MATH 392 Problem Set 6

Sam D. Olson

Exercise from the book

9.1 #1

Q: Let X have the exponential distribution with parameter β. Suppose that we wish to test the hypotheses $H_0 : \beta \geq 1$ versus $H_1 : \beta < 1$. Consider the test procedure δ that rejects H_0 if $X \geq 1$.

(A)

Determine the power function of the test.

Let δ demote a test, specifically:

δ :{Reject H_0 when $X \geq 1$}

Then for $\beta > 0$ we note the power function as follows, $\pi\left(\beta \mid \delta\right) = Pr\left(X \geq 1 \mid \beta\right) = e^{-\beta}$

(B)

Compute the size of the test.

By the definition of size, we have:

$size(\delta) = sup_{\beta \geq 1} \pi\left(\beta \mid \delta\right)$

Note the power function from **(A)**, $\pi\left(\beta \mid \delta\right) = e^{-\beta}$, such that $\pi\left(\beta \mid \delta\right)$ is a decreasing function of β. Thus, the power function takes a maximum at $\beta = 1$, and we may say:

The size of the test is $\pi\left(\beta = 1 \mid \delta\right) = e^{-1}$.

9.1 #2

Q: Suppose that $X_1, ..., X_n$ form a random sample from the uniform distribution on the interval $[0, \theta]$, and that the following hypotheses are to be tested:

$H_0 : \theta \geq 2$

$H_1 : \theta < 2$

Let $Y_n = max\left(X_1, ..., X_n\right)$ and consider a test procedure such that the critical region contains all the outcomes for which $Y_n \leq 1.5$.

(A)

Determine the power function of the test.

Note, for $0 < y < \theta$, then $Pr\left(Y_n \leq y\right) = \left(\frac{y}{\theta}\right)^n$.

Additionally note, if $y \geq \theta$, then $Pr\left(Y_n \leq y\right) = 1$.

Thus, for the stated condition, if $\theta \leq 1.5$, then $\pi(\theta) = Pr(Y_n \leq 1.5) = 1$.

And if $\theta > 1.5$, then $\pi(\theta) = Pr(Y_n \leq 1.5) = \left(\frac{1.5}{\theta}\right)^n$

(B)

Determine the size of the test.

The size of the test, $size(\delta) = \alpha = sup_{\theta \geq 2} \pi\left(\theta\right) = sup_{\theta \geq 2} \left(\frac{1.5}{\theta}\right)^n = \left(\frac{1.5}{2}\right)^n = \left(\frac{3}{4}\right)^n$.

9.1 #14

Plus: plot power functions in R

Q: Let $X_1, ..., X_n$ be i.i.d. with exponential distribution with parameter θ. Suppose that we wish to test the hypotheses:

$H_0 : \theta \geq \theta_0$

$H_1 : \theta < \theta_0$

Let $X = \sum_{i=1}^{n} X_i$. Let δ_c be the test that rejects H_0 if $X \geq c$.

(A)

Show that $\pi\left(\theta \mid \delta_c\right)$ is a decreasing function of θ

We take advantage of the second parameter of the Gamma distribution being being a location parameter. Thus we may note:

The distribution of $X \sim Gamma\left(n, \theta\right)$

The distribution of $Y = \theta X \sim Gamma\left(n, 1\right)$

Let us then note the c.d.f. of Y, G_n.

We then note the power function of δ_c as:

$\pi\left(\theta \mid \delta_c\right) = Pr(X \geq c \mid \theta) = Pr(Y \geq c\theta \mid \theta) = 1 - G_n\left(c\theta\right)$

Note that G_n is an increasing function of θ and $c\theta$ is an increasing function of θ, making $\pi\left(\theta \mid \delta_c\right) = 1 - G_n\left(c\theta\right)$ a decreasing function of θ.

(B)

Find c in order to make δ_c have size α_0.

Note, to find c we set the above power function equal to the size, giving us:

$\pi\left(\theta_0 \mid \delta_c\right) = 1 - G_n\left(c\theta_0\right) = \alpha_0$

Taking the inverse c.d.f. allows us to solve for c, giving us:

$1 - \alpha_0 = G_n\left(c\theta_0\right)$

$G_n^{-1}\left(1 - \alpha_0\right) = c\theta_0$

$c = \frac{G_n^{-1}(1-\alpha_0)}{\theta_0}$

(C)

Let $\theta_0 = 2$, $n = 1$, and $\alpha_0 = 0.1$. Find the precise form of the test δ_c and sketch its power function.

We have:

$G_n(y) = 1 - e^{-y}$,

$G_n^{-1} = -log(1 - p)$

Using our formulation of c from **(B)** gives us:

$c = \frac{-log(0.1)}{2} \approx 1.15$

We may then plot the relationship between θ and the power function, given below.

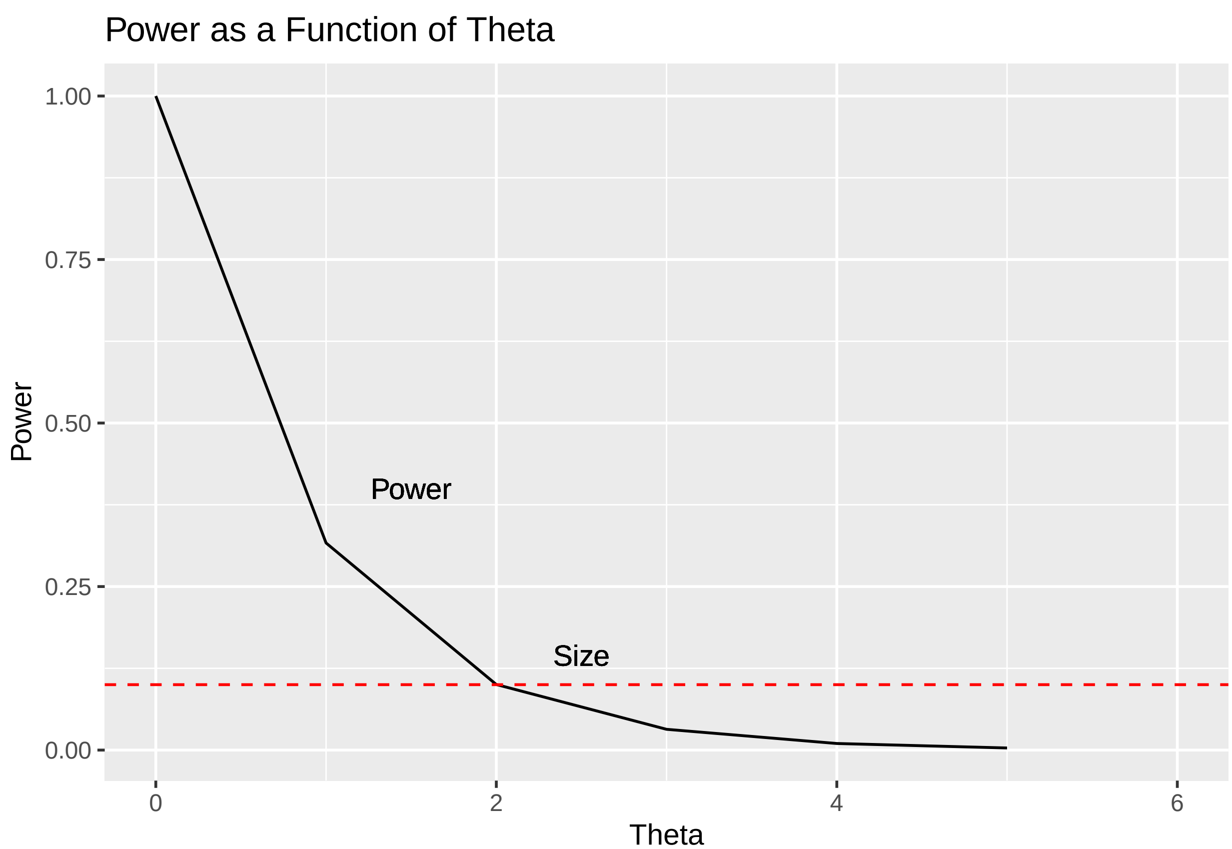
```
theta <- (0:6)

power <- rep(NA, 7)

for(i in 0:6) {
  power[i] <- exp((-1)* 1.15 * theta[i])
}

df <- data.frame(theta,power)

ggplot(df, aes(theta)) +
  geom_line(aes(y=power), colour="black") +
  geom_text(x=1.5, y=0.4, label="Power") +
  ggtitle("Power as a Function of Theta") +
  geom_text(x=2.5, y=0.145, label="Size") +
  geom_hline(yintercept=0.1, linetype="dashed", color = "red") +
  labs(y="Power", x = "Theta")
```



9.2 #2

Q: Consider two p.d.f.'s $f_0(x)$ and $f_1(x)$ that are defined as follows:

$$f_0(x) = \begin{Bmatrix} 1 \ for \ 0 \leq x \leq 1 \\ 0 \ \ otherwise \end{Bmatrix}$$

and

$$f_1(x) = \begin{Bmatrix} 2x \ for \ 0 \le x \le 1 \\ 0 \ \ otherwise \end{Bmatrix}$$

Suppose that a single observation X is taken from a distribution for which the p.d.f. $f(x)$ is either $f_0(x)$ or $f_1(x)$, and the following simple hypotheses are to be tested:

$H_0 : f(x) = f_0(x)$

$H_1 : f(x) = f_0(x)$

(A)

Describe a test procedure δ for which the value of $\alpha(\delta) + 2\beta(\delta)$ is a minimum.

Using Thm. 9.2.1 with $a - 1, b = 2$, we have:

δ :{Do not reject H_0 if $\frac{f_1(x)}{f_0(x)} < \frac{1}{2}$}

Note, as $\frac{f_1(x)}{f_0(x)} = 2x \to 2x < \frac{1}{2}$

Thus we may say the procedure is: Do not reject H_0 if $x < \frac{1}{4}$, and likewise to reject H_0 if $x > \frac{1}{4}$.

(B)

Determine the minimum value of $\alpha(\delta) + 2\beta(\delta)$ attained by that procedure.

Note the following formulations:

$\alpha(\delta) = Pr(RejectH_0 \mid f_0) = \int\limits_{\frac{1}{4}}^{1} f_0(x)dx = \frac{3}{4}$

$\beta(\delta) = Pr(FailtoRejectH_0 \mid f_1) = \int\limits_{0}^{\frac{1}{4}} 2xdx = \frac{1}{16}$

Therefore, we have:

$\alpha(\delta) + 2\beta(\delta) = \frac{3}{4} + 2\frac{1}{16} = \frac{7}{8}$

9.2 #3

Q: Consider again the conditions of Exercise 2 (9.2.2), but suppose now that it is desired to find a test procedure for which the value of $3\alpha(\delta) + \beta(\delta)$ is a minimum.

(A)

Determine the procedure.

Using Thm. 9.2.1. with $a = 3, b = 1$, we have:

δ :{Do not reject H_0 if $\frac{f_1(x)}{f_0(x)} < 3 \to 2x < 3$}

Since all values of X lie in the interval (0, 1), the optimal procedure is to not reject H_0 for every possible observed value (as $max(X) = 1 \to 2 \cdot max(X) = 2 < 3$).

(B)

Determine the minimum value of $3\alpha(\delta) + \beta(\delta)$ attained by the procedure.

Since we never reject H_0 under the stated conditions, we have:

$\alpha(\delta) = 0, \beta(\delta) = 1$ and we thus have:

$3\alpha(\delta) + \beta(\delta) = 1$

9.2 #10

Q: Suppose that $X_1, ..., X_n$ form a random sample from the Poisson distribution with unknown mean λ. Let λ_0 and λ_1 be specified values such that $\lambda_1 > \lambda_0 > 0$, and suppose that it is desired to test the following simple hypotheses:

$H_0 : \lambda = \lambda_0$

$H_1 : \lambda = \lambda_1$

(A)

Show that the value of $\alpha(\delta) + \beta(\delta)$ is minimized by a test procedure which rejects H_0 when $\bar{X}_n > c$.

Applying Thm. 9.2.1 with $a = b = 1$, we know the optimal test procedure is to reject H_0 if $\frac{f_1(\bar{X})}{f_0(\bar{X})} > 1$.

Let $y = \sum_{i=1}^{n} x_i$

We may note the joint p.f. of the data as:

$f_n(\bar{X} \mid \lambda) = f_n(X_1 \mid \lambda)...f_n(X_n \mid \lambda) = \frac{e^{-\lambda}\lambda^{x_1}}{x_1!}...\frac{e^{-\lambda}\lambda^{x_n}}{x_n!} = \frac{e^{-n\lambda}\lambda^{y}}{\prod_{i=1}^{n} x_i!}$

Thus we have:

$f_0(\bar{X}) = \frac{e^{-n\lambda_0}\lambda_0^{y}}{\prod_{i=1}^{n} x_i!}$

$f_1(\bar{X}) = \frac{e^{-n\lambda_1}\lambda_1^{y}}{\prod_{i=1}^{n} x_i!}$

It then follows that:

$\frac{f_1(\bar{X})}{f_0(\bar{X})} = e^{-n(\lambda_2 - \lambda_1)} \cdot \left(\frac{\lambda_2}{\lambda_1}\right)^y$

Taking the log of this gives us:

$log\left(\frac{f_1(\bar{X})}{f_0(\bar{X})}\right) = ylog\left(\frac{\lambda_2}{\lambda_1}\right) - n(\lambda_1 - \lambda_0)$

Note by construction $\lambda_1 > \lambda_0$. Therefore, we know:

$\frac{f_1(\bar{X})}{f_0(\bar{X})} > 1iff\frac{y}{n} > \frac{(\lambda_1 - \lambda_0)}{(log\lambda_1 - log\lambda_0)}$

(B)

Find the value of c.

Note that $\bar{x}_n = \frac{y}{n}$.

Utilizing the above and the formulations from **(A)**, we want a test procedure which rejects H_0 when $\bar{X}_n > c$, giving us: $c = \frac{(\lambda_1 - \lambda_0)}{(log\lambda_1 - log\lambda_0)}$

(C)

For $\lambda_0 = \frac{1}{4}$, $\lambda_1 = \frac{1}{2}$, and $n = 20$, determine the minimum value of $\alpha(\delta) + \beta(\delta)$ that can be attained.

If H_0 is true, then $Y = \sum_{i=1}^{n} X_i \sim Poisson(\lambda_0)$ with mean $n\lambda_0$

And if H_1 is true, then $Y = \sum_{i=1}^{n} X_i \sim Poisson(\lambda_1)$ with mean $n\lambda_1$

Noting the specified conditions, $\lambda_0 = \frac{1}{4}$, $\lambda_1 = \frac{1}{2}$, and $n = 20$, we have:

$\frac{n(\lambda_1 - \lambda_0)}{(log\lambda_1 - log\lambda_0)} = \frac{20 \cdot 0.25}{log(0.50) - log(0.25)} \approx 7.21$

We use this value in the following computations, which we are able to evaluate using Poisson distribution tables, but first we must path our tithe and say:

All Praise the Glorious Tables in the Back of the Book

With tithes out of the way, note to evaluate $\alpha(delta)$ we use a Poisson distribution with mean $5 = 20 \cdot 0.25$.

$\alpha(\delta) = Pr(Y > 7.21 \mid H_0) = Pr(Y \geq 8 \mid H_0) \approx 0.13$

Similarly, to evaluate $\beta(delta)$ we use a Poisson distribution with mean $10 = 20 \cdot 0.50$.

$\beta(\delta) = Pr(Y \leq 7.21 \mid H_1) = Pr(Y \leq 7 \mid H_1) \approx 0.22$

Combining these together, we have:

$\alpha(\delta) + \beta(\delta) = 0.13 + 0.22 = 0.35$ as the minimum value that can be attained.

9.3 #1

Q: Suppose that $X_1, ..., X_n$ form a random sample from the Poisson distribution with unknown mean λ $(\lambda > 0)$. Show that the joint p.f. of $X_1, ..., X_n$ has a monotone likelihood ratio in the statistic $\sum\limits_{i=1}^{n} X_i$.

Let $y = \sum\limits_{i=1}^{n} x_i$

We may note the joint p.f. of the data as:

$f_n(\bar{X} \mid \lambda) = f_n(X_1 \mid \lambda)...f_n(X_n \mid \lambda) = \frac{e^{-\lambda}\lambda^{x_1}}{x_1!}...\frac{e^{-\lambda}\lambda^{x_n}}{x_n!} = \frac{e^{-n\lambda}\lambda^{y}}{\prod\limits_{i=1}^{n} x_i!}$

Thus, for $0 < \lambda_1 < \lambda_2$, we have:

$\frac{f_n(\bar{X}|\lambda_2)}{f_n(\bar{X}|\lambda_1)} = \frac{\frac{e^{-n\lambda_2}\lambda_2^{y}}{\prod\limits_{i=1}^{n} x_i!}}{\frac{e^{-n\lambda_1}\lambda_1^{y}}{\prod\limits_{i=1}^{n} x_i!}}$

Noting the factorials of x_i cancel out, we then have:

$\frac{f_n(\bar{X}|\lambda_2)}{f_n(\bar{X}|\lambda_1)} = e^{-n(\lambda_2 - \lambda_1)} \cdot \left(\frac{\lambda_2}{\lambda_1}\right)^{y}$

Note, as $0 < \lambda_1 < \lambda_2$, this is an increasing function of y and we conclude the joint p.f. of $X_1, ..., X_n$ has a monotone likelihood ratio in the statistic $\sum\limits_{i=1}^{n} X_i$.

9.3 #2

Q: Suppose that $X_1, ..., X_n$ form a random sample from the normal distribution with known mean μ and unknown variance σ^2 $\left(\sigma^2 > 0\right)$. Show that the joint p.d.f. of $X_1, ..., X_n$ has a monotone likelihood ratio in the statistic $\sum\limits_{i=1}^{n} (X_i - \mu)^2$.

Let $y = \sum\limits_{i=1}^{n} (x_i - \mu)^2$

We may then note the joint p.d.f. of the data as:

$f_n(\bar{X} \mid \sigma^2) = f_n(X_1 \mid \sigma^2)...f_n(X_n \mid \sigma^2) = \frac{1}{\sqrt{(2\pi\sigma^2)}} \cdot e^{\frac{-x_1}{2\sigma^2}} ... \frac{1}{\sqrt{(2\pi\sigma^2)}} \cdot e^{\frac{-x_n}{2\sigma^2}}$

Simplifying this expression gives us:

$f_n(\bar{X} \mid \sigma^2) = \frac{1}{\sqrt{(2\pi\sigma^2)}^{n}} \cdot e^{\frac{-y}{2\sigma^2}}$

Thus, noting the condition $0 < \sigma_1^2 < \sigma_2^2$, we have:

$$\frac{f_n(\bar{X}|\sigma_2^2)}{f_n(\bar{X}|\sigma_1^2)} = \frac{\frac{1}{\sqrt{(2\pi\sigma_2^2)}^n} \cdot e^{\frac{-y}{2\sigma_2^2}}}{\frac{1}{\sqrt{(2\pi\sigma_1^2)}^n} \cdot e^{\frac{-y}{2\sigma_1^2}}}$$

Simplifying this expression then gives us:m

$$\frac{f_n(\bar{X}|\sigma_2^2)}{f_n(\bar{X}|\sigma_1^2)} = \left(\frac{\sigma_1}{\sigma_2}\right)^n \cdot e^{\frac{y}{2} \cdot \left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_2^2}\right)}$$

Noting the prior condition that $0 < \sigma_1^2 < \sigma_2^2 \to \frac{1}{\sigma_1^2} > \frac{1}{\sigma_2^2}$, and we may say the above $(\frac{f_n(\bar{X}|\sigma_2^2)}{f_n(\bar{X}|\sigma_1^2)})$ is an increasing function of y.

We may then conclude that the joint p.d.f. of $X_1, ..., X_n$ has a monotone likelihood ratio in the statistic $\sum\limits_{i=1}^{n} (X_i - \mu)^2$.

9.3 #13

Q: Suppose that four observations are taken at random from the normal distribution with unknown mean μ and known variance 1. Suppose also that the following hypotheses are to be tested:

$H_0 : \mu \geq 10$

$H_1 : \mu < 10$

(A)

Determine a UMP test at the level of significance $\alpha_0 = 0.1$

Note: via Exercise 9.3.12, a test which rejects H_0 when $\bar{X}_n \leq c$ will be a UMP test.

Thus, for a significance level $\alpha_0 = 0.1$, we want to choose a value of c that satisfies:

$Pr(\bar{X}_n \leq c \mid \mu = 10) = 0.1$

As we have four observations, we have $n = 4$. For $\mu = 10$, let us note the variable z with the standard normal distribution, specifically:

$z = 2(\bar{X}_n - 10)$

Referencing the above probability formula, we thus have:

$Pr(\bar{X}_n \leq c \mid \mu = 10) = Pr(Z \leq 2(c - 10)) = 0.1$

As the standard normal distribution is a known distribution, we may take advantage of the known formulation, namely:

$Pr(Z \leq 2(c - 10)) = 0.1 = Pr(Z \leq -1.28)$

Solving for c gives us:

$2(c - 10) = -1.28 \to c = \frac{18.72}{2} = 9.36$

(B)

Determine the power of this test when $\mu = 9$.

Using a similar formulation to **(A)**, we note:

$z = 2(\bar{X}_n - 9)\ N(0, 1)$ (the standard normal distribution). We then formulate the power of the test as:

$Pr(\bar{X}_n \leq 9.36 \mid \mu = 9) = Pr(Z \leq 0.718) = \Phi(0.718) \approx 0.76$

(C)

Determine the probability of not rejecting H_0 if $\mu = 11$.

Using a similar formulation to **(A)** and **(A)**, we note:

$z = 2(\bar{X}_n - 11)\ N(0, 1)$ (the standard normal distribution).

We then formulate the probability of rejecting H_0 as:

$Pr(\bar{X}_n \leq 9.36 \mid \mu = 11) = Pr(Z \geq -3.282) = Pr(Z \leq 3.282) = \Phi(3.282) \approx 0.9995$