Homework 2 – STAT 542

Due Friday, Sept 20 by 11:59 PM (to be scanned and uploaded in Canvas under "Assignments")

1. Suppose a random variable X has the following cdf from class (which is neither a step function nor continuous):

$$F(x) = \begin{cases} 0 & x < 0 \\ (1+x)/2 & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \qquad x \in \mathbb{R}.$$

   (a) Find the following probabilities: $P(X > 1/2)$, $P(X \ge 1/2)$, $P(0 < X \le 1/2)$ & $P(0 \le X \le 1/2)$.

   (b) Conditional on the event "$X > 0$," the corresponding conditional cdf of X (i.e., given $X > 0$) is as follows at $x \in \mathbb{R}$:

$$P(X \le x | X > 0) = \frac{P(X \le x, X > 0)}{P(X > 0)} = \frac{P(0 < X \le x)}{P(X > 0)} = \frac{F(x) - F(0)}{1 - F(0)} = \begin{cases} 0 & \text{if } x \le 0 \\ x & \text{if } 0 < x \le 1 \\ 1 & \text{if } x > 1, \end{cases}$$

   Based on the conditional cdf above, show that the distribution of X, conditional on "$X > 0$," is the same (i.e., has the same cdf) as that of a random variable Y which is "uniform" on the interval $(0, 1)$, having a constant pdf $f_Y(y) = 1$ for $0 < y < 1$ (with $f_Y(y) = 0$ for all other $y \in \mathbb{R}$).

2. Statistical reliability involves studying the time to failure of manufactured units. In many reliability textbooks, one can find the exponential distribution

$$f(x) = \begin{cases} \frac{1}{\theta} e^{-x/\theta} & x > 0 \\ 0 & x \le 0 \end{cases}$$

   where $\theta > 0$ is a fixed value, for modeling the time X that a random unit runs until failure (i.e., X is a survival time). Show that if X has an exponential distribution as above, then

$$P(X > s + t | X > t) = P(X > s)$$

   for any values $t, s > 0$; this feature is called the "memoryless" property of the exponential distribution.[1]

3. 2.3, Casella & Berger

4. 2.4, Casella & Berger (Problem (c) is identifying the distribution of $|X|$)

5. 2.6(bc), Casella & Berger
   Just find the pdf in (b)-(c) & skip verifying the pdf part.

6. 2.9, Casella & Berger
   Just find a non-decreasing function $u(x)$ that will work & explain why (i.e., consider the function $u(x) = F(x)$ where $F(\cdot)$ is the cdf of X, which gives a special transformation $u(X)$ when applied to X, called the probability integral transform, having a known distribution).

7. 2.22(a,b), Casella & Berger
   Just find $E(X)$ in (b).
   Use that

$$1 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x^2 e^{-x^2/2} dx = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} x^2 e^{-x^2/2} dx$$

   which we will discuss later in talking about "normal" distributions.

[1] While a useful distribution generally, the exponential distribution is not typically realistic for modeling failure times due to this "memoryless" property (which says that, given a product has survived past a time t (i.e., $X > t$ holds), its conditional probability of surviving another s time units (i.e., $P(X > s + t | X > t)$) is the same as the unconditional probability of surviving past s time units when starting from 0 (i.e., $P(X > s)$). Usually, products wear-out over time so one would expect such conditional probabilities of survival to decrease and be smaller than survival probabilities starting from time zero.)

8. Suppose that a random variable U has a uniform$(0,1)$ distribution (i.e., pdf $f_U(u) = 1$ for $0 < u < 1$)

   (a) Suppose a random variable X has a cdf $F(x)$ which is strictly increasing and continuous on $x \in \mathbb{R}$; this implies that, for any real value of $0 < u < 1$, there is an inverse $F^{-1}(u) = x \in \mathbb{R}$ so that $F(x) = F(F^{-1}(u)) = u$. Define a random variable $Y = F^{-1}(U)$ based on the random variable U. Show that X and Y have the same cdf (i.e., the same distributions).
   Hint: Use that, because F is strictly increasing, $P(Y \leq y) = P(F(Y) \leq F(y))$ holds for any $y \in \mathbb{R}$, i.e., Y can be less than or equal to y if and only if $F(Y)$ is less than or equal to $F(y)$. Note that $F(y) \in (0,1)$ for any real y.

   (b) If there is a computer program (i.e., random number generator) that produces numbers uniformly distributed between zero and one (i.e., according to the pdf $f_U(u)$), explain how these numbers could be used to generate values distributed according to the pdf $f_Z(z) = e^{-|z|}/2$, $-\infty < z < \infty$.
   Hint: Use (a) where F now becomes the cdf of Z; you need to find $F^{-1}(u)$ for a given $0 < u < 1$ by solving the expression $F(z) = u$ for $z \in \mathbb{R}$.