HW9

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Q1

Let $X_1, \dots, X_n$ be iid exponential(θ) and let $\hat{\theta}_n \equiv \bar{X}_n \equiv \sum_{i=1}^n X_i/n$ denote the MLE based on $X_1, \dots, X_n$.

a)

Determine the limiting distribution of $\sqrt{n}(\hat{\theta}_n - \theta)$ as $n \to \infty$.

Answer

Since $X_1, \dots, X_n$ are iid with $X_i \sim \text{Exponential}(\theta)$, we know:

$\mathbb{E}[X_i] = \theta$

And:

$\text{Var}(X_i) = \theta^2$

By the Central Limit Theorem, we also know:

$$\sqrt{n}(\bar{X}_n - \theta) \xrightarrow{d} N(0, \theta^2)$$

Thus,

$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N(0, \theta^2)$$

b)

Find a variance stabilizing transformation (VST) for $\{\hat{\theta}_n\}$ and use this to determine a large sample confidence interval for θ with approximate confidence coefficient $1 - \alpha$.

Answer

Let $X_1, \dots, X_n \overset{iid}{\sim} \text{Exp}(\theta)$, with MLE $\hat{\theta}_n = \bar{X}_n$.

From part (a):

$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N(0, \theta^2).$$

Using the Delta Method, for a differentiable function $g(\cdot)$:

$$\sqrt{n}(g(\hat{\theta}_n) - g(\theta)) \xrightarrow{d} N\left(0, [g'(\theta)]^2\theta^2\right).$$

To stabilize the variance, require:

$$[g'(\theta)]^2\theta^2 = 1.$$

Taking square roots:

$$g'(\theta) = \frac{1}{\theta}.$$

Integrating:

$$g(\theta) = \log\theta + C,$$

where C is an irrelevant additive constant.

Thus, a variance stabilizing transformation is:

$$g(\theta) = \log\theta.$$

By the Delta Method:

$$\sqrt{n}(\log\hat{\theta}_n - \log\theta) \xrightarrow{d} N(0,1).$$

Thus, an approximate $1-\alpha$ confidence interval for $\log(\theta)$ is:

$$\left(\log(\hat{\theta}_n) \pm \frac{z_{\alpha/2}}{\sqrt{n}}\right),$$

where $z_{\alpha/2}$ is the $1-\alpha/2$ standard normal quantile.

Exponentiating, the approximate confidence interval for θ is:

$$\left(\hat{\theta}_n \exp\left(-\frac{z_{\alpha/2}}{\sqrt{n}}\right), \hat{\theta}_n \exp\left(\frac{z_{\alpha/2}}{\sqrt{n}}\right)\right).$$

```r
# Calculate VST CI numerically
x_bar <- 1.835464
n <- 100
z_90 <- 1.6449

lower_vst <- x_bar * exp(-z_90 / sqrt(n))
upper_vst <- x_bar * exp(z_90 / sqrt(n))

c(lower_vst, upper_vst)
```

```
## [1] 1.557072 2.163630
```

c)

Suppose a random sample $X_1, \dots, X_{100}$ of $n = 100$ observations yields $\bar{x}_n = 1.835464$. Use this information to obtain a large sample confidence interval for θ based on a likelihood ratio statistic, which has approximate confidence coefficient 90%. (Use the chi-squared approximation for this; you should be able to then numerically determine the interval.) Using this data, compute also a confidence interval with approximate confidence coefficient 90% using the VST approach from part(b).

Answer

For testing $H_0 : \theta = \theta_0$, the likelihood ratio statistic satisfies:

$$-2 \log \Lambda(\theta) \xrightarrow{d} \chi^2_1,$$

where $\Lambda(\theta) = \frac{L(\theta)}{L(\hat{\theta}_n)}$.

The log-likelihood based on iid Exponential(θ) data is:

$$\ell(\theta) = -n \log \theta - \frac{n\bar{X}_n}{\theta}.$$

Thus, the likelihood ratio test statistic is:

$$-2 \log \Lambda(\theta) = 2n \left[\log \left(\frac{\theta}{\hat{\theta}_n} \right) + \frac{\hat{\theta}_n}{\theta} - 1 \right].$$

We seek values of θ satisfying:

$$2n \left[\log \left(\frac{\theta}{\hat{\theta}_n} \right) + \frac{\hat{\theta}_n}{\theta} - 1 \right] \leq \chi^2_{1,0.90},$$

where $\chi^2_{1,0.90} \approx 2.7055$.

Q3

Suppose $X_1, \dots, X_n$ are a random sample with common cdf given by

$$P(X_1 \leq x|\theta) = \begin{cases} 1 - e^{-(x/\theta)^2} & \text{if } x > 0 \\ 0 & \text{otherwise,} \end{cases} \qquad \theta > 0$$

a)

Use the Mood-Graybill-Boes Method to derive a CI for θ with C.C. $1 - \alpha$ based on the statistic $X_{(1)} = \min_{1 \leq i \leq n} X_i$.

Answer

Since $X_1, \dots, X_n$ are independent:

$$P(X_{(1)} \leq x) = 1 - P(X_1 > x, \dots, X_n > x) = 1 - \left(P(X_1 > x)\right)^n .$$

Given:

$$P(X_1 > x) = e^{-(x/\theta)^2},$$

thus:

$$P(X_{(1)} \leq x) = 1 - e^{-n(x/\theta)^2}.$$

Define:

$$V = n \left(\frac{X_{(1)}}{\theta} \right)^2 .$$

Then:

$$P(V \leq v) = 1 - e^{-v},$$

so $V \sim$ Exponential(1).

Let $q_p = -\log(1 - p)$ denote the p-th quantile of the Exponential(1) distribution.

We want:

$$P\left(q_{\alpha/2} \leq V \leq q_{1-\alpha/2}\right) = 1 - \alpha.$$

In terms of θ, solving:

$$q_{\alpha/2} \leq n \left(\frac{X_{(1)}}{\theta} \right)^2 \leq q_{1-\alpha/2},$$

or equivalently:

$$\sqrt{\frac{q_{\alpha/2}}{n}} \leq \frac{X_{(1)}}{\theta} \leq \sqrt{\frac{q_{1-\alpha/2}}{n}}.$$

Thus:

$$\theta \in \left(\frac{X_{(1)}}{\sqrt{q_{1-\alpha/2}/n}}, \frac{X_{(1)}}{\sqrt{q_{\alpha/2}/n}} \right).$$

Using $q_p = -\log(1-p)$:

$$\left(\frac{X_{(1)}}{\sqrt{-\log(1-\alpha/2)/n}}, \frac{X_{(1)}}{\sqrt{-\log(\alpha/2)/n}} \right)$$

b)

Use the Mood-Graybill-Boes Method to derive a CI for θ with C.C. $1-\alpha$ based on the statistic $T = \sum_{i=1}^n X_i^2$. Express your confidence interval using chi-squared quantiles.

Note: One can show X_i^2 is Exponential(θ^2) distributed so that $2T/\theta^2$ is χ^2_{2n} distributed with $2n$ degrees of freedom.

Answer

Since:

$$X_i^2 \sim \text{Exponential}(\theta^2),$$

then:

$$T = \sum_{i=1}^n X_i^2,$$

and:

$$\frac{2T}{\theta^2} \sim \chi^2_{2n}.$$

Thus:

$$P\left(\chi^2_{2n,\alpha/2} \leq \frac{2T}{\theta^2} \leq \chi^2_{2n,1-\alpha/2} \right) = 1-\alpha.$$

Solving for θ^2:

$$\frac{2T}{\chi^2_{2n,1-\alpha/2}} \leq \theta^2 \leq \frac{2T}{\chi^2_{2n,\alpha/2}}.$$

Taking square roots:

$$\sqrt{\frac{2T}{\chi^2_{2n,1-\alpha/2}}} \leq \theta \leq \sqrt{\frac{2T}{\chi^2_{2n,\alpha/2}}}.$$

Thus, the confidence interval for θ is:

$$\left(\sqrt{\frac{2\sum_{i=1}^n X_i^2}{\chi^2_{2n,1-\alpha/2}}}, \sqrt{\frac{2\sum_{i=1}^n X_i^2}{\chi^2_{2n,\alpha/2}}} \right)$$