

HW5

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Outline

- Q1: SOLID
- Q2: SOLID
- Q3: Iffy
- Q4: Iffy

1.

In the attached article by Prof. M. Ghosh, read pages 509-512 (including example 1), examples 4-6 of Section 3, and Section 5.2 up to and including Examples 17-18. (This is sort of a technical article, so to read a bit of this material is not easy. Also, Example 17 should look like an example from class regarding Basu's theorem.)

In example 18, show that T is a complete and sufficient statistic, while U is an ancillary statistic.

Example 18.

Let X_1, \dots, X_n ($n \geq 2$) be iid with common Weibull pdf

$$f_\theta(x) = \exp(-x^p/\theta)(p/\theta)x^{p-1}; \quad 0 < x < \infty, \quad 0 < \theta < \infty,$$

$p(>0)$ being known. In this case, $T = \sum_{i=1}^n X_i^p$ is complete sufficient for θ , while $U = X_1^p/T$ is ancillary. Also, since X_1^p, \dots, X_n^p are iid exponential with scale parameter θ , $U \sim \text{Beta}(1, n-1)$. Hence, the UMVUE of $P_\theta(X_1 \leq x) = P_\theta(X_1^p \leq x^p)$ is given by

$$k(T) = \begin{cases} 1 - x^{np}/T^n & \text{if } T > x^p, \\ 1 & \text{if } T \leq x^p. \end{cases}$$

Answer

By definition, a statistic T is sufficient if the joint pdf of X_1, \dots, X_n can be factorized into the form:

$$f_\theta(x_1, \dots, x_n) = g(T, \theta)h(x_1, \dots, x_n)$$

where:

$g(T, \theta)$ depends on θ ,

$h(x_1, \dots, x_n)$ does not depend on θ .

Given X_1, \dots, X_n ($n \geq 2$) are iid with Common Weibull pdf, the joint pdf of X_1, \dots, X_n is:

$$f_{\theta}(x_1, \dots, x_n) = \prod_{i=1}^n \left[\exp(-x_i^p/\theta) \cdot \frac{p}{\theta} x_i^{p-1} \right]$$

Where:

$$0 < x_i < \infty \quad \text{and} \quad 0 < \theta < \infty \quad \forall i$$

We can simplify this expression somewhat:

$$f_{\theta}(x_1, \dots, x_n) = \left(\frac{p}{\theta}\right)^n \exp\left(-\frac{T}{\theta}\right) \prod_{i=1}^n x_i^{p-1}$$

Of note:

- The function $g(T, \theta) = \left(\frac{p}{\theta}\right)^n \exp\left(-\frac{T}{\theta}\right)$ depends on T and θ .
- The function $h(x_1, \dots, x_n) = \prod_{i=1}^n x_i^{p-1}$ does not depend on θ at all.

Thus, by the Factorization Theorem, T is sufficient for θ .

We then need to address completeness:

By definition, a statistic T is complete if for any function $g(T)$:

$$E_{\theta}[g(T)] = 0, \quad \forall \theta \quad \Rightarrow \quad P(g(T) = 0) = 1$$

That is, if the expectation of $g(T)$ is zero for all θ , then $g(T)$ must be the zero function.

Since X_1^p, \dots, X_n^p are iid Exponential(θ), following from rescaling the original Weibull-distributed X_i 's, we have the sum:

$$T = \sum_{i=1}^n X_i^p$$

follows a Gamma distribution:

$$T \sim \text{Gamma}(n, \theta)$$

Yes, your statement is appropriate, but it can be slightly refined for clarity and precision. Here's a suggested revision:

We take advantage of the fact that the Gamma family is a complete exponential family in θ , which implies that T is a complete statistic for θ . This follows from the general result that a one-parameter exponential family is complete in θ , and the Gamma distribution is a specific instance of such a family for the scale parameter θ (we cannot find anything with expectation zero for all θ that is not the zero function itself).

Thus, T is both sufficient and complete for θ .

Finally, we address the ancillary statistic. By definition, a statistic U is ancillary if its distribution does not depend on θ .

We are given a hint to try:

$$U = \frac{X_1^p}{T}$$

Since X_1^p, \dots, X_n^p are iid $\text{Exponential}(\theta)$, again following from the initial X_i 's being iid Weibull. At any rate, we can again rescale:

$$\left(\frac{X_1^p}{\theta}, \dots, \frac{X_n^p}{\theta} \right) \sim \text{Exp}(1)$$

While still retaining iid.

Taking the sum, we can express this as another sum of Exponential iid random variables, giving us:

$$T/\theta \sim \text{Gamma}(n, 1)$$

Since:

$$U = \frac{X_1^p}{T} = \frac{X_1^p/\theta}{T/\theta}$$

By multiplying by a “cheeky one”.

We then note that $\frac{X_1^p}{\theta} \sim \text{Gamma}(1, 1)$ and $T/\theta \sim \text{Gamma}(n, 1)$, we know that U is a ratio of two Gamma distributions that are independent is by definition a Beta distribution (the numerator and denominator being independent).

Specifically, U is distributed:

$$U \sim \text{Beta}(1, n-1)$$

Since the $\text{Beta}(1, n-1)$ distribution does not depend on θ , we know the statistic U is ancillary.

Lil Extra Work

I'm pretty sure the above is “enough” (avoiding using the word sufficient explicitly in a non-maths context), but in the event that some more work would help:

For finding the distribution of U :

$$U = \frac{X_1^p}{T} = \frac{X_1^p/\theta}{T/\theta}$$

$$f_{X_1^p/\theta}(x) = \frac{x^{1-1}e^{-x}}{\Gamma(1)} = e^{-x}$$

$$f_{T/\theta}(t) = \frac{t^{n-1}e^{-t}}{\Gamma(n)}$$

Giving joint pdf:

$$f_{X_1^p/\theta, T/\theta}(x, t) = f_{X_1^p/\theta}(x)f_{T/\theta}(t) = e^{-x} \cdot \frac{t^{n-1}e^{-t}}{\Gamma(n)}$$

For:

$$U = \frac{X_1^p}{T}$$

Gives:

$$X_1^p = UT$$

$$T = T$$

With Jacobian:

$$J = \begin{vmatrix} \frac{\partial X_1^p}{\partial U} & \frac{\partial X_1^p}{\partial T} \\ \frac{\partial U}{\partial U} & \frac{\partial U}{\partial T} \end{vmatrix} = \begin{vmatrix} T & U \\ 0 & 1 \end{vmatrix} = T$$

Transformation of the prior joint pdf gives:

$$f_{U,T}(u, t) = f_{X_1^p, T}(ut, t) \cdot |J| = e^{-ut} \cdot \frac{t^{n-1}e^{-t}}{\Gamma(n)} \cdot T = \frac{t^n e^{-t}}{\Gamma(n)} e^{-ut}$$

Getting the marginal distribution of U , integrating over T , we have:

$$f_U(u) = \int_0^\infty f_{U,T}(u, t) dt = \int_0^\infty \frac{t^n e^{-t}}{\Gamma(n)} e^{-ut} dt = \frac{1}{\Gamma(n)} \int_0^\infty t^n e^{-(1+u)t} dt = \frac{\Gamma(n+1)}{\Gamma(n)} \cdot \frac{1}{(1+u)^{n+1}}$$

As $\Gamma(n+1) = n\Gamma(n)$, we can simplify further:

$$f_U(u) = n \frac{1}{(1+u)^{n+1}} = \frac{u^{1-1}(1-u)^{(n-1)-1}}{B(1, n-1)}$$

Where B is the Beta function as is usually defined, and $u^{1-1} = u^0 = 1$ under the constraint $0 < u < 1$ as a “cheeky one”.

This confirms:

$$U \sim \text{Beta}(1, n-1)$$

2.

Problem 7.60, Casella and Berger and the following:

Extra

Let X_1, \dots, X_n be iid $\text{gamma}(\alpha, \beta)$ with α known. Find the best unbiased estimator of $1/\beta$.

Answer

Since $X_i \sim \text{Gamma}(\alpha, \beta)$ are iid, the sum:

$$S_n = \sum_{i=1}^n X_i \sim \text{Gamma}(n\alpha, \beta)$$

Taking expectation of this statistic:

$$E_\beta(S_n) = n\alpha\beta$$

To get just the β term, consider:

$$E_\beta \left[\frac{n\alpha}{S_n} \right] = \frac{n\alpha}{E_\beta(S_n)} = \frac{n\alpha}{n\alpha\beta} = \frac{1}{\beta}$$

This shows that $\frac{n\alpha}{S_n}$ is an unbiased estimator of $\frac{1}{\beta}$

Noting the work shown previously in [.1], via the Factorization Theorem, we know S_n is a sufficient statistic for β (scale parameter of the Gamma).

Similarly, we know that the Gamma family is a specific instance of a complete one-parameter exponential family, meaning we know that S_n is also complete.

Now, we can do something new! Via Lehmann-Scheffé, since $\delta(S_n) = \frac{n\alpha}{S_n}$ is an unbiased function of the complete sufficient statistic, S_n , we know that $\delta(S_n) = \frac{n\alpha}{S_n}$ is the UMVUE of $1/\beta$. Yippee!

a)

Let $S_n = \sum_{i=1}^n X_i$. Using Basu's theorem, show X_1/S_n and S_n are independent.

Answer

By definition, Basu's theorem: If T is a complete sufficient statistic and U is an ancillary statistic, then T and U are independent.

From the prior question, we know that S_n is complete and sufficient for β .

We need to then find an ancillary statistic.

To that end, let:

$$U = \frac{X_1}{S_n}$$

Where:

$$X_1 \sim \text{Gamma}(\alpha, \beta)$$

$$S_n \sim \text{Gamma}(n\alpha, \beta)$$

Using given information, we know U is a ratio of two Gamma random variables. However, this is complicated somewhat by X_1 and S_n not being independent! So we need to do a bit of calculation to identify the underlying structure of their ratios (though it will be Beta-distributed, the parameter values don't follow the typical formula). To that end, using the known pdfs of each statistic:

$$f_{X_1}(x_1) = \frac{x_1^{\alpha-1} e^{-x_1/\beta}}{\beta^\alpha \Gamma(\alpha)}$$

$$f_{S_n}(s) = \frac{s^{n\alpha-1} e^{-s/\beta}}{\beta^{n\alpha} \Gamma(n\alpha)}$$

By the product rule (noting X_1 and S_n are not independent):

$$f_{X_1, S_n}(x_1, s) = f_{X_1|S_n}(x_1|s) f_{S_n}(s)$$

Where:

$$f_{X_1|S_n}(x_1|s) = \frac{x_1^{\alpha-1} (s - x_1)^{(n-1)\alpha-1}}{s^{n\alpha-1} B(\alpha, (n-1)\alpha)}$$

Giving joint pdf:

$$f_{X_1, S_n}(x_1, s) = \frac{x_1^{\alpha-1} (s - x_1)^{(n-1)\alpha-1} e^{-s/\beta} s^{n\alpha-1}}{\beta^{n\alpha} \Gamma(\alpha) \Gamma((n-1)\alpha)}$$

For

$$U = \frac{X_1}{S_n}$$

We have:

$$X_1 = US_n$$

$$S = (1 - U)S_n$$

Calculating the Jacobian determinant:

$$J = \begin{vmatrix} \frac{\partial X_1}{\partial U} & \frac{\partial X_1}{\partial S_n} \\ \frac{\partial S}{\partial U} & \frac{\partial S}{\partial S_n} \end{vmatrix} = \begin{vmatrix} S_n & U \\ -S_n & 1 - U \end{vmatrix} = S_n(1 - U) + S_n U = S_n$$

Via transformation, we have:

$$f_{U, S_n}(u, s) = f_{X_1, S_n}(us, s) |J| = \frac{(us)^{\alpha-1} ((1-u)s)^{(n-1)\alpha-1} e^{-s/\beta} s^{n\alpha-1}}{\beta^{n\alpha} \Gamma(\alpha) \Gamma((n-1)\alpha)} s = \frac{u^{\alpha-1} (1-u)^{(n-1)\alpha-1} s^{n\alpha-1} e^{-s/\beta}}{\beta^{n\alpha} \Gamma(\alpha) \Gamma((n-1)\alpha)}$$

Getting the the marginal distribution of $f_U(u)$:

$$f_U(u) = \int_0^\infty f_{U, S_n}(u, s) ds = \frac{u^{\alpha-1} (1-u)^{(n-1)\alpha-1} \Gamma(n\alpha)}{\Gamma(\alpha) \Gamma((n-1)\alpha)}$$

Thus, we can identify the distribution and parameters from the above! We know that U is Beta-distributed, specifically:

$$U \sim \text{Beta}(\alpha, (n-1)\alpha)$$

Which does not depend on β for any of its parameters! This means we have an ancillary statistic.

As such, by Basu's theorem, $U = X_1/S_n$ and S_n are independent.

b)

Using the result in a) and $E_\theta(S_n) = n\alpha\beta$, find $E_\theta(X_1/S_n)$.

Answer

Using the results in a):

$$E_\theta \left(\frac{X_1}{S_n} \right) = E_\theta(U)$$

Where:

$$U \sim \text{Beta}(\alpha, (n-1)\alpha)$$

Using the properties of a known distribution, we know that:

$$E_\theta \left(\frac{X_1}{S_n} \right) = \frac{\alpha}{\alpha + (n-1)\alpha} = \frac{1}{n}$$

3.

Problem 8.13(a)-(c), Casella and Berger (2nd Edition) and, in place of Problem 8.13(d), consider the following test:

Let X_1, X_2 be iid uniform($\theta, \theta + 1$). For testing $H_0 : \theta = 0$ versus $H_1 : \theta > 0$, we have two competing tests:

$$\phi_1(X_1) : \text{Reject } H_0 \text{ if } X_1 > 0.95,$$

$$\phi_2(X_1, X_2) : \text{Reject } H_0 \text{ if } X_1 + X_2 > C.$$

a)

Find the value of C so that ϕ_2 has the same size as ϕ_1 .

Answer

The size of ϕ_1 is:

$$\alpha_1 = P(X_1 > 0.95 \mid \theta = 0) = 0.05$$

The size of ϕ_2 is:

$$\alpha_2 = P(X_1 + X_2 > C \mid \theta = 0)$$

For $1 \leq C \leq 2$, the probability $P(X_1 + X_2 > C \mid \theta = 0)$ is:

$$\alpha_2 = \int_{1-C}^1 \int_{C-x_1}^1 1 \, dx_2 \, dx_1 = \frac{(2-C)^2}{2}$$

For $\alpha_2 = \alpha_1 = 0.05$ and solving for C :

$$\frac{(2-C)^2}{2} = 0.05 \implies (2-C)^2 = 0.1 \implies C = 2 - \sqrt{0.1} \approx 1.68$$

b)

Calculate the power function of each test. Draw a well-labeled graph of each power function.

Answer

The power function of ϕ_1 is:

$$\beta_1(\theta) = P_\theta(X_1 > 0.95) = \begin{cases} 0 & \text{if } \theta \leq -0.05, \\ \theta + 0.05 & \text{if } -0.05 < \theta \leq 0.95, \\ 1 & \text{if } \theta > 0.95. \end{cases}$$

The distribution of $Y = X_1 + X_2$ is:

$$f_Y(y | \theta) = \begin{cases} y - 2\theta & \text{if } 2\theta \leq y < 2\theta + 1, \\ 2\theta + 2 - y & \text{if } 2\theta + 1 \leq y < 2\theta + 2, \\ 0 & \text{otherwise.} \end{cases}$$

The power function of ϕ_2 is:

$$\beta_2(\theta) = P_\theta(Y > C) = \begin{cases} 0 & \text{if } \theta \leq \frac{C}{2} - 1, \\ \frac{(2\theta + 2 - C)^2}{2} & \text{if } \frac{C}{2} - 1 < \theta \leq \frac{C-1}{2}, \\ 1 - \frac{(C - 2\theta)^2}{2} & \text{if } \frac{C-1}{2} < \theta \leq \frac{C}{2}, \\ 1 & \text{if } \theta > \frac{C}{2}. \end{cases}$$

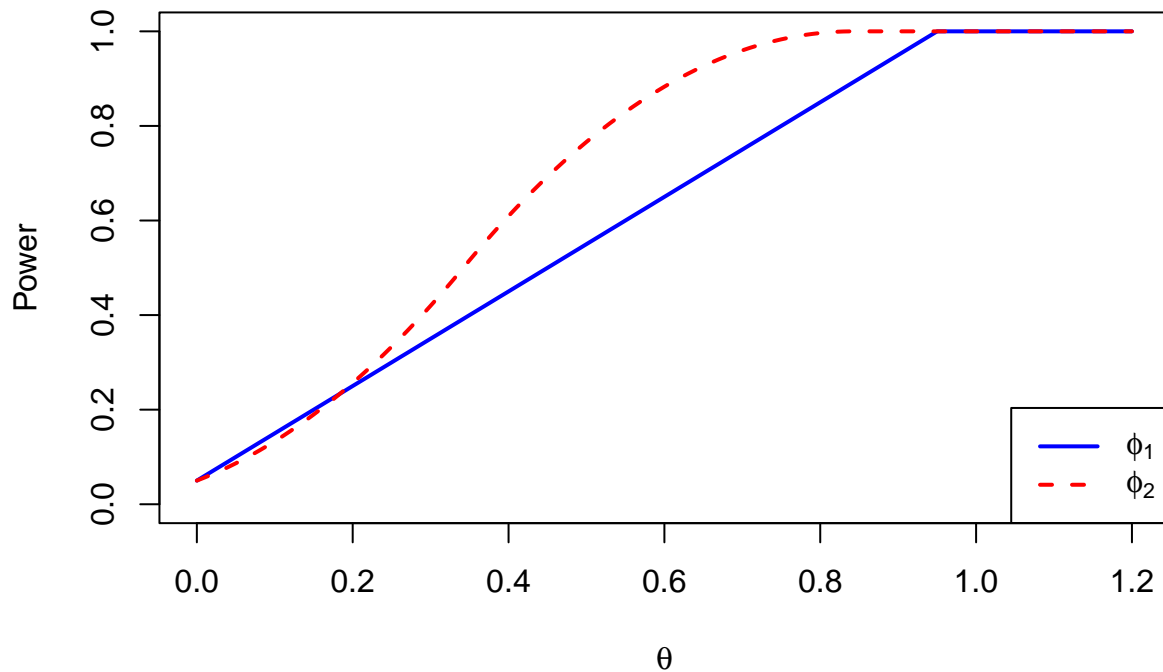
```
theta <- seq(0, 1.2, by = 0.01)
C <- 2 - sqrt(0.1) # Computed value of C

# Power function for phi_1
beta1 <- pmax(0, pmin(1, theta + 0.05))

# Power function for phi_2
beta2 <- ifelse(theta <= (C/2) - 1, 0,
  ifelse(theta <= (C - 1)/2, ((2*theta + 2 - C)^2)/2,
    ifelse(theta <= C/2, 1 - ((C - 2*theta)^2)/2, 1)))

# Plot
plot(theta, beta1, type = "l", col = "blue", lwd = 2, ylim = c(0, 1),
  ylab = "Power", xlab = expression(theta), main = "Power Functions of Phi1 and Phi2")
lines(theta, beta2, col = "red", lwd = 2, lty = 2)
legend("bottomright", legend = c(expression(phi[1]), expression(phi[2])),
  col = c("blue", "red"), lty = c(1, 2), lwd = 2)
```

Power Functions of Phi1 and Phi2



c)

Prove or disprove: ϕ_2 is a more powerful test than ϕ_1 .

Answer

From the graph above, ϕ_1 is more powerful for θ near 0, but ϕ_2 is more powerful for larger values of θ .

To be a more powerful test, you must be uniformly more powerful than the reference test. We do not meet this condition, meaning ϕ_2 is not a more powerful test than ϕ_1 .

Extra

$$\phi_3(X_1, X_2) = \begin{cases} 1 & \text{if } X_{(1)} > 1 - \sqrt{0.05} \text{ or } X_{(2)} > 1 \\ 0 & \text{otherwise} \end{cases}$$

where $X_{(1)}, X_{(2)}$ are the min, max.

Find the size of this test and the power function for $\theta > 0$. Then, graph the power functions of ϕ_3 and ϕ_2 to determine which test is more powerful. (It's enough to graph over the range $\theta \in [0, 1.2]$.)

Answer

Define the test:

$$\phi_3(X_1, X_2) = \begin{cases} 1 & \text{if } X_{(1)} > 1 - \sqrt{0.05} \text{ or } X_{(2)} > 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $X_{(1)}, X_{(2)}$ are the minimum and maximum of X_1, X_2 , respectively.

Under $H_0 : \theta = 0$, the size of ϕ_3 is:

$$\alpha_3 = P(X_{(1)} > 1 - \sqrt{0.05} \mid \theta = 0) = (1 - (1 - \sqrt{0.05}))^2 = 0.05.$$

The power function of ϕ_3 is:

$$\beta_3(\theta) = P_\theta(X_{(1)} > 1 - \sqrt{0.05}) = (1 - (1 - \sqrt{0.05} - \theta))^2.$$

The power functions of ϕ_2 and ϕ_3 are compared below. The results indicate that ϕ_2 is more powerful for small values of θ , while ϕ_3 is more powerful when $\theta > 0.395$.

```
# Define the range of theta values
theta <- seq(0, 1.2, by = 0.01)

# Define the critical value for phi_3
t_crit <- 1 - sqrt(0.05)

# Corrected power function for phi_3
phi3_power <- function(theta) {
  ifelse(theta < (1 - sqrt(0.05)), (sqrt(0.05) + theta)^2, 1)
}

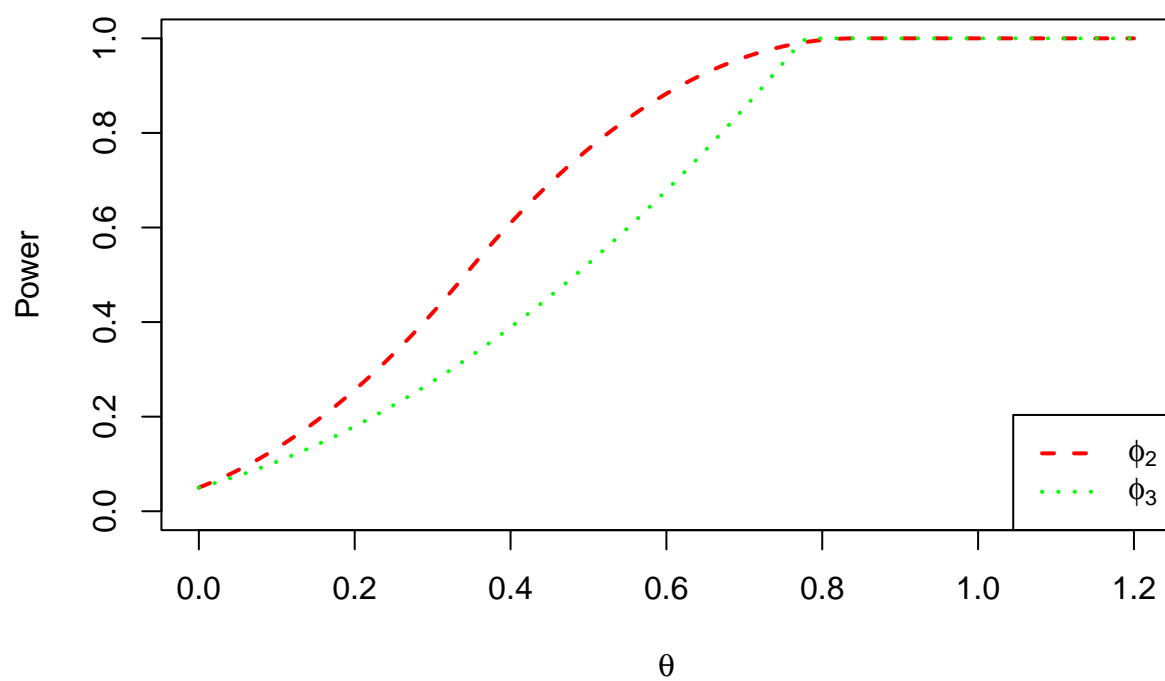
# Compute the power for phi_3
beta3 <- sapply(theta, phi3_power)

# Corrected power function for phi_2
C <- 2 - sqrt(0.1) # The critical value used for phi_2
beta2 <- ifelse(theta <= (C/2) - 1, 0,
  ifelse(theta <= (C - 1)/2, ((2*theta + 2 - C)^2)/2,
  ifelse(theta <= C/2, 1 - ((C - 2*theta)^2)/2, 1)))

# Plot the corrected power functions
plot(theta, beta2, type = "l", col = "red", lwd = 2, ylim = c(0, 1),
  ylab = "Power", xlab = expression(theta), main = "Power Functions of Phi2 and Phi3",
  lty = 2)
lines(theta, beta3, col = "green", lwd = 2, lty = 3)

# Add a legend
legend("bottomright", legend = c(expression(phi[2]), expression(phi[3])),
  col = c("red", "green"), lty = c(2, 3), lwd = 2)
```

Power Functions of Phi2 and Phi3



4.

Problem 8.15, Casella and Berger (2nd Edition), though you can just assume the form given is most powerful (no need to show).

Show that for a random sample X_1, \dots, X_n from a $\mathcal{N}(0, \sigma^2)$ population, the most powerful test of $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma = \sigma_1$, where $\sigma_0 < \sigma_1$, is given by

$$\phi\left(\sum X_i^2\right) = \begin{cases} 1 & \text{if } \sum X_i^2 > c, \\ 0 & \text{if } \sum X_i^2 \leq c. \end{cases}$$

For a given value of α , the size of the Type I Error, show how the value of c is explicitly determined.

Answer

From the Neyman-Pearson lemma, the most powerful (UMP) test rejects H_0 if the likelihood ratio exceeds a threshold k .

The likelihood ratio is:

$$\frac{f(x \mid \sigma_1)}{f(x \mid \sigma_0)} = \frac{(2\pi\sigma_1^2)^{-n/2} e^{-\sum_i x_i^2/(2\sigma_1^2)}}{(2\pi\sigma_0^2)^{-n/2} e^{-\sum_i x_i^2/(2\sigma_0^2)}} = \left(\frac{\sigma_0}{\sigma_1}\right)^n \exp\left\{\frac{1}{2} \sum_i x_i^2 \left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)\right\} > k$$

This simplifies to:

$$\sum_i x_i^2 > \frac{2 \log(k(\sigma_1/\sigma_0)^n)}{\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)} = c$$

where c is a constant.

Noting that: $\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2} > 0$ (since $\sigma_0 < \sigma_1$).

Thus, the UMP test rejects H_0 if:

$$\sum_i X_i^2 > c$$

The critical value c is determined such that the Type I error probability is α :

$$\alpha = P_{\sigma_0} \left(\sum_i X_i^2 > c \right)$$

Under H_0 , $\sum_i X_i^2/\sigma_0^2$ follows a chi-squared distribution with n degrees of freedom:

$$\sum_i X_i^2/\sigma_0^2 \sim \chi_n^2$$

Thus, the probability can be rewritten as:

$$\alpha = P_{\sigma_0} \left(\sum_i X_i^2/\sigma_0^2 > c/\sigma_0^2 \right) = P(\chi_n^2 > c/\sigma_0^2)$$

We then solve for c , taking the $(1 - \alpha)$ -quantile of the χ_n^2 distribution:

$$c = \sigma_0^2 \cdot \chi_{n,1-\alpha}^2$$

where $\chi_{n,1-\alpha}^2$ is the $(1 - \alpha)$ -quantile of the χ_n^2 distribution.