HW5

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1.

In the attached article by Prof. M. Ghosh, read pages 509-512 (including example 1), examples 4-6 of Section 3, and Section 5.2 up to and including Examples 17-18. (This is sort of a technical article, so to read a bit of this material is not easy. Also, Example 17 should look like an example from class regarding Basu's theorem.)

In example 18, show that T is a complete and sufficient statistic, while U is an ancillary statistic.

Example 18.

Let $X_1, \ldots, X_n$ $(n \geq 2)$ be iid with common Weibull pdf

$$f_\theta(x) = \exp(-x^p/\theta)(p/\theta)x^{p-1}; \quad 0 < x < \infty, \quad 0 < \theta < \infty,$$

$p(> 0)$ being known. In this case, $T = \sum_{i=1}^n X_i^p$ is complete sufficient for θ, while $U = X_1^p/T$ is ancillary. Also, since $X_1^p, \ldots, X_n^p$ are iid exponential with scale parameter θ, $U \sim \text{Beta}(1, n-1)$. Hence, the UMVUE of $P_\theta(X_1 \leq x) = P_\theta(X_1^p \leq x^p)$ is given by

$$k(T) = \begin{cases} 1 - x^{np}/T^n & \text{if } T > x^p, \\ 1 & \text{if } T \leq x^p. \end{cases}$$

Answer

By definition, a statistic T is sufficient if the joint pdf of $X_1, \ldots, X_n$ can be factorized into the form:

$$f_\theta(x_1, \ldots, x_n) = g(T, \theta)h(x_1, \ldots, x_n)$$

where:

$g(T, \theta)$ depends on θ,

$h(x_1, \ldots, x_n)$ does not depend on θ.

Given $X_1, ..., X_n$ $(n \geq 2)$ are iid with Common Weibull pdf, the joint pdf of $X_1, \ldots, X_n$ is:

$$f_\theta(x_1, \ldots, x_n) = \prod_{i=1}^n \left[\exp(-x_i^p/\theta) \cdot \frac{p}{\theta} x_i^{p-1}\right]$$

Where:

$$0 < x_i < \infty \quad \text{and} \quad 0 < \theta < \infty \quad \forall i$$

We can simplify this expression, somewhat:

$$f_\theta(x_1, \ldots, x_n) = \left(\frac{p}{\theta}\right)^n \exp\left(-\frac{T}{\theta}\right) \prod_{i=1}^n x_i^{p-1}$$

Of note:

- The function $g(T, \theta) = \left(\frac{p}{\theta}\right)^n \exp\left(-\frac{T}{\theta}\right)$ depends on T and θ.
- The function $h(x_1, \ldots, x_n) = \prod_{i=1}^n x_i^{p-1}$ does not depend on θ.

Thus, by the Factorization Theorem, T is sufficient for θ.

We then need to address completeness:

By definition, a statistic T is complete if for any function $g(T)$:

$$E_\theta[g(T)] = 0, \quad \forall\theta \quad \Rightarrow \quad P(g(T) = 0) = 1$$

That is, if the expectation of $g(T)$ is zero for all θ, then $g(T)$ must be the zero function.

Since $X_1^p, \ldots, X_n^p$ are iid Exponential(θ), following from rescaling the original Weibull-distributed X_i's, we have know the sum:

$$T = \sum_{i=1}^n X_i^p \sim \text{Gamma}(n, \theta)$$

We then note that the Gamma family is a specific instance of the Exponential family, so we note that the above result holds for "a complete exponential family in θ", which implies that T is a complete statistic for θ.

Specifically, this result follows because we treat n as known, meaning the only unknown in the above of the parameter θ, making this an instance of "a one-parameter exponential family is complete in θ", though the general description is we cannot find anything with expectation zero for all θ that is not the zero function itself.

Thus, T is both sufficient and complete for θ.

Finally, we address the ancillary statistic. By definition, a statistic U is ancillary if its distribution does not depend on θ.

We are given a hint to try:

$$U = \frac{X_1^p}{T}$$

Since $X_1^p, \ldots, X_n^p$ are iid Exponential(θ), again following from the initial X_i's being iid Weibull. At any rate, we can again rescale, but with a different linear combination, namely:

$$\left(\frac{X_1^p}{\theta}, \ldots, \frac{X_n^p}{\theta}\right) \sim \text{Exp}(1)$$

And note the above are still iid.

Taking the sum, we can express this as another sum of Exponential iid random variables, giving us:

$$T/\theta \sim \text{Gamma}(n, 1)$$

Since:

$$U = \frac{X_1^p}{T} = \frac{X_1^p/\theta}{T/\theta}$$

By multiplying by a "cheeky one".

We then note that $\frac{X_1^p}{\theta} \sim Gamma(1, 1)$ and $T/\theta \sim \text{Gamma}(n, 1)$, we know that U is a ratio of two Gamma distributions that are independent is by definition a Beta distribution (the numerator and denominator being independent).

Specifically, U is distributed:

$$U \sim \text{Beta}(1, n-1)$$

Since the Beta(1, n-1) distribution does not depend on θ, we know the statistic U is ancillary.

Extra Details

I'm pretty sure the above is "enough" (avoiding using the word sufficient explicitly in a non-maths context), but in the event that some more work would help:

For finding the distribution of U:

$$U = \frac{X_1^p}{T} = \frac{X_1^p/\theta}{T/\theta}$$

$$f_{X_1^p/\theta}(x) = \frac{x^{1-1}e^{-x}}{\Gamma(1)} = e^{-x}$$

$$f_{T/\theta}(t) = \frac{t^{n-1}e^{-t}}{\Gamma(n)}$$

Giving joint pdf:

$$f_{X_1^p/\theta, T/\theta}(x, t) = f_{X_1^p/\theta}(x) f_{T/\theta}(t) = e^{-x} \cdot \frac{t^{n-1}e^{-t}}{\Gamma(n)}$$

For:

$$U = \frac{X_1^p}{T}$$

Gives:

$$X_1^p = UT$$

$$T = T$$

With Jacobian:

$$J = \begin{vmatrix} \frac{\partial X_1^p}{\partial U} & \frac{\partial X_1^p}{\partial T} \\ \frac{\partial T}{\partial U} & \frac{\partial T}{\partial T} \end{vmatrix} = \begin{vmatrix} T & U \\ 0 & 1 \end{vmatrix} = T$$

Transformation of the prior joint pdf gives:

$$f_{U,T}(u,t) = f_{X_1^p,T}(ut,t) \cdot |J| = e^{-ut} \cdot \frac{t^{n-1}e^{-t}}{\Gamma(n)} \cdot T = \frac{t^n e^{-t}}{\Gamma(n)} e^{-ut}$$

Getting the marginal distribution of U, integrating over T, we have:

$$f_U(u) = \int_0^\infty f_{U,T}(u,t)dt = \int_0^\infty \frac{t^n e^{-t}}{\Gamma(n)} e^{-ut} dt = \frac{1}{\Gamma(n)} \int_0^\infty t^n e^{-(1+u)t} dt = \frac{\Gamma(n+1)}{\Gamma(n)} \cdot \frac{1}{(1+u)^{n+1}}$$

As $\Gamma(n+1) = n\Gamma(n)$, we can simplify further:

$$f_U(u) = n \frac{1}{(1+u)^{n+1}} = \frac{u^{1-1}(1-u)^{(n-1)-1}}{B(1, n-1)}$$

Where B is the Beta function as is usually defined, and $u^{1-1} = u^0 = 0$ under the constraint 0<u<1 as a "cheeky one".

This confirms:

$$U \sim \text{Beta}(1, n-1)$$

2.

Problem 7.60, Casella and Berger and the following:

Extra

Let $X_1, \ldots, X_n$ be iid gamma(α, β) with α known. Find the best unbiased estimator of $1/\beta$.

Answer

Since $X_i \sim \text{Gamma}(\alpha, \beta)$ are iid, the sum:

$$S_n = \sum_{i=1}^{n} X_i \sim \text{Gamma}(n\alpha, \beta)$$

Taking expectation of this statistic:

$$E_\beta(S_n) = n\alpha\beta$$

To get $\frac{1}{\beta}$, consider:

$$E_\beta\left[\frac{n\alpha}{S_n}\right] = \frac{n\alpha}{E_\beta(S_n)} = \frac{n\alpha}{n\alpha\beta} = \frac{1}{\beta}$$

This shows that $\frac{n\alpha}{S_n}$ is an unbiased estimator of $\frac{1}{\beta}$

Noting the work shown previously in 1., via the Factorization Theorem, we know S_n is a sufficient statistic for β (scale parameter of the Gamma).

Similarly, we know that the Gamma family is a specific instance of a a complete one-parameter exponential family, meaning we know that S_n is also complete.

Now, we can do something new! Via Lehmann-Scheffé, since $\delta(S_n) = \frac{n\alpha}{S_n}$ is an unbiased function of the complete sufficient statistic,S_n, we know that $\delta(S_n) = \frac{n\alpha}{S_n}$ is the UMVUE of $1/\beta$. Yippee!

a)

Let $S_n = \sum_{i=1}^{n} X_i$. Using Basu's theorem, show X_1/S_n and S_n are independent.

Answer

By definition, Basu's theorem: If T is a complete sufficient statistic and U is an ancillary statistic, then T and U are independent.

From the prior question, we know that S_n is complete and sufficient for β.

We need to then find an ancillary statistic.

To that end, let:

$$U = \frac{X_1}{S_n}$$

Where:

$$X_1 \sim \text{Gamma}(\alpha, \beta)$$

$$S_n \sim \text{Gamma}(n\alpha, \beta)$$

Using given information, we know U is a ratio of two Gamma random variables. However, this is complicated somewhat by X_1 and S_n not being independent! So we need to do a bit of calculation to identify the underlying structure (distribution) of their ratios (though it will be Beta-distributed, the parameter values don't follow the typical formula, i.e. differences between numerator and denominator). To that end, using the known pdfs of each statistic:

$$f_{X_1}(x_1) = \frac{x_1^{\alpha-1}e^{-x_1/\beta}}{\beta^\alpha\Gamma(\alpha)}$$

$$f_{S_n}(s) = \frac{s^{n\alpha-1}e^{-s/\beta}}{\beta^{n\alpha}\Gamma(n\alpha)}$$

By the product rule (noting X_1 and S_n are not independent):

$$f_{X_1,S_n}(x_1, s) = f_{X_1|S_n}(x_1|s)f_{S_n}(s)$$

Where:

$$f_{X_1|S_n}(x_1|s) = \frac{x_1^{\alpha-1}(s-x_1)^{(n-1)\alpha-1}}{s^{n\alpha-1}B(\alpha,(n-1)\alpha)}$$

Giving joint pdf:

$$f_{X_1,S_n}(x_1, s) = \frac{x_1^{\alpha-1}(s-x_1)^{(n-1)\alpha-1}e^{-s/\beta}s^{n\alpha-1}}{\beta^{n\alpha}\Gamma(\alpha)\Gamma((n-1)\alpha)}$$

For

$$U = \frac{X_1}{S_n}$$

We have:

$$X_1 = US_n$$

$$S = (1-U)S_n$$

Caluating the Jacobian determinant:

$$J = \begin{vmatrix} \frac{\partial X_1}{\partial U} & \frac{\partial X_1}{\partial S_n} \\ \frac{\partial S}{\partial U} & \frac{\partial S}{\partial S_n} \end{vmatrix} = \begin{vmatrix} S_n & U \\ -S_n & 1-U \end{vmatrix} = S_n(1-U) + S_nU = S_n$$

Via transformation, we have:

$$f_{U,S_n}(u,s) = f_{X_1,S_n}(us,s)|J| = \frac{(us)^{\alpha-1}((1-u)s)^{(n-1)\alpha-1}e^{-s/\beta}s^{n\alpha-1}}{\beta^{n\alpha}\Gamma(\alpha)\Gamma((n-1)\alpha)}s = \frac{u^{\alpha-1}(1-u)^{(n-1)\alpha-1}s^{n\alpha-1}e^{-s/\beta}}{\beta^{n\alpha}\Gamma(\alpha)\Gamma((n-1)\alpha)}$$

Getting the the marginal distribution of $f_U(u)$, our variable of interest in this problem:

$$f_U(u) = \int_0^\infty f_{U,S_n}(u,s)ds = \frac{u^{\alpha-1}(1-u)^{(n-1)\alpha-1}\Gamma(n\alpha)}{\Gamma(\alpha)\Gamma((n-1)\alpha)}$$

Thus, we can identify the distribution and parameters from the above! We know that U is Beta-distributed, specifically:

$$U \sim \text{Beta}(\alpha, (n-1)\alpha)$$

Which does not depend on β for any of its parameters! This means we have an ancillary statistic.

As such, by Basu's theorem, $U = X_1/S_n$ and S_n are independent.

b)

Using the result in a) and $E_\theta(S_n) = n\alpha\beta$, find $E_\theta(X_1/S_n)$.

Answer

Using the results in a):

$$E_\theta\left(\frac{X_1}{S_n}\right) = E_\theta\left(U\right)$$

Where:

$$U \sim \text{Beta}(\alpha, (n-1)\alpha)$$

Using the properties of a known distribution (distribution of U), we know that:

$$E_\theta\left(\frac{X_1}{S_n}\right) = \frac{\alpha}{\alpha + (n-1)\alpha} = \frac{1}{n}$$

3.

Problem 8.13(a)-(c), Casella and Berger (2nd Edition) and, in place of Problem 8.13(d), consider the following test:

Let X_1, X_2 be iid uniform$(\theta, \theta+1)$. For testing $H_0 : \theta = 0$ versus $H_1 : \theta > 0$, we have two competing tests:

$$\phi_1(X_1) : \text{Reject } H_0 \text{ if } X_1 > 0.95,$$

$$\phi_2(X_1, X_2) : \text{Reject } H_0 \text{ if } X_1 + X_2 > C.$$

a)

Find the value of C so that ϕ_2 has the same size as ϕ_1.

Answer

The size of ϕ_1 is:

$$\alpha_1 = P(X_1 > 0.95 \mid \theta = 0) = 0.05$$

The size of ϕ_2 is:

$$\alpha_2 = P(X_1 + X_2 > C \mid \theta = 0)$$

For $1 \leq C \leq 2$, the probability $P(X_1 + X_2 > C \mid \theta = 0)$ is:

$$\alpha_2 = \int_{1-C}^{1} \int_{C-x_1}^{1} 1 \, dx_2 \, dx_1 = \frac{(2-C)^2}{2}$$

For $\alpha_2 = \alpha_1 = 0.05$, we solve for C:

$$\frac{(2-C)^2}{2} = 0.05 \implies (2-C)^2 = 0.1 \implies C = 2 - \sqrt{0.1} \approx 1.68$$

b)

Calculate the power function of each test. Draw a well-labeled graph of each power function.

Answer

The power function of ϕ_1 is:

$$\beta_1(\theta) = P_\theta(X_1 > 0.95) = \begin{cases} 0 & \text{if } \theta \leq -0.05, \\ \theta + 0.05 & \text{if } -0.05 < \theta \leq 0.95, \\ 1 & \text{if } \theta > 0.95 \end{cases}$$

The distribution of $Y = X_1 + X_2$ is:

$$f_Y(y \mid \theta) = \begin{cases} y - 2\theta & \text{if } 2\theta \le y < 2\theta + 1, \\ 2\theta + 2 - y & \text{if } 2\theta + 1 \le y < 2\theta + 2, \\ 0 & \text{otherwise} \end{cases}$$

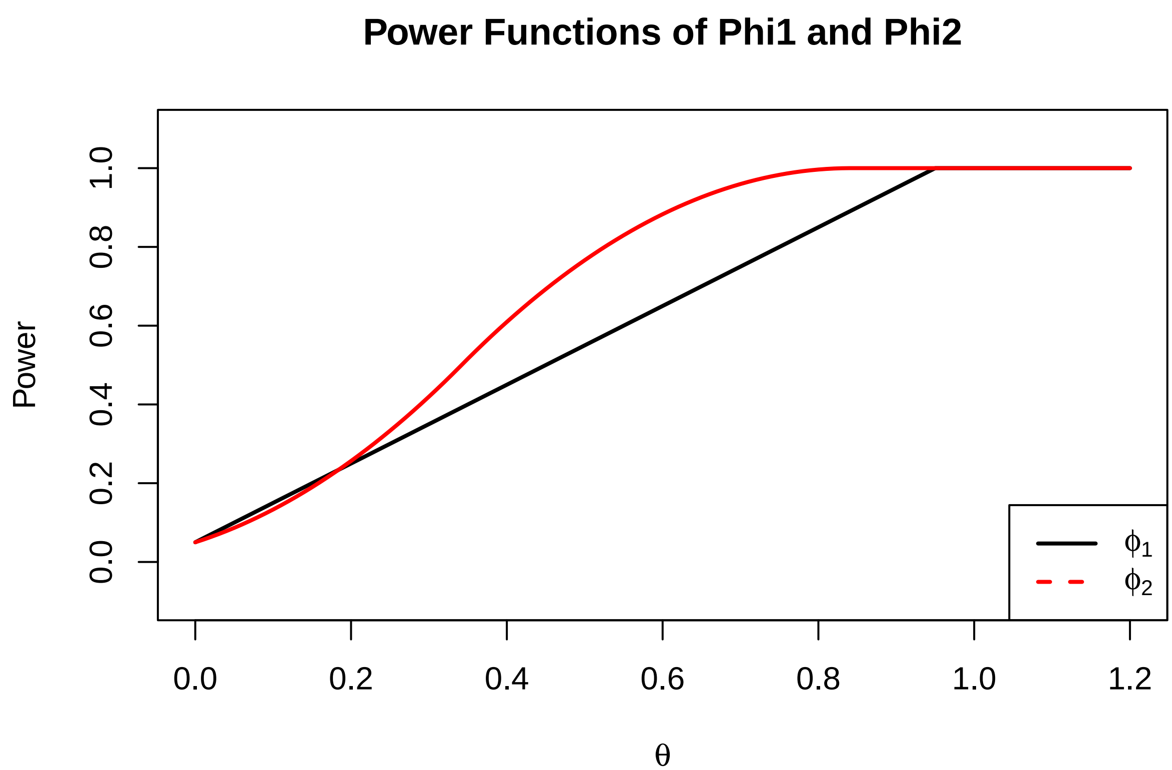
The power function of ϕ_2 is:

$$\beta_2(\theta) = P_\theta(Y > C) = \begin{cases} 0 & \text{if } \theta \le \frac{C}{2} - 1, \\ \frac{(2\theta+2-C)^2}{2} & \text{if } \frac{C}{2} - 1 < \theta \le \frac{C-1}{2}, \\ 1 - \frac{(C-2\theta)^2}{2} & \text{if } \frac{C-1}{2} < \theta \le \frac{C}{2}, \\ 1 & \text{if } \theta > \frac{C}{2} \end{cases}$$

For $C \approx 1.68$:

$$\beta_2(\theta) = \begin{cases} 0 & \text{if } \theta \le -0.16, \\ \frac{(2\theta+0.32)^2}{2} & \text{if } -0.16 < \theta \le 0.34, \\ 1 - \frac{(1.68-2\theta)^2}{2} & \text{if } 0.34 < \theta \le 0.84, \\ 1 & \text{if } \theta > 0.84 \end{cases}$$

```r
theta <- seq(0, 1.2, by = 0.01)
C <- 2 - sqrt(0.1)

# Power function for phi_1
beta1 <- pmax(0, pmin(1, theta + 0.05))

# Power function for phi_2
beta2 <- ifelse(theta <= (C/2) - 1, 0,
         ifelse(theta <= (C - 1)/2, ((2*theta + 2 - C)^2)/2,
         ifelse(theta <= C/2, 1 - ((C - 2*theta)^2)/2, 1)))

plot(theta, beta1, type = "l", col = "black", lwd = 2, ylim = c(-0.1, 1.1),
     ylab = "Power", xlab = expression(theta), main = "Power Functions of Phi1 and Phi2")
lines(theta, beta2, col = "red", lwd = 2)
legend("bottomright", legend = c(expression(phi[1]), expression(phi[2])),
       col = c("black", "red"), lty = c(1, 2), lwd = 2)
```



c)

Prove or disprove: ϕ_2 is a more powerful test than ϕ_1.

Answer

From the graph above, ϕ_1 is more powerful for θ near 0, around 0 to 0.2, but ϕ_2 is more powerful for larger values of θ, particularly around 0.2 to 0.9.

To be a more powerful test, or "uniformly more powerful", the test must be more powerful than the reference test for all values of θ. We do not meet this condition, meaning ϕ_2 is not a more powerful test than ϕ_1 (but also vice versa, neither test is uniformly more powerful than the other for the values of θ being considered.)

Extra

$$\phi_3(X_1, X_2) = \begin{cases} 1 & \text{if } X_{(1)} > 1 - \sqrt{0.05} \text{ or } X_{(2)} > 1 \\ 0 & \text{otherwise} \end{cases}$$

where $X_{(1)}, X_{(2)}$ are the min, max.

Find the size of this test and the power function for $\theta > 0$. Then, graph the power functions of ϕ_3 and ϕ_2 to determine which test is more powerful. (It's enough to graph over the range $\theta \in [0, 1.2]$.)

Answer

With the tests as defined,

where $X_{(1)}$ and $X_{(2)}$ are the minimum and maximum of X_1, X_2, respectively.

Under $H_0 : \theta = 0$, $X_1, X_2 \sim \text{Uniform}(0, 1)$, and the order statistics $X_{(1)}$ and $X_{(2)}$ are random variables with distributions:

$X_{(1)} \sim \text{Beta}(1, 2)$,

$X_{(2)} \sim \text{Beta}(2, 1)$.

Under $H_0 : \theta = 0$, the size of ϕ_3 is:

$$\alpha_3 = P(X_{(1)} > 1 - \sqrt{0.05} \mid \theta = 0) = (1 - (1 - \sqrt{0.05}))^2 = 0.05$$

This is because $X_{(1)} > 1 - \sqrt{0.05}$ requires both X_1 and X_2 to be greater than $1 - \sqrt{0.05}$, and the probability of this event is $(\sqrt{0.05})^2 = 0.05$.

Under $H_1 : \theta > 0$, $X_1, X_2 \sim \text{Uniform}(\theta, \theta + 1)$.

The minimum $X_{(1)}$ follows the CDF:

$$P(X_{(1)} \leq x) = 1 - (1 - (x - \theta))^2 \quad \text{for } \theta \leq x \leq \theta + 1$$

Thus, the power function of ϕ_3 is:

$$\beta_3(\theta) = P_\theta(X_{(1)} > 1 - \sqrt{0.05}) = (1 - (1 - \sqrt{0.05} - \theta))^2$$

For $\theta > 1 - \sqrt{0.05}$, $\beta_3(\theta) = 1$ because $X_{(1)} > 1 - \sqrt{0.05}$ is always true.

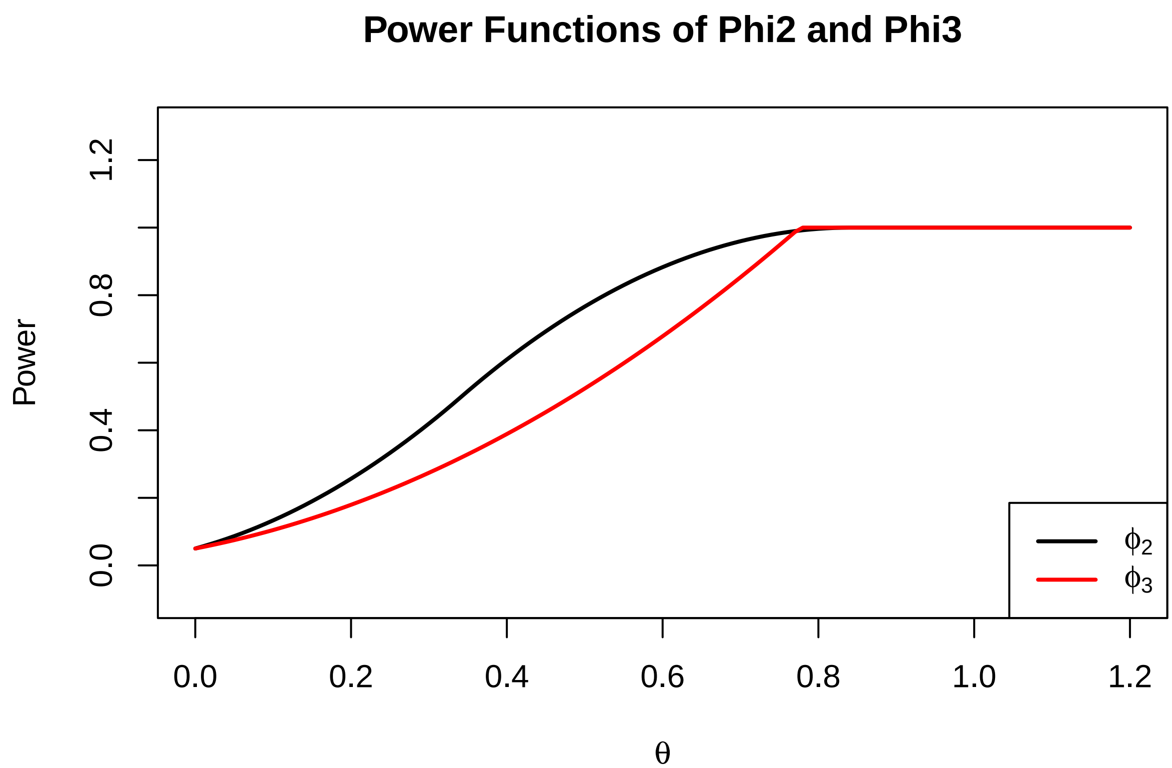
The power function of ϕ_2, as determined previously, is:

$$\beta_2(\theta) = \begin{cases} 0 & \text{if } \theta \leq -0.16, \\ \frac{(2\theta + 0.32)^2}{2} & \text{if } -0.16 < \theta \leq 0.34, \\ 1 - \frac{(1.68 - 2\theta)^2}{2} & \text{if } 0.34 < \theta \leq 0.84, \\ 1 & \text{if } \theta > 0.84 \end{cases}$$

From the graph comparing the two tests, ϕ_2 is more powerful for small values of θ, roughly speaking less than 0.7, and ϕ_3 is more powerful for larger values of θ, roughly greater than 0.7.

```r
# setup
theta <- seq(0, 1.2, by = 0.01)

t_crit <- 1 - sqrt(0.05)

phi3_power <- function(theta) {
    ifelse(theta <= t_crit, (1 - (1 - sqrt(0.05) - theta))^2, 1)
}

beta3 <- sapply(theta, phi3_power)

C <- 2 - sqrt(0.1)
beta2 <- ifelse(theta <= (C - 1)/2, ((2*theta + 2 - C)^2)/2,
```

```r
         ifelse(theta <= C/2, 1 - ((C - 2*theta)^2)/2, 1))

plot(theta, beta2, type = "l", col = "black", lwd = 2, ylim = c(-0.1, 1.3),
     ylab = "Power", xlab = expression(theta), main = "Power Functions of Phi2 and Phi3")
lines(theta, beta3, col = "red", lwd = 2)

legend("bottomright", legend = c(expression(phi[2]), expression(phi[3])),
       col = c("black", "red"), lty = c(1, 1), lwd = 2)
```



4.

Problem 8.15, Casella and Berger (2nd Edition), though you can just assume the form given is most powerful (no need to show).

Show that for a random sample $X_1, \dots, X_n$ from a $\mathcal{N}(0, \sigma^2)$ population, the most powerful test of $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma = \sigma_1$, where $\sigma_0 < \sigma_1$, is given by

$$\phi\left(\sum X_i^2\right) = \begin{cases} 1 & \text{if } \sum X_i^2 > c, \\ 0 & \text{if } \sum X_i^2 \leq c. \end{cases}$$

For a given value of α, the size of the Type I Error, show how the value of c is explicitly determined.

Answer

From the Neyman-Pearson lemma, the most powerful test rejects H_0 if the likelihood ratio exceeds a threshold k.

The likelihood ratio is given by:

$$\Lambda = \frac{f(x \mid \sigma_1)}{f(x \mid \sigma_0)} = \left(\frac{\sigma_0}{\sigma_1}\right)^n \exp\left\{\frac{1}{2}\sum_i x_i^2 \left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)\right\} > k$$

Taking the logarithm (a monotonic function):

$$\log \Lambda = n \log\left(\frac{\sigma_0}{\sigma_1}\right) + \frac{1}{2}\sum_i x_i^2 \left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right) > \log(k)$$

We isolate the term $\sum_i x_i^2$ to one side of the inequality:

$$\frac{1}{2}\sum_i x_i^2 \left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right) > \log k - n \log\left(\frac{\sigma_0}{\sigma_1}\right)$$

$$\sum_i x_i^2 \left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right) > 2\left(\log k - n \log\left(\frac{\sigma_0}{\sigma_1}\right)\right)$$

Solving for $\sum_i x_i^2$:

$$\sum_i x_i^2 > \frac{2\left(\log k - n \log\left(\frac{\sigma_0}{\sigma_1}\right)\right)}{\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}} = c$$

for some constant, real-valued c.

A couple notes:

- The above assumes $\sigma_1 > \sigma_0$, so that $\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2} > 0$, is this switches, then the overall inequailty flips as well.
- Also, the inequality $\sum_i x_i^2 > c$ defines the rejection region for the uniformly most powerful (UMP) test.

That being said, now the critical value c is determined such that the Type I error probability is α:

$$\alpha = P_{\sigma_0}\left(\sum_i X_i^2 > c\right)$$

Under H_0, $\sum_i X_i^2/\sigma_0^2$ follows a chi-squared distribution with n degrees of freedom (squared standard normal, where we achieve a standard normal variable by scaling by σ_0^2):

$$\sum_i X_i^2/\sigma_0^2 \sim \chi_n^2$$

Thus, we can rewrite the expression for α as:

$$\alpha = P_{\sigma_0}\left(\sum_i X_i^2 > c\right) = P_{\sigma_0}\left(\sum_i X_i^2/\sigma_0^2 > c/\sigma_0^2\right) = P\left(\chi_n^2 > c/\sigma_0^2\right)$$

Solving for c:

$$c = \sigma_0^2 \cdot \chi_{n,1-\alpha}^2$$

where $\chi_{n,1-\alpha}^2$ is the $(1-\alpha)$-quantile of the χ_n^2 distribution.

The UMP test rejects H_0 if:

$$\sum_i X_i^2 > c = \sigma_0^2 \cdot \chi_{n,1-\alpha}^2$$

This defines the rejection region for the most powerful test with Type I error probability α.

Note: All the above summations are equivalent to summing from i=1 to n, i.e.

$$\sum_i \equiv \sum_{i=1}^n$$