# HW4

2024-09-29

## Q's

Q7

## Homework 4

Due October 13

### Q1

> Question: 3.6 (a), (b) Casella & Berger

A large number of insects are expected to be attracted to a certain variety of rose plant. A commercial insecticide is advertised as being 99% effective. Suppose 2,000 insects infest a rose garden where the insecticide has been applied and let $X =$ number of surviving insects.

> (a)

What probability distribution might provide a reasonable model for this experiment?

> (b)

Write down, but do not evaluate, an expression for the probability that fewer than 100 insects survive, suing the model in part (a)

> Answer > (a)

We may interpret X as the number of "failures" given an effective rate of 99%, or $p = 1 - 0.99 = 0.01$ (1% chance of failure). As we are counting the number of failures, we have a dicrete random variable. We know $n = 2,000$, or our total number of "trials" for the insecticide.

Taken together, we have X as a Binomial distributed random variable, or:

$X \sim Binomial(n = 2,000, p = 0.01)$

It's worth noting that we can also represent this as a Poisson distributed random variable with parameter $\lambda = np = 2000(0.01) = 20$

> (b)

$P(X < 100) = P(X \leq 99)$

$\sum_{x=0}^{99} P(X = x) = \sum_{x=0}^{99} f(x) = \sum_{x=0}^{99} \binom{2,000}{x} (0.01)^x (0.99)^{2000-x}$

## Q2

Question: 3.13 (a) Casella & Berger

A truncated discrete distribution is one in which a particular class cannot be observed and is eliminated from the sample space. In particular, if X has range 0, 1, 2, ... and the 0 class cannot be observed (as is usually the case), the 0-truncated random variable $X_T$ has pmf:

$$P(X_T = x) = \frac{P(X = x)}{P(X > 0)}$$

for $x = 1, 2, ...$

Find the pmf, mean, and variance of the 0-truncated random variable starting from:

(a)

$X \sim Poisson(\lambda)$

Answer > (a)

The pmf of a Poisson distribution is:

$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$

Also:

$P(X > 0) = 1 - P(X = 0) = 1 - \frac{\lambda^0 e^{-\lambda}}{0!} = 1 - e^{-\lambda}$

Combining these together gives us the truncated pmf:

$$P(X_T = x) = \frac{P(X = x)}{P(X > 0)} = \frac{\lambda^x e^{-\lambda}}{x!}/1 - e^{-\lambda} = \frac{\lambda^x e^{-\lambda}}{x!(1 - e^{-\lambda})}$$

For $x = 1, 2, ...$

Using the above pmf, we may find the mean as:

$$E(X_T) = \sum_{x=1}^{\infty} xP(X_T = x) = \sum_{x=1}^{\infty} x\frac{\lambda^x e^{-\lambda}}{x!(1 - e^{-\lambda})} = \frac{\lambda e^{-\lambda}}{(1 - e^{-\lambda})} \sum_{x \geq 1} \frac{\lambda^{x-1}}{(x-1)!}$$

Let $y = x - 1$, such that we may rewrite the above as:

$$E(X_T) = \frac{\lambda e^{-\lambda}}{(1 - e^{-\lambda})} \sum_{y \geq 0} \frac{\lambda^y}{(y)!}$$

Using the infinite summation for Euler, namely:

$$e^{\lambda} = \sum_{y \geq 0} \lambda^y / y!$$

We may then evaluate this as:

$$E(X_T) = \frac{\lambda e^{-\lambda}}{(1-e^{-\lambda})} e^{\lambda} = \frac{\lambda}{(1-e^{-\lambda})}$$

To then find the variance, let us consider $E(X_T^2)$

$$E(X_T^2) = \sum_{x\geq 1} x^2 P(X_T = x) = \sum_{x \geq 1} x^2 \frac{\lambda^x e^{-\lambda}}{x!(1-e^{-\lambda})} = \frac{\lambda e^{-\lambda}}{(1-e^{-\lambda})} \sum_{x\geq 1} x \frac{\lambda^x}{(x-1)!}$$

$$E(X_T^2) = \frac{\lambda e^{-\lambda}}{(1-e^{-\lambda})} \big( \sum_{x\geq 1} (x-1) \frac{\lambda^{x-1}}{(x-1)!} + \sum_{x\geq 1} \frac{\lambda^{x-1}}{(x-1)!} \big) = \frac{\lambda e^{-\lambda}}{(1-e^{-\lambda})} \big( \lambda \sum_{x\geq 2} \frac{\lambda^{x-2}}{(x-2)!} + \sum_{x\geq 1} \frac{\lambda^{x-1}}{(x-1)!} \big)$$

Then, let $y = x - 2$, $z = x - 1$, we have:

$$E(X_T^2) = \frac{\lambda e^{-\lambda}}{(1-e^{-\lambda})} \big( \lambda \sum_{y\geq 0} \frac{\lambda^y}{(y)!} + \sum_{z\geq 0} \frac{\lambda^z}{(z)!} \big) = \frac{\lambda e^{-\lambda}}{(1-e^{-\lambda})} \big( \lambda e^{\lambda} + e^{\lambda} \big)$$

Thus:

$$E(X_T^2) = \frac{\lambda^2 + \lambda}{(1-e^{-\lambda})}$$

$$Var(E_T) = E(X_T^2) - (E(X_T))^2 = \frac{\lambda^2 + \lambda}{(1-e^{-\lambda})} - (\frac{\lambda}{(1-e^{-\lambda})})^2$$

## Q3

Question: 3.17 Casella & Berger

Establish a formula similar to (3.3.18) for the gamma distribution. If $X \sim Gamma(\alpha, \beta)$, then for any positive constant v,
$EX^v = \frac{\beta^v \Gamma(v+\alpha)}{\Gamma(\alpha)}$

Answer

Formula 3.3.18:

$$EX^n = \frac{B(\alpha+n,\beta)}{B(\alpha,\beta)} = \frac{\Gamma(\alpha+n)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+n)\Gamma(\alpha)}$$

$$EX^v = \int\limits_{x=0}^{\infty} x^v \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx = \frac{1}{\Gamma(\alpha)\beta^\alpha} \int\limits_{x=0}^{\infty} x^{(v+\alpha-1)} e^{-x/\beta} dx$$

Two items of note here:

(1):

$$\frac{\Gamma(v+\alpha)\beta^{v+\alpha}}{\Gamma(v+\alpha)\beta^{v+\alpha}} = 1$$

, such that:

$$\frac{1}{\Gamma(\alpha)\beta^\alpha} \int\limits_{x=0}^{\infty} x^{(v+\alpha-1)} e^{-x/\beta} dx = \frac{\Gamma(v+\alpha)\beta^{v+\alpha}}{\Gamma(v+\alpha)\beta^{v+\alpha}} \frac{1}{\Gamma(\alpha)\beta^\alpha} \int\limits_{x=0}^{\infty} x^{(v+\alpha-1)} e^{-x/\beta} dx = \frac{\Gamma(v+\alpha)\beta^{v+\alpha}}{\Gamma(\alpha)\beta^\alpha} \int\limits_{x=0}^{\infty} \frac{1}{\Gamma(v+\alpha)\beta^{v+\alpha}} x^{(v+\alpha-1)} e^{-x/}$$

(2):

$$\int\limits_{x=0}^{\infty} \frac{1}{\Gamma(v+\alpha)\beta^{v+\alpha}} x^{(v+\alpha-1)} e^{-x/\beta} dx$$

is the evaluation of the pdf of the $\Gamma(v+\alpha)$ distribution over its support (evaluates to 1!)

Taking (1) and (2) together, we may write:

$$EX^v = \frac{\Gamma(v+\alpha)\beta^{v+\alpha}}{\Gamma(\alpha)\beta^\alpha} \int\limits_{x=0}^{\infty} \frac{1}{\Gamma(v+\alpha)\beta^{v+\alpha}} x^{(v+\alpha-1)} e^{-x/\beta} dx = \frac{\Gamma(v+\alpha)\beta^{v+\alpha}}{\Gamma(\alpha)\beta^\alpha}(1)$$

Special thanks to that cheeky: $\frac{\Gamma(v+\alpha)\beta^{v+\alpha}}{\Gamma(v+\alpha)\beta^{v+\alpha}} = 1$

Simplifying gives us:

$$EX^v = \frac{\Gamma(v+\alpha)\beta^{v+\alpha}}{\Gamma(\alpha)\beta^\alpha} = \frac{\Gamma(v+\alpha)\beta^v}{\Gamma(\alpha)}$$

$\forall v > -\alpha$ as The Gamma function is only defined for positive values.

## Q4

Question: 3.19 Casella & Berger

Show that:

$$\int_{x}^{\infty} \frac{1}{\Gamma(\alpha)} z^{\alpha-1} e^{-z} dz = \sum_{y=0}^{\alpha-1} \frac{x^y e^{-x}}{y!}$$

For $\alpha = 1, 2, 3, ...$

Hint: Use integration by parts. Express this formula as a probabilistic relationship between Poisson and Gamma random variables.

Answer

Via Integration by Parts, let us take out the constant $\frac{1}{\Gamma(\alpha)}$, and let $\alpha = n$, such that we have:

$$u = z^{\alpha-1}$$

$$du = (\alpha - 1) z^{\alpha-2}$$

$$dv = e^{-z} dz$$

$$v = -e^{-z}$$

Thus, we have :

$$\int udv = uv - \int vdu$$

Giving us:

$$\int_{z=x}^{\infty} z^{n-1} z^{-z} dz = x^{n-1} e^{-x} - \int_{z=x}^{\infty} z^{n-2}(n-2)(-z^{-z}) dz$$

Simplifying and adding back in the $\frac{1}{\Gamma(n)}$ term gives us:

$$\frac{1}{\Gamma(n)} \int_{z=x}^{\infty} z^{n-1} z^{-z} dz = \frac{1}{\Gamma(n)} \left[ x^{n-1} e^{-x} + \int_{z=x}^{\infty} (n-1) z^{n-2} z^{-z} dz \right]$$

We then note the expansion of $\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2) = ... = (n-1)!\Gamma(1) = (n-1)!(1) = (n-1)!$. Using this relation above gives us:

$$\frac{1}{\Gamma(n)} \int_{z=x}^{\infty} z^{n-1} z^{-z} dz = \frac{x^{n-1} e^{-x}}{(n-1)!} + \int_{z=x}^{\infty} \frac{1}{(n-1)\Gamma(n-1)} (n-1) z^{n-2} z^{-z} dz$$

$$\frac{1}{\Gamma(n)}\int\limits_{z=x}^{\infty} z^{n-1}z^{-z}dz = \frac{x^{n-1}e^{-x}}{(n-1)!} + \frac{1}{\Gamma(n-1)}\int\limits_{z=x}^{\infty} z^{n-2}z^{-z}dz$$

Notice the first term in this evaluation is $\frac{x^y e^{-x}}{y!}$ evaluated at $y = n-1$!

Notice also that we may continue the integration by parts for the term

$$\frac{1}{\Gamma(n-1)}\int\limits_{z=x}^{\infty} z^{n-2}z^{-z}dz$$

,

which will yield two parts, the first being the $\frac{x^y e^{-x}}{y!}$ evaluated at $y = n-2$ and the second part being another integral we may break into parts.

The above process may be repeated until we arrive at an integral in the $\int vdu = 0$

In summary, we can continue breaking the integral into parts which coincide with elements of the summation in reverse order ($\alpha - 1$ or $n - 1$), meaning these two values are equal.

This is all to say that we have shown the relationship between the random variable $X \sim Poisson(x)$ and the random variable $Y \sim Gamma(\alpha, 1)$, as $P(X > x) = P(Y \leq \alpha - 1)$

## Q5

> Question: 3.24 (a), (c) Casella & Berger Note: You can skip the part about showing that the pdf is a pdf; also, in (c), the variance will not exist unless a>2.

Many "named" distributions are special cases of the more common distributions already discussed. For each of the following named distributions derive the form of the pdf, . . . , and calculate the mean and variance.

> (a)

If $X \sim \textit{Exponential}(\beta)$, then $Y = X^{1/\gamma}$ has the Weibull$(\gamma, \beta)$ distribution, where $\gamma > 0$ is a constant.

> (c)

If $X \sim \textit{Gamma}(a, b)$, then $Y = 1/X$ has the inverted Gamma IG(a,b) distribution.

> Answer > (a)

$X \sim \textit{Exponential}(\beta) \rightarrow f_X(x) = \frac{1}{\beta}e^{-x/\beta}$

Using the following general relation for two related random variables:

Thm. 2.1.5:

$$f_X(x) = f_z(g^-1(x))|\frac{d}{dx}g^{-1}(x)|$$

We have:

$$Y = X^{1/\gamma} \rightarrow f_Y(y) = f_x(y^\gamma)(d/dy) = \frac{1}{\beta}e^{-y^\gamma/\beta}\gamma y^{\gamma-1}$$

For $x > 0$, $y > 0$

For simplicity, let us consider $EY^n = \int\limits_{y=0}^{\infty} y^n f(y) = \int\limits_{y=0}^{\infty} y^n \frac{1}{\beta}e^{-y^\gamma/\beta}\gamma y^{\gamma-1}$

Simplifying terms somewhat, and removing terms that don't depend on y, we have:

$$EY^n = \gamma/\beta \int\limits_{y=0}^{\infty} y^{\gamma+n-1}e^{-y^\gamma/\beta}dy$$

Let $z = y^\gamma/\beta \rightarrow dz = \frac{\gamma}{\beta}y^{\gamma-1}dy$ Such that $\frac{\beta}{\gamma y^{\gamma-1}}dz = dy$

Change of variables allows us to rewrite the above as:

$$EY^n = \frac{\gamma}{\beta}\frac{\beta}{\gamma}\beta^{\gamma+n-1/\gamma} \int\limits_{z=0}^{\infty} z^{\frac{\gamma+n-1}{\gamma}} z^{\frac{1}{\gamma}-1}e^{-z}dz$$

Simplifying terms somewhat gives us:

$$EY^n = \beta^{\gamma+n-1/\gamma} \int\limits_{z=0}^{\infty} z^{n/\gamma}e^{-z}dz$$

Recognizing this as the $\Gamma(n/\gamma + 1)$ function and simplifying constants, we thus have:

$$EY^n = \beta^{n/\gamma}\Gamma(n/\gamma + 1)$$

Using the above formula we have:

$EY = \beta^{1/\gamma}\Gamma(1/\gamma + 1)$

and

$EY^2 = \beta^{2/\gamma}\Gamma(2/\gamma + 1)$

$$Var(Y) = EY^2 - (EY)^2 = \beta^{2/\gamma}\Gamma(2/\gamma + 1) - (\beta^{1/\gamma}\Gamma(1/\gamma + 1))^2$$

$$Var(Y) = \beta^{2/\gamma}[\Gamma(2/\gamma + 1) - (\Gamma(1/\gamma + 1))^2]$$

(c)

$X \sim Gamma(a, b) \rightarrow f_X(x) = \frac{1}{\Gamma(a)b^a}x^{a-1}e^{-x/b}$

With note of Thm. 2.1.5:

$$f_X(x) = f_z(g^-1(x))|\frac{d}{dx}g^{-1}(x)|$$

$Y = 1/X \rightarrow f_Y(y) = f_X(1/y)(d/dy) = \frac{1}{\Gamma(a)b^a}(1/y)^{a-1}e^{-1/by}$

$EY = \int\limits_{y=0}^{\infty} y\frac{1}{\Gamma(a)b^a}(1/y)^{a-1}e^{-1/by}dy$

$$EY = \frac{1}{\Gamma(a)b^a}\int\limits_{y=0}^{\infty}(\frac{1}{y})^a e^{-1/by}dy = \frac{\Gamma(a-1)b^{a-1}}{\Gamma(a)b^a} = \frac{1}{(a-1)b}$$

Note: $b \neq 0$ and $a \neq 1$ in order for the above to evaluate.

$EY^2 = \int\limits_{y=0}^{\infty} y^2\frac{1}{\Gamma(a)b^a}(1/y)^{a-1}e^{-1/by}dy$

$$EY^2 = \frac{1}{\Gamma(a)b^a}\int\limits_{y=0}^{\infty}(\frac{1}{y})^{a-1}e^{-1/by}dy = \frac{\Gamma(a-2)b^{a-2}}{\Gamma(a)b^a} = \frac{1}{(a-1)(a-2)b^2}$$

$Var(Y) = EY^2 - (EY)^2 = \frac{1}{(a-1)(a-2)b^2} - (\frac{1}{(a-1)b})^2 = \frac{1}{(a-1)^2(a-2)b^2}$

Note: $Var(Y)$ does not exist unless $a > 2$ as variance must be positive.

## Q6

Question: 3.39 Casella & Berger

Consider the Cauchy family defined in Section 3.3. This family can be extended to a location-scale family yielding pdfs of the form:

$$f(x|\mu,\sigma) = \frac{1}{\sigma\pi(1+(\frac{x-\mu}{\sigma})^2)}$$

For $-\infty < x < \infty$

The mean and variance do not exist for thje Cauchy distribution. So the parameters $\mu$, $\sigma^2$ are not the mean and variance. But they do have important meaning. Show that if X is a random variable with a Cauchy distribution with parameters $\mu$ and $\sigma$, then:

(a)

$\mu$ is the median of the distribution of X, that is, $P(X \geq \mu) = P(X \leq \mu) = \frac{1}{2}$

(b)

$\mu+\sigma$ and $\mu-\sigma$ are the quartiles of the distribution of X, that is $P(X \geq \mu+\sigma) = P(X \leq \mu-\sigma) = \frac{1}{4}$

Hint: Prove this first for $\mu = 0$ and $sigma = 1$ and then use Exercise 3.38.

Note: $\frac{d(arctan x)}{dx} = \frac{1}{1+x^2}$

Answer

(a)

$\mu$ is the median if

$$P(X \geq \mu) = P(X \leq \mu) = \frac{1}{2}$$

Let us consider the random Cauchy variable Z where $\mu = 0$, $\sigma = 1$, $Z \sim Cauchy(\mu = 0, \sigma = 1)$

Let us consider $\mu = 0$, such that: $X = Z + \mu$

$P(Z \geq 0) = \int\limits_{z=0}^{\infty} \frac{1}{\pi}\frac{1}{1+z^2}dz = \frac{1}{\pi}arctan(z)\Big|_{z=0}^{\infty} = \frac{1}{\pi}(\pi/2 - 0) = \frac{1}{2}$

and

$P(Z \leq 0) = \int\limits_{z=-\infty}^{0} \frac{1}{\pi}\frac{1}{1+z^2}dz = \frac{1}{\pi}arctan(z)\Big|_{z=-\infty}^{0} = \frac{1}{\pi}(0 - (-\pi/2)) = \frac{1}{2}$

We have shown then that for the random variable Z, $\mu$ is indeed the median.

Taking advantage of the results of Exercise 3.38 we then have X belongs a location family of Z:

So $P(Z \geq 0) = P(X \geq \mu) = \frac{1}{2}$

And

So $P(Z \leq 0) = P(X \leq \mu) = \frac{1}{2}$

So the $mu$ is the median for the random variable X from the Cauchy family.

(b)

Let us consider the random Cauchy variable Z where $\mu = 0$, $\sigma = 1$, such that: $X = \sigma Z + \mu$, $Z \sim Cauchy(\mu = 0, \sigma = 1)$

$$P(Z \geq \mu + \sigma) = P(Z \geq 1) = \int\limits_{z=1}^{\infty} \frac{1}{\pi} \frac{1}{1+z^2} dz = \frac{1}{\pi} arctanz\Big|_{z=1}^{\infty} = \frac{1}{\pi}(\pi/2 - \pi/4) = \frac{1}{4}$$

$$P(Z \leq \mu - \sigma) = P(Z \leq -1) = \int\limits_{z=-\infty}^{-1} \frac{1}{\pi} \frac{1}{1+z^2} dz = \frac{1}{\pi} arctanz\Big|_{z=-infty}^{-1} = \frac{1}{4}$$

Taking advantage of the results of Exercise 3.38 we then have X belongs to a scale-location family of Z and we may write:

$P(Z \geq 1) = P(X \geq \mu + \sigma)$ $P(Z \leq -1) = P(X \leq \mu - \sigma)$

So the following holds: $P(X \geq \mu + \sigma) = \frac{1}{4}$ $P(X \leq \mu - \sigma) = \frac{1}{4}$

Such that we have shown that $\mu + \sigma$ and $\mu - \sigma$ are the quartiles of the distribution of X.

## Q7

Question:

If $X \sim N(\mu, \sigma^2)$, find values of $\mu$ and $\sigma$ such that $P(|X| < 2) = \frac{1}{2}$. Prove or disprove that the values of $\mu$ and $\sigma$ are unique.

Answer

Note: For any normally distributed random variable X, the standard normal Z may be derived by

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$$

Furthermore, as

$|X| < 2 \rightarrow -2 < X < 2$

Furthermore, note:

$$X = \mu + \sigma Z \rightarrow Z = \frac{X - \mu}{\sigma}$$

Such that:

$$P(|X| < 2) = P(-2 < X < 2) = P\left(\frac{-2 - \mu}{\sigma} < \frac{X - \mu}{\sigma} < \frac{2 - \mu}{\sigma}\right) = P\left(\frac{-2 - \mu}{\sigma} < Z < \frac{2 - \mu}{\sigma}\right)$$

$$P(|X| < 2) = P\left(Z < \frac{2 - \mu}{\sigma}\right) - P\left(Z < \frac{-2 - \mu}{\sigma}\right)$$

For the cdf of the standard normal $\Phi$, we have:

$\Phi(\frac{2-\mu}{\sigma}) - \Phi(\frac{-2-\mu}{\sigma}) = \frac{1}{2}$

Note then that the above may be interpreted as a cdf of the normal distribution. Let us then denote the cdf of the standard normal distribution, $\Phi$, and proceed.

Taking advantage of the normal distribution being symmetrical, we have the following: $\Phi(z) = 1 - \Phi(-z)$

Such that:

$$\Phi(\frac{-2 - \mu}{\sigma}) = 1 - \Phi(-(\frac{-2 - \mu}{\sigma})) = 1 - \Phi(\frac{2 + \mu}{\sigma})$$

Refering back to the above difference in cdfs, we then have:

$\Phi(\frac{2-\mu}{\sigma}) - (1 - \Phi(\frac{2+\mu}{\sigma})) = \frac{1}{2}$

$\Phi(\frac{2-\mu}{\sigma}) + \Phi(\frac{2+\mu}{\sigma})) = \frac{3}{2}$

## One Case

For simplicity, let $\mu = 0$

Simplifying the above equation gives us:

$\Phi(\frac{2}{\sigma}) + \Phi(\frac{2}{\sigma}) = \frac{3}{2}$

Utilizing the inverse cdf of the standard normal then gives us:

$\frac{2}{\sigma} + \frac{2}{\sigma} = \Phi^{-1}(\frac{3}{2})$

Note: We take as a given that the inverse cdf of the standard normal exists.

$\frac{4}{\Phi^{-1}(\frac{3}{4})} = \sigma \approx 4.286$

As $\Phi^{-1}(\frac{3}{2}) \approx 0.9332$, then

$\sigma^2 \approx 18.37$

## Proof by Contradiction - Extending to other values of $\mu$, $\sigma$

$\Phi(\frac{2-\mu}{\sigma}) + \Phi(\frac{2+\mu}{\sigma})) = \frac{3}{2}$

Let us assume that the values of $\mu$, $\sigma$ are unique. Then consider $\mu \neq 0$, specifically $\mu = 1$. We set to prove then that there does not exist a $\sigma$ that satisfies the following relation:

$\Phi(\frac{2-\mu}{\sigma}) + \Phi(\frac{2+\mu}{\sigma})) = \Phi(\frac{2-1}{\sigma}) + \Phi(\frac{2+1}{\sigma})) = \Phi(\frac{1}{\sigma}) + \Phi(\frac{3}{\sigma})) = \frac{3}{2}$

Then we have the following relation to evaluate:

$\Phi(\frac{1}{\sigma}) + \Phi(\frac{3}{\sigma})) = \frac{3}{2}$

We may again utilize the inverse cdf of the standard normal, such that:

$\frac{1}{\sigma} + \frac{3}{\sigma} = \Phi^{-1}(\frac{3}{2})$

We know from the prior calculation that:

$\Phi^{-1}(\frac{3}{4}) \approx 0.9332 \rightarrow \frac{1}{\sigma} + \frac{3}{\sigma} \approx 0.9332$

$\sigma = \frac{(1+3)}{\Phi^{-1}\frac{3}{4}} = \frac{4}{\Phi^{-1}\frac{3}{4}} \approx 4.286$

This is a contradiction though, as we have found a finite value of $\sigma$ that satisfies the relation! Thus we are led to conclude that the values of $\mu, \sigma$ are not unique that satisfy the relation $P(|X| < 2) = \frac{1}{2}$.

## Case for any finite $\mu$

Let $\mu = \alpha$, some finite value. Then the following relation holds:

$$\Phi(\frac{2-\alpha}{\sigma}) + \Phi(\frac{2+\alpha}{\sigma})) = \frac{3}{2}$$

Utilizing the inverse cdf of the standard normal, we then have:

$$\frac{2-\alpha}{\sigma} + \frac{2+\alpha}{\sigma} = \Phi^{-1}(\frac{3}{2})$$

$$\Phi^{-1}(\frac{3}{2})\sigma = 2 - \alpha + 2 + \alpha = 4$$

$$\sigma = \frac{4}{\Phi^{-1}(\frac{3}{2})}$$

So while we find the value of $\sigma$ to be unique, we see the relation holds for any finite $\alpha$!

So while the value of $sigma$ is unique, the value of $\mu$ is not, meaning that the pair of values $\mu, \sigma$ that satisfy $X \sim N(\mu, \sigma^2),\ P(|X| < 2) = \frac{1}{2}$ are not unique! (It holds for a fixed $\sigma$ and any $\mu$!)