Statistics 520, Fall 2025

Assignment 8

Consider a problem of conducting a Bayesian analysis with a one-sample gamma model. Assume that random variables $Y_1, \ldots, Y_n$ are independent and identically distributed with common probability density function, for $\alpha > 0$ and $\beta > 0$,

$$f(y|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} \exp(-\beta y); \quad y > 0$$

Suppose that we will assign a joint prior to α and β as a product form, with particular values of $A > 0$, $\gamma_0 > 0$ and $\lambda_0 > 0$,

$$\begin{aligned} \pi_\alpha(\alpha) &= \frac{1}{A} I(0 < \alpha < A), \\ \pi_\beta(\beta) &= \frac{\lambda_0^{\gamma_0}}{\Gamma(\gamma_0)} \beta^{\gamma_0 - 1} \exp(-\lambda_0 \beta); \quad \beta > 0. \end{aligned}$$

Recall that in the analysis of an actual data set, A, γ_0 and λ_0 will be given specific numerical values. Since this is a simulated example and we have no actual prior information, for this assignment use $A = 20$, $\gamma_0 = 0.5$ and $\lambda_0 = 0.1$. This gives prior expectation of 5.0 and prior variance of 50. The prior does focus probability on smaller values, but still has $Pr(\beta > 10) = 0.16$.

1. (5 pts.) Consider using a Metropolis-Hastings algorithm with independent random walk proposals for α and β. Suppose that our current values are (α_m, β_m) and that the proposal (α^*, β^*) has been generated from the distribution $q(\alpha, \beta|\alpha_m, \beta_m)$ which is the product of independent random walks. Identify the appropriate acceptance probability for the jump proposal (α^*, β^*).

2. (40 pts.) On the course web page is a data set called `gammadat_bayes.txt`. Program a Metropolis-Hastings algorithm and simulate $50,000$ values from the joint posterior of α and β and $\mu = \alpha/\beta$. Provide (with supporting evidence if appropriate),

- information on how you selected a burn-in period. *NOTE: I do not expect you to compute Gelman-Rubin scale reduction factors for this assignment.*
- information on how you tuned the algorithm for acceptance rate and what you ended up with for random walk variances and acceptance rate.
- summaries of the marginal posterior distributions of α and β and $\mu = \alpha/\beta$, including histograms and five-number summaries, 95% central credible intervals, and correlation between α and β in the Markov chain.

3. (15 pts.) Using both the 75^{th} percentile and the range as data characteristics of potential interest, compute posterior predictive p-values from $10,000$ posterior predictive data sets.

4. (40 pts.) Now consider the use of a Gibbs Sampling algorithm to simulate from the joint posterior of α and β and μ. Derive full conditional posterior densities for α and β. Using these distributions, program a Gibbs Sampling algorithm and simulate $50,000$ values from the joint posterior. Provide (with supporting evidence if appropriate),

   - information on how you selected a burn-in period. Again, there is no need to compute Gelman-Rubin scale reduction factors for this assignment.
   - summaries of the marginal posterior distributions of α and β, including histograms and five-number summaries, 95% central credible intervals, and correlation between α and β in the Markov chain.

5. (15 pts.) Using both the 75^{th} percentile and the range as data characteristics of potential interest, compute posterior predictive p-values from $10,000$ simulated posterior predictive data sets.

6. (10 pts.) Compare your results from the use of Metropolis-Hastings and Gibbs Sampling.

7. (5 pts.) On this particular assignment, attach your R code for functions you programmed to do the necessary computations as an APPENDIX – not part of the body of your answer.