

# PS2

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## Assignment 2

In contrast to inverse gamma distributions, an inverse Gaussian distribution is **not** the distribution of the reciprocal of a random variable having a Gaussian distribution. It is an entirely different distribution.

There are any number of ways that people have parameterized inverse Gaussian probability density functions. One way is, for parameters  $\mu > 0$  and  $\lambda > 0$ ,

$$f(y \mid \mu, \lambda) = \left( \frac{\lambda}{2\pi y^3} \right)^{1/2} \exp \left[ -\frac{\lambda}{2\mu^2 y} (y - \mu)^2 \right], \quad y > 0. \quad (1)$$

## Question 1 (5 pts)

Write the density (1) in exponential dispersion family form. Identify the natural parameter  $\theta$  and dispersion parameter  $\phi$  in terms of the original parameters  $\mu$  and  $\lambda$ .

## Answer

Starting from the given density:

$$f(y \mid \mu, \lambda) = \left( \frac{\lambda}{2\pi y^3} \right)^{1/2} \exp \left[ -\frac{\lambda}{2\mu^2 y} (y - \mu)^2 \right], \quad y > 0,$$

we expand the quadratic term in the exponent:

$$f(y \mid \mu, \lambda) = \left( \frac{\lambda}{2\pi y^3} \right)^{1/2} \exp \left[ -\frac{\lambda}{2\mu^2} y + \frac{\lambda}{\mu} - \frac{\lambda}{2y} \right], \quad y > 0,$$

Using  $\log(\exp(x)) = x$ , we have:

$$f(y \mid \mu, \lambda) = \exp \left[ -\frac{\lambda}{2\mu^2} y + \frac{\lambda}{\mu} - \frac{\lambda}{2y} + \frac{1}{2} \log \lambda - \frac{1}{2} \log(2\pi y^3) \right]$$

Note: The general form of the exponential dispersion family is

$$f(y \mid \theta, \phi) = \exp[\phi\{y\theta - b(\theta)\}] + c(y, \phi)$$

For the above density, we have:

$$\theta = -\frac{1}{2\mu^2}, \quad \theta < 0$$

$$\phi = \lambda, \quad \phi > 0$$

$$b(\theta) = -\sqrt{-2\theta},$$

And

$$c(y, \phi) = -\frac{1}{2} \log(2\pi y^3) - \frac{1}{2\phi y}$$

## Question 2 (5 pts)

Using the result from question 1, find the expected value and variance of a random variable  $Y$  that follows an inverse Gaussian distribution. Write these moments in terms of  $\theta$  and  $\phi$  and then also in terms of  $\mu$  and  $\lambda$ .

How is the variance related to the expected value for this distribution?

### Answer

Using Equations 3.8 from Ch. 3.2, we know:

$$E(Y) = \frac{d}{d\theta} b(\theta)$$

And

$$\text{Var}(Y) = \frac{1}{\phi} \frac{d^2}{d\theta^2} b(\theta) = \frac{1}{\phi} V(\mu)$$

With  $b(\theta) = -\sqrt{-2\theta}$  (parameter space:  $\theta < 0$ ),

$$b'(\theta) = 2^{-1/2}(-\theta)^{-1/2}, \quad b''(\theta) = 2^{-3/2}(-\theta)^{-3/2}$$

In terms of  $\theta, \phi$ :

$$E(Y) = (-2\theta)^{-1/2}$$

And

$$\text{Var}(Y) = \frac{1}{\phi} (-2\theta)^{-3/2}$$

In terms of  $\theta, \mu$ , we then have:  $\theta = -\frac{1}{2\mu^2}$  and  $\phi = \frac{1}{\lambda}$ :

$$E(Y) = (-2(-\frac{1}{2\mu^2}))^{-1/2} = \mu$$

And:

$$\text{Var}(Y) = \frac{1}{\lambda} (-2(-\frac{1}{2\mu^2}))^{-3/2} = \frac{\mu^3}{\lambda}$$

Note: The variance is related to the expected value for this distribution in that it is a function of expectation, i.e., the variance is proportional to the cube of the mean,  $\mu$ , which is the expectation as parametrized.

### Question 3 (5 pts)

To get a feel for this distribution with the same location but different values of the dispersion parameter, produce a graph with three overlaid density functions having  $\mu = 1$  and  $\lambda = 4, 8$ , and  $16$ .

Answer

$$f(y \mid \mu, \lambda) = \exp \left( \log \left( \left( \frac{\lambda}{2\pi y^3} \right) \right) \right)^{1/2} \exp \left[ -\frac{\lambda}{2\mu^2 y} (y - \mu)^2 \right], \quad y > 0.$$

```
# lambda, mu density
f_y <- function(y, mu, lambda) {
  dat <- exp(0.5*log(lambda) -
            0.5*log(2*pi*y^3) -
            (lambda/(2*mu^2))*((y - mu)^2)/y )
  dat[!(y > 0)] <- 0
  dat
}

# inputs
mu <- 1
lambdas <- c(4, 8, 16)

# making datapoints
# y > 0
y_dat <- seq(0.01, 4, length.out = 5000)

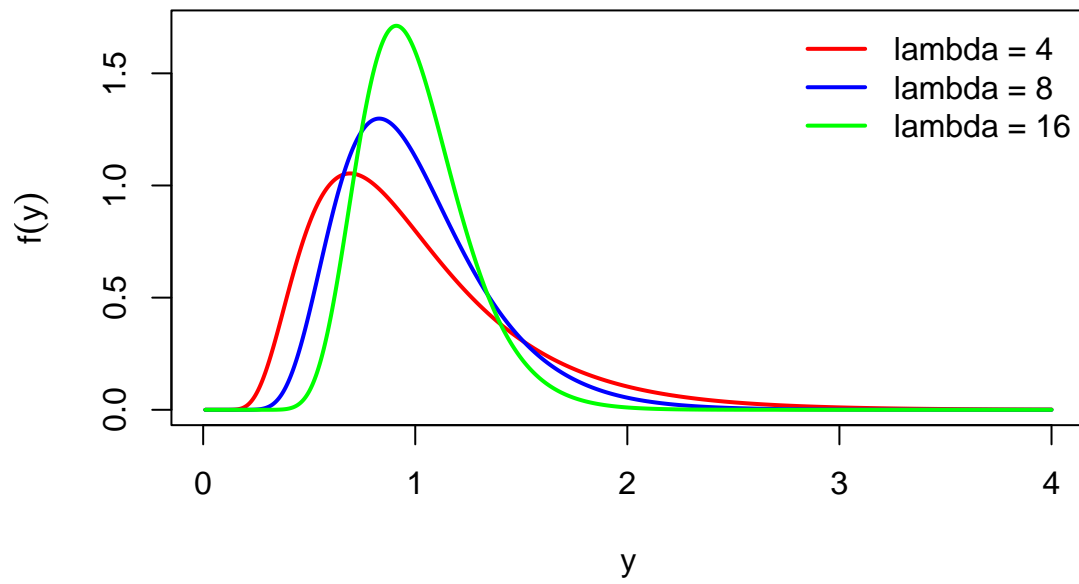
# call function
densities <- lapply(lambdas, function(lam) f_y(y_dat, mu, lam))

# plot
y_max <- max(sapply(densities, max))

# plot
plot(y_dat, densities[[1]], type = "l", col = "red", lwd = 2,
     xlab = "y",
     ylab = expression(f(y)),
     main = "Density curves",
     ylim = c(0, y_max))
lines(y_dat, densities[[2]], col = "blue", lwd = 2)
lines(y_dat, densities[[3]], col = "green", lwd = 2)

# Legend
legend("topright", legend = paste("lambda =", lambdas),
     col = c("red", "blue", "green"), lwd = 2, bty = "n")
```

Density curves



## Question 4 (10 pts)

Now compare the inverse Gaussian distributions from Question 3 with the corresponding gamma distributions, which means gamma distributions having the same mean and variance as the inverse Gaussian distributions.

A gamma density with parameters  $\alpha > 0$  and  $\beta > 0$  is

$$f(y | \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} \exp(-\beta y), \quad y > 0.$$

The expected value and variance for a random variable  $Y$  having this distribution are

$$E(Y) = \frac{\alpha}{\beta}, \quad \text{Var}(Y) = \frac{\alpha}{\beta^2}.$$

To find the corresponding gamma distribution for an inverse Gaussian distribution with parameter values  $\mu$  and  $\lambda$ , first determine the expected value and variance that result from your answer to Question 2, equate these with the expected value and variance for a  $\text{Gamma}(\alpha, \beta)$  distribution as just given, and solve for  $\alpha$  and  $\beta$ .

Finally, produce three graphs with the three inverse Gaussian distributions already computed in Question 3 and overlay the corresponding gamma distribution.

For the three cases of Question 3, do you notice any systematic difference in inverse Gaussian and gamma distributions?

## Answer

*To find the corresponding gamma distribution for an inverse Gaussian distribution with parameter values  $\mu$  and  $\lambda$ , first determine the expected value and variance that result from your answer to Question 2, equate these with the expected value and variance for a  $\text{Gamma}(\alpha, \beta)$  distribution as just given, and solve for  $\alpha$  and  $\beta$ .*

$$E(Y) = \frac{\alpha}{\beta} = \mu$$

$$\text{Var}(Y) = \frac{\alpha}{\beta^2} = \frac{\mu^3}{\lambda}$$

Solving for  $\alpha, \beta$ :

For  $\alpha = \mu\beta$ :

$$\frac{\mu\beta}{\beta^2} = \frac{\mu^3}{\lambda} \rightarrow \frac{\mu}{\beta} = \frac{\mu^3}{\lambda} \rightarrow \beta = \frac{\lambda}{\mu^2}$$

And

$$\frac{\alpha}{\frac{\lambda}{\mu^2}} = \mu \rightarrow \alpha = \frac{\lambda}{\mu}$$

So we have  $(\alpha, \beta) = \left(\frac{\lambda}{\mu}, \frac{\lambda}{\mu^2}\right)$

*Finally, produce three graphs with the three inverse Gaussian distributions already computed in Question 3 and overlay the corresponding gamma distribution.*

```
f_invgauss <- function(y, mu, lambda) {  
  dat <- exp(0.5*log(lambda) - 0.5*log(2*pi*y^3) -  
            (lambda/(2*mu^2)) * ((y - mu)^2)/y )  
  dat[!(y > 0)] <- 0  
  dat
```

```

}

# parameters
mu      <- 1
lambdas <- c(4, 8, 16)
cols    <- c("red", "blue", "green")

# data gen
y_dat <- seq(0.01, 5, length.out = 5000)

# densities
dens_IG <- lapply(lambdas, function(lam) f_invgauss(y_dat, mu, lam))
dens_GA <- lapply(lambdas, function(lam) dgamma(y_dat, shape = lam, rate = lam))

# global
y_max <- max(sapply(c(dens_IG, dens_GA), max))

# base plot
plot(y_dat, dens_IG[[1]], type = "l", lwd = 2, col = cols[1],
     xlab = "y",
     ylab = expression(f(y)),
     main = "IG (solid) vs. Gamma (dashed) (Full Range)",
     ylim = c(0, y_max))

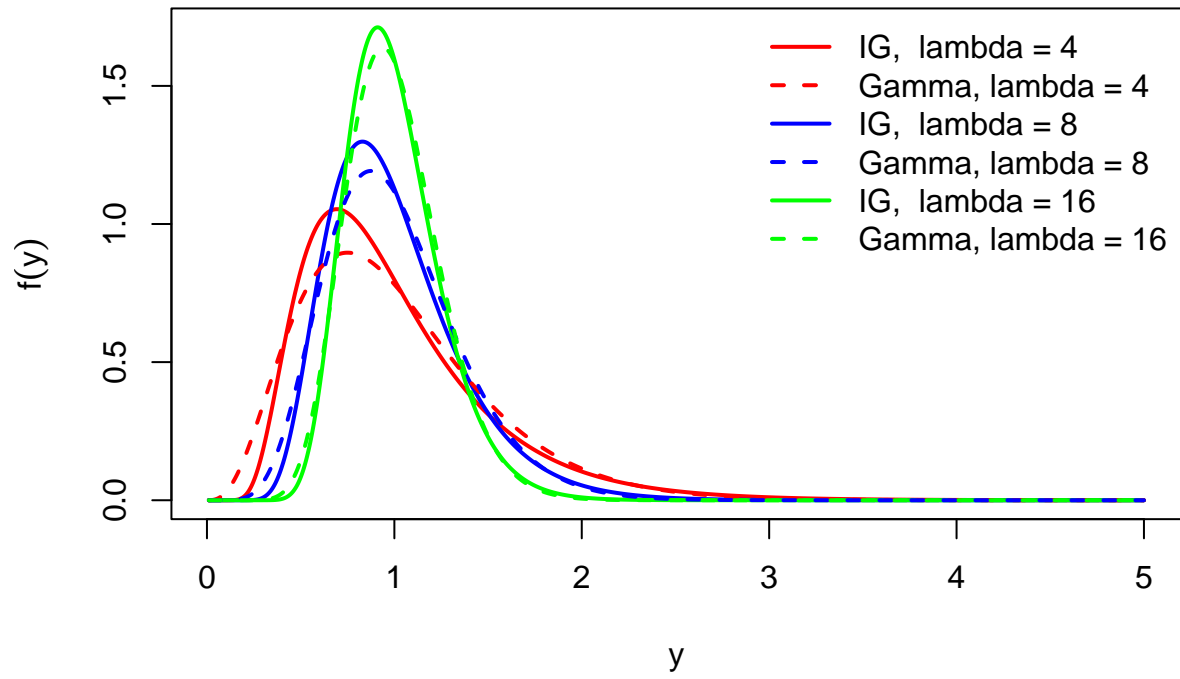
# IG curves
lines(y_dat, dens_IG[[2]], lwd = 2, col = cols[2])
lines(y_dat, dens_IG[[3]], lwd = 2, col = cols[3])

# Gamma curves
lines(y_dat, dens_GA[[1]], lwd = 2, col = cols[1], lty = 2)
lines(y_dat, dens_GA[[2]], lwd = 2, col = cols[2], lty = 2)
lines(y_dat, dens_GA[[3]], lwd = 2, col = cols[3], lty = 2)

legend("topright",
      legend = c("IG, lambda = 4", "Gamma, lambda = 4",
                  "IG, lambda = 8", "Gamma, lambda = 8",
                  "IG, lambda = 16", "Gamma, lambda = 16"),
      col = rep(cols, each = 2),
      lty = rep(c(1, 2), 3), lwd = 2, bty = "n")

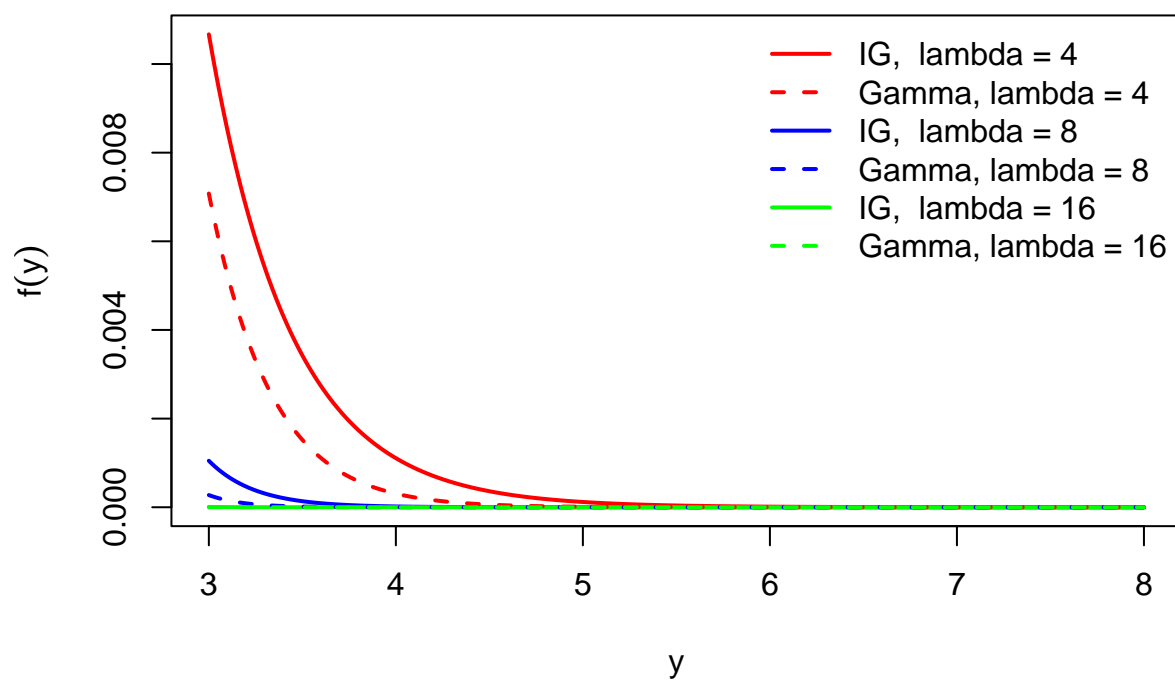
```

**IG (solid) vs. Gamma (dashed) (Full Range)**

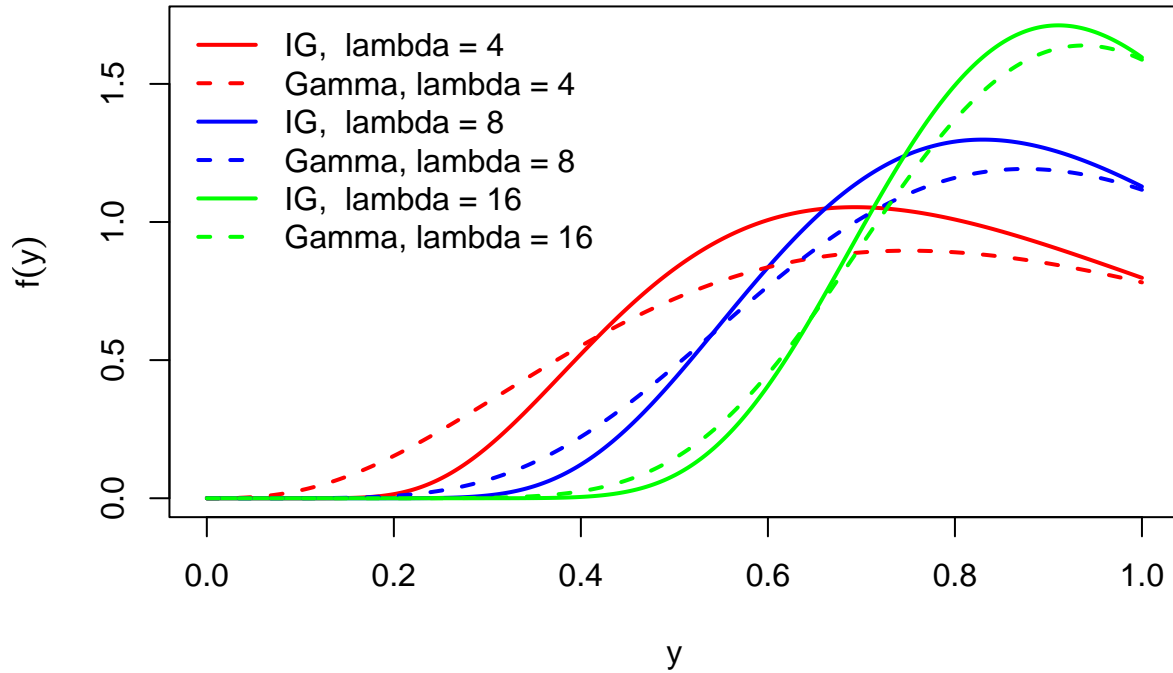




### IG (solid) vs. Gamma (dashed) (Right-Half)



### IG (solid) vs. Gamma (dashed) (Left-Half)



For the three cases of Question 3, do you notice any systematic difference in inverse Gaussian and gamma distributions?

Yes, there appear to be systematic differences between the inverse Gaussian and Gamma distributions despite when they share the same mean and variance. The Gamma distribution is consistently less “peaked” and has different “tail behavior” of their respective distributions than the inverse Gaussian. I believe this is because of differences in higher-order moments (skewness and kurtosis). Generally it seems that, holding  $\mu$  constant, as  $\lambda$  increases, both distributions have greater density about the mean, and the discrepancy between them diminishes, i.e., they more closely “overlap” one another; but the inverse Gaussian remains more sharply peaked with heavier tails compared to the Gamma with the same mean and variance.

Note: The systematic differences above reflect differences in **distributional form** beyond what is summarized by skewness or kurtosis. The two families differ because their variance–mean relationships have different forms:

$$V(Y) \propto \mu^3 \quad (\text{For the Inverse-Gaussian})$$

$$V(Y) \propto \mu^2 \quad (\text{For the Gamma})$$

This distinction between variance functions drives the observed shape differences seen above, i.e., the differences in the “tail behavior” of the distributions, which skewness and kurtosis alone cannot fully account for.