HW6

2024-10-26

Homework 6

Outline:

Q1: DONE Q2: DONE Q3: DONE
Q4: DONE Q5: DONE Q6: WIP Q7: DONE

Q1: 4.17, Casella & Berger

Let X be an exponential(1) random variable, and define Y to be the integer part of X+1, that is:

$$Y = i + 1 \text{ iff } i \leq X < i + 1, i = 0, 1, 2, ...$$

(a)

Find the distribution of Y. What well-known distribution does Y have?

$$P(Y = i + 1) = \int_{i}^{i+1} e^{-x} dx = -e^{-x}\Big|_{x=i}^{i+1} = -e^{-(i+1)} + e^{-i} = e^{-i}(1 - e^{-1})$$

This is a geometric distribution with $p = 1 - e^{-1}$, such that

$Y \sim Geom(1 - e^{-1})$

(b)

Find the conditional distribution of X - 4 given $Y \geq 5$

As defined, $Y = i + 1$, such that

$$Y \geq 5 \iff i + 1 \geq 5 \iff X \geq 4$$

Utilizing the distributions as defined and found, we then have

$$P(X - 4 \leq x \mid Y \geq 5) = P(X - 4 \leq x \mid X \geq 4) = P(X \leq x + 4 \mid X \geq 4)$$

$$P(X - 4 \leq x \mid Y \geq 5) = P(X \leq x + 4 \mid X \geq 4) = 1 - P(X > x + 4 \mid X \geq 4) = 1 - P(X > x) = 1 - e^{-x}$$

This sure looks like the memoryless property we observed previously!

$$P(X - 4 \leq x \mid Y \geq 5) = P(X \leq x) = 1 - e^{-x}$$

Q2: 4.32(a), Casella & Berger

(a)

For a hierarchical model:

$$Y|\Lambda \sim Poisson(\Lambda) \text{ and } \Lambda \sim Gamma(\alpha, \beta)$$

find the marginal distribution, mean, and variance of Y. Show that the marginal distribution of Y is a negative binomial if α is an integer.

For y = 0, 1, . . . , we may write the conditional distribution of Y = y as:

$$P(Y = y|\lambda) = \sum_{n=y}^{\infty} P(Y = y|N = n, \lambda)P(N = n|\lambda) = \sum_{n=y}^{\infty} \binom{n}{y} p^y (1-p)^{n-y} \frac{e^{-\lambda}\lambda^n}{n!}$$

$$P(Y = y|\lambda) = \sum_{n=y}^{\infty} \frac{1}{y!(n-y)!} (\frac{p}{1-p})^y [(1-p)\lambda]^n e^{-\lambda}$$

Define $m = n - y$, such that we may rewrite the above as:

$$P(Y = y|\lambda) = \sum_{n=y}^{\infty} \frac{e^{-\lambda}}{y!m!} (\frac{p}{1-p})^y [(1-p)\lambda)^m] = \sum_{n=y}^{\infty} \frac{e^{-\lambda}}{y!} (\frac{p}{1-p})^y \frac{[(1-p)\lambda)^m]}{m!}$$

After gathering the terms, we see quite a lot of this does not depend on m, such that we may take it out of the summation and write:

$$P(Y = y|\lambda) = \frac{e^{-\lambda}}{y!} (\frac{p}{1-p})^y \sum_{n=y}^{\infty} \frac{[(1-p)\lambda)^m]}{m!}$$

After simplifying, we then take advantage that

$$\sum_{n=y}^{\infty} \frac{[(1-p)\lambda)^m]}{m!} = e^{(1-p)\lambda}$$

And may write:

$$P(Y = y|\lambda) = e^{-\lambda}(p\lambda)^y e^{(1-p)\lambda} = \frac{(p\lambda)^y e^{-p\lambda}}{y!}$$

Note the above is a type of Poisson, specifically:

$$Y|\Lambda \sim Poisson(p\lambda)$$

From this we may "extract" the pmf of Y (pmf as both the conditional of Y and Λ are both Poisson distributed), specifically for $y = 0, 1, ...,$

$$f_Y(y) = \frac{1}{\Gamma(\alpha)y!(p\beta)^\alpha} \Gamma(y + \alpha)(\frac{p\beta}{1 + p\beta})^{y+\alpha}$$

For a positive integer α, the above provides a pmf for a negative binomial distribution, specifically:

$$Y \sim NB(\alpha, \frac{1}{1+p\beta}$$

Q3

Expectation

(a)

Show that any random variable X (with finite mean) has zero covariance with any real constant c, i.e. $Cov(X, c) = 0$

The covariance of X and c may be written:

$$Cov(X, c) = E[(X - E[X])(c - E[c])] = E[(X - E[X])(c - c)] = E[(X - E[X])0] = E[0] = 0$$

Such that we conclude:

$$Cov(X, c) = 0$$

And we must have the condition that X has a finite mean as $\infty * 0$ is undefined.

(b)

Using the definition of conditional expectation, show that

$$E[g(X)h(Y)|X = x] = g(x)E[h(Y)|X = x]$$

for an x with pdf $f_X(x) > 0$ (You may also assume (X, Y) are jointly discrete).

To show that

$$E[g(X)h(Y) \mid X = x] = g(x)E[h(Y) \mid X = x]$$

For jointly discrete random variables X and Y, the conditional expectation of h(Y) | X = x is:

$$E[h(Y) \mid X = x] = \sum_y h(y)P(Y = y \mid X = x)$$

Similarly, the conditional expectation of g(X)h(Y) | X = x is:

$$E[g(X)h(Y) \mid X = x] = \sum_y g(x)h(y)P(Y = y \mid X = x)$$

We can simplify by recognizing there are terms in the above equations that do not depend on the index of the summation, specifically:

$$E[g(X)h(Y) \mid X = x] = \sum_y g(x)h(y)P(Y = y \mid X = x) = g(x)\sum_y h(y)P(Y = y \mid X = x)$$

However, this is a very familiar formula to us!

$$E[g(X)h(Y) \mid X = x] = g(x)\sum_y h(y)P(Y = y \mid X = x) = g(x)E[h(Y) \mid X = x]$$

Note:

The condition $f_X(x) > 0$ is necessary as it ensures that $P(Y = y \mid X = x)$ is defined, because:

$$P(Y = y \mid X = x) = \frac{P(X = x, Y = y)}{P(X = x)} \equiv \frac{P(X = x, Y = y)}{f_X(x)}$$

Q4

Suppose that X_i has mean μ_i and variance σ_i^2, for i = 1, 2, and that the covariance of X_1 and X_2 is σ_{12}. Compute the covariance between $X_1 - 2X_2 + 8$, and then compute the covariance of $3X_1 + X_2$.

(a)

$X_1 - 2X_2 + 8$

$$Var(X_1 - 2X_2 + 8) = Cov(X_1 - 2X_2 + 8, X_1 - 2X_2 + 8) = Cov(X_1 - 2X_2, X_1 - 2X_2)$$

$$Cov(X_1 - 2X_2 + 8, X_1 - 2X_2 + 8) = Cov(X_1, X_1) - 2Cov(X_1, X_2) - 2Cov(X_2, X_1) + 4Cov(X_2, X_2) = \sigma_1^2 - 4\sigma_{12} + 4\sigma_2^2$$

$$Cov(X_1 - 2X_2 + 8, X_1 - 2X_2 + 8) = \sigma_1^2 - 4\sigma_{12} + 4\sigma_2^2.$$

(b)

$3X_1 + X_2$

$$Cov(3X_1 + X_2, 3X_1 + X_2) = Cov(3X_1, 3X_1) + Cov(3X_1, X_2) + Cov(X_2, 3X_1) + Cov(X_2, X_2)$$

$$Cov(3X_1 + X_2, 3X_1 + X_2) = 9\sigma_1^2 + 3\sigma_{12} + 3\sigma_{12} + \sigma_2^2$$

Q5

The joint distribution of X, Y is given by the joint pdf:

$$f(x,y) = 3(x+y) \text{ for } 0 < x < 1, 0 < y < 1, 0 < x+y < 1$$

(a)

Find the marginal distribution of $f_X(x)$

$$f_X(x) = \int_{y \in Y} f(x,y)\,dy = \int_0^1 f(x,y)\,dy = \int_0^1 3(x+y)dy$$

However, the bounds of the integral as given above are not correct, as:

$$0 < x < 1,\ 0 < y < 1,\ 0 < x+y < 1$$

So we actually have:

$$f_X(x) = \int_0^{1-x} 3(x+y)\,dy = 3\int_0^{1-x} (x+y)\,dy = 3\left[\int_0^{1-x} x\,dy + \int_0^{1-x} y\,dy\right]$$

For the sake of simplifcation, these are: ### i.

$$\int_0^{1-x} x\,dy = x(1-x).$$

ii.

$$\int_0^{1-x} y\,dy = \frac{(1-x)^2}{2}.$$

Taken together, we have:

$$f_X(x) = 3\left[x(1-x) + \frac{(1-x)^2}{2}\right] = 3\left[(1-x)\left(x + \frac{1-x}{2}\right)\right] = 3(1-x)\left(\frac{x+1}{2}\right) = \frac{3}{2}(1-x)(x+1)$$

Thus, the marginal distribution of X is:

$$f_X(x) = \frac{3}{2}(1-x)(x+1), \qquad \text{for } 0 < x < 1.$$

(b)

Find the conditional pdf of Y | X = x, given some 0<x<1.

Using the definition of the conditional pdf, we have:

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{3(x+y)}{\frac{3}{2}(1-x)(x+1)}$$

For

$$0 < x < 1,\, 0 < y < 1,\, 0 < x + y < 1$$

Simplifying gives us:

$$f_{Y|X}(y|x) = \frac{2(x+y)}{(1-x)(x+1)} \text{ for } 0 < y < 1 - x$$

(c)

Find $E[Y|X = x]$

Using what we derived in part (b), we have:

$$E[Y \mid X = x] = \int_0^{1-x} y\, f_{Y|X}(y|x)\, dy = \int_0^{1-x} y \frac{2(x+y)}{(1-x)(x+1)} = \frac{2}{(1-x)(x+1)} \int_0^{1-x} y(x{+}y)\, dy = \frac{2}{(1-x)(x+1)} \int_0^{1-x} yx+$$

$$E[Y \mid X = x] = \frac{2}{(1-x)(x+1)}[\left(\int_0^{1-x} xy\, dy\right) + \left(\int_0^{1-x} y^2\, dy\right)]$$

i.

$$\int_0^{1-x} xy\, dy = x \int_0^{1-x} y\, dy = x \left[\frac{(1-x)^2}{2}\right] = \frac{x(1-x)^2}{2}$$

ii.

$$\int_0^{1-x} y^2\, dy = \left[\frac{(1-x)^3}{3}\right]$$

Combining the two parts above, we then have:

$$E[Y \mid X = x] = \frac{2}{(1-x)(x+1)} \left(\frac{x(1-x)^2}{2} + \frac{(1-x)^3}{3}\right) = \frac{2(1-x)^2}{(1-x)(x+1)} \left(\frac{x}{2} + \frac{1-x}{3}\right)$$

Simplify, simplify:

$$E[Y \mid X = x] = \frac{2(1-x)}{x+1} \left(\frac{3x + 2 - 2x}{6}\right) = \frac{2(1-x)}{x+1} \left(\frac{x+2}{6}\right) = \frac{(1-x)(x+2)}{3(x+1)}$$

(d)

Given the results in (a), (b), and (c), explain how you know $E[X|Y = y]$ without any further calculation

Given the above results, we can take advantage of symmetry, since the joint pdf of X and Y involves a simple sum of x + y, and the support of each is the same, i.e.

$$f(x, y) = 3(x + y), \quad \text{for } 0 < x < 1,\, 0 < y < 1,\, 0 < x + y < 1$$

So we can effective "swap" any "x" in the prior calculations with "y" (and similarly if we felt inclined to derive everything again we could/would swap the "y" in our calculations with "x").

Given from (c):

$$E[Y \mid X = x] = \frac{(1-x)(x+2)}{3(x+1)}$$

By symmetry, we know:

$$E[X \mid Y = y] = \frac{(1-y)(y+2)}{3(y+1)}$$

(e)

Find $E[E[2XY - Y|X]]$

From the parts above, we know most everything but $E[XY]$.

$$E[E[2XY - Y \mid X]] = E[2XY - Y] = 2E[XY] - E[Y]$$

Taking advantage of the symmetry property used in part (d), we can easily find the marginal pdf of Y. Namely, as:

$$f_X(x) = \frac{3}{2}(1-x)(x+1), \quad \text{for } 0 < x < 1$$

Then:

$$f_Y(y) = \frac{3}{2}(1-y)(y+1), \quad \text{for } 0 < y < 1$$

However, due to symmetry:

$$E[Y] = E[X]$$

So if we calculate $E[X]$, we effectively get $E[Y]$. Let's do that!

$$E[X] = \int_0^1 x \cdot f_X(x)\, dx = \int_0^1 x \frac{3}{2}(1-x)(x+1)\, dx = \int_0^1 \frac{3x}{2}(1-x^2) = \frac{3}{2}\int_0^1 x(1-x^2)\, dx = \frac{3}{2}\left(\int_0^1 x\, dx - \int_0^1 x^3\, dx\right)$$

o.

$$\int_0^1 x\, dx = \frac{1}{2}$$

$$\int_0^1 x^3\, dx = \frac{1}{4}$$

Taking the above gives us:

$$E[X] = \frac{3}{2}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{3}{2} \cdot \frac{1}{4} = \frac{3}{8}$$

And $E[Y] = \frac{3}{8}$ too.

Last part now, we need to evaluate $E[XY]$:

$$E[XY] = \int_0^1 \int_0^{1-x} xyf(x,y)dydx = \int_0^1 \int_0^{1-x} xy3(x{+}y)\,dy\,dx = 3\int_0^1 \int_0^{1-x} xy(x{+}y) = \,dy\,dx = 3\int_0^1 \int_0^{1-x} (x^2y{+}xy^2)\,dy\,dx$$

Alright, back to it, separating the integrals:

i.

$$\int_0^1 \int_0^{1-x} x^2y\,dy\,dx = \int_0^1 x^2\left(\frac{(1-x)^2}{2}\right)\,dx = \frac{1}{2}\int_0^1 x^2(1{-}x)^2\,dx = \frac{1}{2}\left(\int_0^1 x^2\,dx - 2\int_0^1 x^3\,dx + \int_0^1 x^4\,dx\right)$$

Where:

$$\int_0^1 x^2\,dx = \frac{1}{3}$$

$$\int_0^1 x^3\,dx = \frac{1}{4}$$

$$\int_0^1 x^4\,dx = \frac{1}{5}$$

And our total for the "first" term is then

$$\frac{1}{2}\left(\frac{1}{3} - 2(\frac{1}{4}) + \frac{1}{5}\right) = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = \frac{1}{2}\left(\frac{10}{30} - \frac{15}{30} + \frac{6}{30}\right) = \frac{1}{2}(\frac{1}{30}) = \frac{1}{60}$$

ii.

$$\int_0^1 \int_0^{1-x} xy^2\,dy\,dx = \int_0^1 x\left(\frac{(1-x)^3}{3}\right)\,dx = \frac{1}{3}\int_0^1 x(1-x)^3\,dx = \frac{1}{3}\int_0^1 (x - 3x^2 + 3x^3 - x^4)\,dx$$

$$\frac{1}{3}\left(\frac{1}{2} - 3\cdot\frac{1}{3} + 3\cdot\frac{1}{4} - \frac{1}{5}\right) = \frac{1}{3}\left(\frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5}\right) = \frac{1}{3}\left(\frac{30}{60} - \frac{60}{60} + \frac{45}{60} - \frac{12}{60}\right) = \frac{1}{3}\cdot\frac{3}{60} = \frac{1}{60}$$

Crazy, $\frac{1}{60}$ again…symmetry?

Taking the two parts above, we then have:

$$E[XY] = 3\left(\frac{1}{60} + \frac{1}{60}\right) = \frac{3}{30} = \frac{1}{10}$$

$$2E[XY] = 2(\frac{1}{10}) = \frac{1}{5}$$

And with:

$$E[Y] = \frac{3}{8}$$

We may finally calculate the desired value as:

$$E[E[2XY - Y \mid X]] = \frac{1}{5} - \frac{3}{8} = \frac{8}{40} - \frac{15}{40} = -\frac{7}{40} = -\frac{7}{40}$$

Q6

Suppose that $f(x,y) = e^{-y}$ for $0 < x < y < \infty$

(a)

Find the joint moment generating function for (X, Y).

The joint moment generating function $M_{X,Y}(t_1, t_2)$ may be defined:

$$M_{X,Y}(t_1,t_2) = E\left[e^{t_1 X + t_2 Y}\right] = \int_0^\infty \int_0^y e^{t_1 x + t_2 y} e^{-y}\, dx\, dy = \int_0^\infty \int_0^y e^{t_1 x} e^{(t_2-1)y}\, dx\, dy$$

First, integrate with respect to x. The inner integral is:

$$\int_0^y e^{t_1 x}\, dx = \frac{1}{t_1}\left(e^{t_1 y} - 1\right),$$

assuming $t_1 \neq 0$.

Substitute the result into the outer integral:

$$M_{X,Y}(t_1,t_2) = \frac{1}{t_1}\int_0^\infty \left(e^{(t_1+t_2-1)y} - e^{(t_2-1)y}\right) dy.$$

Now, integrate term by term:

For $e^{(t_1+t_2-1)y}$:

$$\int_0^\infty e^{(t_1+t_2-1)y}\, dy = \frac{1}{1-t_1-t_2} \quad \text{for } t_1 + t_2 < 1.$$

For $e^{(t_2-1)y}$:

$$\int_0^\infty e^{(t_2-1)y}\, dy = \frac{1}{1-t_2} \quad \text{for } t_2 < 1.$$

Now, combine the two results:

$$M_{X,Y}(t_1,t_2) = \frac{1}{t_1}\left(\frac{1}{1-t_1-t_2} - \frac{1}{1-t_2}\right).$$

Thus, the joint moment generating function for (X,Y) is:

$$M_{X,Y}(t_1,t_2) = \frac{1}{t_1}\left(\frac{1}{1-t_1-t_2} - \frac{1}{1-t_2}\right),$$

valid for $t_1 + t_2 < 1$ and $t_2 < 1$.

(b)

Use the joint moment generating function to find the variance of X, the variance of Y, and the covariance of X and Y.

To find the variances of X, Y, and the covariance between X and Y using the joint moment generating function (MGF), we will compute the necessary partial derivatives of the MGF.

The joint MGF we found is:

$$M_{X,Y}(t_1, t_2) = \frac{1}{t_1}\left(\frac{1}{1 - t_1 - t_2} - \frac{1}{1 - t_2}\right),$$

valid for $t_1 + t_2 < 1$ and $t_2 < 1$.

To find the means of X and Y, we use the following formulas for the partial derivatives of the MGF:

- $\mathbb{E}[X] = \frac{\partial}{\partial t_1} M_{X,Y}(t_1, t_2)\Big|_{t_1=0, t_2=0}$,
- $\mathbb{E}[Y] = \frac{\partial}{\partial t_2} M_{X,Y}(t_1, t_2)\Big|_{t_1=0, t_2=0}$.

First, we differentiate $M_{X,Y}(t_1, t_2)$ with respect to t_1:

$$\frac{\partial}{\partial t_1} M_{X,Y}(t_1, t_2) = \frac{-1}{t_1^2}\left(\frac{1}{1 - t_1 - t_2} - \frac{1}{1 - t_2}\right) + \frac{1}{t_1} \cdot \frac{1}{(1 - t_1 - t_2)^2}.$$

Taking the limit as $t_1 \to 0$ and $t_2 \to 0$, we get:

$$\mathbb{E}[X] = \frac{\partial}{\partial t_1} M_{X,Y}(t_1, t_2)\Big|_{t_1=0, t_2=0} = \frac{1}{1^2} = 1.$$

Now, we differentiate $M_{X,Y}(t_1, t_2)$ with respect to t_2:

$$\frac{\partial}{\partial t_2} M_{X,Y}(t_1, t_2) = \frac{1}{t_1}\left(\frac{1}{(1 - t_1 - t_2)^2} - \frac{1}{(1 - t_2)^2}\right).$$

Taking the limit as $t_1 \to 0$ and $t_2 \to 0$, we get:

$$\mathbb{E}[Y] = \frac{\partial}{\partial t_2} M_{X,Y}(t_1, t_2)\Big|_{t_1=0, t_2=0} = 1.$$

The variance of X is given by:

$$\mathrm{Var}(X) = \frac{\partial^2}{\partial t_1^2} M_{X,Y}(t_1, t_2)\Big|_{t_1=0, t_2=0}.$$

From the first derivative:

$$\frac{\partial}{\partial t_1} M_{X,Y}(t_1, t_2) = \frac{-1}{t_1^2}\left(\frac{1}{1 - t_1 - t_2} - \frac{1}{1 - t_2}\right) + \frac{1}{t_1} \cdot \frac{1}{(1 - t_1 - t_2)^2}.$$

The second derivative is:

$$\frac{\partial^2}{\partial t_1^2} M_{X,Y}(t_1,t_2) = \frac{2}{t_1^3}\left(\frac{1}{1-t_1-t_2} - \frac{1}{1-t_2}\right) - \frac{2}{t_1^2}\cdot\frac{1}{(1-t_1-t_2)^2} + \frac{2}{t_1}\cdot\frac{1}{(1-t_1-t_2)^3}.$$

Evaluating at $t_1 = 0$ and $t_2 = 0$, we get:

$$\mathrm{Var}(X) = 1.$$

Similarly, the variance of Y is:

$$\mathrm{Var}(Y) = \frac{\partial^2}{\partial t_2^2} M_{X,Y}(t_1,t_2)\Big|_{t_1=0,t_2=0}.$$

This is:

$$\frac{\partial^2}{\partial t_2^2} M_{X,Y}(t_1,t_2) = \frac{2}{t_1}\left(\frac{1}{(1-t_1-t_2)^3} - \frac{1}{(1-t_2)^3}\right).$$

Evaluating at $t_1 = 0$ and $t_2 = 0$, we get:

$$\mathrm{Var}(Y) = 1.$$

The covariance of X and Y is given by:

$$\mathrm{Cov}(X,Y) = \frac{\partial^2}{\partial t_1 \partial t_2} M_{X,Y}(t_1,t_2)\Big|_{t_1=0,t_2=0}.$$

From the derivative:

$$\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2} M_{X,Y}(t_1,t_2) = \frac{1}{(1-t_1-t_2)^2}.$$

Evaluating at $t_1 = 0$ and $t_2 = 0$, we get:

$$\mathrm{Cov}(X,Y) = 1.$$

So, $Var(X) = 1, Var(Y) = 1,$ and $Cov(X,Y) = 1.$

(c)

Based on the joint moment generating function, identify the marginal distribution of X and the marginal distribution of Y.

Given the joint moment generating function:

$$M_{X,Y}(t_1,t_2) = \frac{1}{t_1}\left(\frac{1}{1-t_1-t_2} - \frac{1}{1-t_2}\right),$$

For the marginal MGF of X, we set $(t_1, t_2) = (t_1, 0)$:

$$M_X(t_1) = M_{X,Y}(t_1,0) = \frac{1}{t_1}\left(\frac{1}{1-t_1} - 1\right) = \frac{1}{t_1}\left(\frac{1}{1-t_1} - 1\right) = \frac{1}{t_1}\left(\frac{1-(1-t_1)}{1-t_1}\right) = \frac{t_1}{t_1(1-t_1)} = \frac{1}{1-t_1}$$

This is the MGF of an Exponential(1) distribution, such that:

$$X \sim \text{Exponential}(1) \rightarrow f_X(x) = e^{-x}$$

Similarly, for the marginal MGF of X, we set $(t_1, t_2) = (0, t_2)$:

$$M_Y(t_2) = M_{X,Y}(0, t_2) = \frac{1}{0}\left(\frac{1}{1-t_2} - \frac{1}{1-t_2}\right),$$

which simplifies directly to:

$$M_Y(t_2) = \frac{1}{1-t_2}.$$

This is also the MGF of an **Exponential(1)** distribution. Therefore, the marginal distribution of Y is:

$$Y \sim \text{Exponential}(1).$$

- The marginal distribution of X is **Exponential(1)**.
- The marginal distribution of Y is **Exponential(1)**.

Both X and Y are independently distributed as **Exponential(1)** random variables.

Q7

Beta-Binomial model: Suppose that the conditional distribution X | P = p is Binomial(n, p) and Suppose P has a Beta(α, β) distribution.

(a)

Using the EVVE formula, find Var(X)

As we know the distribution of X | P = p, we know that its mean and variance are:

$$E[X|P = p] = np$$

$$Var(X|P = p) = np(1 - p).$$

Since we also know the distribution of P, we know it has mean and variance:

$$E[P] = \frac{\alpha}{\alpha + \beta}$$

$$Var(P) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$

Thus, using EVVE, we know:

$$Var(X) = E[Var(X|P)] + Var(E[X|P])$$

But both of these values will need to be evaluated. To that end:

$$E(\mathrm{Var}(X|P)) = E(np(1 - p)) = nE(p(1 - p)) = n\left[E(p) - E(p^2)\right]$$

For a Beta distribution, we know:

$$E(p) = \frac{\alpha}{\alpha + \beta} \quad \text{and} \quad \mathrm{Var}(p) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$

$$E(p^2) = \frac{\alpha(\alpha + 1)}{(\alpha + \beta)(\alpha + \beta + 1)}$$

Additionally, as $E(p(1 - p)) = E(p) - E(p^2)$ we have:

$$E(\mathrm{Var}(X|P)) = n\left(\frac{\alpha}{\alpha + \beta} - \frac{\alpha(\alpha + 1)}{(\alpha + \beta)(\alpha + \beta + 1)}\right) = n\left(\frac{\alpha\beta}{(\alpha + \beta)(\alpha + \beta + 1)}\right)$$

$$\mathrm{Var}(E(X|P)) = \mathrm{Var}(np) = n^2\mathrm{Var}(p) = n^2\frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$

Combining the two parts, we have:

$$\mathrm{Var}(X) = n\left(\frac{\alpha\beta}{(\alpha + \beta)(\alpha + \beta + 1)}\right) + n^2\frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)} = \frac{n\alpha\beta(\alpha + \beta + n)}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$

(b)

Suppose that W has a Binomial(n, $\tilde{p}$) distribution having the same mean as X above. For n > 1, show that X has a larger variance than W by a multiplicative factor of:

$$\frac{\alpha + \beta + n}{\alpha + \beta + 1} > 1$$

From the Beta-Binomial model, we have:

$$\mathbb{E}(X) = n \cdot E(P) = n \cdot \frac{\alpha}{\alpha + \beta}$$

If the RV W has the same mean as X, then:

$$E(W) = n\tilde{p} = n \cdot \frac{\alpha}{\alpha + \beta}$$

Which means that

$$\tilde{p} = \frac{\alpha}{\alpha + \beta}$$

Furthermore:

$$\text{Var}(W) = n\tilde{p}(1 - \tilde{p}) = n\frac{\alpha}{\alpha + \beta}(1 - \frac{\alpha}{\alpha + \beta}) = n(\frac{\alpha}{\alpha + \beta})(\frac{\beta}{\alpha + \beta}) = \frac{n\alpha\beta}{(\alpha + \beta)^2}$$

From (a), we have:

$$\text{Var}(X) = \frac{n\alpha\beta(\alpha + \beta + n)}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$

Looking at the ratio of X/W:

$$\frac{\text{Var}(X)}{\text{Var}(W)} = \frac{\frac{n\alpha\beta(\alpha+\beta+n)}{(\alpha+\beta)^2(\alpha+\beta+1)}}{\frac{n\alpha\beta}{(\alpha+\beta)^2}} = \frac{\alpha + \beta + n}{\alpha + \beta + 1}$$

Assuming n > 1 , we have:

$$(\alpha + \beta + n > \alpha + \beta + 1)$$

Such that X has a larger variance than W by a multiplicative factor of:

$$\frac{\alpha + \beta + n}{\alpha + \beta + 1} > 1$$