HW8

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Q1

Let X_1 and X_2 be independent exponential random variables with mean θ.

(a)

Find the joint moment generating function of X_1 and X_2.

The **moment generating function (MGF)** of a random variable X is defined as:

$$M_X(t) = \mathbb{E}[e^{tX}]$$

For an Exponential(θ) random variable, the mean is θ, and the rate parameter $\lambda = 1/\theta$. The MGF of $X \sim \text{Exponential}(\lambda)$ is:

$$M_X(t) = \frac{\lambda}{\lambda - t}, \quad t < \lambda.$$

Joint MGF of X_1 and X_2:

Since X_1 and X_2 are independent exponential random variables, the joint MGF is the product of the individual MGFs:

$$M_{X_1,X_2}(t_1, t_2) = \mathbb{E}[e^{t_1 X_1 + t_2 X_2}]$$

Using independence:

$$M_{X_1,X_2}(t_1, t_2) = \mathbb{E}[e^{t_1 X_1}] \cdot \mathbb{E}[e^{t_2 X_2}] = M_{X_1}(t_1) \cdot M_{X_2}(t_2).$$

Each $M_{X_i}(t)$ has the same form as the MGF of an exponential random variable. Substituting $\lambda = 1/\theta$, we get:

$$M_{X_1}(t_1) = \frac{\frac{1}{\theta}}{\frac{1}{\theta} - t_1} = \frac{1}{1-\theta t_1}, \quad t_1 < \frac{1}{\theta}.$$

$$M_{X_2}(t_2) = \frac{\frac{1}{\theta}}{\frac{1}{\theta} - t_2} = \frac{1}{1-\theta t_2}, \quad t_2 < \frac{1}{\theta}.$$

Thus, the joint MGF is:

$$M_{X_1,X_2}(t_1,t_2) = \frac{1}{(1-\theta t_1)} \cdot \frac{1}{(1-\theta t_2)} = \frac{1}{(1-\theta t_1)(1-\theta t_2)}, \quad t_1, t_2 < \frac{1}{\theta}.$$

(b)

Give the definition of the moment generating function of $X_1 - X_2$ and show how this can be obtained from part (a).

Definition of the Moment Generating Function

The **moment generating function (MGF)** of a random variable X is defined as:

$$M_X(t) = \mathbb{E}[e^{tX}].$$

For the random variable $X_1 - X_2$, the MGF is given by:

$$M_{X_1-X_2}(t) = \mathbb{E}[e^{t(X_1-X_2)}].$$

Using the Joint MGF to Find $M_{X_1-X_2}(t)$

From part (a), the joint MGF of X_1 and X_2 is:

$$M_{X_1,X_2}(t_1,t_2) = \frac{1}{(1-\theta t_1)(1-\theta t_2)}, \quad t_1, t_2 < \frac{1}{\theta}.$$

To find the MGF of $X_1 - X_2$, substitute $t_1 = t$ and $t_2 = -t$ into the joint MGF, because $t(X_1 - X_2) = tX_1 - tX_2$:

$$M_{X_1-X_2}(t) = M_{X_1,X_2}(t,-t).$$

Substituting into the expression for the joint MGF:

$$M_{X_1-X_2}(t) = \frac{1}{(1-\theta t)(1-\theta(-t))}.$$

Simplify the denominator:

$$M_{X_1-X_2}(t) = \frac{1}{(1-\theta t)(1+\theta t)}.$$

Expand the product in the denominator:

$$M_{X_1-X_2}(t) = \frac{1}{1-(\theta t)^2}, \quad |t| < \frac{1}{\theta}.$$

Final Result

The moment generating function of $X_1 - X_2$ is:

$$M_{X_1-X_2}(t) = \frac{1}{1-(\theta t)^2}, \quad |t| < \frac{1}{\theta}.$$

(c)

Find the distribution of $Y = X_1 - X_2$. Using the mgf, one can find that this is a so-called Laplace or double-exponential distribution.

Finding the Distribution of $Y = X_1 - X_2$

To find the distribution of $Y = X_1 - X_2$, we use the moment generating function (MGF) obtained in part (b):

$$M_Y(t) = \frac{1}{1-(\theta t)^2}, \quad |t| < \frac{1}{\theta}.$$

Step 1: Recognizing the MGF of the Laplace Distribution The MGF of a Laplace (double-exponential) random variable Y with location parameter μ and scale parameter b is:

$$M_Y(t) = \frac{1}{1-b^2t^2}, \quad |t| < \frac{1}{b}.$$

By comparing this with the MGF derived above, we identify that $b = \theta$ and $\mu = 0$. Therefore, Y follows a **Laplace distribution** with location parameter $\mu = 0$ and scale parameter $b = \theta$.

Step 2: The Probability Density Function (PDF) of the Laplace Distribution The probability density function of a Laplace random variable Y with parameters $\mu = 0$ and $b = \theta$ is:

$$f_Y(y) = \frac{1}{2\theta}\exp\left(-\frac{|y|}{\theta}\right), \quad y \in \mathbb{R}.$$

Thus, the distribution of $Y = X_1 - X_2$ is:

$$f_Y(y) = \frac{1}{2\theta}\exp\left(-\frac{|y|}{\theta}\right), \quad y \in \mathbb{R}.$$

Step 3: Interpretation This result confirms that the difference of two independent exponential random variables (with the same mean) follows a Laplace distribution centered at 0, with scale parameter equal to the mean of the exponential distribution. This distribution is often called a **double-exponential distribution** because it has exponential decay in both positive and negative directions.

Q2: 4.30, Casella & Berger

Suppose the distribution of Y, conditional on $X = x$, is $N(x, x^2)$ and that the marginal distribution of X is uniform $(0, 1)$.

(a)

Find $E[Y]$, $\text{Var}[Y]$, and $\text{Cov}(X, Y)$.

To find $E[Y]$, $\text{Var}[Y]$, and $\text{Cov}(X, Y)$, we use the law of total expectation and total variance.

Step 1: Conditional Distribution of $Y|X = x$

We are given that $Y|X = x \sim N(x, x^2)$, so: - The conditional mean is $\mathbb{E}[Y|X = x] = x$. - The conditional variance is $\text{Var}(Y|X = x) = x^2$.

(a) $E[Y]$: Using the Law of Total Expectation

The law of total expectation states:

$$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]].$$

From the conditional mean, $\mathbb{E}[Y|X = x] = x$. Thus:

$$\mathbb{E}[Y] = \mathbb{E}[X].$$

Since $X \sim \text{Uniform}(0, 1)$, we know:

$$\mathbb{E}[X] = \frac{1}{2}.$$

Therefore:

$$\mathbb{E}[Y] = \frac{1}{2}.$$

(b) Var$[Y]$: Using the Law of Total Variance

The law of total variance states:

$$\text{Var}(Y) = \mathbb{E}[\text{Var}(Y|X)] + \text{Var}(\mathbb{E}[Y|X]).$$

- From the conditional variance, $\text{Var}(Y|X = x) = x^2$. Thus:

$$\mathbb{E}[\mathrm{Var}(Y|X)] = \mathbb{E}[x^2].$$

Since $X \sim \text{Uniform}(0,1)$, $\mathbb{E}[x^2]$ is the second moment of a uniform distribution:

$$\mathbb{E}[x^2] = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}.$$

- From the conditional mean, $\mathbb{E}[Y|X = x] = x$, so:

$$\mathrm{Var}(\mathbb{E}[Y|X]) = \mathrm{Var}(X).$$

For $X \sim \text{Uniform}(0,1)$, $\mathrm{Var}(X)$ is:

$$\mathrm{Var}(X) = \frac{1}{12}.$$

Substitute these results into the law of total variance:

$$\mathrm{Var}(Y) = \mathbb{E}[x^2] + \mathrm{Var}(X) = \frac{1}{3} + \frac{1}{12}.$$

Simplify:

$$\mathrm{Var}(Y) = \frac{4}{12} + \frac{1}{12} = \frac{5}{12}.$$

(c) Cov(X, Y): Using the Covariance Definition

The covariance is given by:

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

From $\mathbb{E}[Y|X = x] = x$, we have:

$$\mathbb{E}[XY] = \mathbb{E}[X \cdot \mathbb{E}[Y|X]] = \mathbb{E}[X^2].$$

For $X \sim \text{Uniform}(0,1)$, $\mathbb{E}[X^2] = \frac{1}{3}$. Also, we know $\mathbb{E}[X] = \frac{1}{2}$ and $\mathbb{E}[Y] = \frac{1}{2}$. Thus:

$$\mathrm{Cov}(X,Y) = \mathbb{E}[X^2] - \mathbb{E}[X]\mathbb{E}[Y] = \frac{1}{3} - \frac{1}{2} \cdot \frac{1}{2}.$$

Simplify:

$$\mathrm{Cov}(X,Y) = \frac{1}{3} - \frac{1}{4} = \frac{4}{12} - \frac{3}{12} = \frac{1}{12}.$$

Final Results

1. $\mathbb{E}[Y] = \frac{1}{2}$,
2. $\text{Var}(Y) = \frac{5}{12}$,
3. $\text{Cov}(X, Y) = \frac{1}{12}$.

(b)

Prove that $\frac{Y}{X}$ and X are independent.

To prove that $\frac{Y}{X}$ and X are independent, we need to show that the joint probability density function (PDF) of $(\frac{Y}{X}, X)$ can be written as the product of the marginal PDFs of $\frac{Y}{X}$ and X.

Step 1: Joint PDF of X and Y

The marginal distribution of X is uniform $(0, 1)$, so its PDF is:

$$f_X(x) = \begin{cases} 1, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

The conditional distribution of $Y|X = x$ is $N(x, x^2)$, so the conditional PDF is:

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi x^2}} \exp\left(-\frac{(y-x)^2}{2x^2}\right), \quad -\infty < y < \infty.$$

The joint PDF of X and Y is:

$$f_{X,Y}(x, y) = f_{Y|X}(y|x) f_X(x).$$

Substituting $f_X(x) = 1$ for $0 < x < 1$:

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{\sqrt{2\pi x^2}} \exp\left(-\frac{(y-x)^2}{2x^2}\right), & 0 < x < 1, -\infty < y < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Step 2: Transformation to New Variables

Define $Z = \frac{Y}{X}$ and $X = X$. Then:

$$Y = Z \cdot X.$$

The Jacobian of the transformation is:

$$\begin{vmatrix} \frac{\partial Y}{\partial Z} & \frac{\partial Y}{\partial X} \\ \frac{\partial X}{\partial Z} & \frac{\partial X}{\partial X} \end{vmatrix} = \begin{vmatrix} X & Z \\ 0 & 1 \end{vmatrix} = X.$$

Thus, the joint PDF of (Z, X) is:

$$f_{Z,X}(z, x) = f_{X,Y}(x, y) \cdot |\text{Jacobian}| = f_{X,Y}(x, z \cdot x) \cdot x.$$

Substitute $Y = z \cdot x$ into $f_{X,Y}(x, y)$:

$$f_{X,Y}(x, y) = \frac{1}{\sqrt{2\pi x^2}} \exp\left(-\frac{(y - x)^2}{2x^2}\right),$$

so:

$$f_{Z,X}(z, x) = \frac{1}{\sqrt{2\pi x^2}} \exp\left(-\frac{(z \cdot x - x)^2}{2x^2}\right) \cdot x.$$

Simplify $z \cdot x - x = x(z - 1)$:

$$f_{Z,X}(z, x) = \frac{x}{\sqrt{2\pi x^2}} \exp\left(-\frac{(x(z - 1))^2}{2x^2}\right).$$

Cancel x terms in the denominator:

$$f_{Z,X}(z, x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(z - 1)^2}{2}\right) \cdot f_X(x).$$

Since $f_X(x) = 1$ for $0 < x < 1$, we have:

$$f_{Z,X}(z, x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(z - 1)^2}{2}\right) \cdot 1.$$

Step 3: Marginal PDFs

- The marginal PDF of $Z = \frac{Y}{X}$ is:

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(z - 1)^2}{2}\right), \quad -\infty < z < \infty.$$

- The marginal PDF of X is:

$$f_X(x) = 1, \quad 0 < x < 1.$$

Step 4: Independence

The joint PDF $f_{Z,X}(z, x)$ factors as:

$$f_{Z,X}(z, x) = f_Z(z) \cdot f_X(x).$$

Since the joint PDF is the product of the marginal PDFs, $Z = \frac{Y}{X}$ and X are independent.

Conclusion

$\frac{Y}{X}$ and X are independent.

Q3: 4.54, Casella & Berger

Find the pdf of $\prod_{i=1}^{n} X_i$, where the X_i's are independent uniform $(0,1)$ random variables.

(Hint: Try to calculate the cdf, and remember the relationship between uniforms and exponentials.)

To find the PDF of $W = \prod_{i=1}^{n} X_i$, where the X_i's are independent Uniform$(0,1)$ random variables, we proceed as follows:

Step 1: Understanding the Problem

Each $X_i \sim \text{Uniform}(0,1)$, meaning:

$$f_{X_i}(x) = \begin{cases} 1, & 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

We define $W = \prod_{i=1}^{n} X_i$, which maps the product of the X_i's into a single random variable W on $(0,1)$.

Step 2: CDF of W

To find the PDF of W, we first compute its CDF, $F_W(w) = P(W \leq w)$, and then differentiate.

(a) Start with the definition of the CDF:

$$F_W(w) = P(W \leq w) = P\left(\prod_{i=1}^{n} X_i \leq w\right).$$

(b) Transform the inequality: Take the logarithm on both sides (recall that ln is monotonic and $\ln(w) < 0$ for $w \in (0,1)$):

$$P\left(\prod_{i=1}^{n} X_i \leq w\right) = P\left(\sum_{i=1}^{n} \ln(X_i) \leq \ln(w)\right).$$

Let $Y_i = -\ln(X_i)$. Each Y_i is an **Exponential(1)** random variable because if $X_i \sim \text{Uniform}(0,1)$, then $-\ln(X_i) \sim \text{Exponential}(1)$. Thus, the sum $S = \sum_{i=1}^{n} Y_i$ follows a **Gamma distribution** with shape parameter n and rate parameter 1, denoted $S \sim \text{Gamma}(n, 1)$.

The PDF of S is:

$$f_S(s) = \frac{s^{n-1} e^{-s}}{\Gamma(n)}, \quad s > 0.$$

Thus:

$$F_W(w) = P\left(\sum_{i=1}^{n} Y_i \leq -\ln(w)\right) = P(S \leq -\ln(w)).$$

Step 3: Compute the PDF of W

The CDF of W is:

$$F_W(w) = \int_0^{-\ln(w)} \frac{s^{n-1}e^{-s}}{\Gamma(n)}\, ds, \quad 0 < w \leq 1.$$

The PDF of W is the derivative of the CDF:

$$f_W(w) = \frac{d}{dw}\left[\int_0^{-\ln(w)} \frac{s^{n-1}e^{-s}}{\Gamma(n)}\, ds\right].$$

Using the fundamental theorem of calculus and the chain rule, we get:

$$f_W(w) = \frac{1}{w} \cdot \frac{(-\ln(w))^{n-1}e^{-\ln(w)}}{\Gamma(n)}, \quad 0 < w \leq 1.$$

Simplify $e^{-\ln(w)} = \frac{1}{w}$:

$$f_W(w) = \frac{(-\ln(w))^{n-1}}{\Gamma(n)w^n}, \quad 0 < w \leq 1.$$

Final Result

The PDF of $W = \prod_{i=1}^n X_i$ is:

$$f_W(w) = \frac{(-\ln(w))^{n-1}}{\Gamma(n)w^n}, \quad 0 < w \leq 1.$$

Q4: 4.47, Casella & Berger

(Marginal normality does not imply bivariate normality.)

Let X and Y be independent $N(0,1)$ random variables, and define a new random variable Z by

$$Z = \begin{cases} X & \text{if } XY > 0, \\ -X & \text{if } XY < 0. \end{cases}$$

(a)

Show that Z has a normal distribution.

(b)

Show that the joint distribution of Z and Y is not bivariate normal. *(Hint: Show that*

$$Z$$

and

$$Y$$

always have the same sign.)

Q5: 4.52, Casella & Berger

Bullets are fired at the origin of an (x, y) coordinate system, and the point hit, say (X, Y), is a random variable. The variables X and Y are taken to be independent $N(0, 1)$ random variables. If two bullets are fired independently, what is the distribution of the distance between them?

Problem Overview

Two bullets are fired independently at points (X_1, Y_1) and (X_2, Y_2), where $X_1, Y_1, X_2, Y_2 \sim N(0, 1)$ are independent standard normal random variables. The distance between the two points is given by:

$$R = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2}.$$

We aim to find the distribution of R.

Step 1: Distribution of $X_2 - X_1$ and $Y_2 - Y_1$

Since $X_1, X_2 \sim N(0, 1)$ are independent, the difference $X_2 - X_1$ is also normally distributed:

$$X_2 - X_1 \sim N(0, 2).$$

Similarly, $Y_2 - Y_1 \sim N(0, 2)$.

Step 2: Distribution of the Squared Distance R^2

The squared distance is:

$$R^2 = (X_2 - X_1)^2 + (Y_2 - Y_1)^2.$$

Let $Z_1 = X_2 - X_1$ and $Z_2 = Y_2 - Y_1$. Then $Z_1, Z_2 \sim N(0, 2)$, and they are independent. The squared terms are:

$$Z_1^2 \sim \text{Scaled-Chi-Square}(1, \sigma^2 = 2), \quad Z_2^2 \sim \text{Scaled-Chi-Square}(1, \sigma^2 = 2).$$

For a standard normal variable $Z \sim N(0, 1)$, $Z^2 \sim \chi^2(1)$. Scaling by $\sigma^2 = 2$, Z_1^2 and Z_2^2 are scaled $\chi^2(1)$:

$$Z_1^2 \sim 2 \cdot \chi^2(1), \quad Z_2^2 \sim 2 \cdot \chi^2(1).$$

Since $Z_1^2 + Z_2^2$ is the sum of two independent scaled $\chi^2(1)$ variables, it follows that:

$$Z_1^2 + Z_2^2 \sim 2 \cdot \chi^2(2).$$

A $\chi^2(2)$ distribution is equivalent to an Exponential(1) distribution. Scaling by 2, we have:

$$R^2 \sim \text{Exponential}\left(\frac{1}{2}\right).$$

Step 3: Distribution of R

The random variable $R = \sqrt{R^2}$ is the square root of an Exponential $\left(\frac{1}{2}\right)$ random variable. The PDF of $R^2 \sim$ Exponential $\left(\frac{1}{2}\right)$ is:

$$f_{R^2}(r^2) = \frac{1}{2} e^{-r^2/2}, \quad r^2 \geq 0.$$

To find the PDF of R, we apply the change of variables $R^2 = r^2$ with $R = \sqrt{r^2}$, giving:

$$f_R(r) = f_{R^2}(r^2) \cdot \left| \frac{d(r^2)}{dr} \right| = \frac{1}{2} e^{-r^2/2} \cdot 2r = re^{-r^2/2}, \quad r \geq 0.$$

Final Result

The distance R between the two bullets follows a **Rayleigh distribution** with scale parameter $\sigma = \sqrt{2}$:

$$f_R(r) = re^{-r^2/2}, \quad r \geq 0.$$

Q6: 4.55, Casella & Berger

A **parallel system** is one that functions as long as at least one component of it functions.

A particular parallel system is composed of three independent components, each of which has a lifetime with an exponential (λ) distribution. The lifetime of the system is the maximum of the individual lifetimes.

What is the distribution of the lifetime of the system?

Problem Overview

Let the lifetimes of the three components be X_1, X_2, X_3, where each $X_i \sim \text{Exponential}(\lambda)$, and the lifetimes are independent. The lifetime of the parallel system is the maximum of the individual lifetimes:

$$T = \max(X_1, X_2, X_3).$$

We aim to find the distribution of T.

Step 1: CDF of T

To determine the distribution of T, we first compute its **CDF**, $F_T(t)$, defined as:

$$F_T(t) = P(T \leq t).$$

The maximum $T \leq t$ if and only if all the individual lifetimes satisfy $X_1 \leq t$, $X_2 \leq t$, and $X_3 \leq t$. Because the X_i are independent, the joint probability is the product of the individual probabilities:

$$P(T \leq t) = P(X_1 \leq t) \cdot P(X_2 \leq t) \cdot P(X_3 \leq t).$$

For an exponential random variable $X \sim \text{Exponential}(\lambda)$, the CDF is:

$$P(X \leq t) = 1 - e^{-\lambda t}, \quad t \geq 0.$$

Thus, the CDF of T becomes:

$$F_T(t) = [P(X_1 \leq t)] \cdot [P(X_2 \leq t)] \cdot [P(X_3 \leq t)] = [1 - e^{-\lambda t}]^3, \quad t \geq 0.$$

Step 2: PDF of T

To find the PDF of T, we differentiate the CDF:

$$f_T(t) = \frac{d}{dt} F_T(t) = \frac{d}{dt}[1 - e^{-\lambda t}]^3.$$

Using the chain rule:

$$f_T(t) = 3[1 - e^{-\lambda t}]^2 \cdot \frac{d}{dt}[1 - e^{-\lambda t}].$$

The derivative of $1 - e^{-\lambda t}$ is:

$$\frac{d}{dt}[1 - e^{-\lambda t}] = \lambda e^{-\lambda t}.$$

Substitute this into the expression for $f_T(t)$:

$$f_T(t) = 3[1 - e^{-\lambda t}]^2 \cdot \lambda e^{-\lambda t}, \quad t \geq 0.$$

Final Result

The lifetime of the parallel system $T = \max(X_1, X_2, X_3)$ has the PDF:

$$f_T(t) = 3\lambda[1 - e^{-\lambda t}]^2 e^{-\lambda t}, \quad t \geq 0.$$

Q7: 4.28, Casella & Berger

Let X and Y be independent standard normal random variables.

(a)

Show that $\frac{X}{X+Y}$ has a Cauchy distribution.

(b)

Find the distribution of $\frac{X}{|Y|}$.

(c)

Is the answer to part (b) surprising? Can you formulate a general theorem?

Q8: 4.50, Casella & Berger

If (X, Y) has the bivariate normal probability density function (pdf):

$$f(x, y) = \frac{1}{2\pi(1-\rho^2)^{1/2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(x^2 - 2\rho xy + y^2\right)\right),$$

show that

$$\mathrm{Corr}(X, Y) = \rho$$

and

$$\mathrm{Corr}(X^2, Y^2) = \rho^2.$$

Hint: Conditional expectations will simplify calculations.