MATH 392 Problem Set 3

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Note on PS3

Time Taken

Total Time on Exercises from the Book: X hours Solving Exercises from the Book: 4 1/2 hours LaTeX writeup of Exercises from the Book: 3 1/2 hours

Additional Notes

- This Problem Set does not contain the Case Study. It will be turned in on Monday, 2/17/2020.
- Answers to All 8 problems are included.

Exercises from the book

7.5: 5

Let $y = \sum_{i}^{n} x_i$

Additionally, denote: $\bar{x} = x_1, ..., x_n$

Then, for $y > 0$ the likelihood function is given by:

$$f_n(\bar{x} \mid \theta) = \frac{e^{-n\theta}\theta^y}{\prod_{i=1}^{n} x_i!}$$

Let $L(\theta) = log(f_n(\bar{x} \mid \theta))$

Note that $\prod_{i=1}^{n} x_i!$ does not depend on θ, as such, let $\psi = \prod_{i=1}^{n} x_i!$

Then we may write:

$L(\theta) = -n\theta + ylog(\theta) - \psi$

Then we may take the derivative of $L(\theta)$ with respect to θ, giving us:

$$\frac{\partial L(\theta)}{\partial \theta} = -n + \frac{y}{\theta}$$

To determine the M.L.E. of θ, we maximize this equation and set it equal to zero, such that:

$$0 = -n + \frac{y}{\theta} \to n = \frac{y}{\theta} \to \hat{\theta} = \frac{y}{n}$$

(A): Thus, the M.L.E. of θ is $\hat{\theta} = \frac{y}{n} = \bar{x}_n$.

(B): If $y = 0$, and if $\theta \neq 0$ ($\theta > 0$), then $\theta = 0$ is not in the parameter space, Ω, then the M.L.E. of θ does not exist.

For additional commentary, we may note that if $y = 0$, then the likelihood function $f_n(\bar{x} \mid \theta)$ is a decreasing function of θ. As such, the maximum of the derivative of $L(\theta)$ occurs when $\theta = 0$. However, $\theta > 0$ is a condition of θ, and as such, the M.L.E. of θ does not exist.

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Let $y = \sum_{i}^{n} x_i$

Additionally, denote: $\bar{x} = x_1, ..., x_n$

We note the likelihood function, $f_n(\bar{x} \mid \theta)$, as:

$$f_n(\bar{x} \mid \theta) = \begin{Bmatrix} e^{\theta - y} \; for \; x > \theta \\ 0 \; for \; x \leq \theta \end{Bmatrix}$$

Let $L(\theta) = log(f_n(\bar{x} \mid \theta))$

Then, for $x > \theta$, we may write:

$L(\theta) = \theta - y$

Then we may take the derivative of $L(\theta)$ with respect to θ, setting it equal to 0 to maximize the likelihood function, giving us:

$$\frac{\partial L(\theta)}{\partial \theta} = 0 \rightarrow 1 = 0$$

(A): As this is markedly false, we conclude the M.L.E. of θ does not exist.

Alternatively, we may not that $\forall x_i \in \bar{x}, f_n(\bar{x} \mid \theta)$ will be maximized when θ is maximized (while holding the relation $\theta < x$). As such, $\theta < min(x_1, ..., x_n)$. However, due to the **strict inequality**, the maximum $\theta = min(x_1, ..., x_n)$ cannot be used as an estimate of θ. As such, the M.L.E. of θ does not exist.

We may take advantage of the example provided in the book to create another version of the p.d.f. for which the M.L.E. of θ will exist. To do so, let the **strict inequality** be swapped with the **weak inequality**, such that we may write:

$$f(x \mid \theta) = \begin{Bmatrix} e^{\theta - y} \; for \; x \geq \theta \\ 0 \; for \; x < \theta \end{Bmatrix}$$

We may then note the likelihood function of this p.d.f as:

$$f_n(\bar{x} \mid \theta) = \begin{Bmatrix} e^{\theta - y} \; for \; x \geq \theta \\ 0 \; for \; x < \theta \end{Bmatrix}$$

We may then note: The likelihood function will be non-zero for $\theta \leq min(x_1, ..., x_n)$.

(B): Thus, the M.L.E. of θ, $\hat{\theta} = min(x_1, ..., x_n)$

7.5: 11

Note, as we are dealing with the interval $[\theta_1, \theta_2]$, we know $\theta_1 \leq \theta_2$

We may then write the p.d.f. of each x_i as:

$$f(x_i \mid \theta_1, \theta_2) = \begin{Bmatrix} \frac{1}{\theta_2 - \theta_1} \; for \; \theta_1 \leq x \leq \theta_2 \\ 0 \;\; otherwise \end{Bmatrix}$$

Letting $\bar{x} = x_1, ..., x_n$, we may then write the likelihood function as:

$$f_n\left(\bar{x} \mid \theta_1, \theta_2\right) = \begin{Bmatrix} \frac{1}{(\theta_2 - \theta_1)}^{n} \; for \; \theta_1 \leq min(x_1, ..., x_n) \leq max(x_1, ..., x_n) \leq \theta_2 \\ 0 \;\; otherwise \end{Bmatrix}$$

We may then note that $f_n\left(\bar{x} \mid \theta_1, \theta_2\right)$ is maximized when $\theta_2 - \theta_1$ is minimized.

Thus, we may note:

$min(\theta_2) = max(x_1, ..., x_n)$

And

$max(\theta_1) = min(x_1, ..., x_n)$

Thus, the M.L.E. of θ may be written: $\theta = (\hat{\theta_1} = min(x_1, ..., x_n), \hat{\theta_2} = max(x_1, ..., x_n))$.

7.5: 12

Let $\theta = \theta_1, ..., \theta_n$

Note the following relations for use later: $k \geq 2$, $0 \leq \theta_i \leq 1$, $\theta_1 + ... + \theta_k = 1$, and $n_1 + ... + n_k = n$

Then, the likelihood function is given by:

$$f_n\left(\bar{x} \mid \theta_1, ..., \theta_k\right) = \prod_{i=1}^{k} {\theta_i}^{n_i} = {\theta_1}^{n_1} ... {\theta_k}^{n_k}$$

For ease of computation, let $L\left(\theta\right) = log\left(f_n\left(\bar{x} \mid \theta_1, ..., \theta_k\right)\right)$

Note that: $\theta_k = 1 - \sum\limits_{i=1}^{k-1} \theta_i$

Then we may write:

$L\left(\theta\right) = \sum\limits_{i=1}^{k} n_i log(\theta_i)$

Then we may take the derivative with respect to θ_i, giving us:

$$\frac{\partial L\left(\theta\right)}{\partial \theta_i} = \frac{\partial\left(n_k\left(1 - \sum\limits_{i=1}^{k-1} \theta_i\right) + \sum\limits_{i=1}^{k-1} n_i log(\theta_i)\right)}{\partial \theta_i}$$

This evaluates to:

$$\frac{\partial L\left(\theta\right)}{\partial \theta_i} = -\frac{n_k}{\left(1 - \sum\limits_{i=1}^{k-1} \theta_i\right)} + \frac{n_i}{\theta_i} = -\frac{n_k}{\theta_k} + \frac{n_i}{\theta_i}$$

To determine the M.L.E. of θ_i, we maximize this equation and set it equal to zero, such that:

$0 = -\frac{n_k}{\theta_k} + \frac{n_i}{\theta_i} \rightarrow \frac{n_k}{\theta_k} = \frac{n_i}{\theta_i}$ for $i = 1, ..., k-1$

For $i = 1, ..., k$, if we set $\theta_i = \alpha n_i$, then it follows:

$1 = \sum\limits_{i=1}^{k} \theta_i = \sum\limits_{i=1}^{k} \alpha n_i = \alpha \sum\limits_{i=1}^{k} n_i$

Noting that $\sum\limits_{i=1}^{n} n_i = n$, we may then say:

$1 = \alpha n \to \alpha = \frac{1}{n}$

We may then write, for $i = 1, ..., k$, the M.L.E. of θ_i is:

$\hat{\theta}_i = \alpha n_i = n_i/n$

Intuitively, and similar to other applications of the M.L.E., this results says that the M.L.E. of the individuals of the type i is the proportion of individuals of type i.

7.6: 3

Let m denote the median of the Exponential distribution.

Note the median of the Exponential distribution may be derived from the following equation:

$\int\limits_0^m \beta e^{-\beta x} dx = \frac{1}{2}$

Evaluating this integral gives us:

$\frac{1}{2} = 1 - e^{-\beta m} \to 1 = 2 - e^{-\beta m}$

Solving for m, we take the log of the above equation, giving us:

$0 = log(2) - \beta m \to \beta m = log(2) \to m = \frac{log(2)}{\beta}$

Thus, for $\hat{\beta}$, the sample median, $\hat{m} = \frac{log(2)}{\hat{\beta}}$

All praise the glorious tables at the back of the book For our purposes, note the mean of the Exponential distribution, $\mu = \frac{1}{\beta}$.

Solving for β, we then have $\beta = \frac{1}{\mu}$

For the sample mean, $\bar{x}_n$, note: $\hat{\beta} = \frac{1}{\bar{x}_n}$

Thus, for the sample median, we have:

$\hat{m} = \frac{log(2)}{\frac{1}{\bar{x}_n}} = \bar{x}_n log(2)$

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Note from the prior exercise, the parameter $\hat{\beta} = \frac{1}{\bar{x}_n}$

Additionally, we may once again **praise the glorious tables at the back of the book** and note the mean of the Exponential distribution is given by: $\mu = \frac{1}{\beta}$

Via application of the Law of Large Numbers, we may then write:

$\lim\limits_{n \to \infty} Pr\left(| \ \bar{x}_n - \frac{1}{\beta} \ | \right) = 1 \equiv \bar{x}_n \xrightarrow{p} \frac{1}{\beta} = \mu$

And

$\lim\limits_{n \to \infty} Pr\left(| \ \bar{\beta} - \beta \ | \right) = 1 \equiv \bar{\beta} \xrightarrow{p} \beta$

It then follows that the sequence of M.L.E.s of β, $\hat{\beta}$, is a consistent sequence of M.L.E.s of β

7.6: 14

The M.L.E., $\hat{p}$, is the proportion of butterflies in the sample with the special type of marking on their wings. Of note, this holds regardless of the sampling plan, as noted in the section *Sampling Plans* (pp.439-441) of the book. Therefore, using this method, we have:

(A): $\hat{p} = \frac{5}{43}$

And

(B): $\hat{p} = \frac{3}{58}$

Alternatively, and with greater difficulty, we can take the likelihood function using the Binomial distribution, written:

$$f_n\left(\bar{x} \mid p\right) = \binom{n}{x} p^x \left(1-p\right)^{n-x}$$

Where n denotes the total number of butterflies and x denotes the number of butterflies with the special type of marking on their wings.

Let $L\left(p\right) = log\left(f_n\left(\bar{x} \mid p\right)\right)$

Taking the derivative with respect to p gives us:

$$\frac{\partial L\left(p\right)}{\partial p} = \frac{\partial\left(log\binom{n}{x} + xlog(p) + (n-x)log(1-p)\right)}{\partial p} = \frac{x}{p} - \frac{n-x}{1-p}$$

Setting this equation equal to zero, we then solve for p:

$\frac{n-x}{1-p} = \frac{x}{p} \rightarrow p\frac{n-x}{1-p} = x \rightarrow \frac{1}{p} = \frac{n-x+x}{x} = \frac{n}{x}$
Solving for p gives us:

$p = \frac{x}{n}$, which confirms our above use of the sample proportion for the M.L.E. of p.

7.6: 23

Note, the mean is the first moment and the variance is the second moment. Their respective relations are given as follows:

(1): 1st Moment, $X_i = \frac{\alpha}{\alpha+\beta}$ (2): 2nd Moment, $X_i^2 = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$

We set these, respectively, equal to the sample moments m_1 and m_2, and then use relations (1) and (2) to solve for α and β.