HW8

2024-11-06

Q1

Write a function which takes 2 arguments n and k which are positive integers. It should return the nXn matrix:

$$\begin{pmatrix} k & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & k & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & k & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & k & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & k & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & k \end{pmatrix}$$

Call the function defined above for n = 6 and k = 5, and provide the matrix you obtain.

One with a for loop, one without.

```r
# nkMatFor <- function(n, k) {
#   mat <- matrix(0, n, n)
#
#   diag(mat) <- k
#
#   for (i in 1:(n-1)) {
#     mat[i, i+1] <- 1
#     mat[i+1, i] <- 1
#   }
#
#   return(mat)
# }
#
# n6k5 <- nkMatFor(n = 6,
#                  k = 5)
# n6k5
```

```r
nkMat <- function(n, k) {
  mat <- matrix(0, n, n)

  diag(mat) <- k

  mat[row(mat) == col(mat) + 1] <- 1
  mat[row(mat) == col(mat) - 1] <- 1

  return(mat)
```

```r
}

n6k5 <- nkMat(n = 6,
              k = 5)
n6k5
```

```
##      [,1] [,2] [,3] [,4] [,5] [,6]
## [1,]    5    1    0    0    0    0
## [2,]    1    5    1    0    0    0
## [3,]    0    1    5    1    0    0
## [4,]    0    0    1    5    1    0
## [5,]    0    0    0    1    5    1
## [6,]    0    0    0    0    1    5
```

Q2

Consider the continuous function

$$f(x) = \begin{cases} x^2 + 2x + 3 & \text{if } x < 0 \\ x + 3 & \text{if } 0 \leq x < 2 \\ x^2 + 4x - 7 & \text{if } 2 \leq x \end{cases}$$

Write a function tmpFn which takes a single argument xVec. The function should return the vector of values of the function f(x) evaluated at the values xVec. Plot the function f(x) for -3 < x < 3.

```r
tmpFn <- function(xVec) {
  result <- numeric(length(xVec))

  result[xVec < 0] <- xVec[xVec < 0]^2 + 2 * xVec[xVec < 0] + 3
  result[xVec >= 0 & xVec < 2] <- xVec[xVec >= 0 & xVec < 2] + 3
  result[xVec >= 2] <- xVec[xVec >= 2]^2 + 4 * xVec[xVec >= 2] - 7

  result
}
```

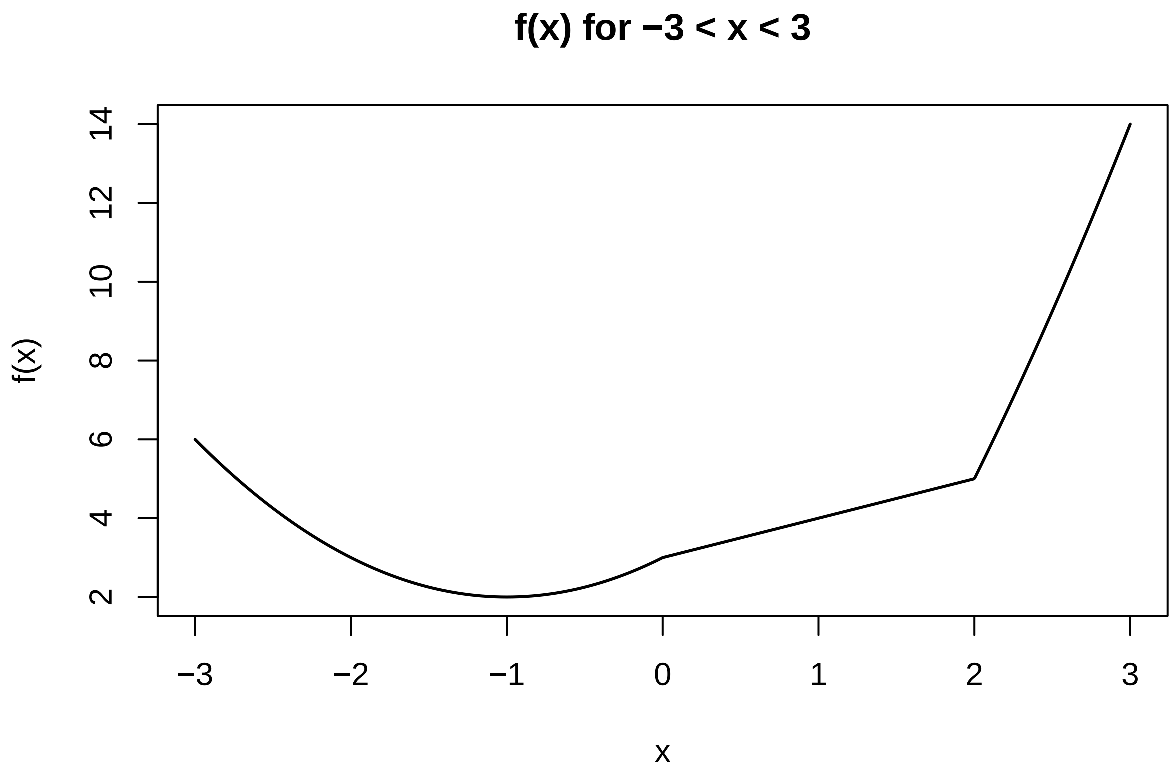
```r
xValues <- seq(from = -3,
               to = 3,
               length.out = 1000)

yValues <- tmpFn(xVec = xValues)

length(xValues) == length(yValues)
```

```
## [1] TRUE
```

```r
plot(x = xValues, y = yValues,
     type = "l",
     col = "black",
     lwd = 1.5,
     xlab = "x",
     ylab = "f(x)",
     main = "f(x) for -3 < x < 3")
```


f(x) for −3 < x < 3
f(x)
2
4
6
8
10
12
14
−3
−2
−1
0
1
2
3
x

Q3

Greatest common divisor of two integers The greatest common divisor (gcd) of two integers m and n can be calculated using Euclid's Algorithm: Divide m by n. If the remainder is zero, the gcd is n. If not, divide n by the remainder. If the remainder is zero, then the previous remainder is the gcd. If not, continue dividing the remainder into previous remainder until a remainder of zero is obtained. The gcd is the value of the last nonzero remainder. Write a function gcd(m,n) using a while loop to find the gcd of two integers m and n.

```r
gcd <- function(m, n) {
  m <- abs(m)
  n <- abs(n)

  while (n != 0) {
    remainder <- m %% n
    m <- n
    n <- remainder
  }

  m
}
```

```r
# Testing examples
gcd(m = 4*7,
    n = 4*4*4*7)
```

```
## [1] 28
```

```r
gcd(m = 47,
    n = 93)
```

```
## [1] 1
```

Q4

eQTL mapping. (The following problem was suggested by Professor Dan Nettleton.) Write a function order.matrix which takes in a matrix x and returns a matrix containing the row and column indices of the sorted values of x. Test this function on a 4X3 matrix of independent χ_1^2 pseudo-random deviates.

```
order.matrix <- function(mat) {
  sorted.indices <- order(mat)

  row.indices <- (sorted.indices - 1) %% nrow(mat) + 1
  col.indices <- (sorted.indices - 1) %/% nrow(mat) + 1

  result <- cbind(row = row.indices,
                  col = col.indices)
  result
}
```

```
set.seed(42)
testMat <- matrix(data = rchisq(4 * 3, df = 1),
                      nrow = 4,
                      ncol = 3)
dim(testMat)
```

```
## [1] 4 3
```

```
testMat
```

```
##          [,1]         [,2]      [,3]
## [1,] 1.521034 4.268462e-01 3.0177684
## [2,] 0.587395 4.222746e-01 0.4038763
## [3,] 3.732687 4.370368e-05 0.5225843
## [4,] 2.963479 1.507955e-04 0.3116673
```

```
testResults <- order.matrix(mat = testMat)
testResults
```

```
##       row col
##  [1,]   3   2
##  [2,]   4   2
##  [3,]   4   3
##  [4,]   2   3
##  [5,]   2   2
##  [6,]   1   2
##  [7,]   3   3
##  [8,]   2   1
##  [9,]   1   1
## [10,]   4   1
## [11,]   1   3
## [12,]   3   1
```

```r
dim(testResults)
```

```
## [1] 12  2
```

Q5

Polar representation of a number. Let $x \in \mathbb{R}^p$. The polar respresentation of $\mathbf{x} = (x_1, x_2, ..., x_p)$ is given by $(R, \theta_1, \theta_2, ..., \theta_{p-1})$, where:

$$\begin{aligned}
x_1 &= R\cos\theta_1 \\
x_2 &= R\sin\theta_1\cos\theta_2 \\
x_3 &= R\sin\theta_1\sin\theta_2\cos\theta_3 \\
\ldots &= \ldots \\
x_{p-1} &= R\prod_{i=1}^{p-2}\sin\theta_i\cos\theta_{p-1} \\
x_p &= R\prod_{i=1}^{p-1}\sin\theta_i,
\end{aligned}$$

where $0 \le R < \infty, 0 \le \theta_1 < 2\pi$ and $0 \le \theta_i < \pi$ i = 2, 3, . . . , p - 1.

(a)

Write a function polaroid which takes in an arbitrary p-dimensional vector $\mathbf{x}$ and provides its polar representation as a vector, with the first element as R and the remainder being $\theta_1, \theta_2, \ldots, \theta_{p-1}$.

```r
polaroid <- function(x) {
  R <- sqrt(sum(x^2))

  if (R == 0) return(c(R))

  theta <- numeric(length(x) - 1)

  theta[1] <- atan2(x[2], x[1])
  if (theta[1] < 0) {
    theta[1] <- theta[1] + 2 * pi
  }

  p <- length(x)
  if (p > 2) {
    for (i in 2:(p - 1)) {
      numerator <- sqrt(sum(x[i:p]^2))
      denominator <- sqrt(sum(x[(i-1):p]^2))
      theta[i] <- acos(numerator / denominator)
    }
  }

  dat <- c(R, theta)
  dat
}
```

```r
x <- c(1, 2, 3)
polarRep <- polaroid(x)
print(polarRep)
```

```
## [1] 3.7416574 1.1071487 0.2705498
```

(b)

Write a function normalize which takes in a matrix and returns its normalized form: i.e., the matrix with rows scaled such that the sum of squares of each row is equal to 1.

```r
normalize <- function(x) {
  rowNorm <- sqrt(rowSums(x^2))
  normMat <- x / rowNorm

  normMat[is.nan(normMat)] <- 0
  normMat
}

# normalize <- function(x) {
#   rowNorm <- sqrt(rowSums(x^2))
#   normMat <- sweep(x, 1, row_norms, FUN = "/")
#   normMat[is.nan(normMat)] <- 0
#   normMat
# }
```

```r
set.seed(42)
testMat <- matrix(data = rnorm(12),
                  nrow = 4,
                  ncol = 3)
testMat
```

```
##            [,1]        [,2]       [,3]
## [1,]  1.3709584  0.40426832  2.0184237
## [2,] -0.5646982 -0.10612452 -0.0627141
## [3,]  0.3631284  1.51152200  1.3048697
## [4,]  0.6328626 -0.09465904  2.2866454
```

```r
rowSums(testMat^2)
```

```
## [1] 6.1169942 0.3340795 4.1192458 5.6382226
```

```r
normMatEx <- normalize(x = testMat)
normMatEx
```

```
##            [,1]        [,2]       [,3]
## [1,]  0.5543132  0.16345593  0.8160999
## [2,] -0.9769930 -0.18360767 -0.1085026
## [3,]  0.1789169  0.74474161  0.6429220
## [4,]  0.2665252 -0.03986493  0.9630032
```

```r
rowSums(normMatEx^2)
```

```
## [1] 1 1 1 1
```

(c)

Obtain a 1000X5 matrix **y** of $N(0,1)$ pseudo-random deviates. Use apply and normalize to obtain the normalized values. Call this matrix **z**. We test whether the columns of **z** are uniform on $U(-1,1)$. One may test whether a sample $x \sim U(-1,1)$ using ks.test(x, "punif", min=-1, max=1) where punif represents the cumulative distribution function of the uniform over range $(-1,1)$. Summarize your results.

```r
set.seed(42)

y <- matrix(data = rnorm(1000 * 5),
            nrow = 1000,
            ncol = 5)

z <- apply(X = y,
           MARGIN = 2,
           FUN = function(col) {
             normalize(x = matrix(col, ncol = 1))
             }
           )

ksTests <- apply(X = z,
                 MARGIN = 2,
                 FUN = function(column) {
                   ks.test(column, "punif", min = -1, max = 1)
                   }
                 )

ksTests
```

```
## [[1]]
##
##  Asymptotic one-sample Kolmogorov-Smirnov test
##
## data:  column
## D = 0.515, p-value < 2.2e-16
## alternative hypothesis: two-sided
##
##
## [[2]]
##
##  Asymptotic one-sample Kolmogorov-Smirnov test
##
## data:  column
## D = 0.503, p-value < 2.2e-16
## alternative hypothesis: two-sided
##
##
## [[3]]
##
##  Asymptotic one-sample Kolmogorov-Smirnov test
##
## data:  column
## D = 0.505, p-value < 2.2e-16
## alternative hypothesis: two-sided
```

```
##
##
## [[4]]
##
##  Asymptotic one-sample Kolmogorov-Smirnov test
##
## data:  column
## D = 0.52, p-value < 2.2e-16
## alternative hypothesis: two-sided
##
##
## [[5]]
##
##  Asymptotic one-sample Kolmogorov-Smirnov test
##
## data:  column
## D = 0.501, p-value < 2.2e-16
## alternative hypothesis: two-sided
```

The KS tests provide small p-values, which is evidence to support rejecting the null hypothesis that the vectors are normally distributed. As we have evidence to suggest the vectors are not normally distributed, then we have evidence that our efforts to normalize the vectors did not effectively transform the data into a uniform distribution over this range (at least when normalizing the sum of squares of each row).

(d)

Obtain polar representations of $\mathbf{y}$ using your function polaroid and test whether $R^2 \sim \chi^2_5$ distribution.

Provide a page of histograms or boxplots of $\theta_1, \theta_2, \theta_3, \theta_4$. Test whether these are from the uniform distributions on their respective ranges, i.e., $[0, 2\pi)$ for θ_1, and $[0, \pi)$ for $\theta_2, \theta_3, \theta_4$.

```
polarRep <- t(apply(X = y,
                    MARGIN = 1,
                    FUN = polaroid))

rVal <- polarRep[, 1]
thetaVal <- polarRep[, -1]
# thetaVal <- if (ncol(polarRep) > 1) polarRep[, -1] else NA
```

```
rSq <- rVal^2
csTest <- ks.test(rSq, "pchisq", df = 5)
csTest
```

```
##
##  Asymptotic one-sample Kolmogorov-Smirnov test
##
## data:  rSq
## D = 0.017123, p-value = 0.9311
## alternative hypothesis: two-sided
```

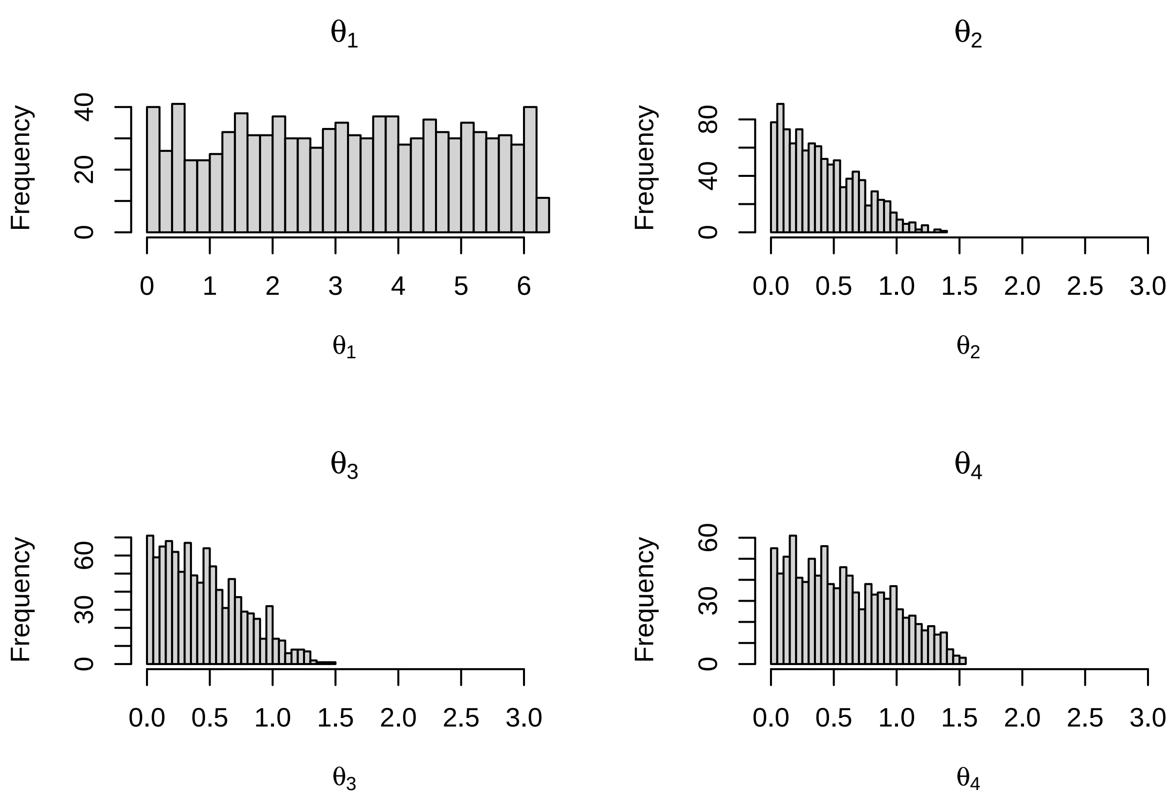
```
par(mfrow = c(2, 2))
```

```
hist(thetaVal[, 1],
     breaks = 30,
     main = expression(theta[1]),
     xlab = expression(theta[1]),
     xlim = c(0, 2 * pi))

hist(thetaVal[, 2],
     breaks = 30,
     main = expression(theta[2]),
     xlab = expression(theta[2]),
     xlim = c(0, pi))

hist(thetaVal[, 3],
     breaks = 30,
     main = expression(theta[3]),
     xlab = expression(theta[3]),
     xlim = c(0, pi))

hist(thetaVal[, 4],
     breaks = 30,
     main = expression(theta[4]),
     xlab = expression(theta[4]),
     xlim = c(0, pi))
```



```r
theta1Test <- ks.test(x = thetaVal[, 1] / (2 * pi),
                      y = "punif",
                      min = 0,
                      max = 1)
theta2Test <- ks.test(x = thetaVal[, 2] / pi,
                      y = "punif",
                      min = 0,
                      max = 1)
theta3Test <- ks.test(x = thetaVal[, 3] / pi,
                      y = "punif",
                      min = 0,
                      max = 1)
theta4Test <- ks.test(x = thetaVal[, 4] / pi,
                      y = "punif",
                      min = 0,
                      max = 1)

list(theta1 = theta1Test,
     theta2 = theta2Test,
     theta3 = theta3Test,
     theta4 = theta4Test)
```

```
## $theta1
##
##  Asymptotic one-sample Kolmogorov-Smirnov test
##
## data:  thetaVal[, 1]/(2 * pi)
## D = 0.017778, p-value = 0.9101
## alternative hypothesis: two-sided
##
##
## $theta2
##
##  Asymptotic one-sample Kolmogorov-Smirnov test
##
## data:  thetaVal[, 2]/pi
## D = 0.65547, p-value < 2.2e-16
## alternative hypothesis: two-sided
##
##
## $theta3
##
##  Asymptotic one-sample Kolmogorov-Smirnov test
##
## data:  thetaVal[, 3]/pi
## D = 0.62241, p-value < 2.2e-16
## alternative hypothesis: two-sided
##
##
## $theta4
##
##  Asymptotic one-sample Kolmogorov-Smirnov test
##
```

```
## data:  thetaVal[, 4]/pi
## D = 0.54649, p-value < 2.2e-16
## alternative hypothesis: two-sided
```