HW4

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Problem 1

Problem 6.2, Casella and Berger (2nd Edition)

6.2 Let $X_1, \dots, X_n$ be independent random variables with densities

$$f_{X_i}(x|\theta) = \begin{cases} e^{\theta - x} & x \geq i\theta \\ 0 & x < i\theta. \end{cases}$$

Prove that $T = \min_i(X_i/i)$ is a sufficient statistic for θ.

Answer

(1): Factorization Theorem, a statistic $T(X)$ is sufficient for θ if the joint pdf can be expressed in the form:

$$f(x_1, \dots, x_n|\theta) = g(T(X), \theta)h(x_1, \dots, x_n),$$

where $g(T(X), \theta)$ is a function depending on θ and the data only through $T(X)$, and $h(x_1, \dots, x_n)$ is a function that does not depend on θ.

We are given that $X_1, \dots, X_n$ are iid.

$$f_{X_i}(x|\theta) = \begin{cases} e^{\theta - x} & x \geq i\theta, \\ 0 & x < i\theta. \end{cases}$$

As given the joint pdf of $X_1, \dots, X_n$ is:

$$f(x_1, \dots, x_n|\theta) = \prod_{i=1}^{n} f_{X_i}(x_i|\theta) = \prod_{i=1}^{n} e^{\theta - x_i} \cdot \mathbb{I}_{[i\theta, +\infty)}(x_i)$$

Substituting the given densities:

$$f(x_1, \ldots, x_n|\theta) = \prod_{i=1}^{n} e^{\theta - x_i} \cdot \mathbb{I}_{[i\theta, +\infty)}(x_i).$$

Rewriting the first product,

$$\prod_{i=1}^{n} e^{\theta - x_i} = e^{n\theta - \sum_{i=1}^{n} x_i}.$$

For the second product, we observe:

$$\prod_{i=1}^{n} \mathbb{I}_{[i\theta, +\infty)}(x_i) = \mathbb{I}_{[\theta, +\infty)} \left(\min_i (x_i/i) \right).$$

This follows because the conditions $x_i \geq i\theta$ for all i are equivalent to requiring $\min_i(x_i/i) \geq \theta$.

Thus, the joint pdf becomes:

$$f(x_1, \ldots, x_n|\theta) = e^{n\theta} \cdot \mathbb{I}_{[\theta, +\infty)} \left(\min_i (x_i/i) \right) \cdot e^{-\sum_{i=1}^{n} x_i}.$$

Defining $T(X) = \min_i(X_i/i)$, we can express this as:

$$f(x_1, \ldots, x_n|\theta) = \underbrace{e^{n\theta} \cdot \mathbb{I}_{[\theta, +\infty)}(T(X))}_{g(T(X), \theta)} \cdot \underbrace{e^{-\sum_{i=1}^{n} x_i}}_{h(x_1, \ldots, x_n)}.$$

Since the factor $g(T(X), \theta)$ depends on θ only through $T(X)$, and $h(x_1, \ldots, x_n)$ is independent of θ, it follows that $T(X) = \min_i(X_i/i)$ is a sufficient statistic for θ.

Problem 2

Example of Rao-Blackwell theorem, which is largely a STAT 5420 problem in computation.

Let X_1 and X_2 be iid Bernoulli(p), $0 < p < 1$.

a)

Show $S = X_1 + X_2$ is Sufficient for p

Answer

By the Factorization Theorem, a statistic S is sufficient for p if the joint pmf can be written in the form:

$$f(x_1, x_2|p) = g(S, p) \cdot h(x_1, x_2).$$

The joint pmf of X_1, X_2, given they are iid Bernoulli(p), is:

$$f(x_1, x_2|p) = p^{x_1}(1-p)^{1-x_1} \cdot p^{x_2}(1-p)^{1-x_2}.$$

Rewriting:

$$f(x_1, x_2|p) = p^{x_1+x_2}(1-p)^{2-(x_1+x_2)}.$$

Letting $S = X_1 + X_2$, we obtain:

$$f(x_1, x_2|p) = p^S(1-p)^{2-S}.$$

Since this is of the form $g(S, p) \cdot h(x_1, x_2)$ with $h(x_1, x_2) = 1$, it follows that S is sufficient for p by the Factorization Theorem.

b)

Identify the conditional probability $P(X_1 = x|S = s)$; you should know which values of x, s to consider.

Answer

We compute:

$$P(X_1 = x|S = s) = \frac{P(X_1 = x, S = s)}{P(S = s)}.$$

For possible values of S:

(0): If $S = 0$, then $X_1 = 0$ and $X_2 = 0$, so:

$$P(X_1 = 0|S = 0) = 1.$$

(1): If $S = 2$, then $X_1 = 1$ and $X_2 = 1$, so:

$$P(X_1 = 1|S = 2) = 1.$$

(2): If $S = 1$, then either:

$X_1 = 0, X_2 = 1$, or$X_1 = 1, X_2 = 0$.

Since both occur with equal probability,

$$P(X_1 = 1|S = 1) = P(X_1 = 0|S = 1) = \frac{1}{2}.$$

c)

Find the conditional expectation $T \equiv E(X_1|S)$, i.e., as a function of the possibilities of S. Note that T is a statistic.

Answer

Using the values from part (b):

$$T = E(X_1|S) = \begin{cases} 0, & S = 0, \\ \frac{1}{2}, & S = 1, \\ 1, & S = 2. \end{cases}$$

T is a statistic, noted.

d)

Show X_1 and T are both unbiased for p.

Answer

For X_1:

$$E_p(X_1) = p.$$

For T, using linearity of expectation:

$$E_p(T) = \sum_{s=0}^{2} E(X_1|S = s)P(S = s).$$

Substituting:

$$E_p(T) = 0 \cdot (1 - p)^2 + \frac{1}{2} \cdot 2p(1 - p) + 1 \cdot p^2.$$

$$= p(1 - p) + p^2 = p.$$

Thus, both X_1 and T are unbiased for p.

e)

Show $\text{Var}_p(T) \leq \text{Var}_p(X_1)$, for any p.

Answer

By the Rao-Blackwell theorem:

$$\text{Var}_p(T) \leq \text{Var}_p(X_1).$$

We verify this explicitly.

Since $X_1 \sim \text{Bernoulli}(p)$, we have:

$$\text{Var}_p(X_1) = p(1-p).$$

For T:

$$\text{Var}_p(T) = E_p(T^2) - (E_p(T))^2.$$

Computing $E_p(T^2)$:

$$E_p(T^2) = 0^2 \cdot (1-p)^2 + \left(\frac{1}{2}\right)^2 \cdot 2p(1-p) + 1^2 \cdot p^2.$$

$$= \frac{p(1-p)}{2} + p^2.$$

Thus,

$$\text{Var}_p(T) = \left(\frac{p(1-p)}{2} + p^2\right) - p^2.$$

$$= \frac{p(1-p)}{2}.$$

Since

$$\frac{p(1-p)}{2} \leq p(1-p),$$

it follows that:

$$\text{Var}_p(T) \leq \text{Var}_p(X_1),$$

as required.

Problem 3

Problem 6.21 a)-b), Casella and Berger (2nd Edition)

6.21 Let X be one observation from the pdf

$$f(x|\theta) = \left(\frac{\theta}{2}\right)^{|x|} (1-\theta)^{1-|x|}, \quad x = -1, 0, 1, \quad 0 \le \theta \le 1.$$

a)

Is X a complete sufficient statistic?

Answer

Since X is the only observation, it is trivially sufficient for θ. To determine whether X is complete, we check whether the only function $g(X)$ satisfying $E[g(X)] = 0$ for all θ is the zero function.

We compute:

$$E[g(X)] = \sum_{x \in \{-1,0,1\}} g(x) f(x|\theta).$$

Using the given density,

$$E[g(X)] = g(-1)P(X = -1) + g(0)P(X = 0) + g(1)P(X = 1).$$

Substituting probabilities,

$$E[g(X)] = \frac{\theta}{2} g(-1) + (1-\theta)g(0) + \frac{\theta}{2} g(1).$$

Since this must be zero for all $\theta \in [0, 1]$, we rewrite:

$$\theta \left(\frac{g(-1) + g(1)}{2} - g(0) \right) + g(0) = 0.$$

For this equation to hold for all θ, both coefficients must be zero:

$$\frac{g(-1) + g(1)}{2} - g(0) = 0, \quad g(0) = 0.$$

From $g(0) = 0$, the first equation simplifies to:

$$\frac{g(-1) + g(1)}{2} = 0 \quad \Rightarrow \quad g(-1) + g(1) = 0.$$

This allows nontrivial solutions, such as $g(-1) = 1, g(1) = -1, g(0) = 0$, showing that X is not complete.

b)

Is $|X|$ a complete sufficient statistic?

Answer

First, we check sufficiency using the Factorization Theorem. The pdf is:

$$f(x|\theta) = \left(\frac{\theta}{2}\right)^{|x|} (1-\theta)^{1-|x|}.$$

This depends on X only through $|X|$, so the conditional distribution of X given $|X|$ does not depend on θ. Thus, $|X|$ is sufficient.

Next, we check completeness. The distribution of $|X|$ follows a Bernoulli distribution:

$$P(|X| = 0) = 1 - \theta, \quad P(|X| = 1) = \theta.$$

It is a standard result (Casella & Berger, Example 6.2.11) that the Bernoulli family is complete. Thus, $|X|$ is complete sufficient.

Problem 4

Problem 6.24, Casella and Berger (2nd Edition)

6.24 Consider the following family of distributions:

$$\mathcal{P} = \{P_\lambda(X = x) : P_\lambda(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}; x = 0, 1, 2, \ldots; \lambda = 0 \text{ or } 1\}.$$

This is a Poisson family with λ restricted to be 0 or 1. Show that the family $\mathcal{P}$ is not complete, demonstrating that completeness can be dependent on the range of the parameter. (See Exercises 6.15 and 6.18.)

Answer

To show that $\mathcal{P}$ is not complete, we must find a nonzero function $h(X)$ such that:

$$E_\lambda[h(X)] = 0, \quad \text{for all } \lambda \in \{0, 1\}.$$

A family of distributions is complete if the only function satisfying this expectation condition is the zero function.

For $\lambda = 0$, the Poisson distribution degenerates to:

$$P_{\lambda=0}(X = x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases}$$

Thus, expectation simplifies to:

$$E_{\lambda=0}[h(X)] = h(0).$$

For $E_{\lambda=0}[h(X)] = 0$, we must have:

$$h(0) = 0.$$

For $\lambda = 1$, $X \sim \text{Poisson}(1)$, so:

$$E_{\lambda=1}[h(X)] = \sum_{x=0}^{\infty} h(x) \cdot \frac{1^x e^{-1}}{x!} = e^{-1} \sum_{x=0}^{\infty} \frac{h(x)}{x!}.$$

Since we already established $h(0) = 0$, this simplifies to:

$$\sum_{x=1}^{\infty} \frac{h(x)}{x!} = 0.$$

We seek a function $h(X)$ that satisfies:

$$\sum_{x=1}^{\infty} \frac{h(x)}{x!} = 0, \quad h(0) = 0.$$

A simple choice is:

$$h(0) = 0, \quad h(1) = 1, \quad h(2) = -2, \quad h(x) = 0 \text{ for } x \geq 3.$$

Computing the sum:

$$\sum_{x=1}^{\infty} \frac{h(x)}{x!} = \frac{h(1)}{1!} + \frac{h(2)}{2!} + \sum_{x=3}^{\infty} \frac{h(x)}{x!}.$$

$$= \frac{1}{1} + \frac{-2}{2} + 0 = 1 - 1 = 0.$$

Thus, $E_\lambda[h(X)] = 0$ for both $\lambda = 0$ and $\lambda = 1$, yet $h(X)$ is not identically zero. This demonstrates that the family $\mathcal{P}$ is not complete.

This result highlights that completeness depends on the range of the parameter. The Poisson family is typically complete when $\lambda > 0$, but when restricted to just two values ($\lambda = 0$ or $\lambda = 1$), the condition $E[h(X)] = 0$ no longer forces $h(X)$ to be identically zero.

Thus, the restriction on λ reduces the richness of the family and leads to incompleteness.

Problem 5

Problem 7.57, Casella and Berger (2nd Edition) You may assume $n \geq 3$.

One has to Rao-Blackwellize on the complete/sufficient statistic here

$$\sum_{i=1}^{n+1} X_i.$$

7.57 Let $X_1, \ldots, X_{n+1}$ be iid Bernoulli(p), and define the function $h(p)$ by

$$h(p) = P\left(\sum_{i=1}^{n} X_i > X_{n+1} \middle| p\right),$$

the probability that the first n observations exceed the $(n+1)$st.

a)

Show that

$$T(X_1, \ldots, X_{n+1}) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} X_i > X_{n+1}, \\ 0 & \text{otherwise,} \end{cases}$$

is an unbiased estimator of $h(p)$.

Answer

We define:

$$T(X_1, \ldots, X_{n+1}) = \begin{cases} 1, & \text{if } \sum_{i=1}^{n} X_i > X_{n+1}, \\ 0, & \text{otherwise.} \end{cases}$$

Since T is an indicator function of the event $\sum_{i=1}^{n} X_i > X_{n+1}$, we compute its expectation:

$$E_p[T] = P_p\left(\sum_{i=1}^{n} X_i > X_{n+1}\right) = h(p).$$

Thus, $T(X)$ is an unbiased estimator of $h(p)$.

b)

Find the best unbiased estimator of $h(p)$.

Answer

Since $\sum_{i=1}^{n+1} X_i$ is a complete sufficient statistic for p, the Rao-Blackwell theorem states that the best unbiased estimator of $h(p)$ is:

$$E\left[T \mid \sum_{i=1}^{n+1} X_i = y\right].$$

$$E\left[T \mid \sum_{i=1}^{n+1} X_i = y\right] = P\left(\sum_{i=1}^{n} X_i > X_{n+1} \middle| \sum_{i=1}^{n+1} X_i = y\right).$$

Since X_{n+1} is binary, we analyze two cases:

(0): $X_{n+1} = 0$*

$\sum_{i=1}^{n} X_i = y - X_{n+1} = y - 0 = y$, which means the event $\sum_{i=1}^{n} X_i > X_{n+1}$ always holds when $y \geq 1$.

$$P\left(\sum_{i=1}^{n} X_i > X_{n+1} \mid \sum_{i=1}^{n+1} X_i = y, X_{n+1} = 0\right) = 1.$$

(1): $X_{n+1} = 1$

Here, $\sum_{i=1}^{n} X_i = y - 1$, so the event $\sum_{i=1}^{n} X_i > X_{n+1}$ holds if $y - 1 > 1$, i.e., $y \geq 2$.

$$P\left(\sum_{i=1}^{n} X_i > X_{n+1} \mid \sum_{i=1}^{n+1} X_i = y, X_{n+1} = 1\right) = \mathbb{I}_{y\geq 2}.$$

Since $X_{n+1} \sim \text{Bernoulli}(p)$, we compute the total probability using law of total probability:

$$P\left(\sum_{i=1}^{n} X_i > X_{n+1} \middle| \sum_{i=1}^{n+1} X_i = y\right)$$

$$= P\left(\sum_{i=1}^{n} X_i > X_{n+1} \mid X_{n+1} = 0\right) P(X_{n+1} = 0 \mid \sum X_i = y)$$

$$+ P\left(\sum_{i=1}^{n} X_i > X_{n+1} \mid X_{n+1} = 1\right) P(X_{n+1} = 1 \mid \sum X_i = y).$$

Using:

$$P(X_{n+1} = 1 \mid \sum_{i=1}^{n+1} X_i = y) = \frac{y}{n+1}, \quad P(X_{n+1} = 0 \mid \sum_{i=1}^{n+1} X_i = y) = \frac{n+1-y}{n+1},$$

we compute:

$$P\left(\sum_{i=1}^{n} X_i > X_{n+1} \mid \sum_{i=1}^{n+1} X_i = y\right) = \left(1 \cdot \frac{n+1-y}{n+1}\right) + \left(\mathbb{I}_{y\geq 2} \cdot \frac{y}{n+1}\right).$$

Simplifying,

$$E\left[T \mid \sum_{i=1}^{n+1} X_i = y\right] = \begin{cases} 0, & y = 0, \\ \frac{n+1-y}{n+1}, & y = 1, \\ 1, & y \geq 2. \end{cases}$$

$$E\left[T \mid \sum_{i=1}^{n+1} X_i = y\right] = \begin{cases} 0, & y = 0, \\ \frac{n+1-y}{n+1}, & y = 1, \\ 1, & y \geq 2. \end{cases}$$

Thus, the best unbiased estimator of $h(p)$ is:

$$\delta(X) = \begin{cases} 0, & y = 0, \\ \frac{n+1-y}{n+1}, & y = 1, \\ 1, & y \geq 2. \end{cases}$$