HW6

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Q1

An ecologist takes data

$$(x_i, Y_i), i = 1, \ldots, n,$$

where $x_i > 0$ is the size of an area and Y_i is the number of moss plants. The data are modeled assuming $x_1, \ldots, x_n$ are fixed; $Y_1, \ldots, Y_n$ are independent; and:

$$Y_i \sim \text{Poisson}(\theta x_i)$$

with parameter θx_i. Suppose that:

$$\sum_{i=1}^{n} x_i = 5$$

is known. Find an exact form of the most powerful (MP) test of size $\alpha = 9e^{-10}$ for testing:

$$H_0 : \theta = 2 \quad \text{vs} \quad H_1 : \theta = 1.$$

Answer

Q2

Problem 8.19:

The random variable X has pdf:

$$f(x) = e^{-x}, \quad x > 0.$$

One observation is obtained on the random variable:

$$Y = X^{\theta},$$

and a test of:

$$H_0 : \theta = 1 \quad \text{versus} \quad H_1 : \theta = 2$$

needs to be constructed.

Find the UMP level $\alpha = 0.10$ test and compute the Type II Error probability.

Hint

Show that the form of the MP test involves rejecting H_0 if:

$$e^{y-\sqrt{y}}/\sqrt{y} > k$$

for some $k > 1$.

(Skip the part involving $\alpha = 0.1$ or the Type II error part.)

Answer

Q3

Problem 8.20, Casella and Berger (2nd Edition).

Let X be a random variable whose pmf under H_0 and H_1 is given by:

x	1	2	3	4	5	6	7
$f(x\|H_0)$	0.01	0.01	0.01	0.01	0.01	0.01	0.94
$f(x\|H_1)$	0.06	0.05	0.04	0.03	0.02	0.01	0.79

Use the Neyman–Pearson Lemma to find the most powerful test for H_0 versus H_1 with size:

$$\alpha = 0.04.$$

Compute the probability of Type II Error for this test.

Hint:

It holds that:

$$\frac{f(x|H_1)}{f(x|H_0)} = 7 - x + \frac{79}{94} I(x = 7)$$

over the support $x = 1, 2, \ldots, 7$, where $I(\cdot)$ denotes the indicator function.

Answer

Q4

Recall Method I for finding Uniformly Most Powerful (UMP) tests:

To find a UMP size α test for $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \notin \Theta_0$, suppose we can fix $\theta_0 \in \Theta_0$ suitably and then use the Neyman–Pearson lemma to find an MP size α test $\varphi(\tilde{X})$ for:

$$H_0 : \theta = \theta_0 \quad \text{vs} \quad H_1 : \theta = \theta_1,$$

where:

a)

$\varphi(\tilde{X})$ does not depend on $\theta_1 \notin \Theta_0$, and

b)

$\max_{\theta \in \Theta_0} E_\theta \varphi(\tilde{X}) = \alpha$.

Show that if (a) and (b) both hold, then $\varphi(\tilde{X})$ must be a UMP size α test for $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \notin \Theta_0$.

Hint:

From b), the size of the test rule $\varphi(\tilde{X})$ is correct. So, by definition of a UMP test, it is necessary to prove that if $\bar{\varphi}(\tilde{X})$ is any other test of $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \notin \Theta_0$ with size:

$$\max_{\theta \in \Theta_0} E_\theta \bar{\varphi}(\tilde{X}) \leq \alpha,$$

then $\varphi(\tilde{X})$ has more power over the parameter subspace of H_1 than $\bar{\varphi}(\tilde{X})$, i.e.,

$$E_\theta \varphi(\tilde{X}) \geq E_\theta \bar{\varphi}(\tilde{X}) \quad \text{for any } \theta \notin \Theta_0.$$

In other words, pick/fix some $\theta_1 \notin \Theta_0$ and argue that:

$$E_{\theta_1} \varphi(\tilde{X}) \geq E_{\theta_1} \bar{\varphi}(\tilde{X})$$

must hold. The way to do this is to take the test $\bar{\varphi}(\tilde{X})$ and apply it to testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$.

Problem 5

Problem 8.23, Casella and Berger (2nd Edition).

Suppose X is one observation from a population with Beta$(\theta, 1)$ pdf.

a)

For testing:

$$H_0 : \theta \leq 1 \quad \text{versus} \quad H_1 : \theta > 1,$$

find the size and sketch the power function of the test that rejects H_0 if:

$$X > \frac{1}{2}.$$

b)

Find the most powerful level-α test of:

$$H_0 : \theta = 1 \quad \text{versus} \quad H_1 : \theta = 2.$$

c)

Is there a UMP test of:

$$H_0 : \theta \leq 1 \quad \text{versus} \quad H_1 : \theta > 1?$$

If so, find it. If not, prove so.