HW3

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HW 3

NAME: SAM OLSON
COLLABORATORS: **Tne Hatman**

Overview

- 1: DONE
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1. 2.23(b)

Question 1

Let X have the pdf

$$f(x) = \frac{1}{2}(1 + x)$$

, $-1 < x < 1$

Define the random variable Y by $Y = X^2$

(b): Find $E(Y)$ and Var(Y).

Answer 1

(b):

(From prior HW) Note from the results of theorem 2.1.8:

$$f_Y(y) = \begin{cases} \sum_{i=1}^{k} f_X(g_i^{-1}(y))|\frac{d}{dy}g_i^{-1}(y)| & y \in \mathbb{Y} \\ 0 & otherwise \end{cases}$$

Note: For $X \in [-1,1]$, we have $Y = X^2 \in [0,1]$

Over the following partitions, we have monotonicity,

$A_1 = (-1,0) \rightarrow X = -\sqrt{y}$, as $g_1(x) = x^2$, and

$A_2 = (0,1) \rightarrow X = \sqrt{y}$, as $g_2(x) = x^2$

Taking the absolute value of the derivatives, $|\frac{d}{dy} g_i^{-1}(y)|$, we have:

$|\frac{d}{dy} g_1^{-1}(y)| = |\frac{d}{dy} g_2^{-1}(y)| = \frac{1}{2} y^{-1/2}$

Thus we have

$f_Y(y) = \frac{1}{2} y^{-1/2} \frac{1}{2}[(1+\sqrt{y}) + (1-\sqrt{y})] = \frac{1}{4} y^{-1/2}[2] = \frac{1}{2} y^{-1/2}$

Such that we have the pdf of Y as:

$f_Y(y) = \frac{1}{2} y^{-1/2}$, $0 < y < 1$

Using this for our Expected value calculation we have:

$E(Y) = \int\limits_{y \in \mathbb{Y}} y f(y) dy = \int\limits_{y=0}^{1} y(\frac{1}{2\sqrt{y}}) dy$

$$E(Y) = \int_{y=0}^{1} \sqrt{y}(\frac{1}{2}) dy = \frac{1}{2}\frac{2}{3} y^{3/2}\Big|_{y=0}^{y=1} = \frac{1}{2}\frac{2}{3}(1) - 0 = \frac{1}{3}$$

To calculate Var(Y), let us consider $E(Y^2)$,

$E(Y^2) = \int\limits_{y \in \mathbb{Y}} y^2 f(y) dy = \int\limits_{y=0}^{1} y^2(\frac{1}{2\sqrt{y}}) dy$

$$E(Y^2) = \int_{y=0}^{1} y^{3/2} \frac{1}{2} dy = \frac{2}{5}(\frac{1}{2}) y^{5/2}\Big|_{y=1}^{y=1} = \frac{2}{5}(\frac{1}{2})(1) - 0 = \frac{2}{10} = \frac{1}{5}$$

Taking Var(Y) $= E(Y^2) - (E(Y))^2$, then,

$Var(Y) = \frac{1}{5} - (\frac{1}{3})^2 = \frac{1}{5} - \frac{1}{9} = \frac{9}{45} - \frac{5}{45} = \frac{4}{45}$

2.

Question 2

A family continues to have children until they have one female child. Suppose, for each birth, a single child is born and the child is equally likely to be male or female. The gender outcomes are independent across births.

(a): Let X be a random variable representing the number of children born to this family. Find the distribution of X.

(b): Find the expected value $E(X)$

(c): Let X_m denote the number of male children in this family and let X_f denote the number of female children. Find the expected value of X_m and the expected value of X_f

Answer 2

(a): We can frame X as the number of children until the family has their first (one) female child. So we can think of X as a Geometric distribution with probability $p = 0.5$ since it is equally likely that they have a male/female for each birth.

Notation-wise we write this as:

$X \sim \text{Geometric}(p = 0.5)$

(b):

Knowing the distribution of X, we know its pmf (discrete!) is given by:

For X number of children, $k = 1, 2, ...,$ we have:

$f_X(x) = P(X = x) = p(1-p)^{x-1}$

$$E(X) = \sum_{x=1}^{\infty} xP(X = x) = \sum_{k=x}^{\infty} x(p(1-p)^{x-1}) = p\sum_{x=1}^{\infty} x((1-p)^{k-1})$$

Note, for the infinite geometric series we have, for $|r| < 1$, k some positive integer, the following holds:

$$\sum_{k=1}^{\infty} r^k = \frac{1}{1-r}$$

Note: Let $q = 1 - p$ for simplicity. As $0 < p < 1 \rightarrow 0 < 1 - p < 1 \rightarrow 0 < q < 1$. For our purposes, we have $|q| < 1$, such that the above relation holds for an infinite geometric series:

$$\sum_{x=1}^{\infty} q^x = \frac{1}{1-q}$$

Note then:

$$\frac{d}{dq}(\sum_{x=1}^{\infty} q^x) = \sum_{x=1}^{\infty}(\frac{d}{dq}q^x) = \sum_{x=1}^{\infty} xq^{x-1}$$

Additionally,

$$\frac{d}{dq}(\frac{1}{1-q}) = \frac{d}{dq}[(1-q)^{-1}] = \frac{1}{(1-q)^2} = (1-q)^{-2}$$

Thus we have:

$$E(X) = p(1-q)^{-2} = p(1-(1-p))^{-2} = p(p)^{-2} = p^{-1} = \frac{1}{p}$$

For $p = 0.5$, we have:

$$E(X) = \frac{1}{p} = \frac{1}{0.5} = 2$$

Or, on average they would have two children before they have their first female.

(c):

Note: X_f and X_m are subsets of the random variable X, specifically $X_f + X_m = X$.

We stop the "experiment" at the first female child, so we will only ever have 1 female child, meaning:

$E(X_f) = 1$

The number of male children then is the number of children we expect to have minus the number of female children, which is:

$E(X_m) = E(X) - E(X_f) = 2 - 1 = 1$

We expect to have two children, one child is female and the other is male.

3. 2.30 (a), (b), (c)

Question 3

Find the moment generating function corresponding to:

(a): $f(x) = \frac{1}{c}$, $0 < x < c$

(b): $f(x) = \frac{2x}{c^2}$, $0 < x < c$

(c): $f(x) = \frac{1}{2\beta}e^{\frac{-|x-\alpha|}{\beta}}$, $-\infty < x < \infty$, $-\infty < \alpha < \infty$, $\beta > 0$

Answer 3

Note, for a continuous random variable X, we may write the moment generating funciton as:

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx$$

Using this method, we then calculate the following:

(a):

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx = \int_{0}^{c} e^{tx} \frac{1}{c}dx = \frac{1}{ct}e^{tx}\Big|_{x=0}^{x=c} = \frac{1}{ct}e^{tc} - \frac{1}{ct}(1)$$

$$M_X(t) = \frac{1}{ct}e^{tc} - \frac{1}{ct}(1) = \frac{1}{ct}(e^{tc} - 1)$$

Note: For $t = 0$, $\frac{1}{ct}$ is not defined, so the above mgf is defined for $t \neq 0$.

(b):

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx = \int_{0}^{c} e^{tx} \frac{2x}{c^2}dx$$

Via integration by parts, let $u = x, du = 1$ such that $dv = \frac{2e^{tx}}{c^2} \implies v = \frac{2e^{tx}}{tc^2}$

So our formula now is

$$M_X(t) = \int udv = uv - \int vdu = x\frac{2e^{tx}}{tc^2} - \frac{2e^{tx}}{t^2c^2} = xt\frac{2e^{tx}}{t^2c^2} - \frac{2e^{tx}}{t^2c^2}$$

Simplifying, we then evaluate over the original range (support of X), giving us:

$$M_X(t) = \frac{2}{c^2t^2}e^{tx}(tx-1)\Big|_{x=0}^{x=c}$$

$$M_X(t) = \frac{2}{c^2t^2}e^{tc}(tc-1) - (\frac{2}{c^2t^2}1(-1)) = \frac{2}{c^2t^2}(tce^{tc} - e^{tc} + 1)$$

Note: For $t = 0$, $\frac{1}{c^2t^2}$ is not defined, so the above mgf is defined for $t \neq 0$.

(c):

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx = \int_{-\infty}^{\infty} e^{tx} \frac{1}{2\beta} e^{\frac{-|x-\alpha|}{\beta}} dx$$

With regards to the usage of $-|x-\alpha|$, when $x < \alpha \to (x-\alpha) < 0$ and when $x \geq \alpha \to (x-\alpha) \geq 0$

Thus, we may break the above integration into two parts, namely:

$$M_X(t) = \int_{-\infty}^{\alpha} e^{tx} \frac{1}{2\beta} e^{\frac{(x-\alpha)}{\beta}} dx + \int_{\alpha}^{\infty} e^{tx} \frac{1}{2\beta} e^{\frac{-(x-\alpha)}{\beta}} dx$$

1.

$$\int_{-\infty}^{\alpha} e^{tx} \frac{1}{2\beta} e^{\frac{(x-\alpha)}{\beta}} dx = \frac{e^{tx+(x-\alpha)/\beta}}{2(t\beta+1)} \Big|_{x=-\infty}^{\alpha} = \frac{e^{tx}}{2(t\beta+1)} - 0 = \frac{e^{t\alpha}}{2(t\beta+1)}$$

2.

$$\int_{\alpha}^{\infty} e^{tx} \frac{1}{2\beta} e^{\frac{-(x-\alpha)}{\beta}} dx = \frac{e^{tx-(x-\alpha)/\beta}}{2(t\beta-1)} \Big|_{x=\alpha}^{\infty} = 0 - \frac{e^{tx}}{2(t\beta-1)} = -\frac{e^{t\alpha}}{2(t\beta-1)}$$

3.

Combining the above (1.) and (2.) together we then have:

$$M_X(t) = \frac{e^{t\alpha}}{2(t\beta+1)} - \frac{e^{t\alpha}}{2(t\beta-1)} = \frac{4e^{\alpha t}}{4-\beta^2 t^2}$$

Note, we need to ensure the above Mgf of X evaluates, so we need to specify the conditions where the denominator is not equal to 0 (divide by zero error!).

For a fixed β, consider: $4 - \beta^2 t^2 = 0 \to 4 = \beta^2 t^2 \to 4/\beta^2 = t^2$

So the denominator is equal to 0 when $t = \sqrt{4/\beta^2} = \pm 2/\beta$

However, we also know that $M_X(t) \geq 0$, (as we assume f(x) is a pdf, hence $f(x) \geq 0$) $\forall t$, so we actually have bounds for t, namely:

The above mgf is defined for $-2/\beta < t < 2/\beta$, where $\beta > 0$.

4. 2.31

Question 4

Does a distribution exist for which $M_X(t) = \frac{t}{(1-t)}$, $|t| < 1$? If yes, find it. If no, prove it.

Answer 4

Let us suppose that the distribution exists.

Then by the definition of a(n) mfg, using the existence of the distribution we have:

$M_X(t) = E(e^{tX})$

We know for $t = 0$ that the relation $|t| = |0| = 0 < 1$ holds.

Thus we know the 0-th moment is defined:

$M_X(0) = E(e^{0X}) = E(e^0) = E(1) = 1$

However, if we evaluate $M_X(t)$ directly using the mgf as given, for $t = 0$ as given, we have:

$M_X(t) = \frac{t}{(1-t)} = \frac{0}{1-0} = 0$

And we arrive at a contradiction. Thus we must conclude that such a distribution does not exist.

5.

Question 5

Suppose that X has the standard normal distribution with pdf:

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}$$

, $-\infty < x < \infty$

Then the random variable Y, $Y = e^X$ has a log-normal distribution.

(a): Find $E(Y^r)$ for any r.

(b): Show the moment generating function of Y does not exist (even though all moments of Y exist).

Answer 5

(a):

$$E(Y^r) = E((e^X)^r) = E(e^{rX})$$

$$E(Y^r) = \int_{-\infty}^{\infty} e^{rx} f(x)dx = \int_{-\infty}^{\infty} e^{rx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$E(Y^r) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{rx-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2-2rx)} dx$$

$$E(Y^r) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}((x-r)^2-r^2)} dx = e^{\frac{r^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-r)^2} dx$$

The below is the normal distribution, which evaluates to $\sqrt{2\pi}$

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-r)^2} = \sqrt{2\pi}$$

So we have:

$$E(Y^r) = \sqrt{2\pi} e^{\frac{r^2}{2}} \frac{1}{\sqrt{2\pi}} = e^{r^2/2}$$

(b):

Part (a) shows that all moments of Y exist. We must then show that the moment generating function of Y does not exist.

To that end, let us consider the moment generating function of Y:

$M_Y(t) = E(e^t Y) = E(e^{te^X})$

$$M_Y(t) = \int\limits_{-\infty}^{\infty} e^{te^x} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$$

$$M_Y(t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{te^x} e^{\frac{-x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{te^x - \frac{x^2}{2}} dx$$

Note: For $x > 0$, define some positive real c, $c > 0$

There exists a sufficiently large x_0 such that for $x > x_0$:

$te^x - \frac{x^2}{2} > 0$

Meaning: $te^x - \frac{x^2}{2} \geq c > 0$

Note the exponential function is a positive monotonic transformation, such that the following holds:

$e^{te^x - \frac{x^2}{2}} > e^c > 0$, such that:

With note that we are working with a non-negative function,

$$M_Y(t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{te^x - \frac{x^2}{2}} dx \geq \frac{1}{\sqrt{2\pi}} \int\limits_{x_0}^{\infty} e^{te^x - \frac{x^2}{2}} dx \geq \frac{1}{\sqrt{2\pi}} \int\limits_{x_0}^{\infty} e^c dx = \infty$$

As the integral does not converge to a finite value, we say the moment generating function does not exist for positive real t.

6.

Question 6

Suppose that X has a normal distribution with pdf:

$$f(x)\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(x-\mu)^2}{\sigma^2 2}}$$

, $-\infty < x < \infty$

The mean of X is μ. Show that the moment generating function of X satisfies $M_X(t) \geq e^{t\mu}$

Answer 6

With note of Jensen's Inequality, we have, for a convex function g, (avoiding confusion of the usage of f with the above pdf),

Let us then consider the moment generation function of X,

$M_X(t) = E(e^{tX})$

Consider then the function e^{tX}, specifically its second derivative:

$$\frac{d^2}{dx^2}e^tX = t^2e^{tX} > 0$$

, $\forall x, t$

We may then note that the mgf of X is convex since its second derivative is positive.

This is advantagous to our purposes, as we know when applying Jensen's inequality that we have a convex function, such that we may write:

Since $E(X) = \mu$,

$$M_X(t) = E[e^{tX}] \geq e^{tE(X)} = e^{t\mu}$$

And we conclude

$M_X(t) \geq e^{t\mu}$

7.

Question 7

Suppose that X has pmf $f(x) = p(1-p)^{x-1}$, for $x = 1, 2, 3, ...$ where $0 < p < 1$. Find the mgf $M_X(t)$ and use this to derive the mean and variance of X.

Answer 7

For deriving the mean and variance of X, we will need to first define the mgf of X as:

$$M_X(t) = E(e^{tX}) = \sum_{x=1}^{\infty} e^{tx} f(x) = \sum_{x=1}^{\infty} e^{tx} p(1-p)^{x-1}$$

The mean $\mu = E(X)$ is equal to the first derivative of the mgf evaluated at $t = 0$:

$E(X) = M_X'(0)$

$$M_X'(t) = \frac{pe^t(1-p)}{\left(1 - e^t(1-p)\right)^2}$$

$$M_X'(0) = \frac{p(1-p)}{(1-(1-p))^2} = \frac{1}{p}$$

$$E(X) = \frac{1}{p}$$

We then derive the variance of X. We know the typical variance formula as: $\mathrm{Var}(X) = E(X^2) - (E(X))^2$

However, we just calculated $E(X)$! Additionally, we know that $E(X^2)$ is equal to the second derivative of the mgf at $t = 0$. As such we write:

$$M_X''(t) = \frac{pe^t(1-p)\left(1 - e^t(1-p) + e^t(1-p)\right)}{\left(1 - e^t(1-p)\right)^3}$$

To make computation easier, let $q = 1 - p$. Then, for $t = 0$,

$$M_X''(0) = \frac{p - pq^2}{(1-q)^4} = \frac{p(1-q^2)}{(1-q)^4} = \frac{p(1+q)}{(1-q)^3} = \frac{1+q}{p^2}$$

Taking this calculation minus the square of the mean gives us:

$$\mathrm{Var}(X) = M_X''(0) - \left(M_X'(0)\right)^2 = \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{q}{p^2} = \frac{1-p}{p^2}$$

And we conclude then

$$\mathrm{Var}(X) = \frac{1-p}{p^2}$$

8.

Question 8

Suppose for one month a company purchases c copies of a software package at a cost of d_1 dollars per copy. The packages are sold to customers for d_2 dollars per copy; any unsold copies are destroyed at the end of the month. Let X represent the demand for this software package in the month. Assume that X is a discrete random variable with pmf $f(x)$ and cdf $F(x)$.

(a): Let $s = \min\{X, c\}$ represent the number of sales during the month. Show that:

$$E(S) = \sum_{x=0}^{c} xf(x) + c(1 - F(c))$$

(b): Let $Y = S * d_2 - cd_1$ represent the profit for the company, the total income from sales minus the total cost of all copies. Find $E(Y)$

(c): As $Y \equiv Y_c$ depends on integer $c \geq 0$, write the expected profit function as $g(c) \equiv E(Y_c)$ from part (b). The company should choose the value of c which maximizes $g(c)$; that is, choose the smallest c such that $g(c+1)$ is less than or equal to $g(c)$. Show that such $c \geq 0$ is the smallest integer with $F(c) \geq \frac{d_2 - d_1}{d_2}$

Answer 8

(a):

Consider two cases: (1): $x < c$, (2): $x > c$ (demand is either greater than or less than or equal to the number of copies sold)

1.

$$E(X) = \sum_{x=0}^{c} xP(X = x) = \sum_{x=0}^{c} xf(x)$$

2.

$$E(c) = \sum_{x=0}^{c} cP(X > c) = \sum_{x=0}^{c} c(1 - F(c))$$

As $F(c) = P(X \leq c) \rightarrow P(X \leq c) + P(X > c) = 1$

3.

We combine parts (1) and (2) together then, for:

$$E(S) = E(X) + E(c) = \sum_{x=0}^{c} xf(x) + \sum_{x=0}^{c} c(1 - F(c)) = \sum_{x=0}^{c} xf(x) + c(1 - F(c))$$

(b):

$Y = Sd_2 - cd_1 \rightarrow E(Y) = E(Sd_2 - cd_1)$

Given linearity of Expectation, we may rewrite this expectation as:

$E(Y) = E(Sd_2) - E(cd_1) = d_2 \cdot E(S) - c \cdot d_1$

$$E(Y) = d_2 \left(\sum_{x=0}^{c} xf(x) + c(1 - F(c)) \right) - cd_1$$

$$E(Y) = d_2 \left(\sum_{x=0}^{c} xf(x) \right) + d_2c(1 - F(c)) - cd_1$$

$$E(Y) = d_2 \left(\sum_{x=0}^{c} xf(x) \right) + c(d_2 - d_1 - F(c))$$

(c):

Using the above calculation for $E(Y)$, we may write the expected profit function as:

$$g(c) = E(Y_c) = d_2 \left(\sum_{x=0}^{c} xf(x) \right) + c(d_2 - d_1 - F(c))$$

$$g(c+1) = E(Y_c) = d_2 \left(\sum_{x=0}^{c+1} xf(x) \right) + (c+1)(d_2 - d_1 - F(c+1))$$

Gathering like terms for simplicity:

$$g(c+1) - g(c) = d_2 \left(\sum_{x=0}^{c+1} xf(x) \right) - d_2 \left(\sum_{x=0}^{c} xf(x) \right) + (c+1)(d_2 - d_1 - F(c+1)) - c(d_2 - d_1 - F(c))$$

$$g(c+1) - g(c) = d_2(c+1)f(c+1) + (d_2 - d_1 + F(c) - F(c+1))$$

Hazy From Here

$$d_2(c+1)f(c+1) + (d_2 - d_1 + F(c) - F(c+1)) \geq 0$$

$g(c+1) \leq g(c) \rightarrow g(c+1) - g(c) \leq 0$

Ending Answer

$$F(c) \geq \frac{d_2 - d_1}{d_2}$$