Statistics 520, Fall 2025

Assignment 6

This assignment is a short exercise in using generalized least squares to estimate the parameters of a nonlinear regression model. There are a number of rather classic nonlinear curves used to model the growth of populations of organisms. Although their use is not strictly tied of one field, different curves have been developed by scientists concerned with the growth of animal populations, plant populations, and microbial populations. A great many papers have been written on the origins, estimation, and interpretation of these curves (e.g., Birch, C. 1999. A new generalized logistic sigmoid growth equation compared with the Richards growth equation. *Annals of Botany* **83**: 713-723). One of these traditional curves is known as the Gompertz model, and there are many versions of the Gompertz curve (e.g., Tjorve, K.M.C. and Tjorve, E. 2017. The use of Gompertz models in growth analysis, and new Gompertz model approach: an addition to the unified Richards family. *PLoS One*, **12** (6):e0178691). We will use a four parameter version of the Gompertz model as the expectation function in a nonlinear regression model.

Define random variables $Y_1, \ldots, Y_n$ connected with the growth of some organism, observed at times $t_1, \ldots, t_n$. Assume the response variables are independent and follow the model,

$$\begin{aligned} Y_i &= \mu_i + \sigma\epsilon_i, \\ \mu_i &= B + A\exp[-\exp\{-k(t_i - T)\}], \\ \epsilon_i &\sim \text{ iid } \mathrm{N}(0,1). \end{aligned} \tag{1}$$

This model describes a sigmoidal curve with both upper and lower asymptotes, but is not constrained to be symmetric about its inflection point in the way a logistic curve is.

The parameters of this model are as follows:

- A: distance between upper and lower asymptotes
- B: lower asymptote
- k: called the growth rate, it is essentially the slope of the linear portion of the curve
- T: the time at which the inflection point occurs

On the course web page in the Data module is a file `growthdat.txt` that contains some data simulated from the Gompertz regression model (1). The variable names in this file are `x` and `y`, where `x` represents time of observation and `y` represents growth in some appropriate units. Your assignment is

1. (5 pts.) Find generalized least squares estimates of the parameters A, B, k, and T, and the associated moment-based estimate of σ^2. Describe how you determined effective starting values for the estimation procedure. Present a scatterplot along with the fitted expectation function.

2. (5 pts.) Compute 95% approximate confidence intervals for the parameters of the expectation function (5 pts.) (A, B, k, and T).

3. (5 pts.) Compute pairwise correlations between $\hat{A}$, $\hat{B}$, $\hat{k}$ and $\hat{T}$.

4. (10 pts.) Two quantities of interest to scientists are called the maximum relative growth rate and the maximum absolute growth rate. These quantities are related to the slope of the growth curve at the inflection point and give the per time unit increase in growth relative to the upper asymptote and in absolute scale at that time point. The maximum absolute growth rate is defined as,

$$k_{abs} = \frac{k(A+B)}{exp(1)}.$$

A plug-in estimate of k_{abs} is then,

$$\hat{k}_{abs} = \frac{\hat{k}(\hat{A} + \hat{B})}{\exp(1)},$$

and note that k_{abs} is an absolute function of its components. Given this, compute a 90% approximate confidence interval for k_{abs} (note a 90% interval – I get tired of using 95% all the time). Outline the procedure you used to calculate the quantities needed.

Computational Options

There are a number of ways to obtain the needed computations for this assignment. There is a function in the base `R` package called `nls` which stands for nonlinear least squares. I'm sure you could get most of what you need from this and associated functions. Alternatively, there is an `R` function called `nonlin` available on the course web page in the Computing Functions module, and a document called "Using nonlin" in the Computing Notes module that explains the syntax and output of `nonlin`.