HW3

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HW 3

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Overview

- 1: DONE
- 2: DONE, but (c) may need a touch-up
- 3: Part (c) left TO-DO
- 4: DONE
- 5:
- 6: DONE
- 7: DONE, but can tidy up a bit
- 8:

1. 2.23(b)

Question 1

Let X have the pdf

$$f(x) = \frac{1}{2}(1+x)$$

, $-1 < x < 1$

Define the random variable Y by $Y = X^2$

(b): Find $E(Y)$ and Var(Y).

Answer 1

(b):

(From prior HW) Note from the results of theorem 2.1.8:

$$f_Y(y) = \begin{cases} \sum_{i=1}^{k} f_X(g_i^{-1}(y))|\frac{d}{dy}g_i^{-1}(y)| & y \in \mathbb{Y} \\ 0 & otherwise \end{cases}$$

Note: For $X \in [-1,1]$, we have $Y = X^2 \in [0,1]$

Over the following partitions, we have monotonicity,

$A_1 = (-1,0) \to X = -\sqrt{y}$, as $g_1(x) = x^2$, and

$A_2 = (0,1) \to X = \sqrt{y}$, as $g_2(x) = x^2$

Taking the absolute value of the derivatives, $|\frac{d}{dy}g_i^{-1}(y)|$, we have:

$|\frac{d}{dy}g_1^{-1}(y)| = |\frac{d}{dy}g_2^{-1}(y)| = \frac{1}{2}y^{-1/2}$

Thus we have

$f_Y(y) = \frac{1}{2}y^{-1/2}\frac{1}{2}[(1+\sqrt{y}) + (1-\sqrt{y})] = \frac{1}{4}y^{-1/2}[2] = \frac{1}{2}y^{-1/2}$

Such that we have the pdf of Y as:

$f_Y(y) = \frac{1}{2}y^{-1/2}$, $0 < y < 1$

Using this for our Expected value calculation we have:

$E(Y) = \int\limits_{y \in \mathbb{Y}} yf(y)dy = \int\limits_{y=0}^{1} y(\frac{1}{2\sqrt{y}})dy$

$$E(Y) = \int_{y=0}^{1} \sqrt{y}(\frac{1}{2})dy = \frac{1}{2}\frac{2}{3}y^{3/2}\Big|_{y=0}^{y=1} = \frac{1}{2}\frac{2}{3}(1) - 0 = \frac{1}{3}$$

To calculate Var(Y), let us consider $E(Y^2)$,

$E(Y^2) = \int\limits_{y \in \mathbb{Y}} y^2 f(y)dy = \int\limits_{y=0}^{1} y^2(\frac{1}{2\sqrt{y}})dy$

$$E(Y^2) = \int_{y=0}^{1} y^{3/2}\frac{1}{2}dy = \frac{2}{5}(\frac{1}{2})y^{5/2}\Big|_{y=1}^{y=1} = \frac{2}{5}(\frac{1}{2})(1) - 0 = \frac{2}{10} = \frac{1}{5}$$

Taking Var(Y) $= E(Y^2) - (E(Y))^2$, then,

$Var(Y) = \frac{1}{5} - (\frac{1}{3})^2 = \frac{1}{5} - \frac{1}{9} = \frac{9}{45} - \frac{5}{45} = \frac{4}{45}$

2.

Question 2

A family continues to have children until they have one female child. Suppose, for each birth, a single child is born and the child is equally likely to be male or female. The gender outcomes are independent across births.

(a): Let X be a random variable representing the number of children born to this family. Find the distribution of X.

(b): Find the expected value $E(X)$

(c): Let X_m denote the number of male children in this family and let X_f denote the number of female children. Find the expected value of X_m and the expected value of X_f

Answer 2

(a): We can frame X as the number of children until the family has their first (one) female child. So we can think of X as a Geometric distribution with probability $p = 0.5$ since it is equally likely that they have a male/female for each birth.

Notation-wise we write this as:

$X \sim \text{Geometric}(p = 0.5)$

(b):

Knowing the distribution of X, we know its pmf (discrete!) is given by:

For X number of children, $k = 1, 2, ...,$ we have:

$f_X(x) = P(X = x) = p(1 - p)^{x-1}$

$$E(X) = \sum_{x=1}^{\infty} xP(X = x) = \sum_{k=x}^{\infty} x(p(1 - p)^{x-1}) = p \sum_{x=1}^{\infty} x((1 - p)^{k-1})$$

Note, for the infinite geometric series we have, for $|r| < 1$, k some positive integer, the following holds:

$$\sum_{k=1}^{\infty} r^k = \frac{1}{1 - r}$$

Note: Let $q = 1 - p$ for simplicity. As $0 < p < 1 \to 0 < 1 - p < 1 \to 0 < q < 1$. For our purposes, we have $|q| < 1$, such that the above relation holds for an infinite geometric series:

$$\sum_{x=1}^{\infty} q^x = \frac{1}{1 - q}$$

Note then:

$$\frac{d}{dq}(\sum_{x=1}^{\infty} q^x) = \sum_{x=1}^{\infty}(\frac{d}{dq} q^x) = \sum_{x=1}^{\infty} xq^{x-1}$$

Additionally,

$$\frac{d}{dq}(\frac{1}{1-q}) = \frac{d}{dq}[(1-q)^{-1}] = \frac{1}{(1-q)^2} = (1-q)^{-2}$$

Thus we have:

$$E(X) = p(1-q)^{-2} = p(1-(1-p))^{-2} = p(p)^{-2} = p^{-1} = \frac{1}{p}$$

For $p = 0.5$, we have:

$$E(X) = \frac{1}{p} = \frac{1}{0.5} = 2$$

Or, on average they would have two children before they have their first female.

(c):

Note: X_f and X_m are subsets of the random variable X.

Both X_f and X_m are Geometrically distributed random variables with parameter p, where $p = 0.5$. Given the derivation in part (b), we have:

$E(X_f) = \frac{1}{p} = \frac{1}{0.5} = 2$

and

$E(X_m) = \frac{1}{p} = \frac{1}{0.5} = 2$

Which we may interpret as meaning the expected number of males/females born before having a child belonging to the other respective gender is 2.

3. 2.30 (a), (b), (c)

Question 3

Find the moment generating function corresponding to:

(a): $f(x) = \frac{1}{c}$, $0 < x < c$

(b): $f(x) = \frac{2x}{c^2}$, $0 < x < c$

(c): $f(x) = \frac{1}{2\beta}e^{\frac{-|x-\alpha|}{\beta}}$, $-\infty < x < \infty$, $-\infty < \alpha < \infty$, $\beta > 0$

Answer 3

Note, for a continuous random variable X, we may write the moment generating funciton as:

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx$$

Using this method, we then calculate the following:

(a):

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx = \int_{0}^{c} e^{tx}\frac{1}{c}dx = \frac{1}{ct}e^{tx}\Big|_{x=0}^{x=c} = \frac{1}{ct}e^{tc} - \frac{1}{ct}(1)$$

$$M_X(t) = \frac{1}{ct}e^{tc} - \frac{1}{ct}(1) = \frac{1}{ct}(e^{tc} - 1)$$

(b):

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx = \int_{0}^{c} e^{tx}\frac{2x}{c^2}dx = \frac{2}{c^2t^2}e^{tx}(tx-1)\Big|_{x=0}^{x=c}$$

$$M_X(t) = \frac{2}{c^2t^2}e^{tc}(tc-1) - (\frac{2}{c^2t^2}1(-1)) = \frac{2}{c^2t^2}(tce^{tc} - e^{tc} + 1)$$

(c):

4. 2.31

Question 4

Does a distribution exist for which $M_X(t) = \frac{t}{(1-t)}$, $|t| < 1$? If yes, find it. If no, prove it.

Answer 4

Let us suppose that the distribution exists.

Then by the definition of a(n) mfg:

$M_X(t) = E(e^{tX})$

We know for $t = 0$ that the relation $|t| = |0| = 0 < 1$ holds.

Thus we know the 0-th moment is defined, as:

$M_X(0) = E(e^{0X}) = E(e^0) = E(1) = 1$

However, if we evaluate $M_X(t)$ directly using the mgf as given, for $t = 0$ as given, we have:

$M_X(t) = \frac{t}{(1-t)} = \frac{0}{1-0} = 0$

And we arrive at a contradiction. Thus we must conclude that such a distribution does not exist.

5.

Question 5

Suppose that X has the standard normal distribution with pdf:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$$

, $-\infty < x < \infty$

Then the random variable Y, $Y = e^X$ has a log-normal distribution.

(a): Find $E(Y^r)$ for any r.

(b): Show the moment generating function of Y does not exist (even though all moments of Y exist).

Answer 5

(a):

(b):

6.

Question 6

Suppose that X has a normal distribution with pdf:

$$f(x)\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(x-\mu)^2}{\sigma^2 2}}$$

, $-\infty < x < \infty$

The mean of X is μ. Show that the moment generating function of X satisfies $M_X(t) \geq e^{t\mu}$

Answer 6

With note of Jensen's Inequality, we have, for a convex function g, (avoiding confusion of the usage of f with the above pdf),

Let us then consider the moment generation function of X,

$M_X(t) = E(e^{tX})$

Consider then the function e^{tX}, specifically its second derivative:

$$\frac{d^2}{dx^2}e^tX = t^2e^{tX} > 0$$

, $\forall x, t$

We may then note that the mgf of X is convex since its second derivative is positive.

This is advantagous to our purposes, as we know when applying Jensen's inequality that we have a convex function, such that we may write:

Since $E(X) = \mu$,

$$M_X(t) = E[e^{tX}] \geq e^{tE(X)} = e^{t\mu}$$

And we conclude

$M_X(t) \geq e^{t\mu}$

7.

Question 7

Suppose that X has pmf $f(x) = p(1-p)^{x-1}$, for $x = 1, 2, 3, \ldots$ where $0 < p < 1$. Find the mgf $M_X(t)$ and use this to derive the mean and variance of X.

Answer 7

For deriving the mean and variance of X, we will need to first define the mgf of X as:

$$M_X(t) = E(e^{tX}) = \sum_{x=1}^{\infty} e^{tx} f(x) = \sum_{x=1}^{\infty} e^{tx} p(1-p)^{x-1}$$

The mean $\mu = E(X)$ is equal to the first derivative of the mgf evaluated at $t = 0$:

$E(X) = M'_X(0)$

$$M'_X(t) = \frac{pe^t(1-p)}{\left(1 - e^t(1-p)\right)^2}$$

$$M'_X(0) = \frac{p(1-p)}{(1-(1-p))^2} = \frac{1}{p}$$

$$E(X) = \frac{1}{p}$$

We then derive the variance of X. We know the typical variance formula as: $\mathrm{Var}(X) = E(X^2) - (E(X))^2$

However, we just calculated $E(X)$! Additionally, we know that $E(X^2)$ is equal to the second derivative of the mgf at $t = 0$. As such we write:

$$M''_X(t) = \frac{pe^t(1-p)\left(1 - e^t(1-p) + e^t(1-p)\right)}{\left(1 - e^t(1-p)\right)^3}$$

To make computation easier, let $q = 1 - p$. Then, for $t = 0$,

$$M''_X(0) = \frac{p - pq^2}{(1-q)^4} = \frac{p(1-q^2)}{(1-q)^4} = \frac{p(1+q)}{(1-q)^3} = \frac{1+q}{p^2}$$

Taking this calculation minus the square of the mean gives us:

$$\mathrm{Var}(X) = M''_X(0) - \left(M'_X(0)\right)^2 = \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{q}{p^2} = \frac{1-p}{p^2}$$

And we conclude then

$$\mathrm{Var}(X) = \frac{1-p}{p^2}$$

8.

Question 8

Suppose for one month a company purfchases c copies of a software package at a cost of d_1 dollars per copy. The packages are sold to customers for d_2 dollars per copy; any unsold copies are destroyed at the end of the month. Let X represent the demand for this software package in the month. Assume that X is a discrete random variable with pmf $f(x)$ and cdf $F(x)$.

(a): Let $s = \min\{X, c\}$ represent the number of sales during the month. Show that:

$$E(S) = \sum_{x=0}^{c} xf(x) + c(1 - F(c))$$

(b): Let $Y = S * d_2 - cd_1$ represent the profit for the company, the total income from sales minus the total cost of all copies. Find $E(Y)$

(c): As $Y \equiv Y_c$ depends on integer $c \geq 0$, write the expected profit function as $g(c) \equiv E(Y_c)$ from part (b). The company should choose the value of c which maximizes $g(c)$; that is, choose the smallest c such that $g(c+1)$ is less than or equal to $g(c)$. Show that such $c \geq 0$ is the smallest integer with $F(c) \geq \frac{d_2 - d_1}{d_2}$

Answer 8

(a):

(b):

(c):