Homework 1 – STAT 542

Due Friday, Sept 13 by 11:59 PM

Some exercises below refer to our text (Casella & Berger, 2nd Edition).

1. Problem 1.12(a), Casella & Berger: Show that the Axiom of Countable Additivity implies Finite Additivity.
   Hint: You assume that, for *infinite* sequence of disjoint events $A_1, A_2, \ldots$, it holds that
   $$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i); \tag{1}$$
   this is axiom 3 of how probability functions $P(\cdot)$ must work. Assuming that (1) holds, you need to show that if $B_1, \ldots, B_n$ is a *finite* sequence of disjoint events, then
   $$P\left(\bigcup_{i=1}^{n} B_i\right) = \sum_{i=1}^{n} P(B_i) \tag{2}$$
   holds. Technically, we have already been using (2) in class; we're proving here that (2) is a just special case of (1), as assumed all along. The trick is to develop an infinite sequence of disjoint events $A_1, A_2, \ldots$ where $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{n} B_i$; consider taking $A_i = B_i$ where possible and "filling in" the rest of the A_i's with nothing (so that the A_i's will be disjoint).
2. Problem 1.13, Casella & Berger: If $P(A) = 1/3$ and $P(B^c) = 1/4$, can A and B be disjoint? Explain.
3. Suppose a family has 4 children, named a, b, c and d, who take turns washing 4 plates denoted p_1, p_2, p_3, p_4. These children are not so careful in their work so, over time, each of the plates will be broken. Suppose any child could break any plate and that the ways in which plates p_1, p_2, p_3, p_4 could be broken by children a, b, c, d are equally likely.
   (a) Find the probability that child a breaks 3 plates.
   (b) Find the probability that one of the four children breaks 3 plates.
4. Problem 1.34, Casella & Berger
5. Problem 1.38, Casella & Berger
6. Problem 1.39, Casella & Berger
7. Problem 1.47(cd), Casella & Berger
   (For purposes of checking cdf conditions, it may be helpful to recall that differentiable (and hence continuous) functions are non-decreasing if their derivatives are non-negative; the derivative of a continuous cdf is a pdf, which needs to be non-negative anyway.)
8. Problem 1.54

9. From the axioms of probability[1], it follows that probability functions $P(\cdot)$ exhibit "monotone continuity from above (mcfa)" (which you don't have to worry about showing), meaning that for any decreasing sequence of sets/events $A_1 \supset A_2 \supset A_3 \supset \cdots$,

$$\lim_{n\to\infty} P(A_n) = P\left(\bigcap_{i=1}^{\infty} A_i\right).$$

By using/applying the mcfa property, show that the cdf F of a random variable X must be right continuous: for any $x \in \mathbb{R}$,

$$\lim_{n\to\infty} F(x + n^{-1}) = F(x)$$

holds.

[1] Actually, probability functions also exhibit "monotone continuity from below (mcfb)" meaning that for any increasing sequence of sets/events $A_1 \subset A_2 \subset A_3 \subset \cdots$ it holds that $\lim_{n\to\infty} P(A_n) = P\left(\bigcup_{i=1}^{\infty} A_i\right)$. This is because one may write disjoint sets $B_1 = A_1$, $B_2 = A_2 \setminus A_1$, $B_3 = A_3 \setminus A_2, \ldots, B_i = A_i \setminus A_{i-1}, \ldots$ where $A_n = \bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i$ for all $n \geq 1$ and $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$. Then, using disjoint set properties,as $n \to \infty$,

$$P(A_n) = P\left(\bigcup_{i=1}^{n} A_i\right) = P\left(\bigcup_{i=1}^{n} B_i\right) = \sum_{i=1}^{n} P(B_i) \to \sum_{i=1}^{\infty} P(B_i) = P\left(\bigcup_{i=1}^{\infty} B_i\right) = P\left(\bigcup_{i=1}^{\infty} A_i\right).$$

One can then show mcfa by applying mcfb to (increasing) set complements and using DeMorgan's rule on complements: if $A_1 \supset A_2 \supset A_3 \supset \cdots$ then $A_1^c \subset A_2^c \subset A_3^c \subset \cdots$, so that

$$P(A_n) = 1 - P(A_n^c) \to 1 - P\left(\bigcup_{i=1}^{\infty} A_i^c\right) = P\left(\left[\bigcup_{i=1}^{n} A_i^c\right]^c\right) = P\left(\bigcap_{i=1}^{\infty} A_i\right)$$

as $n \to \infty$.