HW4

2024-09-29

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Homework 4

Due October 13

Q1

> Question: 3.6 (a), (b) Casella & Berger

A large number of insects are expected to be attracted to a certain variety of rose plant. A commercial insecticide is advertised as being 99% effective. Suppose 2,000 insects infest a rose garden where the insecticide has been applied and let $X =$ number of surviving insects.

> (a)

What probability distribution might provide a reasonable model for this experiment?

> (b)

Write down, but do not evaluate, an expression for the probability that fewer than 100 insects survive, suing the model in part (a)

> Answer > (a)

We may interpret X as the number of "failures" given an effective rate of 99%, or $p = 1 - 0.99 = 0.01$ (1% chance of failure). As we are counting the number of failures, we have a dicrete random variable. We know $n = 2,000$, or our total number of "trials" for the insecticide.

Taken together, we have X as a Binomial distributed random variable, or:

$X \sim Binomial(n = 2,000, p = 0.01)$

It's worth noting that we can also represent this as a Poisson distributed random variable with parameter $\lambda = np = 2000(0.01) = 20$

(b)

$P(X < 100) = P(X \leq 99)$

$\sum_{x=0}^{99} P(X = x) = \sum_{x=0}^{99} f(x) = \sum_{x=0}^{99} \binom{n}{x}(0.01)^x(0.99)^{2000-k}$

Q2

Question: 3.13 (a) Casella & Berger

A truncated discrete distribution is one in which a particular class cannot be observed and is eliminated from the sample space. In particular, if X has range 0, 1, 2, ... and the 0 class cannot be observed (as is usually the case), the 0-truncated random variable X_T has pmf:

$$P(X_T = x) = \frac{P(X = x)}{P(X > 0)}$$

for $x = 1, 2, ...$

Find the pmf, mean, and variance of the 0-truncated random variable starting from:

(a)

$X \sim Poisson(\lambda)$

Answer > (a)

The pmf of a Poisson distribution is:

$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$

Also:

$P(X > 0) = 1 - P(X = 0) = 1 - \frac{\lambda^0 e^{-\lambda}}{0!} = 1 - e^{-\lambda}$

Combining these together gives us the truncated pmf:

$$P(X_T = x) = \frac{P(X = x)}{P(X > 0)} = \frac{\lambda^x e^{-\lambda}}{x!}/1 - e^{-\lambda} = \frac{\lambda^x e^{-\lambda}}{x!(1 - e^{-\lambda})}$$

For $x = 1, 2, ...$

Using the above pmf, we may find the mean as:

$$E(X_T) = \sum_{x=1}^{\infty} xP(X_T = x) = \sum_{x=1}^{\infty} x\frac{\lambda^x e^{-\lambda}}{x!(1 - e^{-\lambda})} = \frac{\lambda e^{-\lambda}}{(1 - e^{-\lambda})} \sum_{x \geq 1} \frac{\lambda^{x-1}}{(x-1)!}$$

Let $y = x - 1$, such that we may rewrite the above as:

$$E(X_T) = \frac{\lambda e^{-\lambda}}{(1 - e^{-\lambda})} \sum_{y \geq 0} \frac{\lambda^y}{(y)!}$$

Using the infinite summation for Euler, namely:

$$e^{\lambda} = \sum_{y \geq 0} \lambda^y / y!$$

We may then evaluate this as:

$$E(X_T) = \frac{\lambda e^{-\lambda}}{(1-e^{-\lambda})} e^{\lambda} = \frac{\lambda}{(1-e^{-\lambda})}$$

To then find the variance, let us consider $E(X_T^2)$

$$E(X_T^2) = \sum_{x \geq 1} x^2 P(X_T = x) = \sum_{x \geq 1} x^2 \frac{\lambda^x e^{-\lambda}}{x!(1-e^{-\lambda})} = \sum_{x \geq 1} x^2 \frac{\lambda^x e^{-\lambda}}{x!(1-e^{-\lambda})}$$

Q3

Question: 3.17 Casella & Berger

Establish a formula simialr to (3.3.18) for the gamma distribution. If $X \sim Gamma(\alpha, \beta)$, then for any positive constant v,

$EX^v = \frac{\beta^v \Gamma(v+\alpha)}{\Gamma(\alpha)}$

Answer

Q4

Question: 3.19 Casella & Berger

Show that:

$$\int_{x}^{\infty} \frac{1}{\Gamma(\alpha)} z^{\alpha-1} e^{-z} dz = \sum_{y=0}^{\alpha-1} \frac{x^y e^{-x}}{y!}$$

For $\alpha = 1, 2, 3, ...$

Hint: Use integration by parts. Express this formula as a probabilistic relationbship between Poisson and Gamma random variables.

Answer

Q5

Question: 3.24 (a), (c) Casella & Berger Note: You can skip the part about showing that the pdf is a pdf; also, in (c), the variance will not exist unless a >2.

Many “named” distributions are special cases of the more common distributions already discussed. For each of the following named distributions derive the form of the pdf, . . . , and calculate the mean and variance.

(a)

If $X \sim \textit{Exponential}(\beta)$, then $Y = X^{1/\gamma}$ has the Weibull(γ, β) distribution, where $\gamma > 0$ is a constant.

(c)

If $X \sim \textit{Gamma}(a, b)$, then $Y = 1/X$ has the inverted Gamma IG(a,b) distribution.

Answer > (a)

(c)

Q6

Question: 3.39 Casella & Berger

Consider the Cauchy family defined in Section 3.3. This family can be extended to a location-scale family yielding pdfs of the form:

$$f(x|\mu, \sigma) = \frac{1}{\sigma\pi(1 + (\frac{x-\mu}{\sigma})^2)}$$

For $-\infty < x < \infty$

The mean and variance do not exist for thje Cauchy distribution. So the parameters μ, σ^2 are not the mean and variance. But they do have important meaning. Show that if X is a random variable with a Cauchy distribution with parameters μ and σ, then:

(a)

μ is the median of the distribution of X, that is, $P(X \geq \mu) = P(X \leq \mu) = \frac{1}{2}$

(b)

$\mu + \sigma$ and $\mu - \sigma$ are the quartiles of the distribution of X, that is $P(X \geq \mu + \sigma) = P(X \leq \mu - \sigma) = \frac{1}{4}$

Hint: Prove this first for $\mu = 0$ and $sigma = 1$ and then use Exercise 3.38.

Note: $\frac{d(arctanx)}{dx} = \frac{1}{1+x^2}$

Answer > (a)

(b)

Q7

Question:

If $X \sim N(\mu, \sigma^2)$, find values of μ and σ such that $P(|X| < 2) = \frac{1}{2}$. Prove or disprove that the values of μ and σ are unique.

Answer