

Homework 9 – STAT 5430

Due Sunday, May 4 by midnight in gradescope;

This is the last homework: you can also submit this on Monday, May 5 by midnight in gradescope with no penalty

Submit a solution to Problem 3 and submit a solution to one exactly problem between Problem 1 or 2 below (you can choose).

1. Let X_1, \dots, X_n be iid exponential(θ) and let $\hat{\theta}_n \equiv \bar{X}_n \equiv \sum_{i=1}^n X_i/n$ denote the MLE based on X_1, \dots, X_n .
 - (a) Determine the limiting distribution of $\sqrt{n}(\hat{\theta}_n - \theta)$ as $n \rightarrow \infty$.
 - (b) Find a variance stabilizing transformation (VST) for $\{\hat{\theta}_n\}$ and use this to determine a large sample confidence interval for θ with approximate confidence coefficient $1 - \alpha$.
 - (c) Suppose a random sample X_1, \dots, X_{100} of $n = 100$ observations yields $\bar{x}_n = 1.835464$. Use this information to obtain a large sample confidence interval for θ based on a likelihood ratio statistic, which has approximate confidence coefficient 90%. (Use the chi-squared approximation for this; you should be able to then numerically determine the interval.) Using this data, compute also a confidence interval with approximate confidence coefficient 90% using the VST approach from part(b).
2. For $\theta > 0$, suppose that X_1, \dots, X_n are iid Uniform($0, \theta$). Consider the MLE of θ , which is given by $\hat{\theta}_n = \max\{X_1, X_2, \dots, X_n\}$.
 - (a) Prove that, in distribution, $n(\theta - \hat{\theta}_n) \xrightarrow{d} \text{Exponential}(\theta)$ as $n \rightarrow \infty$ (i.e., converges in distribution to an exponential with mean θ).
(Hint: Evaluate $P_\theta[n(\theta - \hat{\theta}_n) > t]$ and remember that $\lim_{s \rightarrow \infty} (1 + \frac{a}{s})^s = \exp(a)$.)
 - (b) Argue carefully that $\{\theta > 0 : \theta \leq n\hat{\theta}_n/(n - \log(20))\}$ can be used as a one-sided (upper) confidence interval for θ with approximate confidence coefficient 95% (that is, show that the interval will have an approximate coverage probability of 95% for each $\theta > 0$).
3. Suppose X_1, \dots, X_n are a random sample with common cdf given by

$$P(X_1 \leq x|\theta) = \begin{cases} 1 - e^{-(x/\theta)^2} & \text{if } x > 0 \\ 0 & \text{otherwise,} \end{cases} \quad \theta > 0.$$

- (a) Use the Mood-Graybill-Boes Method to derive a CI for θ with C.C. $1 - \alpha$ based on the statistic $X_{(1)} = \min_{1 \leq i \leq n} X_i$.
- (b) Use the Mood-Graybill-Boes Method to derive a CI for θ with C.C. $1 - \alpha$ based on the statistic $T = \sum_{i=1}^n X_i^2$. Express your confidence interval using chi-squared quantiles.

Note: One can show X_i^2 is Exponential(θ^2) distributed so that $2T/\theta^2$ is χ_{2n}^2 distributed with $2n$ degrees of freedom.