

# MATH 392 Problem Set 6

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## Exercise from the book

### 9.1 #1

**Q:** Let  $X$  have the exponential distribution with parameter  $\beta$ . Suppose that we wish to test the hypotheses  $H_0 : \beta \geq 1$  versus  $H_1 : \beta < 1$ . Consider the test procedure  $\delta$  that rejects  $H_0$  if  $X \geq 1$ .

**(A)**

Determine the power function of the test.

Let  $\delta$  denote a test, specifically:

$$\delta : \{\text{Reject } H_0 \text{ when } X \geq 1\}$$

Then for  $\beta > 0$  we note the power function as follows,  $\pi(\beta | \delta) = Pr(X \geq 1 | \beta) = e^{-\beta}$

**(B)**

Compute the size of the test.

By the definition of size, we have:

$$size(\delta) = \sup_{\beta \geq 1} \pi(\beta | \delta)$$

Note the power function from **(A)**,  $\pi(\beta | \delta) = e^{-\beta}$ , such that  $\pi(\beta | \delta)$  is a decreasing function of  $\beta$ . Thus, the power function takes a maximum at  $\beta = 1$ , and we may say:

The size of the test is  $\pi(\beta = 1 | \delta) = e^{-1}$ .

### 9.1 #2

**Q:** Suppose that  $X_1, \dots, X_n$  form a random sample from the uniform distribution on the interval  $[0, \theta]$ , and that the following hypotheses are to be tested:

$$H_0 : \theta \geq 2$$

$$H_1 : \theta < 2$$

Let  $Y_n = \max(X_1, \dots, X_n)$  and consider a test procedure such that the critical region contains all the outcomes for which  $Y_n \leq 1.5$ .

**(A)**

Determine the power function of the test.

Note, for  $0 < y < \theta$ , then  $Pr(Y_n \leq y) = \left(\frac{y}{\theta}\right)^n$ .

Additionally note, if  $y \geq \theta$ , then  $Pr(Y_n \leq y) = 1$ .

Thus, for the stated condition, if  $\theta \leq 1.5$ , then  $\pi(\theta) = Pr(Y_n \leq 1.5) = 1$ .

And if  $\theta > 1.5$ , then  $\pi(\theta) = Pr(Y_n \leq 1.5) = \left(\frac{1.5}{\theta}\right)^n$

**(B)**

Determine the size of the test.

The size of the test,  $size(\delta) = \alpha = \sup_{\theta \geq 2} \pi(\theta) = \sup_{\theta \geq 2} \left(\frac{1.5}{\theta}\right)^n = \left(\frac{1.5}{2}\right)^n = \left(\frac{3}{4}\right)^n$ .

## 9.1 #14

Plus: plot power functions in R

**Q:** Let  $X_1, \dots, X_n$  be i.i.d. with exponential distribution with parameter  $\theta$ . Suppose that we wish to test the hypotheses:

$$H_0 : \theta \geq \theta_0$$

$$H_1 : \theta < \theta_0$$

Let  $X = \sum_{i=1}^n X_i$ . Let  $\delta_c$  be the test that rejects  $H_0$  if  $X \geq c$ .

(A)

Show that  $\pi(\theta \mid \delta_c)$  is a decreasing function of  $\theta$

(B)

Find  $c$  in order to make  $\delta_c$  have size  $\alpha_0$ .

(C)

Let  $\theta_0 = 2$ ,  $n = 1$ , and  $\alpha_0 = 0.1$ . Find the precise form of the test  $\delta_c$  and sketch its power function.

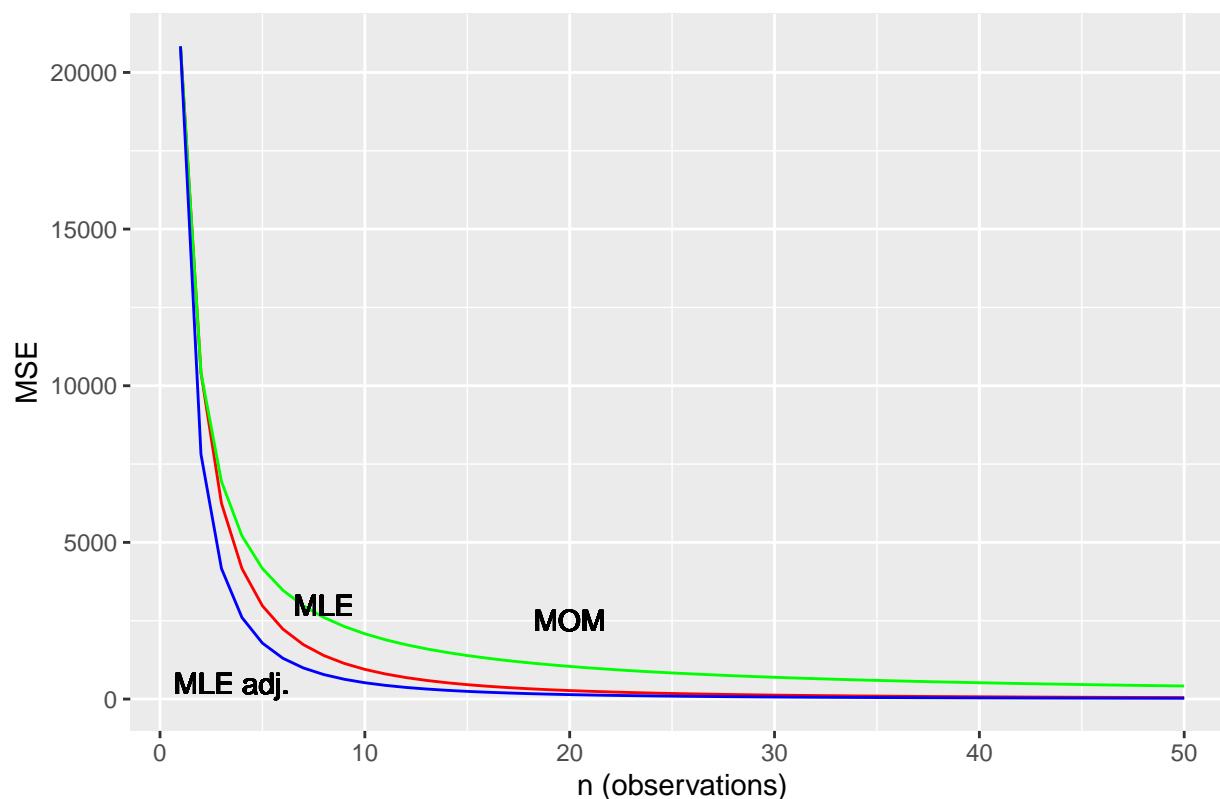
```
theta <- 250
n <- (1:50)

mle_mse <- rep(NA, 50)
mom_mse <- rep(NA, 50)
mle_adj_mse <- rep(NA, 50)
for (i in 1:50){
  mle_mse[i] <- ((theta ^ 2) * 2) / ((n[i] + 1) * (n[i] + 2))
  mom_mse[i] <- (theta ^ 2) / (3 * n[i])
  mle_adj_mse[i] <- (theta ^ 2) / ((n[i] * (n[i] + 2)))
}

df <- data.frame(n,mle_mse,mom_mse, mle_adj_mse)

ggplot(df, aes(n)) +
  geom_line(aes(y=mle_mse), colour="red") +
  geom_line(aes(y=mom_mse), colour="green") +
  geom_line(aes(y=mle_adj_mse), colour="blue") +
  geom_text(x=3.5, y=500, label="MLE adj.") +
  geom_text(x=20, y=2500, label="MOM") +
  geom_text(x=8, y=3000, label="MLE") +
  ggtitle("MSE of MLE, MOM, and adjusted MLE Estimators") +
  labs(y="MSE", x = "n (observations)")
```

MSE of MLE, MOM, and adjusted MLE Estimators



## 9.2 #2

**Q:** Consider two p.d.f.'s  $f_0(x)$  and  $f_1(x)$  that are defined as follows:

$$f_0(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_1(x) = \begin{cases} 2x & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Suppose that a single observation  $X$  is taken from a distribution for which the p.d.f.  $f(x)$  is either  $f_0(x)$  or  $f_1(x)$ , and the following simple hypotheses are to be tested:

$$H_0 : f(x) = f_0(x)$$

$$H_1 : f(x) = f_1(x)$$

**(A)**

Describe a test procedure  $\delta$  for which the value of  $\alpha(\delta) + 2\beta(\delta)$  is a minimum.

Using Thm. 9.2.1 with  $a = 1, b = 2$ , we have:

$$\delta : \{\text{Do not reject } H_0 \text{ if } \frac{f_1(x)}{f_0(x)} < \frac{1}{2}\}$$

Note, as  $\frac{f_1(x)}{f_0(x)} = 2x \rightarrow 2x < \frac{1}{2}$

Thus we may say the procedure is: Do not reject  $H_0$  if  $x < \frac{1}{4}$ , and likewise to reject  $H_0$  if  $x > \frac{1}{4}$ .

**(B)**

Determine the minimum value of  $\alpha(\delta) + 2\beta(\delta)$  attained by that procedure.

Note the following formulations:

$$\alpha(\delta) = Pr(Reject H_0 | f_0) = \int_{\frac{1}{4}}^1 f_0(x) dx = \frac{3}{4}$$

$$\beta(\delta) = Pr(Fail to Reject H_0 | f_1) = \int_0^{\frac{1}{4}} 2x dx = \frac{1}{16}$$

Therefore, we have:

$$\alpha(\delta) + 2\beta(\delta) = \frac{3}{4} + 2 \cdot \frac{1}{16} = \frac{7}{8}$$

### 9.2 #3

**Q:** Consider again the conditions of Exercise 2 (9.2.2), but suppose now that it is desired to find a test procedure for which the value of  $3\alpha(\delta) + \beta(\delta)$  is a minimum.

**(A)**

Determine the procedure.

Using Thm. 9.2.1. with  $a = 3, b = 1$ , we have:

$$\delta : \{\text{Do not reject } H_0 \text{ if } \frac{f_1(x)}{f_0(x)} < 3 \rightarrow 2x < 3\}$$

Since all values of  $X$  lie in the interval  $(0, 1)$ , the optimal procedure is to not reject  $H_0$  for every possible observed value (as  $\max(X) = 1 \rightarrow 2 \cdot \max(X) = 2 < 3$ ).

**(B)**

Determine the minimum value of  $3\alpha(\delta) + \beta(\delta)$  attained by the procedure.

Since we never reject  $H_0$  under the stated conditions, we have:

$$\alpha(\delta) = 0, \beta(\delta) = 1 \text{ and we thus have:}$$

$$3\alpha(\delta) + \beta(\delta) = 1$$

### 9.2 #10

**Q:** Suppose that  $X_1, \dots, X_n$  form a random sample from the Poisson distribution with unknown mean  $\lambda$ . Let  $\lambda_0$  and  $\lambda_1$  be specified values such that  $\lambda_1 > \lambda_0 > 0$ , and suppose that it is desired to test the following simple hypotheses:

$$H_0 : \lambda = \lambda_0$$

$$H_1 : \lambda = \lambda_1$$

**(A)**

Show that the value of  $\alpha(\delta) + \beta(\delta)$  is minimized by a test procedure which rejects  $H_0$  when  $\bar{X}_n > c$ .

**(B)**

Find the value of  $c$ .

**(C)**

For  $\lambda_0 = \frac{1}{4}$ ,  $\lambda_1 = \frac{1}{2}$ , and  $n = 20$ , determine the minimum value of  $\alpha(\delta) + \beta(\delta)$  that can be attained.

### 9.3 #1

**Q:** Suppose that  $X_1, \dots, X_n$  form a random sample from the Poisson distribution with unknown mean  $\lambda$  ( $\lambda > 0$ ). Show that the joint p.f. of  $X_1, \dots, X_n$  has a monotone likelihood ratio in the statistic  $\sum_{i=1}^n X_i$ .

$$\text{Let } y = \sum_{i=1}^n x_i$$

We may note the joint p.f. of the data as:

$$f_n(\bar{X} | \lambda) = f_n(X_1 | \lambda) \dots f_n(X_n | \lambda) = \frac{e^{-\lambda} \lambda^{x_1}}{x_1!} \dots \frac{e^{-\lambda} \lambda^{x_n}}{x_n!} = \frac{e^{-n\lambda} \lambda^y}{\prod_{i=1}^n x_i!}$$

Thus, for  $0 < \lambda_1 < \lambda_2$ , we have:

$$\frac{f_n(\bar{X} | \lambda_2)}{f_n(\bar{X} | \lambda_1)} = \frac{\frac{e^{-n\lambda_2} \lambda_2^y}{\prod_{i=1}^n x_i!}}{\frac{e^{-n\lambda_1} \lambda_1^y}{\prod_{i=1}^n x_i!}}$$

Noting the factorials of  $x_i$  cancel out, we then have:

$$\frac{f_n(\bar{X} | \lambda_2)}{f_n(\bar{X} | \lambda_1)} = e^{-n(\lambda_2 - \lambda_1)} \cdot \left( \frac{\lambda_2}{\lambda_1} \right)^y$$

Note, as  $0 < \lambda_1 < \lambda_2$ , this is an increasing function of  $y$  and we conclude the joint p.f. of  $X_1, \dots, X_n$  has a monotone likelihood ratio in the statistic  $\sum_{i=1}^n X_i$ .

### 9.3 #2

**Q:** Suppose that  $X_1, \dots, X_n$  form a random sample from the normal distribution with known mean  $\mu$  and unknown variance  $\sigma^2$  ( $\sigma^2 > 0$ ). Show that the joint p.d.f. of  $X_1, \dots, X_n$  has a monotone likelihood ratio in the statistic  $\sum_{i=1}^n (X_i - \mu)^2$ .

### 9.3 #13

**Q:** Suppose that four observations are taken at random from the normal distribution with unknown mean  $\mu$  and known variance 1. Suppose also that the following hypotheses are to be tested:

$$H_0 : \mu \geq 10$$

$$H_1 : \mu < 10$$

(A)

Determine a UMP test at the level of significance  $\alpha_0 = 0.1$

(B)

Determine the power of this test when  $\mu = 9$ .

(C)

Determine the probability of not rejecting  $H_0$  if  $\mu = 11$ .