HW6

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Outline

- Q5: Skeleton

Q1

An ecologist takes data
$$(x_i, Y_i), i = 1, \ldots, n,$$
where $x_i > 0$ is the size of an area and Y_i is the number of moss plants. The data are modeled assuming $x_1, \ldots, x_n$ are fixed; $Y_1, \ldots, Y_n$ are independent; and:

$$Y_i \sim \text{Poisson}(\theta x_i)$$

with parameter θx_i. Suppose that:

$$\sum_{i=1}^{n} x_i = 5$$

is known. Find an exact form of the most powerful (MP) test of size $\alpha = 9e^{-10}$ for testing:

$$H_0 : \theta = 2 \quad \text{vs} \quad H_1 : \theta = 1.$$

Answer

To start, we consider the likelihood ratio test statistic. The likelihood function under a general θ is:

$$L(\theta) = \prod_{i=1}^{n} \frac{e^{-\theta x_i}(\theta x_i)^{Y_i}}{Y_i!}$$

The likelihood ratio for testing $H_0 : \theta = 2$ vs $H_1 : \theta = 1$ is then given as a ratio of the likelihood under the alternative over the likelihood over the null:

$$\Lambda = \frac{L(\theta = 1)}{L(\theta = 2)} = \frac{\prod_{i=1}^{n} e^{-x_i} x_i^{Y_i}/Y_i!}{\prod_{i=1}^{n} e^{-2x_i}(2x_i)^{Y_i}/Y_i!} = e^{\sum_i x_i} \cdot 2^{-\sum_i Y_i} = e^5 \cdot 2^{-T}$$

where $T = \sum_{i=1}^{n} Y_i$ and substituting in other known quantities.

Then, via Neyman-Pearson, the MP test rejects H_0 when Λ is large, which corresponds to small values of T (since Λ decreases as T increases).

Thus, the rejection region is of the form:

$$R = \{T \leq c\}$$

for some critical value c.

Under $H_0 : \theta = 2$, we have:

$$T \sim \text{Poisson}(2 \cdot \sum_i x_i) = \text{Poisson}(10)$$

We need to find c such that:

$$P_{H_0}(T \leq c) \leq \alpha = 9 \cdot 10^{-10}$$

We can compute these probabilities for $T \in \mathbb{Z}_0$:

- $P(T = 0) = e^{-10} \approx 4.54 \times 10^{-5}$
- $P(T = 1) = e^{-10} \cdot 10 \approx 4.54 \times 10^{-4}$
- $P(T \leq 1) = P(T = 0) + P(T = 1) \approx 4.99 \times 10^{-4}$

Since $\alpha = 9 \times 10^{-10}$ is much smaller than $P(T \leq 1)$, we see that only $T = 0$ satisfies the size requirement in this problem, i.e.:

$$P(T \leq 0) = e^{-10} \approx 4.54 \times 10^{-5} < \alpha$$

However, $P(T \leq 0) \neq \alpha$, so we must find a suitable $\gamma \in [0, 1]$ to satisfy equality.

To that end, we would need to use a randomized test when $T = 1$, i.e. our test is of the form:

- Reject with probability 1 if $T = 0$
- Reject with probability γ if $T = 1$
- Never reject if $T \geq 2$

Where γ solves:

$$P(T = 0) + \gamma P(T = 1) = \alpha$$

$$e^{-10} + \gamma \cdot 10e^{-10} = 9e^{-10}$$

Solving for γ:

$$\gamma = \frac{9e^{-10} - e^{-10}}{10e^{-10}} = 0.8$$

So we may write the full form of the test now:

$$\phi_{H_0}(X) = \begin{cases} 1 & \text{if } T = \sum Y_i = 0 \\ \gamma = 0.8 & \text{if } T = \sum Y_i = 1 \\ 0 & \text{otherwise} \end{cases}$$

Q2

Problem 8.19:

The random variable X has pdf:

$$f(x) = e^{-x}, \quad x > 0.$$

One observation is obtained on the random variable:

$$Y = X^{\theta},$$

and a test of:

$$H_0 : \theta = 1 \quad \text{versus} \quad H_1 : \theta = 2$$

needs to be constructed.

Find the UMP level $\alpha = 0.10$ test and compute the Type II Error probability.

Hint

Show that the form of the MP test involves rejecting H_0 if:

$$e^{y-\sqrt{y}}/\sqrt{y} > k$$

for some $k > 1$.

(Skip the part involving $\alpha = 0.1$ or the Type II error part.)

Answer

Under the transformation $Y = X^{\theta}$, the inverse is $X = Y^{1/\theta}$, and:

$$\frac{dx}{dy} = \frac{1}{\theta} y^{(1/\theta)-1}$$

The above Jacobian we will need for a change of variables, specifically, using the pdf of X, we have the pdf of Y given by:

$$f_Y(y|\theta) = f_X(y^{1/\theta}) \cdot \left| \frac{dx}{dy} \right| = e^{-y^{1/\theta}} \cdot \frac{1}{\theta} y^{(1/\theta)-1}$$

Where: $y > 0$

Via Neyman-Pearson, the MP test rejects H_0 for large values of the likelihood ratio, given by:

$$\Lambda = \frac{f_Y(y|2)}{f_Y(y|1)}$$

Substituting the pdfs, and simplifying:

$$\Lambda = \frac{\frac{1}{2}y^{-1/2}e^{-y^{1/2}}}{e^{-y}} = \frac{1}{2}y^{-1/2}e^{y-\sqrt{y}}$$

The rejection region is of the form:

$$\Lambda > k \to \frac{e^{y-\sqrt{y}}}{\sqrt{y}} > 2k = k_1$$

Where $k_1 > 1$

Let $g(y) = \frac{e^{y-\sqrt{y}}}{\sqrt{y}}$. Take derivative, with the intent to show monotonicity (Spoiler: Non-monotonicity due to two distinct rejection regions) and also noting log is a monotonic transformation:

$$\frac{d}{dy}\ln g(y) = \frac{d}{dy}\left(y - \sqrt{y} - \frac{1}{2}\ln y\right) = 1 - \frac{1}{2\sqrt{y}} - \frac{1}{2y}$$

Note then:

- For $y \to 0^+$: The derivative $\approx -\frac{1}{2y} \to -\infty$ (decreasing).
- For $y \to \infty$: The derivative ≈ 1 (increasing).
- $g(y)$ is a convex function, such that setting the derivative to zero is a minimum. (Has strictly positive second derivative.)

To that end:

$$1 - \frac{1}{2\sqrt{y}} - \frac{1}{2y} = 0 \implies y = 1$$

So at $y = 1$, $g(y)$ has a minimum. Thus, $\Lambda(y) > k_1$ corresponds to:

$$Y \leq c_0 \quad \text{or} \quad Y \geq c_1$$

where $c_0 < 1 < c_1$.

The UMP level-α test rejects H_0 if:

$$Y \leq c_0 \quad \text{or} \quad Y \geq c_1$$

where c_0, c_1 are chosen such that:

$$P_{H_0}(Y \leq c_0) + P_{H_0}(Y \geq c_1) = \alpha$$

Under H_0 ($\theta = 1$), $Y = X \sim \text{Exp}(1)$, so, these are probabilities we can explicitly calculate::

$$P_{H_0}(Y \leq c_0) = 1 - e^{-c_0}$$

And

$$P_{H_0}(Y \geq c_1) = e^{-c_1}$$

Taken together then, the UMP test for $H_0 : \theta = 1$ vs $H_1 : \theta = 2$ rejects H_0 if:

$$Y \leq c_0 \quad \text{or} \quad Y \geq c_1$$

Written:

$$\phi_{H_0}(Y) = \begin{cases} 1 & Y \leq c_0 \quad \text{or} \quad Y \geq c_1 \\ 0 & \text{otherwise} \end{cases}$$

Noting that $\gamma = 0$ in writing the above test function due to Y being a continuous random variable. Where c_0, c_1 satisfy:

$$(1 - e^{-c_0}) + e^{-c_1} = \alpha$$

And noting to *"Skip the part involving* $\alpha = 0.1$ *or the Type II error part."*

Q3

Problem 8.20, Casella and Berger (2nd Edition).

Let X be a random variable whose pmf under H_0 and H_1 is given by:

x	1	2	3	4	5	6	7
$f(x\|H_0)$	0.01	0.01	0.01	0.01	0.01	0.01	0.94
$f(x\|H_1)$	0.06	0.05	0.04	0.03	0.02	0.01	0.79

Use the Neyman–Pearson Lemma to find the most powerful test for H_0 versus H_1 with size:

$$\alpha = 0.04.$$

Compute the probability of Type II Error for this test.

Hint:

It holds that:

$$\frac{f(x|H_1)}{f(x|H_0)} = 7 - x + \frac{79}{94} I(x = 7)$$

over the support $x = 1, 2, \ldots, 7$, where $I(\cdot)$ denotes the indicator function.

Answer

The likelihood ratio is given by the Hint:

$$\Lambda = \frac{f(x|H_1)}{f(x|H_0)} = 7 - x + \frac{79}{94} I(x = 7),$$

where $I(\cdot)$ is the indicator function.

Notably:

- For $x = 1, \ldots, 6$, the LR simplifies to $\Lambda(x) = 7 - x$.
- For $x = 7$, $\Lambda(7) = \frac{79}{94} \approx 0.84$.
- The likelihood ratio is decreasing in x, so the MP test rejects H_0 for the smallest values of x.
- The smaller the x, the larger the likelihood ratio.

Using the above information, we can directly calculate the following:

x	LR $\Lambda(x)$	$f(x\|H_0)$	Cumulative P_{H_0}
1	6.00	0.01	0.01
2	5.00	0.01	0.02
3	4.00	0.01	0.03
4	3.00	0.01	0.04
5	2.00	0.01	0.05

x	LR $\Lambda(x)$	$f(x\|H_0)$	Cumulative P_{H_0}
6	1.00	0.01	0.06
7	0.84	0.94	1.00

To achieve the desired size, $\alpha = 0.04$, we consider where the cumulative probability, P_{H_0}, achieves α, which is at 4. As this is cumulative then, we have the rejection region given by:

$$R = \{1, 2, 3, 4\}$$

The Type II error probability β is the probability of not rejecting H_0 when H_1 is true:

$$\beta = P_{H_1}(X \notin R) = P_{H_1}(X = 5, 6, 7) = f(5|H_1) + f(6|H_1) + f(7|H_1) = 0.02 + 0.01 + 0.79 = 0.82$$

Giving us a Type II Error Probability of $\beta = 0.82$.

Q4

Recall Method I for finding Uniformly Most Powerful (UMP) tests:

To find a UMP size α test for $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \notin \Theta_0$, suppose we can fix $\theta_0 \in \Theta_0$ suitably and then use the Neyman–Pearson lemma to find an MP size α test $\varphi(\tilde{X})$ for:

$$H_0 : \theta = \theta_0 \quad \text{vs} \quad H_1 : \theta = \theta_1,$$

where:

a)

$\varphi(\tilde{X})$ does not depend on $\theta_1 \notin \Theta_0$, and

b)

$\max_{\theta \in \Theta_0} E_\theta \varphi(\tilde{X}) = \alpha$.

Proof

Show that if a) and b) both hold, then $\varphi(\tilde{X})$ must be a UMP size α test for $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \notin \Theta_0$.

Hint:

From b), the size of the test rule $\varphi(\tilde{X})$ is correct. So, by definition of a UMP test, it is necessary to prove that if $\bar{\varphi}(\tilde{X})$ is any other test of $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \notin \Theta_0$ with size:

$$\max_{\theta \in \Theta_0} E_\theta \bar{\varphi}(\tilde{X}) \leq \alpha,$$

then $\varphi(\tilde{X})$ has more power over the parameter subspace of H_1 than $\bar{\varphi}(\tilde{X})$, i.e.,

$$E_\theta \varphi(\tilde{X}) \geq E_\theta \bar{\varphi}(\tilde{X}) \quad \text{for any } \theta \notin \Theta_0.$$

In other words, pick/fix some $\theta_1 \notin \Theta_0$ and argue that:

$$E_{\theta_1} \varphi(\tilde{X}) \geq E_{\theta_1} \bar{\varphi}(\tilde{X})$$

must hold. The way to do this is to take the test $\bar{\varphi}(\tilde{X})$ and apply it to testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$.

Answer

Assume a) and b) hold. The goal then is to show that $\varphi(\tilde{X})$ is UMP for $H_0 : \theta \in \Theta_0$ vs. $H_1 : \theta \notin \Theta_0$.

Consider testing:

$$H_0 : \theta = \theta_0 \quad \text{vs.} \quad H_1 : \theta = \theta_1$$

Where θ_0 and θ_1 are suitable parameters belonging to Θ_0 and θ_1 respectively.

By Neyman Pearson, $\varphi(\tilde{X})$ is MP at size α for this test.

Let $\bar{\varphi}(\tilde{X})$ be another test with:

$$\sup_{\theta \in \Theta_0} E_\theta \bar{\varphi}(\tilde{X}) \leq \alpha$$

In particular, $E_{\theta_0} \bar{\varphi}(\tilde{X}) \leq \alpha$.

Since $\varphi(\tilde{X})$ is MP for $\theta = \theta_0$ vs. $\theta = \theta_1$, it satisfies:

$$E_{\theta_1} \varphi(\tilde{X}) \geq E_{\theta_1} \bar{\varphi}(\tilde{X})$$

Given condition a) holds then, we know that $\varphi(\tilde{X})$ does not depend on θ_1. Thus, the inequality holds for all $\theta_1 \notin \Theta_0$, proving $\varphi(\tilde{X})$ is UMP.

An Alternative Approach

I believe there is also another approach via a proof by contradiction. To that end:

To start, assume (for contradiction) that $\varphi(\tilde{X})$ is not UMP of size α, yet still meets conditions a) and b).

Then $\exists$ a test $\bar{\varphi}(\tilde{X})$ such that:

- $\sup_{\theta \in \Theta_0} E_\theta[\bar{\varphi}(\tilde{X})] \leq \alpha$ (level α),
- $\exists \theta_1 \notin \Theta_0$ with $E_{\theta_1}[\bar{\varphi}(\tilde{X})] > E_{\theta_1}[\varphi(\tilde{X})]$.

Then, fix $\theta_0 \in \Theta_0$ where size α is attained:

- By condition b), $E_{\theta_0}[\varphi(\tilde{X})] = \alpha$.
- Additionally, we know $E_{\theta_0}[\bar{\varphi}(\tilde{X})] \leq \alpha$.

Via Neyman-Pearson for $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$:

- $\varphi(\tilde{X})$ is MP of size α for this test (via condition a) and Neyman-Pearson).

However, $\bar{\varphi}(\tilde{X})$ has:

- Size $\leq \alpha$ (since $E_{\theta_0}[\bar{\varphi}] \leq \alpha$),
- Higher power at θ_1 (since $E_{\theta_1}[\bar{\varphi}] > E_{\theta_1}[\varphi]$).

This is a contradiction, as Neyman-Pearson guarantees no such $\bar{\varphi}$ can exist (any other MP test with the same size cannot have higher power!)

Thus, we conclude that $\varphi(\tilde{X})$ is UMP of size α.

Q5

Problem 8.23, Casella and Berger (2nd Edition).

Suppose X is one observation from a population with Beta$(\theta, 1)$ pdf.

a)

For testing:

$$H_0 : \theta \leq 1 \quad \text{versus} \quad H_1 : \theta > 1,$$

find the size and sketch the power function of the test that rejects H_0 if:

$$X > \frac{1}{2}.$$

b)

Find the most powerful level-α test of:

$$H_0 : \theta = 1 \quad \text{versus} \quad H_1 : \theta = 2.$$

c)

Is there a UMP test of:

$$H_0 : \theta \leq 1 \quad \text{versus} \quad H_1 : \theta > 1?$$

If so, find it. If not, prove so.