Homework 6 – STAT 5430

Due Monday, March 31 by midnight in gradescope;

1. An ecologist takes data (x_i, Y_i), $i = 1, \ldots, n$, where $x_i > 0$ is the size of an area and Y_i is the number of moss plants. The data are modeled assuming $x_1, \ldots, x_n$ are fixed; $Y_1, \ldots, Y_n$ are independent; and Y_i is Poisson(θx_i) distributed with parameter θx_i. Suppose that $\sum_{i=1}^n x_i = 5$ is known. Find an exact form of the most powerful (MP) test of size $\alpha = 9e^{-10}$ for testing $H_0 : \theta = 2$ vs $H_1 : \theta = 1$.

2. Problem 8.19, Casella and Berger (2nd Edition): Just show the form of the MP test involves rejecting H_0 if $e^{y-\sqrt{y}}/\sqrt{y} > k$ for some $k > 1$ (skip the $\alpha = 0.1$ part or Type II error part of the problem statement)

3. Problem 8.20, Casella and Berger (2nd Edition).
   Hint: It holds that $f(x|H_1)/f(x|H_0) = 7 - x + 79/94 I(x = 7)$ over the support $x = 1, 2, \ldots, 7$, where $I(\cdot)$ denotes an indicator function.

4. Recall Method I for finding Uniformly Most Powerful (UMP) tests:

   To find a UMP size α test for $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \notin \Theta_0$, suppose we can fix $\theta_0 \in \Theta_0$ suitably & then use the Neyman-Pearson lemma to find a MP size α test $\varphi(\underset{\sim}{X})$ for $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$, where

   (a) $\varphi(\underset{\sim}{X})$ does not depend on $\theta_1 \notin \Theta_0$ and

   (b) $\max_{\theta \in \Theta_0} \mathrm{E}_\theta \varphi(\underset{\sim}{X}) = \alpha$

   Show that if (a) and (b) both hold, then $\varphi(\underset{\sim}{X})$ must be a UMP size α test for $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \notin \Theta_0$.

   Hint: From (b), the size of the test rule $\varphi(\underset{\sim}{X})$ is correct. So, by definition of a UMP test, it is necessary to prove that: if $\bar{\varphi}(\underset{\sim}{X})$ is any other test of $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \notin \Theta_0$ with size $\max_{\theta \in \Theta_0} \mathrm{E}_\theta \bar{\varphi}(\underset{\sim}{X}) \leq \alpha$ then $\varphi(\underset{\sim}{X})$ has more power over the parameter subspace of H_1 than $\bar{\varphi}(\underset{\sim}{X})$, i.e.,

   $$\mathrm{E}_\theta \varphi(\underset{\sim}{X}) \geq \mathrm{E}_\theta \bar{\varphi}(\underset{\sim}{X}) \quad \textit{any } \theta \notin \Theta_0.$$

   In other words, pick/fix some $\theta_1 \notin \Theta_0$ and try to argue that $\mathrm{E}_{\theta_1} \varphi(\underset{\sim}{X}) \geq \mathrm{E}_{\theta_1} \bar{\varphi}(\underset{\sim}{X})$ must hold... the way to do this is to take the test $\bar{\varphi}(\underset{\sim}{X})$ and simply apply it to testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$.)

5. Problem 8.23, Casella and Berger (2nd Edition)