + x^2}{1 + (x-1)^2} > k$$

We know that the ratio $\frac{f(x|1)}{f(x|0)}$ has critical points at $x = \frac{1 \pm \sqrt{5}}{2}$, because:

$$\Lambda' = \frac{d\Lambda}{dx} = \frac{(2x)(x^2 - 2x + 2) - (1 + x^2)(2x - 2)}{(x^2 - 2x + 2)^2}$$

$$\Lambda' = 0 \to 2x(x^2 - 2x + 2) - (1 + x^2)(2x - 2) = 0 \to x = \frac{1 \pm \sqrt{5}}{2}$$

At any rate, at $x = 1$ and $x = 3$:

$$\frac{f(1|1)}{f(1|0)} = \frac{f(3|1)}{f(3|0)} = 2$$

And the set $\{x : \frac{f(x|1)}{f(x|0)} > 2\} = (1, 3)$ exactly matches the closed form expression of our test function, $\varphi(x)$.

Since these are one and the same, then $\varphi(x)$ is the most powerful test for its size.

Let us then consider the hypotheses we're dealing with.

Under H_0, the Type I Error Rate is:

$$\alpha = P(1 < X < 3 \mid \theta = 0) = \frac{1}{\pi}\left(\tan^{-1}(3) - \tan^{-1}(1)\right) \approx 0.1476$$

Under H_1, the Type II Error Rate is:

$$\beta = 1 - P(1 < X < 3 \mid \theta = 1) = 1 - \frac{1}{\pi}\left(\tan^{-1}(2) - \tan^{-1}(0)\right) \approx 0.6476$$

So $\varphi(x)$ as defined is MP with $\alpha \approx 0.1476$ (Type I Error Rate) and $\beta \approx 0.6476$ (Type II Error Rate).

Additional Justification For Most Powerful Test

I believe the above is an appropriate solution, but for the sake of completeness I wanted to make the connection a bit more explicit to the hint provided.

To that end:

The Neyman–Pearson Lemma tells us the MP test for testing simple hypotheses H_0 vs H_1 is:

$$\varphi(x) = \begin{cases} 1 & \text{if } \Lambda > k \\ 0 & \text{otherwise} \end{cases}$$

where the likelihood ratio is given by the expression:

$$\Lambda = \frac{f(x \mid \theta = 1)}{f(x \mid \theta = 0)} = \frac{1 + x^2}{1 + (x-1)^2}$$

Given the hint, let us see where this ratio exceeds 2, i.e. when:

$$\frac{1 + x^2}{1 + (x-1)^2} > 2$$

“Solving” this inequality, i.e., finding the appropriate range of x values:

$$\frac{1 + x^2}{x^2 - 2x + 2} > 2 \quad \rightarrow \quad 1 + x^2 > 2(x^2 - 2x + 2) \quad \rightarrow \quad 1 + x^2 > 2x^2 - 4x + 4 \rightarrow 0 > x^2 - 4x + 3$$

We then have:

$$x^2 - 4x + 3 < 0 \rightarrow (x-1)(x-3) < 0 \quad \rightarrow \quad x \in (1, 3)$$

Thus, the likelihood ratio exceeds 2 exactly when $x \in (1, 3)$, matching directly with the hint provided.

Connecting this back to the test function, we then know:

$$\varphi(x) = \begin{cases} 1 & \text{if } 1 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

is equivalent to the Neyman–Pearson test with k = 2, and since the test rejects H_0 when $\Lambda > 2$, with Λ as defined previously. And we know the size of this test is fixed!

Such that we have the Type I and Type II errors derived previously, but now with a more explicit connection to the test equivalency.

Q4

Consider one observation X from the probability density function

$$f(x \mid \theta) = 1 - \theta^2 \left(x - \frac{1}{2}\right), \quad 0 \le x \le 1, \quad 0 \le \theta \le 1.$$

We wish to test:

$$H_0 : \theta = 0 \quad \text{vs.} \quad H_1 : \theta > 0$$

a)

Find the UMP test of size $\alpha = 0.05$ based on X. Carefully justify your answer.

Answer

Under H_0 ($\theta = 0$):

$$f(x \mid 0) = 1 \quad \Rightarrow \quad X \sim \text{Uniform}(0, 1).$$

And under H_1 ($\theta > 0$):

$$f(x \mid \theta) = 1 - \theta^2 \left(x - \frac{1}{2}\right).$$

$x = \frac{1}{2}$ is an "inflection point" of sorts, such that the behavior of the pdf around $\frac{1}{2}$ will provide insight.
When $x < \frac{1}{2}$, $f(x \mid \theta) > 1$. So we observe larger density near $x = 0$ under the alternative.
When $x > \frac{1}{2}$, $f(x \mid \theta) < 1$. So we observe smaller density near $x = 1$ under the alternative.
Turning then to the likelihood ratio for $H_0 : \theta = 0$ vs. $H_1 : \theta = \theta_1 > 0$:

$$\Lambda = \frac{f(x \mid \theta_1)}{f(x \mid 0)} = 1 - \theta_1^2 \left(x - \frac{1}{2}\right)$$

Λ is decreasing in x, maximized at $x = 0$ (where $\Lambda = 1 + \frac{\theta_1^2}{2} > 1$), and minimized at $x = 1$ (where $\Lambda = 1 - \frac{\theta_1^2}{2} < 1$).
Via Neyman-Pearson, the MP test rejects H_0 when $\Lambda > k$, which occurs when small x is observed.
Rejection Region: The MP test rejects for $x < c$, where c is chosen to control the size of a given α, $\alpha = 0.05$.
Under H_0, $X \sim \text{Uniform}(0, 1)$, so we can calculate the probability explicitly:

$$P_{\theta=0}(X < c) = c = 0.05 \to c = 0.05.$$

Via the above rejection region, we may then construct the test function:

$$\varphi(x) = \begin{cases} 1 & \text{if } x < 0.05, \\ 0 & \text{otherwise} \end{cases}$$

As this test function does not depend on θ_1, it is also UMP for all $\theta > 0$.

b)

Find the likelihood ratio test statistic $\lambda(X)$ based on X, expressed as a function of X.

Answer

The LRT is:

$$\lambda(X) = \frac{f(X \mid 0)}{\max_{\theta \in [0,1]} f(X \mid \theta)} = \frac{1}{\max_\theta \left[1 - \theta^2 (X - \frac{1}{2})\right]}$$

Again, our critical value is at $\frac{1}{2}$, so we consider the behavior of the LRT at the value of, less than, and greater than $x = \frac{1}{2}$.

For $X \geq \frac{1}{2}$, the maximum occurs at $\theta = 0$, i.e.,:

$$\max_\theta f(X \mid \theta) = 1$$

For $X < \frac{1}{2}$, the maximum occurs at $\theta = 1$:

$$\max_\theta f(X \mid \theta) = 1 + \left(\frac{1}{2} - X\right) = 1.5 - X$$

Incorporating the two cases together, our LRT is of the form:

$$\lambda(X) = \begin{cases} \frac{1}{1.5 - X} & \text{if } X < \frac{1}{2} \\ 1 & \text{if } X \geq \frac{1}{2} \end{cases}$$

c)

Find the likelihood ratio test (LRT) of size $\alpha = 0.05$ for the above hypotheses.

Answer

Rejection Region: From part b), $\lambda(X) = 1$ for $X \geq \frac{1}{2}$, and is increasing for $X < \frac{1}{2}$. So to make the test most powerful while maintaining the correct size, we reject for large values of X, where the "large values" are determined by the size condition, which is:

$$P_{\theta=0}(X > k) = 1 - k = 0.05 \quad \Rightarrow \quad k = 0.95$$

Taken together, we reject H_0 when $X > 0.95$. So the test of size $\alpha = 0.05$ is given by:

$$\varphi(x) = \begin{cases} 1 & \text{if } x > 0.95 \\ 0 & \text{otherwise} \end{cases}$$