Stat 5000 | Homework #12 |
Fall 2024 | due Fri, Dec 13th @ 11:59 pm | Name: ________________

Directions: Type or clearly handwrite your solutions to each of the following exercises. Partial credit cannot be given unless all work is shown. You may work in groups provided that each person takes responsibility for understanding and writing out the solutions. Additionally, you must give proper credit to your collaborators by providing their names on the line below (if you worked alone, write "No Collaborators"):

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1. **[+5]:** For each of the following models Y_i are the responses, β_i are parameters, X_i are fixed values, and ϵ_i denotes random errors with variance σ^2. Indicate if it is a linear model, a nonlinear model, or an intrinsically linear model (a nonlinear model that can be transformed into a linear model).

   (a) $Y_i = \beta_0 + \beta_1\ X_{1i} + \beta_2\ log(X_{2i}) + \beta_3\ X_{3i} + \epsilon_i$ with $E(\epsilon_i) = 0$

   (b) $Y_i = \beta_0\ exp(\beta_1\ X_{1i}) + \epsilon_i$ with $E(\epsilon_i) = 0$

   (c) $Y_i = [1 + exp(\beta_0 + \beta_1\ X_{1i} + \epsilon_i)]^{-1}$ with $E(\epsilon_i) = 0$

   (d) $Y_i = (\beta_0 + \beta_1\ X_{1i})\epsilon_i$ with $E(\epsilon_i) = 1$

   (e) $Y_i = \epsilon_i\ exp(\beta_0 + \beta_1\ X_{1i} + \beta_2\ X_{2i})$ with $E(\epsilon_i) = 1$

2. [**+2**]**:** Only square, nonsingular matrices have inverses, but every matrix has a generalized inverse. For example, let

$$A = \begin{bmatrix} 1 \\ 2 \\ 5 \\ -2 \end{bmatrix}.$$

Show that $B = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$ satisfies the definition of a generalized inverse for A.

3. **[+8]:** Consider the linear model $\mathbf{Y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon}$ where

$$\mathbf{Y} = \begin{bmatrix} Y_{11} \\ Y_{12} \\ Y_{21} \\ Y_{22} \\ Y_{23} \\ Y_{24} \\ Y_{31} \\ Y_{32} \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}, \quad \boldsymbol{\epsilon} \sim N(\mathbf{0}, \sigma^2\ I).$$

Determine which of the following linear functions, $c_i^T\boldsymbol{\beta}$ of the model parameters are estimable. Briefly justify for your answer. For estimable functions only, find a constant matrix A_i such that $A_i E(\mathbf{Y}) = c_i^T\boldsymbol{\beta}$

(a) $c_1^T\boldsymbol{\beta} = \alpha_1 - \frac{1}{2}(\alpha_2 + \alpha_3)$

(b) $c_2^T\boldsymbol{\beta} = 3\mu + \alpha_1 + 2\alpha_2$

(c) $c_3^T\boldsymbol{\beta} = \alpha_2 + \alpha_3$

(d) $c_4^T\boldsymbol{\beta} = 3\mu - \alpha_1 - \alpha_2 - \alpha_3$

4. **[+10]:** A food scientist performed an experiment to study the effects of combining two different fats and three different surfactants on the specific volume of bread loaves. Two batches of dough were made for each of the six combinations of fat and surfactant. Ten loaves of bread were made from each batch of dough and the average volume of the ten loaves was recorded for each batch. In total, there are 12 observations. Consider the two-way ANOVA model

$$Y_{ijk} = \mu + \alpha_i + \tau_j + (\alpha\tau)_{ij} + \epsilon_{ijk} \quad \text{where } \epsilon_{ijk} \sim N(0, \sigma^2)$$

and Y_{ijk} denotes the average of the volumes of ten loaves of bread made from the k^{th} batch of dough using the i^{th} fat and the j^{th} surfactant. Determine which of the following linear functions of the model parameters are estimable. Briefly justify for your answer.

(a) μ

(b) $\alpha_1 - \alpha_2$

(c) $(\alpha\tau)_{12}$

(d) $(\alpha\tau)_{11} - (\alpha\tau)_{12}$

(e) $(\alpha\tau)_{11} - (\alpha\tau)_{12} - (\alpha\tau)_{21} + (\alpha\tau)_{22}$

Total: 25 points **# correct:** ____________ **%:** ____________