

# HW7

2024-11-03

Outline Q1: PARTIAL check (b), (c) Q2: DONE Q3: DONE Q4: DONE Q5: DONE(ish) Q6: DONE Q7:

## Q1

Consider continuous random vector  $(X, Y)$  where  $X \sim N(0, 1)$  (standard normal) and  $Y|X = x \sim N(x, 1)$  (the conditional distribution of  $Y$  given  $X = x$  is normal with mean  $x$  and variance 1).

### (a)

Find the joint pdf of  $(X, Y)$ .

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-x)^2}{2}}$$

$$f_{X,Y}(x, y) = f_X(x) f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-x)^2}{2}} = \frac{1}{2\pi} e^{-\frac{x^2}{2} - \frac{(y-x)^2}{2}}$$

### (b)

Find the marginal distribution of  $Y$ .

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{x^2}{2} - \frac{(y-x)^2}{2}} dx$$

Simplifying the expression in the exponent, we have:

$$-\frac{x^2}{2} - \frac{(y-x)^2}{2} = -\frac{x^2}{2} - \frac{y^2 - 2xy + x^2}{2} = -\frac{y^2}{2} - \frac{1}{2} [x^2 - 2xy] = -\frac{y^2}{2} - \frac{1}{2} (x-y)^2$$

$$f_Y(y) = \frac{1}{2\pi} e^{-\frac{y^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{2}} dx$$

Looking at just the term(s) in the inside of the integral, we have:

$$\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{2}} dx = \sqrt{2\pi}$$

Such that we can evaluate the full expression now as:

$$f_Y(y) = \frac{1}{2\pi} e^{\left(-\frac{y^2}{2}\right)} \sqrt{2\pi} = \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{y^2}{2}\right)}$$

This pdf is the standard normal! Such that we say the marginal of Y given above describes the random variable Y as:

$$Y \sim N(0, 1)$$

(c)

Consider random vector (V, W) where  $W \sim N(0, 2)$  (normal with mean 0 and variance 2). How should the conditional distribution  $V | W = w$  be chosen so that (V, W) has the same distribution as (X, Y)?

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \right)$$

$$E(V | W = w) = \frac{Cov(V, W)}{Var(W)} \times w = \frac{1}{2}w$$

$$Var(V|W = w) = Var(V) - \frac{Cov(V, W)^2}{Var(W)} = 1 - \frac{1^2}{2} = \frac{1}{2}$$

$$V | W = w \sim N \left( \frac{1}{2}w, \frac{1}{2} \right)$$

$$V | W = w \sim N \left( \frac{1}{2}w, \frac{1}{2} \right)$$

## Q2: 4.7 Casella & Berger

A woman leaves for work between 8AM and 8:30AM and takes between 40 and 50 minutes to get there. Let the random variable  $X$  denote her time of departure, and the random variable  $Y$  the travel time. Assuming that these variables are independent and uniformly distributed, find the probability the woman arrives at work before 9AM.

$$X \sim U(0, 30), Y \sim U(40, 50)$$

Given  $X$  and  $Y$  are both Uniformly distributed and independent, then:

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) = \frac{1}{30}(\frac{1}{50-40}) = \frac{1}{300}$$

Define the random variable  $Z = X + Y$ , then we want to know the probability  $Z < 60$  (1 hour, 60 minutes starting at the earliest time 8:00AM):

$$P(Z < 60) = P(X+Y < 60) = \int_{40}^{50} \int_0^{60-y} f_{X,Y}(x,y) dx dy = \int_{40}^{50} \int_0^{60-y} \frac{1}{300} dx dy = \int_{40}^{50} \frac{60-y}{300} dy = -\frac{(y-120)(y)}{600} \Big|_{y=40}^{50} = \frac{1}{2}$$

### Q3: 4.10 Casella & Berger

The random pair  $(X, Y)$  has the distribution:

		X		
		1	2	3
Y	2	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
	3	$\frac{1}{6}$	0	$\frac{1}{6}$
	4	0	$\frac{1}{3}$	0

Figure 1: Img

(a)

Show that  $X$  and  $Y$  are dependent.

To show that two variables are not independent (are dependent), we need to show that the product of the marginal probabilities is not equal to joint probability. To that end:

$$P(X = 1) = \frac{1}{12} + \frac{1}{6} = \frac{3}{12} = \frac{1}{4}$$

$$P(Y = 4) = \frac{1}{3}$$

$$P(X = 1, Y = 4) = 0$$

$$P(X = 1, Y = 4) = 0 \neq \frac{1}{12} = \frac{1}{4} \left( \frac{1}{3} \right) = P(X = 1)P(Y = 4)$$

(b)

Give a probability table for random variables  $U$  and  $V$  that have the same marginals as  $X$  and  $Y$  but are independent.

We need probabilities of the rows to all be equal to one another. We need  $P(U = 1) = P(U = 3)$ , and  $P(V = 2) = P(V = 3) = P(V = 4)$ . To satisfy these conditions we have:

		U		
		1	2	3
V	1	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
	2	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
	3	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
	4	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$

## Q4

Suppose  $X$  and  $Y$  are independent random variables, where both have the same (marginal) geometric( $p$ ) distribution for  $0 < p < 1$ . Find the conditional distribution of  $X$  given  $X + Y = k$  (integer  $k \geq 2$ ).

$$P(X = x|S = k) = \frac{P(X = x, S = k)}{P(S = k)}$$

For the top part of the equation:

$$P(X = x, Y = k - x) = P(X = x)P(Y = k - x)$$

And:

$$P(X = x) = (1 - p)^{x-1}p$$

And:

$$P(Y = k - x) = (1 - p)^{k-x-1}p$$

Given  $X$  and  $Y$  are independent:

$$P(X = x, S = k) = (1 - p)^{x-1}p(1 - p)^{k-x-1}p = P(X = x, S = k) = (1 - p)^{k-2}p^2$$

Consider then the bottom of the initial formula:

$$P(S = k) = \sum_{x=1}^{k-1} P(X = x, Y = k - x) = (k - 1)(1 - p)^{k-2}p^2$$

Returning the overall, conditional formula, we have:

$$P(X = x|S = k) = \frac{P(X = x, S = k)}{P(S = k)} = \frac{(1 - p)^{k-2}p^2}{(k - 1)(1 - p)^{k-2}p^2} = \frac{1}{k - 1}$$

For  $x = 1, 2, \dots, k - 1$

We then conclude

$$X|(X + Y = k) \sim U(1, k - 1)$$

## Q5: 4.27 Casella & Berger

Let  $X \sim n(\mu, \sigma^2)$ , and let  $Y \sim n(\gamma, \sigma^2)$ . Suppose  $X$  and  $Y$  are independent. Define:  $U = X + Y$  and  $V = X - Y$ . Show that  $U$  and  $V$  are independent normal random variables. Find the distribution of each of them.

$$U \sim n(\mu + \gamma, 2\sigma^2)$$

$$V \sim n(\mu - \gamma, 2\sigma^2)$$

For  $X$  and  $Y$  independent, we may write the joint pdf as the product of the marginals:

$$f_{X,Y}(x, y) = f_X(x)f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(y-\gamma)^2}{2\sigma^2}} = \frac{1}{2\pi\sigma^2}e^{-\frac{(x-\mu)^2+(y-\gamma)^2}{2\sigma^2}}$$

$$U = X + Y \text{ and } V = X - Y \rightarrow X = \frac{1}{2}(U + V), Y = \frac{1}{2}(U - V)$$

We then have the Jacobian:

$$|J| = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \left| -\frac{1}{4} - \frac{1}{4} \right| = \left| -\frac{1}{2} \right| = \frac{1}{2}$$

And we have the new support of  $U, V$  as:

$$(u, v) \in (-\infty, \infty) \times (-\infty, \infty)$$

Using the above relations, we have:

$$f_{UV}(u, v) = \frac{1}{2\pi\sigma^2}e^{-\frac{1}{2\sigma^2} \left[ \left( \frac{u+v}{2} - \mu \right)^2 + \left( \frac{u-v}{2} - \gamma \right)^2 \right]} \frac{1}{2}$$

$$f_{UV}(u, v) = \frac{1}{4\pi\sigma^2}e^{-\frac{1}{2\sigma^2} \left[ 2\left( \frac{u}{2} \right)^2 - u(\mu + \gamma) + \left( \frac{\mu + \gamma}{2} \right)^2 + 2\left( \frac{v}{2} \right)^2 - v(\mu - \gamma) + \left( \frac{\mu - \gamma}{2} \right)^2 \right]}$$

$$f_{UV}(u, v) = g(u) \frac{1}{4\pi\sigma^2}e^{-\frac{1}{2(2\sigma^2)}(u - (\mu + \gamma))^2} h(v) e^{-\frac{1}{2(2\sigma^2)}(v - (\mu - \gamma))^2}$$

Such that we have shown:

For:

$$f_U(u) = g(u) \frac{1}{4\pi\sigma^2}e^{-\frac{1}{2(2\sigma^2)}(u - (\mu + \gamma))^2}$$

And:

$$f_V(v) = h(v) e^{-\frac{1}{2(2\sigma^2)}(v - (\mu - \gamma))^2}$$

$$f_{UV}(u, v) = f_U(u)f_V(v)$$

and we conclude that  $U$  and  $V$  are independent.

## Q6: 4.42 Casella & Berger

Let  $X$  and  $Y$  be independent random variables with means  $\mu_X, \mu_Y$  and variances  $\sigma_X^2, \sigma_Y^2$ . Find an expression for the correlation of  $XY$  and  $Y$  in terms of these means and variances.

Said differently, we want to find:  $\rho_{XY,Y}$

$$\rho_{XY,Y} = \frac{Cov(XY, Y)}{\sigma_{XY}\sigma_Y} = \frac{E(XY^2) - \mu_X\mu_Y}{\sigma_{XY}\sigma_Y} = \frac{EXEY^2 - \mu_X\mu_Y\mu_Y}{\sigma_{XY}\sigma_Y}$$
$$\rho_{XY,Y} = \frac{\mu_X(\sigma_Y^2 + \mu_Y^2) - \mu_X\mu_Y\mu_Y}{\sigma_{XY}\sigma_Y} = \frac{\mu_X(\sigma_Y^2)}{\sigma_{XY}\sigma_Y}$$

With note of  $X$  and  $Y$  being independent, we then have the denominator:

$$\sigma_{XY}^2 = E(XY)^2 - [E(XY)]^2 = EX^2EY^2 - (EX)^2(EY)^2 = (Var(X) + E[X]^2)(Var(Y) + E[Y]^2) - (EX)^2(EY)^2$$

Evaluating with known quantities, we have:

$$\sigma_{XY}^2 = (\sigma_X^2 + \mu_X^2)(\sigma_Y^2 + \mu_Y^2) - \mu_X^2\mu_Y^2 = \sigma_X^2\sigma_Y^2 + \sigma_X^2\mu_Y^2 + \sigma_Y^2\mu_X^2 + \mu_X^2\mu_Y^2 - \mu_X^2\mu_Y^2 = \sigma_X^2\sigma_Y^2 + \sigma_X^2\mu_Y^2 + \sigma_Y^2\mu_X^2$$

Substituting into the original equation gives us:

$$\rho_{XY,Y} = \frac{\mu_X(\sigma_Y^2)}{\sqrt{(\sigma_X^2\sigma_Y^2 + \sigma_X^2\mu_Y^2 + \sigma_Y^2\mu_X^2)}\sigma_Y} = \frac{\mu_X\sigma_Y}{\sqrt{(\mu_X^2\sigma_Y^2 + \mu_Y^2\sigma_X^2 + \sigma_X^2\sigma_Y^2)}}$$

## Q7

Suppose that random variables  $X_1, X_2, X_3$  have joint pdf  $f(x_1, x_2, x_3) = 6$  for  $0 < x_1 < x_2 < x_3 < 1$ .

(a)

Are  $X_1, X_2, X_3$  independent? Give an intuitive explanation.

No they are not independent, as knowing the value of one of the random variables limits the range of possible values that the other random variables can take.

(b)

Find the marginal pdf of  $X_2$  and identify the distribution of  $X_2$  as a member of an important family of distributions.

(c)

Find the conditional pdf  $f(x_1, x_3|x_2)$  of  $X_1, X_3$  given  $X_2 = x_2 \in (0, 1)$ .

(d)

Show that  $X_1, X_3$  are independent given  $X_2 = x_2 \in (0, 1)$  (i.e., think of what must be true of the conditional pdf of  $f(x_1, x_3|x_2)$  in this case).

(e)

Find the covariance of  $X_1$  and  $X_2$  given  $X_2 = x_2$ .