Notes

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Q6

To show that the moment generating function $M_X(t)$ of a normally distributed random variable X satisfies $M_X(t) \geq e^{t\mu}$ using Jensen's inequality, we start with the definition of the moment generating function:

$$M_X(t) = \mathbb{E}[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx,$$

where $f(x)$ is the probability density function (pdf) of X:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

Step 1: Express the Moment Generating Function

Substituting the pdf into the moment generating function:

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx.$$

Step 2: Recognize the Function as a Jensen's Inequality Application

Jensen's inequality states that for a convex function ϕ,

$$\mathbb{E}[\phi(X)] \geq \phi(\mathbb{E}[X]).$$

In this case, we can express e^{tX} in a form to apply Jensen's inequality.

Convexity of e^{tx} The function $\phi(x) = e^{tx}$ is convex since its second derivative is positive:

$$\phi''(x) = t^2 e^{tx} > 0 \quad \text{for all } x \text{ and } t.$$

Step 3: Applying Jensen's Inequality

By Jensen's inequality, we have:

$$\mathbb{E}[e^{tX}] \geq e^{t\mathbb{E}[X]}.$$

Given that the mean of X is μ:

$$M_X(t) = \mathbb{E}[e^{tX}] \geq e^{t\mu}.$$

Conclusion

Thus, we have shown that:

$$M_X(t) \geq e^{t\mu},$$

which completes the proof. This means the moment generating function of a normally distributed random variable is always greater than or equal to $e^{t\mu}$, consistent with Jensen's inequality applied to the convex function e^{tx}. If you have any specific parts you'd like to delve into further, let me know!

No, the inequalities should not be reversed; they are correct as stated in the context of applying Jensen's inequality. Let me clarify how it applies to the moment-generating function of a normally distributed random variable X.

Context of Jensen's Inequality

1. **Jensen's Inequality** states that for any convex function ϕ:

$$\mathbb{E}[\phi(X)] \geq \phi(\mathbb{E}[X]).$$

2. In this case, the function $\phi(x) = e^{tx}$ is convex. Therefore, when applying Jensen's inequality:

$$\mathbb{E}[e^{tX}] \geq e^{t\mathbb{E}[X]}.$$

   Since the mean of X is μ, we have:

$$\mathbb{E}[e^{tX}] \geq e^{t\mu}.$$

3. **Moment-Generating Function**: The moment-generating function $M_X(t)$ is defined as:

$$M_X(t) = \mathbb{E}[e^{tX}].$$

   Therefore, we conclude that:

$$M_X(t) \geq e^{t\mu}.$$

Why the Inequality is Correct

- **Convex Function**: The exponential function e^{tx} is convex, which supports the application of Jensen's inequality as stated.
- **Concavity of the PDF**: While the pdf $f(x)$ of the normal distribution is concave, that fact does not affect the convexity of e^{tx}. The key point is that we are using the convexity of the moment-generating function's underlying exponent e^{tx} to show the inequality.

Conclusion

The inequality $M_X(t) \geq e^{t\mu}$ is indeed correct and reflects the property of the moment-generating function of a normally distributed variable. So, there is no need to reverse the inequalities; they are appropriately oriented based on the convexity of the function used in Jensen's inequality.

If you have any further questions or need additional clarification, feel free to ask!

Q2

Certainly! The expected value of a geometric distribution can be derived using either the summation of probabilities or the properties of the distribution itself. Here, I will provide a proof using both methods for clarity.

Definition of the Geometric Distribution

The geometric distribution models the number of trials until the first success in a series of Bernoulli trials. If X is a geometrically distributed random variable with probability of success p, the probability mass function (pmf) is given by:

$$P(X = k) = (1 - p)^{k-1}p \quad \text{for } k = 1, 2, 3, \dots$$

Method 1: Using the Definition of Expected Value

The expected value $\mathbb{E}[X]$ is defined as:

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} k \cdot P(X = k).$$

Substituting the pmf into this expression gives:

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} k \cdot (1 - p)^{k-1}p.$$

Step 1: Factor Out p

We can factor p out of the summation:

$$\mathbb{E}[X] = p\sum_{k=1}^{\infty} k(1 - p)^{k-1}.$$

Step 2: Use the Formula for the Sum

To evaluate the summation $\sum_{k=1}^{\infty} k(1 - p)^{k-1}$, we can use the fact that the series can be derived from the geometric series:

Let $q = 1 - p$. We know that:

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1 - q} \quad \text{for } |q| < 1.$$

Differentiating both sides with respect to q gives:

$$\sum_{k=1}^{\infty} kq^{k-1} = \frac{1}{(1 - q)^2}.$$

Substituting back $q = 1 - p$:

$$\sum_{k=1}^{\infty} k(1-p)^{k-1} = \frac{1}{p^2}.$$

Step 3: Substitute Back into the Expected Value

Now substituting this result back into the expected value:

$$\mathbb{E}[X] = p \cdot \frac{1}{p^2} = \frac{1}{p}.$$

Conclusion

Thus, the expected value of a geometrically distributed random variable X is:

$$\mathbb{E}[X] = \frac{1}{p}.$$

Method 2: Using the Recursive Property of the Expected Value

Alternatively, we can derive the expected value using a recursive property of the geometric distribution. Let $\mathbb{E}[X] = E$. We can express E in terms of p:

1. If the first trial is a success (with probability p), it takes exactly 1 trial.
2. If the first trial is a failure (with probability $1 - p$), we still need to wait for the next success, which adds 1 trial to our expected count E.

Thus, we can write:

$$E = p \cdot 1 + (1-p)(E+1).$$

Step 1: Expand and Rearrange

Expanding this gives:

$$E = p + (1-p)E + (1-p).$$

Rearranging terms, we have:

$$E - (1-p)E = p + 1 - p.$$

This simplifies to:

$$pE = 1.$$

Step 2: Solve for E

Solving for E gives:

$$E = \frac{1}{p}.$$

Final Result

Both methods lead to the same conclusion:

$$\mathbb{E}[X] = \frac{1}{p}.$$

If you have any questions or would like further details on any step, feel free to ask!

Given that X is a geometric random variable with probability $p = 0.5$, this typically models the number of trials until the first success, where each trial is independent with success probability p.

Setup for male and female children:

- Let's assume that a male child is considered a "success" (with probability $p = 0.5$) and a female child is considered a "failure" (with probability $1 - p = 0.5$).
- X_m is the number of male children before the first female child appears.
- X_f is the number of female children before the first male child appears.

Expected Value of X_m (Number of Male Children):

Since X_m follows a geometric distribution with probability $p = 0.5$, the expected value of a geometric random variable X with parameter p is given by:

$$\mathbb{E}[X_m] = \frac{1}{p}$$

Substituting $p = 0.5$:

$$\mathbb{E}[X_m] = \frac{1}{0.5} = 2$$

Expected Value of X_f (Number of Female Children):

Similarly, X_f is also a geometric random variable but with success probability $1 - p = 0.5$. Therefore, the expected value is:

$$\mathbb{E}[X_f] = \frac{1}{1-p} = \frac{1}{0.5} = 2$$

Final Results:

- $\mathbb{E}[X_m] = 2$
- $\mathbb{E}[X_f] = 2$

Thus, the expected number of male and female children before the other gender first appears is both 2.