

HW6

2024-10-26

Homework 6

Q1: 4.17, Casella & Berger

Let X be an exponential(1) random variable, and define Y to be the integer part of $X+1$, that is:

$$Y = i + 1 \text{ iff } i \leq X < i + 1, i = 0, 1, 2, \dots$$

(a)

Find the distribution of Y . What well-known distribution does Y have?

$$P(Y = i + 1) = \int_i^{i+1} e^{-x} dx = -e^{-x} \Big|_{x=i}^{i+1} = -e^{-(i+1)} + e^{-i} = e^{-i}(1 - e^{-1})$$

This is a geometric distribution with $p = 1 - e^{-1}$, such that

$$Y \sim \text{Geom}(1 - e^{-1})$$

(b)

Find the conditional distribution of $X - 4$ given $Y \geq 5$

As defined, $Y = i + 1$, such that

$$Y \geq 5 \iff i + 1 \geq 5 \iff X \geq 4$$

Utilizing the distributions as defined and found, we then have

$$P(X - 4 \leq x \mid Y \geq 5) = P(X - 4 \leq x \mid X \geq 4) = P(X \leq x + 4 \mid X \geq 4)$$

$$P(X - 4 \leq x \mid Y \geq 5) = P(X \leq x + 4 \mid X \geq 4) = 1 - P(X > x + 4 \mid X \geq 4) = 1 - P(X > x) = 1 - e^{-x}$$

This sure looks like the memoryless property we observed previously!

$$P(X - 4 \leq x \mid Y \geq 5) = P(X \leq x) = 1 - e^{-x}$$

Q2: 4.32(a), Casella & Berger

(a)

For a hierarchical model:

$$Y|\Lambda \sim \text{Poisson}(\Lambda) \text{ and } \Lambda \sim \text{Gamma}(\alpha, \beta)$$

find the marginal distribution, mean, and variance of Y. Show that the marginal distribution of Y is a negative binomial if α is an integer.

For $y = 0, 1, \dots$, the marginal distribution of Y is:

$$\begin{aligned} f_Y(y) &= \int_{\lambda=0}^{\infty} f_Y(y|\lambda) f_{\Lambda}(\lambda) d\lambda = \int_{\lambda=0}^{\infty} \frac{\lambda^y e^{-\lambda}}{y!} \frac{1}{\beta^{\alpha} \Gamma(\alpha)} \lambda^{\alpha-1} e^{-\lambda/\beta} d\lambda \\ f_Y(y) &= \frac{1}{y! \beta^{\alpha} \Gamma(\alpha)} \int_{\lambda=0}^{\infty} \lambda^y e^{-\lambda} \lambda^{\alpha-1} e^{-\frac{\lambda}{\beta}} d\lambda = \frac{1}{y! \beta^{\alpha} \Gamma(\alpha)} \int_{\lambda=0}^{\infty} \lambda^{y+\alpha-1} e^{-\frac{\lambda}{\beta}} d\lambda \end{aligned}$$

Note the right-hand term is a Gamma function, i.e. for $u = \lambda(\frac{1+\beta}{\beta}) \rightarrow \lambda = u(\frac{\beta}{1+\beta}) \rightarrow d\lambda = du(\frac{\beta}{1+\beta})$ and we have:

$$f_Y(y) = \frac{1}{y! \beta^{\alpha} \Gamma(\alpha)} \int_{u=0}^{\infty} u^{y+\alpha-1} e^{-u} \left(\frac{\beta}{1+\beta}\right)^{y+\alpha} du = \frac{1}{y! \beta^{\alpha} \Gamma(\alpha)} \left(\frac{\beta}{1+\beta}\right)^{y+\alpha} \int_{u=0}^{\infty} u^{y+\alpha-1} e^{-u} du = \frac{1}{y! \beta^{\alpha} \Gamma(\alpha)} \left(\frac{\beta}{1+\beta}\right)^{y+\alpha} \Gamma(\alpha+y)$$

Simplifying gives us:

$$f_Y(y) = \binom{y+\alpha-1}{y} \frac{\beta}{1+\beta} \frac{1}{1+\beta}$$

Or, $Y \sim NB(\alpha, \frac{1}{1+\beta})$, where $\alpha \in \mathbb{Z}^+$ for the Gamma function to be evaluated (integral to evaluate/not diverge).

Knowing the distribution of Y allows us to directly compute it's mean and variance, specifically:

$$E[Y] = \alpha \frac{1 - \frac{1}{1+\beta}}{\frac{1}{1+\beta}} = \alpha \frac{\frac{\beta}{1+\beta}}{\frac{1}{1+\beta}} = \alpha\beta$$

However, we may also calculate it using conditional expectation, specifically:

$$E[Y] = E(E[Y|\Lambda]) = E[\Lambda] = \alpha\beta$$

$$\text{Var}(Y) = \alpha \frac{1 - \frac{1}{1+\beta}}{(\frac{1}{1+\beta})^2} = \alpha \frac{\frac{\beta}{1+\beta}}{(\frac{1}{1+\beta})^2} = \alpha\beta(1+\beta)$$

Alternatively we have, via EVVE

$$\text{Var}(Y) = \text{Var}(E[Y|\Lambda]) + E(\text{Var}(Y|\Lambda)) = \text{Var}(\Lambda) + E[\Lambda] = \alpha\beta^2 + \alpha\beta = \alpha\beta(\beta+1)$$

Q3

Expectation

(a)

Show that any random variable X (with finite mean) has zero covariance with any real constant c , i.e. $Cov(X, c) = 0$

The covariance of X and c may be written:

$$Cov(X, c) = E[(X - E[X])(c - E[c])] = E[(X - E[X])(c - c)] = E[(X - E[X])0] = E[0] = 0$$

Such that we conclude:

$$Cov(X, c) = 0$$

And we must have the condition that X has a finite mean as $\infty * 0$ is undefined.

(b)

Using the definition of conditional expectation, show that

$$E[g(X)h(Y)|X = x] = g(x)E[h(Y)|X = x]$$

for an x with pdf $f_X(x) > 0$ (You may also assume (X, Y) are jointly discrete).

To show that

$$E[g(X)h(Y) | X = x] = g(x)E[h(Y) | X = x]$$

For jointly discrete random variables X and Y , the conditional expectation of $h(Y) | X = x$ is:

$$E[h(Y) | X = x] = \sum_y h(y)P(Y = y | X = x)$$

Similarly, the conditional expectation of $g(X)h(Y) | X = x$ is:

$$E[g(X)h(Y) | X = x] = \sum_y g(x)h(y)P(Y = y | X = x)$$

We can simplify by recognizing there are terms in the above equations that do not depend on the index of the summation, specifically:

$$E[g(X)h(Y) | X = x] = \sum_y g(x)h(y)P(Y = y | X = x) = g(x) \sum_y h(y)P(Y = y | X = x)$$

However, this is a very familiar formula to us!

$$E[g(X)h(Y) | X = x] = g(x) \sum_y h(y)P(Y = y | X = x) = g(x)E[h(Y) | X = x]$$

Note:

The condition $f_X(x) > 0$ is necessary as it ensures that $P(Y = y \mid X = x)$ is defined, because:

$$P(Y = y \mid X = x) = \frac{P(X = x, Y = y)}{P(X = x)} \equiv \frac{P(X = x, Y = y)}{f_X(x)}$$

Q4

Suppose that X_i has mean μ_i and variance σ_i^2 , for $i = 1, 2$, and that the covariance of X_1 and X_2 is σ_{12} . Compute the covariance between $X_1 - 2X_2 + 8$, and then compute the covariance of $3X_1 + X_2$.

We want to find:

$$\text{Cov}(X_1 - 2X_2 + 8, 3X_1 + X_2) = \text{Cov}(X_1 - 2X_2, 3X_1 + X_2)$$

$$\text{Cov}(X_1 - 2X_2, 3X_1 + X_2) = \text{Cov}(X_1, 3X_1) + \text{Cov}(X_1, X_2) - \text{Cov}(2X_2, 3X_1) - \text{Cov}(2X_2, X_2) = 3\text{Cov}(X_1, X_1) + \text{Cov}(X_1, X_2) - 6\text{Cov}(X_2, X_1) - \text{Cov}(2X_2, X_2)$$

$$\text{Cov}(X_1 - 2X_2 + 8, 3X_1 + X_2) = 3\sigma_1^2 - 5\sigma_1\sigma_2 - 2\sigma_2^2$$

Q5

The joint distribution of X, Y is given by the joint pdf:

$$f(x, y) = 3(x + y) \text{ for } 0 < x < 1, 0 < y < 1, 0 < x + y < 1$$

(a)

Find the marginal distribution of $f_X(x)$

$$f_X(x) = \int_{y \in Y} f(x, y) dy = \int_0^1 f(x, y) dy = \int_0^1 3(x + y) dy$$

However, the bounds of the integral as given above are not correct, as:

$$0 < x < 1, 0 < y < 1, 0 < x + y < 1$$

So we actually have:

$$f_X(x) = \int_0^{1-x} 3(x + y) dy = 3 \int_0^{1-x} (x + y) dy = 3 \left[\int_0^{1-x} x dy + \int_0^{1-x} y dy \right]$$

For the sake of simplification, these are:

i.

$$\int_0^{1-x} x dy = x(1 - x)$$

ii.

$$\int_0^{1-x} y dy = \frac{(1 - x)^2}{2}$$

Taken together, we have:

$$f_X(x) = 3 \left[x(1 - x) + \frac{(1 - x)^2}{2} \right] = 3 \left[(1 - x) \left(x + \frac{1 - x}{2} \right) \right] = 3(1 - x) \left(\frac{x + 1}{2} \right) = \frac{3}{2}(1 - x)(x + 1)$$

Thus, the marginal distribution of X is:

$$f_X(x) = \frac{3}{2}(1 - x)(x + 1) \quad \text{for } 0 < x < 1$$

(b)

Find the conditional pdf of $Y | X = x$, given some $0 < x < 1$.

Using the definition of the conditional pdf, we have:

$$f_{Y|X}(y|x) = \frac{f(x, y)}{f_X(x)} = \frac{3(x+y)}{\frac{3}{2}(1-x)(x+1)}$$

Again, for constraints:

$$0 < x < 1, 0 < y < 1, 0 < x+y < 1$$

Simplifying gives us:

$$f_{Y|X}(y|x) = \frac{2(x+y)}{(1-x)(x+1)} \text{ for } 0 < y < 1-x$$

(c)

Find $E[Y|X = x]$

Using what we derived in part (b), we have:

$$E[Y | X = x] = \int_0^{1-x} y f_{Y|X}(y|x) dy = \int_0^{1-x} y \frac{2(x+y)}{(1-x)(x+1)} dy = \frac{2}{(1-x)(x+1)} \int_0^{1-x} y(x+y) dy = \frac{2}{(1-x)(x+1)} \int_0^{1-x} yx + y^2 dy$$

$$E[Y | X = x] = \frac{2}{(1-x)(x+1)} \left[\left(\int_0^{1-x} xy dy \right) + \left(\int_0^{1-x} y^2 dy \right) \right]$$

i.

$$\int_0^{1-x} xy dy = x \int_0^{1-x} y dy = x \left[\frac{(1-x)^2}{2} \right] = \frac{x(1-x)^2}{2}$$

ii.

$$\int_0^{1-x} y^2 dy = \left[\frac{(1-x)^3}{3} \right]$$

Combining i. and ii. above, we then have:

$$E[Y | X = x] = \frac{2}{(1-x)(x+1)} \left(\frac{x(1-x)^2}{2} + \frac{(1-x)^3}{3} \right) = \frac{2(1-x)^2}{(1-x)(x+1)} \left(\frac{x}{2} + \frac{1-x}{3} \right)$$

Simplify, simplify:

$$E[Y | X = x] = \frac{2(1-x)}{x+1} \left(\frac{3x+2-2x}{6} \right) = \frac{2(1-x)}{x+1} \left(\frac{x+2}{6} \right) = \frac{(1-x)(x+2)}{3(x+1)}$$

(d)

Given the results in (a), (b), and (c), explain how you know $E[X|Y = y]$ without any further calculation

Given the above results, we can take advantage of symmetry, since the joint pdf of X and Y involves a simple sum of $x + y$, and the support of each is the same, i.e.

$$f(x, y) = 3(x + y) \quad \text{for } 0 < x < 1, 0 < y < 1, 0 < x + y < 1$$

So we can effectively “swap” any “x” in the prior calculations with “y” (and similarly if we felt inclined to derive everything again we could/would swap the “y” in our calculations with “x”).

Given from (c):

$$E[Y | X = x] = \frac{(1-x)(x+2)}{3(x+1)}$$

By symmetry, we know:

$$E[X | Y = y] = \frac{(1-y)(y+2)}{3(y+1)}$$

(e)

Find $E[E[2XY - Y|X]]$

From the parts above, we know (have already calculated) most everything but $E[XY]$.

$$E[E[2XY - Y | X]] = E[2XY - Y] = 2E[XY] - E[Y]$$

Taking advantage of the symmetry property used in part (d), we can easily find the marginal pdf of Y. Namely, as:

$$f_X(x) = \frac{3}{2}(1-x)(x+1), \quad \text{for } 0 < x < 1$$

Then:

$$f_Y(y) = \frac{3}{2}(1-y)(y+1) \quad \text{for } 0 < y < 1$$

However, due to symmetry:

$$E[Y] = E[X]$$

So if we calculate $E[X]$, we effectively get $E[Y]$. Let's do that!

$$E[X] = \int_0^1 x \cdot f_X(x) dx = \int_0^1 x \frac{3}{2}(1-x)(x+1) dx = \int_0^1 \frac{3x}{2}(1-x^2) dx = \frac{3}{2} \int_0^1 x(1-x^2) dx = \frac{3}{2} \left(\int_0^1 x dx - \int_0^1 x^3 dx \right)$$

i.

$$\int_0^1 x \, dx = \frac{1}{2}$$

$$\int_0^1 x^3 \, dx = \frac{1}{4}$$

Taking the above gives us:

$$E[X] = \frac{3}{2} \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{3}{2} \left(\frac{1}{4} \right) = \frac{3}{8}$$

And $E[Y] = \frac{3}{8}$ too.

Last part now, we need to evaluate $E[XY]$:

$$E[XY] = \int_0^1 \int_0^{1-x} xy f(x, y) \, dy \, dx = \int_0^1 \int_0^{1-x} xy 3(x+y) \, dy \, dx = 3 \int_0^1 \int_0^{1-x} xy(x+y) \, dy \, dx = 3 \int_0^1 \int_0^{1-x} (x^2 y + xy^2) \, dy \, dx$$

Separating the integrals:

ii.

$$\int_0^1 \int_0^{1-x} x^2 y \, dy \, dx = \int_0^1 x^2 \left(\frac{(1-x)^2}{2} \right) \, dx = \frac{1}{2} \int_0^1 x^2 (1-x)^2 \, dx = \frac{1}{2} \left(\int_0^1 x^2 \, dx - 2 \int_0^1 x^3 \, dx + \int_0^1 x^4 \, dx \right)$$

Where:

$$\int_0^1 x^2 \, dx = \frac{1}{3}$$

$$\int_0^1 x^3 \, dx = \frac{1}{4}$$

$$\int_0^1 x^4 \, dx = \frac{1}{5}$$

And our total for the “first” term is then

$$\frac{1}{2} \left(\frac{1}{3} - 2 \left(\frac{1}{4} \right) + \frac{1}{5} \right) = \frac{1}{2} \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right) = \frac{1}{2} \left(\frac{10}{30} - \frac{15}{30} + \frac{6}{30} \right) = \frac{1}{2} \left(\frac{1}{30} \right) = \frac{1}{60}$$

iii.

$$\int_0^1 \int_0^{1-x} xy^2 dy dx = \int_0^1 x \left(\frac{(1-x)^3}{3} \right) dx = \frac{1}{3} \int_0^1 x(1-x)^3 dx = \frac{1}{3} \int_0^1 (x - 3x^2 + 3x^3 - x^4) dx$$

$$\frac{1}{3} \left(\frac{1}{2} - 3 \cdot \frac{1}{3} + 3 \cdot \frac{1}{4} - \frac{1}{5} \right) = \frac{1}{3} \left(\frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5} \right) = \frac{1}{3} \left(\frac{30}{60} - \frac{60}{60} + \frac{45}{60} - \frac{12}{60} \right) = \frac{1}{3} \cdot \frac{3}{60} = \frac{1}{60}$$

Crazy, $\frac{1}{60}$ again... symmetry?

Taking the two parts above, ii. and iii., we then have:

$$E[XY] = 3 \left(\frac{1}{60} + \frac{1}{60} \right) = \frac{6}{60} = \frac{1}{10}$$

$$2E[XY] = 2\left(\frac{1}{10}\right) = \frac{1}{5}$$

And with:

$$E[Y] = \frac{3}{8}$$

We may finally calculate the desired value as:

$$E[E[2XY - Y \mid X]] = \frac{1}{5} - \frac{3}{8} = \frac{8}{40} - \frac{15}{40} = -\frac{7}{40} = -\frac{7}{40}$$

Q6

Suppose that $f(x, y) = e^{-y}$ for $0 < x < y < \infty$

(a)

Find the joint moment generating function for (X, Y).

The joint moment generating function $M_{X,Y}(t_1, t_2)$ may be defined:

$$M_{X,Y}(t_1, t_2) = E[e^{t_1 X + t_2 Y}] = \int_0^\infty \int_0^y e^{t_1 x + t_2 y} e^{-y} dx dy = \int_0^\infty \int_0^y e^{t_1 x} e^{(t_2 - 1)y} dx dy = \int_0^\infty e^{(t_2 - 1)y} \left(\int_0^y e^{t_1 x} dx \right) dy = \int_0^\infty$$

$$M_{X,Y}(t_1, t_2) = \frac{1}{t_1} \int_0^\infty e^{(t_2 - 1)y} (e^{t_1 y} - 1) dy = \frac{1}{t_1} \int_0^\infty e^{(t_2 + t_1 - 1)y} - e^{(t_2 - 1)y} dy$$

$$M_{X,Y}(t_1, t_2) = \frac{1}{t_1} \int_0^\infty (e^{(t_1 + t_2 - 1)y} - e^{(t_2 - 1)y}) dy = \frac{1}{t_1} \left[\int_0^\infty e^{(t_1 + t_2 - 1)y} dy - \int_0^\infty e^{(t_2 - 1)y} dy \right]$$

i.

$$\int_0^\infty e^{(t_1 + t_2 - 1)y} dy = \frac{1}{1 - t_1 - t_2} \quad \text{for } t_1 + t_2 < 1$$

ii.

$$\int_0^\infty e^{(t_2 - 1)y} dy = \frac{1}{1 - t_2} \quad \text{for } t_2 < 1$$

Combining the two parts above gives us the joint mgf:

$$M_{X,Y}(t_1, t_2) = \frac{1}{t_1} \left(\frac{1}{1 - t_1 - t_2} - \frac{1}{1 - t_2} \right)$$

(b)

Overview

From (c) we know:

$$X \sim \text{Exponential}(1) \rightarrow f_X(x) = e^{-x} \quad Y \sim \text{Gamma}(\alpha = 2, \beta = 1)$$

So

$$E[X] = 1, \text{Var}(X) = 1$$

$$E[Y] = 2, \text{Var}(Y) = 2$$

This is here for my sanity because I keep messing up the calculations.

Calculations using joint mgf

Use the joint moment generating function to find the variance of X, the variance of Y, and the covariance of X and Y.

For a joint MGF of:

$$M_{X,Y}(t_1, t_2) = \frac{1}{t_1} \left(\frac{1}{1-t_1-t_2} - \frac{1}{1-t_2} \right)$$

$E[X]$

To find the mean of X, we have:

$$M_{X,Y}(t_1, t_2) = \frac{1}{t_1} \left(\frac{1}{1-t_1-t_2} - \frac{1}{1-t_2} \right)$$

$$\frac{\partial}{\partial t_1} M_{X,Y}(t_1, t_2) = \frac{\partial}{\partial t_1} \left(\frac{1}{t_1} \left(\frac{1}{1-t_1-t_2} - \frac{1}{1-t_2} \right) \right) = \frac{-1}{t_1^2} \left(\frac{1}{1-t_1-t_2} - \frac{1}{1-t_2} \right) + \frac{1}{t_1} \frac{1}{(1-t_1-t_2)^2}$$

$$\lim_{t_1 \rightarrow 0, t_2 \rightarrow 0} \frac{\partial}{\partial t_1} M_{X,Y}(t_1, t_2) = \frac{1}{t_1} \left(\frac{-1}{(1-t_1-t_2)(1-t_2)} + \frac{1}{(1-t_1-t_2)^2} \right) = \left(\frac{1}{t_1} \right) \frac{t_1}{(1-t_1-t_2)^2(1-t_2)} = \frac{1}{(1-t_1-t_2)^2(1-t_2)}$$

$$E[X] = \lim_{t_1 \rightarrow 0, t_2 \rightarrow 0} \frac{\partial}{\partial t_1} M_{X,Y}(t_1, t_2) = 1$$

$E[Y]$

For the mean of Y, we have:

$$M_{X,Y}(t_1, t_2) = \frac{1}{t_1} \left(\frac{1}{1-t_1-t_2} - \frac{1}{1-t_2} \right)$$

$$\frac{\partial}{\partial t_2} M_{X,Y}(t_1, t_2) = \frac{1}{t_1} \left(\frac{1}{(1-t_1-t_2)^2} - \frac{1}{(1-t_2)^2} \right)$$

Similar as before, as we cannot directly evaluate $(t_1, t_2) = (0, 0)$, we must then evaluate the limit as $t_1 \rightarrow \infty$, $t_2 \rightarrow \infty$:

$$\frac{\partial}{\partial t_2} M_{X,Y}(t_1, t_2) = \frac{1}{t_1} \left(\frac{1}{(1-t_1-t_2)^2} - \frac{1}{(1-t_2)^2} \right) = \left(\frac{1}{t_1} \right) \frac{2t_1}{(1-t_1-t_2)^2(1-t_2)^2} = \frac{2}{(1-t_1-t_2)^2(1-t_2)^2}$$

$$E[Y] = \lim_{t_1 \rightarrow 0, t_2 \rightarrow 0} \frac{2}{(1-t_1-t_2)^2(1-t_2)^2} = 2$$

Var(X)

We then need to calculate the variance of both X and Y. Starting with X, we have:

$$E[X^2] = \frac{\partial^2}{\partial t_1^2} M_{X,Y}(t_1, t_2) = \frac{2}{t_1^3} \left(\frac{1}{1-t_1-t_2} - \frac{1}{1-t_2} \right) - \frac{2}{t_1^2} \frac{1}{(1-t_1-t_2)^2} + \frac{2}{t_1} \frac{1}{(1-t_1-t_2)^3}$$

$$E[X^2] = \frac{\partial^2}{\partial t_1^2} M_{X,Y}(t_1, t_2) \Big|_{t_1=0, t_2=0} = \frac{2}{t_1^3} \left(\frac{1}{1-t_1-t_2} - \frac{1}{1-t_2} \right) - \frac{2}{t_1^2} \frac{1}{(1-t_1-t_2)^2} + \frac{2}{t_1} \frac{1}{(1-t_1-t_2)^3}$$

Similar as before, as we cannot directly evaluate $(t_1, t_2) = (0, 0)$, we must then evaluate the limit as $t_1 \rightarrow \infty$, $t_2 \rightarrow \infty$:

$$E[X^2] = \frac{\partial^2}{\partial t_1^2} M_{X,Y}(t_1, t_2) = \frac{\partial}{\partial t_1} \left(\frac{1}{(1-t_1-t_2)^2(1-t_2)} \right) = \frac{2}{(1-t_1-t_2)^3(1-t_2)}$$

$$E[X^2] = \frac{\partial^2}{\partial t_1^2} M_{X,Y}(t_1, t_2) \Big|_{t_1=0, t_2=0} = \frac{2}{(1-0-0)^3(1-0)} = 2$$

Then for Var(X):

$$Var(X) = E(X^2) - [E(X)]^2 = 2 - 1 = 1$$

Var(Y)

For the variance of Y we then have:

$$M_{X,Y}(t_1, t_2) = \frac{1}{t_1} \left(\frac{1}{1-t_1-t_2} - \frac{1}{1-t_2} \right)$$

$$E[Y^2] = \frac{\partial^2}{\partial t_2^2} M_{X,Y}(t_1, t_2) \Big|_{t_1=0, t_2=0} = \frac{2}{t_1} \left(\frac{1}{(1-t_1-t_2)^3} - \frac{1}{(1-t_2)^3} \right)$$

Similar as before, as we cannot directly evaluate $(t_1, t_2) = (0, 0)$, we must then evaluate the limit as $t_1 \rightarrow \infty$, $t_2 \rightarrow \infty$:

$$E[Y^2] = \frac{\partial^2}{\partial t_2^2} M_{X,Y}(t_1, t_2) \Big|_{t_1=0, t_2=0} = \frac{2}{t_1} \left(\frac{(1-t_2)^3 - (1-t_1-t_2)^3}{(1-t_1-t_2)^3(1-t_2)^3} \right) = \frac{2}{t_1} \left(\frac{3t_1}{(1-t_1-t_2)^3(1-t_2)^3} \right) = \frac{2}{t_1} \frac{t_1(t_1^2 - 3t_1 + 3 - 6t_2)}{(1-t_1-t_2)^3(1-t_2)^3}$$

$$E[Y^2] = \lim_{t_1 \rightarrow 0, t_2 \rightarrow 0} \frac{2}{t_1} \frac{t_1(t_1^2 - 3t_1 + 3 - 6t_2 + 3t_1t_2 + 3t_2^2)}{(1-t_1-t_2)^3(1-t_2)^3} = \lim_{t_1 \rightarrow 0, t_2 \rightarrow 0} \frac{6}{(1-t_1-t_2)^3(1-t_2)^3} = \lim_{t_1 \rightarrow 0, t_2 \rightarrow 0} \frac{6}{(1-0-0)^3(1-0)^3}$$

$$Var(Y) = E(Y^2) - [E(Y)]^2 = 6 - 2^2 = 6 - 4 = 2$$

Cov(X, Y)

To find Cov(X, Y), we have:

$$M_{X,Y}(t_1, t_2) = \frac{1}{t_1} \left(\frac{1}{1-t_1-t_2} - \frac{1}{1-t_2} \right)$$

$$E[XY] = \frac{\partial^2}{\partial t_1 \partial t_2} M_{X,Y}(t_1, t_2) \Big|_{t_1=0, t_2=0}$$

$$\frac{\partial}{\partial t_1} M_{X,Y}(t_1, t_2) = \frac{-1}{t_1^2} \left(\frac{1}{1-t_1-t_2} - \frac{1}{1-t_2} \right) + \left(\frac{1}{t_1} \right) \frac{1}{(1-t_1-t_2)^2} = \frac{1}{(1-t_1-t_2)^2(1-t_2)}$$

Let us then consider deriving with respect to t_2 :

$$\frac{\partial^2}{\partial t_1 \partial t_2} M_{X,Y}(t_1, t_2) = \frac{\partial}{\partial t_2} \left(\frac{1}{(1-t_1-t_2)^2(1-t_2)} \right) = \frac{2}{(1-t_1-t_2)^3(1-t_2)} + \frac{1}{(1-t_1-t_2)^2(1-t_2)^2}$$

$$E[XY] = \frac{\partial^2}{\partial t_1 \partial t_2} M_{X,Y}(t_1, t_2) \Big|_{t_1=0, t_2=0} = \frac{2}{(1-0-0)^3(1-0)} + \frac{1}{(1-0-0)^2(1-0)^2} = \frac{2}{1} + \frac{1}{1} = 3$$

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y] = 3 - 1(2) = 1$$

So, $\text{Var}(X) = 1$, $\text{Var}(Y) = 2$, and $\text{Cov}(X, Y) = 1$.

(c)

Based on the joint moment generating function, identify the marginal distribution of X and the marginal distribution of Y.

Given the joint moment generating function:

$$M_{X,Y}(t_1, t_2) = \frac{1}{t_1} \left(\frac{1}{1-t_1-t_2} - \frac{1}{1-t_2} \right)$$

For the marginal MGF of X, we set $(t_1, t_2) = (t_1, 0)$:

$$M_X(t_1) = M_{X,Y}(t_1, 0) = \frac{1}{t_1} \left(\frac{1}{1-t_1-0} - \frac{1}{1-0} \right) = \frac{1}{t_1} \left(\frac{1}{1-t_1} - 1 \right) = \frac{1}{t_1} \left(\frac{1}{1-t_1} - 1 \right) = \frac{1}{t_1} \left(\frac{1-(1-t_1)}{1-t_1} \right) = \frac{t_1}{t_1(1-t_1)}$$

This is the MGF of an Exponential distribution with parameter 1, such that:

$$X \sim \text{Exponential}(1) \rightarrow f_X(x) = e^{-x}$$

Similarly, for the marginal MGF of Y, we set $(t_1, t_2) = (0, t_2)$:

$$M_Y(t_2) = M_{X,Y}(0, t_2) = \frac{1}{0} \left(\frac{1}{1-0-t_2} - \frac{1}{1-t_2} \right) = \frac{1}{0} \left(\frac{1}{1-t_2} - \frac{1}{1-t_2} \right)$$

Similar as before, as we cannot directly evaluate $(0, t_2)$, we must then evaluate the limit as $t_1 \rightarrow \infty$:

$$M_Y(t_2) = M_{X,Y}(0, t_2) = \frac{1}{t_1} \left(\frac{1}{1 - t_1 - t_2} - \frac{1}{1 - t_2} \right)$$

$$M_{X,Y}(t_1, t_2) = \frac{1}{t_1} \left(\frac{1}{1 - t_1 - t_2} - \frac{1}{1 - t_2} = \frac{(1 - t_2) - (1 - t_1 - t_2)}{(1 - t_1 - t_2)(1 - t_2)} \right) = \frac{1}{t_1} \left(\frac{1 - t_2 - 1 + t_1 + t_2}{(1 - t_1 - t_2)(1 - t_2)} \right) = \frac{1}{t_1} \frac{t_1}{(1 - t_1 - t_2)(1 - t_2)}$$

$$f_Y(y) = \lim_{t_1 \rightarrow 0} M_Y(t_2) = \lim_{t_1 \rightarrow 0} \frac{1}{(1 - t_1 - t_2)(1 - t_2)} = \frac{1}{(1 - t_2)(1 - t_2)} = \frac{1}{(1 - t_2)^2}$$

Which is the MGF of a Gamma distribution, with parameters 2 and 1, specifically:

$$Y \sim \text{Gamma}(\alpha = 2, \beta = 1)$$

$$X \sim \text{Exponential}(1) \rightarrow f_X(x) = e^{-x} \quad Y \sim \text{Gamma}(\alpha = 2, \beta = 1) \rightarrow f_Y(y) = \frac{1}{\Gamma(2)} y e^{-y} = y e^{-y}$$

Q7

Beta-Binomial model: Suppose that the conditional distribution $X | P = p$ is Binomial(n, p) and Suppose P has a Beta(α, β) distribution.

(a)

Using the EVVE formula, find $\text{Var}(X)$

As we know the distribution of $X | P = p$, we know that its mean and variance are:

$$E[X|P = p] = np$$

$$\text{Var}(X|P = p) = np(1 - p)$$

Since we also know the distribution of P , we know it has mean and variance:

$$E[P] = \frac{\alpha}{\alpha + \beta}$$

$$\text{Var}(P) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$

Using EVVE, we know:

$$\text{Var}(X) = E[\text{Var}(X|P)] + \text{Var}(E[X|P])$$

Both of these values will need to be evaluated. To that end:

$$E(\text{Var}(X|P)) = E(np(1 - p)) = nE(p(1 - p)) = n[E(p) - E(p^2)]$$

For a Beta distribution, we know:

$$E(p) = \frac{\alpha}{\alpha + \beta}$$

$$\text{Var}(p) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$

$$E(p^2) = \frac{\alpha(\alpha + 1)}{(\alpha + \beta)(\alpha + \beta + 1)}$$

Additionally, as $E(p(1 - p)) = E(p) - E(p^2)$ we have:

$$E(\text{Var}(X|P)) = n \left(\frac{\alpha}{\alpha + \beta} - \frac{\alpha(\alpha + 1)}{(\alpha + \beta)(\alpha + \beta + 1)} \right) = n \left(\frac{\alpha\beta}{(\alpha + \beta)(\alpha + \beta + 1)} \right)$$

$$\text{Var}(E(X|P)) = \text{Var}(np) = n^2 \text{Var}(p) = n^2 \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$

Combining the two parts, we have:

$$\text{Var}(X) = n \left(\frac{\alpha\beta}{(\alpha + \beta)(\alpha + \beta + 1)} \right) + n^2 \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)} = \frac{n\alpha\beta(\alpha + \beta + n)}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$

(b)

Suppose that W has a Binomial(n, \tilde{p}) distribution having the same mean as X above. For $n > 1$, show that X has a larger variance than W by a multiplicative factor of:

$$\frac{\alpha + \beta + n}{\alpha + \beta + 1} > 1$$

From the Beta-Binomial model, we have:

$$\mathbb{E}(X) = n \cdot E(P) = n \cdot \frac{\alpha}{\alpha + \beta}$$

If the RV W has the same mean as X, then:

$$E(W) = n\tilde{p} = n \cdot \frac{\alpha}{\alpha + \beta}$$

Which means that:

$$\tilde{p} = \frac{\alpha}{\alpha + \beta}$$

Furthermore:

$$Var(W) = n\tilde{p}(1 - \tilde{p}) = n \frac{\alpha}{\alpha + \beta} (1 - \frac{\alpha}{\alpha + \beta}) = n(\frac{\alpha}{\alpha + \beta})(\frac{\beta}{\alpha + \beta}) = \frac{n\alpha\beta}{(\alpha + \beta)^2}$$

From (a), we have:

$$Var(X) = \frac{n\alpha\beta(\alpha + \beta + n)}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$

Looking at the ratio of X/W:

$$\frac{Var(X)}{Var(W)} = \frac{\frac{n\alpha\beta(\alpha + \beta + n)}{(\alpha + \beta)^2(\alpha + \beta + 1)}}{\frac{n\alpha\beta}{(\alpha + \beta)^2}} = \frac{\alpha + \beta + n}{\alpha + \beta + 1}$$

Assuming $n > 1$, we have:

$$(\alpha + \beta + n > \alpha + \beta + 1)$$

Such that X has a larger variance than W by a multiplicative factor of:

$$\frac{\alpha + \beta + n}{\alpha + \beta + 1} > 1$$