# HW 2

Name: Sam Olson
Collaborators: XXX

## Fig. 1

Used in Q7, part (b)

$$1 = \int\limits_{\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x^2 e^{\frac{-x^2}{2}} dx = \frac{2}{\sqrt{2\pi}} \int\limits_{0}^{\infty} x^2 e^{\frac{-x^2}{2}} dx$$

# 1.

Q: Suppose a random variable X has the following cdf from class (which is neither a step function nor continuous):

$$F(x) = \begin{cases} 1 & x < 0 \\ 1 & 0 \leq x \leq 1 \\ 0 & x > 1 \end{cases}$$

(a): Find the following probabilities: $P(X > \frac{1}{2})$ $P(X \geq \frac{1}{2})$ $P(0 < X \leq \frac{1}{2})$ $P(0 \leq X \leq \frac{1}{2})$

(b): Conditional on the event "X > 0", the corresponding conditional pdf of X (i.e. given X > 0) is as follows at $x \in \mathbb{R}$:

$$P(X \leq x | X > 0) = \frac{P(X \leq x, X > 0)}{P(X > 0)} = \frac{P(0 < X \leq x)}{P(X > 0)} = \frac{F(x) - F(0)}{1 - F(0)}$$

Giving:

$$P(X \leq x | X > 0) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x \leq 1 \\ 1 & x > 1 \end{cases}$$

Based on the conditional cdf above, show that the distribution of X, conditional on "X > 0", is the same (i.e. has the same cdf) as that of a random variable Y which is "uniform" on the interval (0, 1), having constant pdf $f_Y(y) = 1$ for $0 < y < 1$ (with $f_Y(y) = 0$ for all other $y \in \mathbb{R}$)

A:

(a):

(b):

## 2.

Q: Statistical reliability involves studying the time to failure of manufactured units. In many reliability textbooks, one can find the exponential distribution:

$$f(x) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

where $\theta > 0$ is a fixed value, for modeling the time X that a random unit runs until failure (i.e. X is a survival time). Show that if X has an exponential distribution as above, then:

$P(X > s + t | X > t) = P(X > s)$

for any values t, s > 0; this feature is called the "memoryless" property of te exponential distribution.

A:

## 3. 2.3:

Q: Suppose X has the Geometrc pmf: $f_X(x) = \frac{1}{3}(\frac{2}{3})^x$, $x = 0, 1, 2, ...$ Determine the probability distribution of $Y = \frac{X}{X+1}$ Note that here X and Y are discrete random variables. To specify the probability distribution of Y, specify its pmf.

A:

## 4. 2.4:

Q: Let $\lambda$ be a fixed positive constant, and define the function $f(x)$ by: $f(x) = \frac{1}{2}\lambda e^{-\lambda x}$ if $x \geq 0$ and $f(x) = \frac{1}{2}\lambda e^{\lambda x}$ if $x < 0$

(a): Verify that $f(x)$ is a pdf.

(b): If X is a random variable with pdf given by $f(X)$, find $P(X < t)$ $\forall t$. Evaluate all integrals.

(c): Find $P(|X| < t)$ $\forall t$. Evaluate all integrals.

A:

(a) :

(b):

(c):

## 5. 2.6 (b, c):

Q: In each of the following find the pdf of Y. (Do not need to verify the pdf/evaluate the integration, per Instructions).

(b): $f_X(x) = \frac{3}{8}(x+1)^2$, $-1 < x < 1$; $Y = 1 - X^2$

(c): $f_X(x) = \frac{3}{8}(x+1)^2$, $-1 < x < 1$; $Y = 1 - X^2$ if $X \leq 0$ and $Y = 1 - X$ if $X > 0$

A:

(b):

(c):

## 6. 2.9:

Q: If the random variable X has pdf:

$$f(x) = \begin{cases} \frac{x-1}{2} & 1 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

find a monotone function $u(x)$ such that the random variable $Y = u(X)$ has a Uniform(0,1) distribution.

A:

## 7. 2.22 (a, b):

Q: Let X have the pdf:

$f(x) = \frac{4}{\beta^3\sqrt{\pi}} x^2 e^{\frac{-x^2}{\beta^2}}$, $0 < x < \infty$, $\beta > 0$

(a): Verify that $f(x)$ is a pdf.

(b): Find $E(X)$

A:

(a):

(b):

## 8.

Q: Suppose that a random variable U has a Uniform(0,1) distribution

(i.e. pdf $f_U(u) = 1$ for 0 < u < 1)

(a): Suppose a random variable Xhas a cdf $F(x)$ which is strictly increasing and continuous on $x \in \mathbb{R}$; this implies that, for any real value of 0 < u < 1, there is an inverse $F^{-1}(u) = x \in \mathbb{R}$ so that $F(x) = F(F^{-1}(u)) = u$. Define a random variable $Y = F^{-1}(U)$ based on the random variable U. Show that X and Y have the same cdf (i.e. the same distributions).

Hint: Use that, because F is strictly increasing, $P(Y \leq y) = P(F(Y) \leq F(y))$ holds for any $y \in \mathbb{R}$, i.e., Y can be less than or equal to y if and only if F(Y) is less than or equal to $F(y)$. Noe that $F(y) \in (0, 1)$ for any real y.

(b): If there is a computer program (i.e. random number generator) that produces numbers uniformly distributed between zero and one (i.e., according to the pdf $F_U(u)$), explain how these numbers could be used to generate values distributed according to the pdf $f_Z(z) = \frac{e^{-|z|}}{2}$, $-\infty < z < \infty$.

Hint: Use (a) where F now becomes the cdf of Z; you need to find $F^{-1}(u)$ for a given 0 < u < 1 by solving the expression $F(z) = u$ for $z \in \mathbb{R}$

A:

(a):

(b):