HW6

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Q1

An ecologist takes data

$$(x_i, Y_i), i = 1, \ldots, n,$$

where $x_i > 0$ is the size of an area and Y_i is the number of moss plants. The data are modeled assuming $x_1, \ldots, x_n$ are fixed; $Y_1, \ldots, Y_n$ are independent; and:

$$Y_i \sim \text{Poisson}(\theta x_i)$$

with parameter θx_i. Suppose that:

$$\sum_{i=1}^{n} x_i = 5$$

is known. Find an exact form of the most powerful (MP) test of size $\alpha = 9e^{-10}$ for testing:

$$H_0 : \theta = 2 \quad \text{vs} \quad H_1 : \theta = 1.$$

Answer

To start, we consider the likelihood ratio test statistic. The likelihood function under a general θ is:

$$L(\theta) = \prod_{i=1}^{n} \frac{e^{-\theta x_i}(\theta x_i)^{Y_i}}{Y_i!}$$

The likelihood ratio for testing $H_0 : \theta = 2$ vs $H_1 : \theta = 1$ is then given as a ratio of the likelihood under the alternative over the likelihood over the null:

$$\Lambda = \frac{L(\theta = 1)}{L(\theta = 2)} = \frac{\prod_{i=1}^{n} e^{-x_i} x_i^{Y_i} / Y_i!}{\prod_{i=1}^{n} e^{-2x_i} (2x_i)^{Y_i} / Y_i!} = e^{\sum_i x_i} \cdot 2^{-\sum_i Y_i} = e^5 \cdot 2^{-T}$$

where $T = \sum_{i=1}^{n} Y_i$ and substituting in other known quantities.

Then, via Neyman-Pearson, the MP test rejects H_0 when Λ is large, which corresponds to small values of T (since Λ decreases as T increases).

Thus, the rejection region is of the form:

$$R = \{T \leq c\}$$

for some critical value c.

Under $H_0 : \theta = 2$, we have:

$$T \sim \text{Poisson}(2 \cdot \sum_i x_i) = \text{Poisson}(10)$$

We need to find c such that:

$$P_{H_0}(T \leq c) \leq \alpha = 9 \cdot 10^{-10}$$

We can compute these probabilities for $T \in \mathbb{Z}_0$:

- $P(T = 0) = e^{-10} \approx 4.54 \times 10^{-5}$
- $P(T = 1) = e^{-10} \cdot 10 \approx 4.54 \times 10^{-4}$
- $P(T \leq 1) = P(T = 0) + P(T = 1) \approx 4.99 \times 10^{-4}$

Since $\alpha = 9 \times 10^{-10}$ is much smaller than $P(T \leq 1)$, we see that only $T = 0$ satisfies the size requirement in this problem, i.e.:

$$P(T \leq 0) = e^{-10} \approx 4.54 \cdot 10^{-5} < \alpha$$

However, $P(T \leq 0) \neq \alpha$, so we must find a suitable $\gamma \in [0, 1]$ to satisfy equality.

To that end, we would need to use a randomized test when $T = 1$, i.e. our test is of the form:

- Reject with probability 1 if $T = 0$
- Reject with probability γ if $T = 1$
- Never reject if $T \geq 2$

We then need to calculate $\gamma \in [0, 1]$, using:

$$P(T = 0) + \gamma P(T = 1) = \alpha$$

Solving for γ:

$$e^{-10} + \gamma \cdot 10e^{-10} = 9e^{-10}$$

$$\gamma = \frac{9e^{-10} - e^{-10}}{10e^{-10}} = 0.8$$

So we may write the full form of the test as:

$$\phi_{H_0}(X) = \begin{cases} 1 & \text{if } T = \sum Y_i = 0 \\ \gamma = 0.8 & \text{if } T = \sum Y_i = 1 \\ 0 & \text{otherwise} \end{cases}$$

Q2

Problem 8.19:

The random variable X has pdf:

$$f(x) = e^{-x}, \quad x > 0.$$

One observation is obtained on the random variable:

$$Y = X^{\theta},$$

and a test of:

$$H_0 : \theta = 1 \quad \text{versus} \quad H_1 : \theta = 2$$

needs to be constructed.

Find the UMP level $\alpha = 0.10$ test and compute the Type II Error probability.

Hint

Show that the form of the MP test involves rejecting H_0 if:

$$e^{y-\sqrt{y}}/\sqrt{y} > k$$

for some $k > 1$.

(Skip the part involving $\alpha = 0.1$ or the Type II error part.)

Answer

Under the transformation $Y = X^{\theta}$, the inverse is $X = Y^{1/\theta}$, and:

$$\frac{dx}{dy} = \frac{1}{\theta} y^{(1/\theta)-1}$$

The above Jacobian we will need for a change of variables, specifically, using the pdf of X, we have the pdf of Y given by:

$$f_Y(y|\theta) = f_X(y^{1/\theta}) \cdot \left|\frac{dx}{dy}\right| = e^{-y^{1/\theta}} \cdot \frac{1}{\theta} y^{(1/\theta)-1}$$

Where: $y > 0$

Via Neyman-Pearson, the MP test rejects H_0 for large values of the likelihood ratio, given by:

$$\Lambda = \frac{f_Y(y|2)}{f_Y(y|1)}$$

Substituting the pdfs, and simplifying:

$$\Lambda = \frac{\frac{1}{2}y^{-1/2}e^{-y^{1/2}}}{e^{-y}} = \frac{1}{2}y^{-1/2}e^{y-\sqrt{y}}$$

The rejection region is of the form:

$$\Lambda > k \to \frac{e^{y-\sqrt{y}}}{\sqrt{y}} > 2k = k_1$$

Where $k_1 > 1$

Let $g(y) = \frac{e^{y-\sqrt{y}}}{\sqrt{y}}$. Take derivative, with the intent to show monotonicity (Spoiler: Non-monotonicity due to two distinct rejection regions) and also noting log is a monotonic transformation:

$$\frac{d}{dy}\ln g(y) = \frac{d}{dy}\left(y - \sqrt{y} - \frac{1}{2}\ln y\right) = 1 - \frac{1}{2\sqrt{y}} - \frac{1}{2y}$$

For $y \to 0^+$: The derivative

$$\frac{d}{dy}\ln g(y) = 1 - \frac{1}{2\sqrt{y}} - \frac{1}{2y} \to -\infty$$

so $g(y)$ is decreasing as $y \to 0^+$.

For $y \to \infty$: The derivative

$$\frac{d}{dy}\ln g(y) \to 1$$

so $g(y)$ is increasing for large y.

Since

$$\frac{d^2}{dy^2}\ln g(y) = \frac{1}{4y^{3/2}} + \frac{1}{2y^2} > 0 \quad \text{for all } y > 0,$$

it follows that $\ln g(y)$ is strictly convex, so $g(y)$ has a minimum where the derivative is zero.

We then seek to find that minimum! To that end:

$$1 - \frac{1}{2\sqrt{y}} - \frac{1}{2y} = 0 \to y = 1$$

So at $y = 1$, $g(y)$ has a minimum. Thus, $\Lambda > k_1$ corresponds to:

$$Y \leq c_0 \quad \text{or} \quad Y \geq c_1$$

where $c_0 < 1 < c_1$.

The UMP level-α test rejects H_0 if:

$$Y \leq c_0 \quad \text{or} \quad Y \geq c_1$$

where c_0, c_1 are chosen such that:

$$P_{H_0}(Y \leq c_0) + P_{H_0}(Y \geq c_1) = \alpha$$

Under H_0 ($\theta = 1$), $Y = X \sim \text{Exp}(1)$. These are probabilities we can express as:

$$P_{H_0}(Y \leq c_0) = 1 - e^{-c_0}$$

And

$$P_{H_0}(Y \geq c_1) = e^{-c_1}$$

Taken together then, the UMP test for $H_0 : \theta = 1$ vs $H_1 : \theta = 2$ rejects H_0 if:

$$Y \leq c_0 \quad \text{or} \quad Y \geq c_1$$

Written:

$$\phi_{H_0}(Y) = \begin{cases} 1 & Y \leq c_0 \quad \text{or} \quad Y \geq c_1 \\ 0 & \text{otherwise} \end{cases}$$

Noting that $\gamma = 0$ in writing the above test function due to Y being a continuous random variable. Where c_0, c_1 satisfy:

$$(1 - e^{-c_0}) + e^{-c_1} = \alpha$$

And noting to *"Skip the part involving $\alpha = 0.1$ or the Type II error part."*

Q3

Problem 8.20, Casella and Berger (2nd Edition).

Let X be a random variable whose pmf under H_0 and H_1 is given by:

x	1	2	3	4	5	6	7
$f(x\|H_0)$	0.01	0.01	0.01	0.01	0.01	0.01	0.94
$f(x\|H_1)$	0.06	0.05	0.04	0.03	0.02	0.01	0.79

Use the Neyman–Pearson Lemma to find the most powerful test for H_0 versus H_1 with size:

$$\alpha = 0.04.$$

Compute the probability of Type II Error for this test.

Hint:

It holds that:

$$\frac{f(x|H_1)}{f(x|H_0)} = 7 - x + \frac{79}{94} I(x = 7)$$

over the support $x = 1, 2, \ldots, 7$, where $I(\cdot)$ denotes the indicator function.

Answer

The likelihood ratio is given by the Hint:

$$\Lambda = \frac{f(x|H_1)}{f(x|H_0)} = 7 - x + \frac{79}{94} I(x = 7),$$

where $I(\cdot)$ is the indicator function.

Notably:

- For $x = 1, \ldots, 6$, the LR simplifies to $\Lambda = 7 - x$.
- For $x = 7$, $\Lambda = \frac{79}{94} \approx 0.84$.
- The likelihood ratio is decreasing in x, so the MP test rejects H_0 for the smallest values of x.
- The smaller the x, the larger the likelihood ratio.

Using the above information, we can directly calculate the following:

x	Λ	$f(x\|H_0)$	Cumulative P_{H_0}
1	6.00	0.01	0.01
2	5.00	0.01	0.02
3	4.00	0.01	0.03
4	3.00	0.01	0.04
5	2.00	0.01	0.05
6	1.00	0.01	0.06
7	0.84	0.94	1.00

To achieve the desired size, $\alpha = 0.04$, we consider where the cumulative probability, P_{H_0}, achieves α, which is at 4. As this is cumulative then, we have the rejection region given by:

$$R = \{1, 2, 3, 4\}$$

The Type II error probability β is the probability of not rejecting H_0 when H_1 is true:

$$\beta = P_{H_1}(X \notin R) = P_{H_1}(X = 5, 6, 7) = f(5|H_1) + f(6|H_1) + f(7|H_1) = 0.02 + 0.01 + 0.79 = 0.82$$

Giving us a Type II Error Probability of $\beta = 0.82$.

Q4

Recall Method I for finding Uniformly Most Powerful (UMP) tests:

To find a UMP size α test for $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \notin \Theta_0$, suppose we can fix $\theta_0 \in \Theta_0$ suitably and then use the Neyman–Pearson lemma to find an MP size α test $\varphi(\tilde{X})$ for:

$$H_0 : \theta = \theta_0 \quad \text{vs} \quad H_1 : \theta = \theta_1,$$

where:

a)

$\varphi(\tilde{X})$ does not depend on $\theta_1 \notin \Theta_0$, and

b)

$\max_{\theta \in \Theta_0} E_\theta \varphi(\tilde{X}) = \alpha$.

Proof

Show that if a) and b) both hold, then $\varphi(\tilde{X})$ must be a UMP size α test for $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \notin \Theta_0$.

Hint:

From b), the size of the test rule $\varphi(\tilde{X})$ is correct. So, by definition of a UMP test, it is necessary to prove that if $\bar{\varphi}(\tilde{X})$ is any other test of $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \notin \Theta_0$ with size:

$$\max_{\theta \in \Theta_0} E_\theta \bar{\varphi}(\tilde{X}) \leq \alpha,$$

then $\varphi(\tilde{X})$ has more power over the parameter subspace of H_1 than $\bar{\varphi}(\tilde{X})$, i.e.,

$$E_\theta \varphi(\tilde{X}) \geq E_\theta \bar{\varphi}(\tilde{X}) \quad \text{for any } \theta \notin \Theta_0.$$

In other words, pick/fix some $\theta_1 \notin \Theta_0$ and argue that:

$$E_{\theta_1} \varphi(\tilde{X}) \geq E_{\theta_1} \bar{\varphi}(\tilde{X})$$

must hold. The way to do this is to take the test $\bar{\varphi}(\tilde{X})$ and apply it to testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$.

Answer

Assume a) and b) hold. The goal then is to show that $\varphi(\tilde{X})$ is UMP for $H_0 : \theta \in \Theta_0$ vs. $H_1 : \theta \notin \Theta_0$.

We consider fixing the null and alternative hypotheses respectively by:

$$H_0 : \theta = \theta_0 \quad \text{vs.} \quad H_1 : \theta = \theta_1$$

Where θ_0 and θ_1 are suitable parameters belonging to Θ_0 and θ_1, again resp.

By Neyman Pearson, $\varphi(\tilde{X})$ is MP at size α for this test.

Let $\bar{\varphi}(\tilde{X})$ be another test with:

$$\sup_{\theta \in \Theta_0} E_\theta \bar{\varphi}(\tilde{X}) \leq \alpha$$

In particular, $E_{\theta_0} \bar{\varphi}(\tilde{X}) \leq \alpha$.

Since $\varphi(\tilde{X})$ is MP for $\theta = \theta_0$ vs. $\theta = \theta_1$, it satisfies:

$$E_{\theta_1} \varphi(\tilde{X}) \geq E_{\theta_1} \bar{\varphi}(\tilde{X})$$

Given condition a) holds then, we know that $\varphi(\tilde{X})$ does not depend on θ_1. Thus, the inequality holds for all $\theta_1 \notin \Theta_0$, proving $\varphi(\tilde{X})$ is UMP.

An Alternative Approach

I believe there is also another approach via a proof by contradiction. To that end:

To start, assume (for contradiction) that $\varphi(\tilde{X})$ is not UMP of size α, yet still meets conditions a) and b).

Then $\exists$ a test $\bar{\varphi}(\tilde{X})$ such that:

- $\sup_{\theta \in \Theta_0} E_\theta[\bar{\varphi}(\tilde{X})] \leq \alpha$ (level α),
- $\exists \theta_1 \notin \Theta_0$ with $E_{\theta_1}[\bar{\varphi}(\tilde{X})] > E_{\theta_1}[\varphi(\tilde{X})]$.

Fix $\theta_0 \in \Theta_0$ where size α is attained:

- By condition b), $E_{\theta_0}[\varphi(\tilde{X})] = \alpha$.
- We also know $E_{\theta_0}[\bar{\varphi}(\tilde{X})] \leq \alpha$.

Via Neyman-Pearson for $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$, $\varphi(\tilde{X})$ is MP of size α for this test (via condition a) and Neyman-Pearson).

However, $\bar{\varphi}(\tilde{X})$ has:

- Size $\leq \alpha$ (since $E_{\theta_0}[\bar{\varphi}] \leq \alpha$),
- Higher power at θ_1 (since $E_{\theta_1}[\bar{\varphi}] > E_{\theta_1}[\varphi]$).

This is a contradiction, as Neyman-Pearson guarantees no such $\bar{\varphi}$ can exist (any other MP test with the same size cannot have higher power!)

Thus, we conclude that $\varphi(\tilde{X})$ is UMP of size α.

Q5

Problem 8.23, Casella and Berger (2nd Edition).

Suppose X is one observation from a population with Beta$(\theta, 1)$ pdf.

a)

For testing:

$$H_0 : \theta \leq 1 \quad \text{versus} \quad H_1 : \theta > 1,$$

find the size and sketch the power function of the test that rejects H_0 if:

$$X > \frac{1}{2}.$$

Answer

The power function, $\beta(\theta)$, is by definition the probability of rejecting H_0 under a given θ:

$$\beta(\theta) = P_\theta\left(X > \frac{1}{2}\right) = \int_{1/2}^{1} \theta x^{\theta-1}\, dx = x^\theta\big|_{1/2}^{1} = 1 - \left(\frac{1}{2}\right)^\theta = 1 - \frac{1}{2^\theta}$$

Then, the size, is by definition the supremum of $\beta(\theta)$ under H_0 $(\theta \leq 1)$.

Since $\beta(\theta)$ is strictly increasing in θ (because $\beta'(\theta) = \ln(2) \cdot 2^{-\theta} > 0$), the maximum occurs at $\theta = 1$, which is:

$$\sup_{\theta \leq 1} \beta(\theta) = \beta(1) = 1 - \frac{1}{2} = \frac{1}{2}$$

Expectation of the sketch: $(\theta = 1, \beta(1) = 0.5)$.

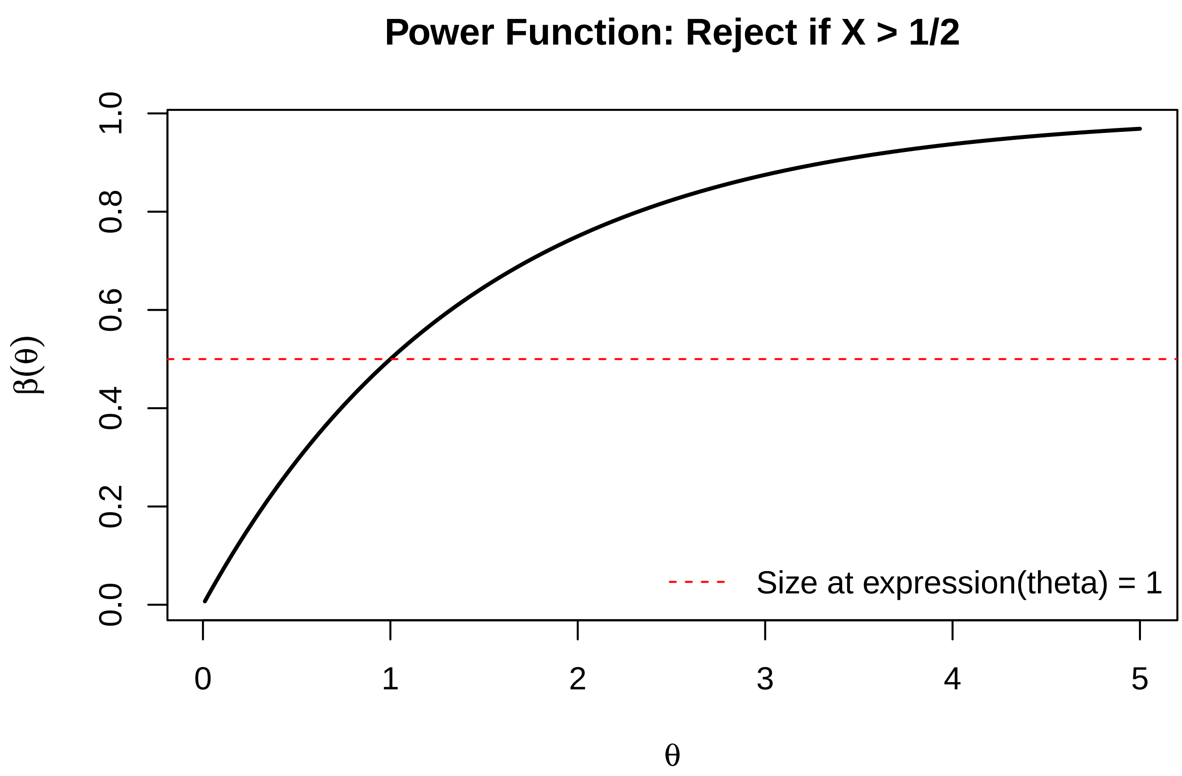
A Sketch

```r
# setup
theta_vals <- seq(0.01, 5, length.out = 300)
power_vals <- 1 - (1 / 2^theta_vals)

# plotting
plot(theta_vals, power_vals, type = "l", lwd = 2,
     xlab = expression(theta), ylab = expression(beta(theta)),
     main = "Power Function: Reject if X > 1/2")

abline(h = 0.5, col = "red", lty = 2)

legend("bottomright", legend = c("Size at expression(theta) = 1"),
       col = c("red"), lty = 2, bty = "n")
```



b)

Find the most powerful level-α test of:

$$H_0 : \theta = 1 \quad \text{versus} \quad H_1 : \theta = 2.$$

Answer

We find the MP test via Neyman-Pearson: The MP test rejects H_0 when the likelihood ratio exceeds a threshold k.

Our likelihood ratio is given by:

$$\Lambda = \frac{f(x \mid \theta = 2)}{f(x \mid \theta = 1)} = \frac{2x^{2-1}}{1x^{1-1}} = 2x$$

The test rejects H_0 when $\Lambda = 2x > k \to x > \frac{k}{2} = t$.

Using the above, the size constraint requires:

$$P_{\theta=1}(X > t) = \alpha$$

For $\theta = 1$, $X \sim \text{Uniform}(0, 1)$, and the probability can be explicitly evaluated and solved for t:

$$P(X > t) = 1 - t = \alpha \to t = 1 - \alpha$$

Taken together, the most powerful level-α test is given by:

$$\phi_{H_0}(X) = \begin{cases} 1 & X > 1 - \alpha \\ 0 & \text{otherwise} \end{cases}$$

Note: The above is under the assumption that U is the continuous normal distribution, meaning we can have $\gamma = 0$ for the "coin toss" scenario in the test function.

c)

Is there a UMP test of:

$$H_0 : \theta \leq 1 \quad \text{versus} \quad H_1 : \theta > 1$$

If so, find it. If not, prove so.

Answer

We start by checking whether the likelihood ratio is monotonic. To that end, for $\theta_2 > \theta_1$, the likelihood ratio is given by:

$$\Lambda = \frac{f(x|\theta_2)}{f(x|\theta_1)} = \frac{\theta_2}{\theta_1} x^{\theta_2 - \theta_1}$$

Since $\theta_2 - \theta_1 > 0$ and $x \in (0, 1)$, the function $x^{\theta_2 - \theta_1}$ is increasing in x, which in turn means the likelihood ratio Λ is increasing in x, which in turn means the family has a monotone likelihood ratio in x.

Importantly, the family having a MLR in x allows us to utilize Karlin-Rubin, i.e. the test that rejects for large values of X is UMP for $H_0 : \theta \leq 1$ vs $H_1 : \theta > 1$.

We choose t such that:

$$\sup_{\theta \leq 1} P_\theta(X > t) = \alpha$$

Under $\theta = 1$ (where the sup is attained), $X \sim \text{Uniform}(0, 1)$, so:

$$P(X > t) = 1 - t = \alpha \to t = 1 - \alpha$$

From part b), the MP test for $\theta = 1$ vs $\theta = 2$ was given by:

$$\phi_{H_0}(X) = \begin{cases} 1 & X > 1 - \alpha \\ 0 & \text{otherwise} \end{cases}$$

Which notably does not include the θ value! Because of this, it is UMP for all $\theta > 1$.

So the UMP level-α test is the same as the MP test in part b):

$$\phi(X) = \begin{cases} 1 & \text{if } X > 1 - \alpha \\ 0 & \text{otherwise} \end{cases}$$