MATH 392 Problem Set 6

Sam D. Olson

Exercise from the book

9.1 #1

Q: Let X have the exponential distribution with parameter β. Suppose that we wish to test the hypotheses $H_0 : \beta \geq 1$ versus $H_1 : \beta < 1$. Consider the test procedure δ that rejects H_0 if $X \geq 1$.

(A)

Determine the power function of the test.

Let δ demote a test, specifically:

δ :{Reject H_0 when $X \geq 1$}

Then for $\beta > 0$ we note the power function as follows, $\pi \left(\beta \mid \delta\right) = Pr\left(X \geq 1 \mid \beta\right) = e^{-\beta}$

(B)

Compute the size of the test.

By the definition of size, we have:

$size(\delta) = sup_{\beta \geq 1} \pi \left(\beta \mid \delta\right)$

Note the power function from **(A)**, $\pi \left(\beta \mid \delta\right) = e^{-\beta}$, such that $\pi \left(\beta \mid \delta\right)$ is a decreasing function of β. Thus, the power function takes a maximum at $\beta = 1$, and we may say:

The size of the test is $\pi \left(\beta = 1 \mid \delta\right) = e^{-1}$.

9.1 #2

Q: Suppose that $X_1, ..., X_n$ form a random sample from the uniform distribution on the interval $[0, \theta]$, and that the following hypotheses are to be tested:

$H_0 : \theta \geq 2$

$H_1 : \theta < 2$

Let $Y_n = max\left(X_1, ..., X_n\right)$ and consider a test procedure such that the critical region contains all the outcomes for which $Y_n \leq 1.5$.

(A)

Determine the power function of the test.

Note, for $0 < y < \theta$, then $Pr\left(Y_n \leq y\right) = \left(\frac{y}{\theta}\right)^n$.

Additionally note, if $y \geq \theta$, then $Pr\left(Y_n \leq y\right) = 1$.

Thus, for the stated condition, if $\theta \leq 1.5$, then $\pi(\theta) = Pr(Y_n \leq 1.5) = 1$.

And if $\theta > 1.5$, then $\pi(\theta) = Pr(Y_n \leq 1.5) = \left(\frac{1.5}{\theta}\right)^n$

(B)

Determine the size of the test.

The size of the test, $size(\delta) = \alpha = sup_{\theta \geq 2} \pi(\theta) = sup_{\theta \geq 2} \left(\frac{1.5}{\theta}\right)^n = \left(\frac{1.5}{2}\right)^n = \left(\frac{3}{4}\right)^n$.

9.1 #14 [To Do]

Plus: plot power functions in R

Q: Let $X_1, ..., X_n$ be i.i.d. with exponential distribution with parameter θ. Suppose that we wish to test the hypotheses:

$H_0 : \theta \geq \theta_0$

$H_1 : \theta < \theta_0$

Let $X = \sum_{i=1}^{n} X_i$. Let δ_c be the test that rejects H_0 if $X \geq c$.

(A)

Show that $\pi(\theta \mid \delta_c)$ is a decreasing function of θ

(B)

Find c in order to make δ_c have size α_0.

(C)

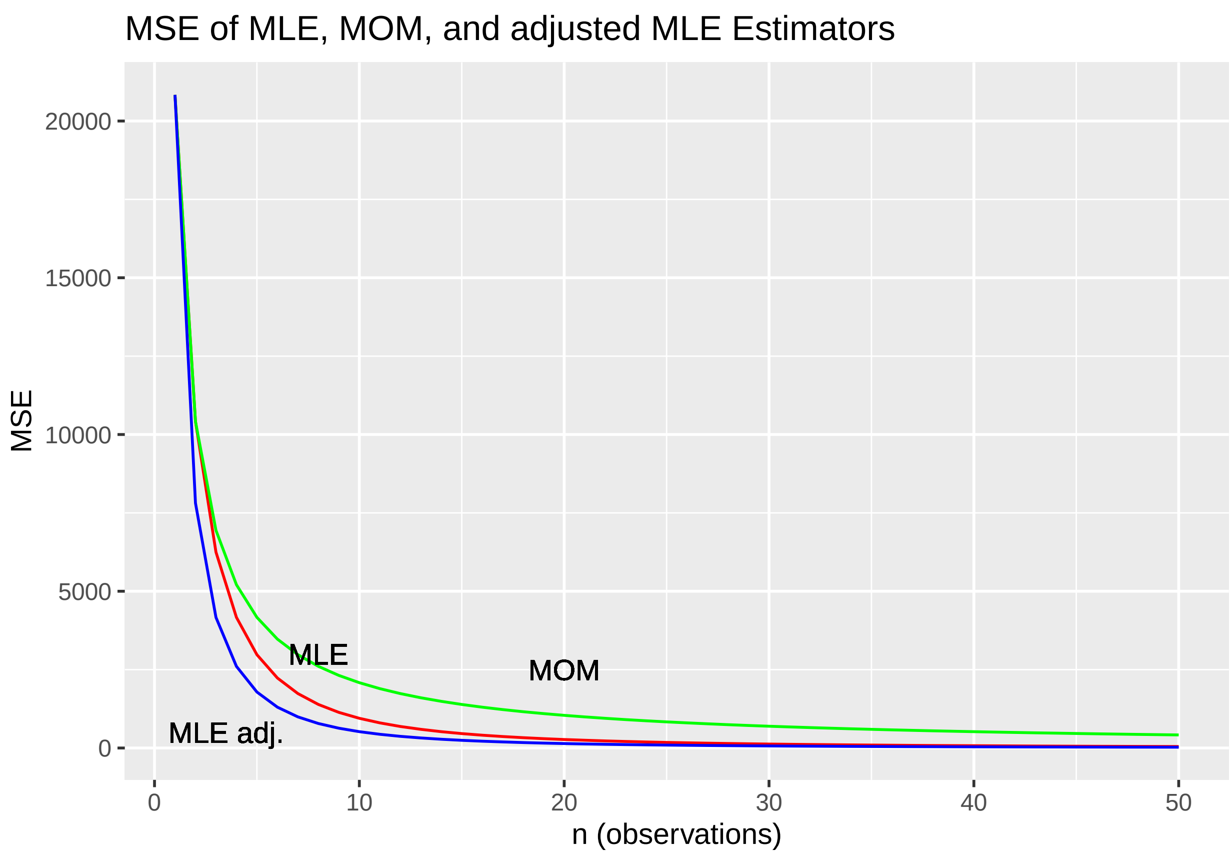
Let $\theta_0 = 2$, $n = 1$, and $\alpha_0 = 0.1$. Find the precise form of the test δ_c and sketch its power function.

```r
theta <- 250
n <- (1:50)

mle_mse <- rep(NA, 50)
mom_mse <- rep(NA, 50)
mle_adj_mse <- rep(NA, 50)
for (i in 1:50){
  mle_mse[i] <- ((theta ^ 2) * 2 )/ ((n[i] + 1) * (n[i] + 2))
  mom_mse[i] <- (theta ^ 2) / (3 * n[i])
  mle_adj_mse[i] <- (theta ^ 2) / ((n[i] * (n[i] +2)))
}

df <- data.frame(n,mle_mse,mom_mse, mle_adj_mse)

ggplot(df, aes(n)) +
  geom_line(aes(y=mle_mse), colour="red") +
  geom_line(aes(y=mom_mse), colour="green") +
  geom_line(aes(y=mle_adj_mse), colour="blue") +
  geom_text(x=3.5, y=500, label="MLE adj.") +
  geom_text(x=20, y=2500, label="MOM") +
  geom_text(x=8, y=3000, label="MLE") +
  ggtitle("MSE of MLE, MOM, and adjusted MLE Estimators") +
  labs(y="MSE", x = "n (observations)")
```



9.2 #2

Q: Consider two p.d.f.'s $f_0(x)$ and $f_1(x)$ that are defined as follows:

$$f_0(x)) = \begin{Bmatrix} 1\ for\ 0 \le x \le 1 \\ 0\ \ otherwise \end{Bmatrix}$$

and

$$f_1(x) = \begin{Bmatrix} 2x\ for\ 0 \le x \le 1 \\ 0\ \ otherwise \end{Bmatrix}$$

Suppose that a single observation X is taken from a distribution for which the p.d.f. $f(x)$ is either $f_0(x)$ or $f_1(x)$, and the following simple hypotheses are to be tested:

$H_0 : f(x) = f_0(x)$

$H_1 : f(x) = f_0(x)$

(A)

Describe a test procedure δ for which the value of $\alpha(\delta) + 2\beta(\delta)$ is a minimum.

Using Thm. 9.2.1 with $a - 1, b = 2$, we have:

δ :{Do not reject H_0 if $\frac{f_1(x)}{f_0(x)} < \frac{1}{2}$}

Note, as $\frac{f_1(x)}{f_0(x)} = 2x \to 2x < \frac{1}{2}$

Thus we may say the procedure is: Do not reject H_0 if $x < \frac{1}{4}$, and likewise to reject H_0 if $x > \frac{1}{4}$.

(B)

Determine the minimum value of $\alpha(\delta) + 2\beta(\delta)$ attained by that procedure.

Note the following formulations:

$\alpha(\delta) = Pr(RejectH_0 \mid f_0) = \int_{\frac{1}{4}}^{1} f_0(x)dx = \frac{3}{4}$

$\beta(\delta) = Pr(FailtoRejectH_0 \mid f_1) = \int_{0}^{\frac{1}{4}} 2xdx = \frac{1}{16}$

Therefore, we have:

$\alpha(\delta) + 2\beta(\delta) = \frac{3}{4} + 2\frac{1}{16} = \frac{7}{8}$

9.2 #3

Q: Consider again the conditions of Exercise 2 (9.2.2), but suppose now that it is desired to find a test procedure for which the value of $3\alpha(\delta) + \beta(\delta)$ is a minimum.

(A)

Determine the procedure.

Using Thm. 9.2.1. with $a = 3, b = 1$, we have:

δ :{Do not reject H_0 if $\frac{f_1(x)}{f_0(x)} < 3 \rightarrow 2x < 3$}

Since all values of X lie in the interval (0, 1), the optimal procedure is to not reject H_0 for every possible observed value (as $max(X) = 1 \rightarrow 2 \cdot max(X) = 2 < 3$).

(B)

Determine the minimum value of $3\alpha(\delta) + \beta(\delta)$ attained by the procedure.

Since we never reject H_0 under the stated conditions, we have:

$\alpha(\delta) = 0, \beta(\delta) = 1$ and we thus have:

$3\alpha(\delta) + \beta(\delta) = 1$

9.2 #10 [To Do]

Q: Suppose that $X_1, ..., X_n$ form a random sample from the Poisson distribution with unknown mean λ. Let λ_0 and λ_1 be specified values such that $\lambda_1 > \lambda_0 > 0$, and suppose that it is desired to test the following simple hypotheses:

$H_0 : \lambda = \lambda_0$

$H_1 : \lambda = \lambda_1$

(A)

Show that the value of $\alpha(\delta) + \beta(\delta)$ is minimized by a test procedure which rejects H_0 when $\bar{X}_n > c$.

(B)

Find the value of c.

(C)

For $\lambda_0 = \frac{1}{4}$, $\lambda_1 = \frac{1}{2}$, and $n = 20$, determine the minimum value of $\alpha(\delta) + \beta(\delta)$ that can be attained.

9.3 #1

Q: Suppose that $X_1, ..., X_n$ form a random sample from the Poisson distribution with unknown mean λ $(\lambda > 0)$. Show that the joint p.f. of $X_1, ..., X_n$ has a monotone likelihood ratio in the statistic $\sum\limits_{i=1}^{n} X_i$.

Let $y = \sum\limits_{i=1}^{n} x_i$

We may note the joint p.f. of the data as:

$f_n(\bar{X} \mid \lambda) = f_n(X_1 \mid \lambda)...f_n(X_n \mid \lambda) = \frac{e^{-\lambda}\lambda^{x_1}}{x_1!}...\frac{e^{-\lambda}\lambda^{x_n}}{x_n!} = \frac{e^{-n\lambda}\lambda^{y}}{\prod\limits_{i=1}^{n} x_i!}$

Thus, for $0 < \lambda_1 < \lambda_2$, we have:

$\frac{f_n(\bar{X}|\lambda_2)}{f_n(\bar{X}|\lambda_1)} = \frac{\frac{e^{-n\lambda_2}\lambda_2^{y}}{\prod\limits_{i=1}^{n} x_i!}}{\frac{e^{-n\lambda_1}\lambda_1^{y}}{\prod\limits_{i=1}^{n} x_i!}}$

Noting the factorials of x_i cancel out, we then have:

$\frac{f_n(\bar{X}|\lambda_2)}{f_n(\bar{X}|\lambda_1)} = e^{-n(\lambda_2 - \lambda_1)} \cdot \left(\frac{\lambda_2}{\lambda_1}\right)^{y}$

Note, as $0 < \lambda_1 < \lambda_2$, this is an increasing function of y and we conclude the joint p.f. of $X_1, ..., X_n$ has a monotone likelihood ratio in the statistic $\sum\limits_{i=1}^{n} X_i$.

9.3 #2

Q: Suppose that $X_1, ..., X_n$ form a random sample from the normal distribution with known mean μ and unknown variance σ^2 $\left(\sigma^2 > 0\right)$. Show that the joint p.d.f. of $X_1, ..., X_n$ has a monotone likelihood ratio in the statistic $\sum\limits_{i=1}^{n} (X_i - \mu)^2$.

Let $y = \sum\limits_{i=1}^{n} (x_i - \mu)^2$

We may then note the joint p.d.f. of the data as:

$f_n(\bar{X} \mid \sigma^2) = f_n(X_1 \mid \sigma^2)...f_n(X_n \mid \sigma^2) = \frac{1}{\sqrt{(2\pi\sigma^2)}} \cdot e^{\frac{-x_1}{2\sigma^2}} ... \frac{1}{\sqrt{(2\pi\sigma^2)}} \cdot e^{\frac{-x_n}{2\sigma^2}}$

Simplifying this expression gives us:

$f_n(\bar{X} \mid \sigma^2) = \frac{1}{\sqrt{(2\pi\sigma^2)}^{n}} \cdot e^{\frac{-y}{2\sigma^2}}$

Thus, noting the condition $0 < \sigma_1^2 < \sigma_2^2$, we have:

$\frac{f_n(\bar{X}|\sigma_2^2)}{f_n(\bar{X}|\sigma_1^2)} = \frac{\frac{1}{\sqrt{\left(2\pi\sigma_2^2\right)}^{n}} \cdot e^{\frac{-y}{2\sigma_2^2}}}{\frac{1}{\sqrt{\left(2\pi\sigma_1^2\right)}^{n}} \cdot e^{\frac{-y}{2\sigma_1^2}}}$

Simplifying this expression then gives us:m

$\frac{f_n(\bar{X}|\sigma_2^2)}{f_n(\bar{X}|\sigma_1^2)} = \left(\frac{\sigma_1}{\sigma_2}\right)^{n} \cdot e^{\frac{y}{2} \cdot \left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_2^2}\right)}$

Noting the prior condition that $0 < \sigma_1^2 < \sigma_2^2 \rightarrow \frac{1}{\sigma_1^2} > \frac{1}{\sigma_2^2}$, and we may say the above $\left(\frac{f_n(\bar{X}|\sigma_2^2)}{f_n(\bar{X}|\sigma_1^2)}\right)$ is an increasing function of y.

We may then conclude that the joint p.d.f. of $X_1, ..., X_n$ has a monotone likelihood ratio in the statistic $\sum\limits_{i=1}^{n} (X_i - \mu)^2$.

9.3 #13 [To Do]

Q: Suppose that four observations are taken at random from the normal distribution with unknown mean μ and known variance 1. Suppose also that the following hypotheses are to be tested:

$H_0 : \mu \geq 10$

$H_1 : \mu < 10$

(A)

Determine a UMP test at the level of significance $\alpha_0 = 0.1$

(B)

Determine the power of this test when $\mu = 9$.

(C)

Determine the probability of not rejecting H_0 if $\mu = 11$.