Homework 3 – STAT 542

Due Sunday, October 6th by 11:59 PM

1. 2.23(b), Casella & Berger
2. A family continues to have children until they have one female child. Suppose, for each birth, a single child is born and the child is equally likely to be male or female. The gender outcomes are independent across births.
   (a) Let X be a random variable representing the number of children born to this family. Find the distribution of X. (Hint: consider a model from class, originally used describe an experiment concerned with certain counts from coin flips; identifying the model form should make part (b) relatively easy too.)
   (b) Find the expected value $\mathrm{E}X$.
   (c) Let X_m denote the number of male children in this family and let X_f denote the number of female children. Find the expected value of X_m and the expected value of X_f.
3. 2.30(a),(b),(c), Casella & Berger. (Note that you may need to separately consider $t = 0$ vs $t \neq 0$.)
4. 2.31, Casella & Berger
5. Suppose that X has a standard normal distribution with pdf

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad -\infty < x < \infty.$$

   Then, $Y = e^X$ has a log-normal distribution (i.e., the logarithm of Y has a normal distribution).
   (a) Find $\mathrm{E}Y^r$ for any r.
   Note: to solve integrals, you need to "complete squares" in exponents to write $e^{rx}e^{-x^2/2} = e^{r^2/2}e^{-(x-r)^2/2}$.
   (b) Show the moment generating function of Y does not exist (even though all moments of Y exist).
   Hint: show $M_Y(t)$ cannot finitely exist for any $t > 0$.
6. Suppose that X has a normal distribution with pdf

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(\sigma^2 2)}, \quad -\infty < x < \infty.$$

   The mean of X is μ. Show that the moment generating function of X satisfies $M_X(t) \geq e^{t\mu}$. (You should not find the mgf of X to show this; it's an application of an inequality.)
7. Suppose that X has pmf $f(x) = p(1-p)^{x-1}$, $x = 1, 2, 3, \dots$ (for some given $0 < p < 1$). Find the mgf $M_X(t)$ and use this to derive the mean and variance of X.
   Note: use the formula $\sum_{k=0}^{\infty} a^k = (1-a)^{-1}$ for $|a| < 1$, which creates constraints on t in $e^t(1-p)$.
8. Suppose for one month a company purchases c copies of a software package at a cost of d_1 dollars per copy. The packages are sold to customers for d_2 dollars per copy; any unsold copies are destroyed at the end of the month. Let X represent the demand for this software package in the month. Assume that X is a discrete random variable with pmf $f(x)$ and cdf $F(x)$.
   (a) Let $S = \min\{X, c\}$ represent the number of sales during the month. Show that $\mathrm{E}(S) = \sum_{x=0}^{c} xf(x) + c(1 - F(c))$.
   Note: It helps to write the pmf of S first, where $P(S = c) = P(X \geq c)$, while $P(S = x) = P(X = x)$ for other $x = 0, 1, \dots, c$.
   (b) Let $Y = S \cdot d_2 - cd_1$ represent the profit for the company, the total income from sales minus the total cost of all copies. Find $\mathrm{E}(Y)$.
   (c) As $Y \equiv Y_c$ depends on integer $c \geq 0$, write the expected profit function as $g(c) \equiv \mathrm{E}(Y_c)$ from part (b). The company should choose the value of c which maximizes $g(c)$; that is, choose the smallest c such that $g(c+1)$ is less or equal to $g(c)$. Show that such $c \geq 0$ is the smallest integer with $F(c) \geq (d_2 - d_1)/d_2$.