HW8

2024-11-19

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Q1

Let X_1 and X_2 be independent exponential random variables with mean θ.

(a)

Find the joint moment generating function of X_1 and X_2.

(b)

Give the definition of the moment generating function of $X_1 - X_2$ and show how this can be obtained from part (a).

(c)

Find the distribution of $Y = X_1 - X_2$. Using the mgf, one can find that this is a so-called Laplace or double-exponential distribution.

Q2: 4.30, Casella & Berger

Suppose the distribution of Y, conditional on $X = x$, is $N(x, x^2)$ and that the marginal distribution of X is uniform $(0, 1)$.

(a)

Find $E[Y]$, $\mathrm{Var}[Y]$, and $\mathrm{Cov}(X, Y)$.

(b)

Prove that $\frac{Y}{X}$ and X are independent.

Q3: 4.54, Casella & Berger

Find the pdf of $\prod_{i=1}^{n} X_i$, where the X_i's are independent uniform $(0, 1)$ random variables.

(Hint: Try to calculate the cdf, and remember the relationship between uniforms and exponentials.)

Q4: 4.47, Casella & Berger

(Marginal normality does not imply bivariate normality.)

Let X and Y be independent $N(0,1)$ random variables, and define a new random variable Z by

$$Z = \begin{cases} X & \text{if } XY > 0, \\ -X & \text{if } XY < 0. \end{cases}$$

(a)

Show that Z has a normal distribution.

(b)

Show that the joint distribution of Z and Y is not bivariate normal. *(Hint: Show that*

$$Z$$

and

$$Y$$

always have the same sign.)

Q5: 4.52, Casella & Berger

Bullets are fired at the origin of an (x, y) coordinate system, and the point hit, say (X, Y), is a random variable. The variables X and Y are taken to be independent $N(0, 1)$ random variables. If two bullets are fired independently, what is the distribution of the distance between them?

Q6: 4.55, Casella & Berger

A **parallel system** is one that functions as long as at least one component of it functions.

A particular parallel system is composed of three independent components, each of which has a lifetime with an exponential (λ) distribution. The lifetime of the system is the maximum of the individual lifetimes.

What is the distribution of the lifetime of the system?

Q7: 4.28, Casella & Berger

Let X and Y be independent standard normal random variables.

(a)

Show that $\frac{X}{X+Y}$ has a Cauchy distribution.

(b)

Find the distribution of $\frac{X}{|Y|}$.

(c)

Is the answer to part (b) surprising? Can you formulate a general theorem?

Q8: 4.50, Casella & Berger

If (X, Y) has the bivariate normal probability density function (pdf):

$$f(x, y) = \frac{1}{2\pi(1-\rho^2)^{1/2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(x^2 - 2\rho xy + y^2\right)\right),$$

show that

$$\mathrm{Corr}(X, Y) = \rho$$

and

$$\mathrm{Corr}(X^2, Y^2) = \rho^2.$$

Hint: Conditional expectations will simplify calculations.