

# PS2

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## Outline

- Q1: Pretty good
- Q2: Pretty good
- Q3: Pretty good
- Q4:
- Q5: part b) needs work

## Problem 1

7.11, Casella & Berger

Let  $X_1, \dots, X_n$  be iid with pdf

$$f(x|\theta) = \theta x^{\theta-1}, \quad 0 \leq x \leq 1, \quad 0 < \theta < \infty.$$

Hint: In part (a), you can assume each observation lies in  $X_i \in (0, 1)$  for finding the MLE (since there is zero probability of “some  $X_i = 0$  or  $1$  for  $i = 1, \dots, n$ ”). To find the variance in part (a), you should be able to show that  $Y_i = -\log(X_i)$  has an exponential distribution with scale parameter  $\beta = 1/\theta > 0$  so that

$$W = \sum_{i=1}^n Y_i$$

has a gamma ( $\alpha = n, \beta$ ) distribution; then, you can compute the variance by finding moments  $E_\theta(W^{-1})$  and  $E_\theta(W^{-2})$ .

**a)**

Find the MLE of  $\theta$ , and show that its variance  $\rightarrow 0$  as  $n \rightarrow \infty$ .

Given the probability density function pdf provided, the likelihood function for is given by:

$$L(\theta) = \prod_{i=1}^n \theta X_i^{\theta-1}$$

To make the evaluation easier, consider the log likelihood:

$$\log(L(\theta)) = \sum_{i=1}^n \log(\theta) + (\theta - 1) \sum_{i=1}^n \log(X_i) = n \log(\theta) + (\theta - 1) \sum_{i=1}^n \log(X_i)$$

We attempt to find a maximum by differentiating with respect to  $\theta$  and setting the expression to 0:

$$\frac{d\log(L(\theta))}{d\theta} = \frac{n}{\theta} + \sum_{i=1}^n \log(X_i) = 0 \rightarrow \hat{\theta}_{\text{MLE}} = -\frac{n}{\sum_{i=1}^n \log(X_i)}$$

To confirm this is a maximum, we then also check that the second derivative is negative:

$$\frac{d^2\log(L(\theta))}{d\theta^2} = -\frac{n}{\theta^2} < 0 \text{ given } n > 0$$

So our calculation is indeed an MLE.

We then must identify the variance of our MLE. To that end, define  $Y_i = -\log(X_i)$ , which follows an exponential distribution:

$$Y_i \sim \text{Exp}(1/\theta)$$

As parametrized, the sum  $W = \sum_{i=1}^n Y_i$  follows a Gamma distribution:

$$W \sim \text{Gamma}(n, 1/\theta)$$

Utilizing some useful properties of the Gamma, we then know:

$$E(W^{-1}) = \frac{\theta}{n-1}, \quad E(W^{-2}) = \frac{\theta^2}{(n-1)(n-2)}$$

Since  $\hat{\theta}_{\text{MLE}} = \frac{n}{W}$ , the variance is:

$$\text{Var}(\hat{\theta}_{\text{MLE}}) = n^2 E(W^{-2}) - (n E(W^{-1}))^2 = \frac{n^2 \theta^2}{(n-1)(n-2)} - \frac{n^2 \theta^2}{(n-1)^2} = \frac{n^2 \theta^2}{(n-1)^2(n-2)}$$

Consider the behavior of this function as  $n$  increases. Namely, since the denominator grows faster than the numerator, noting the expression in the denominator is  $n$  to a degree 3 and for the numerator  $n$  to a degree 2, we conclude:

$$\text{Var}(\hat{\theta}_{\text{MLE}}) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$

**b)**

Find the method of moments estimator of  $\theta$ .

We start by considering the first moment of  $X$  given by:

$$E(X) = \int_0^1 x f(x|\theta) dx = \int_0^1 x \cdot \theta x^{\theta-1} dx = \theta \int_0^1 x^{\theta} dx$$

For the sake of space, breaking this up and evaluating by:

$$E(X) = \theta \left[ \frac{x^{\theta+1}}{\theta+1} \right]_0^1 = \theta \frac{1}{\theta+1} = \frac{\theta}{\theta+1}$$

By definition, the sample mean is:

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$$

Then, by equating the sample mean equal to the population mean, we have:

$$\frac{\theta}{\theta + 1} = \bar{X} \rightarrow \theta = \bar{X}(\theta + 1) = \bar{X}\theta + \bar{X}$$

Our goal is to identify a formula with just  $\theta$  on one side, so to that end, we have:

$$\theta - \bar{X}\theta = \bar{X} \rightarrow \theta(1 - \bar{X}) = \bar{X} \rightarrow \theta = \frac{\bar{X}}{1 - \bar{X}}$$

I'd be remiss not to note one potential issue though. We can't divide by zero, so we cannot have  $\bar{X} = 1$ . Under this condition, the method of moments estimator is not valid.

Bearing in mind that condition then, we say the method of moments estimator of  $\theta$  is:

$$\hat{\theta}_{\text{MM}} = \frac{\bar{X}}{1 - \bar{X}}$$

## Problem 2

7.12(a), Casella & Berger

Let  $X_1, \dots, X_n$  be a random sample from a population with pmf

$$P_\theta(X = x) = \theta^x(1 - \theta)^{1-x}, \quad x = 0 \text{ or } 1, \quad 0 \leq \theta \leq \frac{1}{2}.$$

Hint: Note that the parameter space is  $\Theta \equiv [0, 1/2]$ . In maximizing the likelihood, it might be clearest to consider three data cases:

1.  $\sum_{i=1}^n X_i = 0$ ;
2.  $\sum_{i=1}^n X_i = n$ ; or
3.  $0 < \sum_{i=1}^n X_i < n$ .

In the last case, the derivative of log-likelihood  $L(\theta)$  indicates that  $L(\theta)$  is increasing on  $(0, \bar{X}_n)$  and decreasing on  $(\bar{X}_n, 1)$ .

**a)**

Find the method of moments estimator and MLE of  $\theta$ .

I am starting this problem in the order of what was specified. To that end:

### Method of Moments

As specified, the information above corresponds to a Bernoulli distributed random variable variable. Treating this as a given, the population mean is the parameter, i.e.:

$$E(X) = \theta$$

Equating this with the sample mean, which by definition is given by:

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$$

Solving for  $\theta$ :

$$\hat{\theta} = \bar{X}$$

Note that  $\theta$  is restricted to  $[0, 1/2]$ , we need to make sure this condition is met, therefore we write our method of moments estimator as:

$$\hat{\theta} = \min\left(\bar{X}, \frac{1}{2}\right)$$

### Maximum Likelihood Estimator (MLE)

The likelihood function is:

$$L(\theta) = \prod_{i=1}^n P_{\theta}(X_i) = \prod_{i=1}^n \theta^{X_i} (1 - \theta)^{1-X_i}$$

Define  $S = \sum_{i=1}^n X_i$ , which allows us to simplify the likelihood function as:

$$L(\theta) = \theta^S (1 - \theta)^{n-S}$$

We then take the log of the likelihood function to make evaluation easier:

$$\log(L(\theta)) = S \log \theta + (n - S) \log(1 - \theta)$$

Differentiate and set the expression equal to 0, giving us:

$$\frac{\log(L(\theta))}{d\theta} = \frac{S}{\theta} - \frac{n - S}{1 - \theta} = 0 \rightarrow \frac{S}{\theta} = \frac{n - S}{1 - \theta}$$

We isolate and solve for  $\theta$ :

$$\theta = \frac{S}{n} = \bar{X}$$

To ensure this is the maximum, we evaluate the second derivative:

$$\frac{d^2 \log(L(\theta))}{d\theta^2} = -\frac{S}{\theta^2} - \frac{n - S}{(1 - \theta)^2}$$

Since  $S \geq 0$  and  $n - S \geq 0$ , and  $\theta \in (0, 1)$ , the second derivative is always negative. This ensures that  $\theta = \bar{X}$  is a maximum.

Then, note that the parameter space is  $\Theta \equiv [0, 1/2]$ . If  $\bar{X} \leq 1/2$ , the MLE is  $\hat{\theta}_{\text{MLE}} = \bar{X}$ . If  $\bar{X} > 1/2$ , the likelihood function is decreasing for  $\theta > 1/2$ , so, similar to the method of moments estimator, we need to incorporate these conditions when giving the MLE. To that end, the MLE is given by:

$$\hat{\theta} = \min \left( \bar{X}, \frac{1}{2} \right)$$

### Problem 3

7.14, Casella & Berger

Let  $X$  and  $Y$  be independent exponential random variables, with

$$f(x|\lambda) = \frac{1}{\lambda}e^{-x/\lambda}, \quad x > 0, \quad f(y|\mu) = \frac{1}{\mu}e^{-y/\mu}, \quad y > 0.$$

We observe  $Z$  and  $W$  with

$$Z = \min(X, Y) \quad \text{and} \quad W = \begin{cases} 1 & \text{if } Z = X \\ 0 & \text{if } Z = Y. \end{cases}$$

In Exercise 4.26, the joint distribution of  $Z$  and  $W$  was obtained. Now assume that  $(Z_i, W_i), i = 1, \dots, n$ , are  $n$  iid observations. Find the MLEs of  $\lambda$  and  $\mu$ .

Hint: You may use that the joint density of  $(Z, W)$  is

$$f(z, w|\lambda, \mu) = \frac{dF(z, w)}{dz} = \begin{cases} \mu^{-1}e^{-z(\lambda+\mu^{-1})}, & z > 0, w = 0 \\ \lambda^{-1}e^{-z(\lambda+\mu^{-1})}, & z > 0, w = 1 \end{cases}$$

where

$$F(z, w|\lambda, \mu) = P(Z \leq z, W = w|\lambda, \mu).$$

Then, based on a random sample  $(Z_i, W_i), i = 1, \dots, n$  of pairs, this problem involves using calculus with two variables to find the MLE.

### Answer/Proof, Whatchumacallit

Given the above information, we define the likelihood function as:

$$L(\lambda, \mu) = \prod_{i=1}^n f(z_i, w_i|\lambda, \mu) = \prod_{i=1}^n \left(\frac{1}{\lambda}\right)^{w_i} \left(\frac{1}{\mu}\right)^{1-w_i} e^{-z_i(\frac{1}{\lambda} + \frac{1}{\mu})}$$

Define:

$$S_z = \sum_{i=1}^n Z_i, \quad S_w = \sum_{i=1}^n W_i.$$

Using  $S_z, S_w$ , we may then write our prior the likelihood function as:

$$L(\lambda, \mu) = \lambda^{-S_w} \mu^{-(n-S_w)} e^{-S_z(\lambda^{-1} + \mu^{-1})}$$

To make evaluating easier, we take the log likelihood:

$$\log(L(\lambda, \mu)) = -S_w \log \lambda - (n - S_w) \log \mu - S_z \left( \frac{1}{\lambda} + \frac{1}{\mu} \right)$$

Since we have two parameters of interest, in order to find the MLEs of  $\lambda$  and  $\mu$ , take partial derivatives and set them to zero:

For  $\lambda$

$$\frac{\partial \log(L(\lambda, \mu))}{\partial \lambda} = -\frac{S_w}{\lambda} + \frac{S_z}{\lambda^2} = 0$$

$$\frac{S_w}{\lambda} = \frac{S_z}{\lambda^2} \rightarrow \lambda = \frac{S_z}{S_w}$$

To check it is a maximum, we again check the second derivative is negative:

$$\frac{\partial^2 \log(L(\lambda, \mu))}{\partial \lambda^2} = \frac{S_w}{\lambda^2} - \frac{2S_z}{\lambda^3} = \frac{S_w}{(S_z/S_w)^2} - \frac{2S_z}{(S_z/S_w)^3}$$

Simplifying:

$$\left. \frac{\partial^2 \log(L(\lambda, \mu))}{\partial \lambda^2} \right|_{\lambda=\hat{\lambda}} = \frac{S_w^3}{S_z^2} - \frac{2S_w^3}{S_z^2} = -\frac{S_w^3}{S_z^2} < 0$$

For  $\mu$

$$\frac{\partial \log(L(\lambda, \mu))}{\partial \mu} = -\frac{n - S_w}{\mu} + \frac{S_z}{\mu^2} = 0$$

$$\frac{n - S_w}{\mu} = \frac{S_z}{\mu^2} \rightarrow \mu = \frac{S_z}{n - S_w}$$

To check it is a maximum, we again check the second derivative is negative:

$$\frac{\partial^2 \log(L(\lambda, \mu))}{\partial \mu^2} = \frac{n - S_w}{\mu^2} - \frac{2S_z}{\mu^3} = \frac{(n - S_w)}{(S_z/(n - S_w))^2} - \frac{2S_z}{(S_z/(n - S_w))^3}$$

Simplifying:

$$\left. \frac{\partial^2 \log(L(\lambda, \mu))}{\partial \mu^2} \right|_{\mu=\hat{\mu}} = \frac{(n - S_w)^3}{S_z^2} - \frac{2(n - S_w)^3}{S_z^2} = -\frac{(n - S_w)^3}{S_z^2} < 0$$

Taken together, the maximum likelihood estimators are:

$$\hat{\lambda} = \frac{\sum_{i=1}^n Z_i}{\sum_{i=1}^n W_i}$$

And

$$\hat{\mu} = \frac{\sum_{i=1}^n Z_i}{n - \sum_{i=1}^n W_i}$$

## Problem 4

7.49, Casella & Berger

Let  $X_1, \dots, X_n$  be iid exponential( $\lambda$ ).

**a)**

Find an unbiased estimator of  $\lambda$  based only on  $Y = \min\{X_1, \dots, X_n\}$ .

**b)**

Find a better estimator than the one in part (a). Prove that it is better.

**c)**

The following data are high-stress failure times (in hours) of Kevlar/epoxy spherical vessels used in a sustained pressure environment on the space shuttle:

50.1, 70.1, 137.0, 166.9, 170.5, 152.8, 80.5, 123.5, 112.6, 148.5, 160.0, 125.4.

Failure times are often modeled with the exponential distribution. Estimate the mean failure time using the estimators from parts (a) and (b).



## Problem 5

Suppose someone collects a random sample  $X_1, X_2, \dots, X_n$  from an exponential  $\beta = 1/\theta$  distribution with pdf

$$f(x|\theta) = \theta e^{-\theta x}, \quad x > 0,$$

and a parameter  $\theta > 0$ . However, due to a recording mistake, only truncated integer data  $Y_1, Y_2, \dots, Y_n$  are available for analysis, where  $Y_i$  represents the integer part of  $X_i$  after dropping all digits after the decimal place in  $X_i$ 's representation. (For example, if  $x_1 = 4.9854$  in reality, we would have only  $y_1 = 4$  available.) Then,  $Y_1, \dots, Y_n$  represent a random sample of iid (discrete) random variables with pmf

$$f(y|\theta) = P_\theta(Y_i = y) = e^{-\theta y} - e^{-\theta(1+y)}, \quad y = 0, 1, 2, 3, \dots$$

a)

Show that the likelihood equals

$$L(\theta) = \left[ e^{-\theta \bar{Y}_n} (1 - e^{-\theta}) \right]^n,$$

where  $\bar{Y}_n$  is the sample average.

Given the pmf of  $Y_i$  is:

$$f(y|\theta) = P_\theta(Y_i = y) = e^{-\theta y} - e^{-\theta(1+y)}, \quad y = 0, 1, 2, 3, \dots$$

We note that the likelihood function for the sample  $Y_1, Y_2, \dots, Y_n$  is:

$$L(\theta) = \prod_{i=1}^n f(y_i|\theta) = \prod_{i=1}^n \left( e^{-\theta y_i} - e^{-\theta(1+y_i)} \right)$$

Simplifying a bit gives us:

$$L(\theta) = \prod_{i=1}^n e^{-\theta y_i} (1 - e^{-\theta}) = (1 - e^{-\theta})^n \prod_{i=1}^n e^{-\theta y_i} = (1 - e^{-\theta})^n e^{-\theta \sum_{i=1}^n y_i}$$

By definition, the sample average,  $\bar{Y}_n$ , is defined as:

$$\bar{Y}_n = \frac{1}{n} \sum_{i=1}^n y_i \rightarrow \sum_{i=1}^n y_i = n \bar{Y}_n$$

Using the above, we can incorporate this expression into the simplified likelihood function, and have:

$$L(\theta) = (1 - e^{-\theta})^n e^{-\theta n \bar{Y}_n} = \left[ e^{-\theta \bar{Y}_n} (1 - e^{-\theta}) \right]^n$$

Achieving our desired result. Wa-hoo!

b)

If  $Y_n = \sum_{i=1}^n Y_i/n = 0$ , show that an MLE for  $\theta$  does not exist on the parameter space  $(0, \infty)$ .

(Recall:  $Y_i$  is discrete and this corresponds to a pathological MLE case mentioned in class:  $Y_1 = \dots = Y_n = 0$ . This event can happen but typically with small probability for large  $n$ .)

Note again, the likelihood function as given at the end of part a):

$$L(\theta) = \left[ e^{-\theta \bar{Y}_n} (1 - e^{-\theta}) \right]^n = \left[ e^{-\theta(0)} (1 - e^{-\theta}) \right]^n = \left[ (1 - e^{-\theta}) \right]^n$$

The goal is to find the MLE by maximizing  $L(\theta)$  over  $\theta > 0$ .

The function  $1 - e^{-\theta}$  is increasing in  $\theta$ , approaching 1 as  $\theta \rightarrow \infty$ . Since it is raised to the power  $n$ , we have:

$$\lim_{\theta \rightarrow 0} L(\theta) = (1 - 1)^n = 0$$

And

$$\lim_{\theta \rightarrow \infty} L(\theta) = (1 - 0)^n = 1$$

Since  $L(\theta)$  is strictly increasing in  $\theta$ , it attains its supremum at  $\theta \rightarrow \infty$ . An MLE must be a finite value  $\theta^*$  that maximizes  $L(\theta)$  in the domain  $(0, \infty)$ . However,  $L(\theta)$  is maximized at  $\theta \rightarrow \infty$ , meaning the supremum is not attained at any finite value. Therefore, no finite  $\theta$  maximizes  $L(\theta)$ , implying the MLE does not exist.

c)

If  $0 < \bar{Y}_n$ , show that the MLE  $\hat{\theta}$  is

$$\hat{\theta} = \log(\bar{Y}_n^{-1} + 1)$$

Given the likelihood function of part a):

$$L(\theta) = \left[ e^{-\theta \bar{Y}_n} (1 - e^{-\theta}) \right]^n$$

To find the MLE of  $\theta$ , we use our typical approach of maximizing the log-likelihood function:

$$\log(L(\theta)) = n(-\theta \bar{Y}_n + \log(1 - e^{-\theta}))$$

Differentiating and setting equal to zero gives us:

$$\frac{d\log(L(\theta))}{d\theta} = n \left( -\bar{Y}_n + \frac{e^{-\theta}}{1 - e^{-\theta}} \right) = 0 \rightarrow \frac{e^{-\theta}}{1 - e^{-\theta}} = \bar{Y}_n \rightarrow e^{-\theta} = \bar{Y}_n(1 - e^{-\theta})$$

Simplifying some more, we have:

$$e^{-\theta} + \bar{Y}_n e^{-\theta} = \bar{Y}_n \rightarrow e^{-\theta}(1 + \bar{Y}_n) = \bar{Y}_n$$

Solving for  $\theta$ , with note of the monotonic transformation given from the log function, gives us:

$$e^{-\theta} = \frac{\bar{Y}_n}{1 + \bar{Y}_n} \rightarrow -\theta = \log\left(\frac{\bar{Y}_n}{1 + \bar{Y}_n}\right) \rightarrow \theta = \log\left(\frac{1 + \bar{Y}_n}{\bar{Y}_n}\right)$$

Giving us:

$$\hat{\theta} = \log(\bar{Y}_n^{-1} + 1)$$

However, we need to do one validation! To confirm that  $\hat{\theta}$  is a maximum, we compute the second derivative of the log-likelihood function:

$$\frac{d^2 \log(L(\theta))}{d\theta^2} = n \left( \frac{-e^{-\theta}(1 - e^{-\theta}) + e^{-2\theta}}{(1 - e^{-\theta})^2} \right) = n \left( \frac{-e^{-\theta} + e^{-2\theta} - e^{-2\theta}}{(1 - e^{-\theta})^2} \right) = n \left( \frac{-e^{-\theta}}{(1 - e^{-\theta})^2} \right)$$

Since  $e^{-\theta} > 0$  and  $(1 - e^{-\theta})^2 > 0$ , it follows that:

$$\frac{d^2 \log(L(\theta))}{d\theta^2} = n \left( \frac{-e^{-\theta}}{(1 - e^{-\theta})^2} \right) < 0$$

So we have verified that the MLE we calculated is in fact a maximum. This leads us to conclude that the MLE of  $\theta$  is:

$$\hat{\theta} = \log(\bar{Y}_n^{-1} + 1)$$