HW3

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1.

Suppose $X_1, \ldots, X_n$ are iid Bernoulli(p), $0 < p < 1$.

a)

Find the information number $I_n(p)$ and make a rough sketch of $I_n(p)$ as a function of $p \in (0, 1)$.

Given that $X_1, \ldots, X_n$ are i.i.d. Bernoulli(p), the likelihood function is:

$$L(p) = \prod_{i=1}^{n} p^{X_i}(1-p)^{1-X_i}$$

Taking the log-likelihood,

$$log(L(p)) = \sum_{i=1}^{n} [X_i \log p + (1 - X_i)\log(1-p)]$$

The first derivative is:

$$log(L(p))' = \sum_{i=1}^{n} \left[\frac{X_i}{p} - \frac{1-X_i}{1-p}\right] = \sum_{i=1}^{n} \frac{X_i - p}{p(1-p)}$$

The Fisher information is:

$$I_n(p) = -E\left[log(L((p))''\right]$$

Computing the second derivative:

$$log(L(p))'' = \sum_{i=1}^{n} \left[-\frac{X_i}{p^2} - \frac{1-X_i}{(1-p)^2}\right]$$

Taking expectation:

$$E[log(L(p))''] = \sum_{i=1}^{n} \left[-\frac{E[X_i]}{p^2} - \frac{E[1-X_i]}{(1-p)^2}\right]$$

Given we know the distribution of the random variables, we know $E[X_i] = p$ and $E[1 - X_i] = 1 - p$. This allows us to simplify the expression:

$$-E[log(L(p))''] = -\sum_{i=1}^{n}\left[-\frac{p}{p^2} - \frac{1-p}{(1-p)^2}\right] = -\sum_{i=1}^{n}\left[-\frac{1}{p} - \frac{1}{1-p}\right] = n\left[\frac{1}{p} + \frac{1}{1-p}\right]$$

Simplifying

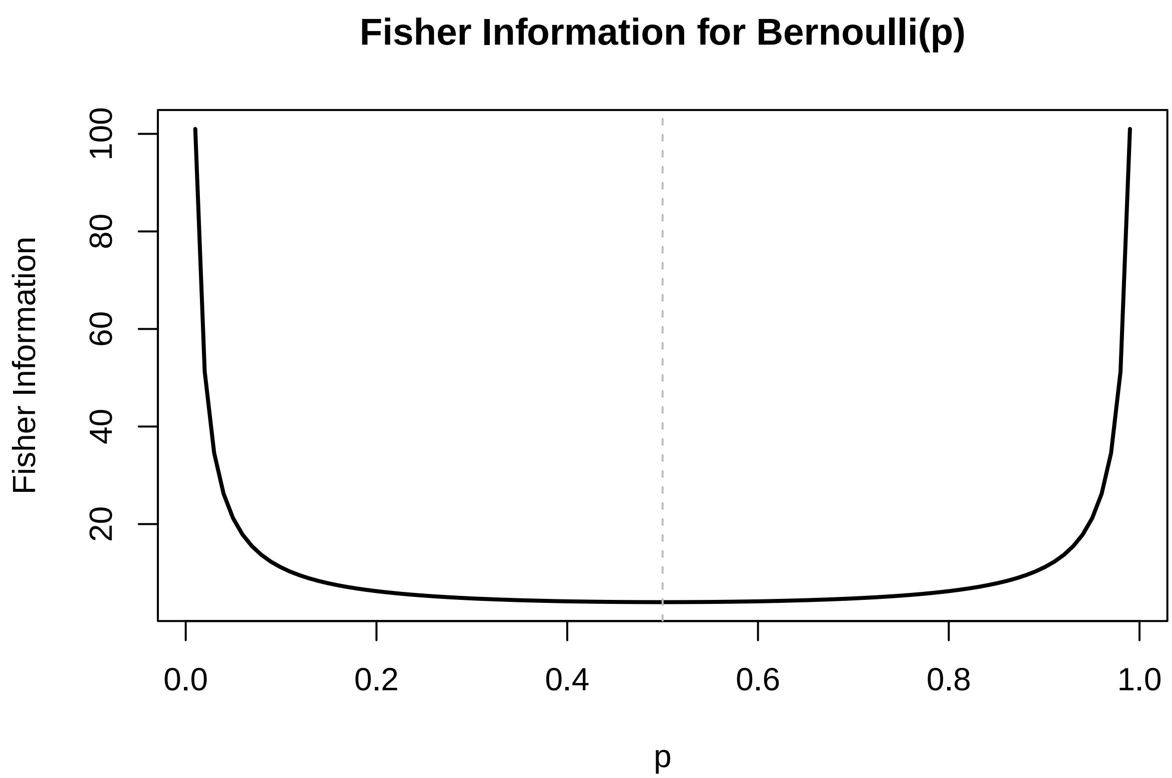
$$I_n(p) = n\left[\frac{1}{p} + \frac{1}{1-p}\right] = n\frac{1}{p(1-p)}$$

Sketch

```r
# functional form
fisher_info <- function(p, n) {
  return(n * (1/p + 1/(1 - p)))
}

# setup
p_values <- seq(0.01, 0.99, length.out = 100)
n <- 1
I_values <- fisher_info(p_values, n)

# plot
plot(x = p_values,
     y = I_values,
     type = "l",
     col = "black", lwd = 2,
     xlab = "p", ylab = "Fisher Information",
     main = "Fisher Information for Bernoulli(p)")
abline(v = 0.5, lty = 2, col = "gray")
```



b)

Find the value of $p \in (0, 1)$ for which $I_n(p)$ is minimal. (This value of p corresponds to the "hardest" case for estimating p. That is, when data are generated under this value of p from the model, the variance of an UE of p is potentially largest.)

To find the value of p that minimizes the Fisher information $I_n(p)$, we use the functional form of the Fisher Information:

$$I_n(p) = n\left[\frac{1}{p} + \frac{1}{1-p}\right]$$

Differentiating $I_n(p)$ with respect to p, and setting equal to zero:

$$I_n(p)' = n\left[-\frac{1}{p^2} + \frac{1}{(1-p)^2}\right] = -\frac{1}{p^2} + \frac{1}{(1-p)^2} = 0$$

This gives us the expression:

$$\frac{1}{p^2} = \frac{1}{(1-p)^2}$$

Taking square roots:

$$\frac{1}{p} = \frac{1}{1-p} \rightarrow p = 1 - p \rightarrow p = \frac{1}{2}$$

To ensure this is a maximum, we also check whether the second derivative is positive (since we are minimizing and not maximizing) at $\frac{1}{2}$:

$$I_n(p)' = n\left[\frac{2}{p^3} + \frac{2}{(1-p)^3}\right]$$

$$I_n\left(\frac{1}{2}\right)'' = n\left[\frac{2}{(1/2)^3} + \frac{2}{(1/2)^3}\right] = n\left[\frac{2}{1/8} + \frac{2}{1/8}\right] = n\,[16+16] = 32n > 0$$

So this is in fact a minimum, hence the Fisher information is minimized at:

$$p = \frac{1}{2}$$

c)

Show that $\hat{X}_n = \sum_{i=1}^{n} X_i/n$ is the UMVUE of p.

Note to self: Uniformly Minimum Variance Unbiased Estimator (UMVUE)

We start by checking if $\hat{X}_n$ is an unbiased estimator of p:

$$E[\hat{X}_n] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[X_i]E[\hat{X}_n] = \frac{1}{n}\cdot np = p$$

$$Bias(\bar{X_n}) = E[\hat{X}_n] - E[X] = p - p = 0$$

So $\hat{X}_n$ is an unbiased estimator of p.

Now as far as the "Uniformly Minimum Variance" part of the question:

Note again the Fisher Information formula we've found:

$$I_n(p) = \frac{np}{p^2} + \frac{n(1-p)}{(1-p)^2} = \frac{n}{p(1-p)}$$

By the definition, the Cramér-Rao Lower Bound, for any unbiased estimator T of p:

$$\mathrm{Var}_p(T) \geq \frac{(\gamma'(p))^2}{I_n(p)}$$

Here, we are estimating $\gamma(p) = p$, so $\gamma'(p) = 1$. Therefore:

$$\mathrm{Var}_p(T) \geq \frac{1^2}{I_n(p)} = \frac{p(1-p)}{n}$$

We compute the variance of $\hat{X}_n = S_n/n$:

$$E[\hat{X}_n] = E\left[\frac{S_n}{n}\right] = \frac{1}{n}E[S_n] = \frac{np}{n} = p$$

$$\mathrm{Var}(\hat{X}_n) = \mathrm{Var}\left(\frac{S_n}{n}\right) = \frac{1}{n^2}\mathrm{Var}(S_n)$$

Since $S_n \sim \text{Binomial}(n, p)$, we know:

$$\text{Var}(S_n) = np(1-p)$$

Thus:

$$\text{Var}(\hat{X}_n) = \frac{np(1-p)}{n^2} = \frac{p(1-p)}{n}$$

Comparing with the CRLB:

$$\text{Var}(\hat{X}_n) = \frac{p(1-p)}{n} = \frac{1}{I_n(p)}$$

Since $\hat{X}_n$ is unbiased and attains the CRLB, it is the UMVUE.

2.

Suppose that the random variables $Y_1, \ldots, Y_n$ satisfy

$$Y_i = \beta x_i + \varepsilon_i, \quad i = 1, \ldots, n$$

where $x_1, \ldots, x_n$ are fixed constants and $\varepsilon_1, \ldots, \varepsilon_n$ are iid $N(0, \sigma^2)$; here we assume $\sigma^2 > 0$ is known.

a)

Find the MLE of β.

To find the Maximum Likelihood Estimator (MLE) of β, we first write the likelihood function.

Since $Y_i = \beta x_i + \varepsilon_i$ with $\varepsilon_i \sim N(0, \sigma^2)$, we have:

$$Y_i \sim N(\beta x_i, \sigma^2)$$

Thus, the joint density function of $Y_1, \ldots, Y_n$ is:

$$L(\beta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(Y_i - \beta x_i)^2}{2\sigma^2}\right)$$

Taking the log-likelihood:

$$log(L(\beta)) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(Y_i - \beta x_i)^2$$

To find the MLE of β, we take the derivative with respect to β and set to zero:

$$\frac{d}{d\beta}log(L(\beta)) = -\frac{1}{2\sigma^2} \cdot (-2)\sum_{i=1}^{n} x_i(Y_i - \beta x_i) = \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i(Y_i - \beta x_i) \to \sum_{i=1}^{n} x_i Y_i - \beta \sum_{i=1}^{n} x_i^2 = 0$$

Solving for β, we get our MLE of β as::

$$\hat{\beta} = \frac{\sum_{i=1}^{n} x_i Y_i}{\sum_{i=1}^{n} x_i^2}$$

To ensure this is a maximum, we take the second derivative at the MLE and see if it is negative:

$$log(L(\beta))'' = -\frac{1}{\sigma^2}\sum_{i=1}^{n} x_i^2 < 0$$

So this is in fact the maximum.

b)

Find the distribution of the MLE.

From part a), the MLE of β is:

$$\hat{\beta} = \frac{\sum_{i=1}^{n} x_i Y_i}{\sum_{i=1}^{n} x_i^2}$$

To determine the distribution of $\hat{\beta}$, determine its expectation and variance, noting that since $\hat{\beta}$ is a linear combination of the normal random variables ε_i, it follows that $\hat{\beta}$ itself is normally distributed.

That being said, given $Y_i = \beta x_i + \epsilon_i$, we may write:

$$\hat{\beta} = \frac{\sum_{i=1}^{n} x_i(\beta x_i + \varepsilon_i)}{\sum_{i=1}^{n} x_i^2} = \frac{\beta \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} x_i \varepsilon_i}{\sum_{i=1}^{n} x_i^2}$$

Taking the expectation, noting our data is treated as “fixed”, we may write:

$$E[\hat{\beta}] = \frac{\beta \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} x_i E[\varepsilon_i]}{\sum_{i=1}^{n} x_i^2} = \frac{\beta \sum_{i=1}^{n} x_i^2}{\sum_{i=1}^{n} x_i^2} = \beta$$

Noting $E[\varepsilon_i] = 0$

Because $E[\hat{\beta}] = \beta$, it has zero bias and $\hat{\beta}$ is an unbiased estimator of β. Not needed for the distribution, but will need this note for later.

Let us then analyze the variance. We start again with definitions:

$$\text{Var}(\hat{\beta}) = \text{Var}(\beta + \frac{\sum_{i=1}^{n} x_i \varepsilon_i}{\sum_{i=1}^{n} x_i^2}) = \text{Var}(\beta) + \text{Var}(\frac{\sum_{i=1}^{n} x_i \varepsilon_i}{\sum_{i=1}^{n} x_i^2})$$

Simplifying:

$$\text{Var}(\hat{\beta}) = \text{Var}\left(\frac{\sum_{i=1}^{n} x_i \varepsilon_i}{\sum_{i=1}^{n} x_i^2}\right) = \text{Var}\left(\sum_{i=1}^{n} \frac{(x_i^2 \sigma^2)}{(x_i^2)^2}\right) = \frac{\sigma^2}{\sum_{i=1}^{n} x_i^2}$$

We thus conclude:

$$\hat{\beta} \sim N\left(\beta, \frac{\sigma^2}{\sum_{i=1}^{n} x_i^2}\right)$$

c)

Find the CRLB for estimating β. (Hint: you’ll have to work with the joint distribution $f(y_1, \ldots, y_n|\beta)$ directly, since $Y_1, \ldots, Y_n$ are not iid.)

To find the CRLB, we first calculate the Fisher information.

Note the joint density:

$$f(Y_1, \ldots, Y_n|\beta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(Y_i - \beta x_i)^2}{2\sigma^2}\right)$$

Taking the log-likelihood:

$$log(L(\beta)) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(Y_i - \beta x_i)^2$$

We take the derivative:

$$log(L(\beta))' = -\frac{1}{2\sigma^2} \cdot (-2)\sum_{i=1}^{n} x_i(Y_i - \beta x_i) = \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i(Y_i - \beta x_i)$$

The Fisher information is then:

$$I(\beta) = -E[log(L(\beta))''] = -E\left[-\frac{1}{\sigma^2}\sum_{i=1}^{n} x_i^2\right] = \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i^2$$

We then have what we need to calculate the CRLB using the information we've gathered.

The CRLB is:

$$\frac{1}{I(\beta)} = \frac{1}{\frac{1}{\sigma^2}\sum_{i=1}^{n} x_i^2} = \frac{\sigma^2}{\sum_{i=1}^{n} x_i^2}$$

d)

Show the MLE is the UMVUE of β.

Now we just need to compare the variance of our MLE of β to the value calculated in part c). To that end:

We have already calculated the expectation of $\hat{\beta}_{MLE}$, which is β, so via Bias calculation:

$$\text{Bias}(\hat{\beta}_{MLE}) = E[\hat{\beta}_{MLE}] - \beta = \beta - \beta = 0$$

Hence it is unbiased. We then just need to determine if our MLE attains the CRLB. If so, then the MLE is the UMVUE.

Recall the variance of the MLE:

$$\frac{\sigma^2}{\sum_{i=1}^{n} x_i^2}$$

And the CRLB:

$$\frac{\sigma^2}{\sum_{i=1}^{n} x_i^2}$$

These are one and the same! So we do indeed satisfy:

$$\text{Var}(\hat{\beta}_{MLE}) = CRLB$$

Such that the MLE is the UMVUE.

3.

Suppose $X_1, \ldots, X_n$ are iid normal $N(0,1)$, where $\theta \in \mathbb{R}$. It turns out that $T = (\bar{X}_n)^2 - n^{-1}$ is the UMVUE of $\gamma(\theta) = \theta^2$. (We can show this later in the course; our goal here is to show that the UMVUE can exist without obtaining the CRLB.)

a)

Show T is an UE of $\gamma(\theta) = \theta^2$ and find the variance $\text{Var}_\theta(T)$ of T. (Note $Z = \sqrt{n}(\bar{X}_n - \theta) \sim N(0,1)$ and one can write $T = (Z^2/n) + (2\theta Z/\sqrt{n}) + \theta^2 - n^{-1}$, where $Z^2 \sim \chi_1^2$, $E_\theta Z^2 = 1$, $\text{Var}_\theta(Z^2) = 2$.)

Given:

$$Z = \sqrt{n}(\bar{X}_n - \theta) \sim N(0,1)$$

we can rewrite T in terms of Z, specifically:

$$T = \frac{Z^2}{n} + \frac{2\theta Z}{\sqrt{n}} + \theta^2 - \frac{1}{n}$$

Taking expectation:

$$E_\theta[T] = E_\theta\left[\frac{Z^2}{n} + \frac{2\theta Z}{\sqrt{n}} + \theta^2 - \frac{1}{n}\right] = \frac{1}{n} + \frac{2\theta}{\sqrt{n}}(0) + \theta^2 - \frac{1}{n} = \theta^2$$

Thus, T is an unbiased estimator of θ^2.

We then must calculate the variance of T, to that end, we find $E[T^2]$:

As defined:

$$T^2 = \left(\frac{Z^2}{n} + \frac{2\theta Z}{\sqrt{n}} + \theta^2 - \frac{1}{n}\right)^2 = \frac{Z^4}{n^2} + \frac{4\theta Z^3}{n^{3/2}} + \frac{4\theta^2 Z^2}{n} + \theta^4 + \frac{1}{n^2} + \frac{4\theta^3 Z}{\sqrt{n}} - \frac{2Z^2}{n^2} - \frac{4\theta Z}{n^{3/2}} - \frac{2\theta^2}{n}$$

Though that's quite a lot, we can actually simplify it quite a bit when taking expectation, noting the distribution of Z aids in these calculations

(Note: $E_\theta[Z] = 0$, $E_\theta[Z^2] = 1$, $E_\theta[Z^3] = 0$, and $E_\theta[Z^4] = \text{Var}(Z^2) + (E_\theta[Z^2])^2 = 2 + 1 = 3$.)

Thus,

$$E_\theta[T^2] = \frac{3}{n^2} + \frac{4\theta^2}{n} + \theta^4 - \frac{2}{n^2} - \frac{2\theta^2}{n} = \theta^4 + \frac{2\theta^2}{n} + \frac{1}{n^2}$$

Now we can calculate the variance:

$$\text{Var}_\theta(T) = \text{Var}(\frac{Z^2}{n} + \frac{2\theta Z}{\sqrt{n}} + \theta^2 - \frac{1}{n}) = \text{Var}(\frac{Z^2}{n} + \frac{2\theta Z}{\sqrt{n}}) = \frac{1}{n^2}\text{Var}(Z^2) + \frac{4\theta^2}{n}\text{Var}(Z) + \frac{2\theta}{n^{(3/2)}}\text{Cov}(Z^2, Z)$$

Simplifying:

$$\text{Var}_\theta(T) = \frac{2}{n^2} + \frac{4\theta^2}{n}$$

b)

Find the CRLB for an UE of $\gamma(\theta) = \theta^2$.

Since $X_1, \ldots, X_n$ are i.i.d. normal $N(\theta, 1)$, the likelihood function is:

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(X_i - \theta)^2}{2}\right)$$

Taking the log-likelihood:

$$log(L(\theta)) = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^{n}(X_i - \theta)^2$$

Differentiating with respect to θ, and getting expectation to derive the Fisher Information:

$$log(L(\theta))' = \sum_{i=1}^{n}(X_i - \theta) \rightarrow I(\theta) = -E[log(L(\theta))'] = -E\left[-\sum_{i=1}^{n} 1\right] = -(-n) = n$$

The CRLB by definition is given by:

$$\frac{(\gamma'(\theta))^2}{I(\theta)}$$

We just need now to calculate the numerator. To that end, note that $\gamma(\theta) = \theta^2$, making its derivative:

$$\gamma'(\theta) = 2\theta$$

Thus the CRLB is:

$$\frac{(\gamma'(\theta))^2}{n} = \frac{(2\theta)^2}{n} = \frac{4\theta^2}{n}$$

c)

Show that $\text{Var}_\theta(T) > \text{CRLB}$ for all values of $\theta \in \mathbb{R}$.

We compare the variance we calcualted from part a) with the CRLB from part b). To that end, note:

From part a):

$$\text{Var}_\theta(T) = \frac{2}{n^2} + \frac{4\theta^2}{n}$$

From part b), the CRLB (for any unbiased estimator of θ^2) is:

$$\text{CRLB} = \frac{4\theta^2}{n}$$

Comparing these two quantities directly, their difference is given by:

$$\mathrm{Var}_\theta(T) - \mathrm{CRLB} = \frac{2}{n^2} + \frac{4\theta^2}{n} - \frac{4\theta^2}{n} = \frac{2}{n^2} > 0$$

So, for n > 0, and $\forall \theta \in \mathbb{R}$, it holds that:

$$\mathrm{Var}_\theta(T) > \mathrm{CRLB}$$

4. Casella & Berger 7.58

("better" here refers to MSE as a criterion.)

Let X be an observation from the pdf

$$f(x|\theta) = \left(\frac{\theta}{2}\right)^{|x|} (1-\theta)^{1-|x|}, \quad x = -1, 0, 1; \quad 0 \le \theta \le 1$$

a)

Find the MLE of θ.

Given that X takes values in $\{-1, 0, 1\}$, it is discrete, so we note the pmf:

$$f(x|\theta) = \begin{cases} \left(\frac{\theta}{2}\right)^{|x|} (1-\theta)^{1-|x|} & x = -1, 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

For a sample $X_1, X_2, \ldots, X_n$, the likelihood function is:

$$L(\theta) = \prod_{i=1}^{n} \left(\frac{\theta}{2}\right)^{|X_i|} (1-\theta)^{1-|X_i|}$$

Let $S_n = \sum_{i=1}^{n} |X_i|$. We may then rewrite the likelihood function as:

$$L(\theta) = \left(\frac{\theta}{2}\right)^{S_n} (1-\theta)^{n-S_n}$$

Using our log-likelihood technique:

$$log(L(\theta)) = S_n \log\left(\frac{\theta}{2}\right) + (n-S_n)\log(1-\theta) = S_n \log\theta - S_n \log 2 + (n-S_n)\log(1-\theta) = S_n \log\theta + (n-S_n)\log(1-\theta)$$

We find the maximum the typical route, i.e., taking the derivative with respect to θ and setting equal to zero:

$$log(L(\theta))' = \frac{S_n}{\theta} - \frac{n-S_n}{1-\theta} = 0 \to \frac{S_n}{\theta} = \frac{n-S_n}{1-\theta}$$

After some simplifying:

$$S_n(1-\theta) = (n-S_n)\theta \to S_n - S_n\theta = n\theta - S_n\theta \to S_n = n\theta$$

And we arrive at our "MLE" (in quotes because there's our second check to account for):

$$\hat{\theta} = \frac{S_n}{n} = \frac{1}{n}\sum_{i=1}^{n} |X_i|$$

To double check, we take the second derivative (at the MLE) and see if it is negative:

$$log(L(\theta))'' = -\frac{S_n}{\theta^2} - \frac{n - S_n}{(1-\theta)^2}$$

$$log(L(\hat{\theta}))'' = -\frac{S_n n^2}{S_n^2} - \frac{(n-S_n)n^2}{(n-S_n)^2} = -\frac{n^2}{S_n} - \frac{n^2}{n-S_n} < 0$$

Noting: $S_n > 0$ and $n - S_n > 0$

So yes, this is our maximum and our MLE!

Jetzt zock' ich Fortnite und trink' Cola! Yipee!

b)

Define the estimator $T(X)$ by

$$T(X) = \begin{cases} 2 & \text{if } x = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Show that $T(X)$ is an unbiased estimator of θ.

To test for bias, we find the the expectation of $T(X)$:

$$E[T(X)] = \sum_{x \in \{-1,0,1\}} T(x)P(X = x)$$

Using the pmf from part a), each possible outcome/observation of X has its associated probability given by:

$$P(X=1) = \frac{\theta}{2}, \quad P(X=0) = 1-\theta, \quad P(X=-1) = \frac{\theta}{2}$$

So we need to do the more "manual" calculation of expectation:

$$E[T(X)] = 2P(X=1) + 0P(X=0) + 0P(X=-1)$$

Since $T(X) = 2$ when $X = 1$ and 0 otherwise from the initial definition of T.

Thus, we calculate:

$$E[T(X)] = 2 \cdot \frac{\theta}{2} + 0 + 0 = \theta$$

So, via Bias calculation, we know T(X) is an unbiased estimator of θ because $E[T(X)] = \theta$.

c)

Find a better estimator than $T(X)$ and prove that it is better.

By "better" we are making note of the "hint" to compare MSE, and "better" corresponding to smaller MSE compared to $T(X)$.

By definition, the MSE of the estimator $T(X)$ is:

$$\text{MSE}(T) = E[(T(X)-\theta)^2] = E[T^2(X)] - 2\theta E[T(X)] + \theta^2$$

From part b), we know that $T(X)$ is unbiased, so the unknown quantity in the above expression is $E[T^2(X)]$. Solving for that:

$$E[T^2(X)] = \sum_{x\in\{-1,0,1\}} T^2(x)P(X=x) = 2^2 P(X=1) = 4\cdot\frac{\theta}{2} = 2\theta$$

Returning to the MSE, our goal is to then find a better (smaller) MSE than T(X), which is:

$$\text{MSE}(T) = 2\theta - 2\theta^2 + \theta^2 = 2\theta - \theta^2 = \theta(2-\theta)$$

Our first guess will be to use the sample mean, the MLE from part a):

$$\hat{\theta} = \frac{S_n}{n} = \frac{1}{n}\sum_{i=1}^{n}|X_i|$$

To find the relevant quantities to find its MSE, we note/derive expectation for a discrete random variable: $E[|X|]$:

$$E[|X|] = 1\cdot P(X=1) + 0\cdot P(X=0) + 1\cdot P(X=-1) = \frac{\theta}{2} + 0 + \frac{\theta}{2} = \theta$$

Next, $E[|X|^2]$:

$$E[|X|^2] = 1^2\cdot P(X=1) + 0^2\cdot P(X=0) + 1^2\cdot P(X=-1) = \frac{\theta}{2} + 0 + \frac{\theta}{2} = \theta$$

So, the variance is:

$$\text{Var}(|X|) = E[|X|^2] - (E[|X|])^2 = \theta - \theta^2 \rightarrow \text{Var}(\hat{\theta}) = \frac{\theta-\theta^2}{n}$$

Since $\hat{\theta}$ is unbiased, i.e.

$$E[\hat{\theta}] = E[\frac{S_n}{n}] = E[\frac{1}{n}\sum_{i=1}^{n}|X_i|] = \frac{1}{n}E[\sum_{i=1}^{n}|X_i|] = \frac{n\theta}{n} = \theta$$

The MSE of $\hat{\theta}$ is:

$$\text{MSE}(\hat{\theta}) = \frac{\theta-\theta^2}{n}$$

We now comparing the two estimators:

$$\text{MSE}(T) = 2\theta - \theta^2$$

$$\text{MSE}(\hat{\theta}) = \frac{\theta-\theta^2}{n}$$

Since $n > 0$:

$$\text{MSE}(T) - \text{MSE}(\hat{\theta}) = 2\theta - \theta^2 - \left(\frac{\theta - \theta^2}{n}\right) = \frac{2n\theta - n\theta^2 - \theta + \theta^2}{n}$$

for n = 1:

$$\text{MSE}(T) - \text{MSE}(\hat{\theta}) = \theta(2-\theta) - \frac{\theta(1-\theta)}{n} = \theta(2-\theta) - \theta(1-\theta) = \theta \to \text{MSE}(\hat{\theta}) < \text{MSE}(T)$$

for n > 1, noting $0 \leq \theta \leq 1 \to 0 \leq (1-\theta) \leq 1)$ and $\theta(2-\theta) > \theta(1-\theta)$:

$$\text{MSE}(T) - \text{MSE}(\hat{\theta}) = \theta(2-\theta) - \frac{\theta(1-\theta)}{n} = \frac{n\theta(2-\theta) - \theta(1-\theta)}{n} > 0 \to \text{MSE}(\hat{\theta}) < \text{MSE}(T)$$

Taken together:

$$\text{MSE}(\hat{\theta}) < \text{MSE}(T)$$

So the MLE $\hat{\theta} = \frac{1}{n}\sum |X_i|$ is a “better” estimator than $T(X)$ because it has a lower Mean Squared Error for all values of θ (while also being unbiased!)

5.

Let $X_1, \ldots, X_n$ be iid Bernoulli(θ), $\theta \in (0,1)$. Find the Bayes estimator of θ with respect to the uniform$(0,1)$ prior under the loss function

The likelihood function is:

$$L(\theta) = \prod_{i=1}^{n} \theta^{X_i}(1-\theta)^{1-X_i}$$

Let $S_n = \sum_{i=1}^{n} X_i$, we may then write the above expression as:

$$L(\theta) \propto \theta^{S_n}(1-\theta)^{n-S_n}$$

Noting the prior, $\theta \sim$ Uniform$(0,1)$, the posterior is:

$$\theta|S_n \sim \text{Beta}(S_n+1, n-S_n+1)$$

Since Beta is a known distribution, we may write the posterior density:

$$f_{\theta|S_n}(\theta) = \frac{\theta^{S_n}(1-\theta)^{n-S_n}}{B(S_n+1, n-S_n+1)}$$

By definition, our Bayes estimate minimizes the posterior expected loss:

$$E_{\theta|S_n} L(h(X), \theta) = \int_0^1 \frac{(h(X)-\theta)^2}{\theta(1-\theta)} f_{\theta|S_n}(\theta) d\theta$$

Using the expressions detailed previously:

$$E_{\theta|S_n} L(h(X), \theta) = \int_0^1 \frac{h(X)^2 - 2h(X)\theta + \theta^2}{\theta(1-\theta)} f_{\theta|S_n}(\theta) d\theta$$

For simplicity, we can break us this evaluation:

$$E_{\theta|S_n} L(h(X), \theta) = h(X)^2 \int_0^1 \frac{1}{\theta(1-\theta)} f_{\theta|S_n}(\theta) d\theta - 2h(X) \int_0^1 \frac{\theta}{\theta(1-\theta)} f_{\theta|S_n}(\theta) d\theta + \int_0^1 \frac{\theta^2}{\theta(1-\theta)} f_{\theta|S_n}(\theta) d\theta$$

(1):

$$E_{\theta|S_n}\left[\frac{1}{\theta(1-\theta)}\right] = \int_0^1 \frac{1}{\theta(1-\theta)} f_{\theta|S_n}(\theta) d\theta = \frac{n(n-1)}{S_n(n-S_n)}$$

(2):

$$E_{\theta|S_n}\left[\frac{\theta}{\theta(1-\theta)}\right] = \int_0^1 \frac{\theta}{\theta(1-\theta)} f_{\theta|S_n}(\theta) d\theta. = \frac{(n-1)}{(n-S_n)}$$

(3):

$$E_{\theta|S_n}\left[\frac{\theta^2}{\theta(1-\theta)}\right] = \int_0^1 \frac{\theta^2}{\theta(1-\theta)} f_{\theta|S_n}(\theta)d\theta = \frac{(S_n+1)(n+2)}{(S_n+1)(n-S_n+1)}$$

Combining the results of (1) through (3) gives us:

$$E_{\theta|S_n}L(h(X),\theta) = h(X)^2\frac{n(n-1)}{S_n(n-S_n)} - 2h(X)\frac{(n-1)}{(n-S_n)} + \frac{(S_n+1)(n+2)}{(S_n+1)(n-S_n+1)}$$

Similar to the MLE method, we derive and set equal to zero:

$$\frac{d}{dh(X)}E_{\theta|S_n}L(h(X),\theta) = 2h(X)\frac{n(n-1)}{S_n(n-S_n)} - 2\frac{(n-1)}{(n-S_n)} = 0 \to 2h(X)\frac{n(n-1)}{S_n(n-S_n)} = 2\frac{(n-1)}{(n-S_n)}$$

Isolating h(X):

$$h(X) = \frac{\frac{(n-1)}{(n-S_n)}}{\frac{n(n-1)}{S_n(n-S_n)}} = \frac{S_n}{n}$$

Givins us our Bayes estimator of θ:

$$\hat{\theta}_{\text{Bayes}} = \frac{S_n}{n} = \bar{X}_n$$

Extra check time!

To confirm this is a minimum, we compute the second derivative, just like the MLE method.

Evaluating for the Bayes estimator gives us:

$$\frac{d^2}{dh(X)^2}E_{\theta|S_n}L(h(X),\theta) = 2\frac{n(n-1)}{S_n(n-S_n)} > 0$$

So our Bayes estimator, $\hat{\theta}_{\text{Bayes}} = \bar{X}_n$ does in fact minimize!