MATH 392 Problem Set 2

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Time Spent on PS 2: 7 hours

Exercises from the book

7.2.1

$$f\left(\boldsymbol{x} \mid \theta\right) = \theta^n e^{-\theta y}$$

Where $y = \sum_{i=1}^{n} x_i$

As Example 7.2.8 provides five observed values, we have

$$y = x_1 + x_2 + ... + x_5 = 16,178$$

If we believe, as the question supposes, that the prior distribution of θ is the gamma distribution with parameters 1 and 5000, we then have a new posterior distribution of θ, $\xi\left(\boldsymbol{x} \mid \theta\right)$ as the gamma distribution with parameters $\alpha_{new} = \alpha_{prior} + 1 = 5 + 1 = 6$ and $\beta_{new} = \beta_{prior} + 5000 = 16178 + 5000 = 21178$

When we evaluate the conditional joint p.d.f., we have

$$f\left(x_n \mid \boldsymbol{x}\right) = \int_{\Omega} \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^n e^{-\theta y} \theta^{\alpha-1} e^{-\beta\theta} d\theta$$

We thus have

$$f\left(x_6 \mid \boldsymbol{x}\right) = \int_{0}^{\infty} (7.518 \cdot 10^{23}) \beta^6 e^{-\theta(21178 + x_6)} d\theta$$

For $x_6 > 0$, we may simplify this expression by evaluating this integral, yielding:

$$f\left(x_6 \mid \boldsymbol{x}\right) = (7.518 \cdot 10^{23}) \frac{\Gamma(7)}{(21178 + x_6)^7} = \frac{5.413 \cdot 10^{26}}{(21178 + x_6)^7}$$

We may now compute $Pr(X_6 > 3000 \mid \boldsymbol{x})$ using the above, integrating from 3000 to ∞ with respect to x_6. Once we simplify the integral, we may substitute $x_6 = 3000$ and write:

$$Pr(X_6 > 3000 \mid \boldsymbol{x}) = \int_{3000}^{\infty} \frac{5.413 \cdot 10^{26}}{(21178 + x_6)^7} dx_6 = \int_{3000}^{\infty} 5.413 \cdot 10^{26} \cdot (21178 + x_6)^{-7} dx_6 = 0.4516$$

7.2.2

Note: via Equation 7.2.11: $f_n(x \mid \theta) = \theta^y(1-\theta)^{n-y}$

For $n = 8, y = 2$ we then have:

$f_n(x \mid \theta\ right) = \theta^2(1-\theta)^6$

Given, $\xi(0.1) = 0.7$ and $\xi(0.2) = 0.3$

$$\begin{aligned}\xi(0.1 \mid \boldsymbol{x}) = Pr(\theta = 0.1 \mid \boldsymbol{x}) &= \frac{\xi(0.1)f_n(x \mid 0.1)}{\xi(0.1)f_n(x \mid 0.1) + \xi(0.2)f_n(x \mid 0.2)} \\ &= \frac{(0.7)(0.1)^2(0.9)^2}{(0.7)(0.1)^2(0.9)^2 + (0.3)(0.2)^2(0.8)^2} \\ &= 0.5418\end{aligned}$$

Note: $\xi(0.2 \mid x) + \xi(0.1 \mid x) = 1$

Hence:

$$\begin{aligned}\xi(0.2 \mid x) &= 1 - \xi(0.1 \mid x) \\ &= 1 - \frac{(0.7)(0.1)^2(0.9)^2}{(0.7)(0.1)^2(0.9)^2 + (0.3)(0.2)^2(0.8)^2} \\ &= 1 - 0.5418 \\ &= 0.4582\end{aligned}$$

Thus, the posterior pdf of θ is

$\xi\left(0.1\right) = 0.5418$

$\xi\left(0.2\right) = 0.4582$

Intuitively, given the information we are less sure the rate of defective items is 0.1 and more sure the rate is 0.2.

7.2.3

Let X denote the number of defects on the roll of tape.

For λ, the p.f. of X, we have, for $x \in \mathbb{Z}^{\geq 0}$:

$f\left(x \mid \lambda\right) = \frac{e^{-\lambda}\lambda^x}{x!}$

$$\begin{aligned}\xi\left(1.0 \mid x = 3\right) &= Pr\left(1.0 \mid x = 3\right) \\ &= \frac{\xi\left(1.0\right) f\left(3 \mid 1.0\right)}{\xi\left(1.0\right) f\left(3 \mid 1.0\right) + \xi\left(1.5\right) f\left(3 \mid 1.5\right)}\end{aligned}$$

Note, using the Poisson table in the back of the book, we have:

$f\left(3 \mid 1.0\right) = 0.0613)$

$f\left(3 \mid 1.5\right) = 0.1255)$

We then note: $\xi\left(1.0 \mid 3\right) + \xi\left(1.5 \mid 3\right) = 1$

Hence: $\xi\left(1.5 \mid 3\right) = 1 - \xi\left(1.0 \mid 3\right)$

Evaluating yields: $\xi\left(1.0 \mid 3\right) = 0.2456$

$\xi\left(1.5 \mid 3\right) = 0.7544$

Thus the posterior p.f. of λ is given by: $\xi(1.0) = 0.2456$ and $\xi(1.5) = 0.7544$

Intuitively, we're now more sure $\lambda = 0.6$ and less sure $\lambda = 0.4$.

7.2.5

Note, the mean and variance of the beta distribution is respectively given by the following:

Mean: $\frac{\alpha}{\alpha+\beta} = \frac{1}{3}$

Variance: $\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{1}{45}$

Given the formula for the mean, we have:

$\alpha + \beta = 3\alpha$

Hence,

$\beta = 2\alpha$

It follows, then

$\frac{\alpha}{\alpha+\beta} = \frac{2}{3}$

We may then rewrite the variance as:

$\frac{2}{9(\alpha+\beta+1)} = \frac{1}{45}$

Thus:

$90 = 9(\alpha + \beta + 1)$

And

$10 = (\alpha + \beta + 1)$

$9 = (\alpha + \beta)$

As

$\beta = 2\alpha$

$9 = (3\alpha)$

Hence

$\alpha = 3$, $\beta = 6$

As we have established the parameters of the prior p.d.f. of θ, it follows:

$\theta \sim Beta(\alpha = 3, \beta = 6)$

Or, alternatively, the prior p.d.f. of θ, is:

$$\xi(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}$$

$$= \frac{\Gamma(9)}{\Gamma(3)\Gamma(6)}\theta^{2}(1-\theta)^{5}$$

7.2.8

Per 7.2.14:

$\xi\,(\theta \mid x_1) \propto f(x_1 \mid \theta)\xi(\theta)$

Extending this to x_2 via 7.2.15, we have

$\xi\,(\theta \mid x_1, x_2) \propto f(x_2 \mid \theta)\xi(\theta \mid x_1) = f(x_2 \mid \theta)f(x_1 \mid \theta)\xi(\theta)$

Per 7.2.16, this holds for $x_1, x_2, ..., x_n$, written

$\xi\,(\theta \mid \boldsymbol{x}) \propto f(x_n \mid \theta)\xi(\theta \mid x_1, x_2, ..., x_{n-1})$

Via Eq. 7.2.4, we note:

$f_{n-1}(x_1, x_2, ..., x_n) = f(x_1 \mid \theta)f(x_2 \mid \theta)...f(x_{n-1} \mid \theta)$

Thus,

$\xi\,(\theta \mid \boldsymbol{x}) \propto f(x_n \mid \theta)f(x_{n-1} \mid \theta)...f(x_1 \mid \theta)\xi(\theta)$

As $\boldsymbol{x} = x_1, x_2, ..., x_n$

$\xi\,(\theta \mid \boldsymbol{x}) \propto f(\boldsymbol{x} \mid \theta)\xi(\theta)$

However, as this is proportional, to make this an equality we must add an appropriate factor.

Let: $\psi = g_n(\boldsymbol{x})^{-1}$, where $g_n(\boldsymbol{x}) = \int_\Omega f_n(\boldsymbol{x} \mid \theta)\xi(\theta)d\theta$

Thus, per 7.2.7, we have:

$\xi\,(\theta \mid \boldsymbol{x}) = f(\boldsymbol{x} \mid \theta)\xi(\theta)\psi = \frac{f(\boldsymbol{x}|\theta)\xi(\theta)}{g_n(\boldsymbol{x})}$

7.2.10

Note, $f(x \mid \theta) = 1$ for $\theta - \frac{1}{2} < x < \theta - \frac{1}{2}$ and 0 otherwise.

And similarly, $\xi(\theta) = \frac{1}{10}$ for $10 < \theta < 20$ and 0 otherwise.

If we observe $x = 12$, we then have

$\xi\,(\theta \mid x = 12)$ is positive for $11.5 < \theta < 12.5$

Additionally, given X is taken from the uniform distribution, and as θ is similarly taken from a uniform distribution, the posterior prior distribution is from the uniform distribution.

Thus, as:

$\xi\,(\theta \mid x = 12) \propto f(x \mid \theta)\xi(\theta)$

We know the posterior distribution $\xi\,(\theta \mid x = 12)$ is uniform for $11.5 < \theta < 12.5$

7.3.10

For the random sample taken from the normal distribution with θ unknown, and standard deviation, $\sigma = 2$, $\sigma^2 = 4$.

Similarly, as the prior distribution of θ is a normal distribution with standard deviation $v = 1$, $v^2 = 1$.

Note the variance of the posterior distribution, τ^2 which also follows the normal distribution, is given by the following equation:

$$\tau^2 = \frac{\sigma^2 v^2}{\sigma^2 + nv^2}$$
$$= \frac{4}{4+n}$$

For $\tau = 0.1$, we have:

$\tau^2 = 0.01 = \frac{4}{4+n}$

Solving for n, we have $n = 396$. Thus, we need at least 396 observations to reduce the standard deviation of the posterior distribution of θ, τ to the value 0.1.

7.3.21

We know the posterior distritubution of θ with the p.d.f. of the prior distribution of θ given by $\frac{1}{\theta}$ (for $\theta > 0$) is proportional to:

$$\xi\left(\theta \mid \boldsymbol{x}\right) \propto \theta^n e^{-\theta\sum\limits_{i=1}^{n} x_i} \frac{1}{\theta} \propto \theta^{n-1} e^{-\theta \bar{x_n} n}$$

Note the posterior distribution of θ is proportional to the Gamma distribution with parameters n and $n\bar{x_n}$. Hence:

$\xi\left(\theta \mid \boldsymbol{x}\right) \sim Gamma\left(\alpha = n, \beta = n\bar{x_n}\right)$.

Note, the mean of the Gamma distribution is $fract\alpha\beta$, thus the mean of the posterior distribution of θ is:

$$\frac{\alpha}{\beta} = \frac{n}{n\bar{x_n}} = \frac{1}{\bar{x_n}}$$

7.4.5

Per Ex. 5 of Section 7.3, specifically with note of Thm. 7.3.2,

The posterior distribution of θ is the Gamma distribution with parameters $\alpha + y$ and $\beta + n$.

Thus:

$\alpha = 3 + \sum\limits_{i=1}^{n} x_i = 3 + 2 + 2 + 6 + 0 + 3 = 16$

and

$\beta = 1 + n = 1 + 5 = 6$

As we now know the parameters of the posterior distribution of θ, we note the Bayes estimate is the mean of the posterior distribution. As such, we nnote the mean of the Gamma distribution is given by:

$\frac{\alpha}{\beta} = \frac{16}{6} = \frac{8}{3}$.

7.4.12

Prior to finding the posterior distribution of θ for each statistican, note, for $0 < \theta < 1$:

Statistican A: $\xi_A\left(\theta\right) = 2\theta \rightarrow$ A's prior distribution for $\theta \sim Beta(2, 1)$

Statistican B: $\xi_A\left(\theta\right) = 4\theta^3 \rightarrow$ B's prior distribution for $\theta \sim Beta(4, 1)$

A. Note Thm. 7.3.4 such that the posterior distribution of θ is as follows for statisticans A and B respectively.

(A): $\xi(\theta \mid \boldsymbol{x}) \sim Beta(\alpha + y, \beta + n - y) = Beta(2 + 710, 1 + 290) = Beta(712, 291)$

(B): $\xi(\theta \mid \boldsymbol{x}) \sim Beta(\alpha + y, \beta + n - y) = Beta(4 + 710, 1 + 290) = Beta(714, 291)$

B. Now that we know the parameters of the posterior distribution of θ for each statistican, note the Bayes estimate for each statistician is equal to the mean of the posterior distribution.

Thus, for the mean of the Beta distribution, $\frac{\alpha}{\alpha+\beta}$, we have:

(A): $\frac{\alpha}{\alpha+\beta} = \frac{712}{712+291} = \frac{712}{1003}$

(B): $\frac{\alpha}{\alpha+\beta} = \frac{714}{714+291} = \frac{714}{1005}$

C. The difference between the Bayes estimates of statisticans A and B are given by the following equation:

(A): $\frac{2+y}{2+y+1+1000-y} = \frac{2+y}{1003}$

(B): $\frac{4+y}{4+y+1+1000-y} = \frac{4+y}{1005}$

Difference, (A)-(B): $\frac{2+y}{1003} - \frac{4+y}{1005} = \frac{1003(4+y)-1005(2+y)}{1003(1005)} = \frac{2002-2y}{1003(1005)} = \frac{2(1001-y)}{1003(1005)}$

Note, we maximize the difference between the Bayes estimates of A and B when $y = 0$, i.e. no one votes in favor of the proposition. For $y = 0$, we have:

$\frac{2002}{1003(1005)} = 0.001986 \approx 0.002$

It follows that the Bayes estimates of statisticans A and B could not differ by more than 0.002

Additional Exercise

For 7.4.12, use the `prop_model()` function in the slides to visualize the prior and posterior distributions of statisticians A and B.

For 7.4.12, use the `prop_model()` function in the slides to visualize the prior and posterior distributions of statisticians A and B. Note: for this exercise, the order of the 710 'TRUE' values and 290 'FALSE' values is arbitrary. To illustrate this point, the prop_model of Statistican A is illustrated with 'TRUE' values first in the first output, and 'TRUE' values last in the second output.

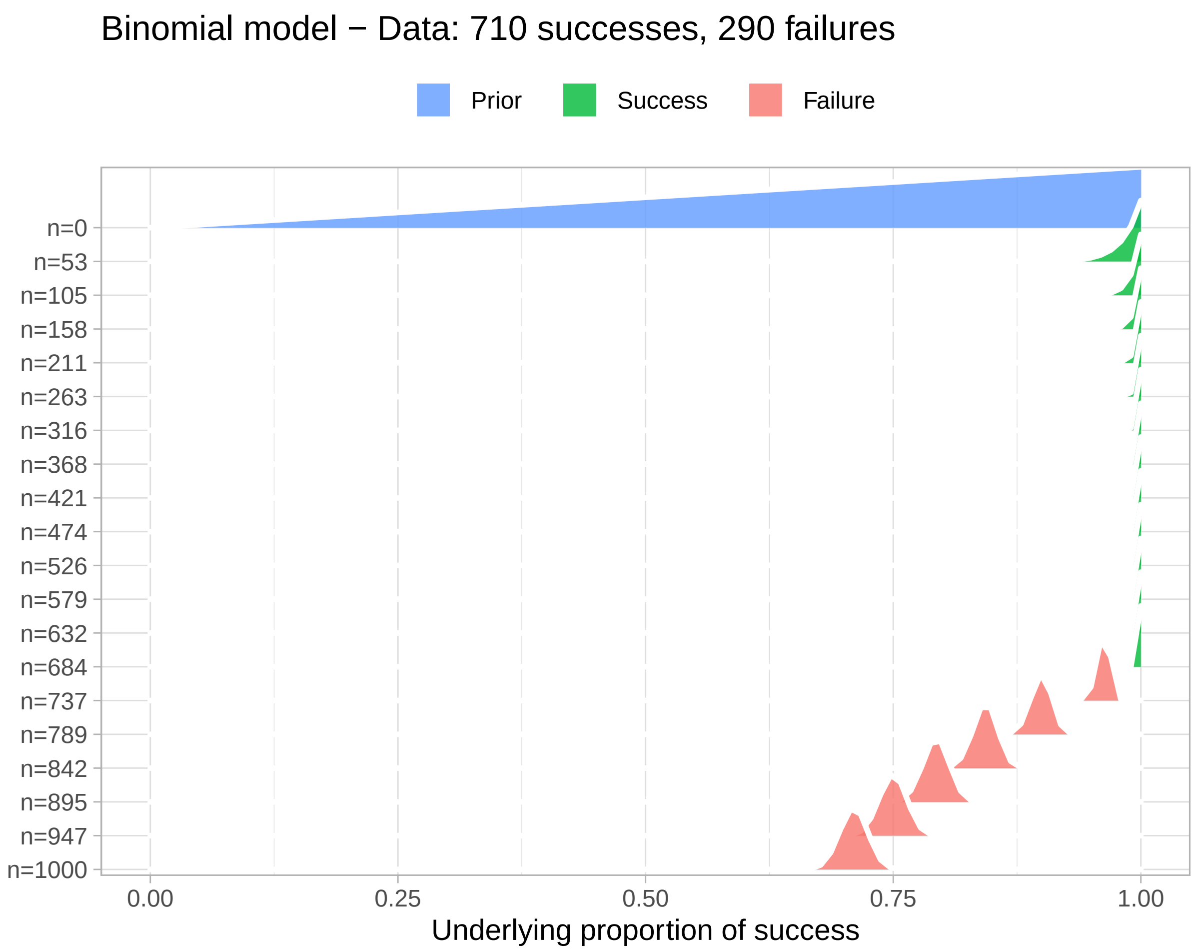
–

```
trues <- rep(TRUE, times=710)
falses <- rep(FALSE, times=290)
sampledata <- c(trues, falses)

prop_model(sampledata, prior_prop = c(2, 1))
```

Stat A Pre-Post ('TRUE' first) –

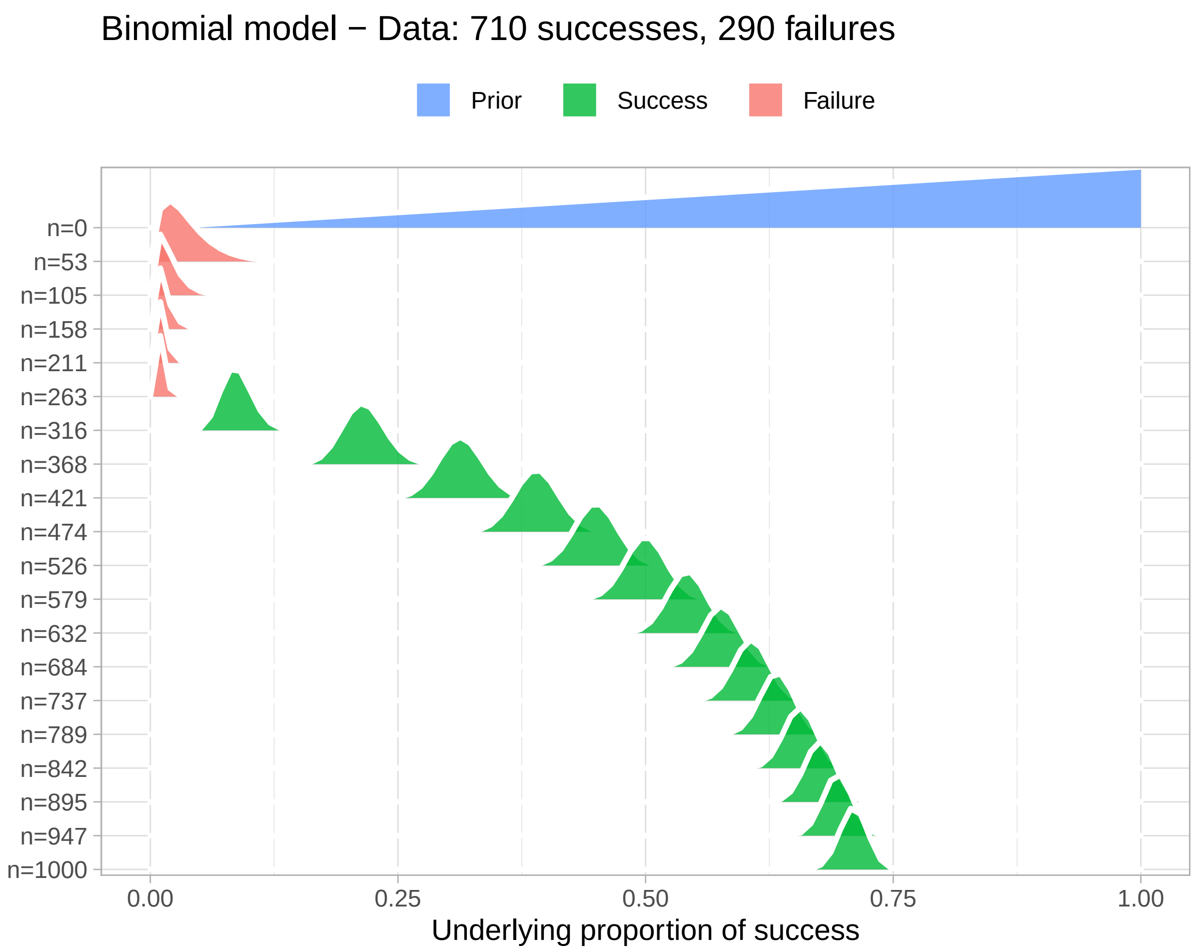
```
## Warning: `data_frame()` is deprecated, use `tibble()`.
## This warning is displayed once per session.
```




```
trues <- rep(TRUE, times=710)
falses <- rep(FALSE, times=290)
sampledata <- c(falses, trues)

prop_model(sampledata, prior_prop = c(2, 1))
```

Stat A Pre-Post ('TRUE' second) –

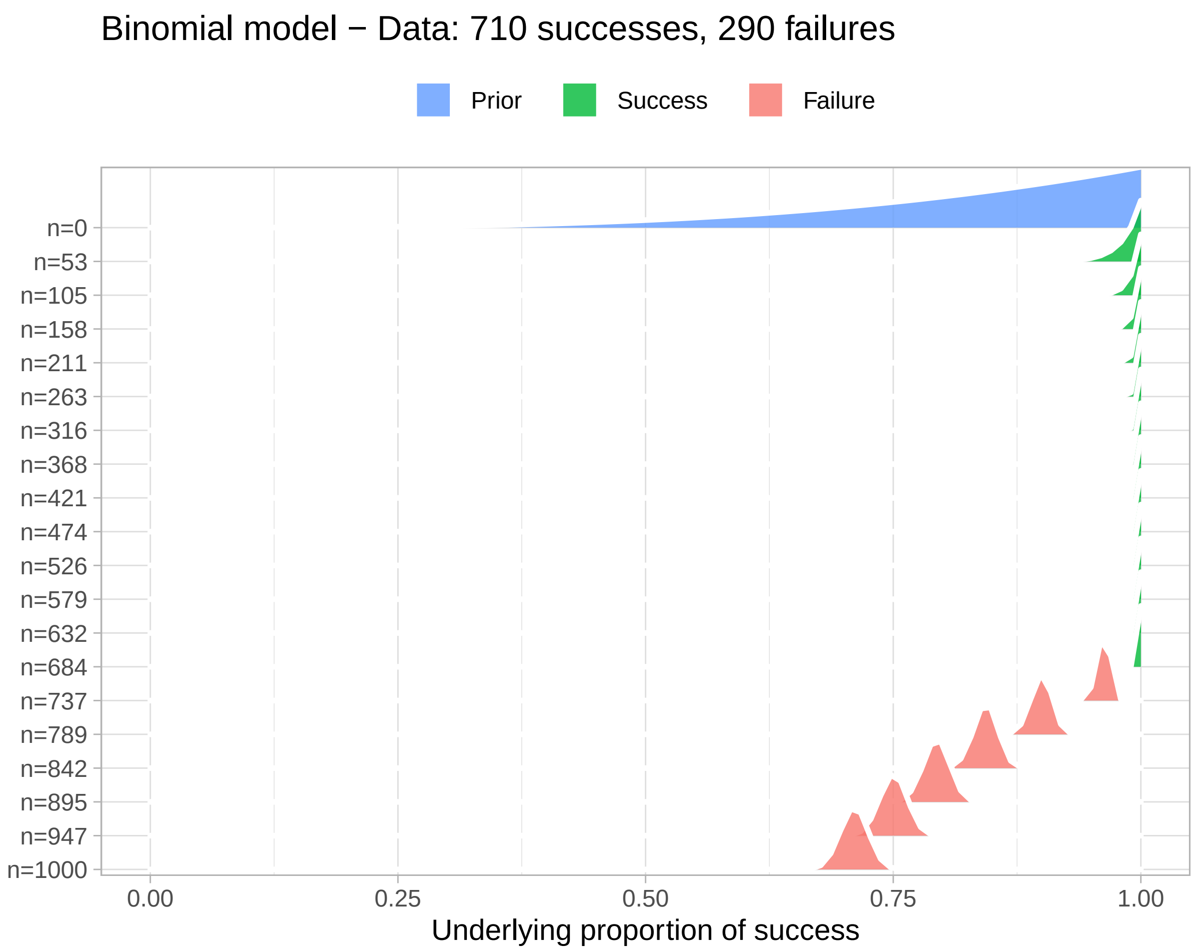


Stat B Pre-Post –

```
trues <- rep(TRUE, times=710)
falses <- rep(FALSE, times=290)
sampledata <- c(trues, falses)

prop_model(sampledata, prior_prop = c(4, 1))
```

–


Binomial model − Data: 710 successes, 290 failures
Prior
Success
Failure
n=0
n=53
n=105
n=158
n=211
n=263
n=316
n=368
n=421
n=474
n=526
n=579
n=632
n=684
n=737
n=789
n=842
n=895
n=947
n=1000
0.00
0.25
0.50
0.75
1.00
Underlying proportion of success