MATH 392 Problem Set 5

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Exercise from the book

8.5 #4

(Q): Suppose that $X_1, ..., X_n$ form a random sample from the normal distribution with unkown mean μ and unknown variance σ^2. How large a random sample must be taken in order that there will be a confidence interval for μ with confidence coefficient 0.95 and length less than 0.01σ?

(A):

Note: $\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma}$ has a standard normal distribution.

It then follows:

$Pr\left(-1.96 < \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} < 1.96\right) = 0.95$

We isolate the μ in the above relation, giving us:

$Pr\left(\bar{X}_n - 1.96\frac{\sigma}{\sqrt{n}} < \mu < \bar{X}_n + 1.96\frac{\sigma}{\sqrt{n}}\right) = 0.95$

THus, we now have the confidence interval: $\left(\bar{X}_n - 1.96\frac{\sigma}{\sqrt{n}}, \bar{X}_n + 1.96\frac{\sigma}{\sqrt{n}}\right)$

As the above interval included $\bar{X}_n$ on both ends of the interval (taking it as a constant for our following observation), we may then note the length of this interval is $1.96\frac{\sigma}{\sqrt{n}} \cdot 2 = 3.92\frac{\sigma}{\sqrt{n}}$

If we want a length less than 0.01σ, then we have:

$3.92\frac{\sigma}{\sqrt{n}} < 0.01\sigma \to \frac{3.92}{\sqrt{n}} < 0.01$

Thus we have, for n positive (given the context of the problem):

$3.92 < \sqrt{n} \cdot 0.01 \to 392 < \sqrt{n} \to 392^2 < n$

Taking note of the strict inequality (and the fact $392^2 = 153,664$), we need an n sample of size *at least* 153,665 to satisfy the stated condition.

Case Study: German Tank Problem

Let's pick up the example that we began in class but make the simplifying assumption that we're studying a process where our sample is drawn from the continuous distribution, $X_1, X_2, \ldots, X_n \sim \text{Unif}(0, \theta)$, but we're still interested in estimating θ. The MLE and Method of Moments estimators are the same:

$$\hat{\theta}_{MLE} = \max(X_1, X_2, \ldots, X_n) = X_{max}$$
$$\hat{\theta}_{MOM} = 2\bar{X}$$

1. Calculate the bias of each estimator. If either one is biased, propose an additional estimator that corrects that bias (in the spirit of how s^2 is the bias-corrected version of $\hat{\sigma}^2$) . What happens to the bias of these estimators as sample size grows? Plot the relationship between sample size and bias for each estimator (two lines on one plot).

(A):

For θ_{MLE} we have:

Note, the sampling distribution to θ_{MLE} is given by:

$$f_{\theta_{MLE}}(x) = \begin{Bmatrix} n\frac{x^{n-1}}{\theta^n} & for\ 0 \leq x \leq \theta \\ 0 & otherwise \end{Bmatrix}$$

Thus, we may evaluate $E_{\theta_{MLE}}(X) = \int\limits_0^\theta x f_{\theta_{MLE}} dx = \int\limits_0^\theta xn\frac{x^{n-1}}{\theta^n}dx = \frac{nx^{n+1}}{(n+1)\theta^n}\Big|_0^\theta = \frac{n}{n+1}\theta$

Thus, we have:

$Bias_{\theta_{MLE}} = E_{\theta_{MLE}}\left(\hat{\theta}\right) - \theta = \frac{n}{n+1}\theta - \theta = -\frac{1}{n+1}\theta$

(B):

For θ_{MOM} we have:

$E_{\theta_{MOM}}(X) = \int\limits_0^\theta x\frac{1}{\theta}dx = \frac{x^2}{2\theta}\Big|_0^\theta = \frac{\theta}{2}$

$Bias_{\theta_{MOM}} = E_{\theta_{MOM}}\left(\hat{\theta}\right) - \theta = \frac{2\theta}{2} - \theta = \theta - \theta = 0$

(C): Unbiased MLE estimator

Let θ_{adj} be the adjusted MLE estimator. We aim to show θ_{MLE} is bias adjusted.

Define θ_{adj} as follows:

$\theta_{adj} = \frac{n+1}{n}max(X_1, ..., X_n) = \frac{n+1}{n}X_{max}$

Note, and referring back to the Bias formulation in (A):

$Bias_{\theta_{adj}} = E_\theta\left(\theta_{adj}\right) - \theta = \frac{n+1}{n}\frac{n}{n+1}\theta - \theta = \theta - \theta = 0$

Thus, we note θ_{adj} is the bias adjusted MLE estimator.

Commentary

The above results indicate the MLE estimate is biased (but unbiased asymptotically as n increases) and the MOM estimate is unbiased.

2. Calculate the variance of each estimator (including any new bias-corrected ones). What happens as sample size grows? Create an analogous plot to the one above.

Note

The pdf of X_{max} is $nF^{n-1}(x)f(x)$.

2. Calculate the Variance What happens in the asymptote?

(A):

For θ_{MLE} we have:

$E_{\theta_{MLE}}(X^2) = \int\limits_0^\theta x^2 f_{\theta_{MLE}} dx = \int\limits_0^\theta x^2 n \frac{x^{n-1}}{\theta^n} dx = \frac{nx^{n+2}}{(n+2)\theta^n}\Big|_0^\theta = \frac{n}{n+2}\theta^2$

$Var_{\theta_{MLE}} = E_{\theta_{MLE}}\left(\hat{\theta}^2\right) - \left(E_{\theta_{MLE}}\left(\hat{\theta}\right)\right)^2$

$= \frac{n}{n+2}\theta^2 - \left(\frac{n}{n+1}\theta\right)^2 = \frac{n}{(n+1)^2(n+2)}\theta^2$

(B):

For θ_{MOM} we have:

$E_{\theta_{MOM}}(X^2) = \int\limits_0^\theta x^2 \frac{1}{\theta} dx = \frac{x^3}{3\theta}\Big|_0^\theta = \frac{\theta}{3}$

Thus, we have:

$Var_{\theta_{MOM}} = E_{\theta_{MOM}}\left(\hat{\theta}^2\right) - \left(E_{\theta_{MOM}}\left(\hat{\theta}\right)\right)^2$

$= \frac{\theta^2}{3} - \left(\frac{\theta}{2}\right)^2 = \frac{\theta^2}{3} - \frac{\theta^2}{4} = \frac{\theta^2}{12}$

Thus:

$V_\theta\left(\hat{\theta}_n\right) = \frac{4}{n^2} n V_\theta\left(X\right) = \frac{4}{n^2} n \frac{\theta^2}{12} = \frac{\theta^2}{3n}$

(C):

For θ_{adj} we have:

$Var_\theta\left(\theta_{adj}\right) = Var_\theta\left(\frac{n+1}{n}\theta_{MLE}\right) = \left(\frac{n+1}{n}\right)^2 Var_\theta(\theta_{MLE})$

As we again solved the variance of the initial MLE estimator above, we have:

$Var_\theta\left(\theta_{adj}\right) = \left(\frac{n+1}{n}\right)^2 \cdot \frac{n}{(n+1)^2(n+2)}\theta^2 = \frac{1}{n(n+2)}\theta^2$

3. Combine the notions of bias and variance into a third plot that shows how the Mean Squared Error changes as a function of sample size. Based on this plot, which estimator would you use and why?

Note, the formula for the M.S.E. is given as follows:

$M.S.E.\left(\hat{\theta}\right) = Var(\hat{\theta}) + Bias(\hat{\theta})^2$

Thus, using the above relations we have:

(A):

$M.S.E.(\theta_{MLE}) = Var(\theta_{MLE}) + Bias(\theta_{MLE})^2 = \frac{n}{(n+1)(n+2)}\theta^2 + \left(-\frac{1}{n+1}\theta\right)^2 = \frac{n\theta^2}{(n+1)(n+2)} + \frac{\theta^2}{(n+1)^2} = \frac{2\theta^2}{(n+1)(n+2)}$

(B):

Similarly, for the MOM we have:

$M.S.E.\,(\theta_{MOM}) = Var(\theta_{MOM}) + Bias(\theta_{MOM})^2 = \frac{\theta^2}{3n} + 0^2 = \frac{\theta^2}{3n}$

(C):

For the adjusted MLE estimator, we first note it has no bias, like the MOM estimator in (B). Thus, it follows:

$M.S.E.\,(\theta_{adj}) = Var(\theta_{adj}) + Bias(\theta_{adj})^2 = \frac{1}{n(n+2)}\theta^2 + 0^2 = \frac{1}{n(n+2)}\theta^2$

4. Using the method that we saw in class based on Markov's Inequality, assess whether each of these estimators is consistent.

5. What is the sampling distribution of each statistic? For the MOM, consider both the Irwin-Hall distribution and a sensible approximation based on the Central Limit Theorem.

6. Create a plot of the sampling distribution of each estimator using $n = 10$. Construct the empirical distribution via simulation and overlay the appropriate exact or approximate analytical form (each plot should be a curve overlayed on a histogram. See slides.)

7. Form two different 95% confidence intervals for θ by using pivotal statistics inspired by each estimator.