# HW3

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## 1.

Suppose $X_1, \ldots, X_n$ are iid Bernoulli($p$), $0 < p < 1$.

### a)

Find the information number $I_n(p)$ and make a rough sketch of $I_n(p)$ as a function of $p \in (0, 1)$.

### b)

Find the value of $p \in (0, 1)$ for which $I_n(p)$ is minimal. (This value of $p$ corresponds to the "hardest" case for estimating $p$. That is, when data are generated under this value of $p$ from the model, the variance of an UE of $p$ is potentially largest.)

### c)

Show that $\hat{X}_n = \sum_{i=1}^{n} X_i/n$ is the UMVUE of $p$.

## 2.

Suppose that the random variables $Y_1, \ldots, Y_n$ satisfy

$$Y_i = \beta x_i + \varepsilon_i, \quad i = 1, \ldots, n$$

where $x_1, \ldots, x_n$ are fixed constants and $\varepsilon_1, \ldots, \varepsilon_n$ are iid $N(0, \sigma^2)$; here we assume $\sigma^2 > 0$ is known.

### a)

Find the MLE of $\beta$.

### b)

Find the distribution of the MLE.

### c)

Find the CRLB for estimating $\beta$. (Hint: you'll have to work with the joint distribution $f(y_1, \ldots, y_n|\beta)$ directly, since $Y_1, \ldots, Y_n$ are not iid.)

### d)

Show the MLE is the UMVUE of $\beta$.

## 3.

Suppose $X_1, \ldots, X_n$ are iid normal $N(0, 1)$, where $\theta \in \mathbb{R}$. It turns out that $T = (\bar{X}_n)^2 - n^{-1}$ is the UMVUE of $\gamma(\theta) = \theta^2$. (We can show this later in the course; our goal here is to show that the UMVUE can exist without obtaining the CRLB.)

### a)

Show $T$ is an UE of $\gamma(\theta) = \theta^2$ and find the variance $\mathrm{Var}_\theta(T)$ of $T$. (Note $Z = \sqrt{n}(\bar{X}_n - \theta) \sim N(0, 1)$ and one can write $T = (Z^2/n) + (2\theta Z/\sqrt{n}) + \theta^2 - n^{-1}$, where $Z^2 \sim \chi^2_1$, $E_\theta Z^2 = 1$, $\mathrm{Var}_\theta(Z^2) = 2$.)

### b)

Find the CRLB for an UE of $\gamma(\theta) = \theta^2$.

### c)

Show that $\mathrm{Var}_\theta(T) > \mathrm{CRLB}$ for all values of $\theta \in \mathbb{R}$.

## 4.

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(“better” here refers to MSE as a criterion.)

Let $X$ be an observation from the pdf

$$f(x|\theta) = \left(\frac{\theta}{2}\right)^{|x|} (1-\theta)^{1-|x|}, \quad x = -1, 0, 1; \quad 0 \leq \theta \leq 1.$$

### a)

Find the MLE of $\theta$.

### b)

Define the estimator $T(X)$ by

$$T(X) = \begin{cases} 2 & \text{if } x = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Show that $T(X)$ is an unbiased estimator of $\theta$.

### c)

Find a better estimator than $T(X)$ and prove that it is better.

## 5.

Let $X_1, \ldots, X_n$ be iid Bernoulli$(\theta)$, $\theta \in (0,1)$. Find the Bayes estimator of $\theta$ with respect to the uniform$(0,1)$ prior under the loss function

$$L(t,\theta) = \frac{(t-\theta)^2}{\theta(1-\theta)}.$$