

HW1

2024-09-07

Note

Exercises start ~ pg 67

Progress

1. DONE - CHECK STATUS: Does this proof work or can it be infinite?
2. DONE
3. IN PROGRESS
4. DONE
5. DONE
6. DONE
7. IN PROGRESS
8. DONE
9. IN PROGRESS

1. 1.12 It was noted in Section 1.2.1 that statisticians who follow the deFinetti school do not accept the Axiom of Countable Additivity, instead adhering to the Axiom of Finite Additivity.

(a)

Show that the Axiom of Countable Additivity implies Finite Additivity.

(1) Definition of Axiom of Countable Additivity:

If $A_1, A_2, \dots \in \mathbb{B}$ are pairwise disjoint, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

(2) Goal:

We wish to prove the Axiom of Finite Additivity given Countable Additivity, which is defined as:

$A \in \mathbb{B}, B \in \mathbb{B}$ disjoint, then:

$$P(A \cup B) = P(A) + P(B)$$

(3) Method:

Let $A, B \in \mathbb{B}$ and assume Countable Additivity (1).

Define $A \equiv A_1$ and $B \equiv A_2 \cup A_3 \cup \dots$

As A_1, A_2, A_3, \dots are pairwise disjoint, A, B are disjoint sets.

Thus, using Theorem 1.2.9, we have:

$$1.2.9: P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B) = P(A) + P(B) - 0 = P(A) + P(B) \quad .//$$

2. 1.13: If $P(A) = \frac{1}{3}$ and $P(B^c) = \frac{1}{4}$, can A and B be disjoint?

A:

$$P(A) = \frac{1}{3} \rightarrow P(A^c) = 1 - \frac{1}{3} = \frac{2}{3} \text{ and } P(B^c) = \frac{1}{4} \rightarrow P(B) = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\text{Let us consider } P(A) + P(B) = \frac{1}{3} + \frac{3}{4} = \frac{4}{12} + \frac{9}{12} = \frac{13}{12}$$

For two events to be disjoint, then $P(A \cap B) = 0$, or they cannot occur jointly. Furthermore, the following condition must hold:

$$P(A \cup B) = P(A) + P(B)$$

$$\text{However, as shown, } P(A) + P(B) = \frac{13}{12} > 1$$

However, the probability over the sample space of all possible events must be at most 1 (between 0 and 1). Thus, if we assume events A and B are disjoint, we arrive at a contradiction.

3.

Suppose a family has 4 children, named a, b, c and d, who take turns washing 4 plates denoted p_1, p_2, p_3, p_4 . These children are not so careful in their work, so, over time, each of the plates will be broken. Suppose any child could break any plate and that the ways in which plates p_1, p_2, p_3, p_4 could be broken by children a, b, c, d are equally likely.

Notes: <https://math.stackexchange.com/questions/2419485/probability-question-in-a-certain-family-four-girls-take-turns-at-washing-dish>

The probability that three from four dishes were randomly broken by the same girl is $\frac{3}{16}$

The probability that three from four dishes were randomly broken by the youngest girl is $\frac{3}{64}$

4^4 possible arrangements

(a) Find the probability that child a breaks 3 plates.

4 choose 3: 4 children, choose one child to break 3 dishes Probability(one child breaks 3 dishes) $(\frac{1}{4} \wedge 3)$
Probability(any of the other children break 1 dishes) $(\frac{3}{4})$

(b) Find the probability that one of the four children breaks 3 plates.

4 choose 1: 4 children, choose 1 4 choose 3 : 4 children, choose 3 Probability(one child breaks 3 dishes) $(\frac{1}{4} \wedge 3)$ Probability(any of the other children break 1 dishes) $(\frac{3}{4})$

4. 1.34

Two litters of a particular rodent species have been born, one with two brown-haired and one gray-haired (litter 1), and the other with three brown-haired and two gray-haired (litter 2). We select a litter at random and then select an offspring at random from the selected litter.

- (a) What is the probability that the animal chosen is brown-haired?

$$P(\text{brown-haired}) = P(\text{brown-haired}|\text{litter 1})P(\text{litter 1}) + P(\text{brown-haired}|\text{litter 2})P(\text{litter 2})$$

$$P(\text{brown-haired}) = \left(\frac{2}{3}\right)\frac{1}{2} + \left(\frac{3}{5}\right)\frac{1}{2}$$

$$P(\text{brown-haired}) = \frac{2}{6} + \frac{3}{10} = \frac{10}{30} + \frac{9}{30}$$

$$P(\text{brown-haired}) = \frac{19}{30}$$

- (b) Give that a brown-haired offspring was selected, what is the probability that the sampling was from litter 1?

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

where $P(A) = P(\text{litter 1}) = \frac{1}{2}$, $P(B) = P(\text{brown-haired}) = \frac{19}{30}$, and $P(B|A) = P(\text{brown-haired}|\text{litter 1}) = \frac{2}{3}$

$$P(\text{litter 1}|\text{brown-haired}) = \frac{\frac{2}{3} \cdot \frac{1}{2}}{\frac{19}{30}}$$

$$P(\text{litter 1}|\text{brown-haired}) = \frac{\frac{2}{6}}{\frac{19}{30}}$$

$$P(\text{litter 1}|\text{brown-haired}) = \frac{1}{3} * \frac{30}{19}$$

$$P(\text{litter 1}|\text{brown-haired}) = \frac{10}{19}$$

Prove each of the following statements. (Assume that any conditioning event has positive probability.)

(a) If $P(B) = 1$, then $P(A|B) = P(A)$ for any A.

$$P(B) = 1 \rightarrow P(B^c) = 1 - 1 = 0$$

$$\text{By definition: } P(A|B) = \frac{P(A \cap B)}{P(B)}$$

So we find $P(A \cap B)$ through: $P(A \cap B^c) = P(A) - P(A \cap B)$, $P(A \cap B) = P(A) - P(A \cap B^c)$.

However, as $P(B^c) = 0$, $P(A \cap B) = P(A) - 0 = P(A)$

Giving us:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{P(A)}{1}$$

Thus, for $P(B) = 1$, $P(A|B) = P(A)$.//

(b) If $A \subset B$, then $P(B|A) = 1$ and $P(A|B) = \frac{P(A)}{P(B)}$

$$\text{By definition: } P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Thus for $P(B|A) = 1$, we have:

$$P(A|B) = \frac{1 * P(A)}{P(B)} = \frac{P(A)}{P(B)}$$

(c) If A and B are mutually exclusive, then:

$$P(A|A \cup B) = \frac{P(A)}{P(A) + P(B)}$$

If A and B are mutually exclusive, then:

$$P(A \cup B) = P(A) + P(B)$$

$$\text{and } P(A \cap B) = 0$$

So using the formula for conditional probability we have:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \rightarrow P(A|A \cup B) = \frac{P(A \cup (A \cap B))}{P(A \cup B)}$$

Using Distributive Laws so have:

$$P(A \cup (A \cap B)) = P(A \cup A) \cap P(A \cup B) = P(A) \cap P(A \cup B) = P(A)$$

So we have:

$$P(A|A \cup B) = \frac{P(A)}{P(A) + P(B)}$$

(d) $P(A \cap B \cap C) = P(A|B \cup C)P(B|C)P(C)$

$$\text{Base Formula: } P(A|B) = \frac{P(A \cap B)}{P(B)} \rightarrow P(A|B)P(B) = P(A \cap B)$$

$$P(A \cap B \cap C) = P(A \cap (B \cap C))$$

$$P(A \cap B \cap C) = P(A|B \cap C)P(B \cap C)$$

$$P(A \cap B \cap C) = P(A|B \cap C)(P(B|C)P(C))$$

6. 1.39

A pair of events A and B cannot be simultaneously mutually exclusive and independent. Prove that $P(A) > 0$ and $P(B) > 0$ then:

(1) If events A and B are mutually exclusive, then: $P(A \cap B) = 0$

(2) If events A and B are independent, then: $P(A|B) = P(A)$

(a)

If A and B are mutually exclusive, they cannot be independent.

Let A and B be mutually exclusive events.

Let us then assume A and B are independent and $P(A) > 0$ and $P(B) > 0$.

By their mutual exclusivity, (1) $P(A \cap B) = 0$ Let us then consider $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0}{P(B)} = 0$ This implies that $P(A|B) = 0$, However, due to independence, $P(A|B) = P(A) > 0$.

We thus have a contradiction via our assumption of independence.

(b)

If A and B are independent, they cannot be mutually exclusive.

Let A and B now be independent events.

Furthermore define $P(A) > 0$ and $P(B) > 0$ and assume A and B are mutually exclusive.

By independence (2), $P(A|B) = P(A)$ and $P(A \cap B) = P(A)P(B)$.

However, by presuming mutual exclusivity (1), we have

$$P(A \cap B) = 0 \rightarrow P(A)P(B) = 0$$

However, this requires $P(A)$, $P(B)$, or both to be zero! Thus we've reached a contradiction from our presumption of mutual exclusivity.

7. 1.47 Prove that the following functions are cdfs.

To prove a function is a (valid, proper) cdf, we must verify:

(1): $\lim_{x \rightarrow -\infty} F(x) = 0$ and $\lim_{x \rightarrow \infty} F(x) = 1$

(2): $F(x)$ is a nondecreasing function of x .

(3): $F(x)$ is right-continuous; that is, for every number x_0 , $\lim_{x \downarrow x_0} F(x) = F_o$

1.47 (c)

$$F_X(x) = e^{-e^{-x}} \text{ for } x \in (-\infty, \infty)$$

(1): $\lim_{x \rightarrow -\infty} F(x) = 0$

(2): $\lim_{x \rightarrow \infty} F(x) = 1$

(3): Derivative > 0

1.47 (d)

$$F_X(x) = 1 - e^{-x} \text{ for } x \in (0, \infty)$$

(1): $\lim_{x \rightarrow -\infty} F(x) = 0$

(2): $\lim_{x \rightarrow \infty} F(x) = 1$

(3): Derivative > 0

8. 1.54

For each of the following, determine the value of c that makes $f(x)$ a pdf.

To prove a function is a (valid, proper) pdf (for the continuous cases) we must verify:

(1): $f_X(x) \geq 0$ for all x .

(2): $\int_{-\infty}^{\infty} f_X(x) dx = 1$

(a)

$$f(x) = c * \sin(x) \text{ for } 0 < x < \frac{\pi}{2}$$

$$\int c * \sin(x) dx = c \int \sin(x) dx = c(-\cos(x))$$

So evaluating over the range we have:

$$\int_0^{\frac{\pi}{2}} c * \sin(x) dx = c * (-\cos(x)) \Big|_0^{\frac{\pi}{2}}$$

$$\int_0^{\frac{\pi}{2}} c * \sin(x) dx = c(-\cos(\frac{\pi}{2}) + \cos(0)) = c(1) = c$$

So for the integral to evaluate to 1 over the support, we have

$$c = 1$$

(b)

$$f(x) = c * e^{-|x|} \text{ for } -\infty < x < \infty$$

As the absolute value does not have a defined derivative, we must separate $f(x)$ into two distinct integrals.

$$\int_{-\infty}^{\infty} e^{-|x|} = \int_{-\infty}^0 e^x + \int_0^{\infty} e^{-x}$$

So we have:

$$(1): \int e^x dx = e^x$$

$$(2): \int e^{-x} dx = -e^{-x}$$

Over the respective range of x values, we may evaluate:

$$(1): \int_{-\infty}^0 e^x = e^x \Big|_{-\infty}^0 = 1 - 0 = 1$$

$$(2): \int_0^{\infty} e^{-x} = -e^{-x} \Big|_0^{\infty} = 0 - (-1) = 1$$

Combining (1) and (2) we have:

$$\int_{-\infty}^{\infty} e^{-|x|} = 1 + 1 = 2$$

So if we include the constant c (which can be factored out of the integral), we have:

$$\int_{-\infty}^{\infty} c * e^{-|x|} = c * \int_{-\infty}^{\infty} e^{-|x|} = c(1 + 1) = c(2)$$

To have this evaluate to 1, we must set $c = \frac{1}{2}$

9.

From the axioms of probability, it follows that probability functions $P(\cdot)$ exhibit “monotone continuity from above (mcfa)” (which you don’t have to worry about showing), meaning that for any decreasing sequence of sets/events

$$A_1 \supset A_2 \supset A_3 \supset \dots,$$

$$\lim_{n \rightarrow \infty} P(A_n) = P\left(\bigcap_{i=1}^{\infty} A_i\right)$$

By using/applying the mcfa property, show that the cdf F of a random variable X must be right continuous for any $x \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} F(x + n^{-1}) = F(x)$$

holds.

Notes: <https://math.stackexchange.com/questions/480042/prove-that-the-cdf-of-a-random-variable-is-always-right-continuous>

A: