HW5

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Outline

- Q1: Edited
- Q2: Edited
- Q3: Edited
- Q4: Edited

1.

In the attached article by Prof. M. Ghosh, read pages 509-512 (including example 1), examples 4-6 of Section 3, and Section 5.2 up to and including Examples 17-18. (This is sort of a technical article, so to read a bit of this material is not easy. Also, Example 17 should look like an example from class regarding Basu's theorem.)

In example 18, show that T is a complete and sufficient statistic, while U is an ancillary statistic.

Example 18.

Let $X_1, \ldots, X_n$ $(n \geq 2)$ be iid with common Weibull pdf

$$f_\theta(x) = \exp(-x^p/\theta)(p/\theta)x^{p-1}; \quad 0 < x < \infty, \quad 0 < \theta < \infty,$$

$p(> 0)$ being known. In this case, $T = \sum_{i=1}^n X_i^p$ is complete sufficient for θ, while $U = X_1^p/T$ is ancillary. Also, since $X_1^p, \ldots, X_n^p$ are iid exponential with scale parameter θ, $U \sim \text{Beta}(1, n-1)$. Hence, the UMVUE of $P_\theta(X_1 \leq x) = P_\theta(X_1^p \leq x^p)$ is given by

$$k(T) = \begin{cases} 1 - x^{np}/T^n & \text{if } T > x^p, \\ 1 & \text{if } T \leq x^p. \end{cases}$$

Answer

A statistic T is sufficient for θ if its conditional distribution given the sample does not depend on θ. By the **Factorization Theorem**, a statistic T is sufficient if the joint pdf can be factorized as:

$$f_\theta(x_1, \ldots, x_n) = g(T, \theta)h(x_1, \ldots, x_n).$$

For our case, the joint pdf of $X_1, \ldots, X_n$ is:

$$f_\theta(x_1, \ldots, x_n) = \prod_{i=1}^{n} \left[\exp(-x_i^p/\theta)(p/\theta)x_i^{p-1}\right].$$

Rewriting,

$$f_\theta(x_1, \ldots, x_n) = \left(\frac{p}{\theta}\right)^n \exp\left(-\frac{T}{\theta}\right) \prod_{i=1}^{n} x_i^{p-1}.$$

Here, $g(T, \theta) = \left(\frac{p}{\theta}\right)^n \exp\left(-\frac{T}{\theta}\right)$ depends on the data only through T, and $h(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i^{p-1}$ does not depend on θ. By the Factorization Theorem, T is **sufficient** for θ.

A statistic T is complete if for any function $g(T)$,

$$E_\theta[g(T)] = 0 \quad \forall\theta \Rightarrow P(g(T) = 0) = 1.$$

Since $T = \sum_{i=1}^{n} X_i^p$ follows a **gamma distribution**:

$$T \sim \text{Gamma}(n, \theta),$$

and the gamma family is a **complete exponential family**, T is **complete** for θ.

We now turn to the question of U being ancillary. A statistic U is ancillary if its distribution does not depend on θ. Consider

$$U = \frac{X_1^p}{T}.$$

Since $X_1^p, \ldots, X_n^p$ are **iid** exponential(θ), we can write

$$\left(\frac{X_1^p}{\theta}, \ldots, \frac{X_n^p}{\theta}\right) \sim \text{iid Exp}(1).$$

Thus, $T/\theta \sim \text{Gamma}(n, 1)$, and $U = X_1^p/T$ follows a **Beta(1, n-1)** distribution, which **does not depend on θ**. Hence, U is **ancillary**.

2.

Problem 7.60, Casella and Berger and the following:

Base

Let $X_1, \ldots, X_n$ be iid gamma(α, β) with α known. Find the best unbiased estimator of $1/\beta$.

Answer

Let $X_1, \ldots, X_n$ be iid gamma(α, β) with α known. We want to find the best unbiased estimator of $1/\beta$.
Since $X_i \sim \text{Gamma}(\alpha, \beta)$, the sum

$$S_n = \sum_{i=1}^{n} X_i$$

follows a **Gamma** distribution:

$$S_n \sim \text{Gamma}(n\alpha, \beta).$$

The expectation is:

$$E_\theta(S_n) = n\alpha\beta.$$

A natural unbiased estimator for $1/\beta$ is:

$$\frac{n\alpha}{S_n},$$

since:

$$E_\theta\left[\frac{n\alpha}{S_n}\right] = \frac{n\alpha}{E_\theta(S_n)} = \frac{n\alpha}{n\alpha\beta} = \frac{1}{\beta}.$$

Since S_n is a **complete sufficient statistic** for β (by the **Factorization Theorem** and **Lehmann-Scheffé Theorem**), any unbiased estimator that is a function of S_n is **UMVUE**. Thus,

$$\frac{n\alpha}{S_n}$$

is the **best unbiased estimator** of $1/\beta$.

a)

Let $S_n = \sum_{i=1}^{n} X_i$. Using Basu's theorem, show X_1/S_n and S_n are independent.

Answer

Using Basu's theorem, we show that X_1/S_n and S_n are independent.

Basu's theorem states that if T is a **complete sufficient statistic** and U is an **ancillary statistic**, then T and U are **independent**.

- We already know that S_n is **complete and sufficient** for β.
- Consider the ratio:

$$U = \frac{X_1}{S_n}.$$

The distribution of U does not depend on β. Specifically,

$$U \sim \text{Beta}(\alpha, (n-1)\alpha),$$

which is **free of** β and hence **ancillary**.

By **Basu's theorem**, $U = X_1/S_n$ and S_n are **independent**.

b)

Using the result in a) and $E_\theta(S_n) = n\alpha\beta$, find $E_\theta(X_1/S_n)$.

Answer

Using the results in a), we compute:

$$E_\theta\left(\frac{X_1}{S_n}\right).$$

Since $X_1/S_n \sim \text{Beta}(\alpha, (n-1)\alpha)$, we use the expectation formula for a Beta distribution:

$$E\left(\text{Beta}(a, b)\right) = \frac{a}{a+b}.$$

Thus,

$$E_\theta\left(\frac{X_1}{S_n}\right) = \frac{\alpha}{\alpha + (n-1)\alpha} = \frac{1}{n}.$$

3.

Problem 8.13(a)-(c), Casella and Berger (2nd Edition) and, in place of Problem 8.13(d), consider the following test:

Let X_1, X_2 be iid uniform$(\theta, \theta+1)$. For testing $H_0 : \theta = 0$ versus $H_1 : \theta > 0$, we have two competing tests:

$$\phi_1(X_1) : \text{Reject } H_0 \text{ if } X_1 > 0.95,$$

$$\phi_2(X_1, X_2) : \text{Reject } H_0 \text{ if } X_1 + X_2 > C.$$

a)

Find the value of C so that ϕ_2 has the same size as ϕ_1.

Answer

The size of ϕ_1 is:

$$\alpha_1 = P(X_1 > 0.95 \mid \theta = 0) = 0.05.$$

The size of ϕ_2 is:

$$\alpha_2 = P(X_1 + X_2 > C \mid \theta = 0).$$

For $1 \le C \le 2$, the probability $P(X_1 + X_2 > C \mid \theta = 0)$ is computed as:

$$\alpha_2 = \int_{1-C}^{1} \int_{C-x_1}^{1} 1 \, dx_2 \, dx_1 = \frac{(2-C)^2}{2}.$$

Setting $\alpha_2 = \alpha_1 = 0.05$ and solving for C:

$$\frac{(2-C)^2}{2} = 0.05 \implies (2-C)^2 = 0.1 \implies C = 2 - \sqrt{0.1} \approx 1.68.$$

Thus, the value of C such that ϕ_2 has the same size as ϕ_1 is:

$$C = 2 - \sqrt{0.1} \approx 1.68.$$

b)

Calculate the power function of each test. Draw a well-labeled graph of each power function.

Answer

Power Function of ϕ_1 The power function of ϕ_1 is:

$$\beta_1(\theta) = P_\theta(X_1 > 0.95) = \begin{cases} 0 & \text{if } \theta \le -0.05, \\ \theta + 0.05 & \text{if } -0.05 < \theta \le 0.95, \\ 1 & \text{if } \theta > 0.95. \end{cases}$$

Power Function of ϕ_2 The distribution of $Y = X_1 + X_2$ is:

$$f_Y(y \mid \theta) = \begin{cases} y - 2\theta & \text{if } 2\theta \le y < 2\theta + 1, \\ 2\theta + 2 - y & \text{if } 2\theta + 1 \le y < 2\theta + 2, \\ 0 & \text{otherwise.} \end{cases}$$

The power function of ϕ_2 is:

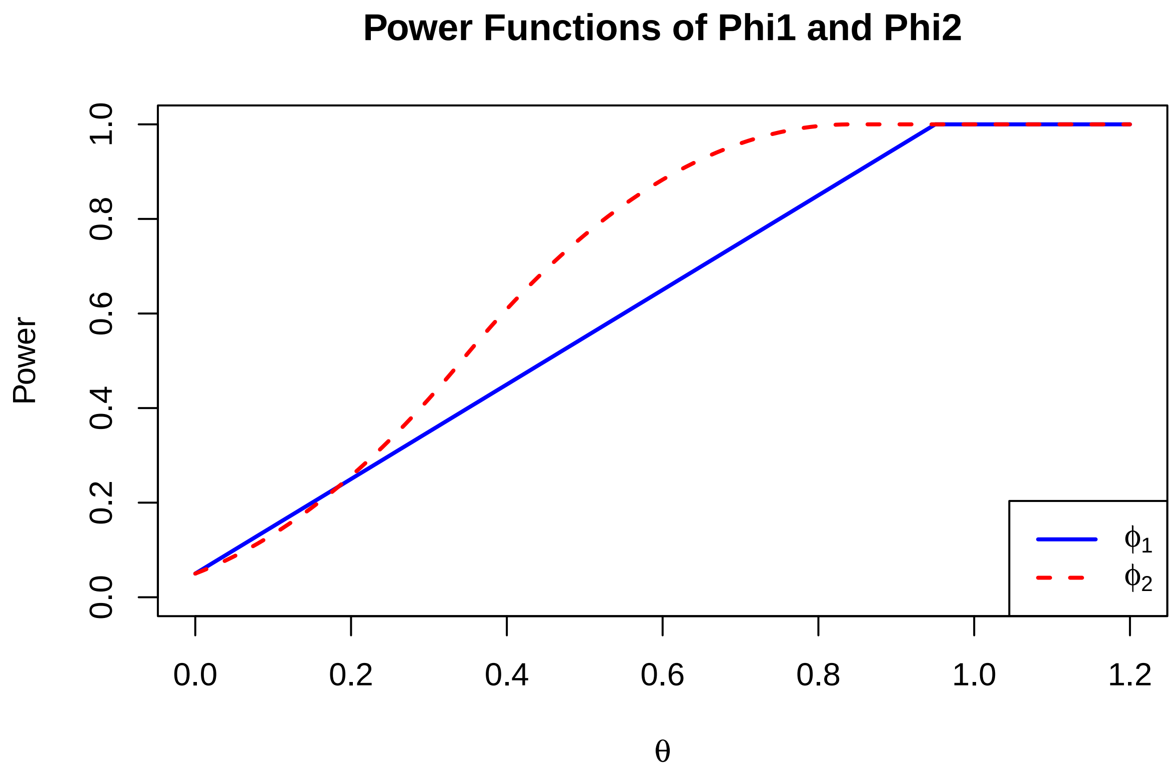
$$\beta_2(\theta) = P_\theta(Y > C) = \begin{cases} 0 & \text{if } \theta \le \frac{C}{2} - 1, \\ \frac{(2\theta+2-C)^2}{2} & \text{if } \frac{C}{2} - 1 < \theta \le \frac{C-1}{2}, \\ 1 - \frac{(C-2\theta)^2}{2} & \text{if } \frac{C-1}{2} < \theta \le \frac{C}{2}, \\ 1 & \text{if } \theta > \frac{C}{2}. \end{cases}$$

```r
theta <- seq(0, 1.2, by = 0.01)
C <- 2 - sqrt(0.1)  # Computed value of C

# Power function for phi_1
beta1 <- pmax(0, pmin(1, theta + 0.05))

# Power function for phi_2
beta2 <- ifelse(theta <= (C/2) - 1, 0,
         ifelse(theta <= (C - 1)/2, ((2*theta + 2 - C)^2)/2,
         ifelse(theta <= C/2, 1 - ((C - 2*theta)^2)/2, 1)))

# Plot
plot(theta, beta1, type = "l", col = "blue", lwd = 2, ylim = c(0, 1),
     ylab = "Power", xlab = expression(theta), main = "Power Functions of Phi1 and Phi2")
lines(theta, beta2, col = "red", lwd = 2, lty = 2)
legend("bottomright", legend = c(expression(phi[1]), expression(phi[2])),
       col = c("blue", "red"), lty = c(1, 2), lwd = 2)
```



c)

Prove or disprove: ϕ_2 is a more powerful test than ϕ_1.

Answer

From the graph in Part (b), we observe that:

- ϕ_1 is more powerful for θ near 0.
- ϕ_2 is more powerful for larger values of θ.

Thus, ϕ_2 is **not uniformly more powerful** than ϕ_1.

Extra

$$\phi_3(X_1, X_2) = \begin{cases} 1 & \text{if } X_{(1)} > 1 - \sqrt{0.05} \text{ or } X_{(2)} > 1 \\ 0 & \text{otherwise} \end{cases}$$

where $X_{(1)}, X_{(2)}$ are the min, max.

Find the size of this test and the power function for $\theta > 0$. Then, graph the power functions of ϕ_3 and ϕ_2 to determine which test is more powerful. (It's enough to graph over the range $\theta \in [0, 1.2]$.)

Answer

Define the test:

$$\phi_3(X_1, X_2) = \begin{cases} 1 & \text{if } X_{(1)} > 1 - \sqrt{0.05} \text{ or } X_{(2)} > 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $X_{(1)}, X_{(2)}$ are the minimum and maximum of X_1, X_2, respectively.

Size of ϕ_3 Under $H_0 : \theta = 0$, the size of ϕ_3 is:

$$\alpha_3 = P(X_{(1)} > 1 - \sqrt{0.05} \mid \theta = 0) = (1 - (1 - \sqrt{0.05}))^2 = 0.05.$$

Power Function of ϕ_3 The power function of ϕ_3 is:

$$\beta_3(\theta) = P_\theta(X_{(1)} > 1 - \sqrt{0.05}) = (1 - (1 - \sqrt{0.05} - \theta))^2.$$

Plot Via the below:

- ϕ_3 is more powerful than ϕ_2 for all $\theta > 0$.

Thus, ϕ_3 is **uniformly more powerful** than ϕ_2.

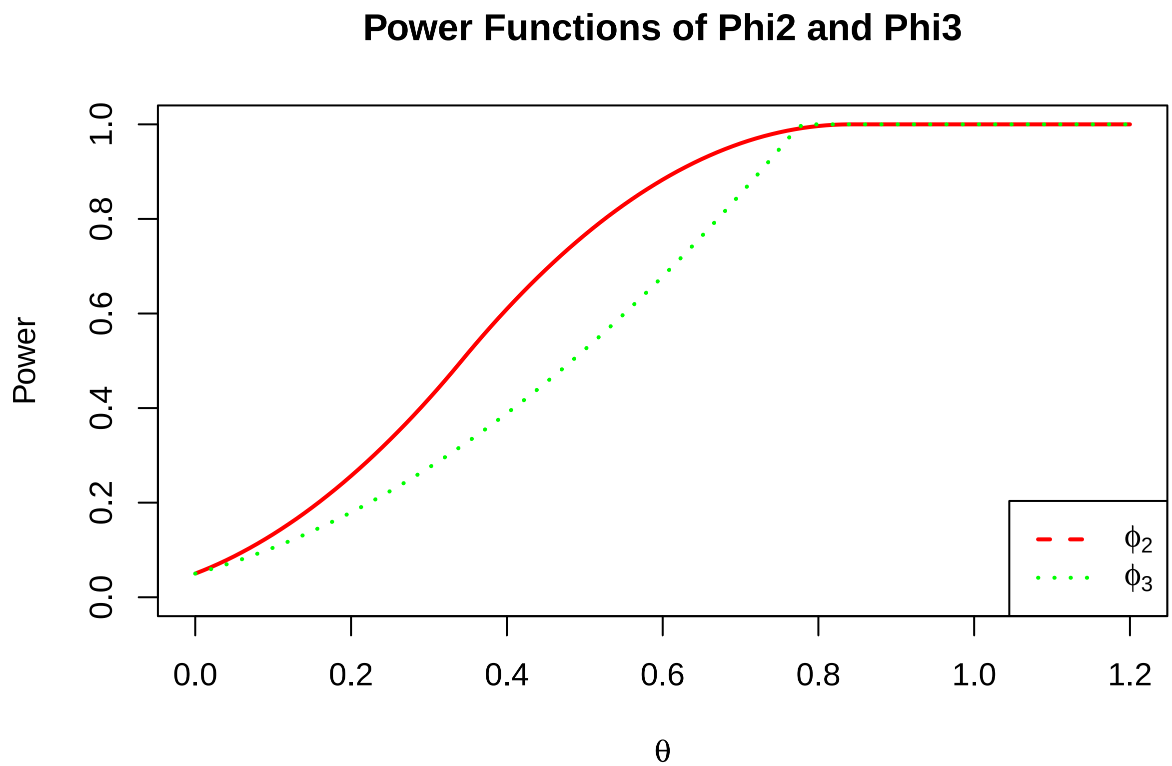
```r
# Define theta range
theta <- seq(0, 1.2, by = 0.01)
C <- 2 - sqrt(0.1)  # Computed value of C for phi_2

# Power function for phi_2
beta2 <- ifelse(theta <= (C/2) - 1, 0,
         ifelse(theta <= (C - 1)/2, ((2*theta + 2 - C)^2)/2,
         ifelse(theta <= C/2, 1 - ((C - 2*theta)^2)/2, 1)))

# Power function for phi_3
phi3_power <- function(theta) {
  pmin(1, (1 - pmax(0, 1 - sqrt(0.05) - theta))^2)
}

beta3 <- sapply(theta, phi3_power)

# Plot
plot(theta, beta2, type = "l", col = "red", lwd = 2, ylim = c(0, 1),
     ylab = "Power", xlab = expression(theta), main = "Power Functions of Phi2 and Phi3")
lines(theta, beta3, col = "green", lwd = 2, lty = 3)
legend("bottomright", legend = c(expression(phi[2]), expression(phi[3])),
       col = c("red", "green"), lty = c(2, 3), lwd = 2)
```


Power Functions of Phi2 and Phi3
Power
0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.2
0.4
0.6
0.8
1.0
1.2
θ
ϕ_2
ϕ_3

4.

Problem 8.15, Casella and Berger (2nd Edition), though you can just assume the form given is most powerful (no need to show).

Show that for a random sample $X_1, \ldots, X_n$ from a $\mathcal{N}(0, \sigma^2)$ population, the most powerful test of $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma = \sigma_1$, where $\sigma_0 < \sigma_1$, is given by

$$\phi\left(\sum X_i^2\right) = \begin{cases} 1 & \text{if } \sum X_i^2 > c, \\ 0 & \text{if } \sum X_i^2 \leq c. \end{cases}$$

For a given value of α, the size of the Type I Error, show how the value of c is explicitly determined.

Answer

From the **Neyman-Pearson lemma**, the most powerful (UMP) test rejects H_0 if the likelihood ratio exceeds a threshold k. The likelihood ratio is:

$$\frac{f(x \mid \sigma_1)}{f(x \mid \sigma_0)} = \frac{(2\pi\sigma_1^2)^{-n/2} e^{-\sum_i x_i^2/(2\sigma_1^2)}}{(2\pi\sigma_0^2)^{-n/2} e^{-\sum_i x_i^2/(2\sigma_0^2)}} = \left(\frac{\sigma_0}{\sigma_1}\right)^n \exp\left\{\frac{1}{2}\sum_i x_i^2 \left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)\right\} > k.$$

After some algebra, this inequality simplifies to:

$$\sum_i x_i^2 > \frac{2\log\left(k\left(\sigma_1/\sigma_0\right)^n\right)}{\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)} = c,$$

where c is a constant. This is because $\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2} > 0$ (since $\sigma_0 < \sigma_1$).

Thus, the UMP test rejects H_0 if:

$$\sum_i X_i^2 > c.$$

The critical value c is determined such that the Type I error probability is α, i.e.,

$$\alpha = P_{\sigma_0}\left(\sum_i X_i^2 > c\right).$$

Under H_0, $\sum_i X_i^2/\sigma_0^2$ follows a chi-squared distribution with n degrees of freedom:

$$\sum_i X_i^2/\sigma_0^2 \sim \chi_n^2.$$

Thus, the probability can be rewritten as:

$$\alpha = P_{\sigma_0}\left(\sum_i X_i^2/\sigma_0^2 > c/\sigma_0^2\right) = P\left(\chi_n^2 > c/\sigma_0^2\right).$$

To find c, we solve for the $(1-\alpha)$-quantile of the χ_n^2 distribution:

$$c = \sigma_0^2 \cdot \chi^2_{n,1-\alpha},$$

where $\chi^2_{n,1-\alpha}$ is the $(1-\alpha)$-quantile of the χ^2_n distribution.