Homework 1 – STAT 5430

Due Monday, Feb 3 by midnight in gradescope;

1. Find the method of moment estimators (MMEs) of the unknown parameters based on a random sample $X_1, X_2, \ldots, X_n$ of size n from the following distributions:

   (a) Negative Binomial $(3, p)$, unknown p

   (b) Double Exponential (μ, σ), unknown μ and σ

   See "Table of Common Distributions" in Casella & Berger (pages 623-623) for the definitions/properties of the above distributions.

2. Problem 7.1, Casella & Berger
   Hint: For context, there is only one (discrete) data observation X which has possible outcomes as $0, 1, 2, 3, 4$. For a given outcome x of X, the likelihood $L(\theta) \equiv f(x|\theta)$ is given by the pmf as a function of $\theta \in \Theta \equiv \{1, 2, 3\}$.

3. An indicator function $I(A)$ of an event A as the form $I(A) = 1$ if event A holds true and $I(A) = 0$ otherwise. Suppose that $A_1, \ldots, A_n$ are n separate events and show that

$$\prod_{i=1}^{n} I(A_i) = I(B)$$

   where B is the event that $B = \bigcap_{i=1}^{n} A_i$.

4. Maximum-likelihood & indicator functions:
   Given a random sample $X_1, \ldots, X_n$ from a pdf/pmf $f(x|\theta)$, $\theta \in \Theta \subset \mathbb{R}$, we know that the likelihood function will generically be

$$L(\theta) = \prod_{i=1}^{n} f(x_i|\theta), \quad \theta \in \Theta,$$

   but there's one subtle point to again highlight about how to exactly write the likelihood expression depending on the support of $f(x|\theta) > 0$.

   - Recall the support or range of $f(x|\theta)$ is a set

$$S_\theta = \{x \in \mathbb{R} : f(x|\theta) > 0\},$$

     which could possibly depend on $\theta \in \Theta$. For example, an exponential distribution has a pdf

$$f(x|\theta) = \begin{cases} \frac{1}{\theta} e^{-x/\theta} & \text{if } x > 0 \\ 0 & \text{otherwise,} \end{cases}$$

     with a parameter $\theta > 0$, and in this case the support $S_\theta = (0, \infty)$ doesn't depend on $\theta \in \Theta = (0, \infty)$. On the other hand, the pdf

$$f(x|\theta) = \begin{cases} 2x/\theta^2 & \text{if } 0 < x \leq \theta \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

     with parameter $\theta > 0$, does have a support $S_\theta = (0, \theta]$ depending on $\theta \in \Theta = (0, \infty)$.

   - It's always true that $f(x|\theta) = f(x|\theta) I(x \in S_\theta)$ for all $x \in \mathbb{R}$ and so always true that

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{n} [f(x_i|\theta) I(x_i \in S_\theta)] && (2) \\ &= \left(\prod_{i=1}^{n} f(x_i|\theta)\right) \left(\prod_{i=1}^{n} I(x_i \in S_\theta)\right) \\ &= \left(\prod_{i=1}^{n} f(x_i|\theta)\right) I(x_1, \ldots, x_n \text{ are all in } S_\theta). \end{aligned}$$

- Here's the point again: *when the support S_θ of $f(x|\theta)$ depends on θ, be sure to write the likelihood to specifically include an indicator $I(x_1, \ldots, x_n$ are all in $S_\theta)$.* Complete the following two problems (a)-(b) to re-enforce this idea.

(a) If $X_1, \ldots, X_n$ are a random sample from an exponential pdf $f(x|\theta)$, $\theta > 0$ (and so $X_1, \ldots, X_n$ are positive values), show that the likelihood function (2) can be written as

$$L(\theta) = \frac{1}{\theta^n} e^{-\sum_{i=1}^n x_i/\theta}$$

and that the MLE of θ is $\bar{X}_n$. (Message here: The support of an exponential doesn't depend on θ, so we don't have to worry about indicating the support.)

(b) If $X_1, \ldots, X_n$ are a random sample from the pdf (1), $\theta > 0$ (and so $X_1, \ldots, X_n > 0$ are less than or equal to θ), show that the likelihood function (2) can be written as

$$\begin{aligned} L(\theta) &= \left(\frac{2^n}{\theta^{2n}} \prod_{i=1}^n x_i \right) I\left(\max_{1 \le i \le n} x_i \le \theta \right) \\ &= \begin{cases} \frac{2^n}{\theta^{2n}} \prod_{i=1}^n x_i & \text{if } \theta \ge \max_{1 \le i \le n} x_i \\ 0 & \text{if } \theta < \max_{1 \le i \le n} x_i \end{cases} \end{aligned}$$

as a function of $\theta > 0$ and that the MLE of θ is $\max_{1 \le i \le n} X_i$. (Message here: The support in this case depends on θ, so we should think about indicator functions in writing the likelihood.)

5. Problem 7.6(b)-(c), Casella & Berger (Skip part (a).)