

HW8

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Outline

- Q1: Draft
- Q2: Skeleton
- Q3: All but part e) Draft
- Q4: Example 1,2 Draft

Q1

Refer to slide set 12 titled The ANOVA Approach to the Analysis of Linear Mixed-Effects Models, slides 52 – 55. Note that the BLUE $\hat{\beta}_\Sigma$ depends on the variance components σ_e^2 and σ_u^2 . Specifically, the weights of \tilde{y}_{11} , and y_{121} are functions of σ_e^2 and σ_u^2 . On slide 54, we also state that the weights are proportional to the inverse variances of the linear unbiased estimators.

Given the underlying model, show that

$$\frac{\frac{1}{\text{Var}(\tilde{y}_{11.})}}{\frac{1}{\text{Var}(\tilde{y}_{11.})} + \frac{1}{\text{Var}(y_{121})}} = \frac{2\sigma_e^2 + 2\sigma_u^2}{3\sigma_e^2 + 4\sigma_u^2}$$

and consequently

$$\frac{\frac{1}{\text{Var}(y_{121})}}{\frac{1}{\text{Var}(\tilde{y}_{11.})} + \frac{1}{\text{Var}(y_{121})}} = \frac{\sigma_e^2 + 2\sigma_u^2}{3\sigma_e^2 + 4\sigma_u^2}.$$

Answer

First, refer back to the slides being referenced:

Second Example

$$\mathbf{y} = \begin{bmatrix} y_{111} \\ y_{112} \\ y_{121} \\ y_{211} \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{Z} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Figure 1: Slide 52

In this case, it can be shown that

$$\begin{aligned} \hat{\beta}_{\Sigma} &= (\mathbf{X}^{\top} \Sigma^{-1} \mathbf{X})^{-1} \mathbf{X}^{\top} \Sigma^{-1} \mathbf{y} \\ &= \begin{bmatrix} \frac{\sigma_e^2 + \sigma_u^2}{3\sigma_e^2 + 4\sigma_u^2} & \frac{\sigma_e^2 + \sigma_u^2}{3\sigma_e^2 + 4\sigma_u^2} & \frac{\sigma_e^2 + 2\sigma_u^2}{3\sigma_e^2 + 4\sigma_u^2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{111} \\ y_{112} \\ y_{121} \\ y_{211} \end{bmatrix} \\ &= \begin{bmatrix} \frac{2\sigma_e^2 + 2\sigma_u^2}{3\sigma_e^2 + 4\sigma_u^2} \bar{y}_{11\cdot} & + & \frac{\sigma_e^2 + 2\sigma_u^2}{3\sigma_e^2 + 4\sigma_u^2} y_{121} \\ & & y_{211} \end{bmatrix}. \end{aligned}$$

Figure 2: Slide 53

It can be shown that the weights on $\bar{y}_{11\cdot}$ and y_{121} are

$$\frac{\frac{1}{\text{Var}(\bar{y}_{11\cdot})}}{\frac{1}{\text{Var}(\bar{y}_{11\cdot})} + \frac{1}{\text{Var}(y_{121})}} \text{ and } \frac{\frac{1}{\text{Var}(y_{121})}}{\frac{1}{\text{Var}(\bar{y}_{11\cdot})} + \frac{1}{\text{Var}(y_{121})}}, \text{ respectively.}$$

This is a special case of a more general phenomenon: the BLUE is a weighted average of independent linear unbiased estimators with weights for the linear unbiased estimators proportional to the inverse variances of the linear unbiased estimators.

Figure 3: Slide 54

Of course, in this case and in many others,

$$\hat{\beta}_{\Sigma} = \left[\begin{array}{cc} \frac{2\sigma_e^2 + 2\sigma_u^2}{3\sigma_e^2 + 4\sigma_u^2} \bar{y}_{11\cdot} & + \frac{\sigma_e^2 + 2\sigma_u^2}{3\sigma_e^2 + 4\sigma_u^2} y_{121} \\ & y_{211} \end{array} \right]$$

is not an estimator because it is a function of unknown parameters.

Thus, we use $\hat{\beta}_{\hat{\Sigma}}$ as our estimator (i.e., we replace σ_e^2 and σ_u^2 by estimates in the expression above).

Figure 4: Slide 55

The BLUE $\hat{\beta}_\Sigma$ weights $\bar{y}_{11.}$ and y_{121} proportionally to their inverse variances. From the slides, we have:
For the average $\bar{y}_{11.} = \frac{y_{111} + y_{112}}{2}$:

$$\text{Var}(\bar{y}_{11.}) = \frac{\sigma_e^2}{2} + \sigma_u^2$$

since observations share the same random effect u_1 .

For the single observation y_{121} :

$$\text{Var}(y_{121}) = \sigma_e^2 + \sigma_u^2$$

with its own random effect u_2 .

The weights are proportional to inverse variances:

Weight for $\bar{y}_{11.}$:

$$\frac{\frac{1}{\text{Var}(\bar{y}_{11.})}}{\frac{1}{\text{Var}(\bar{y}_{11.})} + \frac{1}{\text{Var}(y_{121})}} = \frac{\frac{1}{\frac{\sigma_e^2}{2} + \sigma_u^2}}{\frac{1}{\frac{\sigma_e^2}{2} + \sigma_u^2} + \frac{1}{\sigma_e^2 + \sigma_u^2}}$$

Simplifying numerator and denominator:

$$= \frac{\frac{2}{\sigma_e^2 + 2\sigma_u^2}}{\frac{2}{\sigma_e^2 + 2\sigma_u^2} + \frac{1}{\sigma_e^2 + \sigma_u^2}} = \frac{2(\sigma_e^2 + \sigma_u^2)}{3\sigma_e^2 + 4\sigma_u^2}$$

Weight for y_{121} :

$$\frac{\frac{1}{\text{Var}(y_{121})}}{\frac{1}{\text{Var}(\bar{y}_{11.})} + \frac{1}{\text{Var}(y_{121})}} = \frac{\frac{1}{\sigma_e^2 + \sigma_u^2}}{\frac{1}{\frac{\sigma_e^2}{2} + \sigma_u^2} + \frac{1}{\sigma_e^2 + \sigma_u^2}} = \frac{\sigma_e^2 + 2\sigma_u^2}{3\sigma_e^2 + 4\sigma_u^2}$$

Thus, the weights match the given expressions:

$$\frac{2\sigma_e^2 + 2\sigma_u^2}{3\sigma_e^2 + 4\sigma_u^2} \quad \text{and} \quad \frac{\sigma_e^2 + 2\sigma_u^2}{3\sigma_e^2 + 4\sigma_u^2}$$

Q2

In SAS Studio in the Stat 510 folder you can find a data set called Machines.xlsx and a SAS program called Proc Mixed Machines Data.sas. Open the SAS program and follow the instructions to read in the data.

a)

How many machines and how many persons are accounted for in the data set? How many unique machine \times person combinations are there?

Answer

b)

Run the proc glm SAS code associated with Model 1. What model does SAS fit? Write out the model using mathematical/statistical notation. Be sure to define all variables and parameters. Use appropriate subscripts where necessary.

Answer

c)

Report the MSE.

Answer

1.

Look at the table containing the Type III SS and explain what information this table provides to us about the model we fit. Provide appropriate interpretations about any terms you deem significant.

Answer

d)

Look at the Interaction plot SAS provides. Based in the interaction plot, what can you conclude about the effect of machine and person?

Answer

e)

Run the proc mixed SAS code associated with Model 2. What model does SAS fit? Write out the model using mathematical/statistical notation. Be sure to define all variables and parameters. Use appropriate subscripts where necessary.

Answer

f)

Report the MSE for Model 2 and compare it to the MSE for Model 1.

Answer

g)

How does the evidence for the fixed effect associated with Machines change? Why does this make sense?

Answer

h)

Report the estimated variance components for this model – there should be two.

Answer

i)

Run the proc mixed SAS code associated with Model 3. What model does SAS fit? Write out the model using mathematical/statistical notation. Be sure to define all variables and parameters. Use appropriate subscripts where necessary.

Answer

j)

Report the MSE for Model 3 and compare it to the MSE for Models 1 and 2. Describe your findings.

Answer

k)

Explain the main difference between Models 2 and 3. Hint: Looking at the table called “Dimensions” in the SAS output might be helpful.

Answer

l)

How does the evidence for the fixed effect associated with Machines change in Model 3 compared to Models 1 and 2? Why does this make sense?

Answer

Q3

In Chapter 12 we discussed two examples illustrating imbalanced designs. For this question we will focus on the second example introduced on slide 52 and compare its analysis to the analysis of the first example.

Relevant SAS code can be found in SAS Studio in a file called 13 Cochran-Satterthwaite Approximation Assignment 8.sas.

First Example

First Example

$$\mathbf{y} = \begin{bmatrix} y_{111} \\ y_{121} \\ y_{211} \\ y_{212} \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{Z} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{X}_1 = \mathbf{1}, \quad \mathbf{X}_2 = \mathbf{X}, \quad \mathbf{X}_3 = \mathbf{Z}$$

$$\mathbf{P}_1 \mathbf{y} = \begin{bmatrix} \bar{y}_{...} \\ \bar{y}_{...} \\ \bar{y}_{...} \\ \bar{y}_{...} \end{bmatrix}, \quad \mathbf{P}_2 \mathbf{y} = \begin{bmatrix} \bar{y}_{1\cdot} \\ \bar{y}_{1\cdot} \\ \bar{y}_{21\cdot} \\ \bar{y}_{21\cdot} \end{bmatrix}, \quad \mathbf{P}_3 \mathbf{y} = \begin{bmatrix} y_{111} \\ y_{121} \\ \bar{y}_{21\cdot} \\ \bar{y}_{21\cdot} \end{bmatrix}$$

Figure 5: Slide 40

Second Example

$$\mathbf{P}_1 \mathbf{y} = \begin{bmatrix} \bar{y}_{...} \\ \bar{y}_{...} \\ \bar{y}_{...} \\ \bar{y}_{...} \end{bmatrix}, \quad \mathbf{P}_2 \mathbf{y} = \begin{bmatrix} \bar{y}_{1\cdot1} \\ \bar{y}_{1\cdot1} \\ \bar{y}_{21\cdot} \\ \bar{y}_{21\cdot} \end{bmatrix}, \quad \mathbf{P}_3 \mathbf{y} = \begin{bmatrix} y_{111} \\ y_{121} \\ \bar{y}_{21\cdot} \\ \bar{y}_{21\cdot} \end{bmatrix}$$

Thus,

$$\begin{aligned} SS_{trt} &= \mathbf{y}^\top (\mathbf{P}_2 - \mathbf{P}_1) \mathbf{y} = \|\mathbf{P}_2 \mathbf{y} - \mathbf{P}_1 \mathbf{y}\|^2 \\ &= (\bar{y}_{1\cdot1} - \bar{y}_{...})^2 + (\bar{y}_{1\cdot1} - \bar{y}_{...})^2 + (\bar{y}_{21\cdot} - \bar{y}_{...})^2 + (\bar{y}_{21\cdot} - \bar{y}_{...})^2 \\ &= 2(\bar{y}_{1\cdot1} - \bar{y}_{...})^2 + 2(\bar{y}_{21\cdot} - \bar{y}_{...})^2 = (\bar{y}_{1\cdot1} - \bar{y}_{21\cdot})^2, \end{aligned}$$

where the last line follows from

$$\bar{y}_{1\cdot1} - \bar{y}_{...} = \bar{y}_{1\cdot1} - (\bar{y}_{1\cdot1} + \bar{y}_{21\cdot})/2 = (\bar{y}_{1\cdot1} - \bar{y}_{21\cdot})/2$$

and

$$\bar{y}_{21\cdot} - \bar{y}_{...} = \bar{y}_{21\cdot} - (\bar{y}_{1\cdot1} + \bar{y}_{21\cdot})/2 = -(\bar{y}_{1\cdot1} - \bar{y}_{21\cdot})/2.$$

Figure 6: Slide 41

Deriving the other sums of squares similarly and noting that $r_1 = 1$, $r_2 = 2$, and $r_3 = 3$ so that the degrees of freedom for each sum of squares is 1, we have



$$\begin{aligned} MS_{trt} &= \mathbf{y}^\top (\mathbf{P}_2 - \mathbf{P}_1) \mathbf{y} = 2(\bar{y}_{1.1} - \bar{y}_{...})^2 + 2(\bar{y}_{21.} - \bar{y}_{...})^2 \\ &= (\bar{y}_{1.1} - \bar{y}_{21.})^2 \end{aligned}$$



$$\begin{aligned} MS_{xu(trt)} &= \mathbf{y}^\top (\mathbf{P}_3 - \mathbf{P}_2) \mathbf{y} = (y_{111} - \bar{y}_{1.1})^2 + (y_{121} - \bar{y}_{1.1})^2 \\ &= \frac{1}{2}(y_{111} - y_{121})^2 \end{aligned}$$



$$\begin{aligned} MS_{ou(xu, trt)} &= \mathbf{y}^\top (\mathbf{I} - \mathbf{P}_3) \mathbf{y} = (y_{211} - \bar{y}_{21.})^2 + (y_{212} - \bar{y}_{21.})^2 \\ &= \frac{1}{2}(y_{211} - y_{212})^2. \end{aligned}$$



Figure 7: Slide 42

$$\begin{aligned} E(MS_{trt}) &= E(\bar{y}_{1.1} - \bar{y}_{21.})^2 \\ &= E(\tau_1 - \tau_2 + \bar{u}_{1.} - u_{21} + \bar{e}_{1.1} - \bar{e}_{21.})^2 \\ &= (\tau_1 - \tau_2)^2 + \text{Var}(\bar{u}_{1.}) + \text{Var}(u_{21}) + \text{Var}(\bar{e}_{1.1}) \\ &\quad + \text{Var}(\bar{e}_{21.}) \\ &= (\tau_1 - \tau_2)^2 + \frac{\sigma_u^2}{2} + \sigma_u^2 + \frac{\sigma_e^2}{2} + \frac{\sigma_e^2}{2} \\ &= (\tau_1 - \tau_2)^2 + 1.5\sigma_u^2 + \sigma_e^2 \end{aligned}$$

Figure 8: Slide 43

$$\begin{aligned}
E(MS_{xu(trt)}) &= \frac{1}{2}E(y_{111} - y_{121})^2 \\
&= \frac{1}{2}E(u_{11} - u_{12} + e_{111} - e_{121})^2 \\
&= \frac{1}{2}(2\sigma_u^2 + 2\sigma_e^2) \\
&= \sigma_u^2 + \sigma_e^2 \\
E(MS_{ou(xu, trt)}) &= \frac{1}{2}E(y_{211} - y_{212})^2 \\
&= \frac{1}{2}E(e_{211} - e_{212})^2 \\
&= \sigma_e^2
\end{aligned}$$

Figure 9: Slide 44

Second Example

$$\mathbf{y} = \begin{bmatrix} y_{111} \\ y_{112} \\ y_{121} \\ y_{211} \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{Z} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Figure 10: Slide 52

a)

Review the derivations of the mean squares and expected mean squares we did for the first example on slides 41–44. Repeat the same steps for the second example. Start out with deriving P_1y , P_2y and P_3y . Write out the corresponding sums of squares/mean squares before taking the expectation of each in the final step.

Answer

For the second example with:

$$\mathbf{y} = \begin{bmatrix} y_{111} \\ y_{112} \\ y_{121} \\ y_{211} \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{Z} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We define the projection matrices:

Overall Mean (P_1):

$$P_1\mathbf{y} = \begin{bmatrix} \bar{y}_{...} \\ \bar{y}_{...} \\ \bar{y}_{...} \\ \bar{y}_{...} \end{bmatrix}, \quad \bar{y}_{...} = \frac{y_{111} + y_{112} + y_{121} + y_{211}}{4}$$

Treatment Means (P_2):

$$P_2\mathbf{y} = \begin{bmatrix} \bar{y}_{1..} \\ \bar{y}_{1..} \\ \bar{y}_{1..} \\ \bar{y}_{2..} \end{bmatrix}, \quad \bar{y}_{1..} = \frac{y_{111} + y_{112} + y_{121}}{3}, \quad \bar{y}_{2..} = y_{211}$$

Subject Means (P_3):

$$P_3\mathbf{y} = \begin{bmatrix} \bar{y}_{11.} \\ \bar{y}_{11.} \\ y_{121} \\ y_{211} \end{bmatrix}, \quad \bar{y}_{11.} = \frac{y_{111} + y_{112}}{2}$$

Sums of Squares

Treatment SS:

$$SS_{\text{trt}} = \mathbf{y}^\top (P_2 - P_1)\mathbf{y} = 3(\bar{y}_{1..} - \bar{y}_{...})^2 + (\bar{y}_{2..} - \bar{y}_{...})^2$$

Subject(Treatment) SS:

$$SS_{\text{subj(trt)}} = \mathbf{y}^\top (P_3 - P_2)\mathbf{y} = 2(\bar{y}_{11.} - \bar{y}_{1..})^2 + (y_{121} - \bar{y}_{1..})^2$$

Error SS:

$$SS_{\text{error}} = \mathbf{y}^\top (I - P_3)\mathbf{y} = (y_{111} - \bar{y}_{11.})^2 + (y_{112} - \bar{y}_{11.})^2$$

Expected Mean Squares

Expected MS for Treatment:

$$E[MS_{\text{trt}}] = (\tau_1 - \tau_2)^2 + \frac{1}{3}(\sigma_u^2 + \sigma_e^2)$$

Expected MS for Subject(Treatment):

$$E[MS_{\text{subj}(\text{trt})}] = \sigma_u^2 + \sigma_e^2$$

Expected MS for Error:

$$E[MS_{\text{error}}] = \sigma_e^2$$

b)

Set up a table similar to the one seen on slide 45 containing the Source of variation and the corresponding expected mean squares.

SOURCE EMS

trt	$(\tau_1 - \tau_2)^2 + 1.5\sigma_u^2 + \sigma_e^2$
$xu(trt)$	$\sigma_u^2 + \sigma_e^2$
$ou(xu, trt)$	σ_e^2

Figure 11: Slide 45

Answer

Source	df	Expected Mean Square
Treatment	1	$(\tau_1 - \tau_2)^2 + 1.5\sigma_u^2 + \sigma_e^2$
Subject(Treatment)	2	$\sigma_u^2 + \sigma_e^2$
Error	1	σ_e^2

c)

Based on the table, what linear combination of expected mean squares provides an unbiased estimator for the variance components in the numerator of the test statistic that we can use to test for a treatment effect?

Answer

To test for a treatment effect, we use the test statistic:

$$F = \frac{MS_{\text{trt}}}{\text{Estimator of variance components in } E[MS_{\text{trt}}]}$$

The expected mean square for treatment includes:

$$E[MS_{\text{trt}}] = (\tau_1 - \tau_2)^2 + 1.5\sigma_u^2 + \sigma_e^2$$

We approximate the variance component part $1.5\sigma_u^2 + \sigma_e^2$ using a linear combination of mean squares:

$$\hat{V} = 1.5 \cdot MS_{\text{subj}(\text{trt})} - 0.5 \cdot MS_{\text{error}}$$

This combination eliminates the fixed treatment effect and gives an unbiased estimator of the variance component portion of the treatment EMS.

Therefore, the test statistic becomes:

$$F = \frac{MS_{\text{trt}}}{1.5 \cdot MS_{\text{subj}(\text{trt})} - 0.5 \cdot MS_{\text{error}}}$$

d)

Calculate the error of using the Cochran-Satterthwaite approximation as done on slide 17 of Chapter 13.

The Cochran-Satterthwaite formula for the approximate degrees of freedom associated with the linear combination of mean squares defined by M is

$$d = \frac{M^2}{\sum_{i=1}^k a_i^2 M_i^2 / d_i} = \frac{\left(\sum_{i=1}^k a_i M_i \right)^2}{\sum_{i=1}^k a_i^2 M_i^2 / d_i}.$$

.

Figure 12: Slide 4

Answer

To test the treatment effect, we use:

$$F = \frac{MS_{\text{trt}}}{1.5 \cdot MS_{\text{subj}(\text{trt})} - 0.5 \cdot MS_{\text{error}}}$$

$$\begin{aligned}
d &= \frac{(1.5MS_{xu(trt)} - 0.5MS_{ou(xu,trt)})^2}{(1.5)^2 [MS_{xu(trt)}]^2 + (-0.5)^2 [MS_{ou(xu,trt)}]^2} \\
&= \frac{(1.5 \times 2.42 - 0.5 \times 0.18)^2}{(1.5)^2 [2.42]^2 + (-0.5)^2 [0.18]^2} \\
&= 0.9504437
\end{aligned}$$

Figure 13: Slide 17

This denominator estimates the variance component portion of the treatment EMS:

$$1.5\sigma_u^2 + \sigma_e^2$$

We apply the Cochran–Satterthwaite approximation to this linear combination:

$$d = \frac{(1.5MS_1 - 0.5MS_2)^2}{(1.5)^2 \cdot \frac{MS_1^2}{df_1} + (-0.5)^2 \cdot \frac{MS_2^2}{df_2}}$$

With:

- $MS_1 = MS_{\text{subj}(trt)} = 2.42$, $df_1 = 2$
- $MS_2 = MS_{\text{error}} = 0.18$, $df_2 = 1$

We compute:

$$d = \frac{(1.5 \cdot 2.42 - 0.5 \cdot 0.18)^2}{(1.5)^2 \cdot \frac{2.42^2}{2} + (-0.5)^2 \cdot \frac{0.18^2}{1}} = \boxed{1.1}$$

Thus, the approximate degrees of freedom for the denominator is:

$$\boxed{d = 1.1}$$

e)

Run all the code in SAS. Verify the work you derived in parts b), c) and d).

Answer

Q4

You have the SAS code to analyze the two mini examples discussed in Chapters 12 and 13. Write R code that replicates these analyses.

Answer

```
library(lme4)
```

```
## Loading required package: Matrix
```

```
library(lmerTest)
```

```
## Warning: package 'lmerTest' was built under R version 4.4.3
```

```
##
```

```
## Attaching package: 'lmerTest'
```

```
## The following object is masked from 'package:lme4':
```

```
##
```

```
##      lmer
```

```
## The following object is masked from 'package:stats':
```

```
##
```

```
##      step
```

Note on Example 1

In Example 1, each subject appears only once. This results in a saturated random effects structure where the subject random effect fully explains the variability in the data. Because no replication is available to estimate residual variance, the model cannot be fit using `lmer()`.

This limitation is expected and highlights a key modeling constraint: REML-based mixed model fitting requires replication to distinguish random effect variance from residual error.

However, we can still proceed analytically by calculating the expected mean squares and manually applying the Cochran–Satterthwaite formula to approximate degrees of freedom.

```
d1 <- data.frame(
  trt = factor(c(1, 1, 1, 2)),
  subject = factor(c("1_1", "1_2", "1_3", "2_1")), # All subjects unique
  y = c(6.4, 4.2, 1.5, 0.9)
)

# Fit using subject as random effect
model1 <- lmer(y ~ trt + (1 | subject), data = d1)
summary(model1)
anova(model1)
```

```
d2 <- data.frame(
  trt = factor(c(1, 1, 1, 2)),
  subject = factor(c("1_1", "1_1", "1_2", "2_1")),
  y = c(6.4, 4.2, 1.5, 0.9)
)

# Fit the mixed model with subject as random effect
model2 <- lmer(y ~ trt + (1 | subject), data = d2)
summary(model2)
```

```
## Linear mixed model fit by REML. t-tests use Satterthwaite's method [
## lmerModLmerTest]
## Formula: y ~ trt + (1 | subject)
## Data: d2
##
## REML criterion at convergence: 9.9
##
## Scaled residuals:
##      Min       1Q   Median       3Q      Max
## -0.5024 -0.4326 -0.2047  0.2280  0.9118
##
## Random effects:
## Groups Name Variance Std.Dev.
## subject (Intercept) 5.405 2.325
## Residual 2.420 1.556
## Number of obs: 4, groups: subject, 3
##
## Fixed effects:
## Estimate Std. Error df t value Pr(>|t|)
## (Intercept) 3.5592 1.8933 0.9721 1.880 0.317
## trt2 -2.6592 3.3778 1.1019 -0.787 0.565
##
## Correlation of Fixed Effects:
## (Intr)
## trt2 -0.561
```

```
anova(model2)
```

```
## Type III Analysis of Variance Table with Satterthwaite's method
## Sum Sq Mean Sq NumDF DenDF F value Pr(>F)
## trt 1.4998 1.4998 1 1.1019 0.6198 0.5653
```

```
VarCorr(model2)
```

```
## Groups Name Std.Dev.
## subject (Intercept) 2.3249
## Residual 1.5556
```