HW5

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1.

In the attached article by Prof. M. Ghosh, read pages 509-512 (including example 1), examples 4-6 of Section 3, and Section 5.2 up to and including Examples 17-18. (This is sort of a technical article, so to read a bit of this material is not easy. Also, Example 17 should look like an example from class regarding Basu's theorem.)

In example 18, show that T is a complete and sufficient statistic, while U is an ancillary statistic.

Example 18.

Let $X_1, \dots, X_n$ $(n \geq 2)$ be iid with common Weibull pdf

$$f_\theta(x) = \exp(-x^p/\theta)(p/\theta)x^{p-1}; \quad 0 < x < \infty, \quad 0 < \theta < \infty,$$

$p(> 0)$ being known. In this case, $T = \sum_{i=1}^n X_i^p$ is complete sufficient for θ, while $U = X_1^p/T$ is ancillary. Also, since $X_1^p, \dots, X_n^p$ are iid exponential with scale parameter θ, $U \sim \text{Beta}(1, n-1)$. Hence, the UMVUE of $P_\theta(X_1 \leq x) = P_\theta(X_1^p \leq x^p)$ is given by

$$k(T) = \begin{cases} 1 - x^{np}/T^n & \text{if } T > x^p, \\ 1 & \text{if } T \leq x^p. \end{cases}$$

2.

Problem 7.60, Casella and Berger and the following:

Base

Let $X_1, \ldots, X_n$ be iid gamma(α, β) with α known. Find the best unbiased estimator of $1/\beta$.

a)

Let $S_n = \sum_{i=1}^n X_i$. Using Basu's theorem, show X_1/S_n and S_n are independent.

b)

Using the result in (a) and $E_\theta(S_n) = n\alpha\beta$, find $E_\theta(X_1/S_n)$.

3.

Problem 8.13(a)-(c), Casella and Berger (2nd Edition) and, in place of Problem 8.13(d), consider the following test:

Let X_1, X_2 be iid uniform$(\theta, \theta+1)$. For testing $H_0 : \theta = 0$ versus $H_1 : \theta > 0$, we have two competing tests:

$$\phi_1(X_1) : \text{Reject } H_0 \text{ if } X_1 > 0.95,$$

$$\phi_2(X_1, X_2) : \text{Reject } H_0 \text{ if } X_1 + X_2 > C.$$

a)

Find the value of C so that ϕ_2 has the same size as ϕ_1.

b)

Calculate the power function of each test. Draw a well-labeled graph of each power function.

c)

Prove or disprove: ϕ_2 is a more powerful test than ϕ_1.

Extra

$$\phi_3(X_1, X_2) = \begin{cases} 1 & \text{if } X_{(1)} > 1 - \sqrt{0.05} \text{ or } X_{(2)} > 1 \\ 0 & \text{otherwise} \end{cases}$$

where $X_{(1)}, X_{(2)}$ are the min, max.

Find the size of this test and the power function for $\theta > 0$. Then, graph the power functions of ϕ_3 and ϕ_2 to determine which test is more powerful. (It's enough to graph over the range $\theta \in [0, 1.2]$.)

4.

Problem 8.15, Casella and Berger (2nd Edition), though you can just assume the form given is most powerful (no need to show).

Show that for a random sample $X_1, \dots, X_n$ from a $\mathcal{N}(0, \sigma^2)$ population, the most powerful test of $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma = \sigma_1$, where $\sigma_0 < \sigma_1$, is given by

$$\phi\left(\sum X_i^2\right) = \begin{cases} 1 & \text{if } \sum X_i^2 > c, \\ 0 & \text{if } \sum X_i^2 \le c. \end{cases}$$

For a given value of α, the size of the Type I Error, show how the value of c is explicitly determined.