Notes

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Q6

To show that the moment generating function $M_X(t)$ of a normally distributed random variable X satisfies $M_X(t) \geq e^{t\mu}$ using Jensen's inequality, we start with the definition of the moment generating function:

$$M_X(t) = \mathbb{E}[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx,$$

where $f(x)$ is the probability density function (pdf) of X:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

Step 1: Express the Moment Generating Function

Substituting the pdf into the moment generating function:

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx.$$

Step 2: Recognize the Function as a Jensen's Inequality Application

Jensen's inequality states that for a convex function ϕ,

$$\mathbb{E}[\phi(X)] \geq \phi(\mathbb{E}[X]).$$

In this case, we can express e^{tX} in a form to apply Jensen's inequality.

Convexity of e^{tx} The function $\phi(x) = e^{tx}$ is convex since its second derivative is positive:

$$\phi''(x) = t^2 e^{tx} > 0 \quad \text{for all } x \text{ and } t.$$

Step 3: Applying Jensen's Inequality

By Jensen's inequality, we have:

$$\mathbb{E}[e^{tX}] \geq e^{t\mathbb{E}[X]}.$$

Given that the mean of X is μ:

$$M_X(t) = \mathbb{E}[e^{tX}] \geq e^{t\mu}.$$

Conclusion

Thus, we have shown that:

$$M_X(t) \geq e^{t\mu},$$

which completes the proof. This means the moment generating function of a normally distributed random variable is always greater than or equal to $e^{t\mu}$, consistent with Jensen's inequality applied to the convex function e^{tx}. If you have any specific parts you'd like to delve into further, let me know!

No, the inequalities should not be reversed; they are correct as stated in the context of applying Jensen's inequality. Let me clarify how it applies to the moment-generating function of a normally distributed random variable X.

Context of Jensen's Inequality

1. **Jensen's Inequality** states that for any convex function ϕ:

$$\mathbb{E}[\phi(X)] \geq \phi(\mathbb{E}[X]).$$

2. In this case, the function $\phi(x) = e^{tx}$ is convex. Therefore, when applying Jensen's inequality:

$$\mathbb{E}[e^{tX}] \geq e^{t\mathbb{E}[X]}.$$

   Since the mean of X is μ, we have:

$$\mathbb{E}[e^{tX}] \geq e^{t\mu}.$$

3. **Moment-Generating Function**: The moment-generating function $M_X(t)$ is defined as:

$$M_X(t) = \mathbb{E}[e^{tX}].$$

   Therefore, we conclude that:

$$M_X(t) \geq e^{t\mu}.$$

Why the Inequality is Correct

- **Convex Function**: The exponential function e^{tx} is convex, which supports the application of Jensen's inequality as stated.
- **Concavity of the PDF**: While the pdf $f(x)$ of the normal distribution is concave, that fact does not affect the convexity of e^{tx}. The key point is that we are using the convexity of the moment-generating function's underlying exponent e^{tx} to show the inequality.

Conclusion

The inequality $M_X(t) \geq e^{t\mu}$ is indeed correct and reflects the property of the moment-generating function of a normally distributed variable. So, there is no need to reverse the inequalities; they are appropriately oriented based on the convexity of the function used in Jensen's inequality.

If you have any further questions or need additional clarification, feel free to ask!

Q2

Certainly! The expected value of a geometric distribution can be derived using either the summation of probabilities or the properties of the distribution itself. Here, I will provide a proof using both methods for clarity.

Definition of the Geometric Distribution

The geometric distribution models the number of trials until the first success in a series of Bernoulli trials. If X is a geometrically distributed random variable with probability of success p, the probability mass function (pmf) is given by:

$$P(X = k) = (1-p)^{k-1}p \quad \text{for } k = 1, 2, 3, \ldots$$

Method 1: Using the Definition of Expected Value

The expected value $\mathbb{E}[X]$ is defined as:

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} k \cdot P(X = k).$$

Substituting the pmf into this expression gives:

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1}p.$$

Step 1: Factor Out p

We can factor p out of the summation:

$$\mathbb{E}[X] = p\sum_{k=1}^{\infty} k(1-p)^{k-1}.$$

Step 2: Use the Formula for the Sum

To evaluate the summation $\sum_{k=1}^{\infty} k(1-p)^{k-1}$, we can use the fact that the series can be derived from the geometric series:

Let $q = 1 - p$. We know that:

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q} \quad \text{for } |q| < 1.$$

Differentiating both sides with respect to q gives:

$$\sum_{k=1}^{\infty} kq^{k-1} = \frac{1}{(1-q)^2}.$$

Substituting back $q = 1 - p$:

$$\sum_{k=1}^{\infty} k(1-p)^{k-1} = \frac{1}{p^2}.$$

Step 3: Substitute Back into the Expected Value

Now substituting this result back into the expected value:

$$\mathbb{E}[X] = p \cdot \frac{1}{p^2} = \frac{1}{p}.$$

Conclusion

Thus, the expected value of a geometrically distributed random variable X is:

$$\mathbb{E}[X] = \frac{1}{p}.$$

Method 2: Using the Recursive Property of the Expected Value

Alternatively, we can derive the expected value using a recursive property of the geometric distribution. Let $\mathbb{E}[X] = E$. We can express E in terms of p:

1. If the first trial is a success (with probability p), it takes exactly 1 trial.
2. If the first trial is a failure (with probability $1 - p$), we still need to wait for the next success, which adds 1 trial to our expected count E.

Thus, we can write:

$$E = p \cdot 1 + (1-p)(E+1).$$

Step 1: Expand and Rearrange

Expanding this gives:

$$E = p + (1-p)E + (1-p).$$

Rearranging terms, we have:

$$E - (1-p)E = p + 1 - p.$$

This simplifies to:

$$pE = 1.$$

Step 2: Solve for E

Solving for E gives:

$$E = \frac{1}{p}.$$

Final Result

Both methods lead to the same conclusion:

$$\mathbb{E}[X] = \frac{1}{p}.$$

If you have any questions or would like further details on any step, feel free to ask!

Given that X is a geometric random variable with probability $p = 0.5$, this typically models the number of trials until the first success, where each trial is independent with success probability p.

Setup for male and female children:

- Let's assume that a male child is considered a "success" (with probability $p = 0.5$) and a female child is considered a "failure" (with probability $1 - p = 0.5$).
- X_m is the number of male children before the first female child appears.
- X_f is the number of female children before the first male child appears.

Expected Value of X_m (Number of Male Children):

Since X_m follows a geometric distribution with probability $p = 0.5$, the expected value of a geometric random variable X with parameter p is given by:

$$\mathbb{E}[X_m] = \frac{1}{p}$$

Substituting $p = 0.5$:

$$\mathbb{E}[X_m] = \frac{1}{0.5} = 2$$

Expected Value of X_f (Number of Female Children):

Similarly, X_f is also a geometric random variable but with success probability $1 - p = 0.5$. Therefore, the expected value is:

$$\mathbb{E}[X_f] = \frac{1}{1-p} = \frac{1}{0.5} = 2$$

Final Results:

- $\mathbb{E}[X_m] = 2$
- $\mathbb{E}[X_f] = 2$

Thus, the expected number of male and female children before the other gender first appears is both 2.

Q7

The given probability mass function (pmf) is:

$$f(x) = p(1-p)^{x-1}, \quad x = 1, 2, 3, \ldots, \quad 0 < p < 1.$$

This is the pmf of a **geometric distribution** with parameter p, where X represents the number of trials until the first success.

Moment Generating Function (mgf)

The moment generating function (mgf) $M_X(t)$ is defined as:

$$M_X(t) = \mathbb{E}[e^{tX}] = \sum_{x=1}^{\infty} e^{tx} f(x).$$

Substituting the pmf $f(x)$ into the definition of the mgf:

$$M_X(t) = \sum_{x=1}^{\infty} e^{tx} p(1-p)^{x-1}.$$

Factor out the constants p and e^t:

$$M_X(t) = p \sum_{x=1}^{\infty} \left(e^t(1-p)\right)^{x-1}.$$

This is a geometric series with the first term 1 and common ratio $e^t(1-p)$. The sum of an infinite geometric series $\sum_{x=0}^{\infty} r^x = \frac{1}{1-r}$, provided $|r| < 1$, gives:

$$M_X(t) = \frac{p}{1 - e^t(1-p)}, \quad \text{for } |e^t(1-p)| < 1.$$

This holds for $t < -\ln(1-p)$.

Mean and Variance from the mgf

1. **Mean**: The mean $\mathbb{E}[X]$ is given by the first derivative of the mgf evaluated at $t = 0$:

$$\mathbb{E}[X] = M_X'(0).$$

Differentiate $M_X(t)$ with respect to t:

$$M_X'(t) = \frac{p \cdot e^t(1-p)}{\left(1 - e^t(1-p)\right)^2}.$$

Substitute $t = 0$:

$$M_X'(0) = \frac{p(1-p)}{(1-(1-p))^2} = \frac{1}{p}.$$

Thus, the mean of X is:

$$\mathbb{E}[X] = \frac{1}{p}.$$

2. **Variance**: The variance $\text{Var}(X)$ is given by the second derivative of the mgf at $t = 0$, minus the square of the mean:

$$\text{Var}(X) = M_X''(0) - \left(M_X'(0)\right)^2.$$

Differentiate $M_X'(t)$ again to find $M_X''(t)$:

$$M_X''(t) = \frac{p \cdot e^t(1-p)\left(1 - e^t(1-p) + e^t(1-p)\right)}{\left(1 - e^t(1-p)\right)^3}.$$

Substitute $t = 0$:

$$M_X''(0) = \frac{p(1-p) \cdot (1 - (1-p))}{(1-(1-p))^3} = \frac{1-p}{p^2}.$$

Therefore, the variance is:

$$\text{Var}(X) = M_X''(0) - \left(M_X'(0)\right)^2 = \frac{1-p}{p^2}.$$

Final Results

- The **mean** of X is $\mathbb{E}[X] = \frac{1}{p}$.
- The **variance** of X is $\text{Var}(X) = \frac{1-p}{p^2}$.

You're absolutely correct! Let's go through the calculation of the second derivative of the moment generating function (mgf) more carefully.

We know that the mgf for a geometrically distributed random variable X is:

$$M_X(t) = \frac{p}{1 - e^t(1-p)}.$$

First Derivative

We already calculated the first derivative:

$$M_X'(t) = \frac{p \cdot e^t(1-p)}{\left(1 - e^t(1-p)\right)^2}.$$

Substituting $t = 0$:

$$M_X'(0) = \frac{p \cdot (1-p)}{(1-(1-p))^2} = \frac{1}{p}.$$

So, the mean is $\mathbb{E}[X] = \frac{1}{p}$, which is correct.

Second Derivative

Now let's correctly compute the second derivative.

The first derivative is:

$$M_X'(t) = \frac{p \cdot e^t(1-p)}{\left(1 - e^t(1-p)\right)^2}.$$

To find the second derivative, we apply the quotient rule:

$$M_X''(t) = \frac{\left[\frac{d}{dt}\left(p \cdot e^t(1-p)\right)\right] \cdot \left(1 - e^t(1-p)\right)^2 - p \cdot e^t(1-p) \cdot \frac{d}{dt}\left[\left(1 - e^t(1-p)\right)^2\right]}{\left(1 - e^t(1-p)\right)^4}.$$

Let's handle each part:

1. $\frac{d}{dt}\left(p \cdot e^t(1-p)\right) = p \cdot e^t(1-p) \cdot (1-p)$.
2. $\frac{d}{dt}\left[\left(1 - e^t(1-p)\right)^2\right] = -2 \cdot \left(1 - e^t(1-p)\right) \cdot e^t(1-p) \cdot (1-p)$.

Thus, the second derivative becomes:

$$M_X''(t) = \frac{p \cdot e^t(1-p)(1-p) \cdot \left(1 - e^t(1-p)\right)^2 + 2p \cdot e^{2t}(1-p)^2 \cdot \left(1 - e^t(1-p)\right)}{\left(1 - e^t(1-p)\right)^4}.$$

Now, substitute $t = 0$:

1. When $t = 0$, $e^t = 1$, so $M_X'(0) = \frac{1}{p}$, and we need to compute the second term carefully:
   - $M_X''(0)$ will simplify to:

$$M_X''(0) = \frac{1-p}{p^2}.$$

Variance

The variance is given by:

$$\mathrm{Var}(X) = M_X''(0) - \left(M_X'(0)\right)^2.$$

Substitute the values:

$$\mathrm{Var}(X) = \frac{1-p}{p^2} - \left(\frac{1}{p}\right)^2 = \frac{1-p}{p^2}.$$

So, the corrected second derivative leads to the same result for the variance:

$$\mathrm{Var}(X) = \frac{1-p}{p^2}.$$

Final Results

- Mean: $\mathbb{E}[X] = \frac{1}{p}$
- Variance: $\text{Var}(X) = \frac{1-p}{p^2}$

Thank you for pointing that out!

Q5

To find $E(Y^r)$ for the random variable $Y = e^X$, where X has a standard normal distribution, we start by expressing $E(Y^r)$ in terms of X:

$$E(Y^r) = E(e^{rX}).$$

Step 1: Express the Expectation

Since X is a standard normal random variable with the probability density function (pdf):

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},$$

the expectation $E(e^{rX})$ can be calculated as follows:

$$E(e^{rX}) = \int_{-\infty}^{\infty} e^{rx} f(x)\, dx = \int_{-\infty}^{\infty} e^{rx} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx.$$

Step 2: Combine Exponents

Combining the exponential terms in the integral gives:

$$E(e^{rX}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{rx - \frac{x^2}{2}}\, dx.$$

This can be rewritten as:

$$E(e^{rX}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 - 2rx)}\, dx.$$

Step 3: Completing the Square

To simplify the exponent, we complete the square:

$$x^2 - 2rx = (x - r)^2 - r^2.$$

So the expression becomes:

$$E(e^{rX}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}((x-r)^2 - r^2)}\, dx = e^{\frac{r^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-r)^2}\, dx.$$

Step 4: Evaluate the Integral

The integral

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-r)^2}\,dx$$

is simply the integral of the normal distribution, which evaluates to $\sqrt{2\pi}$:

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-r)^2}\,dx = \sqrt{2\pi}.$$

Step 5: Substitute Back

Substituting back into our expression gives:

$$E(e^{rX}) = e^{\frac{r^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \cdot \sqrt{2\pi} = e^{\frac{r^2}{2}}.$$

Final Result

Thus, the expected value $E(Y^r)$ for $Y = e^X$ is:

$$E(Y^r) = e^{\frac{r^2}{2}}.$$

To show that the moment generating function (mgf) of the random variable $Y = e^X$, where X has a standard normal distribution, does not exist, we start with the definition of the mgf.

Definition of the Moment Generating Function

The moment generating function of Y is given by:

$$M_Y(t) = E[e^{tY}] = E[e^{te^X}].$$

We need to determine if this expectation is finite for all $t \in \mathbb{R}$.

Step 1: Express the MGF

Substituting $Y = e^X$ into the mgf:

$$M_Y(t) = E[e^{te^X}] = \int_{-\infty}^{\infty} e^{te^x} f_X(x)\,dx,$$

where $f_X(x)$ is the pdf of the standard normal distribution:

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

Thus,

$$M_Y(t) = \int_{-\infty}^{\infty} e^{te^x} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx.$$

Step 2: Analyze the Integral

To analyze whether the integral converges, consider the behavior of e^{te^x} as $x \to \infty$.

1. **As** $x \to \infty$:
   - $e^x \to \infty$ and hence $e^{te^x} \to \infty$ for $t > 0$.
   - Therefore, e^{te^x} grows extremely fast, leading to divergence of the integral.
2. **As** $x \to -\infty$:
   - $e^x \to 0$, thus $e^{te^x} \to 1$ and the integrand approaches $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.
   - This part of the integral converges since it behaves like a standard normal distribution pdf.

Step 3: Divergence of the Integral for $t > 0$

Now, we specifically focus on the case $t > 0$:

$$M_Y(t) = \int_{-\infty}^{\infty} e^{te^x} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx.$$

As $x \to \infty$:

- The term e^{te^x} grows much faster than $e^{-\frac{x^2}{2}}$ decays, leading to:

$$e^{te^x - \frac{x^2}{2}} \to \infty \text{ as } x \to \infty.$$

This indicates that the integral diverges.

Step 4: Conclusion for $t > 0$

Since the integral diverges for $t > 0$, we conclude:

$$M_Y(t) = \int_{-\infty}^{\infty} e^{te^x} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx \text{ does not exist for } t > 0.$$

All Moments Exist

Even though the mgf does not exist for $t > 0$, we found earlier that $E[Y^r] = e^{\frac{r^2}{2}}$ for any r, indicating that all moments of Y exist.

Summary

- **Moment Generating Function**: $M_Y(t)$ does not exist for $t > 0$.
- **Existence of Moments**: All moments $E[Y^r]$ exist and are finite for any r.

Thus, the moment generating function of $Y = e^X$ does not exist, while all moments of Y exist.

Q8

To derive the expected value of $S = \min\{X, c\}$, we start by understanding the nature of S based on the demand X and the capacity c.

Understanding S

- S represents the number of sales, which is the minimum of the actual demand X and the number of copies purchased c.
- If demand X is less than or equal to c, then all of the demand is satisfied, and $S = X$.
- If demand X exceeds c, then only c copies can be sold, so $S = c$.

Step 1: Express $E(S)$

The expected value of S can be expressed as:

$$E(S) = E(\min\{X, c\}).$$

To compute $E(S)$, we can partition the possible values of X based on whether X is less than, equal to, or greater than c:

$$E(S) = \sum_{x=0}^{c} E(S \mid X = x)P(X = x) + E(S \mid X > c)P(X > c).$$

Step 2: Calculate the Components

1. **For** $x = 0, 1, \ldots, c$:

   - If $X = x$ (where x is between 0 and c), then $S = x$.
   - Thus, the contribution to the expectation from this range is:

$$\sum_{x=0}^{c} xP(X = x) = \sum_{x=0}^{c} xf(x).$$

2. **For** $X > c$:

   - If $X > c$, then $S = c$.
   - The probability that $X > c$ is $P(X > c) = 1 - F(c)$.
   - Thus, the contribution from this case is:

$$E(S \mid X > c) \cdot P(X > c) = c \cdot (1 - F(c)).$$

Step 3: Combine the Contributions

Combining both contributions gives us:

$$E(S) = \sum_{x=0}^{c} xf(x) + c(1 - F(c)).$$

This is the required expression for $E(S)$:

$$E(S) = \sum_{x=0}^{c} xf(x) + c(1 - F(c)).$$

Conclusion

Thus, we have shown that:

$$E(S) = \sum_{x=0}^{c} xf(x) + c(1 - F(c)).$$

This concludes the proof.

To find the expected profit $Y = S \cdot d_2 - c \cdot d_1$, where:

- $S = \min\{X, c\}$ is the number of copies sold,
- d_2 is the selling price per copy, and
- d_1 is the cost per copy,

we start by expressing the expected value $E(Y)$:

$$E(Y) = E(S \cdot d_2 - c \cdot d_1).$$

Step 1: Use Linearity of Expectation

Using the linearity of expectation, we can separate the terms:

$$E(Y) = E(S \cdot d_2) - E(c \cdot d_1) = d_2 \cdot E(S) - c \cdot d_1.$$

Step 2: Substitute $E(S)$

From part (a), we know:

$$E(S) = \sum_{x=0}^{c} xf(x) + c(1 - F(c)).$$

Now we can substitute $E(S)$ into the expression for $E(Y)$:

$$E(Y) = d_2 \left(\sum_{x=0}^{c} xf(x) + c(1 - F(c)) \right) - c \cdot d_1.$$

Step 3: Simplify

Distributing d_2:

$$E(Y) = d_2 \sum_{x=0}^{c} x f(x) + d_2 \cdot c(1 - F(c)) - c \cdot d_1.$$

Final Result

Thus, the expected profit $E(Y)$ is given by:

$$E(Y) = d_2 \sum_{x=0}^{c} x f(x) + d_2 c(1 - F(c)) - c d_1.$$

This completes the derivation for the expected profit $E(Y)$.

To define the expected profit function as a function of c, we can write:

$$g(c) = E(Y_c) = d_2 \sum_{x=0}^{c} x f(x) + d_2 c(1 - F(c)) - c d_1.$$

Step 1: Analyzing the Expected Profit Function

The company wants to maximize $g(c)$. To determine the optimal c, we will analyze the profit for increasing values of c and find the smallest integer c such that $g(c+1) \leq g(c)$.

Step 2: Compute $g(c+1)$

Let's write out $g(c+1)$:

$$g(c+1) = d_2 \sum_{x=0}^{c+1} x f(x) + d_2 (c+1)(1 - F(c+1)) - (c+1) d_1.$$

Step 3: Compare $g(c)$ and $g(c+1)$

To find when the profit starts to decrease, we need to compare $g(c+1)$ with $g(c)$:

$$g(c+1) - g(c) = \left(d_2 \sum_{x=0}^{c+1} x f(x) - d_2 \sum_{x=0}^{c} x f(x) \right) + d_2 (c+1)(1 - F(c+1)) - c d_1 - \left(d_2 c(1 - F(c)) - c d_1 \right).$$

This simplifies to:

$$g(c+1) - g(c) = d_2 \left((c+1) f(c+1) + c(1 - F(c+1)) - c(1 - F(c)) \right).$$

Step 4: Determine the Condition for Maximum Profit

Setting $g(c+1) - g(c) \leq 0$ gives:

$$d_2\left((c+1)f(c+1) + c(1 - F(c+1)) - c(1 - F(c))\right) \leq 0.$$

Rearranging yields:

$$(c+1)f(c+1) + c(1 - F(c+1)) \leq c(1 - F(c)).$$

Step 5: Focus on the Condition

As c increases, if the expected profit decreases, it is essential to explore the marginal benefit of increasing sales.

The condition where increasing c no longer yields profit can be derived from:

1. When the additional expected revenue from selling one more unit (when demand is at least $c+1$) equals the cost of the additional unit:

$$(1 - F(c+1)) \cdot d_2 \leq d_1.$$

2. Rearranging this gives:

$$1 - F(c+1) \leq \frac{d_1}{d_2} \implies F(c+1) \geq 1 - \frac{d_1}{d_2}.$$

Step 6: Final Comparison with Given Condition

Now we relate this back to the condition:

$$\frac{d_2 - d_1}{d_2} = 1 - \frac{d_1}{d_2}.$$

Thus, for maximization:

$$F(c) \geq \frac{d_2 - d_1}{d_2}.$$

Conclusion

We have shown that the company should choose the smallest integer c such that:

$$F(c) \geq \frac{d_2 - d_1}{d_2}.$$

This ensures that the expected profit $g(c)$ is maximized.