HW6

2024-10-26

Homework 6

Howdy! I realized my equations were getting cut off at times. To address this I've added some redundancy and extra lines to most of the longer equations. If anything is still cut off, please let me know–I will happily provide the RMD or attempt further modifications.

Q1: 4.17, Casella & Berger

Let X be an exponential(1) random variable, and define Y to be the integer part of X+1, that is:

$$Y = i + 1 \text{ iff } i \leq X < i + 1, i = 0, 1, 2, ...$$

(a)

Find the distribution of Y. What well-known distribution does Y have?

$$P(Y = i + 1) = \int_{i}^{i+1} e^{-x} dx = -e^{-x}\Big|_{x=i}^{i+1} = -e^{-(i+1)} + e^{-i} = e^{-i}(1 - e^{-1})$$

This is a geometric distribution with $p = 1 - e^{-1}$, such that

$Y \sim Geom(1 - e^{-1})$

(b)

Find the conditional distribution of X - 4 given $Y \geq 5$

As defined, $Y = i + 1$, such that

$$Y \geq 5 \iff i + 1 \geq 5 \iff X \geq 4$$

Utilizing the distributions as defined and found, we then have

$$P(X - 4 \leq x \mid Y \geq 5) = P(X - 4 \leq x \mid X \geq 4) = P(X \leq x + 4 \mid X \geq 4)$$

$$P(X - 4 \leq x \mid Y \geq 5) = P(X \leq x + 4 \mid X \geq 4) = 1 - P(X > x + 4 \mid X \geq 4) = 1 - P(X > x) = 1 - e^{-x}$$

This sure looks like the memoryless property we observed previously!

$$P(X - 4 \leq x \mid Y \geq 5) = P(X \leq x) = 1 - e^{-x}$$

Q2: 4.32(a), Casella & Berger

(a)

For a hierarchical model:

$$Y|\Lambda \sim Poisson(\Lambda) \text{ and } \Lambda \sim Gamma(\alpha, \beta)$$

find the marginal distribution, mean, and variance of Y. Show that the marginal distribution of Y is a negative binomial if α is an integer.

For y = 0, 1, ..., the marginal distribution of Y is:

$$f_Y(y) = \int_{\lambda=0}^{\infty} f_Y(y|\lambda) f_\Lambda(\lambda) d\lambda = \int_{\lambda=0}^{\infty} \frac{\lambda^y e^{-\lambda}}{y!} \frac{1}{\beta^\alpha \Gamma(\alpha)} \lambda^{\alpha-1} e^{-\lambda/\beta} d\lambda$$

$$f_Y(y) = \frac{1}{y!\beta^\alpha \Gamma(\alpha)} \int_{\lambda=0}^{\infty} \lambda^y e^{-\lambda} \lambda^{\alpha-1} e^{-\frac{\lambda}{\beta}} d\lambda$$

$$f_Y(y) = \frac{1}{y!\beta^\alpha \Gamma(\alpha)} \int_{\lambda=0}^{\infty} \lambda^{y+\alpha-1} e^{\frac{\frac{-\lambda}{\beta}}{(1+\beta)}} d\lambda$$

Note the right-hand term is a Gamma function, i.e. for $u = \lambda(\frac{1+\beta}{\beta}) \to \lambda = u(\frac{\beta}{1+\beta}) \to d\lambda = du(\frac{\beta}{1+\beta})$ and we have:

$$f_Y(y) = \frac{1}{y!\beta^\alpha \Gamma(\alpha)} \int_{u=0}^{\infty} u^{y+\alpha-1} e^{-u} (\frac{\beta}{1+\beta})^{y+\alpha} du$$

$$f_Y(y) = \frac{1}{y!\beta^\alpha \Gamma(\alpha)} (\frac{\beta}{1+\beta})^{y+\alpha} \int_{u=0}^{\infty} u^{y+\alpha-1} e^{-u} du = \frac{1}{y!\beta^\alpha \Gamma(\alpha)} (\frac{\beta}{1+\beta})^{y+\alpha} \Gamma(\alpha+y)$$

Simplifying gives us:

$$f_Y(y) = \binom{y+\alpha-1}{y} \frac{\beta}{1+\beta} \frac{1}{1+\beta}$$

Or, $Y \sim NB(\alpha, \frac{1}{1+\beta})$, where $\alpha \in \mathbb{Z}^+$ for the Gamma function to be evaluated (integral to evaluate/not diverge).

Knowing the distribution of Y allows us to directly compute it's mean and variance, specifically:

$$E[Y] = \alpha \frac{1 - \frac{1}{1+\beta}}{\frac{1}{1+\beta}} = \alpha \frac{\frac{\beta}{1+\beta}}{\frac{1}{1+\beta}} = \alpha\beta$$

However, we may also calculate it using conditional expectation, specifically:

$$E[Y] = E(E[Y|\Lambda]) = E[\Lambda] = \alpha\beta$$

$$Var(Y) = \alpha \frac{1 - \frac{1}{1+\beta}}{(\frac{1}{1+\beta})^2} = \alpha \frac{\frac{\beta}{1+\beta}}{(\frac{1}{1+\beta})^2} = \alpha\beta(1+\beta)$$

Alternatively we have, via EVVE

$$Var(Y) = Var(E[Y|\Lambda]) + E(Var(Y|\Lambda)) = Var(\Lambda) + E[\Lambda] = \alpha\beta^2 + \alpha\beta = \alpha\beta(\beta + 1)$$

Q3

Expectation

(a)

Show that any random variable X (with finite mean) has zero covariance with any real constant c, i.e. $Cov(X, c) = 0$

The covariance of X and c may be written:

$$Cov(X, c) = E[(X - E[X])(c - E[c])] = E[(X - E[X])(c - c)] = E[(X - E[X])0] = E[0] = 0$$

Such that we conclude:

$$Cov(X, c) = 0$$

And we must have the condition that X has a finite mean as $\infty * 0$ is undefined.

(b)

Using the definition of conditional expectation, show that

$$E[g(X)h(Y)|X = x] = g(x)E[h(Y)|X = x]$$

for an x with pdf $f_X(x) > 0$ (You may also assume (X, Y) are jointly discrete).

To show that

$$E[g(X)h(Y) \mid X = x] = g(x)E[h(Y) \mid X = x]$$

For jointly discrete random variables X and Y, the conditional expectation of h(Y) | X = x is:

$$E[h(Y) \mid X = x] = \sum_y h(y)P(Y = y \mid X = x)$$

Similarly, the conditional expectation of g(X)h(Y) | X = x is:

$$E[g(X)h(Y) \mid X = x] = \sum_y g(x)h(y)P(Y = y \mid X = x)$$

We can then evaluate and simplify the expression:

$$E[g(X)h(Y) \mid X = x] = \sum_y g(x)h(y)P(Y = y \mid X = x) = g(x)\sum_y h(y)P(Y = y \mid X = x)$$

However, this is a very familiar formula to us!

$$E[g(X)h(Y) \mid X = x] = g(x)\sum_y h(y)P(Y = y \mid X = x) = g(x)E[h(Y) \mid X = x]$$

Note:

The condition $f_X(x) > 0$ is necessary as it ensures that $P(Y = y \mid X = x)$ is defined, because:

$$P(Y = y \mid X = x) = \frac{P(X = x, Y = y)}{P(X = x)} \equiv \frac{P(X = x, Y = y)}{f_X(x)}$$

Q4

Suppose that X_i has mean μ_i and variance σ_i^2, for i = 1, 2, and that the covariance of X_1 and X_2 is σ_{12}. Compute the covariance between $X_1 - 2X_2 + 8$, and then compute the covariance of $3X_1 + X_2$.

We want to find:

$$Cov(X_1 - 2X_2 + 8, 3X_1 + X_2) = Cov(X_1 - 2X_2, 3X_1 + X_2)$$

$$Cov(X_1 - 2X_2, 3X_1 + X_2) = Cov(X_1, 3X_1) + Cov(X_1, X_2) - Cov(2X_2, 3X_1) - Cov(2X_2, X_2)$$

$$Cov(X_1 - 2X_2, 3X_1 + X_2) = 3Cov(X_1, X_1) + Cov(X_1, X_2) - 6Cov(X_1, X_2) - 2Cov(X_2, X_2)$$

$$Cov(X_1 - 2X_2 + 8, 3X_1 + X_2) = 3\sigma_1^2 - 5\sigma_1\sigma_2 - 2\sigma_2^2$$

Q5

The joint distribution of X, Y is given by the joint pdf:

$$f(x,y) = 3(x+y) \text{ for } 0<x<1, 0<y<1, 0<x+y<1$$

(a)

Find the marginal distribution of $f_X(x)$

$$f_X(x) = \int_{y \in Y} f(x,y)\,dy = \int_0^1 f(x,y)\,dy = \int_0^1 3(x+y)dy$$

However, the bounds of the integral as given above are not correct, as:

$$0<x<1,\ 0<y<1,\ 0<x+y<1$$

So we actually have:

$$f_X(x) = \int_0^{1-x} 3(x+y)\,dy = 3\int_0^{1-x} (x+y)\,dy = 3\left[\int_0^{1-x} x\,dy + \int_0^{1-x} y\,dy\right]$$

For the sake of simplifcation, these are:

i.

$$\int_0^{1-x} x\,dy = x(1-x)$$

ii.

$$\int_0^{1-x} y\,dy = \frac{(1-x)^2}{2}$$

Taken together, we have:

$$f_X(x) = 3\left[x(1-x) + \frac{(1-x)^2}{2}\right] = 3\left[(1-x)\left(x + \frac{1-x}{2}\right)\right] = 3(1-x)\left(\frac{x+1}{2}\right) = \frac{3}{2}(1-x)(x+1)$$

Thus, the marginal distribution of X is:

$$f_X(x) = \frac{3}{2}(1-x)(x+1) \quad \text{for } 0<x<1$$

(b)

Find the conditional pdf of Y | X = x, given some 0<x<1.

Using the definition of the conditional pdf, we have:

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{3(x+y)}{\frac{3}{2}(1-x)(x+1)}$$

Again, for constraints:

$$0 < x < 1,\ 0 < y < 1,\ 0 < x + y < 1$$

Simplifying gives us:

$$f_{Y|X}(y|x) = \frac{2(x+y)}{(1-x)(x+1)} \text{ for } 0 < y < 1 - x$$

(c)

Find $E[Y|X = x]$

Using what we derived in part (b), we have:

$$E[Y \mid X = x] = \int_0^{1-x} y\, f_{Y|X}(y|x)\, dy = \int_0^{1-x} y \frac{2(x+y)}{(1-x)(x+1)}$$

$$E[Y \mid X = x] = \frac{2}{(1-x)(x+1)} \int_0^{1-x} y(x+y)\, dy = \frac{2}{(1-x)(x+1)} \int_0^{1-x} yx + y^2 dy$$

$$E[Y \mid X = x] = \frac{2}{(1-x)(x+1)} [\left(\int_0^{1-x} xy\, dy\right) + \left(\int_0^{1-x} y^2 dy\right)]$$

i.

$$\int_0^{1-x} xy\, dy = x \int_0^{1-x} y\, dy = x \left[\frac{(1-x)^2}{2}\right] = \frac{x(1-x)^2}{2}$$

ii.

$$\int_0^{1-x} y^2\, dy = \left[\frac{(1-x)^3}{3}\right]$$

Combining i. and ii. above, we then have:

$$E[Y \mid X = x] = \frac{2}{(1-x)(x+1)} \left(\frac{x(1-x)^2}{2} + \frac{(1-x)^3}{3}\right) = \frac{2(1-x)^2}{(1-x)(x+1)} \left(\frac{x}{2} + \frac{1-x}{3}\right)$$

Simplify, simplify:

$$E[Y \mid X = x] = \frac{2(1-x)}{x+1} \left(\frac{3x + 2 - 2x}{6}\right) = \frac{2(1-x)}{x+1} \left(\frac{x+2}{6}\right) = \frac{(1-x)(x+2)}{3(x+1)}$$

(d)

Given the results in (a), (b), and (c), explain how you know $E[X|Y=y]$ without any further calculation

Given the above results, we can take advantage of symmetry, since the joint pdf of X and Y involves a simple sum of x + y, and the support of each is the same, i.e.

$$f(x,y) = 3(x+y) \quad \text{for } 0<x<1,\, 0<y<1,\, 0<x+y<1$$

So we can effective "swap" any "x" in the prior calculations with "y" (and similarly if we felt inclined to derive everything again we could/would swap the "y" in our calculations with "x").

Given from (c):

$$E[Y \mid X=x] = \frac{(1-x)(x+2)}{3(x+1)}$$

By symmetry, we know:

$$E[X \mid Y=y] = \frac{(1-y)(y+2)}{3(y+1)}$$

(e)

Find $E[E[2XY-Y|X]]$

From the parts above, we know (have already calculated) most everything but $E[XY]$.

$$E[E[2XY-Y \mid X]] = E[2XY-Y] = 2E[XY]-E[Y]$$

Taking advantage of the symmetry property used in part (d), we can easily find the marginal pdf of Y. Namely, as:

$$f_X(x) = \frac{3}{2}(1-x)(x+1), \quad \text{for } 0<x<1$$

Then:

$$f_Y(y) = \frac{3}{2}(1-y)(y+1) \quad \text{for } 0<y<1$$

However, due to symmetry:

$$E[Y] = E[X]$$

So if we calculate $E[X]$, we effectively get $E[Y]$. Let's do that!

$$E[X] = \int_0^1 x{\cdot}f_X(x)\,dx = \int_0^1 x\frac{3}{2}(1-x)(x+1)\,dx = \int_0^1 \frac{3x}{2}(1-x^2) = \frac{3}{2}\int_0^1 x(1-x^2)\,dx = \frac{3}{2}\left(\int_0^1 x\,dx - \int_0^1 x^3\,dx\right)$$

i.

$$\int_0^1 x\,dx = \frac{1}{2}$$

$$\int_0^1 x^3\,dx = \frac{1}{4}$$

Taking the above gives us:

$$E[X] = \frac{3}{2}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{3}{2}(\frac{1}{4}) = \frac{3}{8}$$

And $E[Y] = \frac{3}{8}$ too.

Last part now, we need to evaluate $E[XY]$:

$$E[XY] = \int_0^1 \int_0^{1-x} xyf(x,y)dydx = \int_0^1 \int_0^{1-x} xy3(x+y)dydx$$

$$E[XY] = 3\int_0^1 \int_0^{1-x} xy(x+y) = dydx = 3\int_0^1 \int_0^{1-x} (x^2y + xy^2)dydx$$

Separating the integrals:

ii.

$$\int_0^1 \int_0^{1-x} x^2 y\,dy\,dx = \int_0^1 x^2 \left(\frac{(1-x)^2}{2}\right) dx = \frac{1}{2}\int_0^1 x^2(1-x)^2\,dx = \frac{1}{2}\left(\int_0^1 x^2\,dx - 2\int_0^1 x^3\,dx + \int_0^1 x^4\,dx\right)$$

Where:

$$\int_0^1 x^2\,dx = \frac{1}{3}$$

$$\int_0^1 x^3\,dx = \frac{1}{4}$$

$$\int_0^1 x^4\,dx = \frac{1}{5}$$

And our total for the "first" term is then

$$\frac{1}{2}\left(\frac{1}{3} - 2(\frac{1}{4}) + \frac{1}{5}\right) = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = \frac{1}{2}\left(\frac{10}{30} - \frac{15}{30} + \frac{6}{30}\right) = \frac{1}{2}(\frac{1}{30}) = \frac{1}{60}$$

iii.

$$\int_0^1 \int_0^{1-x} xy^2 \, dy \, dx = \int_0^1 x \left(\frac{(1-x)^3}{3} \right) dx = \frac{1}{3} \int_0^1 x(1-x)^3 \, dx = \frac{1}{3} \int_0^1 (x - 3x^2 + 3x^3 - x^4) \, dx$$

$$\frac{1}{3} \left(\frac{1}{2} - 3 \cdot \frac{1}{3} + 3 \cdot \frac{1}{4} - \frac{1}{5} \right) = \frac{1}{3} \left(\frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5} \right) = \frac{1}{3} \left(\frac{30}{60} - \frac{60}{60} + \frac{45}{60} - \frac{12}{60} \right) = \frac{1}{3} \cdot \frac{3}{60} = \frac{1}{60}$$

Crazy, $\frac{1}{60}$ again... symmetry?

Taking the two parts above, ii. and iii., we then have:

$$E[XY] = 3 \left(\frac{1}{60} + \frac{1}{60} \right) = \frac{6}{60} = \frac{1}{10}$$

$$2E[XY] = 2(\frac{1}{10}) = \frac{1}{5}$$

And with:

$$E[Y] = \frac{3}{8}$$

We may finally calculate the desired value as:

$$E[E[2XY - Y \mid X]] = \frac{1}{5} - \frac{3}{8} = \frac{8}{40} - \frac{15}{40} = -\frac{7}{40} = -\frac{7}{40}$$

Q6

Suppose that $f(x,y) = e^{-y}$ for $0 < x < y < \infty$

(a)

Find the joint moment generating function for (X, Y).

The joint moment generating function $M_{X,Y}(t_1, t_2)$ may be defined:

$$M_{X,Y}(t_1,t_2) = E\left[e^{t_1X+t_2Y}\right] = \int_0^\infty \int_0^y e^{t_1x+t_2y}e^{-y}\,dx\,dy = \int_0^\infty \int_0^y e^{t_1x}e^{(t_2-1)y}\,dx\,dy$$

$$M_{X,Y}(t_1,t_2) = E\left[e^{t_1X+t_2Y}\right] = \int_0^\infty e^{(t_2-1)y}\left(\int_0^y e^{t_1x}\,dx\right)dy = \int_0^\infty e^{(t_2-1)y}(\frac{1}{t_1}(e^{t_1y}-1))dy$$

$$M_{X,Y}(t_1,t_2) = \frac{1}{t_1}\int_0^\infty e^{(t_2-1)y}(e^{t_1y}-1)dy = \frac{1}{t_1}\int_0^\infty e^{(t_2+t_1-1)y} - e^{(t_2-1)y}dy$$

$$M_{X,Y}(t_1,t_2) = \frac{1}{t_1}\int_0^\infty \left(e^{(t_1+t_2-1)y} - e^{(t_2-1)y}\right)\,dy = \frac{1}{t_1}\left[\int_0^\infty \left(e^{(t_1+t_2-1)y}dy\right) - \int_0^\infty \left(e^{(t_2-1)y}\right)\right]dy$$

i.

$$\int_0^\infty e^{(t_1+t_2-1)y}dy = \frac{1}{1-t_1-t_2} \qquad \text{for } t_1+t_2<1$$

ii.

$$\int_0^\infty e^{(t_2-1)y}dy = \frac{1}{1-t_2} \qquad \text{for } t_2<1$$

Combining the two parts above gives us the joint mgf:

$$M_{X,Y}(t_1,t_2) = \frac{1}{t_1}\left(\frac{1}{1-t_1-t_2} - \frac{1}{1-t_2}\right)$$

(b)

Overview

From (c) we know:

$X \sim \text{Exponential}(1) \to f_X(x) = e^{-x}$ $Y \sim \text{Gamma}(\alpha = 2, \beta = 1)$

So

$E[X] = 1, Var(X) = 1$

$E[Y] = 2, Var(Y) = 2$

This is here for my sanity because I keep messing up the calculations.

Calculations using joint mgf

Use the joint moment generating function to find the variance of X, the variance of Y, and the covariance of X and Y.

For a joint MGF of:

$$M_{X,Y}(t_1, t_2) = \frac{1}{t_1}\left(\frac{1}{1-t_1-t_2} - \frac{1}{1-t_2}\right)$$

E[X]

To find the mean of X, we have:

$$M_{X,Y}(t_1, t_2) = \frac{1}{t_1}\left(\frac{1}{1-t_1-t_2} - \frac{1}{1-t_2}\right)$$

$$\frac{\partial}{\partial t_1} M_{X,Y}(t_1, t_2) = \frac{\partial}{\partial t_1}\left(\frac{1}{t_1}\left(\frac{1}{1-t_1-t_2} - \frac{1}{1-t_2}\right)\right) = \frac{-1}{t_1^2}\left(\frac{1}{1-t_1-t_2} - \frac{1}{1-t_2}\right) + \frac{1}{t_1}\frac{1}{(1-t_1-t_2)^2}$$

$$\lim_{t_1 \to 0, t_2 \to 0} \frac{\partial}{\partial t_1} M_{X,Y}(t_1, t_2) = \frac{1}{t_1}\left(\frac{-1}{(1-t_1-t_2)(1-t_2)} + \frac{1}{(1-t_1-t_2)^2}\right)$$

$$\lim_{t_1 \to 0, t_2 \to 0} \frac{\partial}{\partial t_1} M_{X,Y}(t_1, t_2) = (\frac{1}{t_1})\frac{t_1}{(1-t_1-t_2)^2(1-t_2)} = \frac{1}{(1-t_1-t_2)^2(1-t_2)} = \frac{1}{(1-0-0)^2(1-0)} = \frac{1}{1^2 \cdot 1} = 1$$

$$E[X] = \lim_{t_1 \to 0, t_2 \to 0} \frac{\partial}{\partial t_1} M_{X,Y}(t_1, t_2) = 1$$

E[Y]

For the mean of Y, we have:

$$M_{X,Y}(t_1, t_2) = \frac{1}{t_1}\left(\frac{1}{1-t_1-t_2} - \frac{1}{1-t_2}\right)$$

$$\frac{\partial}{\partial t_2} M_{X,Y}(t_1, t_2) = \frac{1}{t_1}\left(\frac{1}{(1-t_1-t_2)^2} - \frac{1}{(1-t_2)^2}\right)$$

Similar as before, as we cannot directly evaluate $(t_1, t_2) = (0, 0)$, we must then evaluate the limit as $t_1 \to \infty$, $t_2 \to \infty$:

$$\frac{\partial}{\partial t_2} M_{X,Y}(t_1, t_2) = \frac{1}{t_1}\left(\frac{1}{(1-t_1-t_2)^2} - \frac{1}{(1-t_2)^2}\right) = (\frac{1}{t_1})\frac{2t_1}{(1-t_1-t_2)^2(1-t_2)^2} = \frac{2}{(1-t_1-t_2)^2(1-t_2)^2}$$

$$E[Y] = \lim_{t_1 \to 0, t_2 \to 0} \frac{2}{(1-t_1-t_2)^2(1-t_2)^2} = 2$$

Var(X)

We then need to calculate the variance of both X and Y. Starting with X, we have:

$$E[X^2] = \frac{\partial^2}{\partial t_1^2} M_{X,Y}(t_1,t_2) = \frac{2}{t_1^3}\left(\frac{1}{1-t_1-t_2} - \frac{1}{1-t_2}\right) - \frac{2}{t_1^2}\frac{1}{(1-t_1-t_2)^2} + \frac{2}{t_1}\frac{1}{(1-t_1-t_2)^3}$$

$$E[X^2] = \frac{\partial^2}{\partial t_1^2} M_{X,Y}(t_1,t_2)\Big|_{t_1=0,t_2=0} = \frac{2}{t_1^3}\left(\frac{1}{1-t_1-t_2} - \frac{1}{1-t_2}\right) - \frac{2}{t_1^2}\frac{1}{(1-t_1-t_2)^2} + \frac{2}{t_1}\frac{1}{(1-t_1-t_2)^3}$$

Similar as before, as we cannot directly evaluate $(t_1,t_2) = (0,0)$, we must then evaluate the limit as $t_1 \to \infty$, $t_2 \to \infty$:

$$E[X^2] = \frac{\partial^2}{\partial t_1^2} M_{X,Y}(t_1,t_2) = \frac{\partial}{\partial t_1}\left(\frac{1}{(1-t_1-t_2)^2(1-t_2)}\right) = \frac{2}{(1-t_1-t_2)^3(1-t_2)}$$

$$E[X^2] = \frac{\partial^2}{\partial t_1^2} M_{X,Y}(t_1,t_2)\Big|_{t_1=0,t_2=0} \frac{2}{(1-t_1-t_2)^3(1-t_2)} = \frac{2}{(1-0-0)^3(1-0)} = 2$$

Then for Var(X):

$$Var(X) = E(X^2) - [E(X)]^2 = 2 - 1 = 1$$

Var(Y)

For the variance of Y we then have:

$$M_{X,Y}(t_1,t_2) = \frac{1}{t_1}\left(\frac{1}{1-t_1-t_2} - \frac{1}{1-t_2}\right)$$

$$E[Y^2] = \frac{\partial^2}{\partial t_2^2} M_{X,Y}(t_1,t_2)\Big|_{t_1=0,t_2=0} \frac{2}{t_1}\left(\frac{1}{(1-t_1-t_2)^3} - \frac{1}{(1-t_2)^3}\right)$$

Similar as before, as we cannot directly evaluate $(t_1,t_2) = (0,0)$, we must then evaluate the limit as $t_1 \to \infty$, $t_2 \to \infty$:

$$E[Y^2] = \frac{\partial^2}{\partial t_2^2} M_{X,Y}(t_1,t_2)\Big|_{t_1=0,t_2=0} \frac{2}{t_1}\left(\frac{(1-t_2)^3 - (1-t_1-t_2)^3}{(1-t_1-t_2)^3(1-t_2)^3}\right)$$

$$E[Y^2] = \frac{2}{t_1}\left(\frac{3t_1}{(1-t_1-t_2)^3(1-t_2)^3}\right) = \frac{2}{t_1}\frac{t_1(t_1^2 - 3t_1 + 3 - 6t_2 + 3t_1t_2 + 3t_2^2)}{(1-t_1-t_2)^3(1-t_2)^3}$$

$$E[Y^2] = \lim_{t_1\to 0, t_2\to 0} \frac{2}{t_1}\frac{t_1(t_1^2 - 3t_1 + 3 - 6t_2 + 3t_1t_2 + 3t_2^2)}{(1-t_1-t_2)^3(1-t_2)^3}$$

$$E[Y^2] = \lim_{t_1\to 0, t_2\to 0} \frac{6}{(1-t_1-t_2)^3(1-t_2)^3} = \lim_{t_1\to 0, t_2\to 0} \frac{6}{(1-0-0)^3(1-0)^3} = \frac{6}{1} = 6$$

$$Var(Y) = E(Y^2) - [E(Y)]^2 = 6 - 2^2 = 6 - 4 = 2$$

Cov(X, Y)

To find Cov(X, Y), we have:

$$M_{X,Y}(t_1,t_2) = \frac{1}{t_1}\left(\frac{1}{1-t_1-t_2} - \frac{1}{1-t_2}\right)$$

$$E[XY] = \frac{\partial^2}{\partial t_1 \partial t_2} M_{X,Y}(t_1,t_2)\Big|_{t_1=0,t_2=0}$$

$$\frac{\partial}{\partial t_1} M_{X,Y}(t_1,t_2) = \frac{-1}{t_1^2}\left(\frac{1}{1-t_1-t_2} - \frac{1}{1-t_2}\right) + (\frac{1}{t_1})\frac{1}{(1-t_1-t_2)^2} = \frac{1}{(1-t_1-t_2)^2(1-t_2)}$$

Let us then consider deriving with respect to t_2:

$$\frac{\partial^2}{\partial t_1 \partial t_2} M_{X,Y}(t_1,t_2) = \frac{\partial}{\partial t_2}\left(\frac{1}{(1-t_1-t_2)^2(1-t_2)}\right) = \frac{2}{(1-t_1-t_2)^3(1-t_2)} + \frac{1}{(1-t_1-t_2)^2(1-t_2)^2}$$

$$E[XY] = \frac{\partial^2}{\partial t_1 \partial t_2} M_{X,Y}(t_1,t_2)\Big|_{t_1=0,t_2=0} \frac{2}{(1-t_1-t_2)^3(1-t_2)} + \frac{1}{(1-t_1-t_2)^2(1-t_2)^2} = \frac{2}{1} + \frac{1}{1} = 3$$

$$Cov(X,Y) = E[XY] - E[X]E[Y] = 3 - 1(2) = 1$$

So, $Var(X) = 1, Var(Y) = 2,$ and $Cov(X,Y) = 1$.

(c)

Based on the joint moment generating function, identify the marginal distribution of X and the marginal distribution of Y.

Given the joint moment generating function:

$$M_{X,Y}(t_1,t_2) = \frac{1}{t_1}\left(\frac{1}{1-t_1-t_2} - \frac{1}{1-t_2}\right)$$

For the marginal MGF of X, we set $(t_1,t_2) = (t_1,0)$:

$$M_X(t_1) = M_{X,Y}(t_1,0) = \frac{1}{t_1}\left(\frac{1}{1-t_1-t_2} - \frac{1}{1-t_2}\right) = \frac{1}{t_1}\left(\frac{1}{1-t_1} - 1\right)$$

$$M_X(t_1) = M_{X,Y}(t_1,0) = \frac{1}{t_1}\left(\frac{1}{1-t_1} - 1\right) = \frac{1}{t_1}\left(\frac{1-(1-t_1)}{1-t_1}\right) = \frac{t_1}{t_1(1-t_1)} = \frac{1}{1-t_1}$$

This is the MGF of an Exponential distribution with parameter 1, such that:

$$X \sim \text{Exponential}(1) \to f_X(x) = e^{-x}$$

Similarly, for the marginal MGF of Y, we set $(t_1,t_2) = (0,t_2)$:

$$M_Y(t_2) = M_{X,Y}(0,t_2) = \frac{1}{t_1}\left(\frac{1}{1-t_1-t_2} - \frac{1}{1-t_2}\right) = \frac{1}{0}\left(\frac{1}{1-t_2} - \frac{1}{1-t_2}\right)$$

Similar as before, as we cannot directly evaluate $(0, t_2)$, we must then evaluate the limit as $t_1 \to \infty$:

$$M_Y(t_2) = M_{X,Y}(0,t_2) = \frac{1}{t_1}\left(\frac{1}{1-t_1-t_2} - \frac{1}{1-t_2}\right)$$

$$M_{X,Y}(t_1,t_2) = \frac{1}{t_1}\left(\frac{1}{1-t_1-t_2} - \frac{1}{1-t_2} = \frac{(1-t_2)-(1-t_1-t_2)}{(1-t_1-t_2)(1-t_2)}\right)$$

$$M_{X,Y}(t_1,t_2) = \frac{1}{t_1}\left(\frac{1-t_2-1+t_1+t_2}{(1-t_1-t_2)(1-t_2)}\right) = \frac{1}{t_1}\frac{t_1}{(1-t_1-t_2)(1-t_2)} = \frac{1}{(1-t_1-t_2)(1-t_2)}$$

$$f_Y(y) = \lim_{t_1\to 0} M_Y(t_2) = \lim_{t_1\to 0}\frac{1}{(1-t_1-t_2)(1-t_2)} = \frac{1}{(1-t_2)(1-t_2)} = \frac{1}{(1-t_2)^2}$$

Which is the MGF of a Gamma distribution, with parameters 2 and 1, specifically:

$$Y \sim \text{Gamma}(\alpha = 2, \beta = 1)$$

$X \sim \text{Exponential}(1) \to f_X(x) = e^{-x}$ $Y \sim \text{Gamma}(\alpha = 2, \beta = 1) \to f_Y(y) = \frac{1}{\Gamma(2)} y e^{-y} = y e^{-y}$

Q7

Beta-Binomial model: Suppose that the conditional distribution X | P = p is Binomial(n, p) and Suppose P has a Beta(α, β) distribution.

(a)

Using the EVVE formula, find Var(X)

As we know the distribution of X | P = p, we know that its mean and variance are:

$$E[X|P=p] = np$$

$$Var(X|P=p) = np(1-p)$$

Since we also know the distribution of P, we know it has mean and variance:

$$E[P] = \frac{\alpha}{\alpha+\beta}$$

$$Var(P) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Using EVVE, we know:

$$Var(X) = E[Var(X|P)] + Var(E[X|P])$$

Both of these values will need to be evaluated. To that end:

$$E(Var(X|P)) = E(np(1-p)) = nE(p(1-p)) = n\left[E(p) - E(p^2)\right]$$

For a Beta distribution, we know:

$$E(p) = \frac{\alpha}{\alpha+\beta}$$

$$Var(p) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$E(p^2) = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

Additionally, as $E(p(1-p)) = E(p) - E(p^2)$ we have:

$$E(Var(X|P)) = n\left(\frac{\alpha}{\alpha+\beta} - \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}\right) = n\left(\frac{\alpha\beta}{(\alpha+\beta)(\alpha+\beta+1)}\right)$$

$$Var(E(X|P)) = Var(np) = n^2 Var(p) = n^2\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Combining the two parts, we have:

$$Var(X) = n\left(\frac{\alpha\beta}{(\alpha+\beta)(\alpha+\beta+1)}\right) + n^2\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{n\alpha\beta(\alpha+\beta+n)}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

(b)

Suppose that W has a Binomial(n, $\tilde{p}$) distribution having the same mean as X above. For n > 1, show that X has a larger variance than W by a multiplicative factor of:

$$\frac{\alpha+\beta+n}{\alpha+\beta+1} > 1$$

From the Beta-Binomial model, we have:

$$\mathbb{E}(X) = n \cdot E(P) = n \cdot \frac{\alpha}{\alpha+\beta}$$

If the RV W has the same mean as X, then:

$$E(W) = n\tilde{p} = n \cdot \frac{\alpha}{\alpha+\beta}$$

Which means that:

$$\tilde{p} = \frac{\alpha}{\alpha+\beta}$$

Furthermore:

$$Var(W) = n\tilde{p}(1-\tilde{p}) = n\frac{\alpha}{\alpha+\beta}(1-\frac{\alpha}{\alpha+\beta}) = n(\frac{\alpha}{\alpha+\beta})(\frac{\beta}{\alpha+\beta}) = \frac{n\alpha\beta}{(\alpha+\beta)^2}$$

From (a), we have:

$$Var(X) = \frac{n\alpha\beta(\alpha+\beta+n)}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Looking at the ratio of X/W:

$$\frac{Var(X)}{Var(W)} = \frac{\frac{n\alpha\beta(\alpha+\beta+n)}{(\alpha+\beta)^2(\alpha+\beta+1)}}{\frac{n\alpha\beta}{(\alpha+\beta)^2}} = \frac{\alpha+\beta+n}{\alpha+\beta+1}$$

Assuming n > 1 , we have:

$$(\alpha+\beta+n > \alpha+\beta+1)$$

Such that X has a larger variance than W by a multiplicative factor of:

$$\frac{\alpha+\beta+n}{\alpha+\beta+1} > 1$$