

MATH 392 Problem Set 6

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Time Spent

Total: 7 hours

Book Exercises: $6\frac{1}{2}$ hours

R Coding: $\frac{1}{2}$ hours

Exercise from the book

9.1 #1

Q: Let X have the exponential distribution with parameter β . Suppose that we wish to test the hypotheses $H_0 : \beta \geq 1$ versus $H_1 : \beta < 1$. Consider the test procedure δ that rejects H_0 if $X \geq 1$.

(A)

Determine the power function of the test.

Let δ denote a test procedure, and define δ as follows:

$\delta : \{\text{Reject } H_0 \text{ when } X \geq 1\}$

Then for $\beta > 0$ we note the power function as follows:

$$\pi(\beta | \delta) = Pr(X \geq 1 | \beta) = e^{-\beta}$$

(B)

Compute the size of the test.

By the definition of size, we have:

$$size(\delta) = \sup_{\beta \geq 1} \pi(\beta | \delta)$$

Note the power function from (A), $\pi(\beta | \delta) = e^{-\beta}$.

Note that $\pi(\beta | \delta)$ is a decreasing function of β . Thus, given the condition $\beta \geq 1$, the power function takes a maximum at $\beta = 1$, and we may say:

The size of the test is $\pi(\beta = 1 | \delta) = e^{-1} \approx 0.37$.

9.1 #2

Q: Suppose that X_1, \dots, X_n form a random sample from the uniform distribution on the interval $[0, \theta]$, and that the following hypotheses are to be tested:

$$H_0 : \theta \geq 2$$

$$H_1 : \theta < 2$$

Let $Y_n = \max(X_1, \dots, X_n)$ and consider a test procedure such that the critical region contains all the outcomes for which $Y_n \leq 1.5$.

(A)

Determine the power function of the test.

Note, for $0 < y < \theta$, then $Pr(Y_n \leq y) = \left(\frac{y}{\theta}\right)^n$.

Additionally note, if $y \geq \theta$, then $Pr(Y_n \leq y) = 1$.

Thus, for the stated condition, if $\theta \leq 1.5$, then $\pi(\theta) = Pr(Y_n \leq 1.5) = 1$.

And if $\theta > 1.5$, then $\pi(\theta) = Pr(Y_n \leq 1.5) = \left(\frac{1.5}{\theta}\right)^n$

(B)

Determine the size of the test.

The size of the test is given as follows: $size(\delta) = \alpha = \sup_{\theta \geq 2} \pi(\theta) = \sup_{\theta \geq 2} \left(\frac{1.5}{\theta}\right)^n = \left(\frac{1.5}{2}\right)^n = \left(\frac{3}{4}\right)^n$.

9.1 #14

Q: Let X_1, \dots, X_n be i.i.d. with exponential distribution with parameter θ . Suppose that we wish to test the hypotheses:

$$H_0 : \theta \geq \theta_0$$

$$H_1 : \theta < \theta_0$$

Let $X = \sum_{i=1}^n X_i$. Let δ_c be the test that rejects H_0 if $X \geq c$.

(A)

Show that $\pi(\theta | \delta_c)$ is a decreasing function of θ

We take advantage of the location parameters, noting:

As $X_1, \dots, X_n \sim Exp(\theta)$,

$X \sim Gamma(n, \theta)$, and

$Y = \theta X \sim Gamma(n, 1)$

Let us then note the c.d.f. of Y , G_n .

We then note the power function of δ_c as:

$$\pi(\theta | \delta_c) = Pr(X \geq c | \theta) = Pr(Y \geq c\theta | \theta) = 1 - G_n(c\theta)$$

Note that G_n is an increasing function of θ and $c\theta$ is an increasing function of θ , making $\pi(\theta | \delta_c) = 1 - G_n(c\theta)$ a decreasing function of θ .

(B)

Find c in order to make δ_c have size α_0 .

Note, to find c we set the above power function equal to the size, giving us:

$$\pi(\theta_0 | \delta_c) = 1 - G_n(c\theta_0) = \alpha_0$$

Taking the inverse c.d.f. allows us to solve for c , giving us:

$$1 - \alpha_0 = G_n(c\theta_0)$$

$$G_n^{-1}(1 - \alpha_0) = c\theta_0$$

$$c = \frac{G_n^{-1}(1 - \alpha_0)}{\theta_0}$$

(C)

Let $\theta_0 = 2$, $n = 1$, and $\alpha_0 = 0.1$. Find the precise form of the test δ_c and sketch its power function.

We have:

$$G_n(y) = 1 - e^{-y},$$

$$G_n^{-1} = -\log(1 - p)$$

Using our formulation of c from (B) gives us:

$$c = \frac{-\log(0.1)}{2} \approx 1.15$$

We may then plot the relationship between θ and the power function, given below.

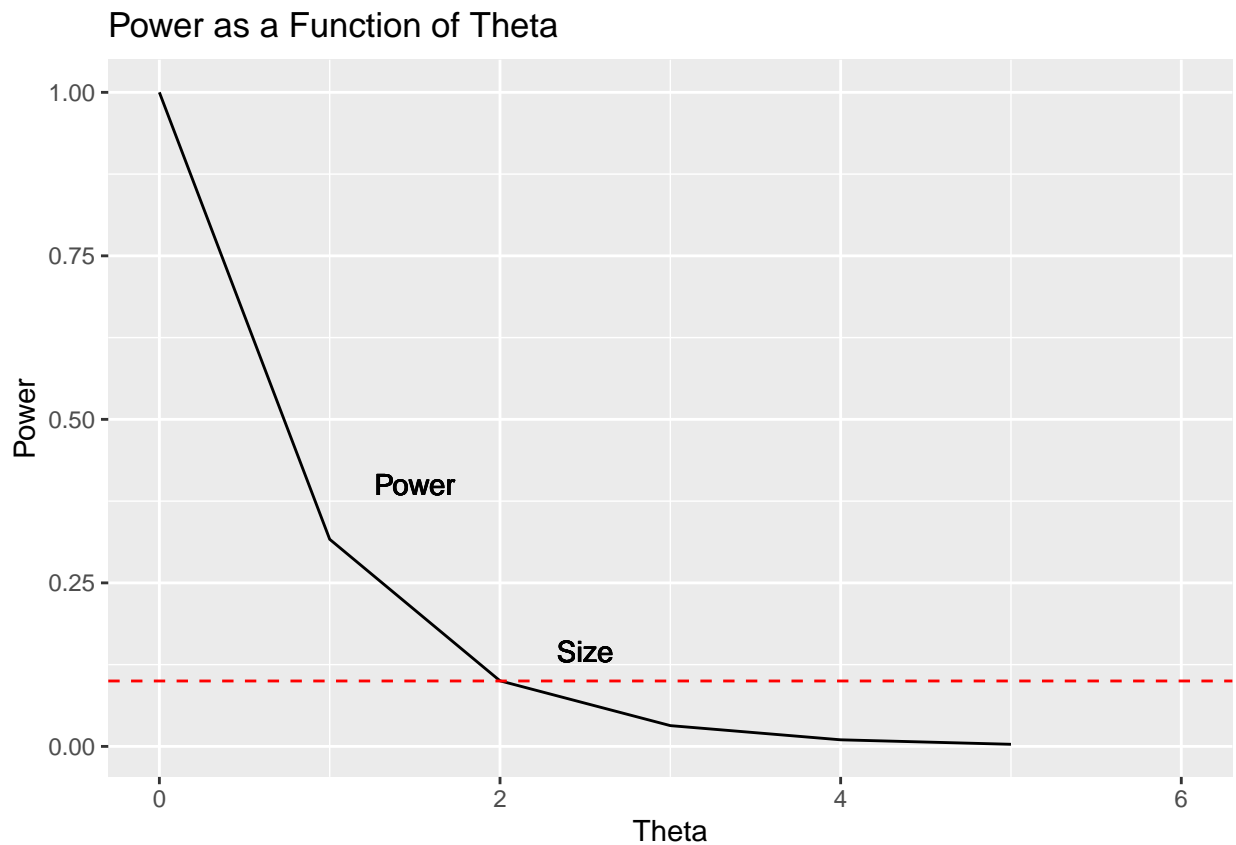
```
theta <- (0:6)

power <- rep(NA, 7)

for(i in 0:6) {
  power[i] <- exp((-1)* 1.15 * theta[i])
}

df <- data.frame(theta,power)

ggplot(df, aes(theta)) +
  geom_line(aes(y=power), colour="black") +
  geom_text(x=1.5, y=0.4, label="Power") +
  ggtitle("Power as a Function of Theta") +
  geom_text(x=2.5, y=0.145, label="Size") +
  geom_hline(yintercept=0.1, linetype="dashed", color = "red") +
  labs(y="Power", x = "Theta")
```



9.2 #2

Q: Consider two p.d.f.'s $f_0(x)$ and $f_1(x)$ that are defined as follows:

$$f_0(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_1(x) = \begin{cases} 2x & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Suppose that a single observation X is taken from a distribution for which the p.d.f. $f(x)$ is either $f_0(x)$ or $f_1(x)$, and the following simple hypotheses are to be tested:

$$H_0 : f(x) = f_0(x)$$

$$H_1 : f(x) = f_1(x)$$

(A)

Describe a test procedure δ for which the value of $\alpha(\delta) + 2\beta(\delta)$ is a minimum.

Using Thm. 9.2.1 with $a = 1, b = 2$, we have:

$$\delta : \{\text{Do not reject } H_0 \text{ if } \frac{f_1(x)}{f_0(x)} < \frac{a}{b} = \frac{1}{2}\}$$

$$\text{Note, as } \frac{f_1(x)}{f_0(x)} = 2x \rightarrow 2x < \frac{1}{2}$$

Thus we may say the procedure is: Do not reject H_0 if $x < \frac{1}{4}$, and likewise to reject H_0 if $x > \frac{1}{4}$.

(B)

Determine the minimum value of $\alpha(\delta) + 2\beta(\delta)$ attained by that procedure.

Note the following formulations:

$$\alpha(\delta) = Pr(\text{Reject } H_0 \mid f_0) = \int_{\frac{1}{4}}^1 f_0(x) dx = \frac{3}{4}$$

$$\beta(\delta) = Pr(\text{Fail to Reject } H_0 \mid f_1) = \int_0^{\frac{1}{4}} 2x dx = \frac{1}{16}$$

Therefore, we have:

$$\alpha(\delta) + 2\beta(\delta) = \frac{3}{4} + 2 \cdot \frac{1}{16} = \frac{7}{8}$$

9.2 #3

Q: Consider again the conditions of Exercise 2 (9.2.2), but suppose now that it is desired to find a test procedure for which the value of $3\alpha(\delta) + \beta(\delta)$ is a minimum.

(A)

Determine the procedure.

Using Thm. 9.2.1. with $a = 3, b = 1$, we have:

$$\delta : \{\text{Do not reject } H_0 \text{ if } \frac{f_1(x)}{f_0(x)} < \frac{a}{b} = 3 \rightarrow 2x < 3 \rightarrow x < \frac{3}{2}\}$$

Since all values of X lie in the interval $(0, 1)$, and since $x < \frac{3}{2}$, the optimal procedure is to not reject H_0 for every possible observed value, noting as $\max(X) = 1 \rightarrow 2 \cdot \max(X) = 2 < 3$.

(B)

Determine the minimum value of $3\alpha(\delta) + \beta(\delta)$ attained by the procedure.

Since we never reject H_0 under **(A)**, we have:

$\alpha(\delta) = 0, \beta(\delta) = 1$ and we thus have:

$$3\alpha(\delta) + \beta(\delta) = 0 + 1 = 1$$

9.2 #10

Q: Suppose that X_1, \dots, X_n form a random sample from the Poisson distribution with unknown mean λ . Let λ_0 and λ_1 be specified values such that $\lambda_1 > \lambda_0 > 0$, and suppose that it is desired to test the following simple hypotheses:

$$H_0 : \lambda = \lambda_0$$

$$H_1 : \lambda = \lambda_1$$

(A)

Show that the value of $\alpha(\delta) + \beta(\delta)$ is minimized by a test procedure which rejects H_0 when $\bar{X}_n > c$.

Applying Thm. 9.2.1 with $a = b = 1$, we know the optimal test procedure is to reject H_0 if $\frac{f_1(\bar{X})}{f_0(\bar{X})} > 1$.

$$\text{Let } y = \sum_{i=1}^n x_i$$

We may note the joint p.f. of the data as:

$$f_n(\bar{X} | \lambda) = f_n(X_1 | \lambda) \dots f_n(X_n | \lambda) = \frac{e^{-\lambda} \lambda^{x_1}}{x_1!} \dots \frac{e^{-\lambda} \lambda^{x_n}}{x_n!} = \frac{e^{-n\lambda} \lambda^y}{\prod_{i=1}^n x_i!}$$

Thus we have:

$$f_0(\bar{X}) = \frac{e^{-n\lambda_0} \lambda_0^y}{\prod_{i=1}^n x_i!}$$

$$f_1(\bar{X}) = \frac{e^{-n\lambda_1} \lambda_1^y}{\prod_{i=1}^n x_i!}$$

It then follows that:

$$\frac{f_1(\bar{X})}{f_0(\bar{X})} = e^{-n(\lambda_1 - \lambda_0)} \cdot \left(\frac{\lambda_1}{\lambda_0} \right)^y$$

Taking the log of this gives us:

$$\log \left(\frac{f_1(\bar{X})}{f_0(\bar{X})} \right) = y \log \left(\frac{\lambda_1}{\lambda_0} \right) - n(\lambda_1 - \lambda_0)$$

Note by construction $\lambda_1 > \lambda_0$. Therefore, we know:

$$\frac{f_1(\bar{X})}{f_0(\bar{X})} > 1 \iff y > \frac{n(\lambda_1 - \lambda_0)}{\log \lambda_1 - \log \lambda_0}$$

(B)

Find the value of c.

$$\text{Note that } \bar{x}_n = \frac{y}{n}.$$

Utilizing the above and the formulations from **(A)**, we want a test procedure which rejects H_0 when $\bar{X}_n > c$, giving us: $\log(c) = y \log \left(\frac{\lambda_1}{\lambda_0} \right) - n(\lambda_1 - \lambda_0)$

Giving us a value of c defined as:

$$c = \left(\frac{\lambda_1}{\lambda_0} \right)^y - (\lambda_1 - \lambda_0)^n$$

(C)

For $\lambda_0 = \frac{1}{4}$, $\lambda_1 = \frac{1}{2}$, and $n = 20$, determine the minimum value of $\alpha(\delta) + \beta(\delta)$ that can be attained.

If H_0 is true, then $Y = \sum_{i=1}^n X_i \sim \text{Poisson}(\lambda_0)$ with mean $n\lambda_0$

And if H_1 is true, then $Y = \sum_{i=1}^n X_i \sim \text{Poisson}(\lambda_1)$ with mean $n\lambda_1$

Noting the specified conditions, $\lambda_0 = \frac{1}{4}$, $\lambda_1 = \frac{1}{2}$, and $n = 20$, we have:

$$\frac{n(\lambda_1 - \lambda_0)}{(\log \lambda_1 - \log \lambda_0)} = \frac{20 \cdot 0.25}{\log(0.50) - \log(0.25)} \approx 7.21$$

We use this value in the following computations, which we are able to evaluate using Poisson distribution tables, but first we must path our tithe and say:

All Praise the Glorious Tables in the Back of the Book

With tithes out of the way, note to evaluate $\alpha(\text{delta})$ we use a Poisson distribution with mean $5 = 20 \cdot 0.25$.

$$\alpha(\delta) = \Pr(Y > 7.21 \mid H_0) = \Pr(Y \geq 8 \mid H_0) \approx 0.13$$

Note: We go from 7.21 to 8 in the above formulation as the Poisson is a discrete distribution.

Similarly, to evaluate $\beta(\text{delta})$ we use a Poisson distribution with mean $10 = 20 \cdot 0.50$.

$$\beta(\delta) = \Pr(Y \leq 7.21 \mid H_1) = \Pr(Y \leq 7 \mid H_1) \approx 0.22$$

Note: Similar to the prior formulation, we go from 7.21 to 7 in the above formulation as the Poisson is a discrete distribution.

Combining these together, we have:

$$\alpha(\delta) + \beta(\delta) = 0.13 + 0.22 = 0.35 \text{ as the minimum value that can be attained.}$$

Note: For 9.3.1 and 9.3.2, we need to show that as a parameter of interest increases (e.g. λ, σ^2), then y increases as well, where $y = \sum_{i=1}^n x_i$. This is sufficient to show the joint p.f. (or p.d.f.) of X_1, \dots, X_n has a monotone likelihood ratio in the specified statistic(s).

9.3 #1

Q: Suppose that X_1, \dots, X_n form a random sample from the Poisson distribution with unknown mean λ ($\lambda > 0$). Show that the joint p.f. of X_1, \dots, X_n has a monotone likelihood ratio in the statistic $\sum_{i=1}^n X_i$.

$$\text{Let } y = \sum_{i=1}^n x_i$$

We may note the joint p.f. of the data as:

$$f_n(\bar{X} \mid \lambda) = f_n(X_1 \mid \lambda) \dots f_n(X_n \mid \lambda) = \frac{e^{-\lambda} \lambda^{x_1}}{x_1!} \dots \frac{e^{-\lambda} \lambda^{x_n}}{x_n!} = \frac{e^{-n\lambda} \lambda^y}{\prod_{i=1}^n x_i!}$$

Thus, for $0 < \lambda_1 < \lambda_2$, we have:

$$\frac{f_n(\bar{X} \mid \lambda_2)}{f_n(\bar{X} \mid \lambda_1)} = \frac{\frac{e^{-n\lambda_2} \lambda_2^y}{\prod_{i=1}^n x_i!}}{\frac{e^{-n\lambda_1} \lambda_1^y}{\prod_{i=1}^n x_i!}}$$

Noting the factorials of x_i cancel out, we then have:

$$\frac{f_n(\bar{X} \mid \lambda_2)}{f_n(\bar{X} \mid \lambda_1)} = e^{-n(\lambda_2 - \lambda_1)} \cdot \left(\frac{\lambda_2}{\lambda_1} \right)^y$$

Note, as $0 < \lambda_1 < \lambda_2$, this is an increasing function of y and we conclude the joint p.f. of X_1, \dots, X_n has a monotone likelihood ratio in the statistic $\sum_{i=1}^n X_i$.

9.3 #2

Q: Suppose that X_1, \dots, X_n form a random sample from the normal distribution with known mean μ and unknown variance σ^2 ($\sigma^2 > 0$). Show that the joint p.d.f. of X_1, \dots, X_n has a monotone likelihood ratio in the statistic $\sum_{i=1}^n (X_i - \mu)^2$.

$$\text{Let } y = \sum_{i=1}^n (x_i - \mu)^2$$

We may then note the joint p.d.f. of the data as:

$$f_n(\bar{X} | \sigma^2) = f_n(X_1 | \sigma^2) \dots f_n(X_n | \sigma^2) = \frac{1}{\sqrt{(2\pi\sigma^2)}} \cdot e^{-\frac{x_1}{2\sigma^2}} \dots \frac{1}{\sqrt{(2\pi\sigma^2)}} \cdot e^{-\frac{x_n}{2\sigma^2}}$$

Simplifying this expression gives us:

$$f_n(\bar{X} | \sigma^2) = \frac{1}{\sqrt{(2\pi\sigma^2)^n}} \cdot e^{-\frac{y}{2\sigma^2}}$$

Thus, noting the condition $0 < \sigma_1^2 < \sigma_2^2$, we have:

$$\frac{f_n(\bar{X} | \sigma_2^2)}{f_n(\bar{X} | \sigma_1^2)} = \frac{\frac{1}{\sqrt{(2\pi\sigma_2^2)^n}} \cdot e^{-\frac{y}{2\sigma_2^2}}}{\frac{1}{\sqrt{(2\pi\sigma_1^2)^n}} \cdot e^{-\frac{y}{2\sigma_1^2}}}$$

Simplifying this expression then gives us:

$$\frac{f_n(\bar{X} | \sigma_2^2)}{f_n(\bar{X} | \sigma_1^2)} = \left(\frac{\sigma_1}{\sigma_2}\right)^n \cdot e^{\frac{y}{2} \cdot \left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_2^2}\right)}$$

Noting the prior condition that $0 < \sigma_1^2 < \sigma_2^2 \rightarrow \frac{1}{\sigma_1^2} > \frac{1}{\sigma_2^2}$, and we may say the above $\left(\frac{f_n(\bar{X} | \sigma_2^2)}{f_n(\bar{X} | \sigma_1^2)}\right)$ is an increasing function of y .

We may then conclude that the joint p.d.f. of X_1, \dots, X_n has a monotone likelihood ratio in the statistic $\sum_{i=1}^n (X_i - \mu)^2$.

9.3 #13

Q: Suppose that four observations are taken at random from the normal distribution with unknown mean μ and known variance

Suppose also that the following hypotheses are to be tested:

$$H_0 : \mu \geq 10$$

$$H_1 : \mu < 10$$

(A)

Determine a UMP test at the level of significance $\alpha_0 = 0.1$

Note: via Exercise 9.3.12, a test which rejects H_0 when $\bar{X}_n \leq c$ will be a UMP test.

Thus, for a significance level $\alpha_0 = 0.1$, we want to choose a value of c that satisfies:

$$Pr(\bar{X}_n \leq c | \mu = 10) = 0.1$$

As we have four observations, we have $n = 4$. For $\mu = 10$, let us note the variable z with the standard normal distribution, specifically:

$$z = 2(\bar{X}_n - 10)$$

Referencing the above probability formula, we thus have:

$$Pr(\bar{X}_n \leq c \mid \mu = 10) = Pr(Z \leq 2(c - 10)) = 0.1$$

As the standard normal distribution is a known distribution, we may take advantage of the known formulation, namely:

$$Pr(Z \leq 2(c - 10)) = 0.1 = Pr(Z \leq -1.28)$$

Solving for c gives us:

$$2(c - 10) = -1.28 \rightarrow c = \frac{18.72}{2} = 9.36$$

(B)

Determine the power of this test when $\mu = 9$.

Using a similar formulation to **(A)**, we note:

$z = 2(\bar{X}_n - 9) \sim N(0, 1)$ (the standard normal distribution). We then formulate the power of the test as:

$$Pr(\bar{X}_n \leq 9.36 \mid \mu = 9) = Pr(Z \leq 0.718) = \Phi(0.718) \approx 0.76$$

(C)

Determine the probability of not rejecting H_0 if $\mu = 11$.

Using a similar formulation to **(A)** and **(B)**, we note:

$$z = 2(\bar{X}_n - 11) \sim N(0, 1) \text{ (the standard normal distribution).}$$

We then formulate the probability of rejecting H_0 as:

$$Pr(\bar{X}_n \leq 9.36 \mid \mu = 11) = Pr(Z \geq -3.282) = Pr(Z \leq 3.282) = \Phi(3.282) \approx 0.9995 \text{ (More digits shown due to inappropriateness of rounding)}$$