Stat 5100 Assignment 1

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Problem 3

Let $\mathbf{A}$ be an $m \times m$ idempotent matrix. Show that:

a) $\underset{m\times m}{\mathbf{I}} - \mathbf{A}$ is idempotent.

Note, by the definition of idempotent:

$$\mathbf{AA} = \mathbf{A}$$

Let $\mathbf{B} = \mathbf{I} - \mathbf{A}$. Then:

$$\mathbf{BB} = (\mathbf{I} - \mathbf{A})^2 = \mathbf{B}^2 = \mathbf{I}^2 - 2\mathbf{IA} + \mathbf{A}$$

Note the identity matrix, $\mathbf{I}$, is also idempotent, such that we may simplify, noting our initial assumption of $\mathbf{A}$ is idempotent:

$$(\mathbf{I} - \mathbf{A})(\mathbf{I} - \mathbf{A}) = \mathbf{BB} = \mathbf{I} - 2\mathbf{A} + \mathbf{A} = \mathbf{I} - \mathbf{A}$$

And we conclude that $\mathbf{I} - \mathbf{A}$ is idempotent.

b) $\mathbf{BAB}^{-1}$ is idempotent, where $\mathbf{B}$ is any $m \times m$ nonsingular matrix.

To prove idempotence, we must show:

$$(\mathbf{BAB}^{-1})(\mathbf{BAB}^{-1}) = \mathbf{BAB}^{-1}$$

We start by assuming that the matrices $\mathbf{A}$ and $\mathbf{B}$ are compatible matrices.

Noting associativity of matrix multiplication, we have:

$$(\mathbf{BAB}^{-1})(\mathbf{BAB}^{-1}) = \mathbf{BA}(\mathbf{B}^{-1}\mathbf{B})\mathbf{AB}^{-1}$$

By the definition of an inverse matrix, and given our assumption that $\mathbf{B}$ is a nonsingular matrix, $\mathbf{B}^{-1}\mathbf{B} = \mathbf{I}$:

$$(\mathbf{BAB}^{-1})(\mathbf{BAB}^{-1}) = \mathbf{BA}(\mathbf{I})\mathbf{AB}^{-1} = \mathbf{BAAB}^{-1}$$

Then with note of $\mathbf{A}$ being idempotent, we have:

$$(\mathbf{BAB}^{-1})(\mathbf{BAB}^{-1}) = \mathbf{BAB}^{-1}$$

And we conclude that $\mathbf{BAB}^{-1}$ is idempotent.

Problem 4

A matrix $\mathbf{A}$ is symmetric if $\mathbf{A} = \mathbf{A}^\top$. Determine the truth of the following statements:

a) If $\mathbf{A}$ and $\mathbf{B}$ are symmetric, then their product $\mathbf{AB}$ is symmetric.

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \text{ and } \mathbf{B} = \begin{bmatrix} 4 & 5 \\ 5 & 6 \end{bmatrix}.$$

Note, both $\mathbf{A}$ and $\mathbf{B}$ are symmetric.

Now, consider $\mathbf{AB}$:

$$\mathbf{AB} = \begin{bmatrix} 1(4) + 2(5) & 1(5) + 2(6) \\ 2(4) + 3(5) & 2(5) + 3(6) \end{bmatrix} = \begin{bmatrix} 14 & 17 \\ 23 & 28 \end{bmatrix}$$

Notably, the transpose of $\mathbf{AB}$ is:

$$(\mathbf{AB})^\top = \begin{bmatrix} 14 & 23 \\ 17 & 28 \end{bmatrix}.$$

Such that $\mathbf{AB} \neq (\mathbf{AB})^\top$. So $\mathbf{AB}$ is not symmetric.

This counterexample shows the statement in question is false.

b) If $\mathbf{A}$ is not symmetric, then $\mathbf{A}^{-1}$ is not symmetric.

Given the definition of an inverse, we have:

$$\mathbf{AA}^{-1} = \mathbf{I}$$

From the property of transposes, we then may write:

$$(\mathbf{AA}^{-1})^\top = \mathbf{I}^\top$$

Assuming conformal for post-multiplication, we may write this:

$$(\mathbf{A}^{-1})^\top(\mathbf{A}^\top) = \mathbf{I}$$

This implies that:

$$(\mathbf{A}^\top)^{-1} = (\mathbf{A}^{-1})^\top$$

Which we will then turn to investigate. To that end,

Let us consider: If $\mathbf{A}^{-1}$ were symmetric, then clearly:

$$\mathbf{A}^{-1} = (\mathbf{A}^{-1})^\top$$

However, if we assume that $\mathbf{A}$ is **not** symmetric, which means $\mathbf{A} \neq \mathbf{A}^\top$, then it would still follow from the above relation that:

$$(\mathbf{A}^\top)^{-1} = \mathbf{A}^{-1}$$

If we then apply the inverse (or take the inverse of both sides) of the above relation, with note that $(\mathbf{A}^{-1})^{-1} = A$, we would then have:

$$\mathbf{A} = \mathbf{A}^\top$$

However, this would be a contradiction! This means that if $\mathbf{A}$ is not symmetric, then $\mathbf{A}^{-1}$ cannot be symmetric. This means that the statement is true.

c) When $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are symmetric, the transpose of $\mathbf{ABC}$ is $\mathbf{CBA}$.

Using the transpose property:

$$(\mathbf{ABC})^\top = \mathbf{C}^\top \mathbf{B}^\top \mathbf{A}^\top$$

Let $\mathbf{D} = \mathbf{AB}$, such that we may write the above as:

$$(\mathbf{ABC})^\top = (\mathbf{DC})^\top$$

Then via our typical matrix arithmetic of transposes, we have:

$$(\mathbf{DC})^\top = \mathbf{C}^\top \mathbf{D}^\top$$

Simplifying further we have:

Since $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are symmetric, this simplifies to:

$$(\mathbf{ABC})^\top = \mathbf{C}^\top (\mathbf{AB})^\top = \mathbf{C}^\top \mathbf{B}^\top \mathbf{A}^\top$$

However, as the matrices are all respectively symmetric, we then have:

$$(\mathbf{ABC})^\top = \mathbf{C}^\top \mathbf{B}^\top \mathbf{A}^\top = \mathbf{CBA}$$

And the original statement is indeed true.

Section Break

If $\mathbf{A} = \mathbf{A}^\top$ and $\mathbf{B} = \mathbf{B}^\top$, which of these matrices are certainly symmetric?

Again, for each of the following we will assume necessarily that all matrices involved are compatible for the purposes of matrix multiplication.

d) $\mathbf{A}^2 - \mathbf{B}^2$:

Note the properties of summing/subtracting two matrices, and the property that $\mathbf{A}$ and $\mathbf{B}$ being symmetric implies their square (multiplied by itself) is also symmetric:

$$(\mathbf{A}^2 - \mathbf{B}^2)^\top = (\mathbf{A}^2)^\top - (\mathbf{B}^2)^\top = \mathbf{A}^2 - \mathbf{B}^2$$

So we conclude that this matrix is certainly symmetric.

e) $\mathbf{ABA}$:

With note of the results of the above problem, part c), we may simplify this as:

$$(\mathbf{ABA})^\top = \mathbf{A}^\top \mathbf{B}^\top \mathbf{A}^\top = \mathbf{ABA}$$

And with note of the symmetry of matrices $\mathbf{A}$ and $\mathbf{B}$,

we conclude that this matrix is certainly symmetric.

f) $\mathbf{ABAB}$:

Again with note of the results of the above problem, part c), we may extend these results and write:

$$(\mathbf{ABAB})^\top = \mathbf{B}^\top \mathbf{A}^\top \mathbf{B}^\top \mathbf{A}^\top = \mathbf{BABA}$$

However, to say that

$$(\mathbf{ABAB})^\top = \mathbf{BABA} = \mathbf{ABAB}$$

and conclude this matrix is certainly symmetric, we would require that the matrices $\mathbf{A}$ and $\mathbf{B}$ are commutative, which we do not have a guarantee of. So we cannot conclude this matrix is certainly symmetric.

g) $(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B})$:

$$(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B}) = \mathbf{A}^2 + \mathbf{BA} - \mathbf{AB} + \mathbf{B}^2$$

And:

$$((\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B}))^\top = \left(\mathbf{A}^2 + \mathbf{BA} - \mathbf{AB} + \mathbf{B}^2\right)^\top = (\mathbf{A}^2)^\top + (\mathbf{BA})^\top - (\mathbf{AB})^\top + (\mathbf{B}^2)^\top$$

However, to say that:

$$\mathbf{A}^2 + \mathbf{BA} - \mathbf{AB} + \mathbf{B}^2 = (\mathbf{A}^2)^\top + (\mathbf{BA})^\top - (\mathbf{AB})^\top + (\mathbf{B}^2)^\top$$

Which is to say:

$$(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B}) = ((\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B}))^\top$$

and conclude this matrix is certainly symmetric, we would require that the matrices $\mathbf{A}$ and $\mathbf{B}$ are commutative, such that $\mathbf{AB} = \mathbf{BA} \rightarrow (\mathbf{AB})^\top = (\mathbf{BA})^\top$

However, we do not have a guarantee or presumption of commutivity, so we cannot conclude this matrix is certainly symmetric.

Problem 5

Consider the matrix

$$\mathbf{X} = \begin{bmatrix} 1 & -3 & 0 & -3 \\ 1 & -2 & -1 & 2 \\ 2 & -5 & -1 & -1 \end{bmatrix}$$

a) Show that the columns of **X** are linearly dependent.

To prove linear dependence, we must find some $\mathbf{a} \in \mathbb{R}^4$ that satisfies the following relation:

$$\mathbf{Xa} = \sum_{i=1}^{4} a_i \mathbf{x}_i = 0$$

where a_i is the i-th element of **a**.

We have the following system of equations:

$$\begin{cases} a_1 1 + a_2(-3) + a_3(0) + a_4(-3) = 0, \\ a_1 1 + a_2(-2) + a_3(-1) + a_4 2 = 0, \\ a_1 2 + a_2(-5) + a_3(-1) + a_4(-1) = 0 \end{cases}$$

Solving this system yields:

$$a_1 = -12t + 3s,\ a_2 = -5t + s,\ a_3 = s,\ \text{and } a_4 = t$$

where $s, t \in \mathbb{R}$ (some real-valued scalars).

Then, for the above, if we set $s = 0, t = 1$,

the associated solution for **a** is:

$$\mathbf{a} = \begin{bmatrix} -12 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

Which we may write as:

$$-12\mathbf{x}_1 - 5\mathbf{x}_2 + 0\mathbf{x}_3 + 1\mathbf{x}_4 = \mathbf{0}$$

However, there are many possible solutions. For example we could have had $s = 1, t = 0$ and had another valid **a**. As such we know that **X** is linearly dependent.

Additional Note

If we use part b), then we know the matrix $\mathbf{X}$ does not have full rank, and as such is linearly dependent. This is the easiest answer, but I didn't know if we could/should presume it given the question followed below.

b) Find the rank of $\mathbf{X}$.

Via row reduction of $\mathbf{X}$, it follows:

$$\mathbf{X} = \begin{bmatrix} 1 & -3 & 0 & -3 \\ 1 & -2 & -1 & 2 \\ 2 & -5 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 0 & -3 \\ 0 & 1 & -1 & 5 \\ 0 & 1 & -1 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -3 & 12 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since rank is the maximum number of linearly independent rows or columns of the matrix $\mathbf{X}$, is follows that the rank of $\mathbf{X}$ is 2.

c) Use the generalized inverse algorithm in Slide Set 1 to find a generalized inverse of $\mathbf{X}$.

(1): Find any $n \times n$ nonsingular submatrix of $\mathbf{X}$, where $n = \text{rank}(\mathbf{X}) = 2$ and call if $\mathbf{W}$.

$$W = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix}$$

To verify $\mathbf{W}$ is nonsingular, I calculated:

$det(\mathbf{W}) = 1$, which is nonsingular (not zero).

(2): Invert and transpose $\mathbf{W}$, i.e. compute $(W^{-1})^\top$:

$$W^{-1} = \begin{bmatrix} -2 & 3 \\ -1 & 1 \end{bmatrix}$$

$$(W^{-1})^\top = \begin{bmatrix} -2 & -1 \\ 3 & 1 \end{bmatrix}$$

(3): Replace the elements of W in $\mathbf{X}$ with the corresponding elements of $(W^{-1})^\top$. Then:

$$\mathbf{X} = \begin{bmatrix} -2 & -1 & 0 & -3 \\ 3 & 1 & -1 & 2 \\ 2 & -5 & -1 & -1 \end{bmatrix}$$

(4): Replace all other elements in $\mathbf{X}$ with zeros:

$$\mathbf{X} = \begin{bmatrix} -2 & -1 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(5): Transpose the matrix to obtain $\mathbf{G}$, a generalized inverse of $\mathbf{X}$:

$$\mathbf{G} = \begin{bmatrix} -2 & 3 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Quick Validation

The matrix $\mathbf{G}$ is a generalized inverse if it satisfies the relation:

$$\mathbf{XGX} = \mathbf{X}$$

Using

$$\mathbf{G} = \begin{bmatrix} -2 & 3 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

And

$$\mathbf{X} = \begin{bmatrix} 1 & -3 & 0 & -3 \\ 1 & -2 & -1 & 2 \\ 2 & -5 & -1 & -1 \end{bmatrix}$$

To start that verification, we have:

$$\mathbf{GX} = \begin{bmatrix} -2 & 3 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 & -3 \\ 1 & -2 & -1 & 2 \\ 2 & -5 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 & 12 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Followed by:

$$\mathbf{XGX} = \begin{bmatrix} 1 & -3 & 0 & -3 \\ 1 & -2 & -1 & 2 \\ 2 & -5 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 & 12 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 & -3 \\ 1 & -2 & -1 & 2 \\ 2 & -5 & -1 & -1 \end{bmatrix} = \mathbf{X}$$

So a large collective sigh of relief was heard, and it is verified that this is an appropriate generalized inverse!

d) Use the R function `ginv` in the `MASS` package to find a generalized inverse of $\mathbf{X}$.

- To load the `MASS` package into your R workspace, use the command `library(MASS)`.
- If the `MASS` package is not already installed, use `install.packages("MASS")` to install it.

$$\mathbf{X} = \begin{bmatrix} 1 & -3 & 0 & -3 \\ 1 & -2 & -1 & 2 \\ 2 & -5 & -1 & -1 \end{bmatrix}$$

```r
library(MASS)
X <- matrix(c(1,1,2,
              -3,-2,-5,
              0,-1,-1,
              -3,2,-1), ncol = 4)
massX <- MASS::ginv(X)
X
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    1   -3    0   -3
## [2,]    1   -2   -1    2
## [3,]    2   -5   -1   -1
```

```
massX
```

```
##             [,1]        [,2]        [,3]
## [1,]  0.00000000  0.04761905  0.04761905
## [2,] -0.03703704 -0.07407407 -0.11111111
## [3,]  0.03703704 -0.06878307 -0.03174603
## [4,] -0.18518519  0.20105820  0.01587302
```

e) Provide one matrix $\mathbf{X}^*$ that satisfies both of the following characteristics:
   - $\mathbf{X}^*$ has full-column rank.
   - $\mathbf{X}^*$ has column space equal to the column space of $\mathbf{X}$.

Note: The rank of $\mathbf{X}$ is 2.

Since $\mathbf{x}_1$ and $\mathbf{x}_3$ are linearly independent, and $\mathbf{x}_2$ and $\mathbf{x}_4$ can be generated by linear combinations of $\mathbf{x}_1$ and $\mathbf{x}_3$, we have:

$$C([\mathbf{x}_1, \mathbf{x}_3]) = C([\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4])$$

For:

$$\mathbf{X} = \begin{bmatrix} 1 & -3 & 0 & -3 \\ 1 & -2 & -1 & 2 \\ 2 & -5 & -1 & -1 \end{bmatrix}$$

We can construct (one of many possible) solutions, such as:

$$\mathbf{X}^* = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 2 & -1 \end{bmatrix}$$

Any column of $\mathbf{X}^*$ can be written as a linear combination of the columns of $\mathbf{X}$, and any column of $\mathbf{X}$ can be written as a linear combination of the columns of $\mathbf{X}^*$, meaning:

$\mathbf{X}^*$ has full-column rank.

Furthermore, we have:

$$C(\mathbf{X}) = C([\mathbf{x}_1, \mathbf{x}_3]) = \{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \} = C(\mathbf{X}^*)$$

So we have in effect shown that the following holds by construction: - $\mathbf{X}^*$ has full-column rank. - $\mathbf{X}^*$ has column space equal to the column space of $\mathbf{X}$.

Note:

$$\mathbf{X}^* = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 2 & -1 \end{bmatrix}$$

is one of many possible solutions. Other solutions could be obtained by multiplying $\mathbf{X}^*$ by any nonsingular 2×2 matrix.

Problem 6

Prove the following result:

Suppose the set of $m \times 1$ vectors $\mathbf{x}_1, \dots, \mathbf{x}_n$ is a basis for the vector space $\mathcal{S}$. Then any vector $\mathbf{x} \in \mathcal{S}$ has a unique representation as a linear combination of the vectors $\mathbf{x}_1, \dots, \mathbf{x}_n$.

Since $\mathbf{x}_1, \dots, \mathbf{x}_n$ is a basis for $\mathcal{S}$, we know:

(1): The vectors $\mathbf{x}_1, \dots, \mathbf{x}_n$ are linearly independent.

(2): The span of $\mathbf{x}_1, \dots, \mathbf{x}_n$ equals $\mathcal{S}$, written:

$$\mathcal{S} = \text{span}\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$$

Bearing the above in mind, let $\mathbf{x} \in \mathcal{S}$.

By definition, $\mathbf{x}$ can be written as a linear combination of $\mathbf{x}_1, \dots, \mathbf{x}_n$ (the vector space generated by $\mathbf{x}_1, \dots, \mathbf{x}_n$):

$$\mathbf{x} = \sum_{i=1}^{n} c_i \mathbf{x}_i$$

For some $c_1, ..., c_n \in \mathbb{R}$.

Suppose there exists another representation of $\mathbf{x}$:

$$\mathbf{x} = \sum_{i=1}^{n} d_i \mathbf{x}_i$$

For some $d_1, ..., d_n \in \mathbb{R}$.

Then by subtracting the two, we have:

$$\sum_{i=1}^{n} (c_i \mathbf{x}_i) - (d_i \mathbf{x}_i) = \sum_{i=1}^{n} (c_i - d_i) \mathbf{x}_i = \mathbf{x} - \mathbf{x} = \mathbf{0}$$

However, as $\mathbf{x}_1, \dots, \mathbf{x}_n$ are linearly independent of one another, the only solution to this equation is:

$$(c_i - d_i) = 0, \forall i$$

Which is to say, $\forall i$, $c_i - d_i$, implying uniqueness.

Therefore, the representation of $\mathbf{x}$ as a linear combination of $\mathbf{x}_1, \dots, \mathbf{x}_n$ is unique.

Problem 7

Am I a vector space? (The basic question here is whether every linear combination is in the space. If there is no zero, then I'm for sure not a vector space.)

a) All vectors in $\mathbb{R}^n$ whose entries sum to 0.

Let $\mathbf{v} \in \mathbb{R}^n$ satisfy $\sum_{i=1}^n v_i = 0$, and let $\mathbf{w} \in \mathbb{R}^n$ satisfy $\sum_{i=1}^n w_i = 0$.

We then consider a linear combination:

$$\mathbf{u} = a\mathbf{v} + b\mathbf{w}$$

where $a, b \in \mathbb{R}$ (some real-valued scalars).

It follows then, that:

$$\sum_{i=1}^n u_i = \sum_{i=1}^n (av_i + bw_i) = a\sum_{i=1}^n v_i + b\sum_{i=1}^n w_i = a(0) + b(0) = 0$$

Thus, $\mathbf{u} \in \mathbb{R}^n$ also satisfies $\sum_{i=1}^n u_i = 0$, so the set is closed under linear combinations, and this set is a vector space (as the set of all vectors in $\mathbb{R}^n$ whose entries sum to 0 is a vector space).

Additionally, the zero vector $\mathbf{0} \in \mathbb{R}^n$ also satisfies $\sum_{i=1}^n 0 = 0$, so the set contains the zero vector.

b) All matrices in $\mathbb{R}^{m\times n}$ whose entries, when squared, sum to 1.

One (of a number) of properties of a vector space is that it must contain the zero vector. In this instance, the vector space must contain the zero matrix: $\mathbf{0} \in \mathbb{R}^{m\times n}$.

Consider the zero matrix $\mathbf{0}$. Its entries are all zero, so is the sum of their squares, i.e.:

$$\sum_{i=1}^m \sum_{j=1}^n 0^2 = 0$$

However, as defined, we are considering all matrices in $\mathbb{R}^{m\times n}$ whose entries, when squared, sum to 1, i.e.:

$$\sum_{i=1}^m \sum_{j=1}^n A_{ij}^2 = 1$$

Since $0 \neq 1$, the zero matrix $\mathbf{0}$ is not in the set. Consequently, the set fails to satisfy one of the fundamental axioms of a vector space and is as a result not a vector space.

Problem 8

Let $\mathbf{A}$ represent any $m \times n$ matrix, and let $\mathbf{B}$ represent any $n \times q$ matrix. Prove that for any choices of generalized inverses $\mathbf{A}^-$ and $\mathbf{B}^-$, $\mathbf{B}^-\mathbf{A}^-$ is a generalized inverse of $\mathbf{AB}$ if and only if $\mathbf{A}^-\mathbf{ABB}^-$ is idempotent.

Structure of Proof: Iff $\iff$ means we must provide proof of both directions of the argument. To that end:

Direction 1

generalized inverse $\rightarrow$ idempotent

Let us then assume that $\mathbf{B}^-\mathbf{A}^-$ is a generalized inverse of $\mathbf{AB}$.

Generally, a matrix $\mathbf{C}$ is a generalized inverse of $\mathbf{D}$ if:

$$\mathbf{DCD} = \mathbf{D}$$

By definition then, we may write:

$$\mathbf{AB}(\mathbf{B}^-\mathbf{A}^-)\mathbf{AB} = \mathbf{AB}$$

We may then consider that:

$$\mathbf{AB}(\mathbf{B}^-\mathbf{A}^-)\mathbf{AB} = \mathbf{AB} = \mathbf{A}(\mathbf{BB}^-)(\mathbf{A}^-\mathbf{A})\mathbf{B} = \mathbf{AB}$$

Multiplying terms on both sides of the equation above gives us:

$$(\mathbf{A}^-\mathbf{ABB}^-)(\mathbf{A}^-\mathbf{ABB}^-) = \mathbf{A}^-\mathbf{ABB}^-$$

Such that we may conclude that $\mathbf{A}^-\mathbf{ABB}^-$ is idempotent.

Direction 2

idempotent $\rightarrow$ generalized inverse

We start by assuming that $\mathbf{A}^-\mathbf{ABB}^-$ is idempotent.

By definition, this means:

$$(\mathbf{A}^-\mathbf{ABB}^-)(\mathbf{A}^-\mathbf{ABB}^-) = \mathbf{A}^-\mathbf{ABB}^-$$

The objective is to show that:

$$\mathbf{AB}(\mathbf{B}^-\mathbf{A}^-)\mathbf{AB} = \mathbf{AB}$$

To that end, consider the expression:

$$\mathbf{AB}(\mathbf{B}^-\mathbf{A}^-)\mathbf{AB}$$

Via associativity of matrix multiplication, we write this expression as:

$$\mathbf{AB}(\mathbf{B}^-\mathbf{A}^-)\mathbf{AB} = \mathbf{A}(\mathbf{BB}^-\mathbf{A}^-\mathbf{A})\mathbf{B}$$

Key Aside

My proof relies upon the expression $\mathbf{B}\mathbf{B}^-\mathbf{A}^-\mathbf{A} = \mathbf{A}^-\mathbf{A}\mathbf{B}\mathbf{B}^-$ being valid.

To show this, we start with our base assumption that $\mathbf{A}^-\mathbf{A}\mathbf{B}\mathbf{B}^-$ is idempotent, meaning:

$$(\mathbf{A}^-\mathbf{A}\mathbf{B}\mathbf{B}^-)(\mathbf{A}^-\mathbf{A}\mathbf{B}\mathbf{B}^-) = \mathbf{A}^-\mathbf{A}\mathbf{B}\mathbf{B}^-$$

Noting new here, yet, but necessarily this means that:

$$\mathbf{B}\mathbf{B}^-\mathbf{A}^-\mathbf{A}\mathbf{B}\mathbf{B}^- = \mathbf{B}\mathbf{B}^-$$

So, by multiplying both sides (on the left) by $\mathbf{A}^-\mathbf{A}$, we then have:

$$\mathbf{A}^-\mathbf{A}\mathbf{B}\mathbf{B}^-\mathbf{A}^-\mathbf{A}\mathbf{B}\mathbf{B}^- = \mathbf{A}^-\mathbf{A}\mathbf{B}\mathbf{B}^-$$

Of note, this means that there is commutivity, specifically that $\mathbf{A}^-\mathbf{A}\mathbf{B}\mathbf{B}^-$ commutes with $\mathbf{B}\mathbf{B}^-\mathbf{A}^-\mathbf{A}$.

It then follows that:

$$\mathbf{B}\mathbf{B}^-\mathbf{A}^-\mathbf{A} = \mathbf{A}^-\mathbf{A}\mathbf{B}\mathbf{B}^-$$

Ok, back to the proof

Substituting this back into the initial proof, we have:

$$\mathbf{A}(\mathbf{B}\mathbf{B}^-\mathbf{A}^-\mathbf{A})\mathbf{B} = \mathbf{A}(\mathbf{A}^-\mathbf{A}\mathbf{B}\mathbf{B}^-)\mathbf{B}$$

Simplifying further:

$$\mathbf{A}(\mathbf{A}^-\mathbf{A}\mathbf{B}\mathbf{B}^-)\mathbf{B} = (\mathbf{A}\mathbf{A}^-\mathbf{A})(\mathbf{B}\mathbf{B}^-\mathbf{B}) = \mathbf{A}\mathbf{B}$$

So, in summary we have shown:

$$\mathbf{A}\mathbf{B}(\mathbf{B}^-\mathbf{A}^-)\mathbf{A}\mathbf{B} = \mathbf{A}\mathbf{B}$$

Which proves that $\mathbf{A}^-\mathbf{A}\mathbf{B}\mathbf{B}^-$ is idempotent $\rightarrow$ $\mathbf{B}^-\mathbf{A}^-$ is a generalized inverse of $\mathbf{A}\mathbf{B}$.

Conclusion

Taken together, having shown the proof works for both directions, we conclude: for any $\mathbf{A}^-$ and $\mathbf{B}^-$, $\mathbf{B}^-\mathbf{A}^-$ is a generalized inverse of $\mathbf{A}\mathbf{B}$ if and only if $\mathbf{A}^-\mathbf{A}\mathbf{B}\mathbf{B}^-$ is idempotent.