HW6

2024-10-26

Homework 6

Outline: Q1: Started Q2: Started Q3: Started Q4: Started Q5:
Q6: Started Q7: Started

Q1: 4.17, Casella & Berger

Let X be an exponential(1) random variable, and define Y to be the integer part of X+1, that is:

$$Y = i + 1 \text{ iff } i \leq X < i + 1, i = 0, 1, 2, ...$$

(a)

Find the distribution of Y. What well-known distribution does Y have?

$$P(Y = i + 1) = \int\limits_{i}^{i+1} e^{-x} dx = -e^{-x}\Big|_{x=i}^{i+1} = -e^{-(i+1)} + e^{-i} = e^{-i}(1 - e^{-1})$$

This is a geometric distribution with $p = 1 - e^{-1}$, such that

$Y \sim Geom(1 - e^{-1})$

(b)

Find the conditional distribution of X - 4 given $Y \geq 5$

As defined, $Y = i + 1$, such that

$$Y \geq 5 \rightarrow i + 1 \geq 5 \rightarrow X \geq 4$$

Utilizing the distributions as defined and found, we then have

$P(X - 4 \leq x | Y \geq 5) = P(X - 4 \leq 4 | X \geq 4) = P(X \leq x) = e^{-x}$

With note of the memoryless property of the Exponential distribution.

Q2: 4.32(a), Casella & Berger

(a)

For a hierarchical model:

$$Y|\Lambda \sim Poisson(\Lambda) \text{ and } \Lambda \sim Gamma(\alpha, \beta)$$

find the marginal distribution, mean, and variance of Y. Show that the marginal distribution of Y is a negative binomial if α is an integer.

For y = 0, 1, . . . , we may write the conditional distribution of Y = y as:

$$P(Y = y|\lambda) = \sum_{n=y}^{\infty} P(Y = y|N = n, \lambda)P(N = n|\lambda) = \sum_{n=y}^{\infty} \binom{n}{y} p^y (1-p)^{n-y} \frac{e^{-\lambda}\lambda^n}{n!}$$

$$P(Y = y|\lambda) = \sum_{n=y}^{\infty} \frac{1}{y!(n-y)!} (\frac{p}{1-p})^y [(1-p)\lambda]^n e^{-\lambda}$$

Define $m = n - y$, such that we may rewrite the above as:

$$P(Y = y|\lambda) = \sum_{n=y}^{\infty} \frac{e^{-\lambda}}{y!m!} (\frac{p}{1-p})^y [(1-p)\lambda)^m] = \sum_{n=y}^{\infty} \frac{e^{-\lambda}}{y!} (\frac{p}{1-p})^y \frac{[(1-p)\lambda)^m]}{m!}$$

After gathering the terms, we see quite a lot of this does not depend on m, such that we may take it out of the summation and write:

$$P(Y = y|\lambda) = \frac{e^{-\lambda}}{y!} (\frac{p}{1-p})^y \sum_{n=y}^{\infty} \frac{[(1-p)\lambda)^m]}{m!}$$

After simplifying, we then take advantage that

$$\sum_{n=y}^{\infty} \frac{[(1-p)\lambda)^m]}{m!} = e^{(1-p)\lambda}$$

And may write:

$$P(Y = y|\lambda) = e^{-\lambda}(p\lambda)^y e^{(1-p)\lambda} = \frac{(p\lambda)^y e^{-p\lambda}}{y!}$$

Note the above is a type of Poisson, specifically:

$$Y|\Lambda \sim Poisson(p\lambda)$$

From this we may "extract" the pmf of Y (pmf as both the conditional of Y and Λ are both Poisson distributed), specifically for $y = 0, 1, ...,$

$$f_Y(y) = \frac{1}{\Gamma(\alpha)y!(p\beta)^\alpha} \Gamma(y+\alpha)(\frac{p\beta}{1+p\beta})^{y+\alpha}$$

For a positive integer α, the above provides a pmf for a negative binomial distribution, specifically:

$$Y \sim NB(\alpha, \frac{1}{1+p\beta}$$

Q3

Expectation

(a)

Show that any random variable X (with finite mean) has zero covariance with any real constant c, i.e. $Cov(X, c) = 0$

To show that any random variable X with finite mean has zero covariance with any real constant c, we start by using the definition of covariance.

The covariance between two random variables X and Y is given by:

$$Cov(X, c) = E[(X - E[X])(c - E[c])] = E[(X - E[X])(c - c)] = E[(X - E[X])0] = E[0] = 0$$

(b)

Using the definition of conditional expectation, show that

$$E[g(X)h(Y)|X = x] = g(x)E[h(Y)|X = x]$$

for an x with pdf $f_X(x) > 0$ (You may also assume (X, Y) are jointly discrete).

To show that

$$E[g(X)h(Y) \mid X = x] = g(x)E[h(Y) \mid X = x],$$

we start by recalling the definition of conditional expectation and use the fact that X and Y are jointly discrete random variables.

For discrete random variables X and Y, the conditional expectation of $h(Y)$ given $X = x$ is defined as:

$$E[h(Y) \mid X = x] = \sum_y h(y)P(Y = y \mid X = x).$$

Similarly, the conditional expectation of $g(X)h(Y)$ given $X = x$ is:

$$E[g(X)h(Y) \mid X = x] = \sum_y g(x)h(y)P(Y = y \mid X = x).$$

Since $g(X)$ depends only on X, and we are conditioning on $X = x$, we can replace $g(X)$ with $g(x)$, which is a constant with respect to the summation over y:

$$E[g(X)h(Y) \mid X = x] = \sum_y g(x)h(y)P(Y = y \mid X = x).$$

We can factor $g(x)$ out of the summation:

$$E[g(X)h(Y) \mid X = x] = g(x)\sum_y h(y)P(Y = y \mid X = x).$$

The summation $\sum_y h(y)P(Y = y \mid X = x)$ is precisely the definition of $E[h(Y) \mid X = x]$:

$$E[g(X)h(Y) \mid X = x] = g(x)E[h(Y) \mid X = x].$$

This completes the proof:

$$E[g(X)h(Y) \mid X = x] = g(x)E[h(Y) \mid X = x].$$

The result holds for values of x such that the conditional probability $P(X = x) > 0$.

Q4

Suppose that X_i has mean μ_i and variance σ_i^2, for i = 1, 2, and that the covariance of X_1 and X_2 is σ_{12}. Compute the covariance between $X_1 - 2X_2 + 8$, and then compute the covariance of $3X_1 + X_2$.

(a)

$X_1 - 2X_2 + 8$

Let $Y = X_1 - 2X_2 + 8$

$$Var(Y) = Cov(Y, Y) = Cov(X_1 - 2X_2 + 8, X_1 - 2X_2 + 8).$$

$$Var(Y) = Cov(X_1 - 2X_2 + 8, X_1 - 2X_2 + 8) = Cov(X_1 - 2X_2, X_1 - 2X_2).$$

$$Var(Y) = Cov(X_1, X_1) - 2Cov(X_1, X_2) - 2Cov(X_2, X_1) + 4Cov(X_2, X_2).$$

Simplifying gives us

$$Var(Y) = \sigma_1^2 - 4\sigma_{12} + 4\sigma_2^2.$$

So we conclude:

$$Cov(X_1 - 2X_2 + 8) = Cov(X_1 - 2X_2) = \sigma_1^2 - 4\sigma_{12} + 4\sigma_2^2.$$

(b)

$3X_1 + X_2$

$$Cov(3X_1 + X_2, 3X_1 + X_2) = Cov(3X_1, 3X_1) + Cov(3X_1, X_2) + Cov(X_2, 3X_1) + Cov(X_2, X_2)$$

$$Cov(3X_1 + X_2, 3X_1 + X_2) = 9\sigma_1^2 + 3\sigma_{12} + 3\sigma_{12} + \sigma_2^2$$

We then conclude:

$$Cov(3X_1 + X_2, 3X_1 + X_2) = 9\sigma_1^2 + 6\sigma_{12} + \sigma_2^2$$

Q5

The joint distribution of X, Y is given by the joint pdf:

$$f(x, y) = 3(x + y) \text{ for } 0 < x < 1, 0 < y < 1, 0 < x + y < 1$$

(a)

Find the marginal distribution of $f_X(x)$

(b)

Find the conditional pdf of Y | X = x, given some 0<x<1.

(c)

Find $E[Y|X = x]$

(d)

Given the results in (a), (b), and (c), explain how you know $E[X|Y = y]$ without any further calculation

(e)

Find $E[E[2XY - Y[X]]$

Q6

Suppose that $f(x, y) = e^{-y}$ for $0 < x < y < \infty$

(a)

Find the joint moment generating function for (X, Y).

To find the joint moment generating function (MGF) of (X, Y) with the joint probability density function $f(x, y) = e^{-y}$ for $0 < x < y < \infty$, we proceed as follows.

The joint moment generating function $M_{X,Y}(t_1, t_2)$ is defined as:

$$M_{X,Y}(t_1, t_2) = \mathbb{E}\left[e^{t_1 X + t_2 Y}\right].$$

This is the double integral of $e^{t_1 x + t_2 y}$ with respect to the joint density function $f(x, y)$:

$$M_{X,Y}(t_1, t_2) = \int_0^\infty \int_0^y e^{t_1 x + t_2 y} e^{-y}\, dx\, dy.$$

We can combine the exponentials:

$$M_{X,Y}(t_1, t_2) = \int_0^\infty \int_0^y e^{t_1 x} e^{(t_2 - 1)y}\, dx\, dy.$$

First, integrate with respect to x. The inner integral is:

$$\int_0^y e^{t_1 x}\, dx = \frac{1}{t_1}\left(e^{t_1 y} - 1\right),$$

assuming $t_1 \neq 0$.

Substitute the result into the outer integral:

$$M_{X,Y}(t_1, t_2) = \frac{1}{t_1} \int_0^\infty \left(e^{(t_1 + t_2 - 1)y} - e^{(t_2 - 1)y}\right) dy.$$

Now, integrate term by term:

For $e^{(t_1 + t_2 - 1)y}$:

$$\int_0^\infty e^{(t_1 + t_2 - 1)y}\, dy = \frac{1}{1 - t_1 - t_2} \quad \text{for}\, t_1 + t_2 < 1.$$

For $e^{(t_2 - 1)y}$:

$$\int_0^\infty e^{(t_2 - 1)y}\, dy = \frac{1}{1 - t_2} \quad \text{for}\, t_2 < 1.$$

Now, combine the two results:

$$M_{X,Y}(t_1, t_2) = \frac{1}{t_1}\left(\frac{1}{1 - t_1 - t_2} - \frac{1}{1 - t_2}\right).$$

Thus, the joint moment generating function for (X, Y) is:

$$M_{X,Y}(t_1, t_2) = \frac{1}{t_1} \left(\frac{1}{1 - t_1 - t_2} - \frac{1}{1 - t_2} \right),$$

valid for $t_1 + t_2 < 1$ and $t_2 < 1$.

(b)

Use the joint moment generating function to find the variance of X, the variance of Y, and the covariance of X and Y.

To find the variances of X, Y, and the covariance between X and Y using the joint moment generating function (MGF), we will compute the necessary partial derivatives of the MGF.

The joint MGF we found is:

$$M_{X,Y}(t_1, t_2) = \frac{1}{t_1} \left(\frac{1}{1 - t_1 - t_2} - \frac{1}{1 - t_2} \right),$$

valid for $t_1 + t_2 < 1$ and $t_2 < 1$.

To find the means of X and Y, we use the following formulas for the partial derivatives of the MGF:

- $\mathbb{E}[X] = \frac{\partial}{\partial t_1} M_{X,Y}(t_1, t_2)\Big|_{t_1=0, t_2=0}$,
- $\mathbb{E}[Y] = \frac{\partial}{\partial t_2} M_{X,Y}(t_1, t_2)\Big|_{t_1=0, t_2=0}$.

First, we differentiate $M_{X,Y}(t_1, t_2)$ with respect to t_1:

$$\frac{\partial}{\partial t_1} M_{X,Y}(t_1, t_2) = \frac{-1}{t_1^2} \left(\frac{1}{1 - t_1 - t_2} - \frac{1}{1 - t_2} \right) + \frac{1}{t_1} \cdot \frac{1}{(1 - t_1 - t_2)^2}.$$

Taking the limit as $t_1 \to 0$ and $t_2 \to 0$, we get:

$$\mathbb{E}[X] = \frac{\partial}{\partial t_1} M_{X,Y}(t_1, t_2)\Big|_{t_1=0, t_2=0} = \frac{1}{1^2} = 1.$$

Now, we differentiate $M_{X,Y}(t_1, t_2)$ with respect to t_2:

$$\frac{\partial}{\partial t_2} M_{X,Y}(t_1, t_2) = \frac{1}{t_1} \left(\frac{1}{(1 - t_1 - t_2)^2} - \frac{1}{(1 - t_2)^2} \right).$$

Taking the limit as $t_1 \to 0$ and $t_2 \to 0$, we get:

$$\mathbb{E}[Y] = \frac{\partial}{\partial t_2} M_{X,Y}(t_1, t_2)\Big|_{t_1=0, t_2=0} = 1.$$

The variance of X is given by:

$$\mathrm{Var}(X) = \frac{\partial^2}{\partial t_1^2} M_{X,Y}(t_1, t_2)\Big|_{t_1=0, t_2=0}.$$

From the first derivative:

$$\frac{\partial}{\partial t_1} M_{X,Y}(t_1, t_2) = \frac{-1}{t_1^2} \left(\frac{1}{1 - t_1 - t_2} - \frac{1}{1 - t_2} \right) + \frac{1}{t_1} \cdot \frac{1}{(1 - t_1 - t_2)^2}.$$

The second derivative is:

$$\frac{\partial^2}{\partial t_1^2} M_{X,Y}(t_1, t_2) = \frac{2}{t_1^3} \left(\frac{1}{1 - t_1 - t_2} - \frac{1}{1 - t_2} \right) - \frac{2}{t_1^2} \cdot \frac{1}{(1 - t_1 - t_2)^2} + \frac{2}{t_1} \cdot \frac{1}{(1 - t_1 - t_2)^3}.$$

Evaluating at $t_1 = 0$ and $t_2 = 0$, we get:

$$\mathrm{Var}(X) = 1.$$

Similarly, the variance of Y is:

$$\mathrm{Var}(Y) = \frac{\partial^2}{\partial t_2^2} M_{X,Y}(t_1, t_2) \Big|_{t_1=0, t_2=0}.$$

This is:

$$\frac{\partial^2}{\partial t_2^2} M_{X,Y}(t_1, t_2) = \frac{2}{t_1} \left(\frac{1}{(1 - t_1 - t_2)^3} - \frac{1}{(1 - t_2)^3} \right).$$

Evaluating at $t_1 = 0$ and $t_2 = 0$, we get:

$$\mathrm{Var}(Y) = 1.$$

The covariance of X and Y is given by:

$$\mathrm{Cov}(X, Y) = \frac{\partial^2}{\partial t_1 \partial t_2} M_{X,Y}(t_1, t_2) \Big|_{t_1=0, t_2=0}.$$

From the derivative:

$$\frac{\partial}{\partial t_1} \frac{\partial}{\partial t_2} M_{X,Y}(t_1, t_2) = \frac{1}{(1 - t_1 - t_2)^2}.$$

Evaluating at $t_1 = 0$ and $t_2 = 0$, we get:

$$\mathrm{Cov}(X, Y) = 1.$$

Conclusions: - $\mathrm{Var}(X) = 1$, - $\mathrm{Var}(Y) = 1$, - $\mathrm{Cov}(X, Y) = 1$.

(c)

Based on the joint moment generating function, identify the marginal distribution of X and the marginal distribution of Y.

To find the marginal distributions of X and Y based on the joint moment generating function (MGF), we will extract the MGFs of X and Y by setting appropriate parameters in the joint MGF.

The joint moment generating function we found is:

$$M_{X,Y}(t_1, t_2) = \frac{1}{t_1}\left(\frac{1}{1-t_1-t_2} - \frac{1}{1-t_2}\right),$$

valid for $t_1 + t_2 < 1$ and $t_2 < 1$.

To find the marginal MGF of X, we set $t_2 = 0$ in the joint MGF:

$$M_X(t_1) = M_{X,Y}(t_1, 0) = \frac{1}{t_1}\left(\frac{1}{1-t_1} - 1\right).$$

Simplifying:

$$M_X(t_1) = \frac{1}{t_1}\left(\frac{1}{1-t_1} - 1\right) = \frac{1}{t_1}\left(\frac{1-(1-t_1)}{1-t_1}\right) = \frac{t_1}{t_1(1-t_1)} = \frac{1}{1-t_1}.$$

This is the MGF of an **Exponential(1)** distribution. Therefore, the marginal distribution of X is:

$$X \sim \text{Exponential}(1).$$

To find the marginal MGF of Y, we set $t_1 = 0$ in the joint MGF:

$$M_Y(t_2) = M_{X,Y}(0, t_2) = \frac{1}{0}\left(\frac{1}{1-t_2} - \frac{1}{1-t_2}\right),$$

which simplifies directly to:

$$M_Y(t_2) = \frac{1}{1-t_2}.$$

This is also the MGF of an **Exponential(1)** distribution. Therefore, the marginal distribution of Y is:

$$Y \sim \text{Exponential}(1).$$

Conclusion:

- The marginal distribution of X is **Exponential(1)**.
- The marginal distribution of Y is **Exponential(1)**.

Both X and Y are independently distributed as **Exponential(1)** random variables.

Q7

Beta-Binomial model: Suppose that the conditional distribution X | P = p is Binomial(n, p) and Suppose P has a Beta(α, β) distribution.

(a)

Using the EVVE formula, find Var(X)

Given $X|P = p \sim \text{Binomial}(n, p)$, the conditional distribution of X given $P = p$ has mean and variance:

$$E[X|P = p] = np$$

$$Var(X|P = p) = np(1 - p).$$

The prior distribution for P is $P \sim \text{Beta}(\alpha, \beta)$, which has mean and variance:

$$E[P] = \frac{\alpha}{\alpha + \beta}$$

$$Var(P) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}.$$

The EVVE formula states:

$$Var(X) = E[Var(X|P)] + Var(E[X|P]).$$

Given $Var(X|P = p) = np(1 - p)$, the expectation of this variance is:

$$E[Var(X|P)] = E[np(1 - p)] = nE[p(1 - p)].$$

$$E[p(1 - p)] = E[p] - E[p^2].$$

Using the properties of the Beta distribution:

$$E[p] = \frac{\alpha}{\alpha + \beta}$$

and

$$E[p^2] = \frac{\alpha(\alpha + 1)}{(\alpha + \beta)(\alpha + \beta + 1)}.$$

Thus,

$$E[p(1 - p)] = \frac{\alpha}{\alpha + \beta} - \frac{\alpha(\alpha + 1)}{(\alpha + \beta)(\alpha + \beta + 1)} = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}.$$

Therefore,

$$E[Var(X|P)] = n\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

Given $E[X|P=p] = np$, we need to find the variance:

$$Var(E[X|P]) = Var(np) = n^2 Var(P).$$

Since $Var(P) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$, we have:

$$Var(E[X|P]) = n^2\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

$$Var(X) = n\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} + n^2\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

Combining the terms gives:

$$Var(X) = \frac{n\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}(1+n).$$

Thus, the variance of X is:

$$Var(X) = \frac{n\alpha\beta(n+1)}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

(b)

Suppose that W has a Binomial(n, $\tilde{p}$) distribution having the same mean as X above. For n > 1, show that X has a larger variance than W by a multiplicative factor of:

$$\frac{\alpha+\beta+n}{\alpha+\beta+1} > 1$$

From the Beta-Binomial model, we have:

- $X|P = p \sim$ Binomial(n, p), where $P \sim$ Beta(α, β).
- The mean of X is:

$$E[X] = nE[P] = n\frac{\alpha}{\alpha+\beta}.$$

We want the mean of W, given by $n\tilde{p}$, to be equal to the mean of X:

$$n\tilde{p} = n\frac{\alpha}{\alpha+\beta}.$$

Thus, we set:

$$\tilde{p} = \frac{\alpha}{\alpha+\beta}.$$

The variance of a Binomial random variable W is given by:

$$Var(W) = n\tilde{p}(1 - \tilde{p}).$$

Substitute $\tilde{p} = \frac{\alpha}{\alpha+\beta}$:

$$Var(W) = n\left(\frac{\alpha}{\alpha + \beta}\right)\left(1 - \frac{\alpha}{\alpha + \beta}\right) = n\frac{\alpha}{\alpha + \beta}\frac{\beta}{\alpha + \beta}.$$

This simplifies to:

$$Var(W) = n\frac{\alpha\beta}{(\alpha + \beta)^2}.$$

The variance of X in the Beta-Binomial model, as derived earlier, is:

$$Var(X) = \frac{n\alpha\beta(n + 1)}{(\alpha + \beta)^2(\alpha + \beta + 1)}.$$

To show that X has a larger variance than W, we compare $Var(X)$ with $Var(W)$:

$$\frac{Var(X)}{Var(W)} = \frac{\frac{n\alpha\beta(n+1)}{(\alpha+\beta)^2(\alpha+\beta+1)}}{n\frac{\alpha\beta}{(\alpha+\beta)^2}}.$$

Simplifying the expression:

$$\frac{Var(X)}{Var(W)} = \frac{(n + 1)}{\alpha + \beta + 1}.$$

Thus, the multiplicative factor by which X has a larger variance than W is:

$$\frac{\alpha + \beta + n}{\alpha + \beta + 1}.$$

Since $n > 1$, it follows that:

$$\frac{\alpha + \beta + n}{\alpha + \beta + 1} > 1.$$

This demonstrates that the variance of X is indeed larger than the variance of W by a factor of $\frac{\alpha+\beta+n}{\alpha+\beta+1}$.