

HW4

2024-09-29

Status

- Q1: DONE
- Q2: DONE
- Q3: DONE
- Q4: DONE
- Q5: DONE
- Q6: DONE
- Q7: WIP

Homework 4

Due October 13

Q1

Question: 3.6 (a), (b) Casella & Berger

A large number of insects are expected to be attracted to a certain variety of rose plant. A commercial insecticide is advertised as being 99% effective. Suppose 2,000 insects infest a rose garden where the insecticide has been applied and let X = number of surviving insects.

(a)

What probability distribution might provide a reasonable model for this experiment?

(b)

Write down, but do not evaluate, an expression for the probability that fewer than 100 insects survive, using the model in part (a)

Answer > (a)

We may interpret X as the number of “failures” given an effective rate of 99%, or $p = 1 - 0.99 = 0.01$ (1% chance of failure). As we are counting the number of failures, we have a discrete random variable. We know $n = 2,000$, or our total number of “trials” for the insecticide.

Taken together, we have X as a Binomial distributed random variable, or:

$$X \sim \text{Binomial}(n = 2,000, p = 0.01)$$

It’s worth noting that we can also represent this as a Poisson distributed random variable with parameter $\lambda = np = 2000(0.01) = 20$

(b)

$$P(X < 100) = P(X \leq 99)$$

$$\sum_{x=0}^{99} P(X = x) = \sum_{x=0}^{99} f(x) = \sum_{x=0}^{99} \binom{n}{x} (0.01)^x (0.99)^{2000-x}$$

Q2

Question: 3.13 (a) Casella & Berger

A truncated discrete distribution is one in which a particular class cannot be observed and is eliminated from the sample space. In particular, if X has range $0, 1, 2, \dots$ and the 0 class cannot be observed (as is usually the case), the 0-truncated random variable X_T has pmf:

$$P(X_T = x) = \frac{P(X = x)}{P(X > 0)}$$

for $x = 1, 2, \dots$

Find the pmf, mean, and variance of the 0-truncated random variable starting from:

(a)

$$X \sim \text{Poisson}(\lambda)$$

Answer > (a)

The pmf of a Poisson distribution is:

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

Also:

$$P(X > 0) = 1 - P(X = 0) = 1 - \frac{\lambda^0 e^{-\lambda}}{0!} = 1 - e^{-\lambda}$$

Combining these together gives us the truncated pmf:

$$P(X_T = x) = \frac{P(X = x)}{P(X > 0)} = \frac{\lambda^x e^{-\lambda}}{x!} / 1 - e^{-\lambda} = \frac{\lambda^x e^{-\lambda}}{x!(1 - e^{-\lambda})}$$

For $x = 1, 2, \dots$

Using the above pmf, we may find the mean as:

$$E(X_T) = \sum_{x=1}^{\infty} x P(X_T = x) = \sum_{x=1}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!(1 - e^{-\lambda})} = \frac{\lambda e^{-\lambda}}{(1 - e^{-\lambda})} \sum_{x \geq 1} \frac{\lambda^{x-1}}{(x-1)!}$$

Let $y = x - 1$, such that we may rewrite the above as:

$$E(X_T) = \frac{\lambda e^{-\lambda}}{(1 - e^{-\lambda})} \sum_{y \geq 0} \frac{\lambda^y}{y!}$$

Using the infinite summation for Euler, namely:

$$e^{\lambda} = \sum_{y \geq 0} \lambda^y / y!$$

We may then evaluate this as:

$$E(X_T) = \frac{\lambda e^{-\lambda}}{(1 - e^{-\lambda})} e^{\lambda} = \frac{\lambda}{(1 - e^{-\lambda})}$$

To then find the variance, let us consider $E(X_T^2)$

$$E(X_T^2) = \sum_{x \geq 1} x^2 P(X_T = x) = \sum_{x \geq 1} x^2 \frac{\lambda^x e^{-\lambda}}{x! (1 - e^{-\lambda})} = \frac{\lambda e^{-\lambda}}{(1 - e^{-\lambda})} \sum_{x \geq 1} x \frac{\lambda^x}{(x - 1)!}$$

$$E(X_T^2) = \frac{\lambda e^{-\lambda}}{(1 - e^{-\lambda})} \left(\sum_{x \geq 1} (x - 1) \frac{\lambda^{x-1}}{(x - 1)!} + \sum_{x \geq 1} \frac{\lambda^{x-1}}{(x - 1)!} \right) = \frac{\lambda e^{-\lambda}}{(1 - e^{-\lambda})} \left(\lambda \sum_{x \geq 2} \frac{\lambda^{x-2}}{(x - 2)!} + \sum_{x \geq 1} \frac{\lambda^{x-1}}{(x - 1)!} \right)$$

Then, let $y = x - 2$, $z = x - 1$, we have:

$$E(X_T^2) = \frac{\lambda e^{-\lambda}}{(1 - e^{-\lambda})} \left(\lambda \sum_{y \geq 0} \frac{\lambda^y}{(y)!} + \sum_{z \geq 0} \frac{\lambda^z}{(z)!} \right) = \frac{\lambda e^{-\lambda}}{(1 - e^{-\lambda})} (\lambda e^{\lambda} + e^{\lambda})$$

Thus:

$$E(X_T^2) = \frac{\lambda^2 + \lambda}{(1 - e^{-\lambda})}$$

$$Var(E_T) = E(X_T^2) - (E(X_T))^2 = \frac{\lambda^2 + \lambda}{(1 - e^{-\lambda})} - \left(\frac{\lambda}{(1 - e^{-\lambda})} \right)^2$$

Q3

Question: 3.17 Casella & Berger

Establish a formula similar to (3.3.18) for the gamma distribution. If $X \sim \text{Gamma}(\alpha, \beta)$, then for any positive constant v ,

$$EX^v = \frac{\beta^v \Gamma(v+\alpha)}{\Gamma(\alpha)}$$

Answer

Formula 3.3.18:

$$\begin{aligned} EX^n &= \frac{B(\alpha + n, \beta)}{B(\alpha, \beta)} = \frac{\Gamma(\alpha + n)\Gamma(\alpha + \beta)}{\Gamma(\alpha + \beta + n)\Gamma(\alpha)} \\ EX^v &= \int_{x=0}^{\infty} x^v \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx = \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_{x=0}^{\infty} x^{(v+\alpha-1)} e^{-x/\beta} dx \\ EX^v &= \frac{\Gamma(v+\alpha)\beta^{v+\alpha}}{\Gamma(\alpha)\beta^\alpha} \int_{x=0}^{\infty} \frac{1}{\Gamma(v+\alpha)\beta^{v+\alpha}} x^{(v+\alpha-1)} e^{-x/\beta} dx = \frac{\Gamma(v+\alpha)\beta^{v+\alpha}}{\Gamma(\alpha)\beta^\alpha} (1) \end{aligned}$$

With note of a cheeky: $\frac{\Gamma(v+\alpha)\beta^{v+\alpha}}{\Gamma(v+\alpha)\beta^{v+\alpha}} = 1$

Simplifying gives us:

$$EX^v = \frac{\Gamma(v+\alpha)\beta^{v+\alpha}}{\Gamma(\alpha)\beta^\alpha} = \frac{\Gamma(v+\alpha)\beta^v}{\Gamma(\alpha)}$$

$\forall v > -\alpha$ as The Gamma function is only defined for positive values.

Q4

Question: 3.19 Casella & Berger

Show that:

$$\int_x^\infty \frac{1}{\Gamma(\alpha)} z^{\alpha-1} e^{-z} dz = \sum_{y=0}^{\alpha-1} \frac{x^y e^{-x}}{y!}$$

For $\alpha = 1, 2, 3, \dots$

Hint: Use integration by parts. Express this formula as a probabilistic relationship between Poisson and Gamma random variables.

Answer

Via Integration by Parts, let us take out the constant $\frac{1}{\Gamma(\alpha)}$, and let $\alpha = n$, such that we have:

$$u = z^{\alpha-1}$$

$$du = (\alpha - 1)z^{\alpha-2}$$

$$dv = e^{-z} dz$$

$$v = -e^{-z}$$

Thus, we have :

$$\int u dv = uv - \int v du$$

Giving us:

$$\int_{z=x}^\infty z^{n-1} z^{-z} dz = x^{n-1} e^{-x} - \int_{z=x}^\infty z^{n-2} (n-2)(-z^{-z}) dz$$

Simplifying and adding back in the $\frac{1}{\Gamma(n)}$ term gives us:

$$\frac{1}{\Gamma(n)} \int_{z=x}^\infty z^{n-1} z^{-z} dz = \frac{1}{\Gamma(n)} [x^{n-1} e^{-x} + \int_{z=x}^\infty (n-1) z^{n-2} z^{-z} dz]$$

We then note the expansion of $\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2) = \dots = (n-1)!\Gamma(1) = (n-1)!(1) = (n-1)!$. Using this relation above gives us:

$$\frac{1}{\Gamma(n)} \int_{z=x}^\infty z^{n-1} z^{-z} dz = \frac{x^{n-1} e^{-x}}{(n-1)!} + \int_{z=x}^\infty \frac{1}{(n-1)\Gamma(n-1)} (n-1) z^{n-2} z^{-z} dz$$

$$\frac{1}{\Gamma(n)} \int_{z=x}^{\infty} z^{n-1} z^{-z} dz = \frac{x^{n-1} e^{-x}}{(n-1)!} + \frac{1}{\Gamma(n-1)} \int_{z=x}^{\infty} z^{n-2} z^{-z} dz$$

Notice the first term in this evaluation is $\frac{x^y e^{-x}}{y!}$ evaluated at $y = n - 1$!

Notice also that we may continue the integration by parts for the term

$$\frac{1}{\Gamma(n-1)} \int_{z=x}^{\infty} z^{n-2} z^{-z} dz$$

,

which will yield two parts, the first being the $\frac{x^y e^{-x}}{y!}$ evaluated at $y = n - 2$ and the second part being another integral we may break into parts.

The above process may be repeated until we arrive at an integral in the $\int v du = 0$

In summary, we can continue breaking the integral into parts which coincide with elements of the summation in reverse order ($\alpha - 1$ or $n - 1$), meaning these two values are equal.

This is all to say that we have shown the relationship between the random variable $X \sim \text{Poisson}(x)$ and the random variable $Y \sim \text{Gamma}(\alpha, 1)$, as $P(X > x) = P(Y \leq \alpha - 1)$

Q5

Question: 3.24 (a), (c) Casella & Berger Note: You can skip the part about showing that the pdf is a pdf; also, in (c), the variance will not exist unless $a > 2$.

Many “named” distributions are special cases of the more common distributions already discussed. For each of the following named distributions derive the form of the pdf, . . . , and calculate the mean and variance.

(a)

If $X \sim \text{Exponential}(\beta)$, then $Y = X^{1/\gamma}$ has the Weibull(γ, β) distribution, where $\gamma > 0$ is a constant.

(c)

If $X \sim \text{Gamma}(a, b)$, then $Y = 1/X$ has the inverted Gamma IG(a, b) distribution.

Answer > (a)

$$X \sim \text{Exponential}(\beta) \rightarrow f_X(x) = \frac{1}{\beta} e^{-x/\beta}$$

Using the following general relation for two related random variables:

Thm. 2.1.5:

$$f_X(x) = f_Z(g^{-1}(x)) \left| \frac{d}{dx} g^{-1}(x) \right|$$

We have:

$$Y = X^{1/\gamma} \rightarrow f_Y(y) = f_X(y^\gamma) (d/dy) = \frac{1}{\beta} e^{-y^\gamma/\beta} \gamma y^{\gamma-1}$$

For $x > 0, y > 0$

$$\text{For simplicity, let us consider } EY^n = \int_{y=0}^{\infty} y^n f(y) = \int_{y=0}^{\infty} y^n \frac{1}{\beta} e^{-y^\gamma/\beta} \gamma y^{\gamma-1}$$

Simplifying terms somewhat, and removing terms that don't depend on y, we have:

$$EY^n = \gamma/\beta \int_{y=0}^{\infty} y^{\gamma+n-1} e^{-y^\gamma/\beta} dy$$

Let $z = y^\gamma/\beta \rightarrow dz = \frac{\gamma}{\beta} y^{\gamma-1} dy$ Such that $\frac{\beta}{\gamma y^{\gamma-1}} dz = dy$

Change of variables allows us to rewrite the above as:

$$EY^n = \frac{\gamma}{\beta} \frac{\beta}{\gamma} \beta^{\gamma+n-1/\gamma} \int_{z=0}^{\infty} z^{\frac{\gamma+n-1}{\gamma}} z^{\frac{1}{\gamma}-1} e^{-z} dz$$

Simplifying terms somewhat gives us:

$$EY^n = \beta^{\gamma+n-1/\gamma} \int_{z=0}^{\infty} z^{n/\gamma} e^{-z} dz$$

Recognizing this as the $\Gamma(n/\gamma + 1)$ function and simplifying constants, we thus have:

$$EY^n = \beta^{n/\gamma} \Gamma(n/\gamma + 1)$$

Using the above formula we have:

$$EY = \beta^{1/\gamma} \Gamma(1/\gamma + 1)$$

and

$$EY^2 = \beta^{2/\gamma} \Gamma(2/\gamma + 1)$$

$$Var(Y) = EY^2 - (EY)^2 = \beta^{2/\gamma} \Gamma(2/\gamma + 1) - (\beta^{1/\gamma} \Gamma(1/\gamma + 1))^2$$

$$Var(Y) = \beta^{2/\gamma} [\Gamma(2/\gamma + 1) - (\Gamma(1/\gamma + 1))^2]$$

(c)

$$X \sim \text{Gamma}(a, b) \rightarrow f_X(x) = \frac{1}{\Gamma(a)b^a} x^{a-1} e^{-x/b}$$

With note of Thm. 2.1.5:

$$f_X(x) = f_z(g^{-1}(x)) \left| \frac{d}{dx} g^{-1}(x) \right|$$

$$Y = 1/X \rightarrow f_Y(y) = f_X(1/y) (d/dy) = \frac{1}{\Gamma(a)b^a} (1/y)^{a-1} e^{-1/by}$$

$$EY = \int_{y=0}^{\infty} y \frac{1}{\Gamma(a)b^a} (1/y)^{a-1} e^{-1/by} dy$$

$$EY = \frac{1}{\Gamma(a)b^a} \int_{y=0}^{\infty} \left(\frac{1}{y}\right)^a e^{-1/by} dy = \frac{\Gamma(a-1)b^{a-1}}{\Gamma(a)b^a} = \frac{1}{(a-1)b}$$

Note: $b \neq 0$ and $a \neq 1$ in order for the above to evaluate.

$$EY^2 = \int_{y=0}^{\infty} y^2 \frac{1}{\Gamma(a)b^a} (1/y)^{a-1} e^{-1/by} dy$$

$$EY^2 = \frac{1}{\Gamma(a)b^a} \int_{y=0}^{\infty} \left(\frac{1}{y}\right)^{a-1} e^{-1/by} dy = \frac{\Gamma(a-2)b^{a-2}}{\Gamma(a)b^a} = \frac{1}{(a-1)(a-2)b^2}$$

$$Var(Y) = EY^2 - (EY)^2 = \frac{1}{(a-1)(a-2)b^2} - \left(\frac{1}{(a-1)b}\right)^2 = \frac{1}{(a-1)^2(a-2)b^2}$$

Note: $Var(Y)$ does not exist unless $a > 2$ as variance must be positive.

Q6

Question: 3.39 Casella & Berger

Consider the Cauchy family defined in Section 3.3. This family can be extended to a location-scale family yielding pdfs of the form:

$$f(x|\mu, \sigma) = \frac{1}{\sigma\pi(1 + (\frac{x-\mu}{\sigma})^2)}$$

For $-\infty < x < \infty$

The mean and variance do not exist for the Cauchy distribution. So the parameters μ, σ^2 are not the mean and variance. But they do have important meaning. Show that if X is a random variable with a Cauchy distribution with parameters μ and σ , then:

(a)

μ is the median of the distribution of X , that is, $P(X \geq \mu) = P(X \leq \mu) = \frac{1}{2}$

(b)

$\mu + \sigma$ and $\mu - \sigma$ are the quartiles of the distribution of X , that is $P(X \geq \mu + \sigma) = P(X \leq \mu - \sigma) = \frac{1}{4}$

Hint: Prove this first for $\mu = 0$ and $\sigma = 1$ and then use Exercise 3.38.

Note: $\frac{d(\arctan x)}{dx} = \frac{1}{1+x^2}$

Answer

(a)

μ is the median if

$$P(X \geq \mu) = P(X \leq \mu) = \frac{1}{2}$$

Let us consider the random Cauchy variable Z where $\mu = 0, \sigma = 1, Z \sim \text{Cauchy}(\mu = 0, \sigma = 1)$

Let us consider $\mu = 0$, such that: $X = Z + \mu$

$$P(Z \geq 0) = \int_{z=0}^{\infty} \frac{1}{\pi} \frac{1}{1+z^2} dz = \frac{1}{\pi} \arctan(z) \Big|_{z=0}^{\infty} = \frac{1}{\pi} (\pi/2 - 0) = \frac{1}{2}$$

and

$$P(Z \leq 0) = \int_{z=-\infty}^0 \frac{1}{\pi} \frac{1}{1+z^2} dz = \frac{1}{\pi} \arctan(z) \Big|_{z=-\infty}^0 = \frac{1}{\pi} (0 - (-\pi/2)) = \frac{1}{2}$$

We have shown then that for the random variable Z , μ is indeed the median.

Taking advantage of the results of Exercise 3.38 we then have X belongs a location family of Z :

$$\text{So } P(Z \geq 0) = P(X \geq \mu) = \frac{1}{2}$$

And

$$\text{So } P(Z \leq 0) = P(X \leq \mu) = \frac{1}{2}$$

So the μ is the median for the random variable X from the Cauchy family.

(b)

Let us consider the random Cauchy variable Z where $\mu = 0, \sigma = 1$, such that: $X = \sigma Z + \mu, Z \sim \text{Cauchy}(\mu = 0, \sigma = 1)$

$$P(Z \geq \mu + \sigma) = P(Z \geq 1) = \int_{z=1}^{\infty} \frac{1}{\pi} \frac{1}{1+z^2} dz = \frac{1}{\pi} \arctan z \Big|_{z=1}^{\infty} = \frac{1}{\pi} (\pi/2 - \pi/4) = \frac{1}{4}$$

$$P(Z \leq \mu - \sigma) = P(Z \leq -1) = \int_{z=-\infty}^{-1} \frac{1}{\pi} \frac{1}{1+z^2} dz = \frac{1}{\pi} \arctan z \Big|_{z=-\infty}^{-1} = \frac{1}{4}$$

Taking advantage of the results of Exercise 3.38 we then have X belongs to a scale-location family of Z and we may write:

$$P(Z \geq 1) = P(X \geq \mu + \sigma) \quad P(Z \leq -1) = P(X \leq \mu - \sigma)$$

$$\text{So the following holds: } P(X \geq \mu + \sigma) = \frac{1}{4} \quad P(X \leq \mu - \sigma) = \frac{1}{4}$$

Such that we have shown that $\mu + \sigma$ and $\mu - \sigma$ are the quartiles of the distribution of X .

Q7

Question:

If $X \sim N(\mu, \sigma^2)$, find values of μ and σ such that $P(|X| < 2) = \frac{1}{2}$. Prove or disprove that the values of μ and σ are unique.

Answer

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$$

$$|X| < 2 \rightarrow -2 < X < 2$$

$$P(-2 < X < 2) = P\left(\frac{-2 - \mu}{\sigma} < Z < \frac{2 - \mu}{\sigma}\right)$$

$$a = \frac{-2 - \mu}{\sigma} \quad \text{and} \quad b = \frac{2 - \mu}{\sigma}$$

$$P(a < Z < b) = \Phi(b) - \Phi(a) = \frac{1}{2}$$

$$\Phi(b) = \Phi(a) + \frac{1}{2}$$

$$b = a + d$$

$$\Phi(b) = \Phi(a) + \frac{1}{2}$$

Symmetry

Thus

$$\Phi(z) - \Phi(-z) = \frac{1}{2}$$

$$\Phi(-z) = 1 - \Phi(z)$$

$$\Phi(z) - (1 - \Phi(z)) = \frac{1}{2} \Rightarrow 2\Phi(z) - 1 = \frac{1}{2} \Rightarrow \Phi(z) = \frac{3}{4}$$

$$\Phi(z) = \frac{3}{4} \Rightarrow z \approx 0.674$$

$$a = \frac{-2 - \mu}{\sigma} = -z \Rightarrow \frac{-2 - \mu}{\sigma} = -0.674 \Rightarrow -2 - \mu = -0.674\sigma \Rightarrow \mu = -2 + 0.674\sigma$$

$$b = \frac{2 - \mu}{\sigma} = z \Rightarrow \frac{2 - \mu}{\sigma} = 0.674 \Rightarrow 2 - \mu = 0.674\sigma \Rightarrow \mu = 2 - 0.674\sigma$$

$$-2 + 0.674\sigma = 2 - 0.674\sigma$$

$$0.674\sigma + 0.674\sigma = 4 \Rightarrow 1.348\sigma = 4 \Rightarrow \sigma = \frac{4}{1.348} \approx 2.96$$

$$\mu = -2 + 0.674 \cdot 2.96 \approx -2 + 2.0 \approx 0$$

$$\mu \approx 0, \quad \sigma \approx 2.96$$

To check the uniqueness, consider the functions $\Phi(b) - \Phi(a)$ as continuous functions of μ and σ . The conditions specified yield a unique pair (μ, σ) as long as $\sigma > 0$ since each probability corresponds to a unique interval in the normal distribution. Thus, the solution is indeed unique:

$$\mu = 0, \quad \sigma \approx 2.96 \text{ (to satisfy } P(|X| < 2) = \tfrac{1}{2})$$