VIT-AP UNIVERSITY

Computer Vision

(Course Code: 4047)

Module-2:Lecture-12: Wavelets

Gundimeda Venugopal, Professor of Practice, SCOPE

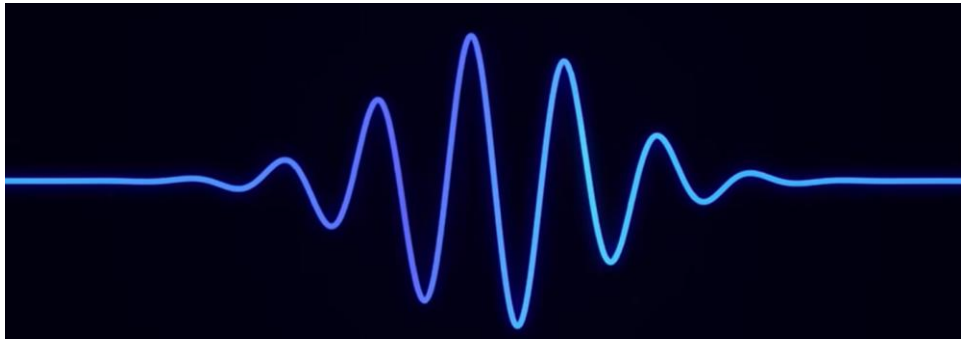
Wavelet Definition

"The wavelet transform is a tool that cuts up data, functions or operators into different frequency components, and then studies each component with a resolution matched to its scale"

Dr. Ingrid Daubechies, Lucent, Princeton U

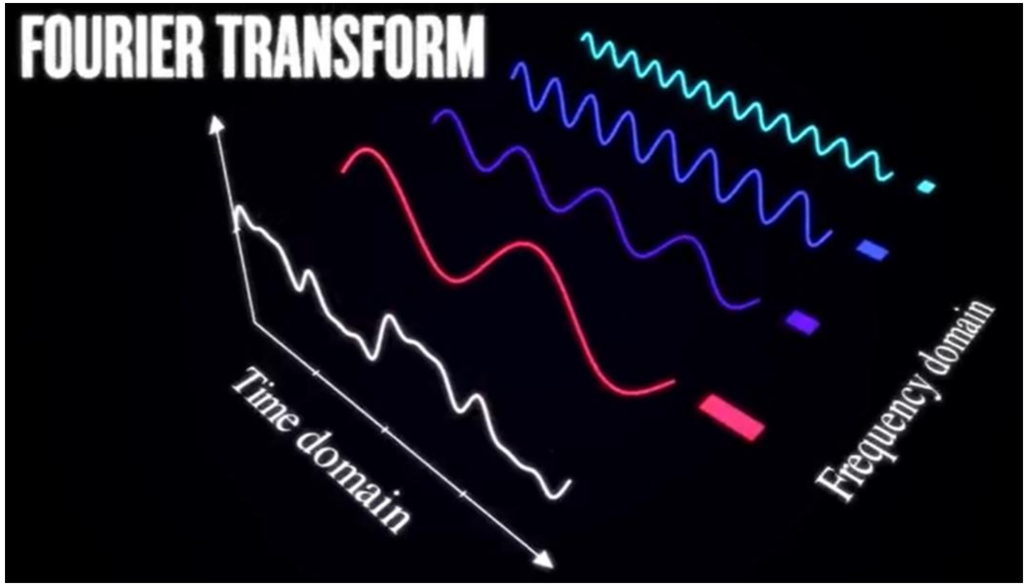
Wavelet Transform

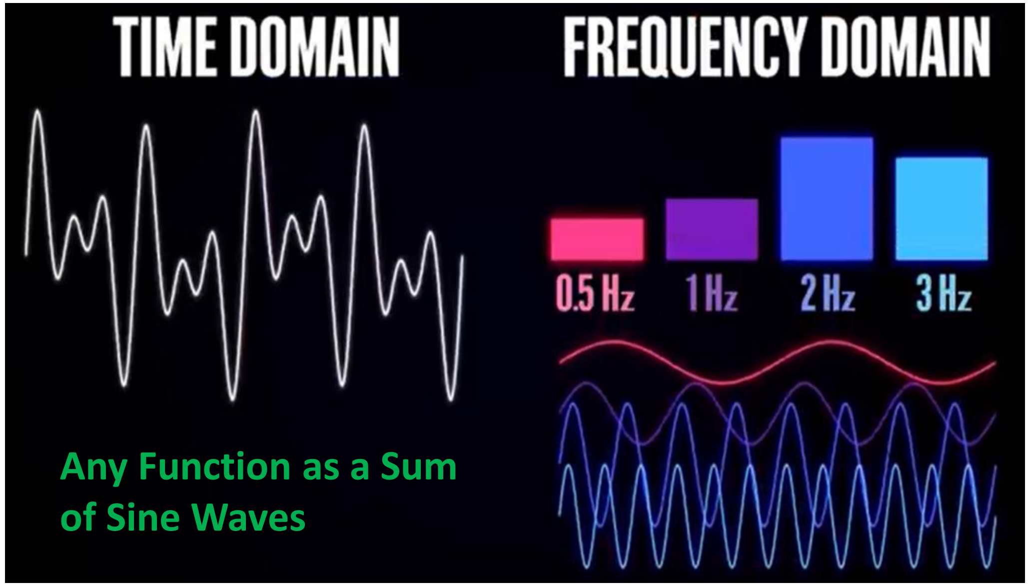
- Scale and shift original Wavelet
- Compare to the Input Signal
- Assign a coefficient of similarity
- Signal becomes a set of wavelet coefficients
- Coefficients represent features of signal
- Signal can be completely reconstructed from all the coefficients
- Signal can be partially reconstructed from some coefficients

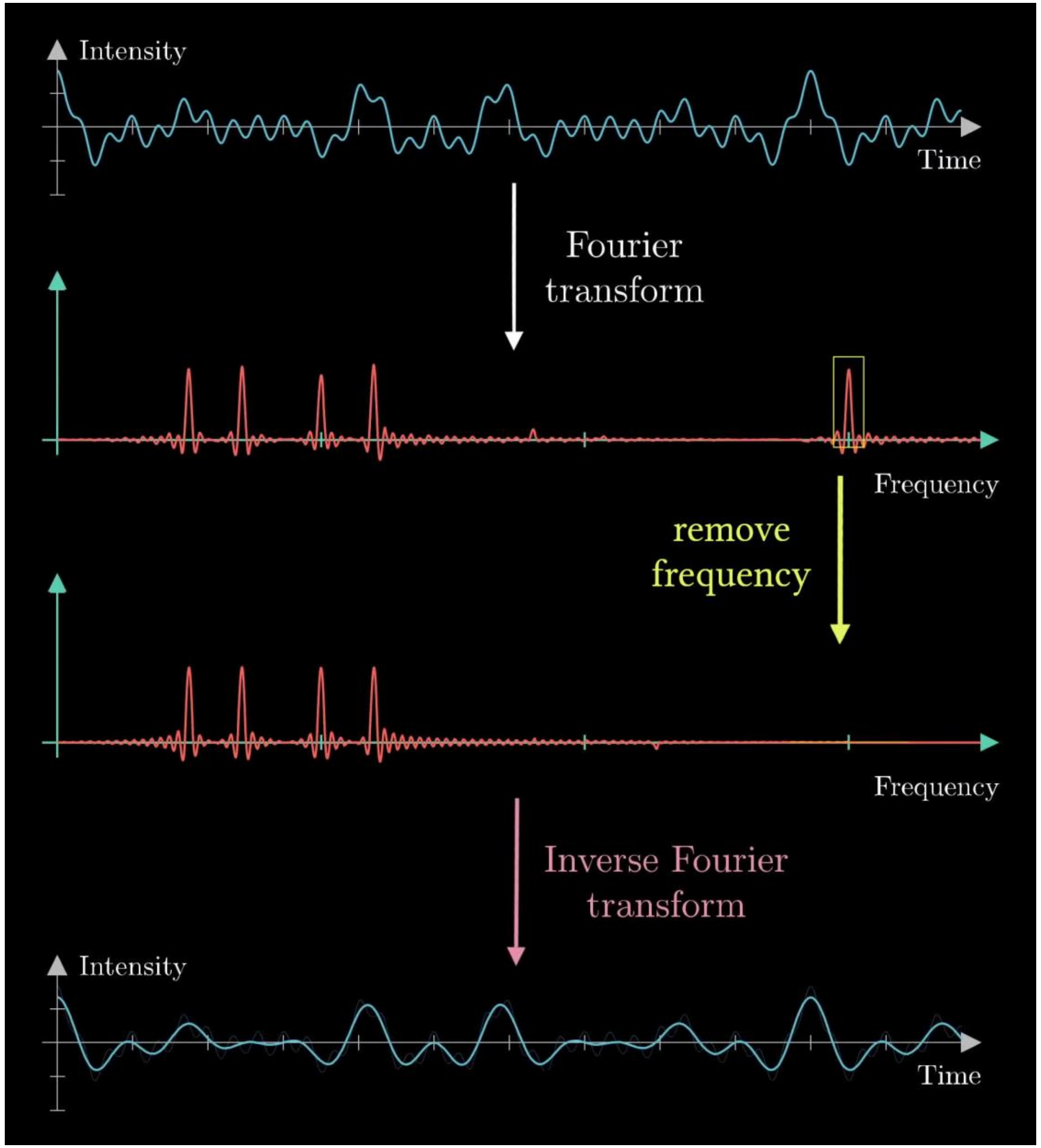


Morlet Wavelet

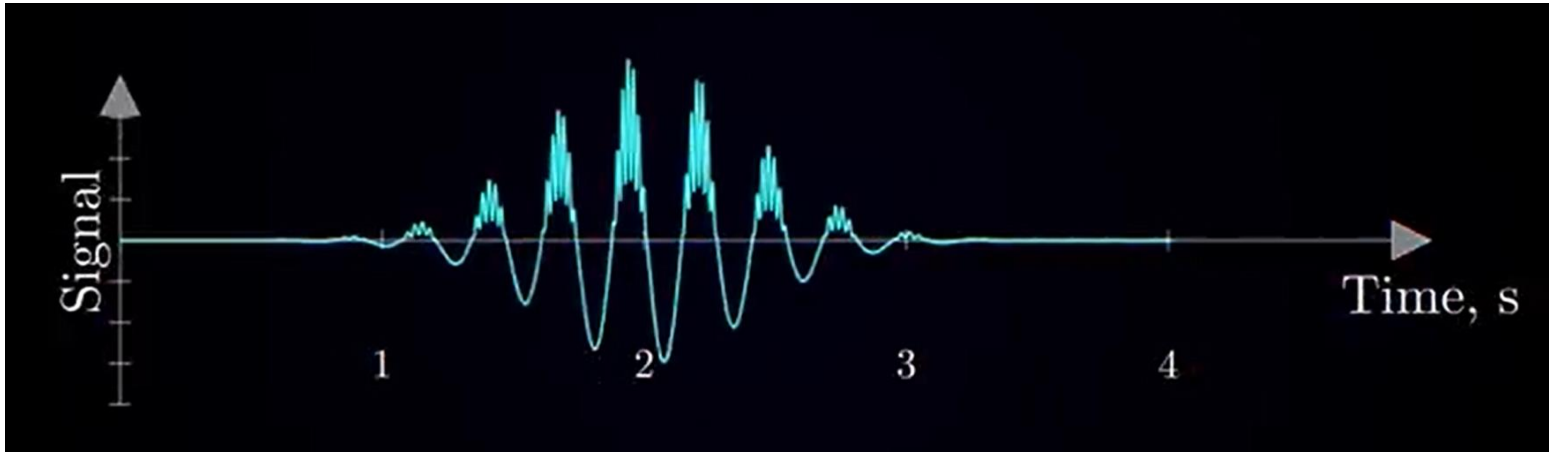
Fourier Transform: Recap

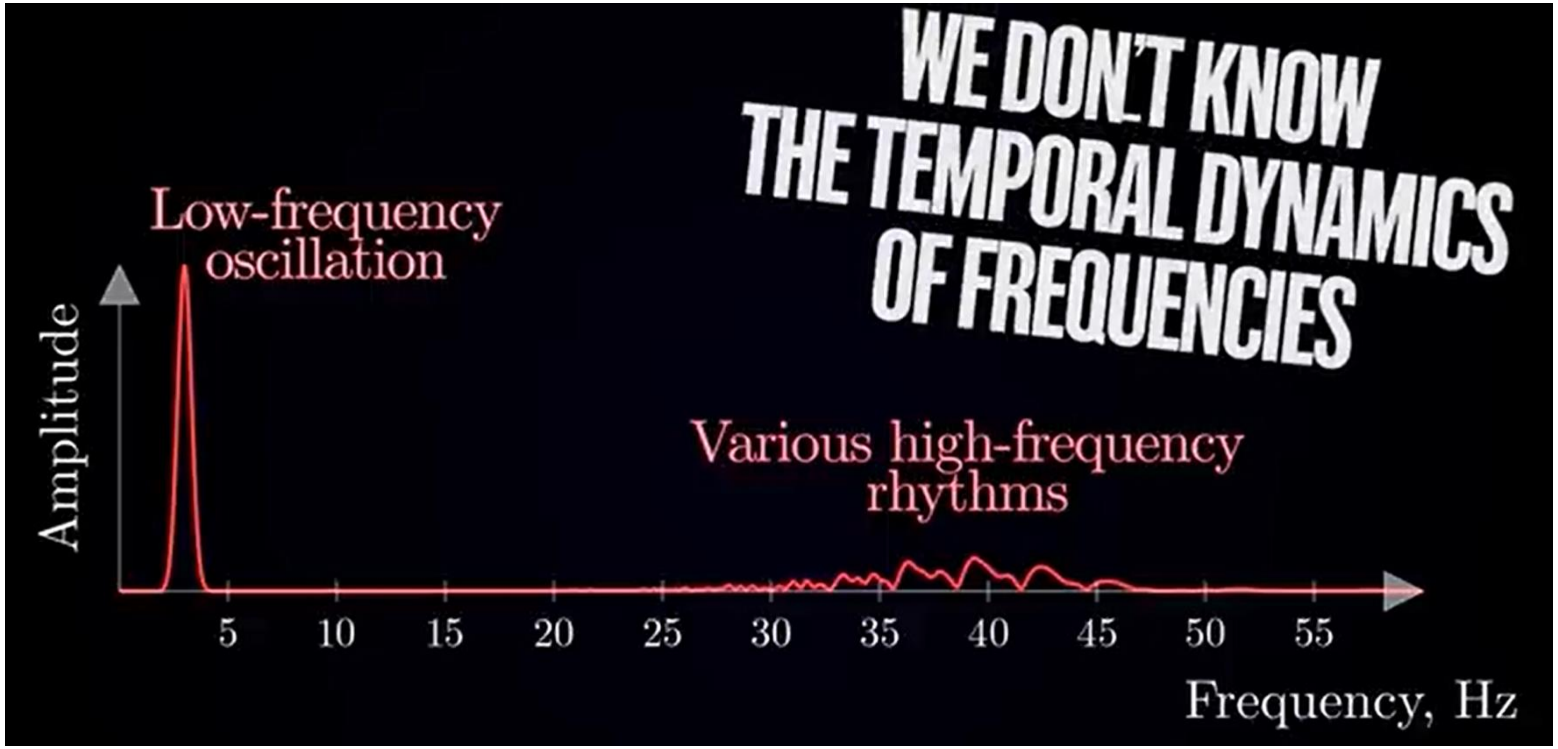






Fourier Transform: Signals with abrupt changes





- ❖ Do a very poor job in approximating sharp spikes
- ❖ Does not capture any events over time. (Temporal Dynamics)

At what time the frequency components occur? Fourier Transform cannot tell.

Stock Market Analysis: A Time series example

Long term trends
Short term oscillations

2024 Sensex (last 1 year)

2022 Sensex

What is Localization?

- Localization
  - In time: you can tell (more or less exactly) when something happened
  - In space: you can tell (more or less exactly) where something happened
  - In frequency: you can tell (more or less exactly) the frequencies that make up an event
- If you have a signal as a function of time (or space), you have near perfect temporal/spatial localisation of features, but can't really tell the frequency.
- If you Fourier transform the signal, you have perfect information on the involved frequencies, but can't easily tell when/where exactly a particular feature is.

Stationary and Non-stationary Signals

❖ Stationary Signal

- ➢ Signals with frequency content unchanged in time
- ➢ All frequency components exist at all times
- ➢ E.g., Frictionless pendulum, Single frequency sine waves, Red traffic light and White noise

❖ Non-stationary Signal

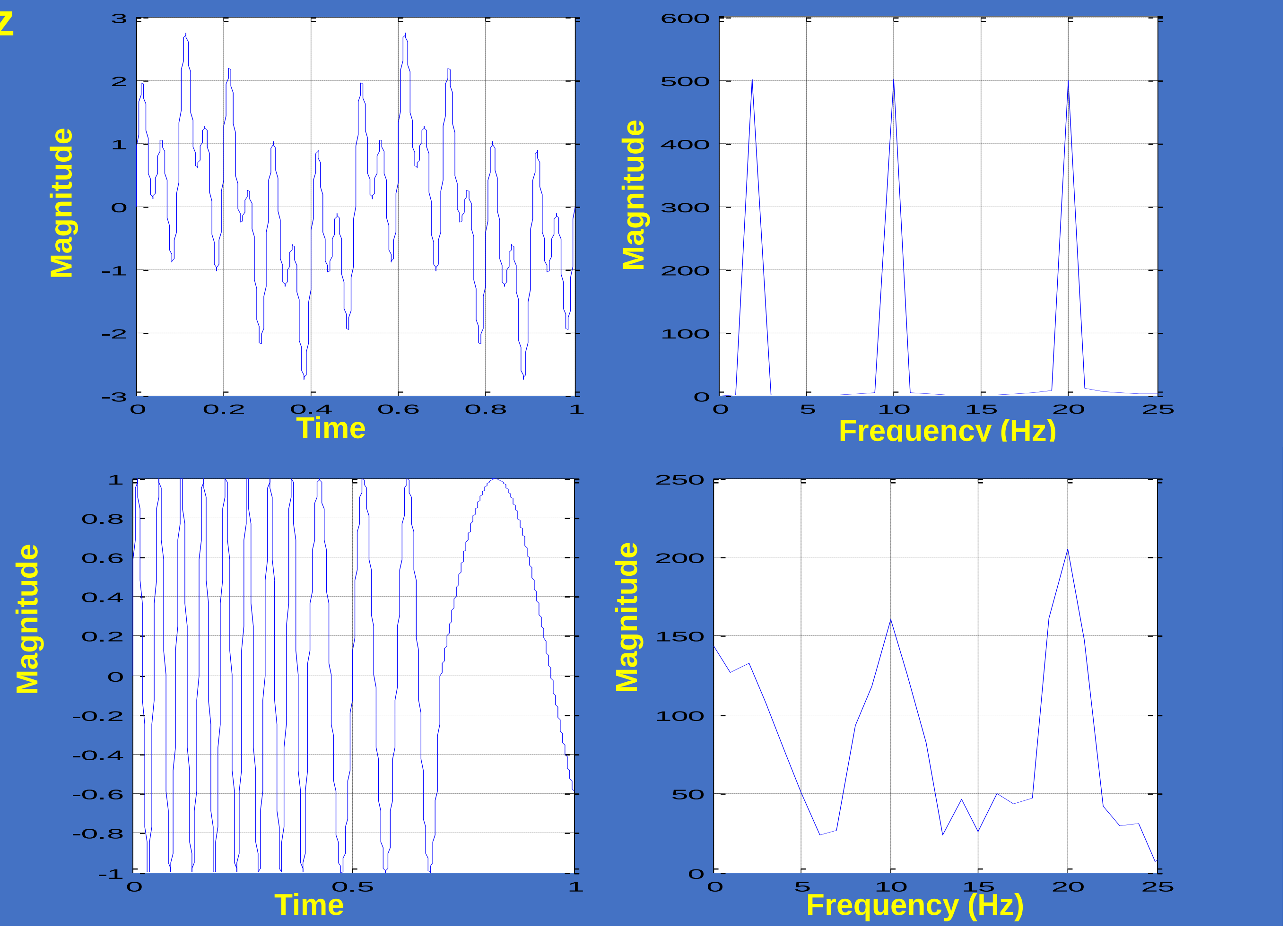
- ➢ Frequency changes in time
- ➢ E.g., The "Chirp Signal", Speech, EEG, ECG, PPG recordings, Machine vibration, Sonar, Communication, and Biological signals, stock market indices (e.g., Sensex)

2 Hz + 10 Hz + 20Hz

Stationary

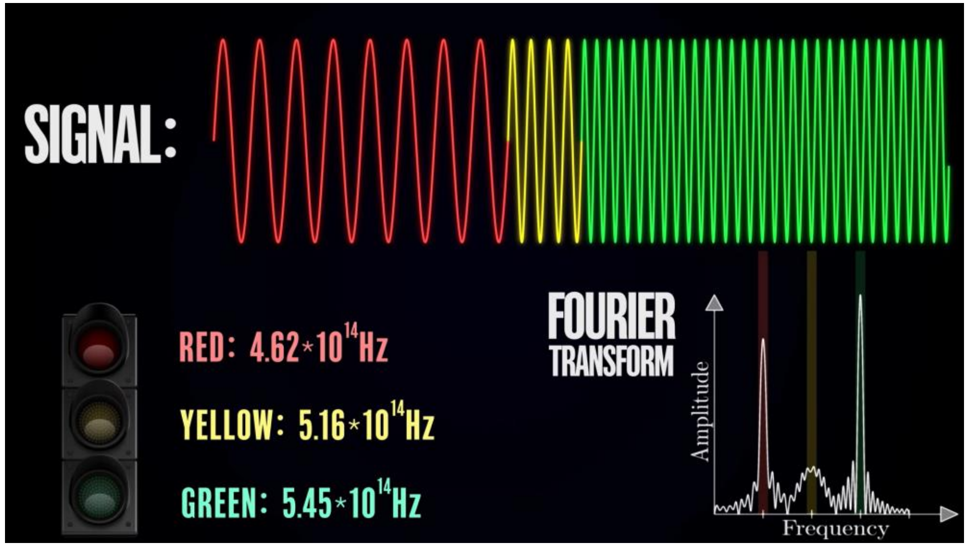
0.0-0.4: 2 Hz +
0.4-0.7: 10 Hz +
0.7-1.0: 20Hz

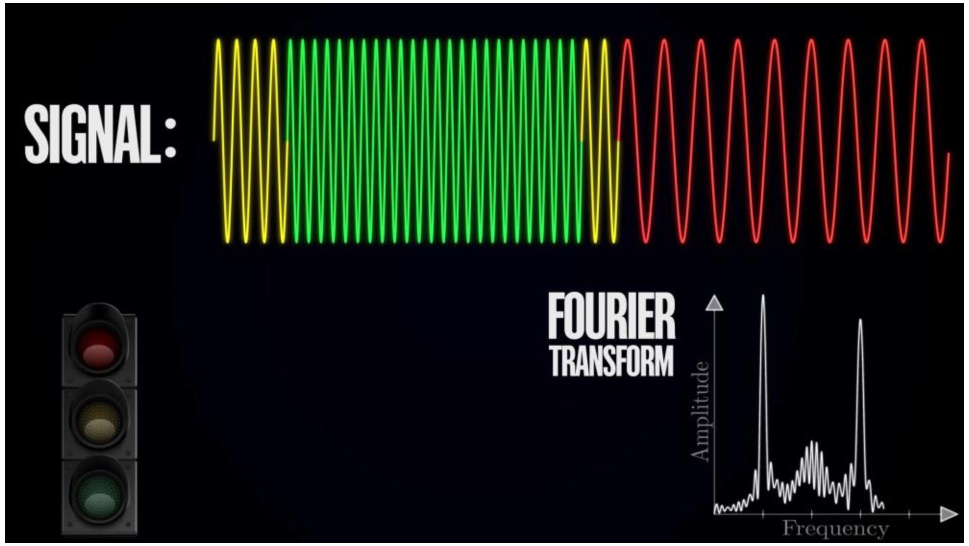
Non-Stationary

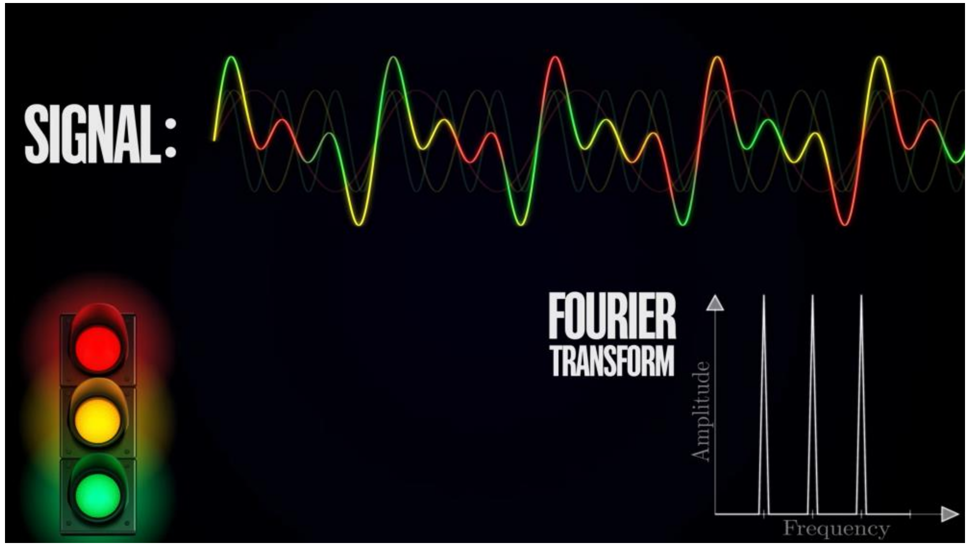


Fourier Transform- Not Localised in time: Traffic Light Example

Fourier Transform is completely blind through time. Does not capture any traffic light events over time.







Scenario 1: Red 20 sec, Yellow 5 sec and Green 30 sec
Scenario 2: Yellow 5 sec, Green 22 sec Yellow 3 sec and Red 20 sec
Scenario 3: Red, Green and yellow all are on.

All three scenarios produce the same Fourier transform
Showcasing the frequency signals for R, Y and G

Most of Transportation Signals are Non-stationary.

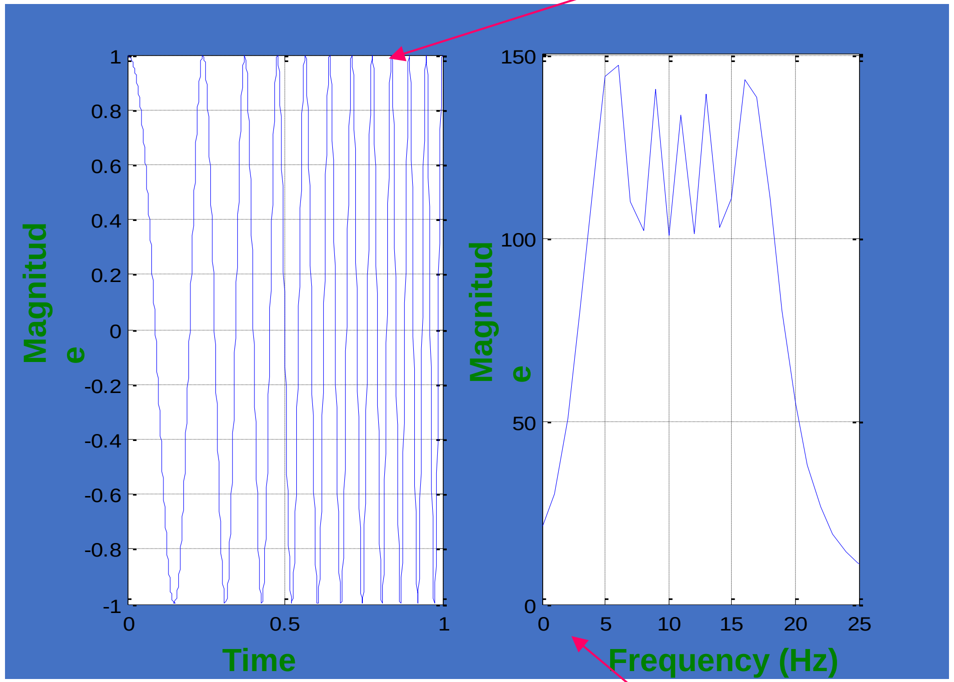
(We need to know whether and also when an incident was happened.)

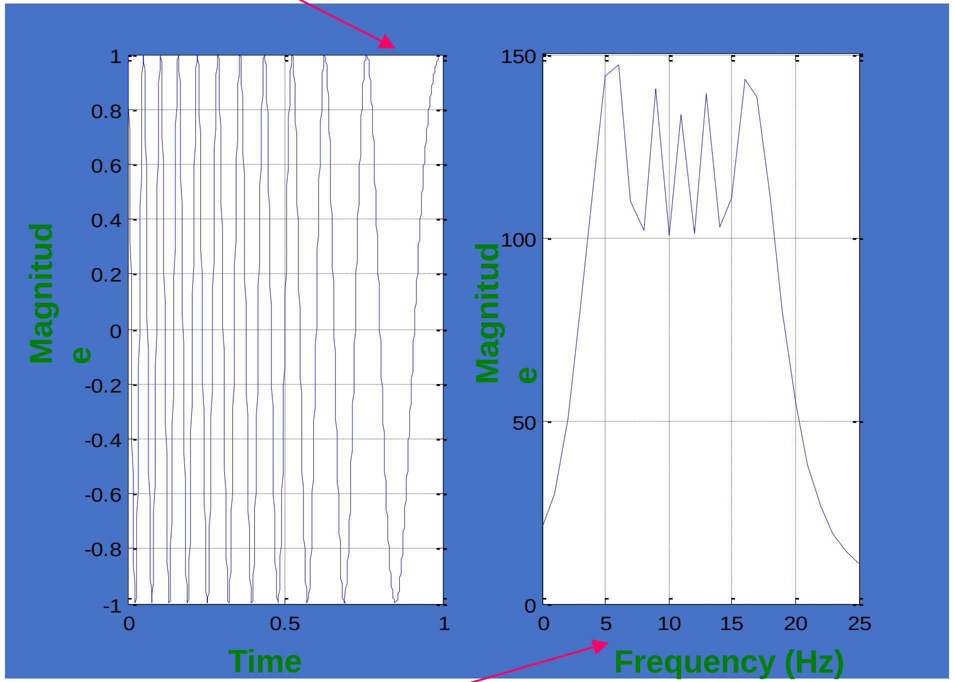
Chirp Signals

Frequency: 2 Hz to 20 Hz

Different in Time Domain

Frequency: 20 Hz to 2 Hz

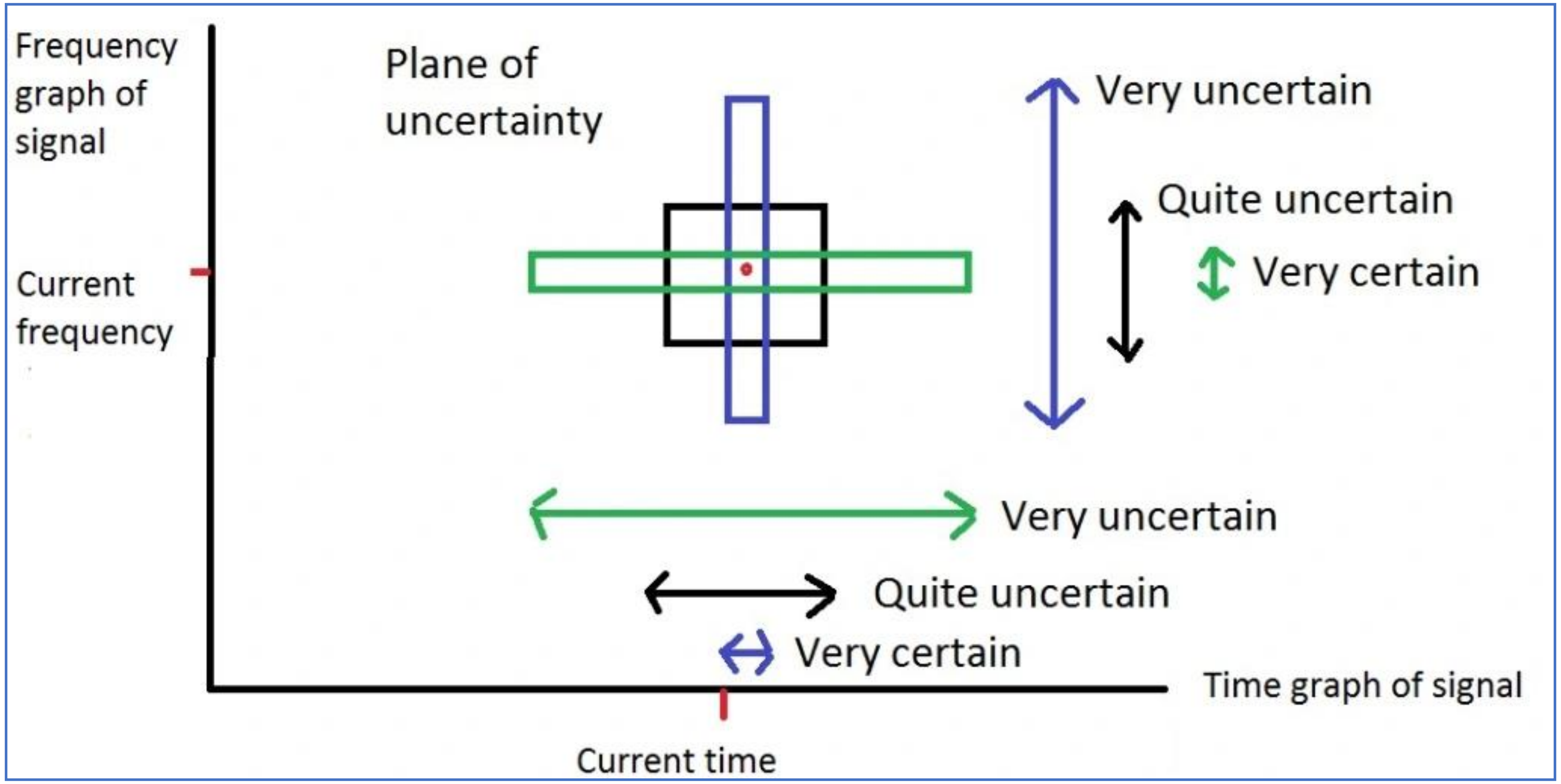




Same in Frequency Domain

At what time the frequency components occur? FT can not tell!

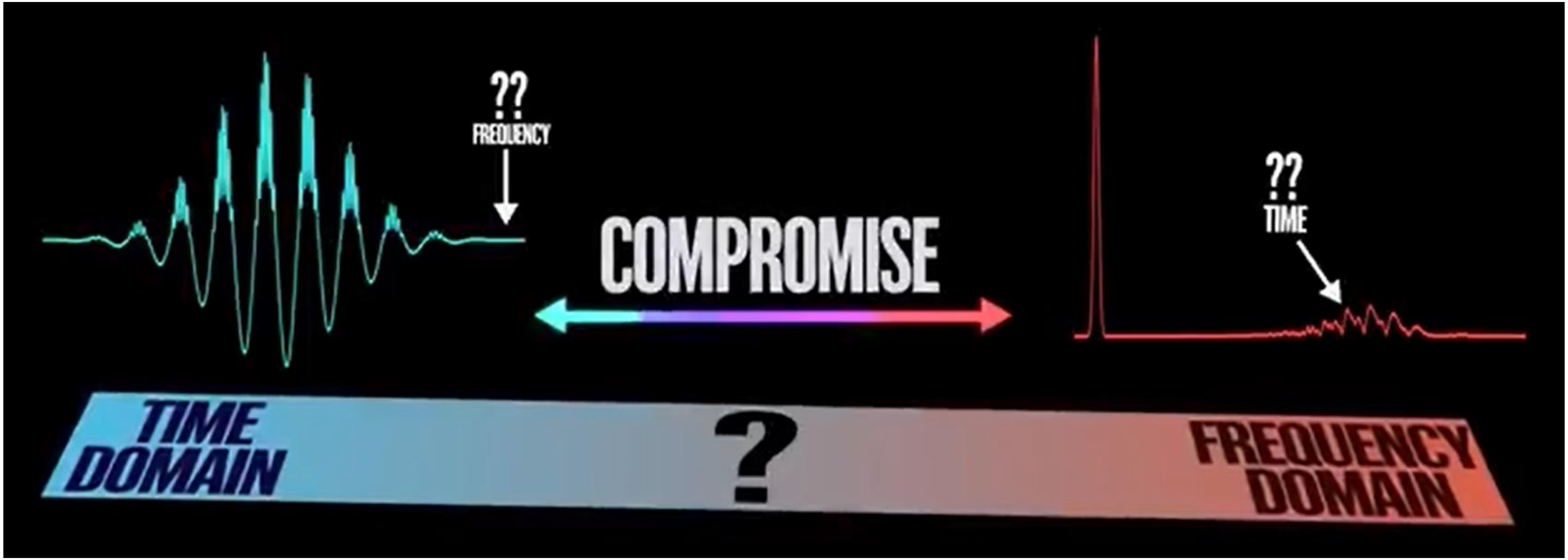
Fourier Transform: Localization in time and frequency



- The Fourier Transform has the problem of resolution. You can either be sure of the frequency or the time of a signal, but not both.
- When you are studying a real signal it would be useful to know what the 'instantaneous frequency' of the signal is. The instantaneous frequency is the exact frequency of a signal at an exact moment in time.
- For instance if I was listening to a music track I would like to be able to say 'at 1 min 59.0423 seconds into the music track the sound is 1563.2 Hz'.
- Unfortunately the Fourier Transform cannot do this because there exists a minimum amount of uncertainty between the frequency and time domains.
- You can know the moment in time you want to find the frequency for (like the blue box), but because there is a minimum uncertainty, the box of has to stretch out across frequency, meaning that you are unsure of the frequency at that moment in time.

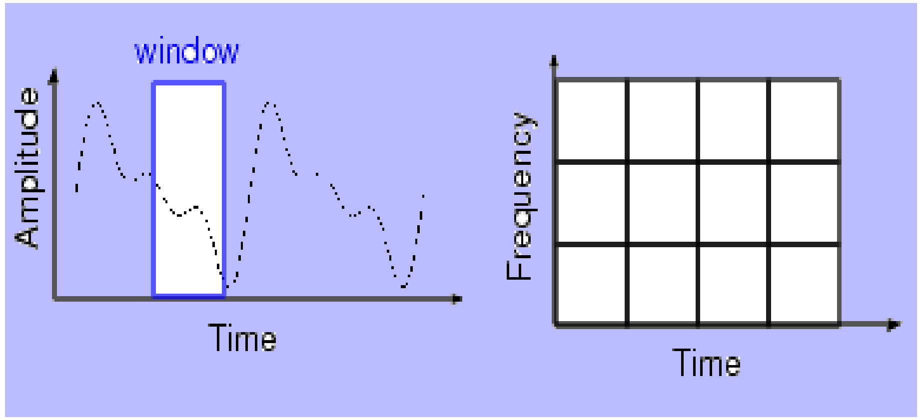
- FT Only Gives what Frequency Components Exist in the Signal
- The Time and Frequency Information can not be Seen at the Same Time
- Time-frequency Representation of the Signal is Needed

Time – Frequency Duality



Short Time Fourier Transform (STFT)

- Dennis Gabor (1946) Used STFT
  - To analyze only a small section of the signal at a time -- a technique called *Windowing the Signal*.
- The Segment of Signal is Assumed *Stationary*
- A 3D transform



$$\mathrm{STFT}_X^{(\omega)}(t', f) = \int_t \left[x(t) \bullet \omega^*(t - t')\right] \bullet e^{-j2\pi ft} dt$$

$\omega(t)$: the window function

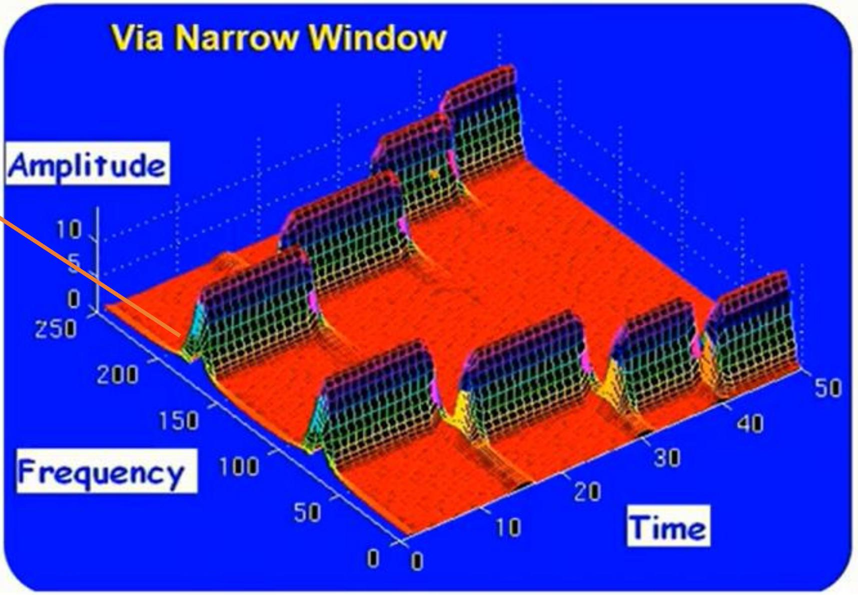
A function of time and frequency

Drawbacks of Short Time Fourier Transform (STFT)

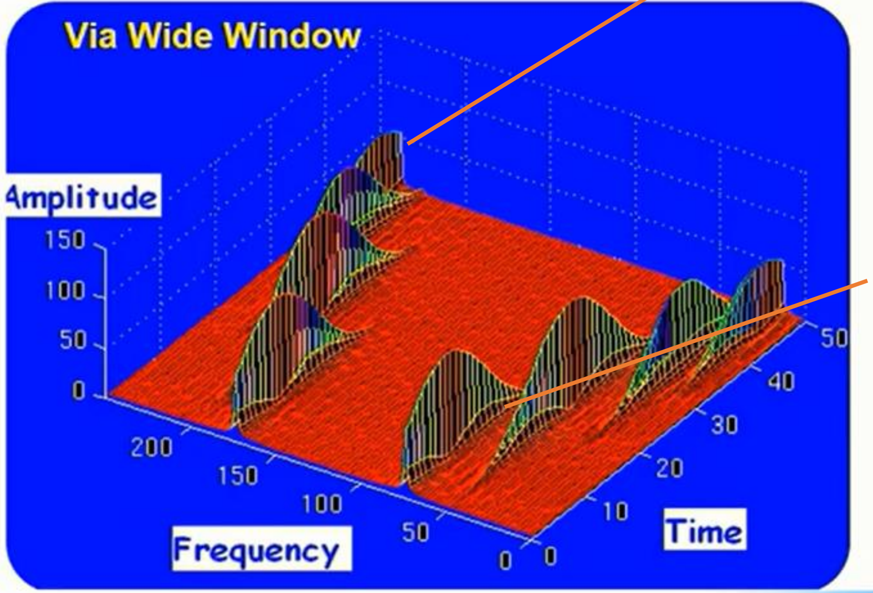
- **Unchanged Window**
- **Dilemma of Resolution**
  - ➢ Narrow window (good time resolution) -> Poor frequency resolution.
  - ➢ Wide window (poor time resolution) -> Good frequency resolution.
- **Uncertainty?**

Cannot know what frequency exists at what time intervals.

Band frequency response (frequency resolution less)

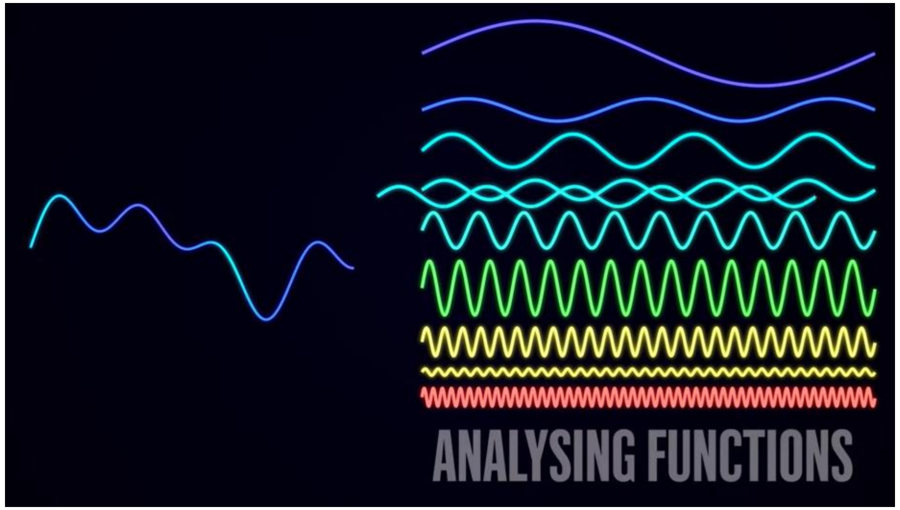


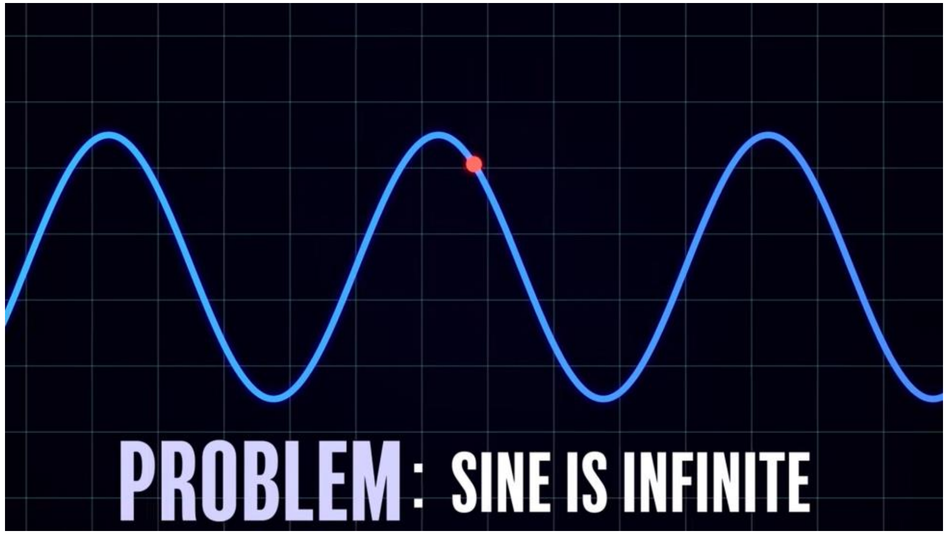
Sharp frequency response

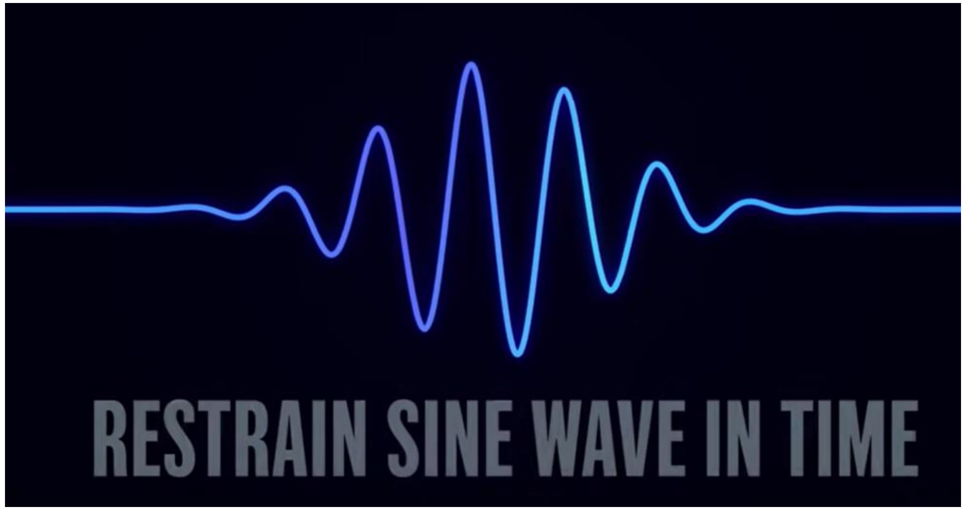


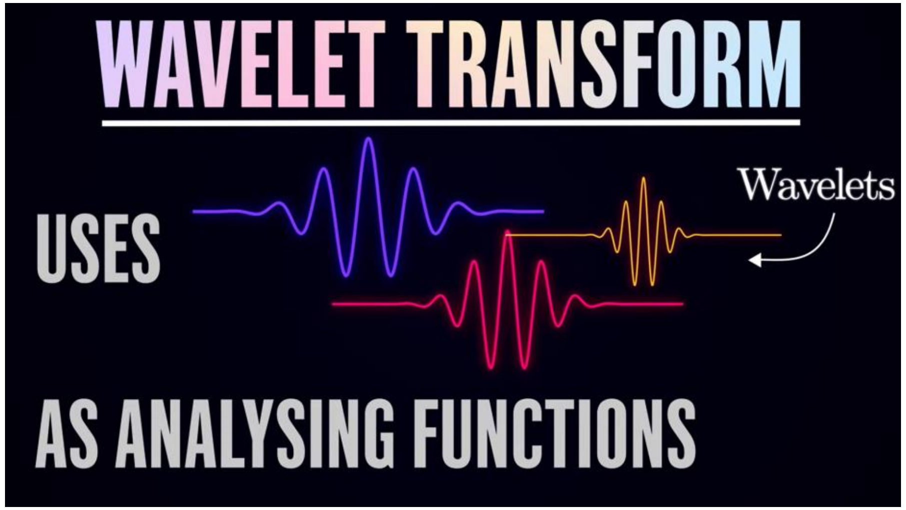
Overlapping frequencies responses. Burst, spike related data not visible

Evolution of a Wavelet

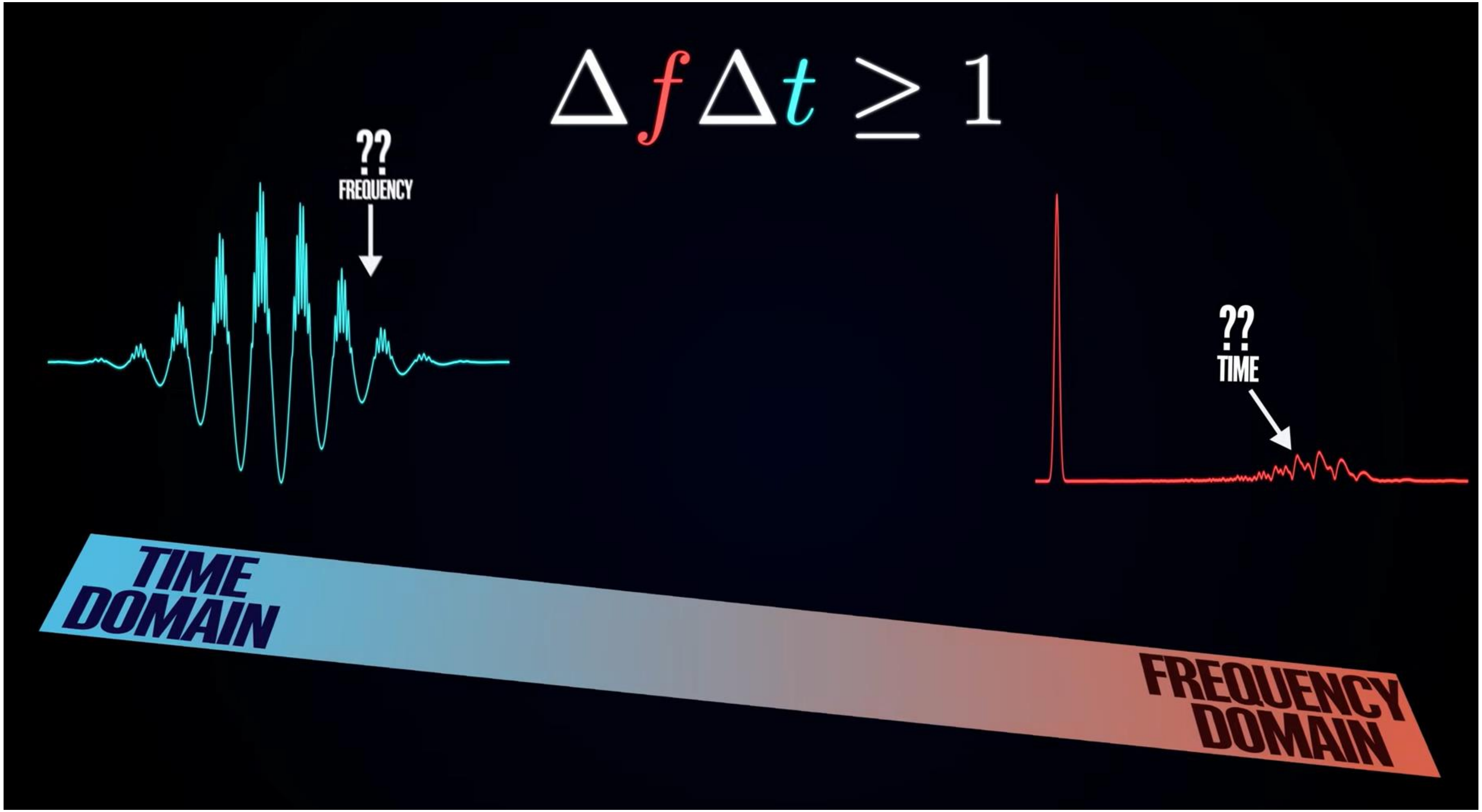






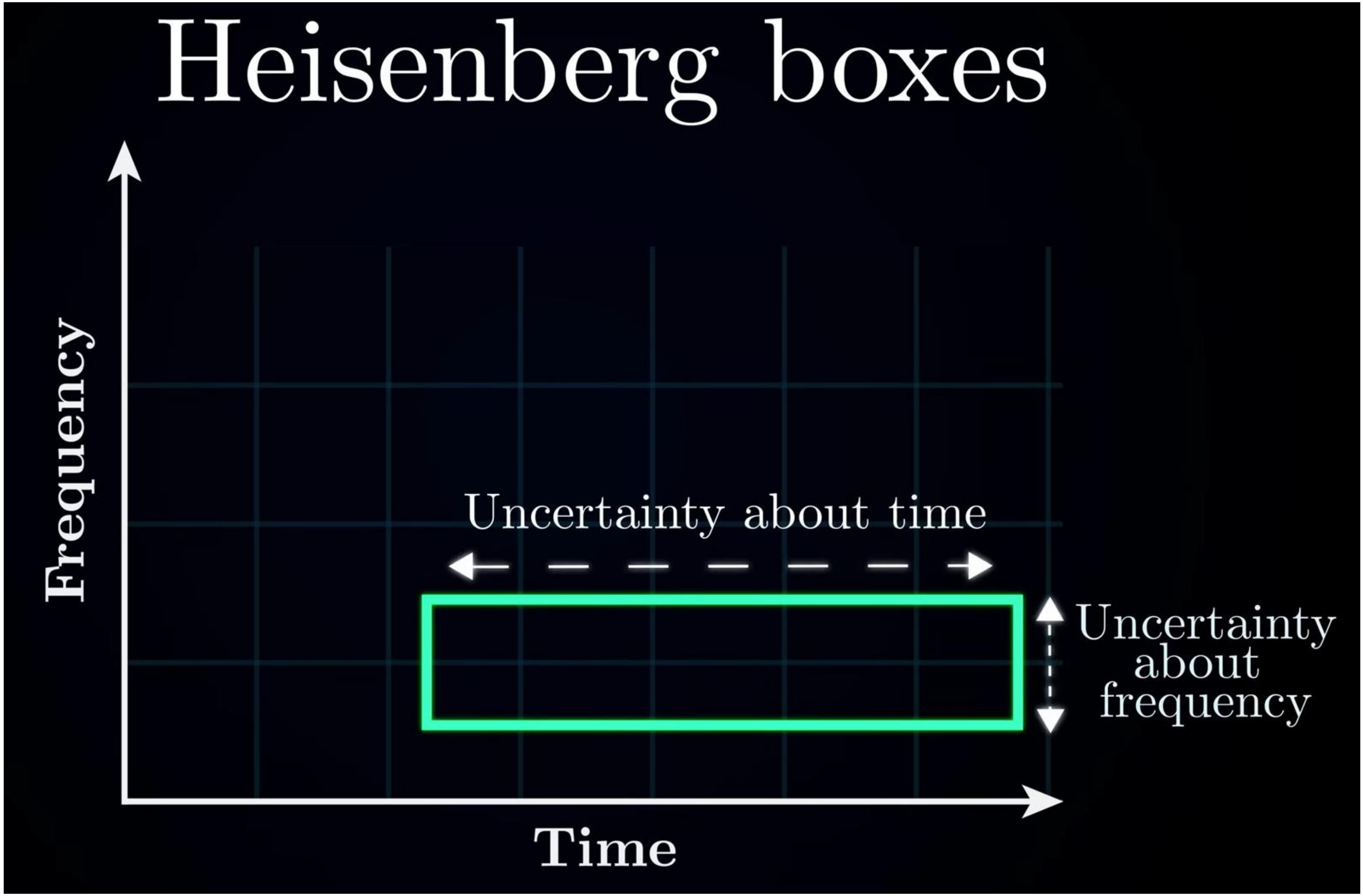


Time Frequency Resolution Trade-off



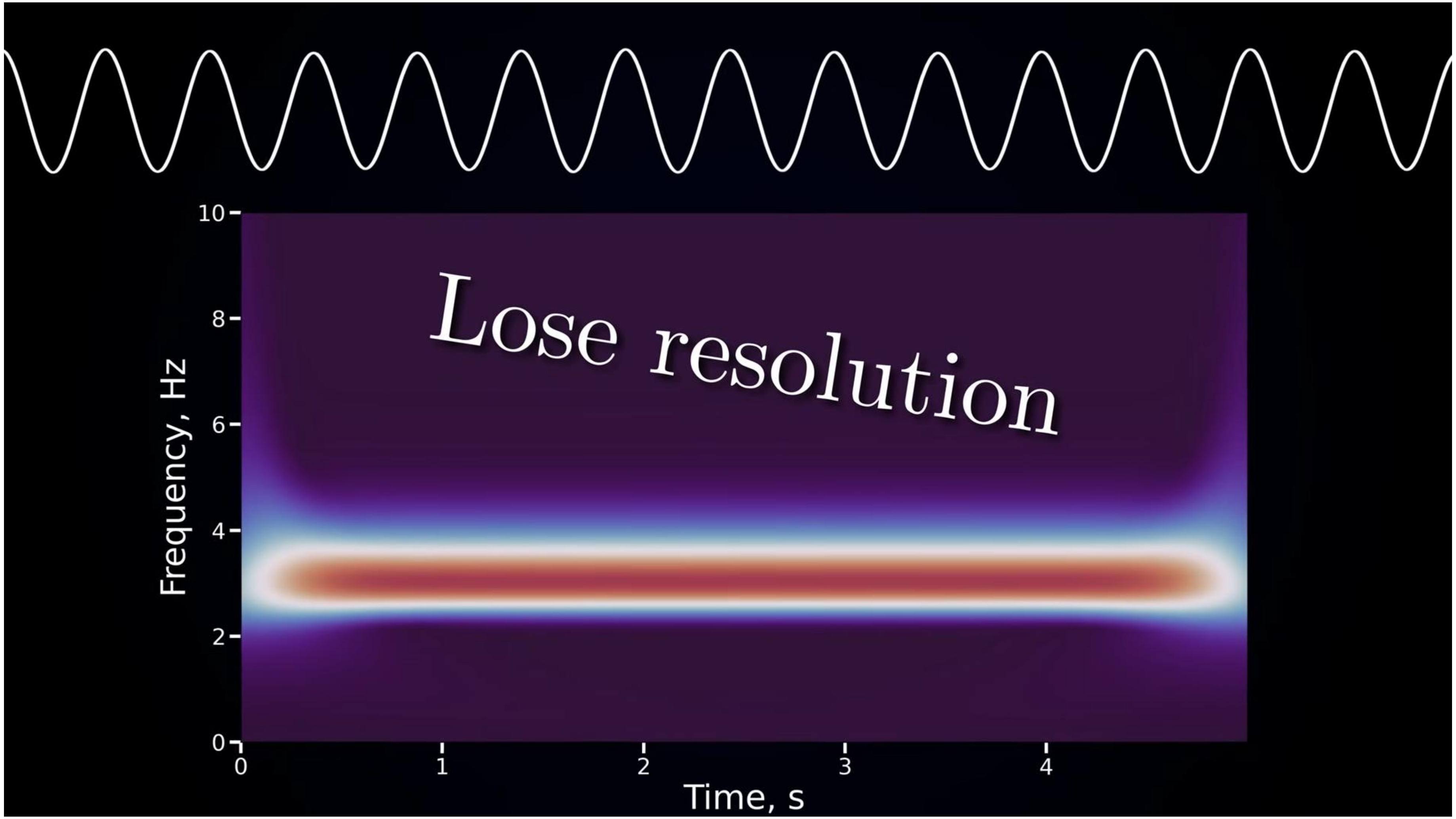
You cannot perfectly know frequency and time at the same time

Heisenberg boxes



Wavelets: Lose resolution in both frequency and time

Do wavelets violate Heisenberg Uncertainty principle? No.

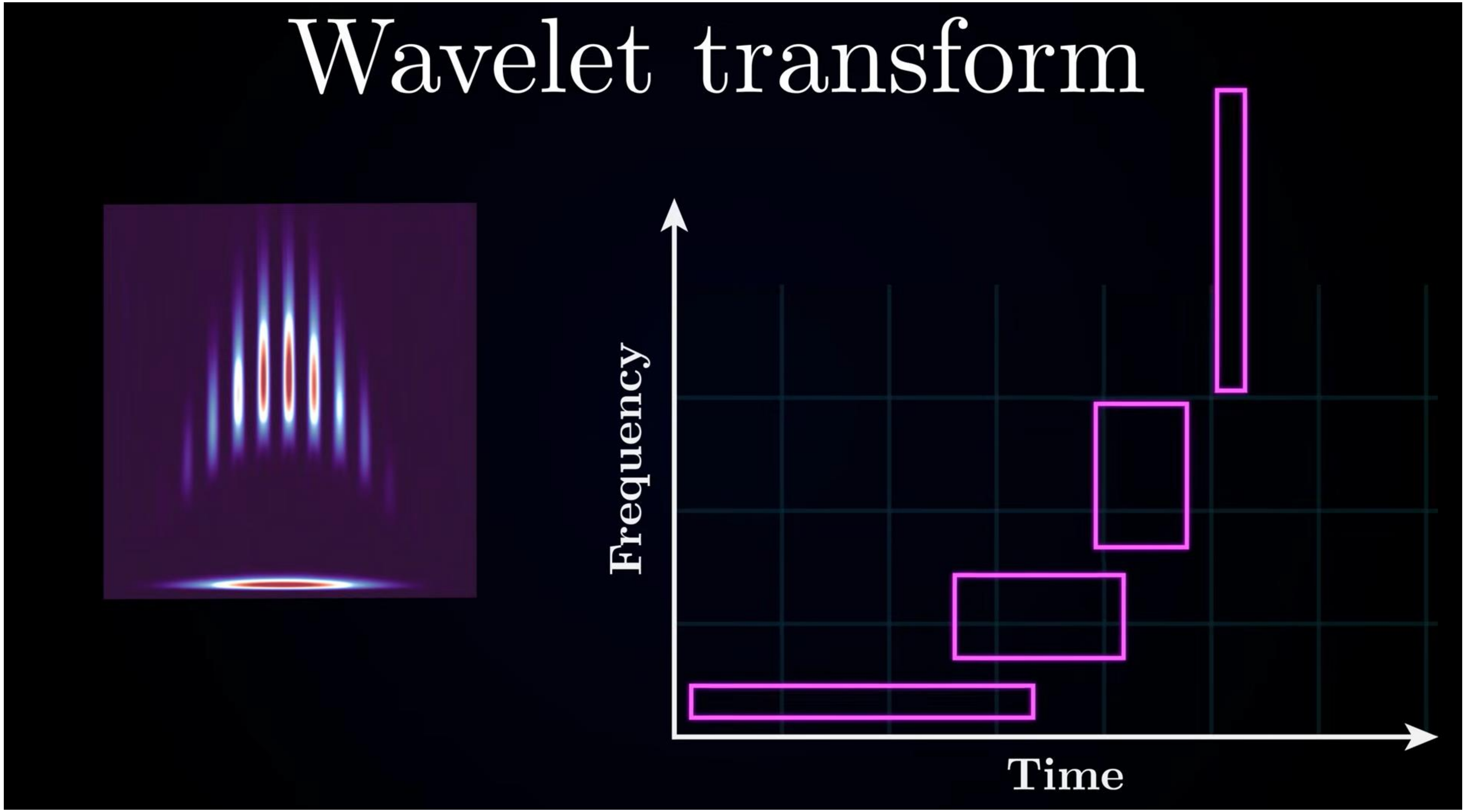


Analyzing a sine wave using wavelets: We lose resolution in both time and frequency to know something about both.

Wavelet Transform: Heisenberg Boxes

At **higher** frequencies, we can see poor frequency resolution and good time resolution

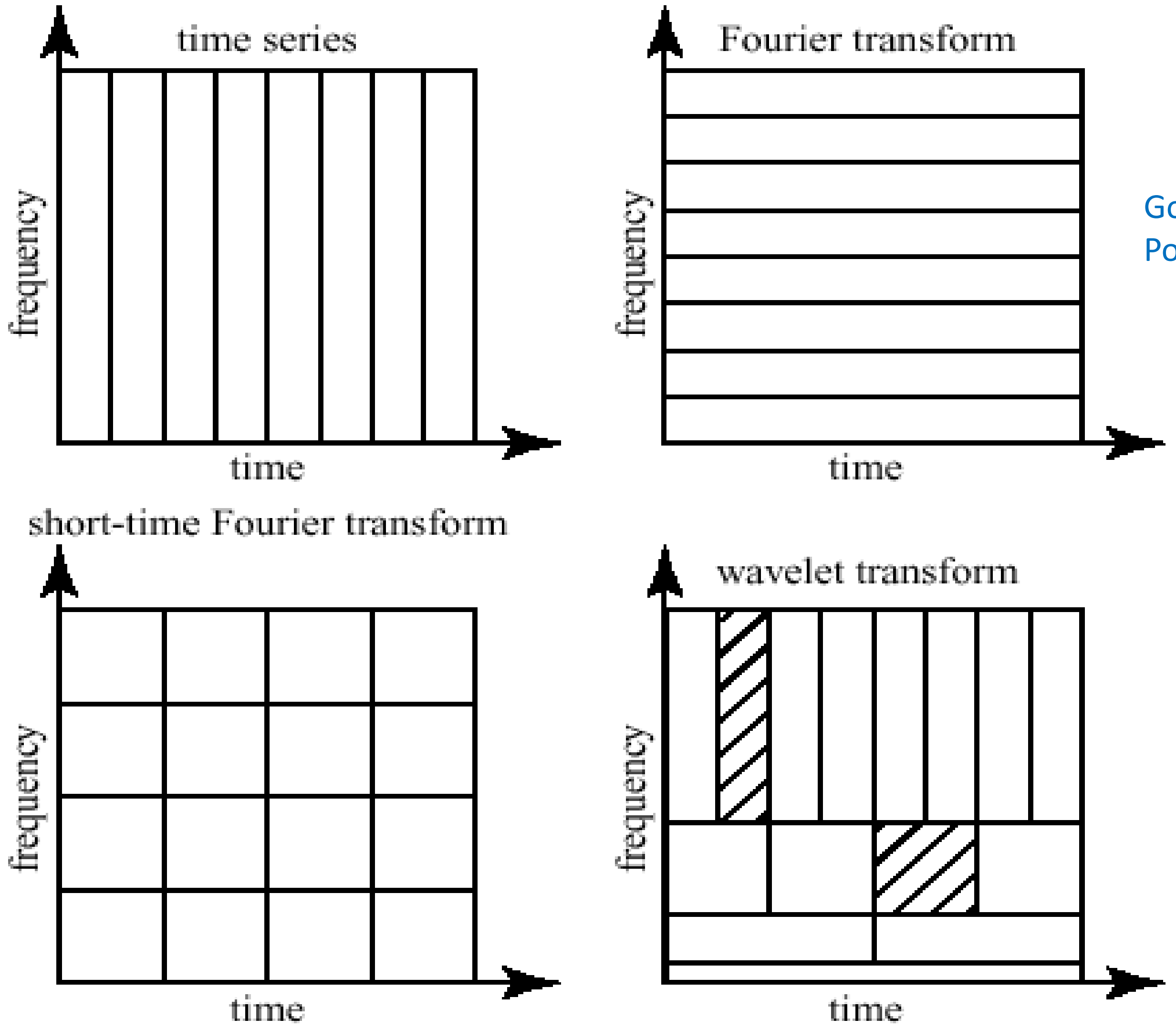
At **low** frequencies, we can see the high frequency resolution and poor time resolution



Time Resolution vs Frequency Resolution

Good Time Resolution
Poor Frequency Resolution

Good Frequency Resolution
Poor Time Resolution



Narrow Window:
Good Time Resolution
Poor Frequency Resolution

Wide Window:
Good Frequency Resolution
Poor time Resolution

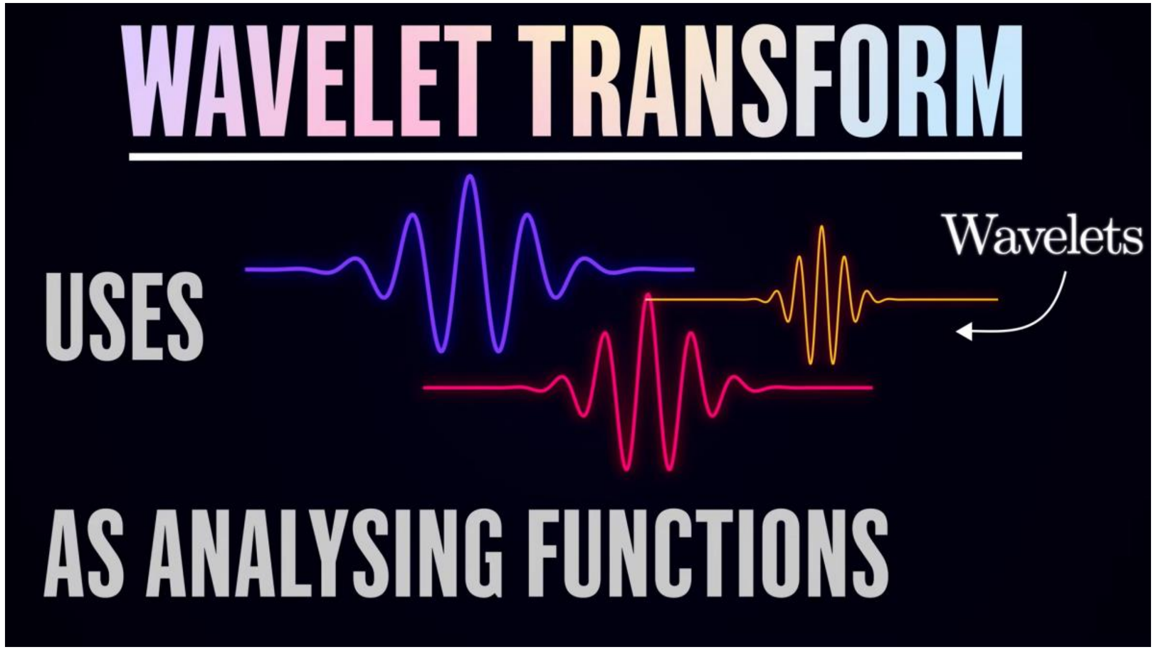
Low Scale (High frequencies)
Good Time Resolution
Poor Frequency Resolution

High Scale (Low Frequencies):
Good Frequency Resolution
Poor time Resolution

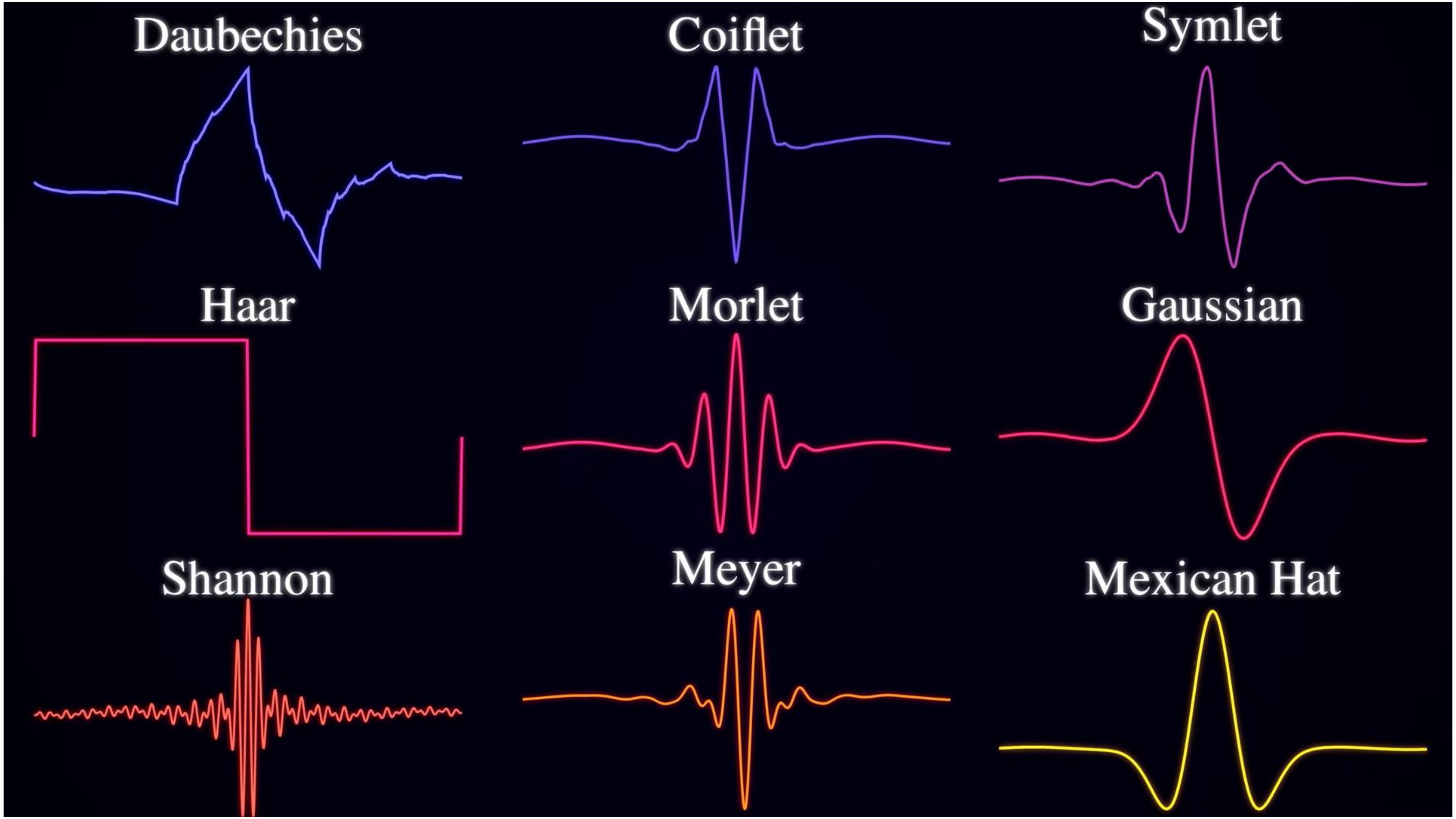
Multi-resolution Analysis (MRA)

- Wavelet Transform
  - An alternative approach to the short time Fourier transform to overcome the resolution problem
  - Similar to STFT: signal is multiplied with a function
- Multiresolution Analysis
  - Analyze the signal at different frequencies with different resolutions
  - Good time resolution and poor frequency resolution at high frequencies
  - Good frequency resolution and poor time resolution at low frequencies
  - More suitable for short duration of higher frequency; and longer duration of lower frequency components

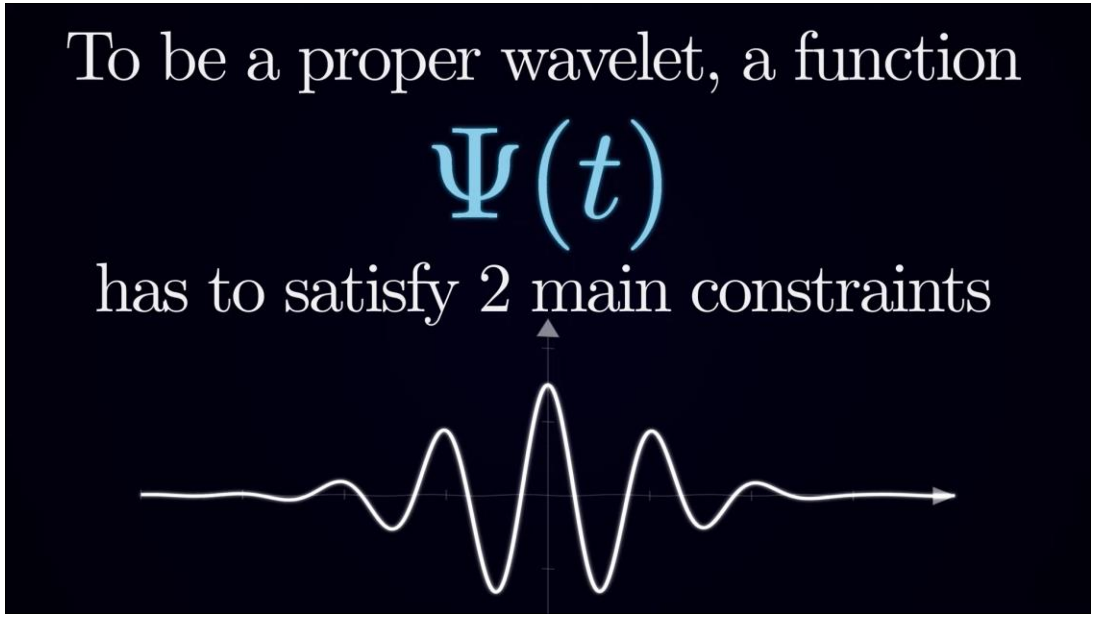
What is a Wavelet?



Different Wavelet functions



Wavelet Definition



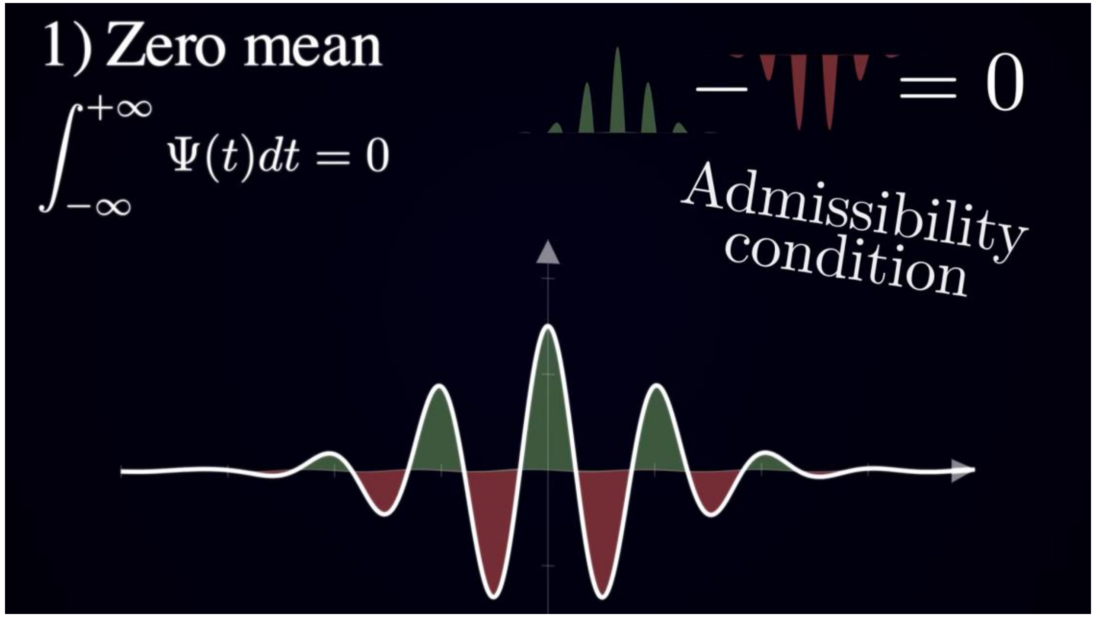
A function has to satisfy the following 2 constraints to be a Wavelet:

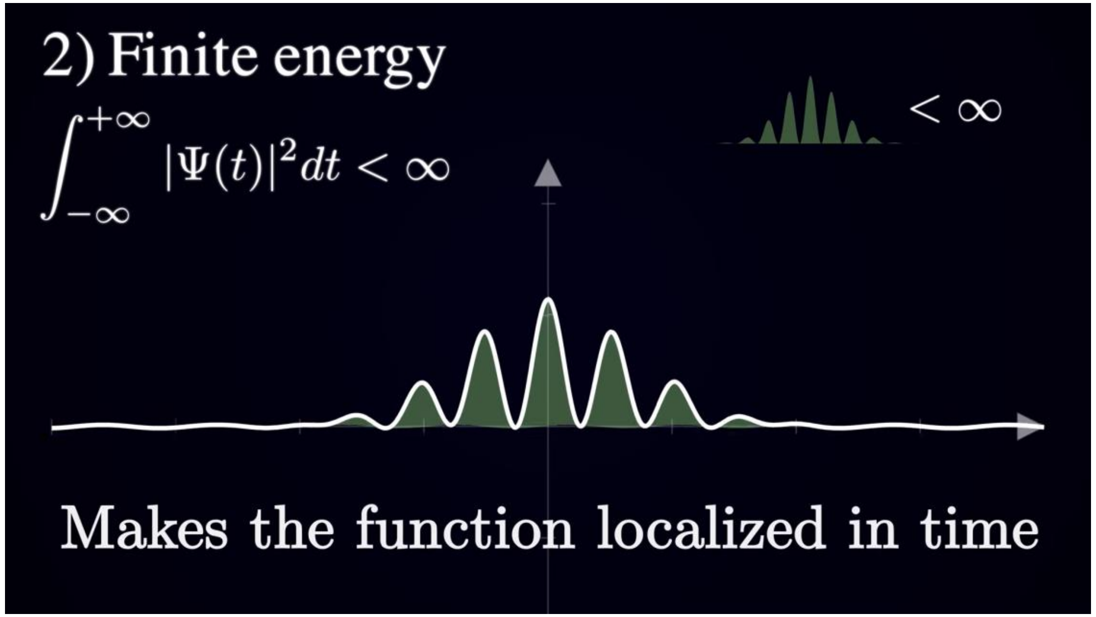
1) The function should have a Zero mean

The area under the curve of the function from $-\infty$ to $+\infty$

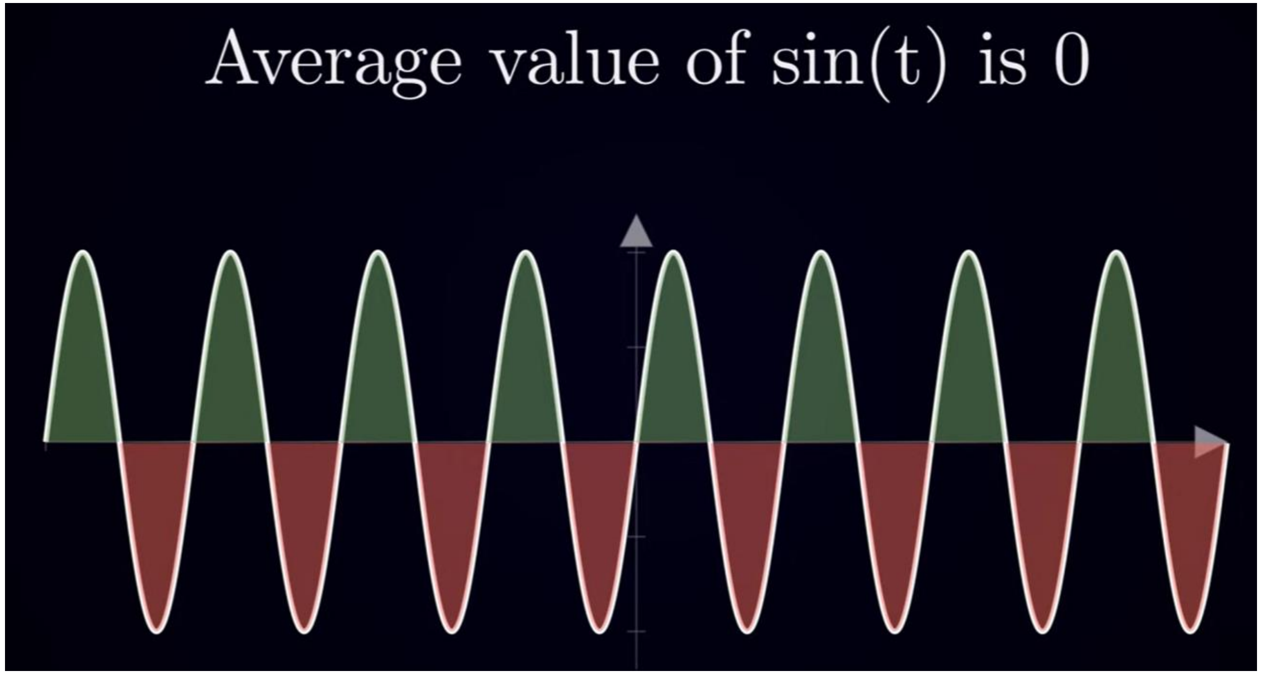
2) The function should have finite energy

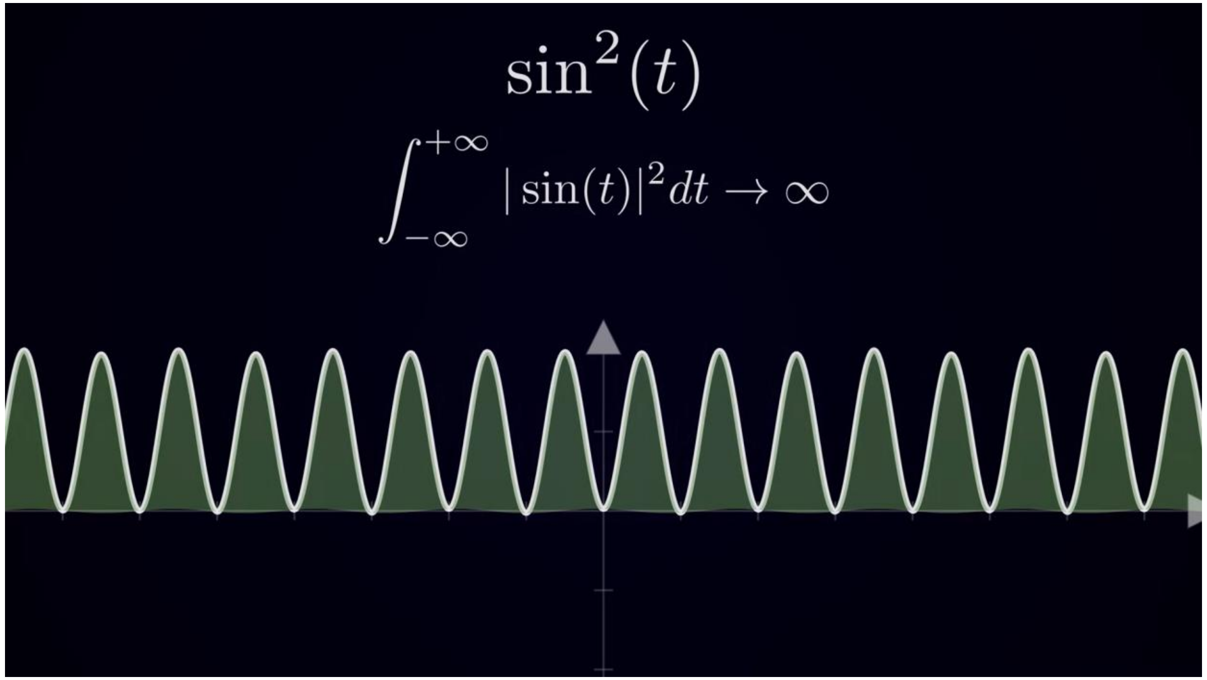
Energy of a Function: Square the function and find the area under the curve from $-\infty$ to $+\infty$





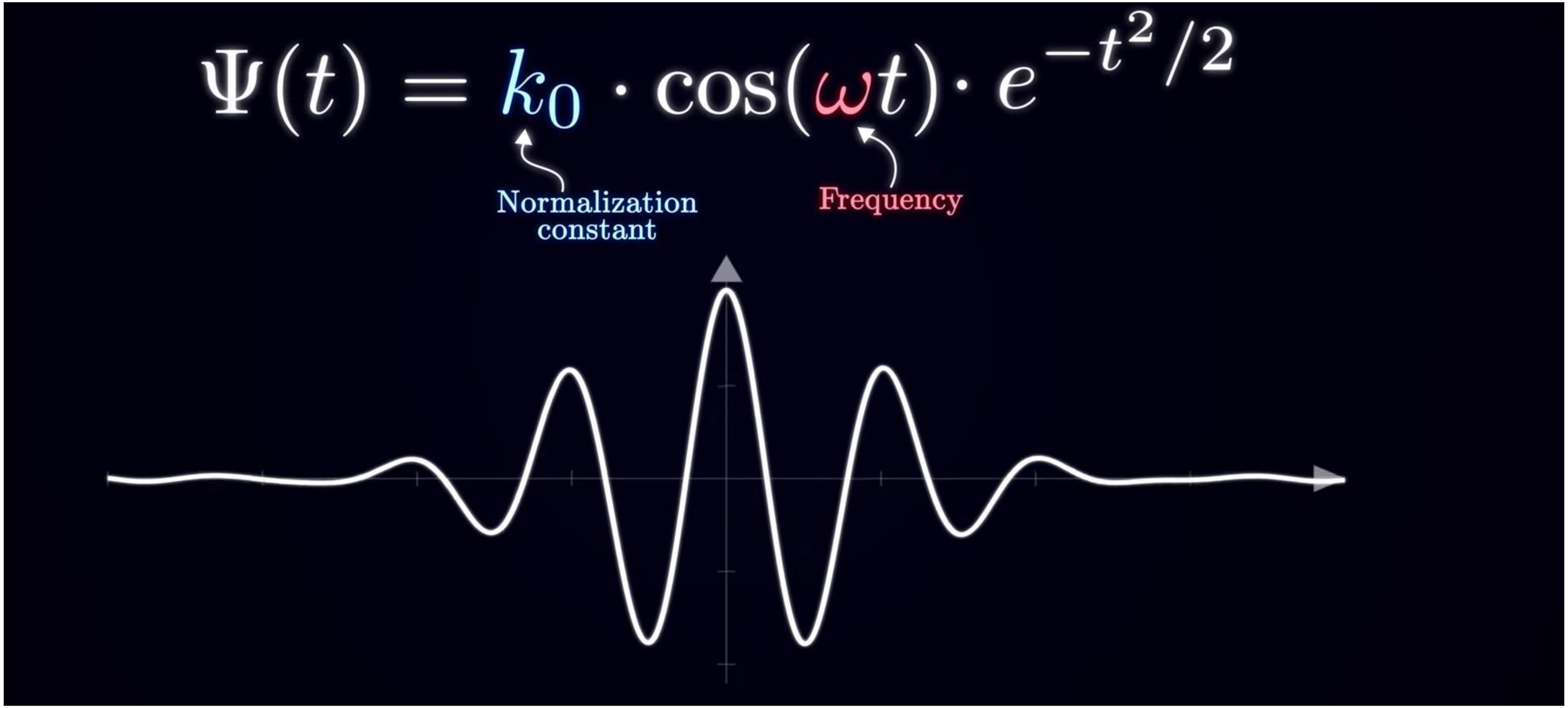
Sinusoid Signal





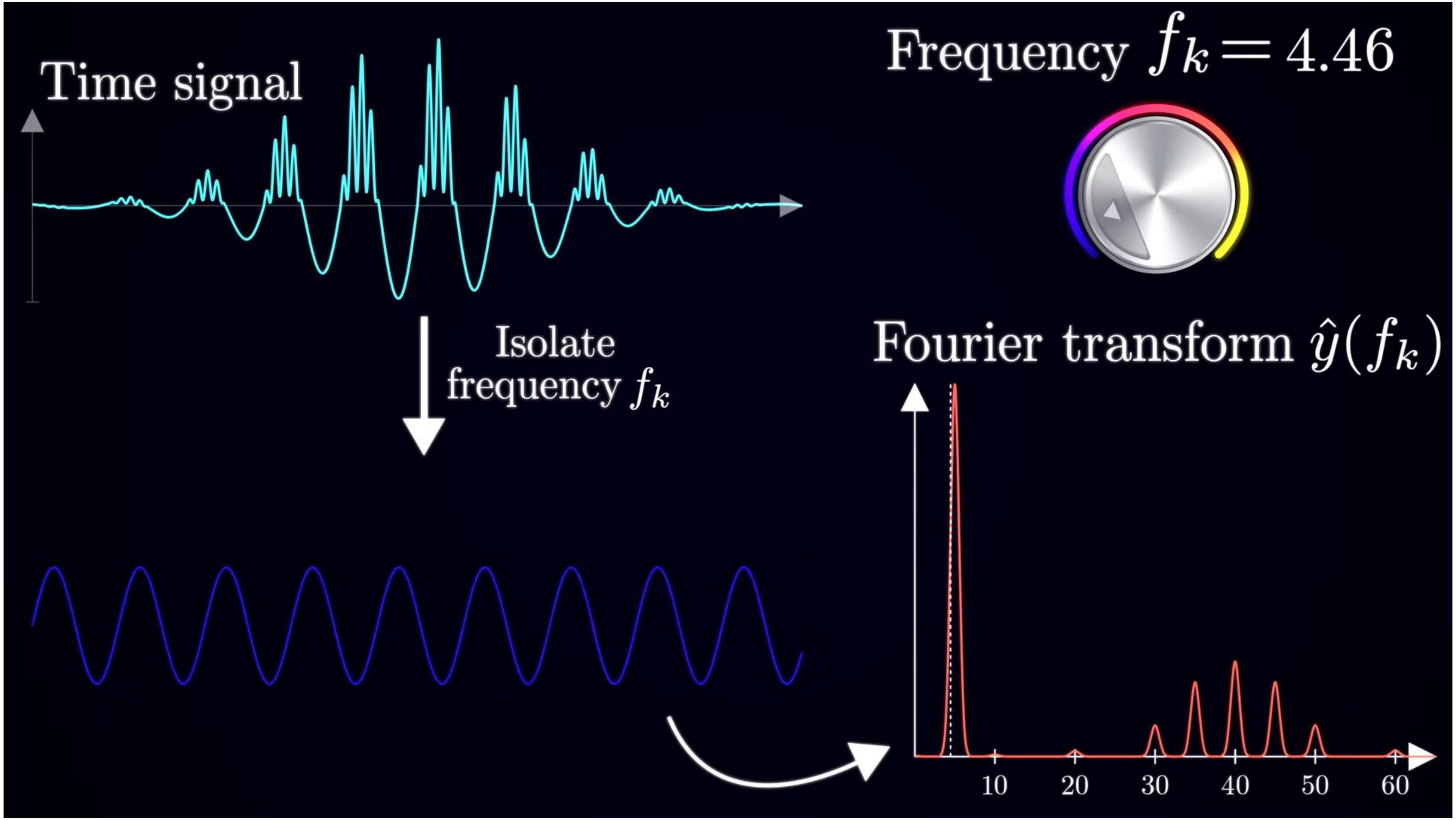
Sine function fails the second constraint. Energy is infinite +∞

Real Component of Morlet Wavelet

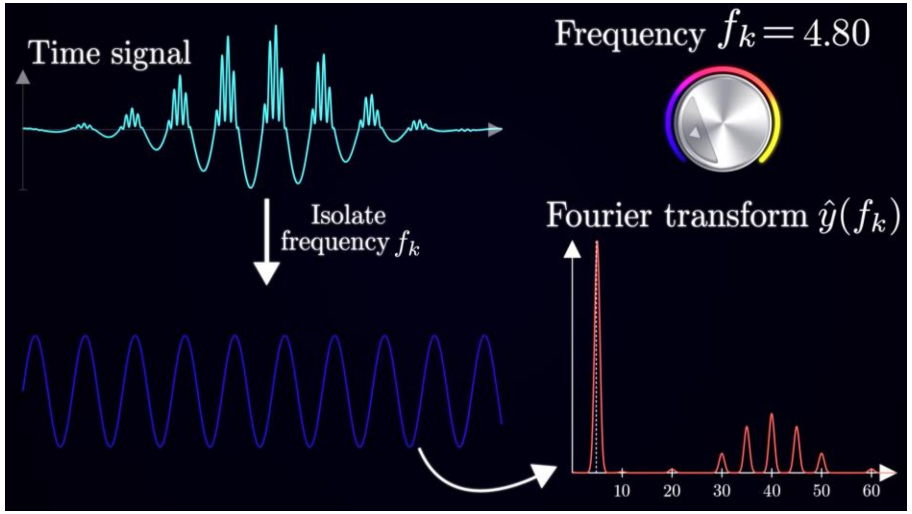


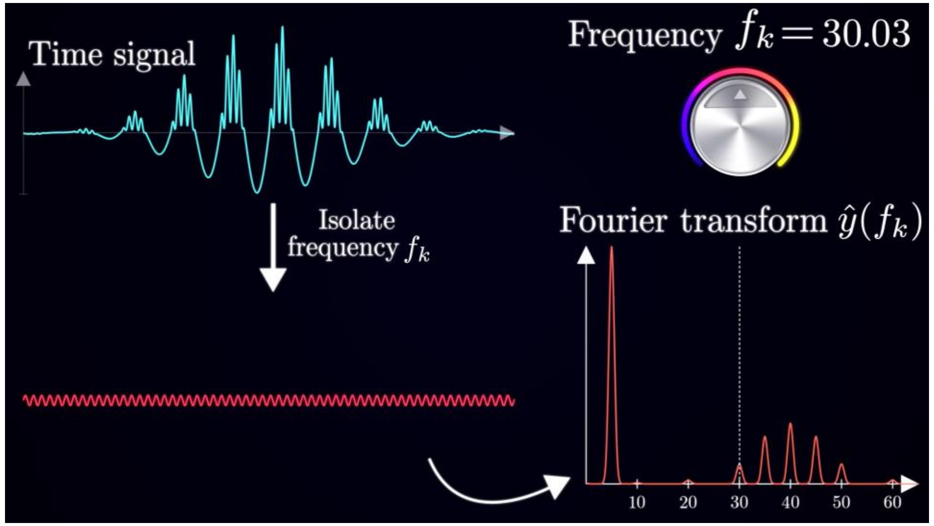
The Morlet wavelet is a cosine function dampened by multiplying with a Gaussian bell curve.

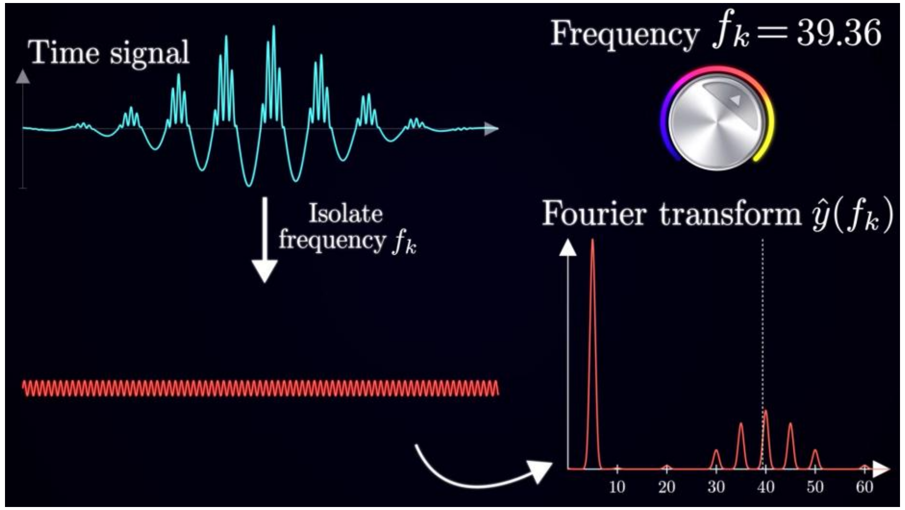
Fourier Transform: A Frequency Tuner

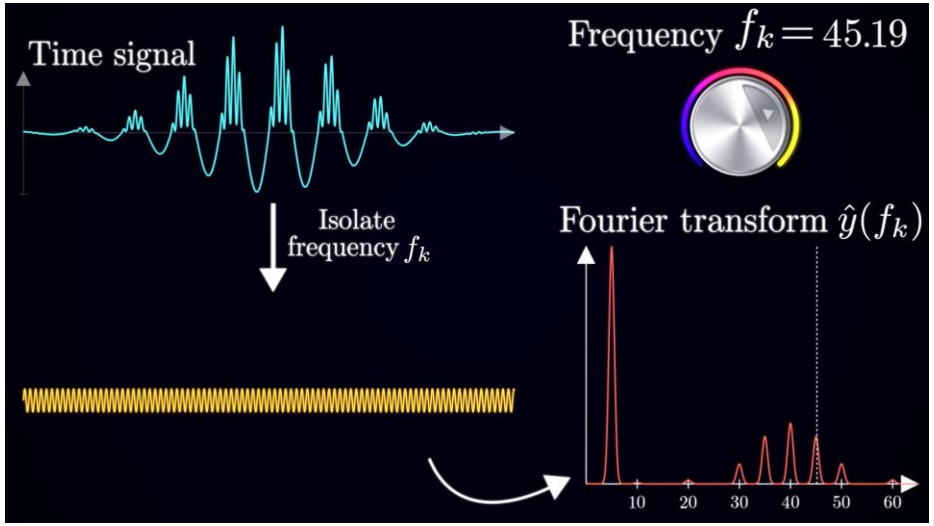


Fourier Transform of a signal





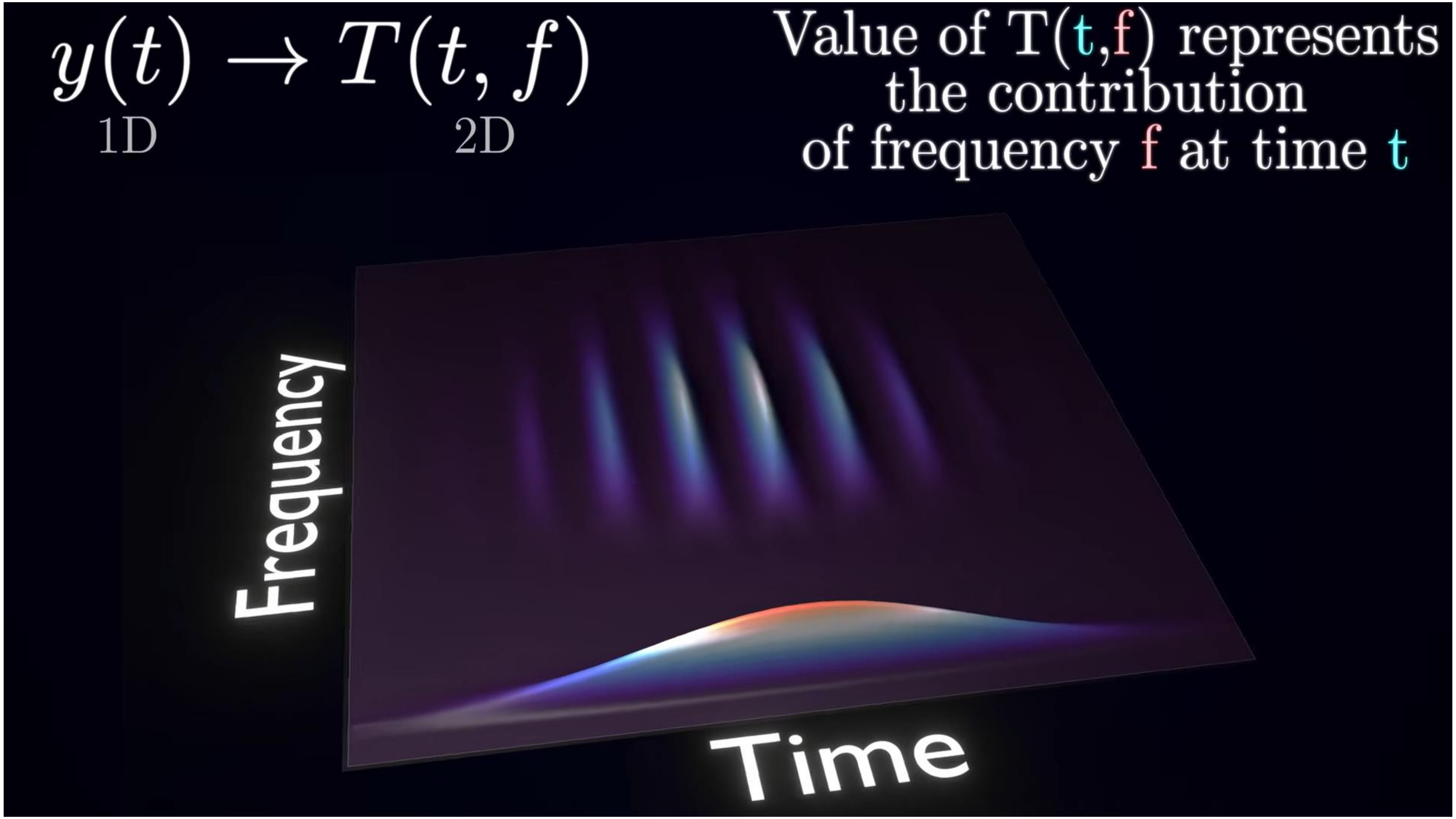




Tune for frequencies

• • •

Wavelet Transform of a signal

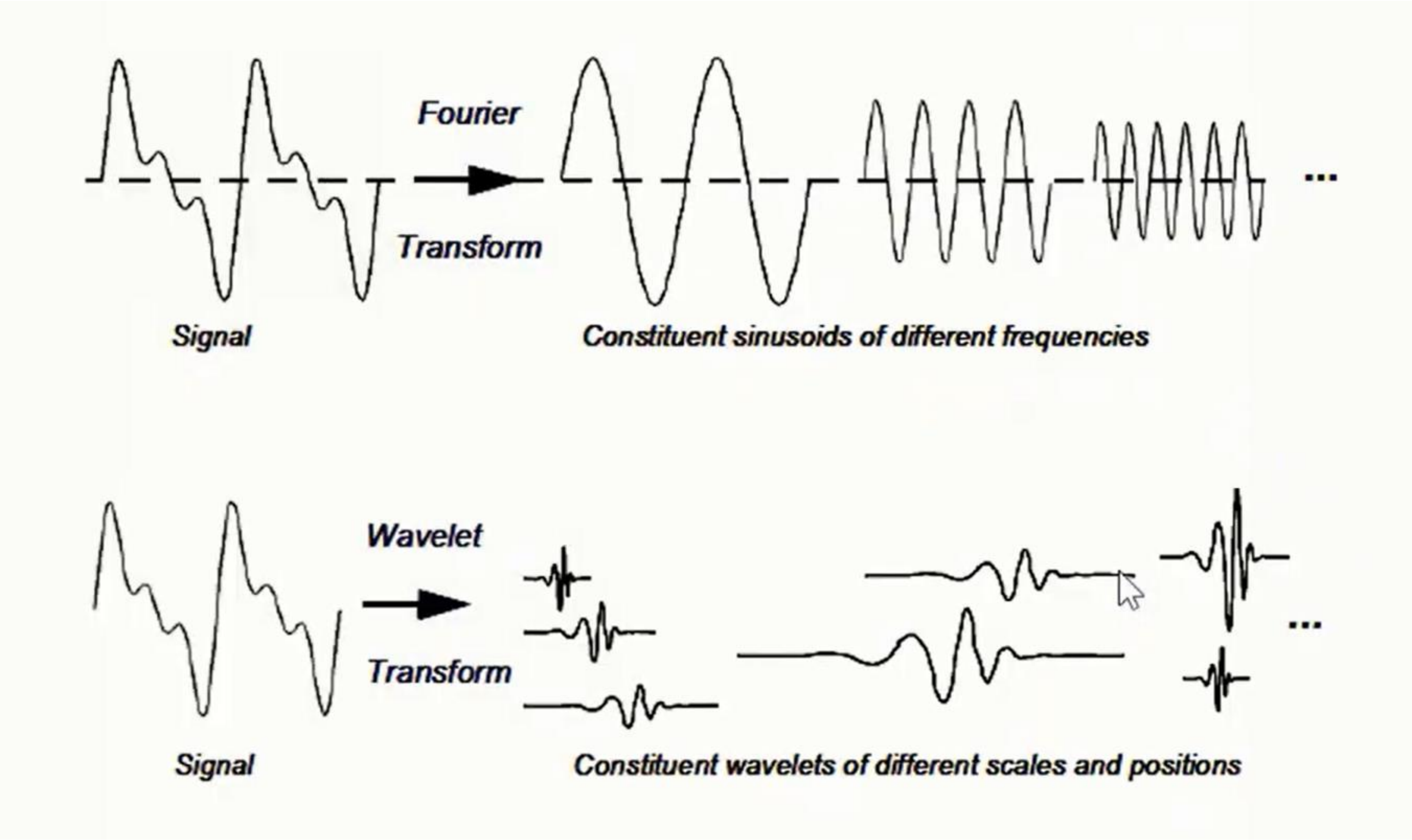


Wavelet vs Fourier Transform

Basis Functions
Sine
Cosine

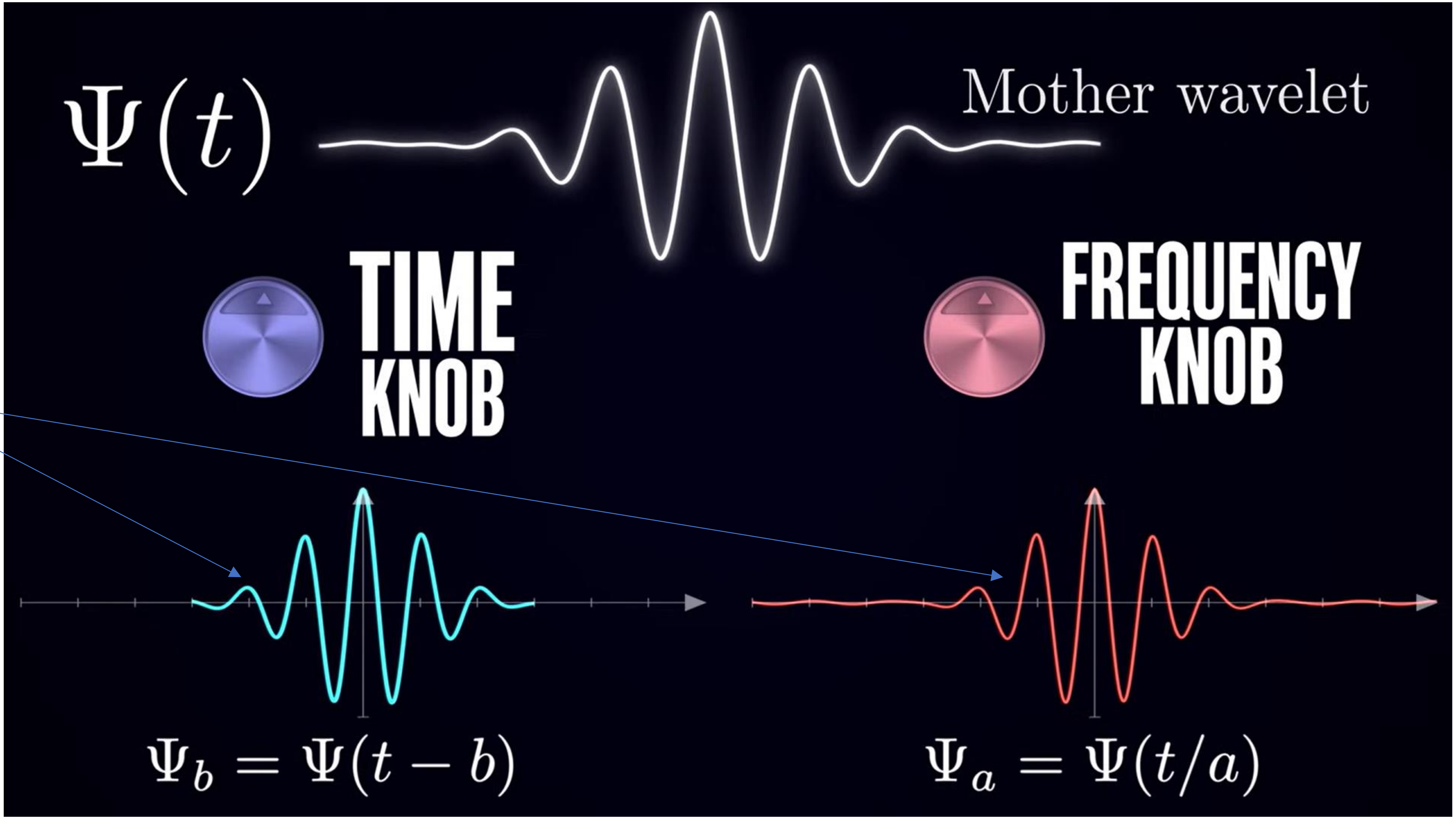
Basis Functions
Morlet

Haar
Symlet
Mexican Hat
Gaussian



Wavelet Transform

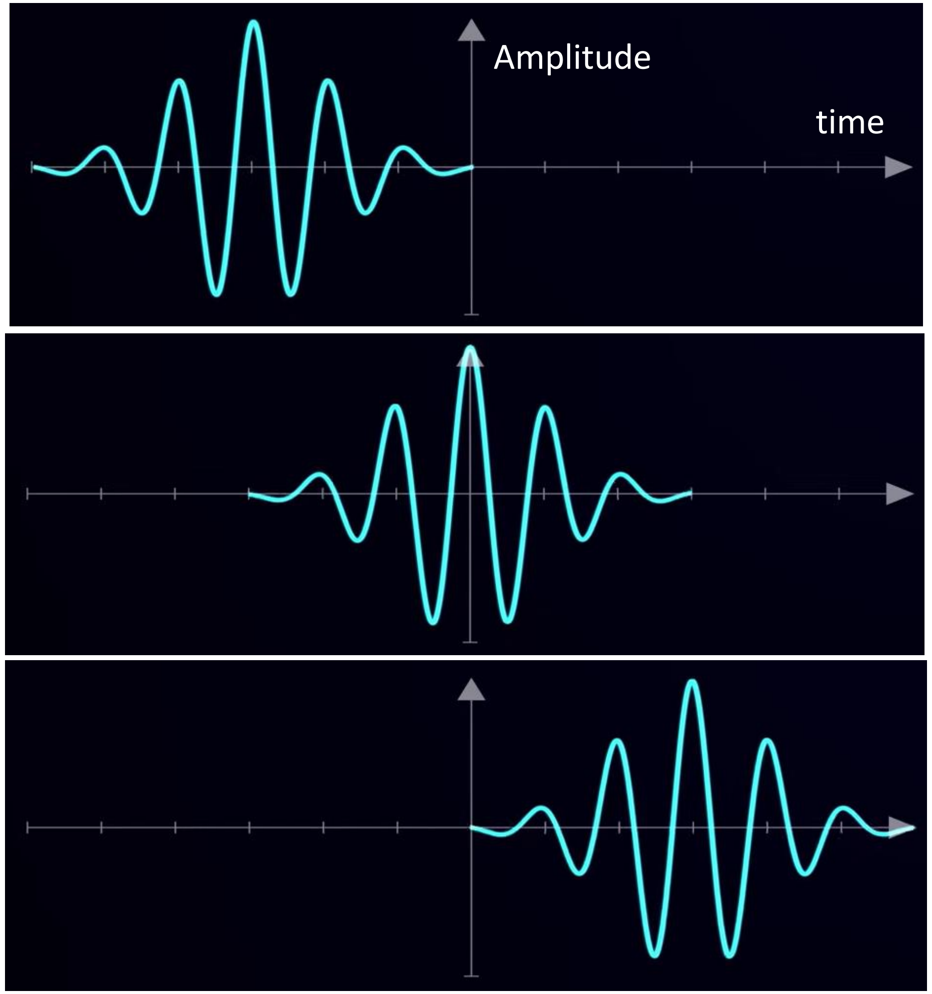
Daughter wavelets



Shift (Position in time: Translate or move Mother wavelet along the time axis)

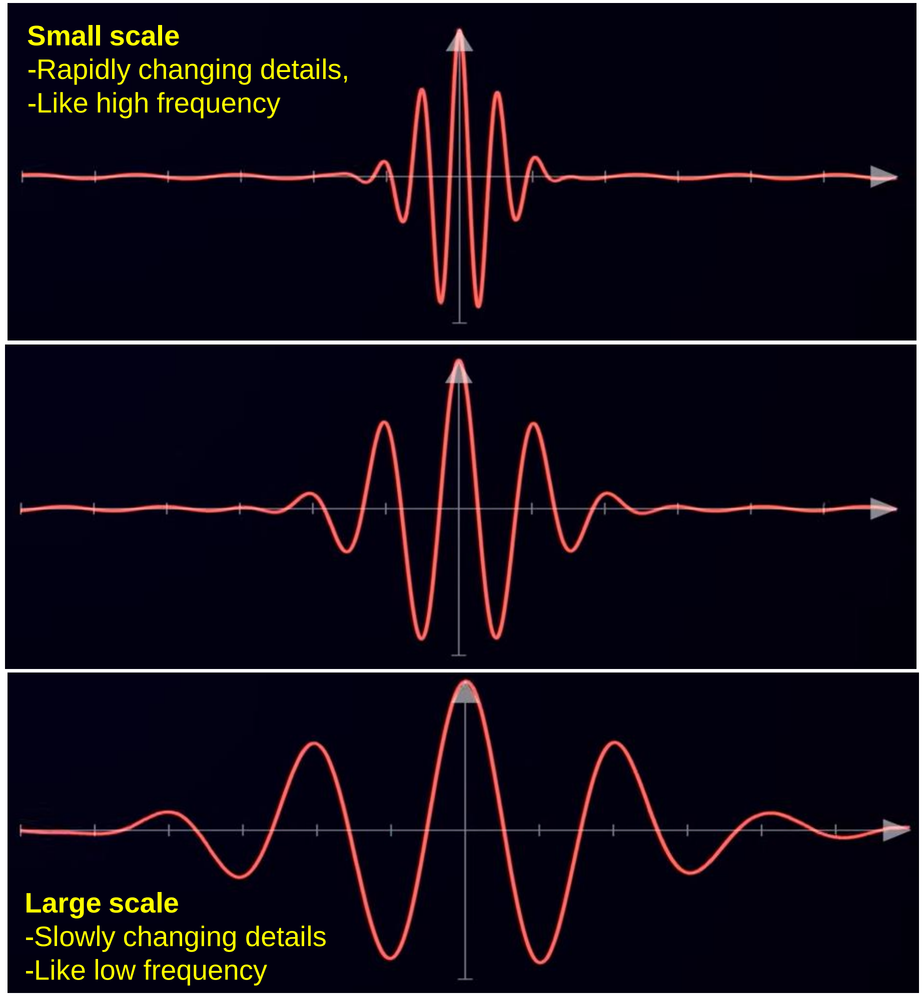
Scale (Shrink or Stretch the mother wavelet to vary the frequencies)

Wavelets: Shifting and Scaling



$$\Psi_b = \Psi(t - b)$$

Shift (Position in time: Change Shift parameter ***b*** to translate or move Mother wavelet along the time axis)

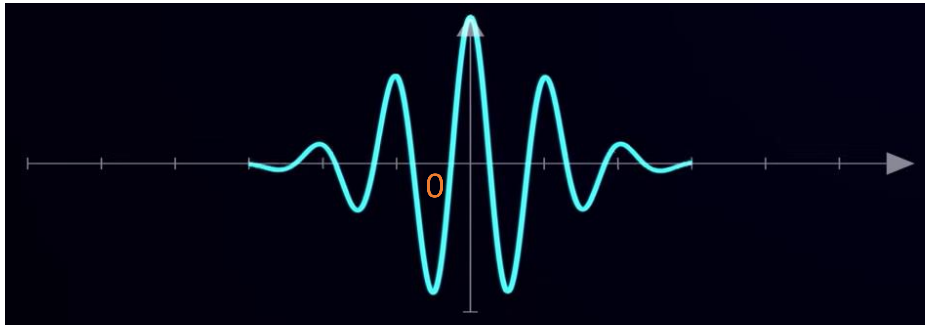


$$\Psi_a = \Psi(t/a)$$

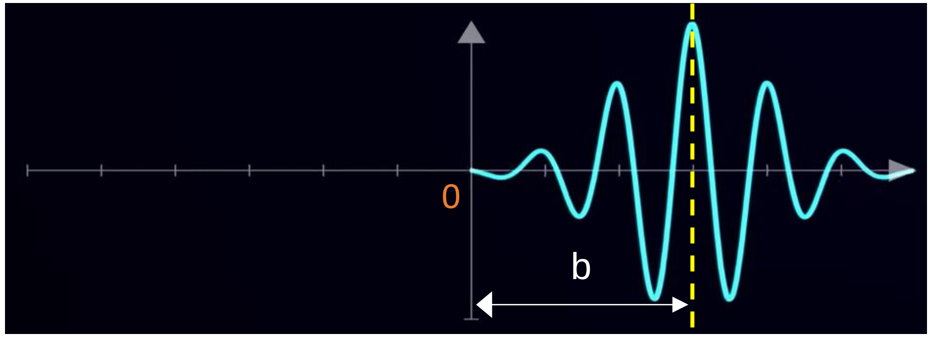
Scale (Change Scale factor ***a*** Shrink/Compress or Stretch the mother wavelet to vary the frequencies)

Shifting of a wavelet

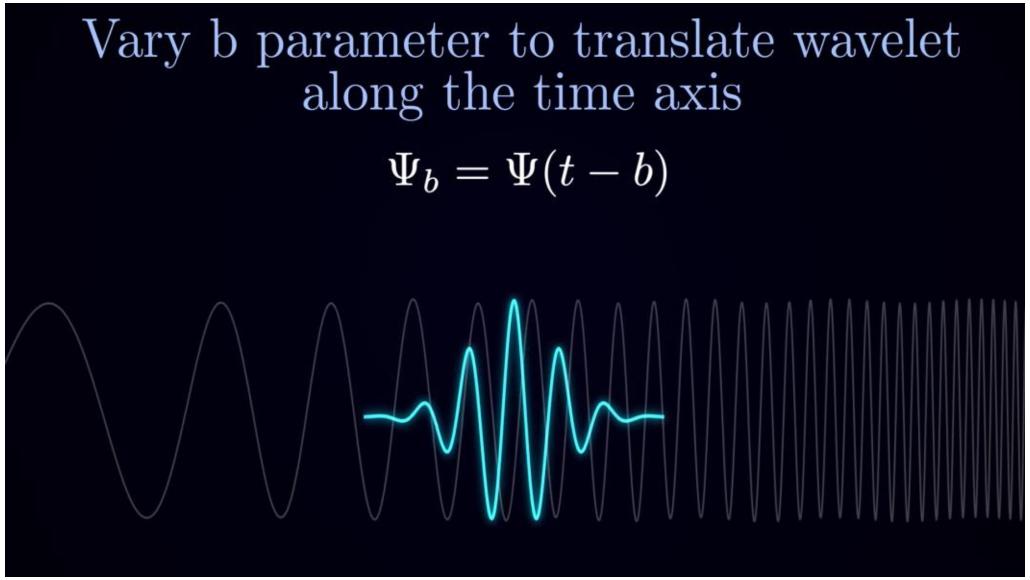
❖ Shifting a wavelet simply means delaying (or hastening) its onset. Mathematically, delaying a function f(t) by *b* is represented by f(t-b)



Wavelet function Ψ(t)

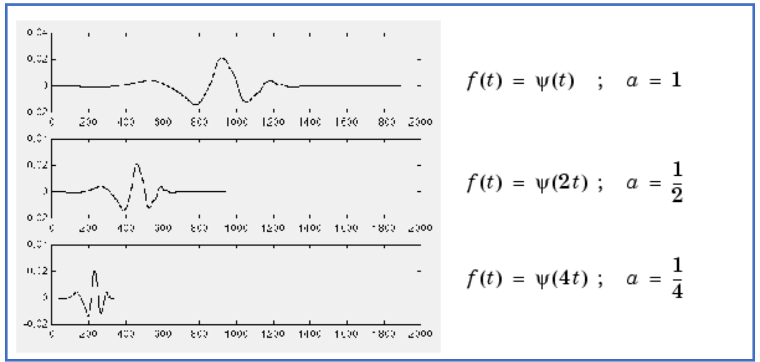


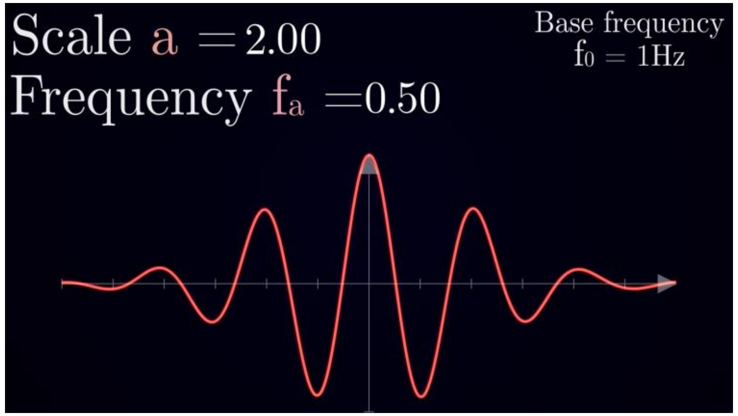
Wavelet function Ψ(t-b)

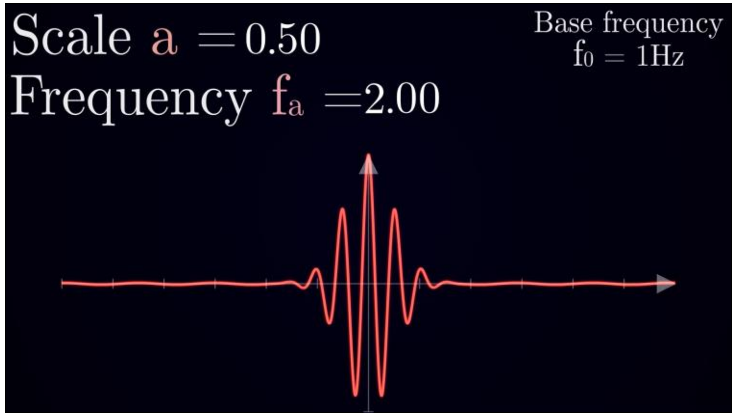


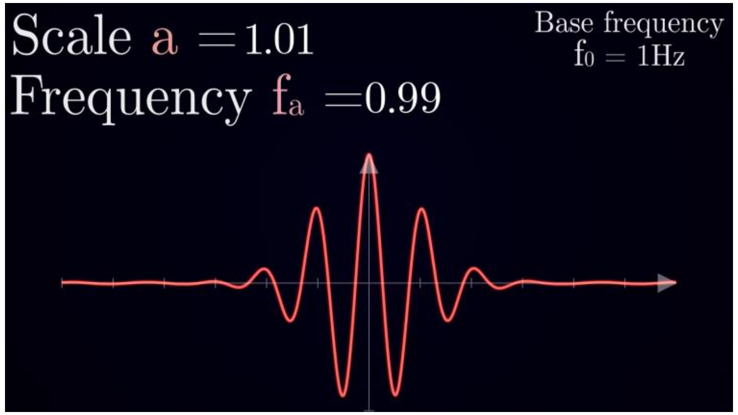
Scaling of a wavelet

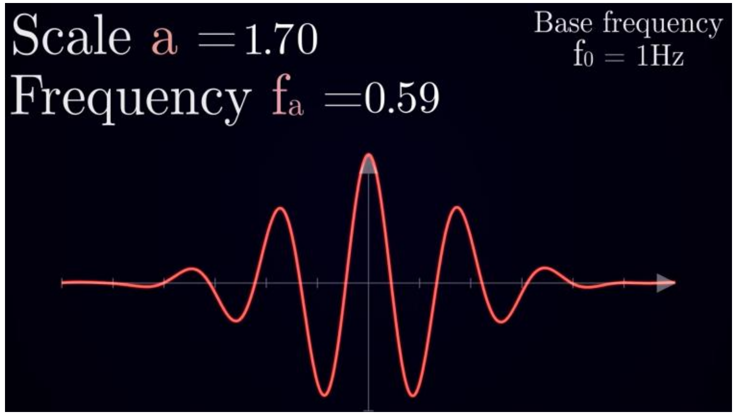
- **Scaling** lets you either narrow down the frequency band of interest, or determine the frequency content in a narrower time interval
- Scaling = frequency band
- Good for non-stationary data

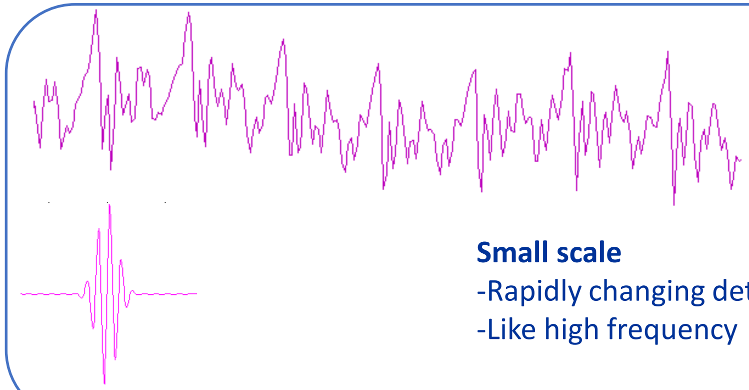




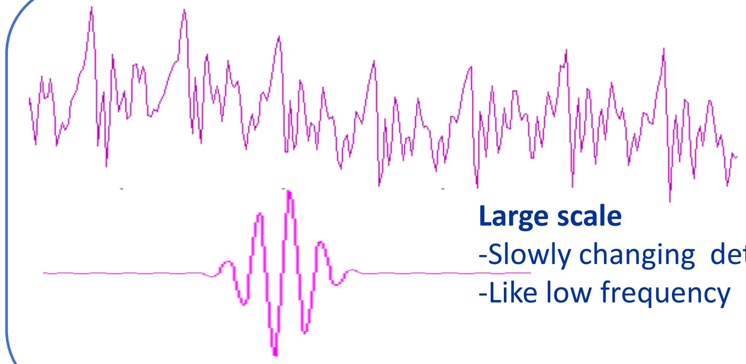








Small scale
-Rapidly changing details
-Like high frequency



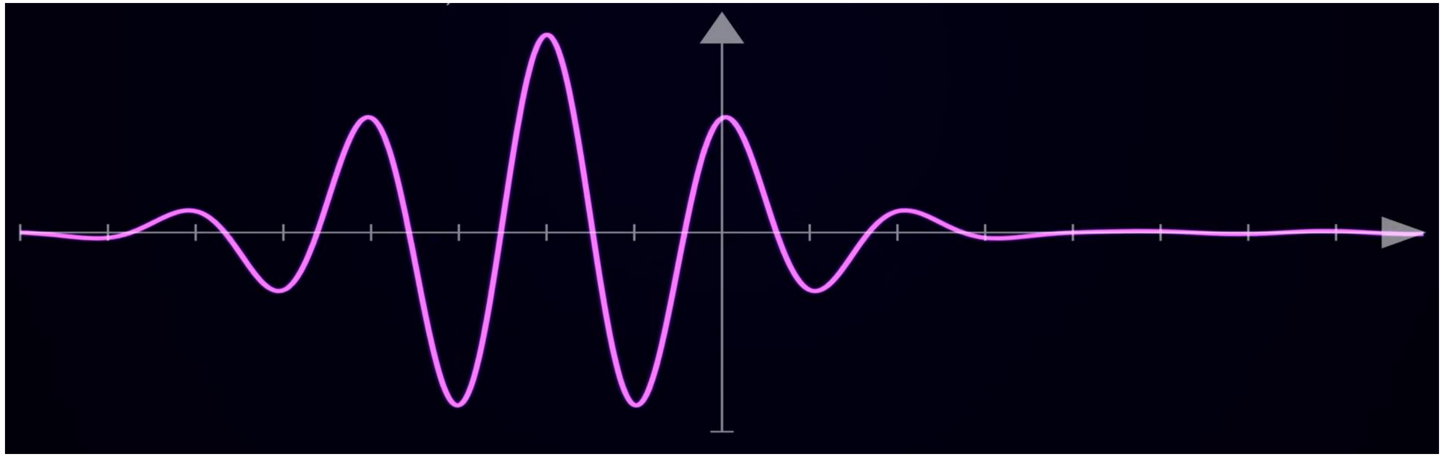
Large scale
-Slowly changing details
-Like low frequency

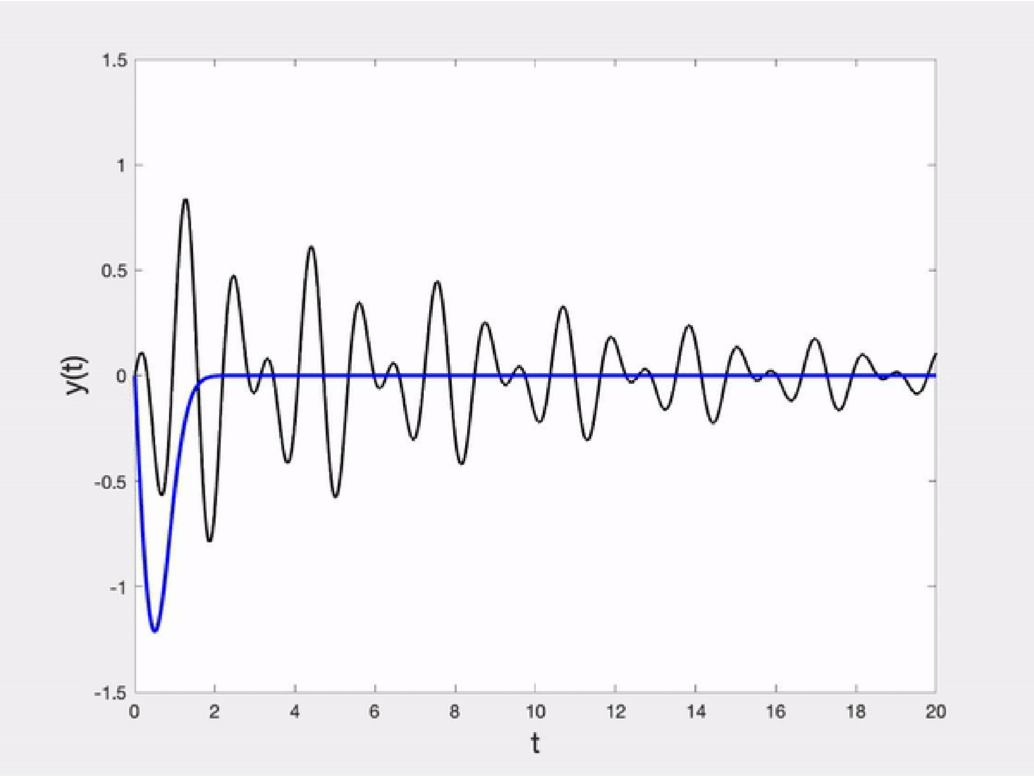
- Low scale → *a* Compressed wavelet→ Rapidly changing details→High frequency
- High scale →*a* Stretched wavelet → Slowly changing, coarse features → Low frequency

The scale factor works exactly the same with wavelets. The smaller the scale factor, the more "compressed" the wavelet.

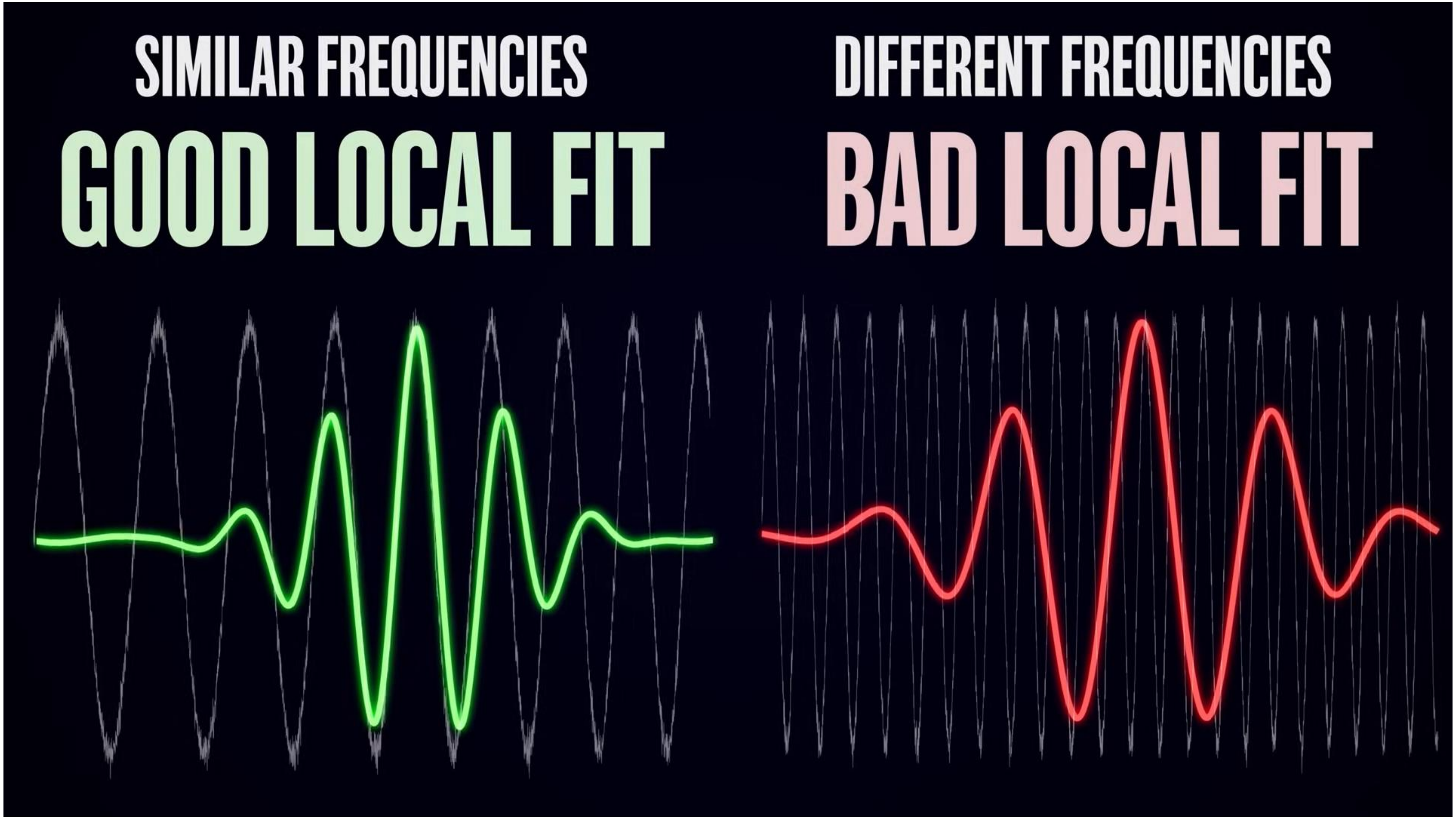
Wavelets: Scaling and Shifting

Scaled and translated wavelet: $\Psi_{a,b} = \Psi(\frac{t-b}{a})$

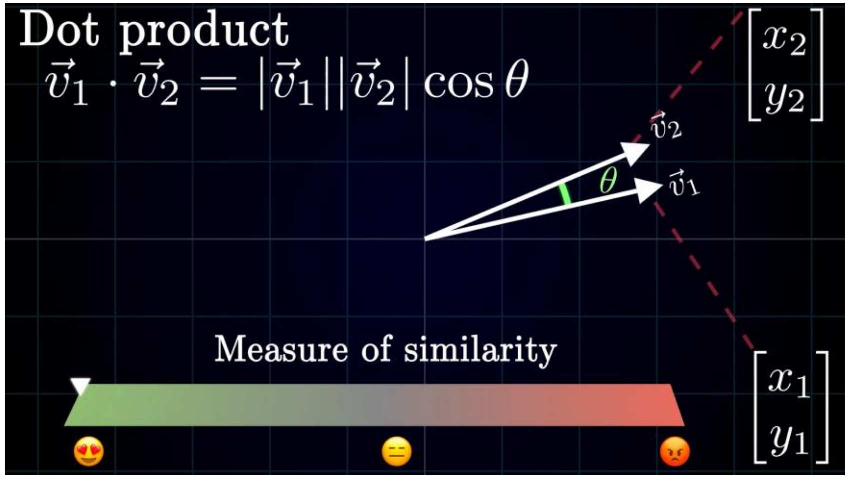


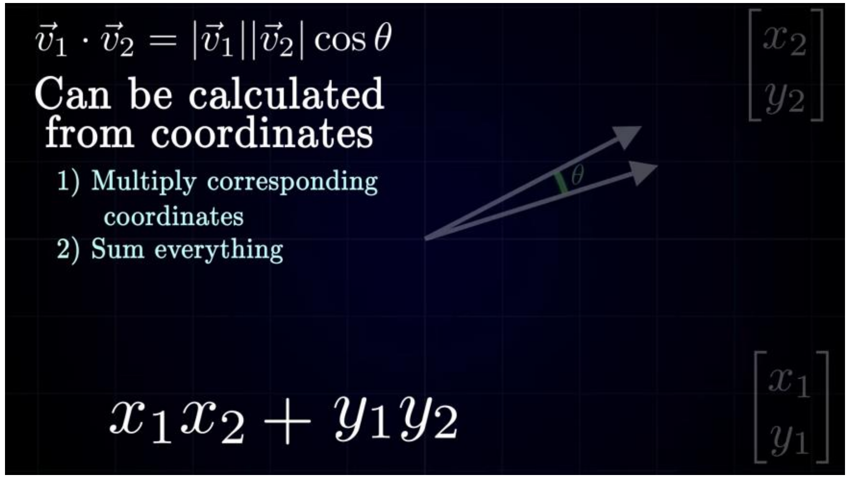


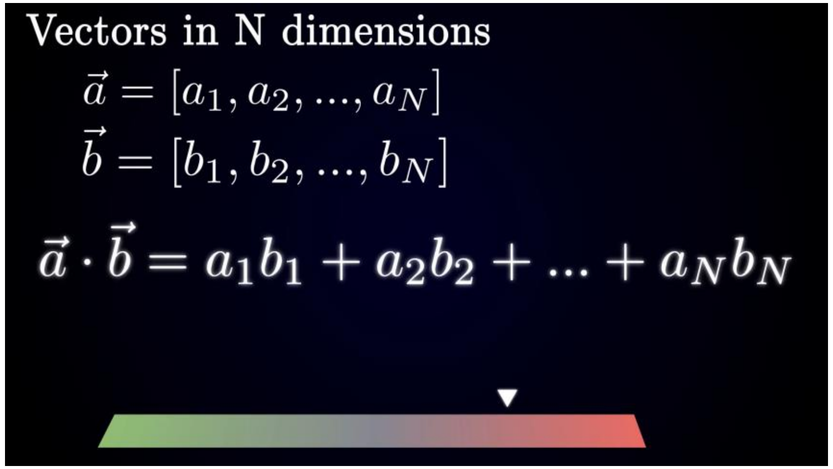
By Changing ***a*** (scale parameter) and ***b*** (shift parameter), you can scale and translate the wavelet

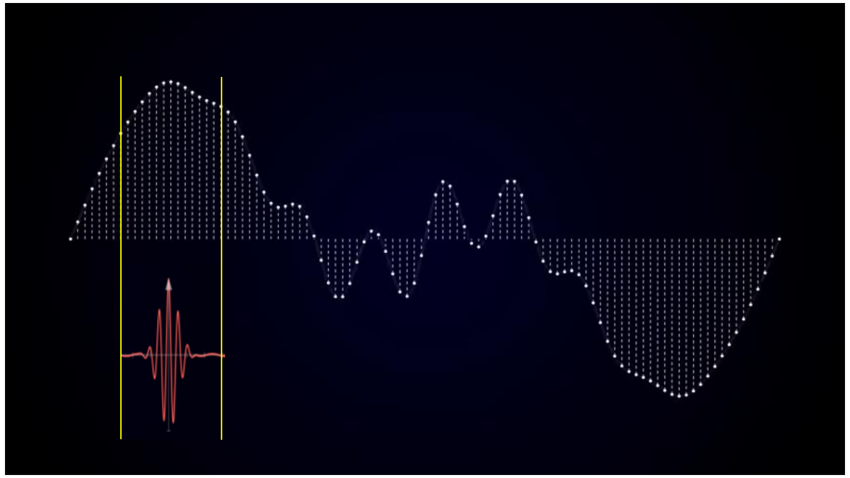

SIMILAR FREQUENCIES
GOOD LOCAL FIT
DIFFERENT FREQUENCIES
BAD LOCAL FIT

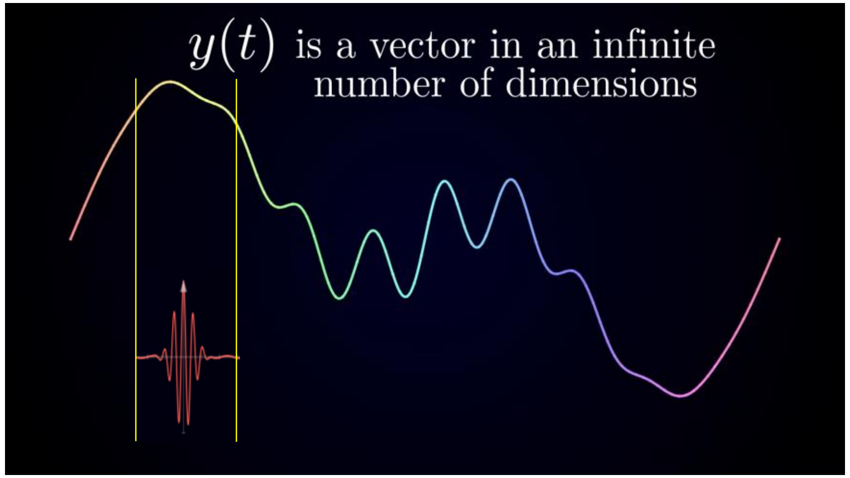
Similarity between two functions (using Dot Product)

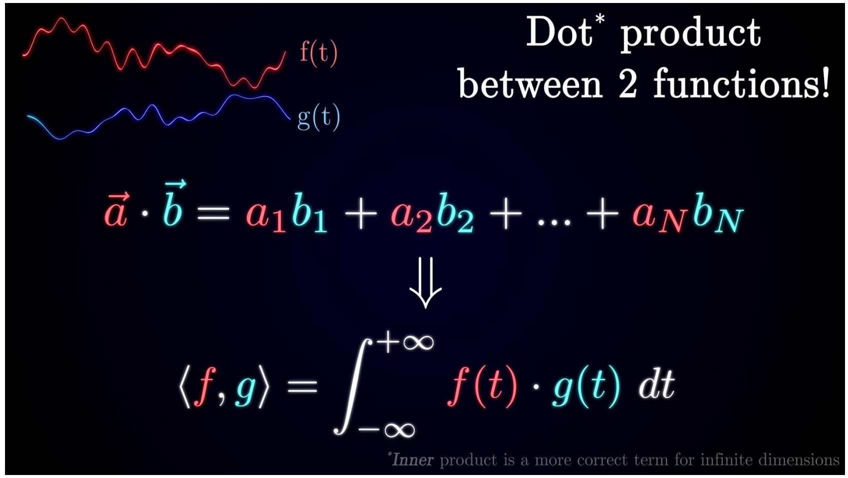






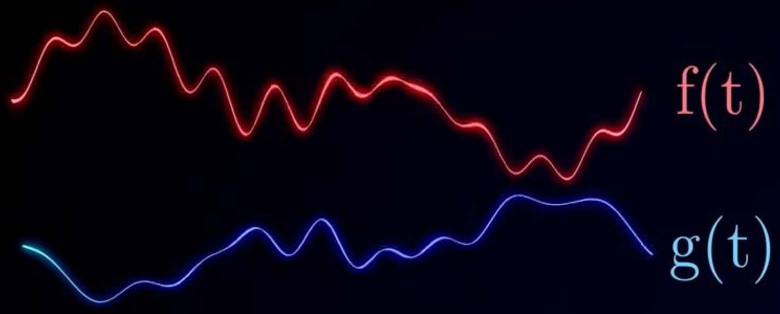






In our case, we are finding the Local Similarity between the Signal and the wavelet (at a position in time and scale).
The scale refers to the scale of the wavelet.

Dot Product between two functions



Dot* product between 2 functions!

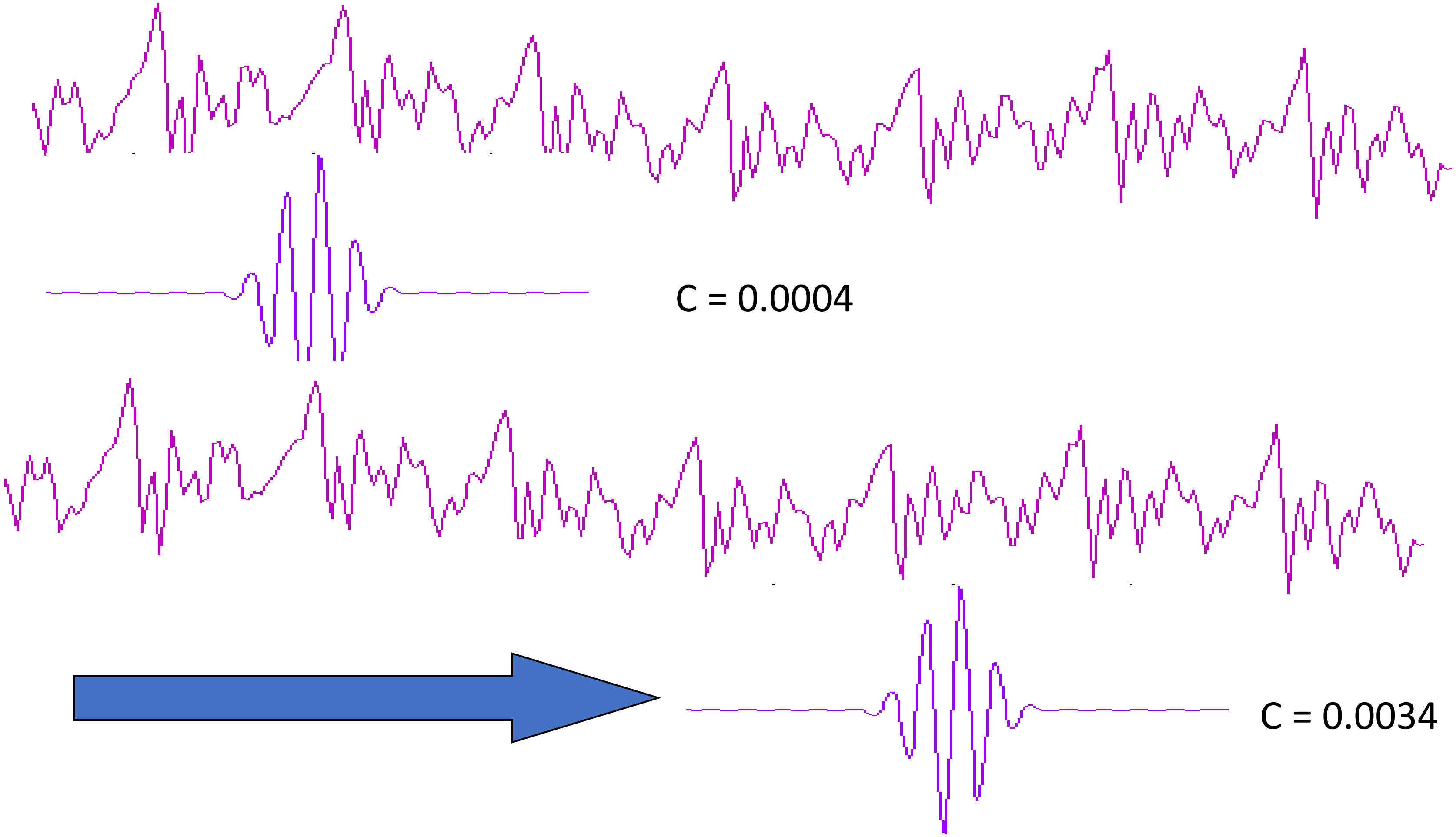
$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \ldots + a_N b_N$$

$$\Downarrow$$

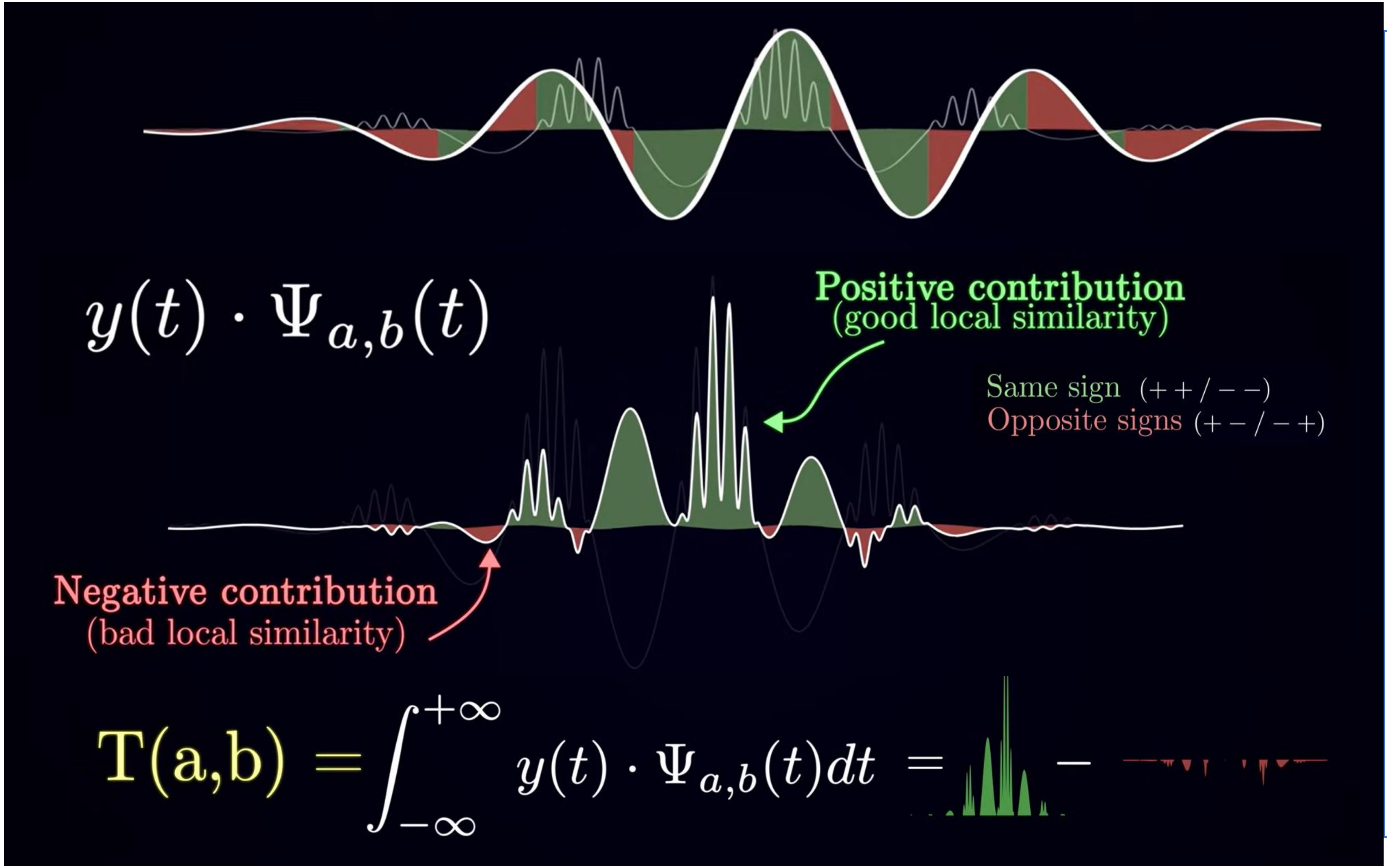
$$\langle f, g \rangle = \int_{-\infty}^{+\infty} f(t) \cdot g(t)\, dt$$

*Inner product is a more correct term for infinite dimensions

Wavelet Coefficients through Shifting

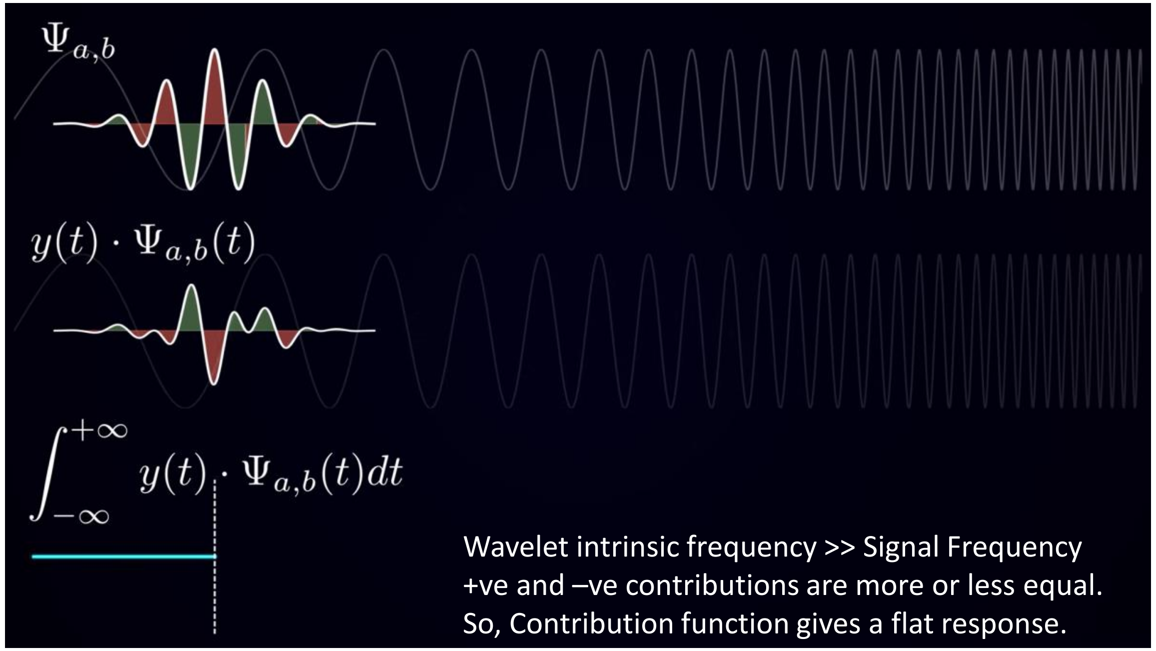


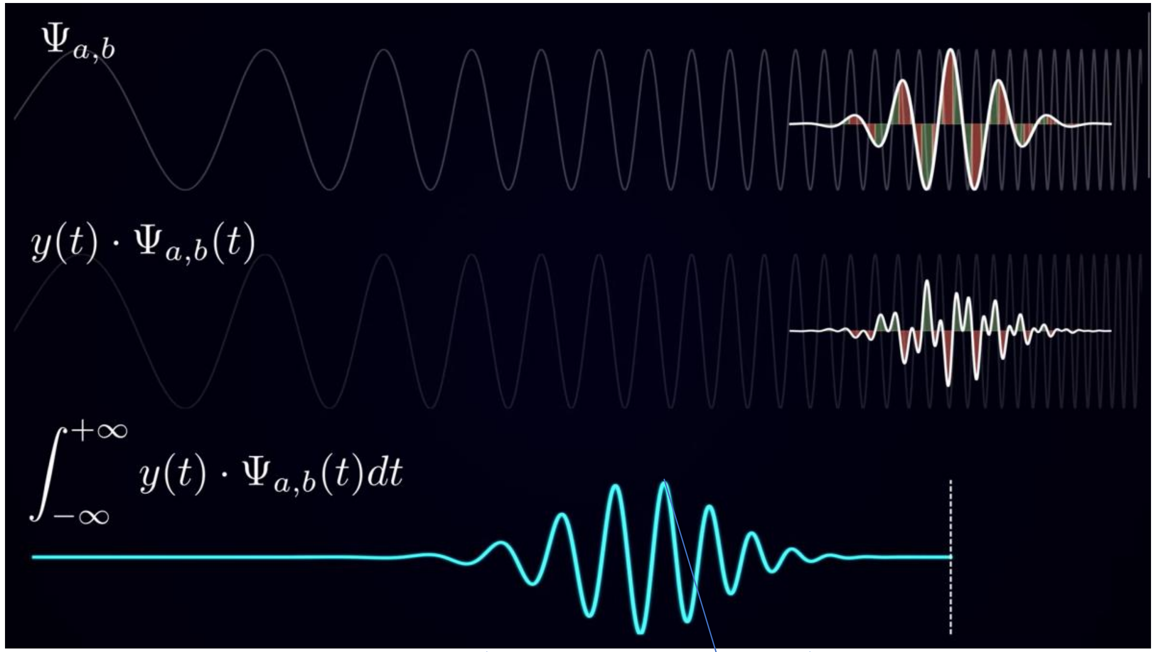
Wavelet Coefficient for given position in Time and Scale



- The value of T (a,b) is equal to the **contribution of $\Psi_{a,b}$ to the input signal at a given time (position) and scale**.
- T (a,b) can also be thought as how well the wavelet $\Psi_{a,b}$ **matches** the input signal
- What we did is to multiply of two functions (input signal and wavelet) and **compute the integral of the product**.
- The integral is essentially the sum of infinitely many narrow rectangles with the height equal to the result of multiplication at that point
- This is nothing but doing a set of pairwise multiplications and summing thing together. This is nothing but the **Dot Product** between the two functions (input signal and wavelet) at a given time (position) and scale.
- As Dot Product is a measure of similarity, the Dot product we calculated is the **Local Similarity** between the input signal and wavelet at a given time (position) and scale.

Signal Frequency Component Identification (1)





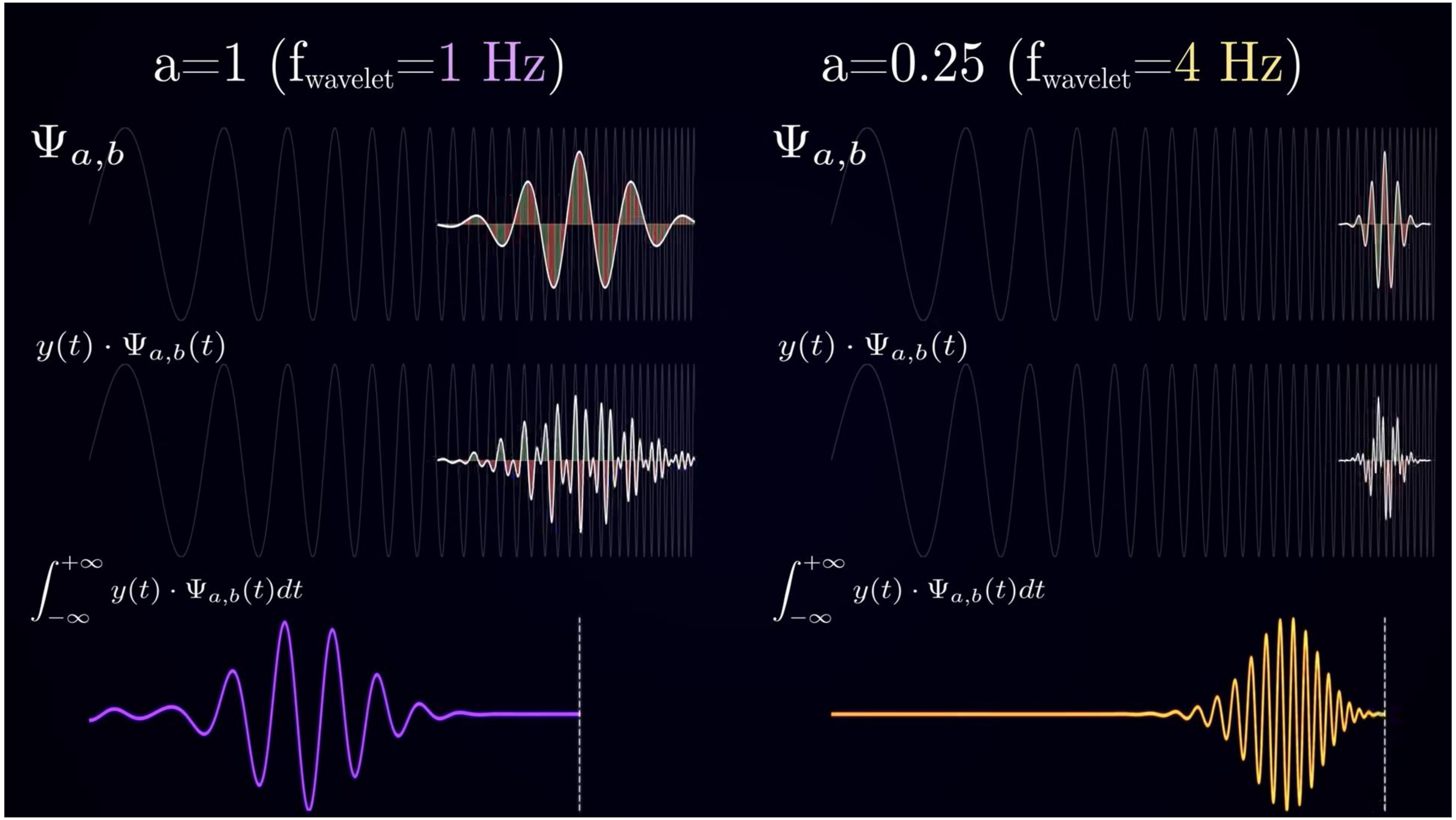
As the Wavelet intrinsic frequency approaches Signal Frequency, contribution function begins to resonate and you get significant overall positive contributions they are in phase and significant overall contributions when they are out of phase.

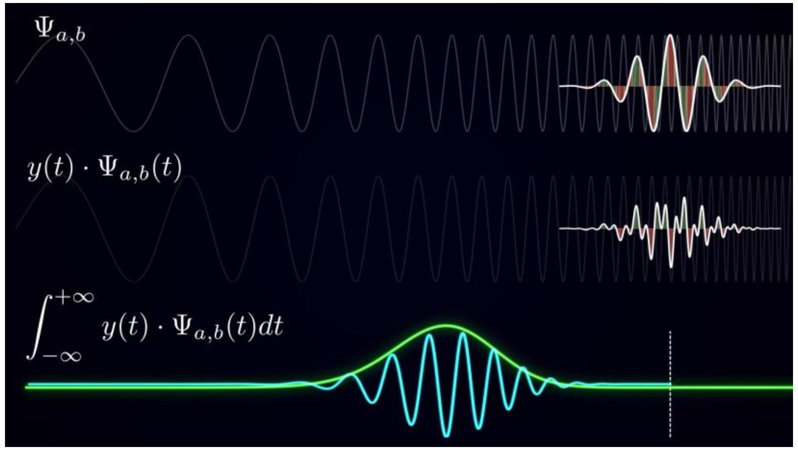
This is why the dot product wobbles around zero and Amplitude is highest when the frequencies of the wavelet and the signal match exactly.

So, We are able to identify / pull out a signal frequency component f_{a1} from the signal.

Contribution is peak when Wavelet intrinsic frequency = Signal Frequency (say f_{a1})

Signal Frequency Component Identification (2)



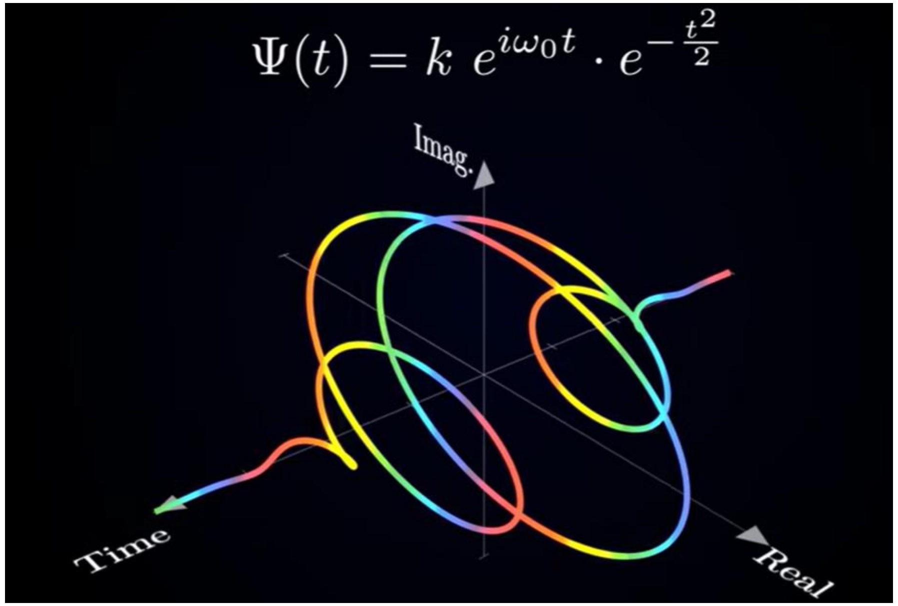


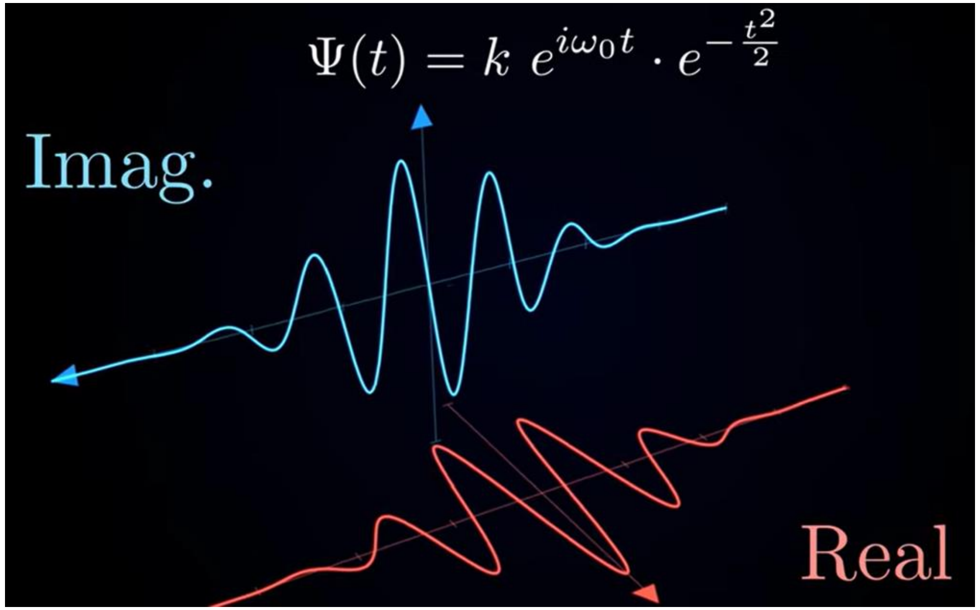
Take the envelope of the resulting oscillation to understand when a frequency component begins, reaches its peak and when it ends.

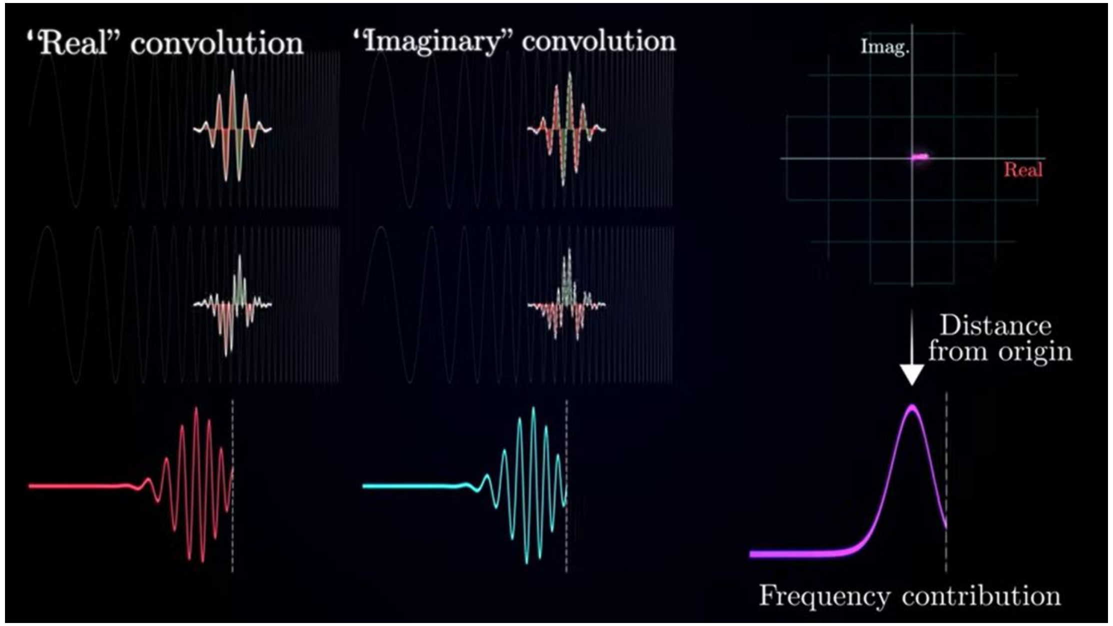
We can pull out other frequency components too.

Wavelet transform: By scaling and translation parameters, we can scan the signal with analysing wavelets of different scales to see what frequencies are most prominent at any time point in the signal.

Morlet Wavelet







Morlet wavelet is a complex exponent which spins around the circle with a constant frequency and whose amplitude is modulated by a Bell curve

To find the wavelet contrition to the signal,

a) Convolution of the signal with both real and imaginary parts.
b) Vary the parameter a to Analyze the signal at different scales.
c) The Power of frequency, the intensity of its contribution at each point in time is given by the distance from the resulting point to the origin. Also known as Absolute value of the complex number

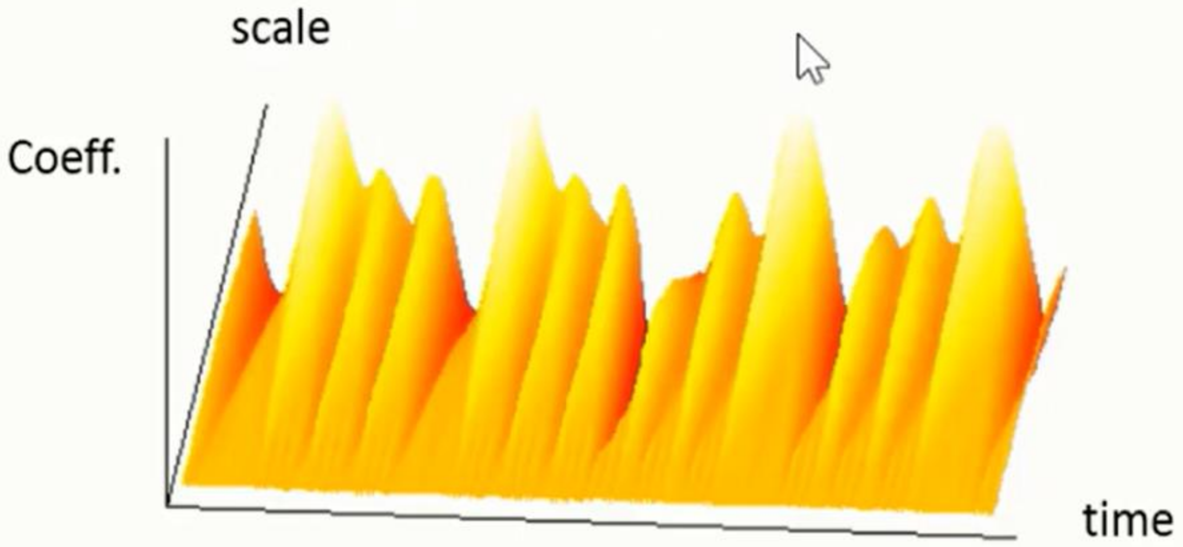
Continuous Wavelet Transform (CWT)

The Continuous Wavelet Transform (CWT) of a signal $f(t)$ is then given by the equation,

$$\boxed{CWT(a,b) = \langle f,\, \psi_{a,b} \rangle = \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} f(t).\, \psi^{*}\left(\frac{t-b}{a}\right) dt}$$

Here, $\langle f, \psi_{a,b} \rangle$ is the $\mathbb{L}^2$ inner product.

The results of the CWT are many wavelet coefficients, which are a function of a (scale) and b (position).



Coefficient Plots

Continuous Wavelet transform

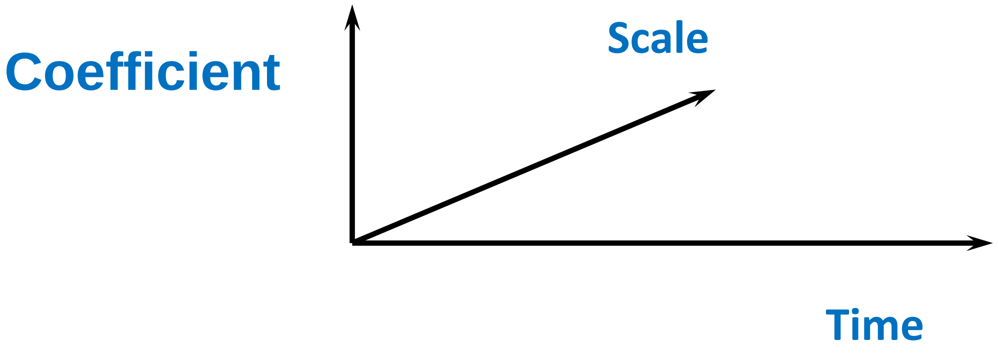
For each Scale

for each Position

$$\text{Coefficient}\ (S,P) = \int_{\text{all time}} \text{Signal} \times \text{Wavelet}\ (S,P)$$

end

end



Continuous Wavelet Transform: Five Easy Steps

1) Take a wavelet and compare it to a section at the start of the original signal.
2) Calculate a correlation coefficient c   (see Figure 1)
3) Shift the wavelet to the right and repeat steps 1 and 2 until you've covered the whole signal. (see Figure 2)
4) Scale (stretch) the wavelet and repeat steps 1 through 3.   (See Figure 3)
5) Repeat steps 1 through 4 for all scales.

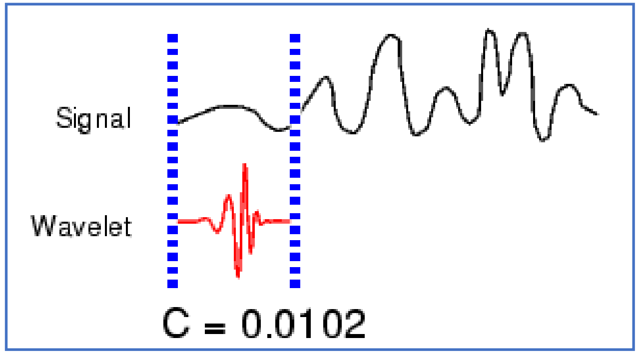


Figure 1

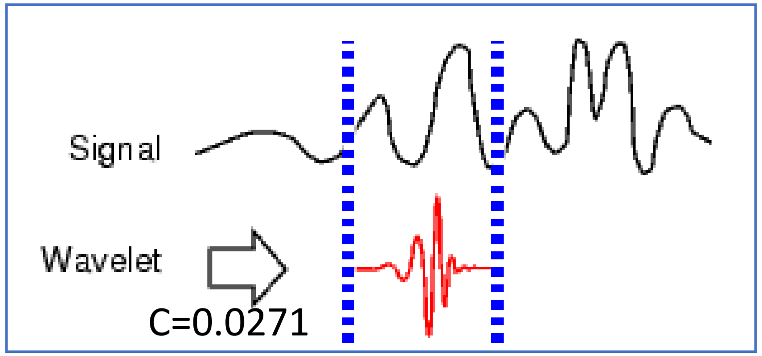


Figure 2

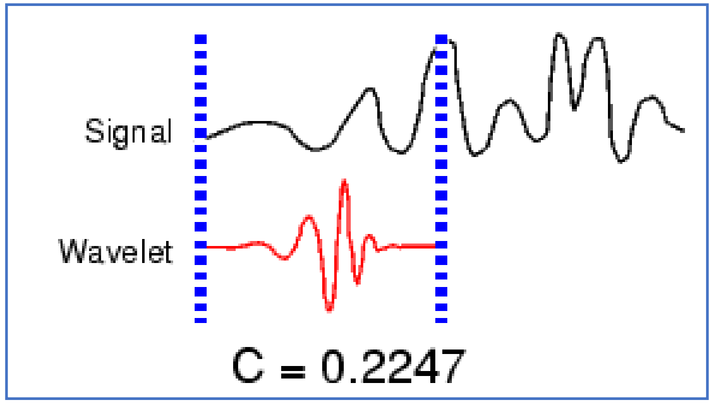
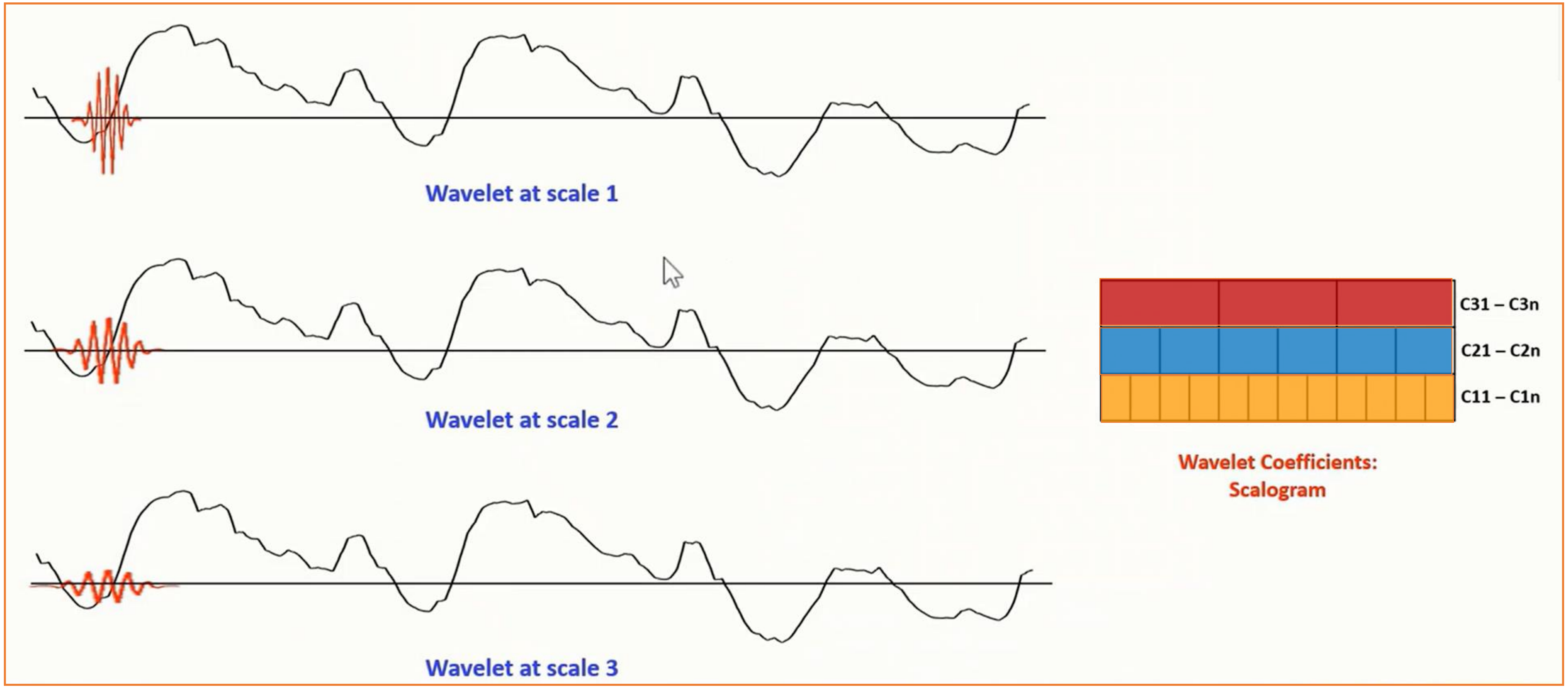


Figure 3

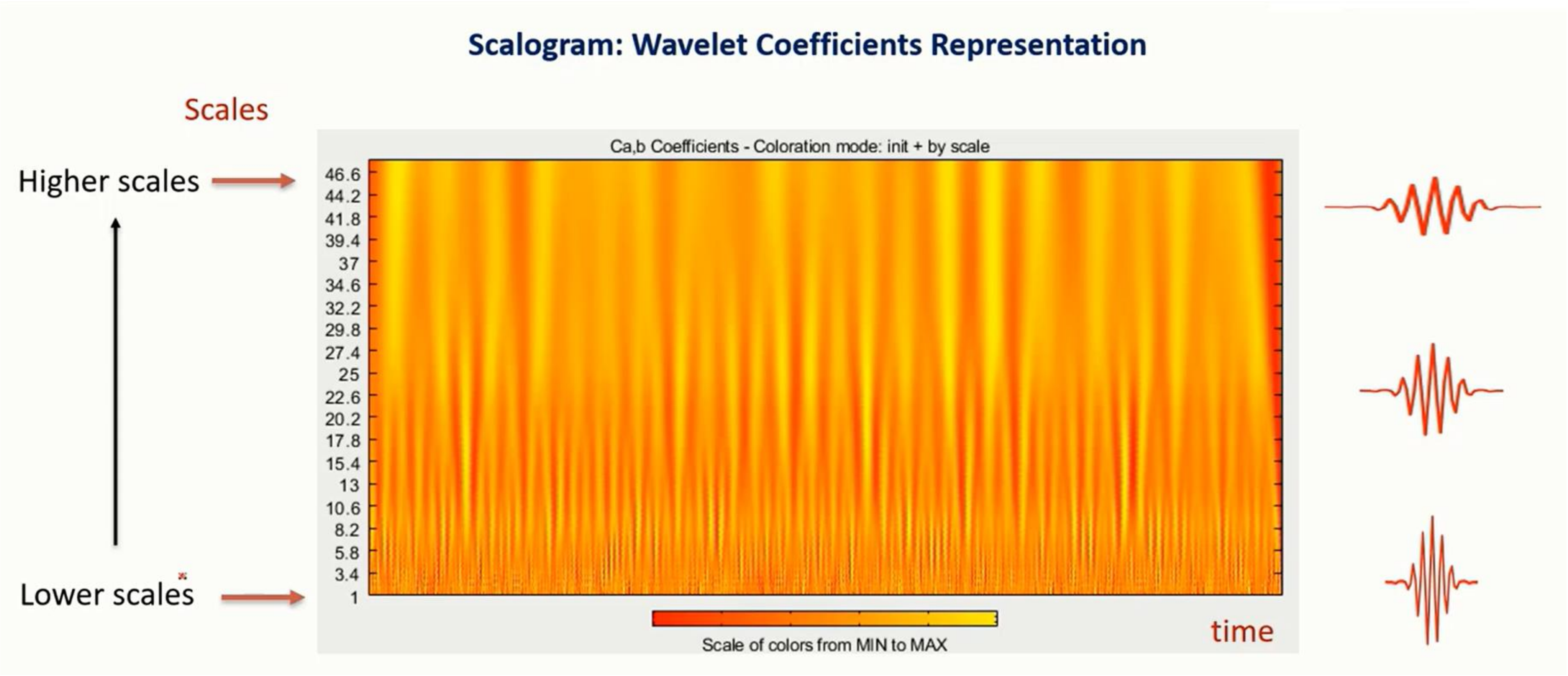
Continuous Wavelet Transform (CWT)

Capture High frequency components
(bursts, sharp spikes, discontinuities)

Capture Low frequency components



Continuous Wavelet Transform (CWT)



Discrete Wavelet Transform

- In CWT, calculating wavelet coefficients at every possible scale is a fair amount of work, and it generates an awful lot of data.

- If scales (a) and positions (b) are chosen to be discrete then analysis will be much easier and will not generate the huge data.

- This idea of choosing discrete values of dilation (a) and translation (b) parameters is implemented in,
  - ✓ **Redundant Wavelet Transform (Frames)** and
  - ✓ **Orthonormal bases for wavelets or Multi Resolution Analysis (MRA)**.

Discrete Wavelet Transform: Redundant Wavelet Transform (Frames)

means scaling

- The ***a*** is chosen to be an integer powers of one fixed dilation parameter $a_0 > 1$, i.e. $a = a_0^m$.

- The different values of ***m*** correspond to wavelets of different widths.

- The narrow wavelets are translated by small steps, while wider wavelets are translated by larger steps. Therefore, ***b*** is discretized by $b = nb_0 a_0^m$, where $b_0 > 0$ is fixed and $n \in \mathbb{Z}$.

- The corresponding discretely labeled wavelets are therefore,

$$\psi_{m,n}(k) = a_0^{-\frac{m}{2}} \psi\left(a_0^{-m}(k - nb_0 a_0^m)\right) \qquad m, n \in \mathbb{Z}$$

- For a given function $f(k)$, the inner product $\langle f, \psi_{m,n} \rangle$ then gives the discrete wavelet transform as given as,

$$\boxed{DWT(m,n) = \langle f, \psi_{m,n} \rangle = a_0^{-\frac{m}{2}} \sum_{k=-\infty}^{\infty} f(k).\psi^*(a_0^{-m}\, k - nb_0)}$$

Discrete Wavelet Transform: Multi-Resolution Analysis (MRA)

- If scales and positions are chosen based on powers of two, so-called Dyadic scales and positions, then analysis becomes much more efficient and just as accurate.

- It was developed in 1988 by S. Mallat. For some very special choice of $\psi(k)$ and a_0, b_0, the $\psi_{m,n}(k)$ constitute an orthonormal basis for $\mathbb{L}^2(\mathbb{R})$.

- In particular, if $a_0 = 2, b_0 = 1$, then there exist ψ(k) with good time-frequency localization properties, such that the,

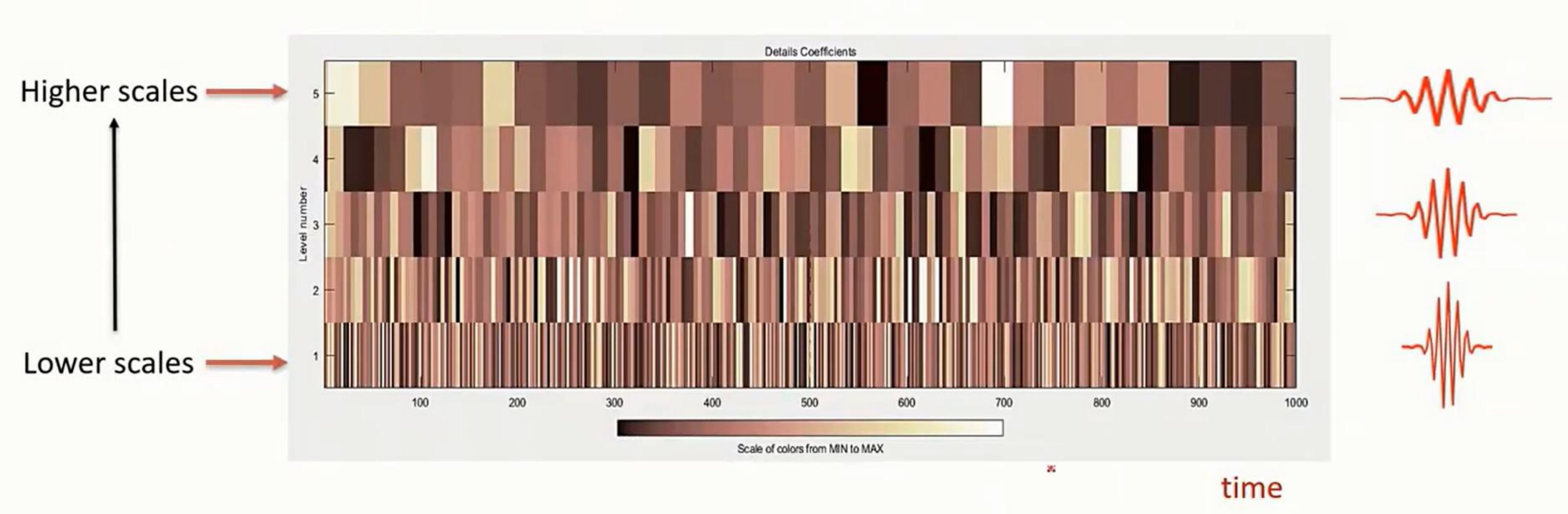
$$\psi_{m,n}(k) = 2^{-\frac{m}{2}}\psi(2^{-m}k - n) \quad m, n \in \mathbb{Z}$$

, constitutes an orthonormal basis for $\mathbb{L}^2$ $(\mathbb{R})$.

- For a given function $f(k)$, the inner product $\langle f, \psi m,n\rangle$ then gives the discrete wavelet transform as given as,

$$\boxed{DWT(m,n) = \langle f, \psi_{m,n}\rangle = 2^{-\frac{m}{2}} \sum_{k=-\infty}^{\infty} f(k).\psi^*(2^{-m}k - n)}$$

Scalogram: Discrete Wavelet Coefficients Representation



Why using wavelets?

- The use of wavelets allows the resolutions of the query and target images to be different .
- **Wavelet decompositions** are fast to compute and yield a small number of coefficients.
  - **Haar wavelets** are the **fastest to compute** and simplest to implement.
- The signature can be extracted from a **wavelet-compressed version** of the image directly.
  - Standard decomposition worked best

Haar Father Wavelet

Motivation: suppose we have a basic function (Haar Father Wavelet or Scaling Function)

$$\phi(x) = \begin{cases} 1 \text{ if } 0 \leq x \leq 1 \\ 0 \text{ otherwise} \end{cases} = \text{basic "pixel"}.$$

We wish to build all other functions out of this pixel and translates $\phi(x - k)$

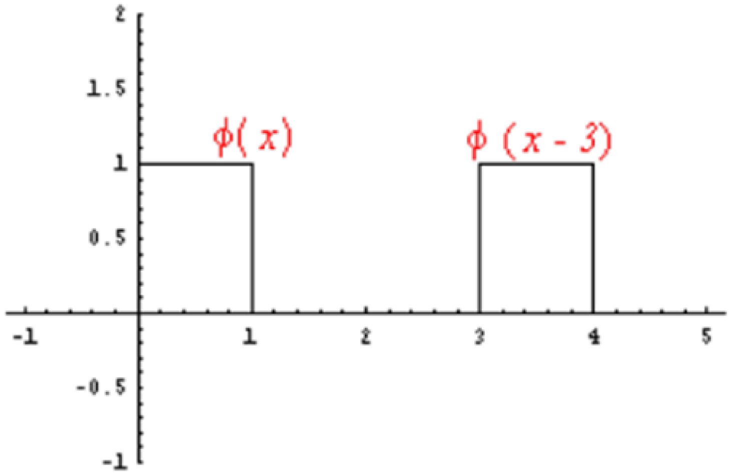
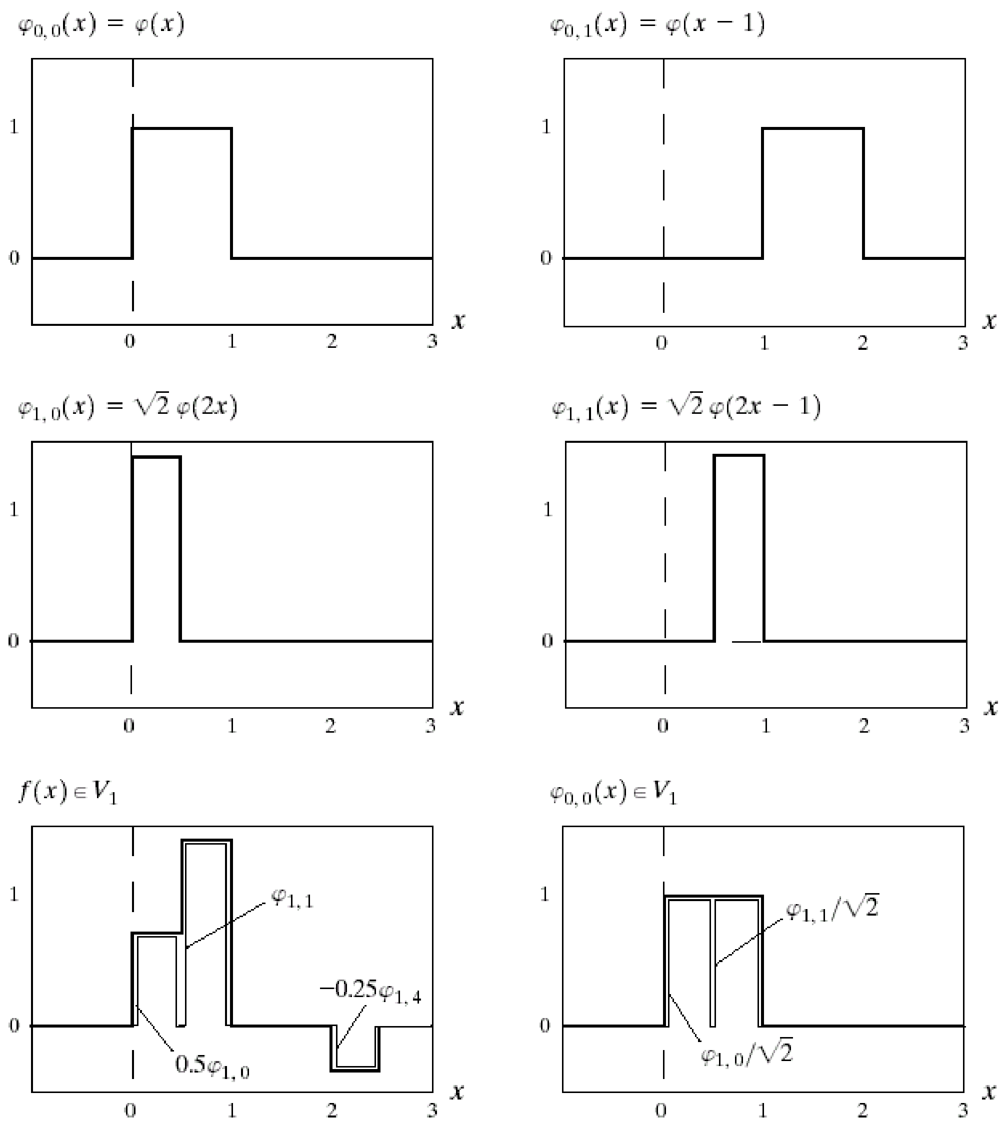


fig : ϕ and its translates

Haar Scaling Function



a b
c d
e f

FIGURE 7.9 Haar scaling functions in V_0 in V_1.

Functional approximation using coefficients (1/4)

Linear combinations of the $\phi(x-k)$:

$$f(x) = 2\phi(x) + 3\phi(x-1) - 2\phi(x-2) + 4\phi(x-3)$$

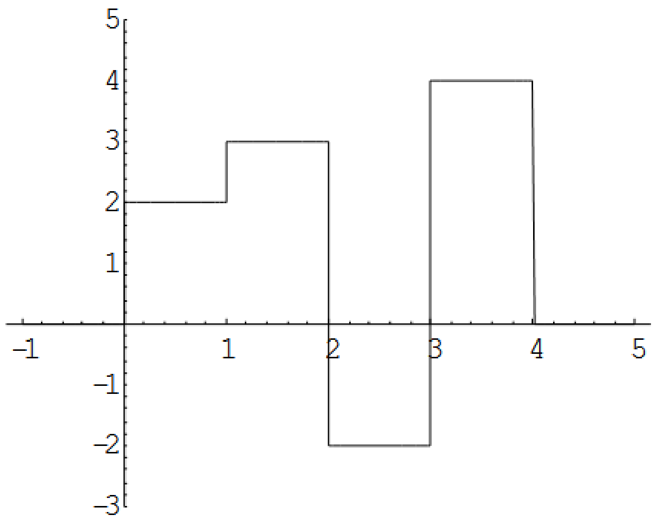


fig 8: a linear combination of $\phi(x-k)$

[Note that any function which is constant on the integers can be written in such a form:]

Quantized to limited resolution

Functional approximation using coefficients (2/4)

Given function $f(x)$, approximate $f(x)$ by a linear combination of $\phi(x-k)$:

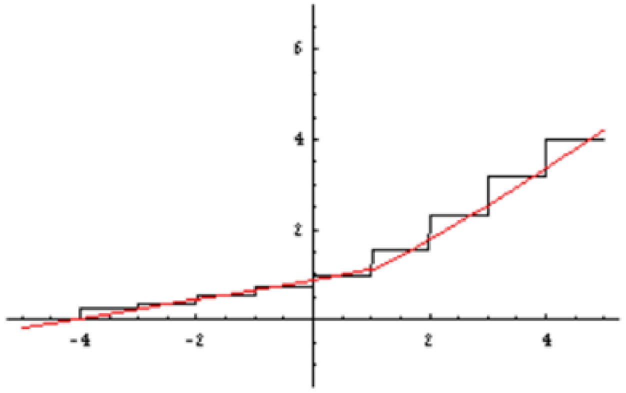


fig : approximation of $f(x)$ using the pixel $\phi(x)$ and its translates.

Define V_0 = all square integrable functions of the form

$$g(x) = \sum_k a_k \phi(x-k)$$

= all square integrable functions which are constant on integer intervals

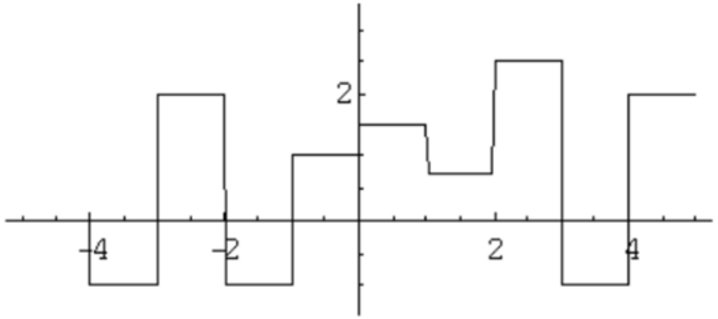


fig : a function in V_0

Father Wavelet: Better Approximation using Scale Support

To get better approximations, shrink the pixel :

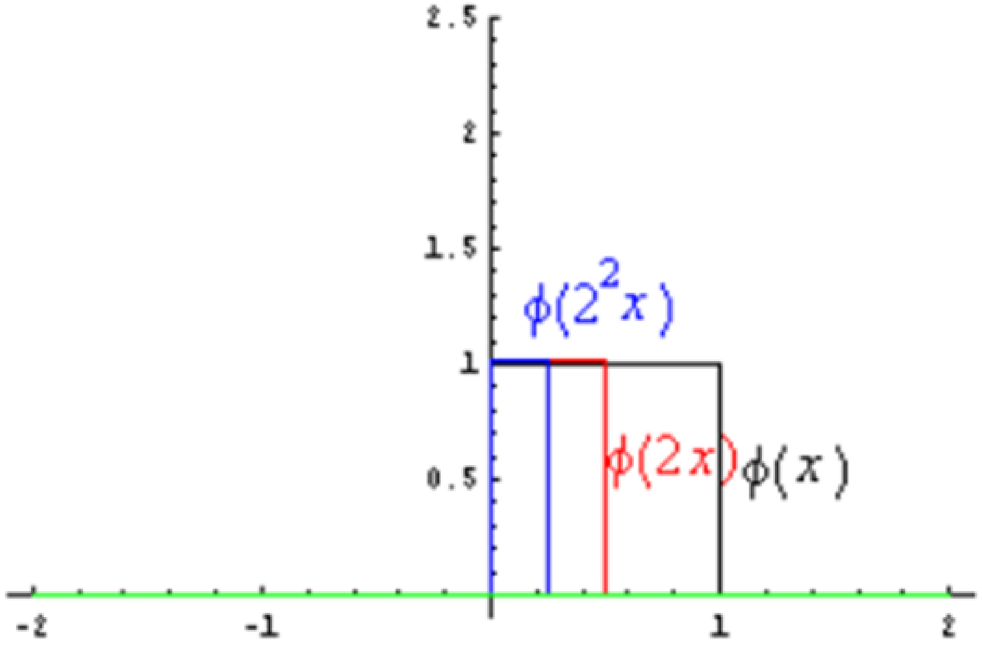


fig : $\phi(x), \phi(2x)$, and $\phi(2^2x)$

Functional approximation using coefficients (3/4)

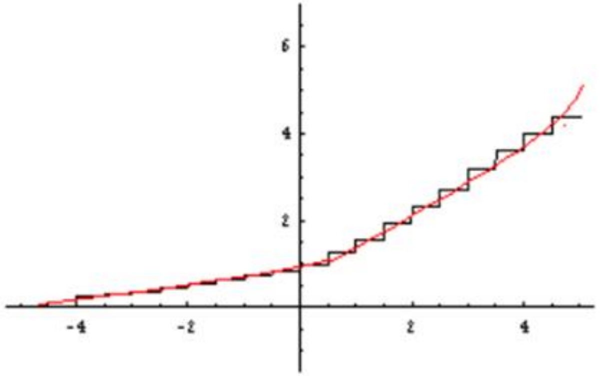


fig: approximation of $f(x)$ by translates of $\phi(2x)$.

Define

V_1 = all square integrable functions of the form

$$g(x) = \sum_k a_k \phi(2x - k)$$

= all square integrable functions which are constant on all half-integers

Slightly better Approximation

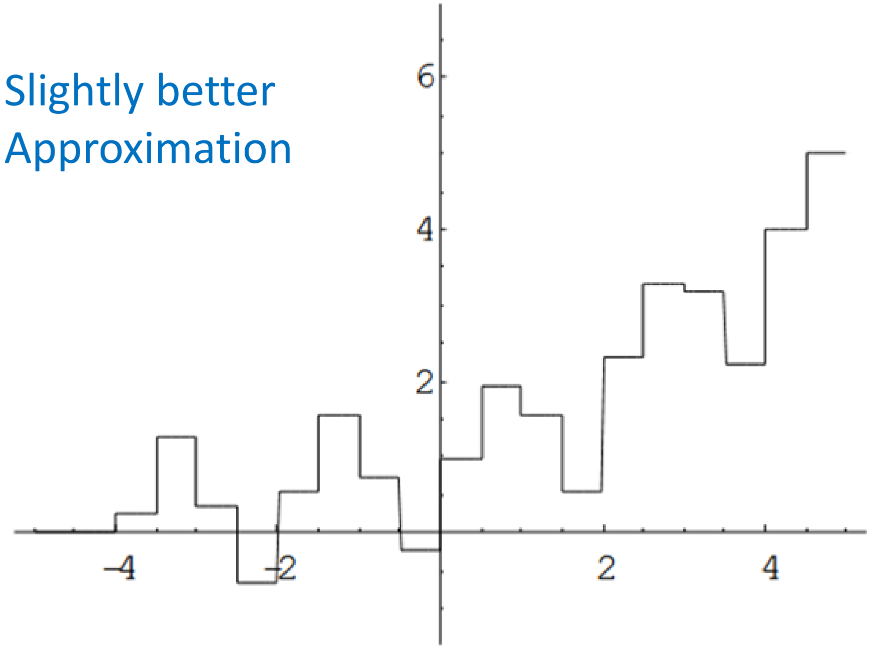


fig: Function in V_1

Functional approximation using coefficients (3/4)

Define V_2 = sq. int. functions

$$g(x) = \sum_k a_k \phi(2^2 x - k)$$

= sq. int. fns which are constant on quarter integer intervals

Better Approximation

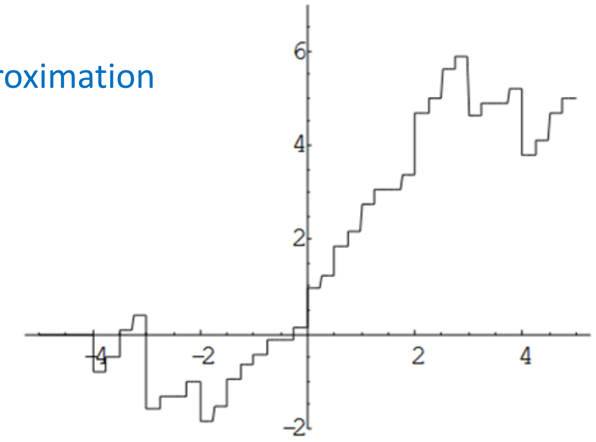


fig : function in V_2

Generally define V_j = all square integrable functions of the form

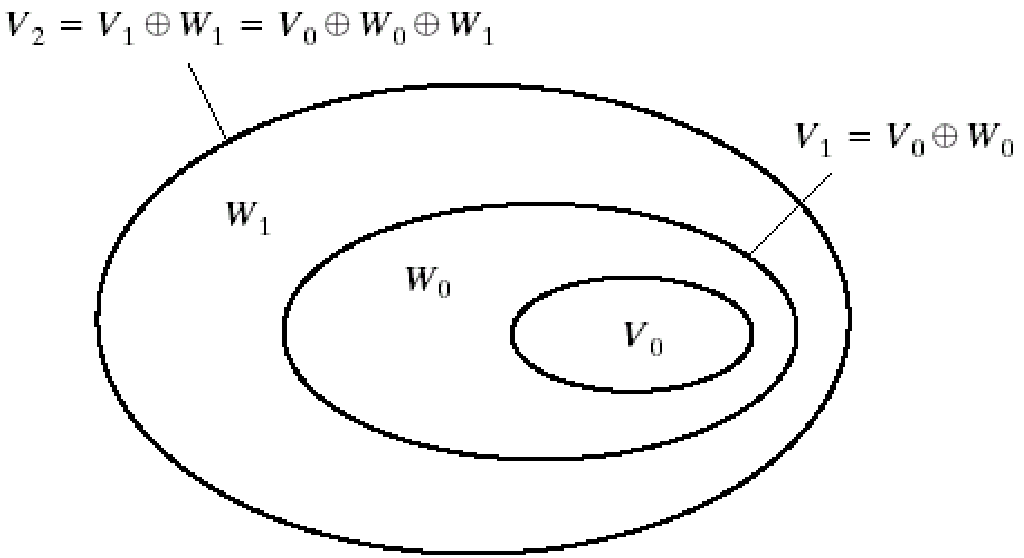
$$g(x) = \sum_k a_k \phi(2^j x - k)$$

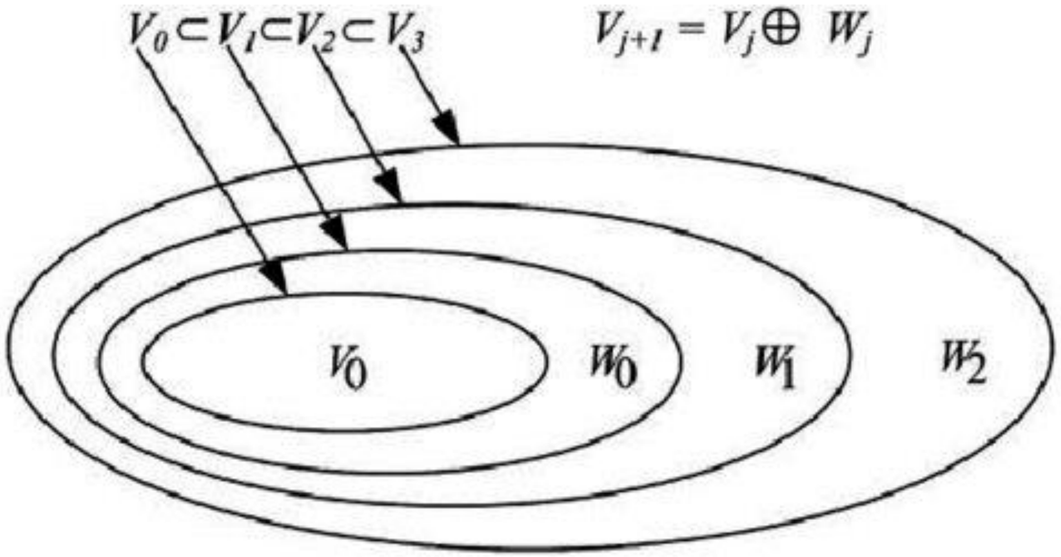
= all square integrable functions which are constant on 2^{-j} length intervals

[note if j is negative the intervals are of length greater than 1].

Nested Subspaces of MRA (Multi-resolution analysis)

The relationship between scaling and wavelet function subspaces





Haar Wavelet

Now define the desired wavelet $\psi(x)$

$$\equiv \begin{cases} 1 & \text{if } 0 \le x \le 1/2 \quad -1 \text{ if } 1/2 \le x < 1 \\ & 0 \quad \text{otherwise} \end{cases}$$

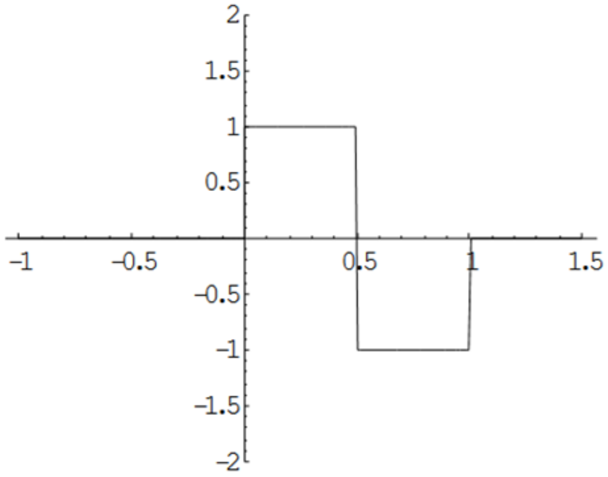


fig : $\psi(x)$

Define a family of Haar Wavelets

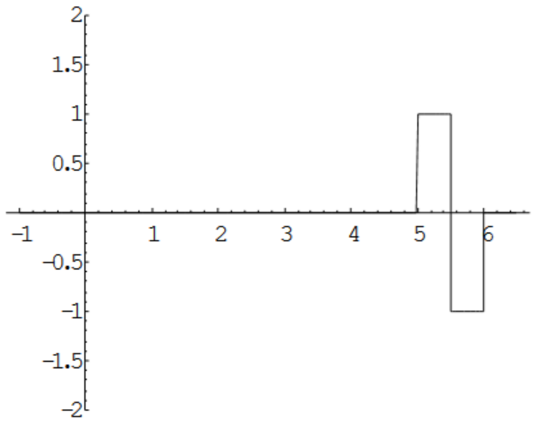
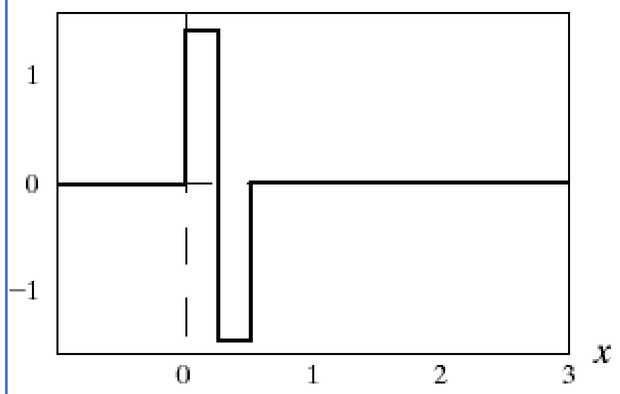


fig : $\psi(x-5) = \psi_{0,5}$

Translation

$\psi_{1,0}(x) = \sqrt{2}\,\psi(2x)$

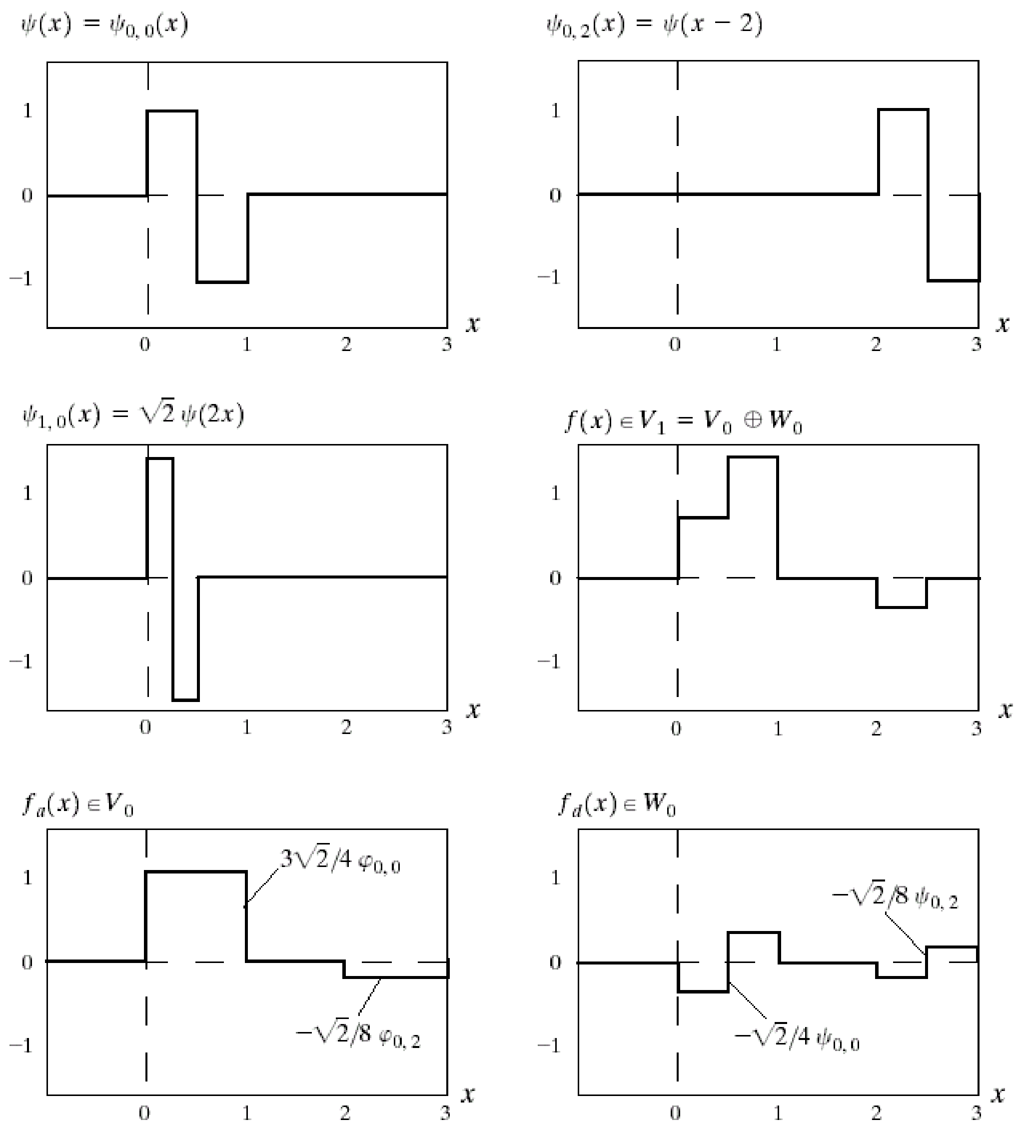


Scale (Shrink / Stretch)

In general: Haar Wavelet function $\psi_{jk} \equiv 2^{j/2}\,\psi(2^j x - k)$

for *j* a nonnegative integer and $0 \le k \le 2^j - 1$

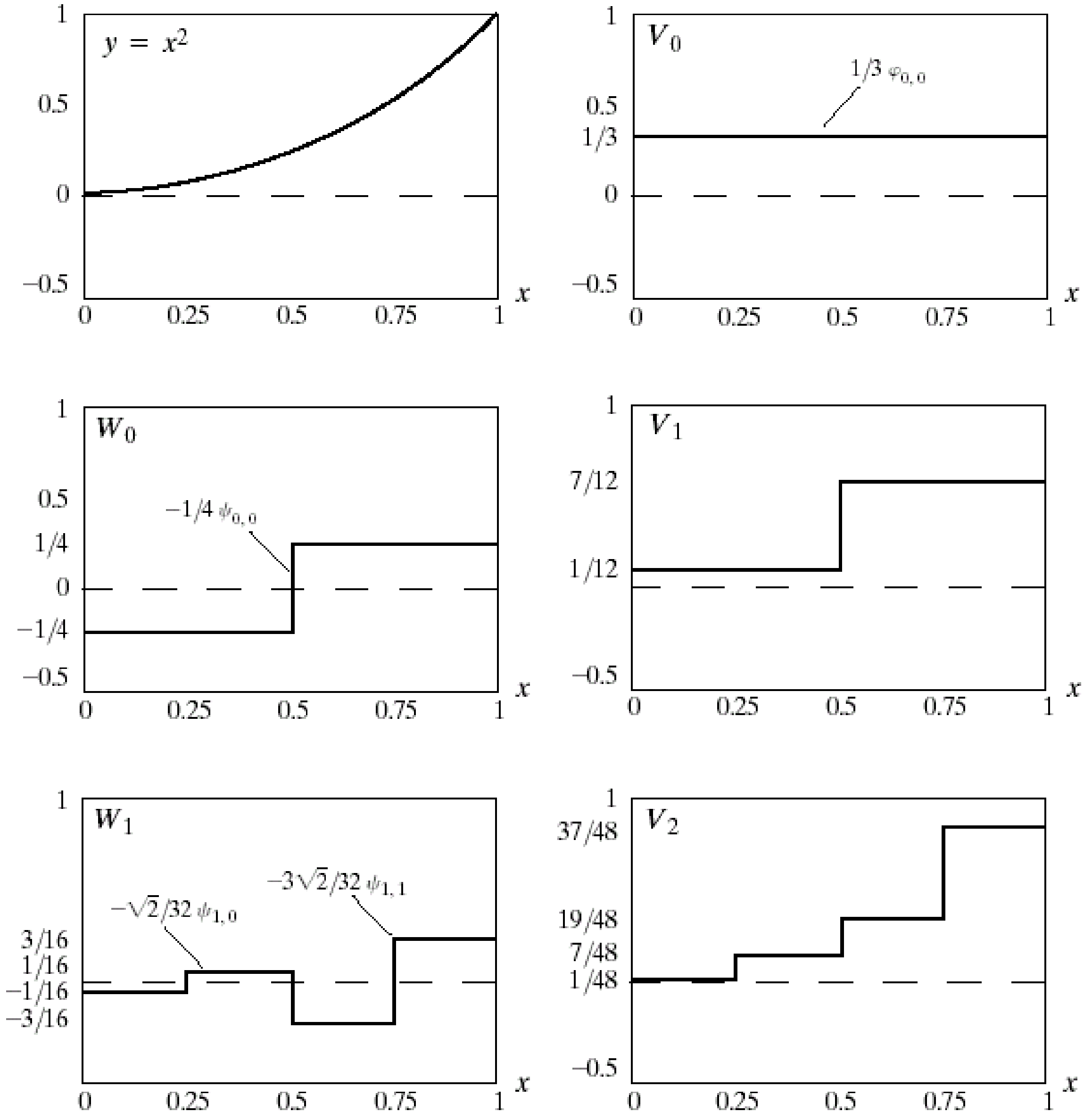
Wavelet Functions



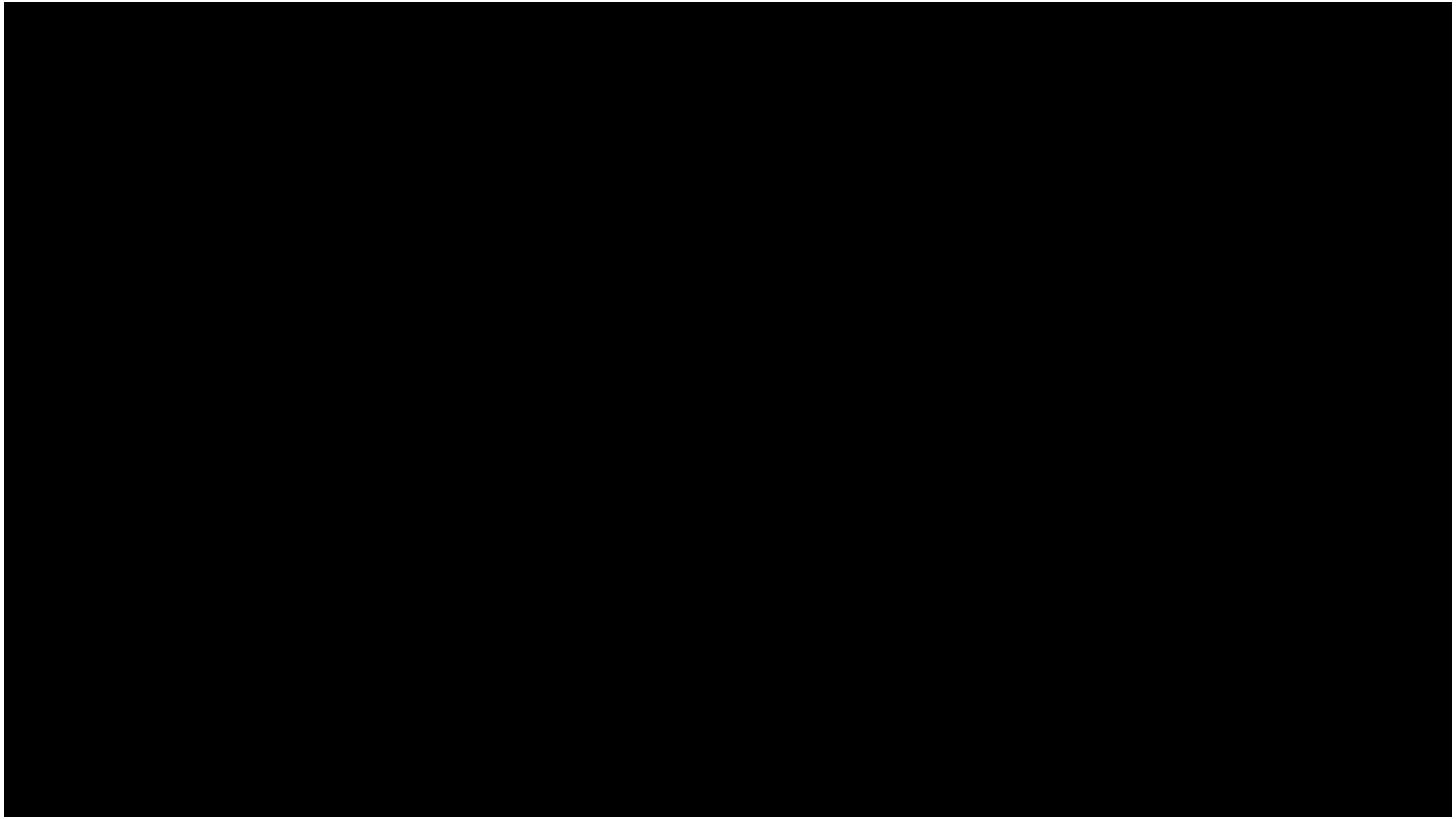
a b
c d
e f

FIGURE 7.12 Haar wavelet functions in W_0 and W_1.

Wavelet Function

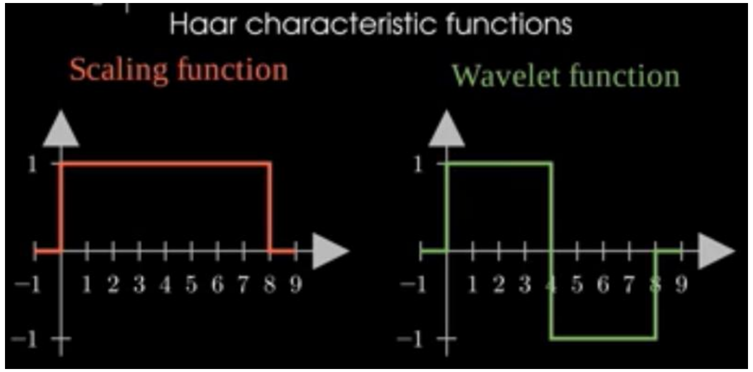


Haar Wavelet: Signal Decomposition and Reconstruction



Haar Wavelet: Multi-level Signal Decomposition

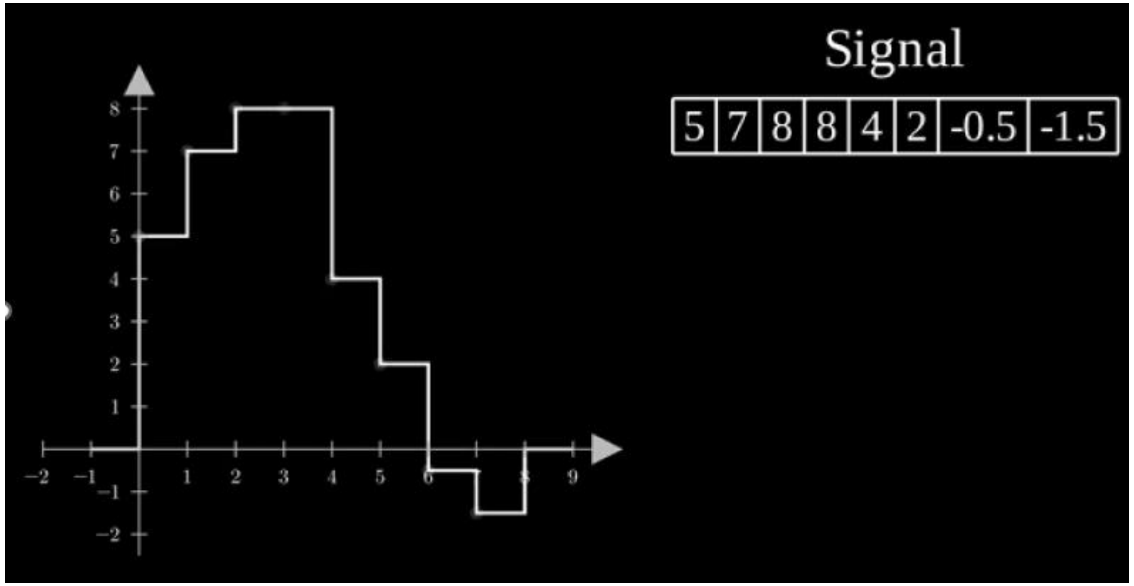
Multi-level Haar Wavelet Transform Example

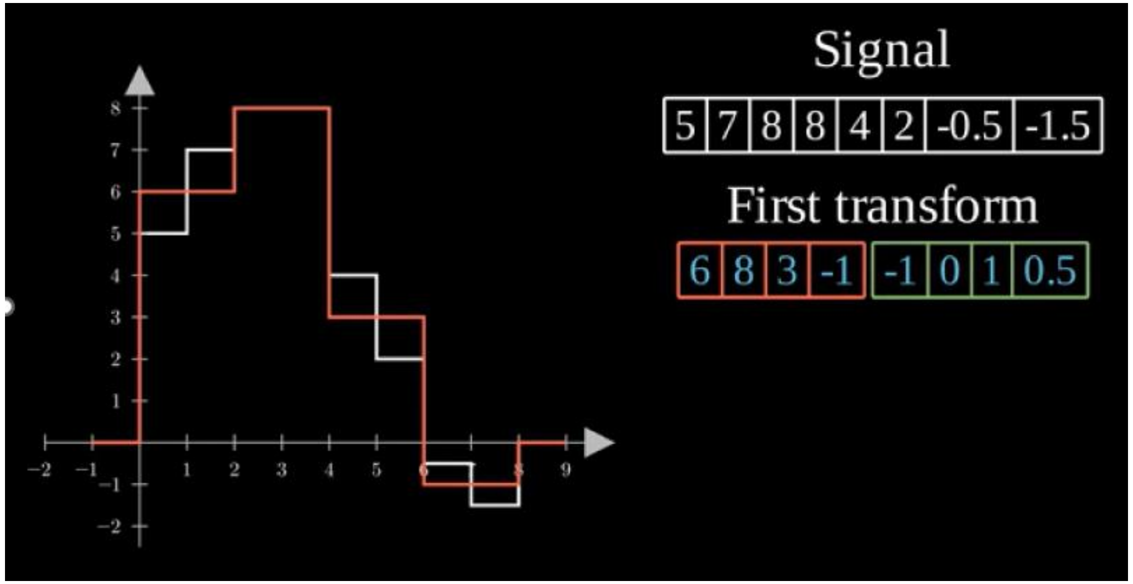


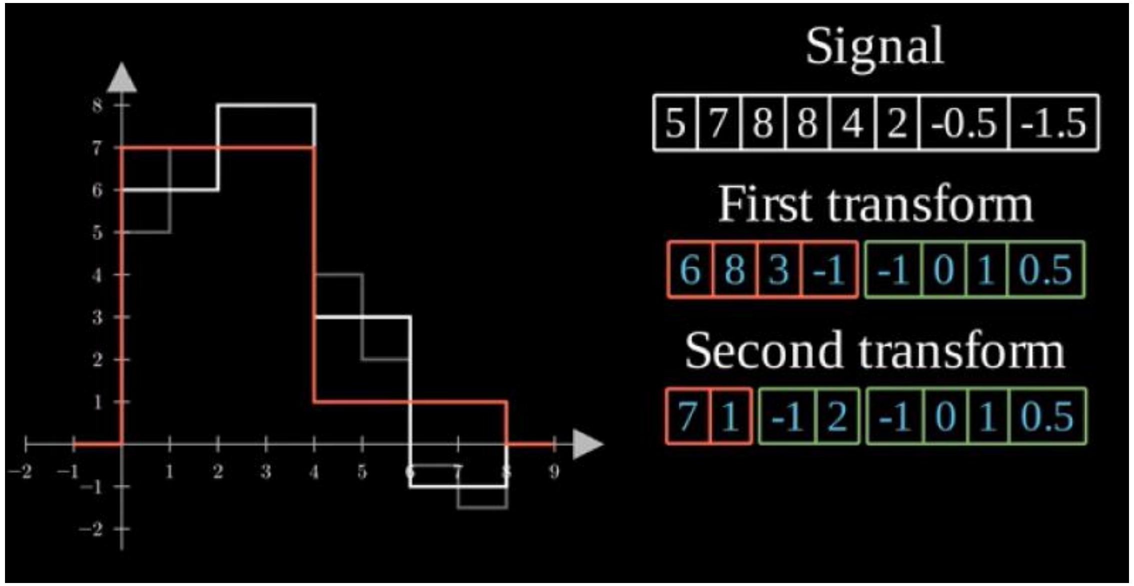
Scaling Function helps in reconstructing the Approximation Space

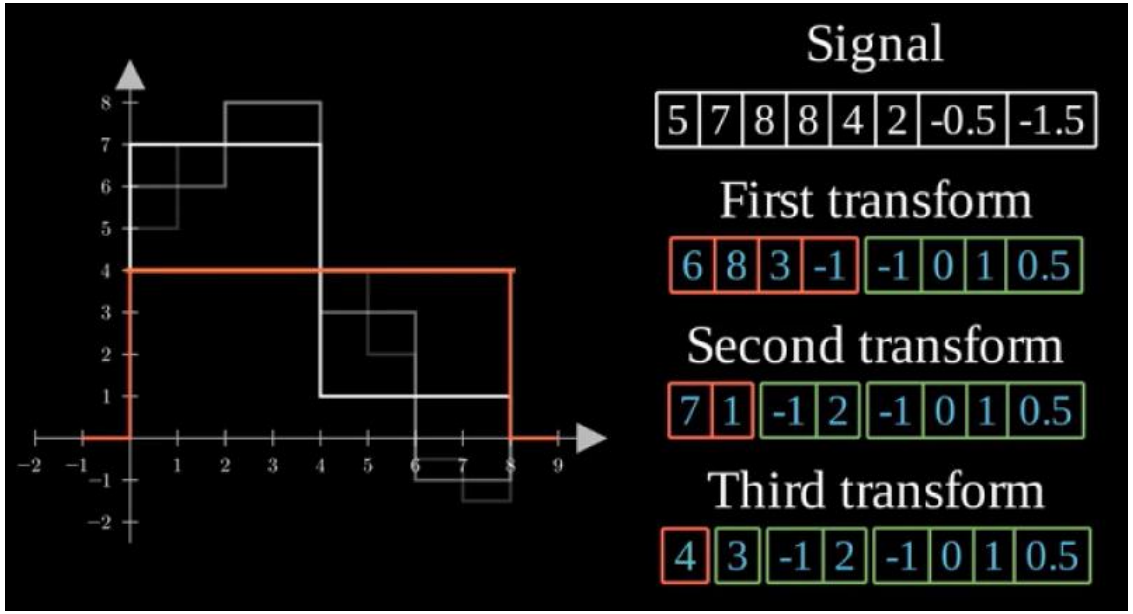
Wavelet Function helps in reconstructing the Detail Spaces

Haar Example: Multi-level Signal Decomposition

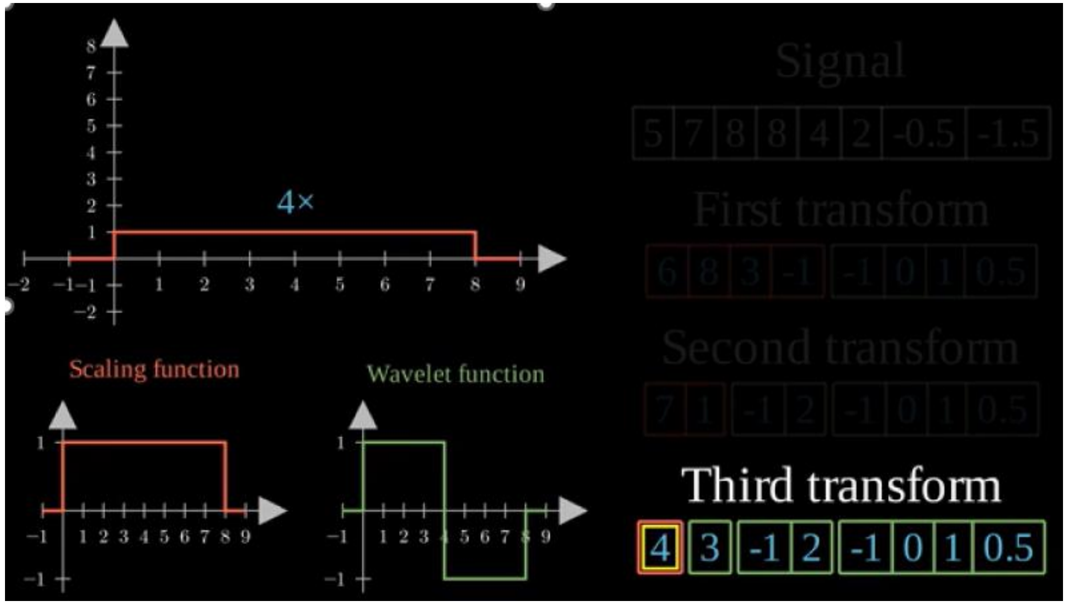


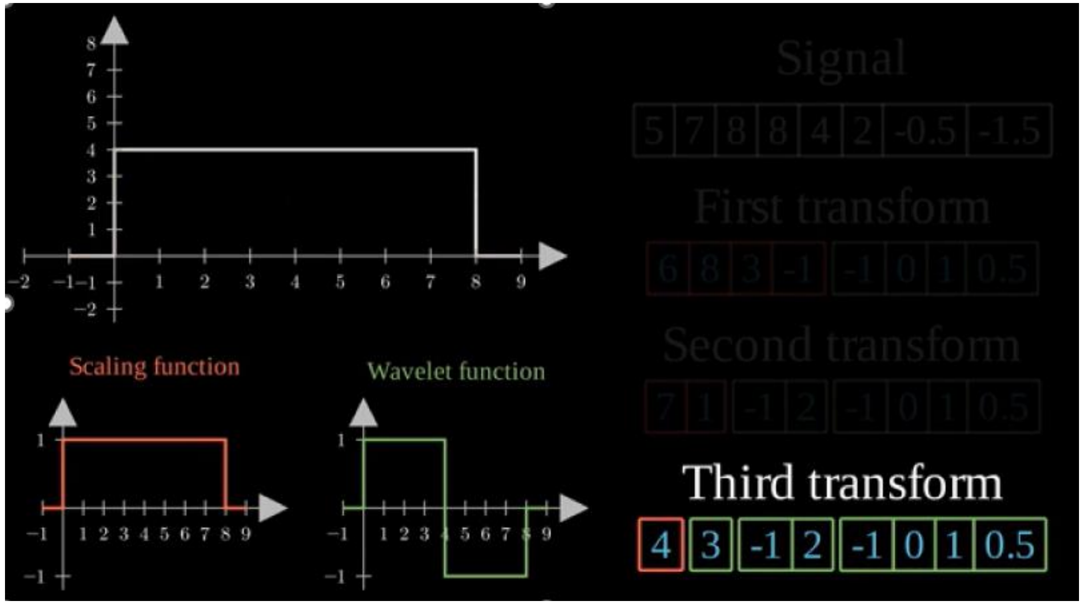


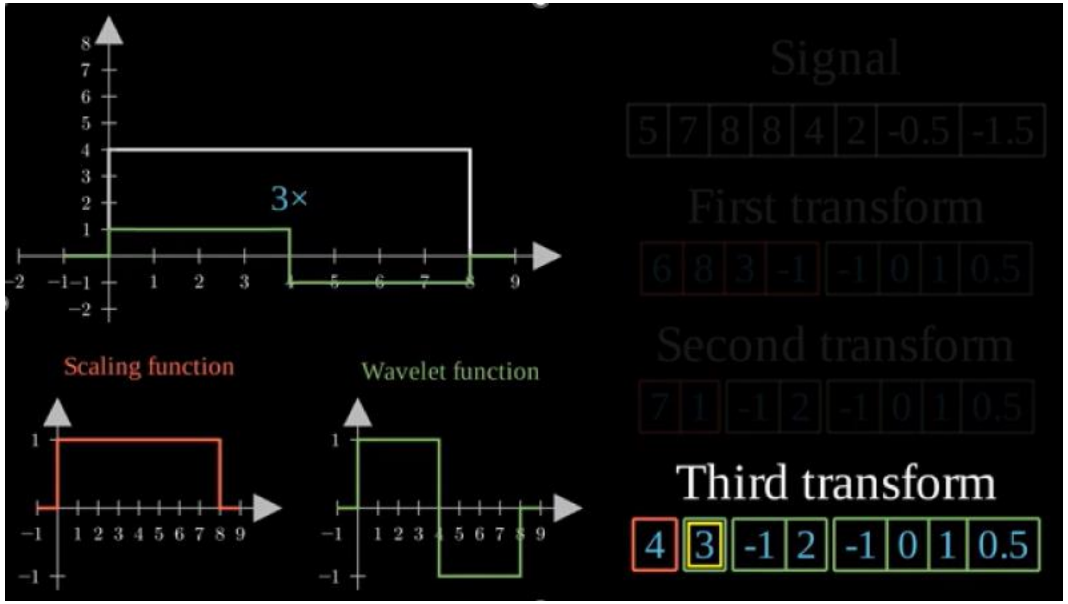


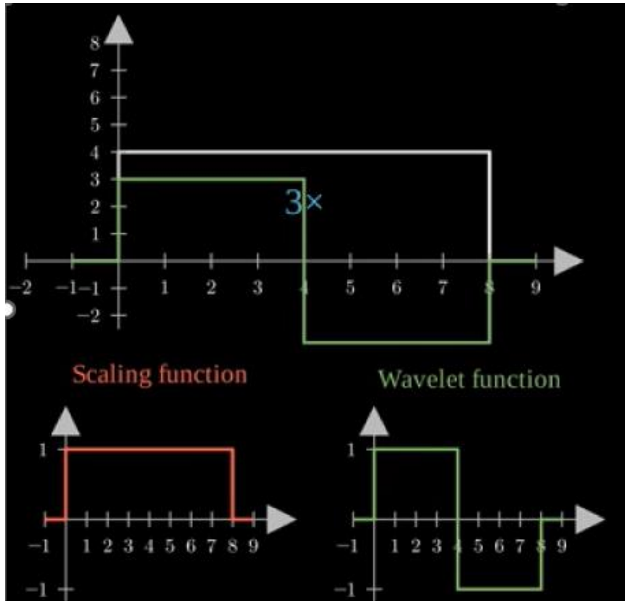


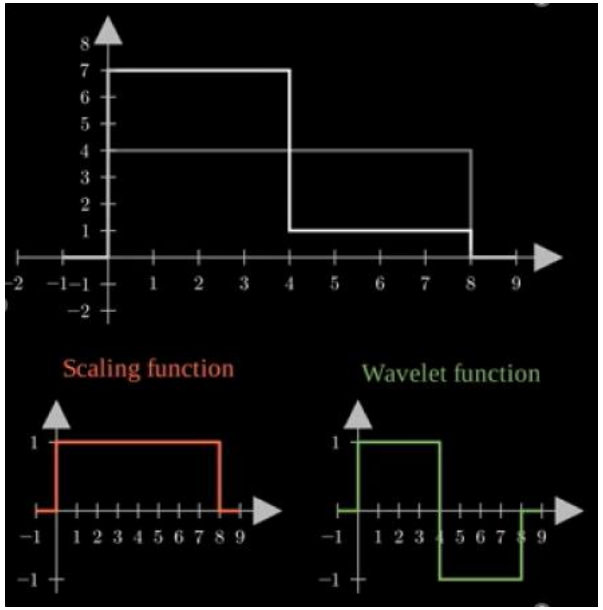
Haar Example: Signal Reconstruction (Inverse Transform) 1/3



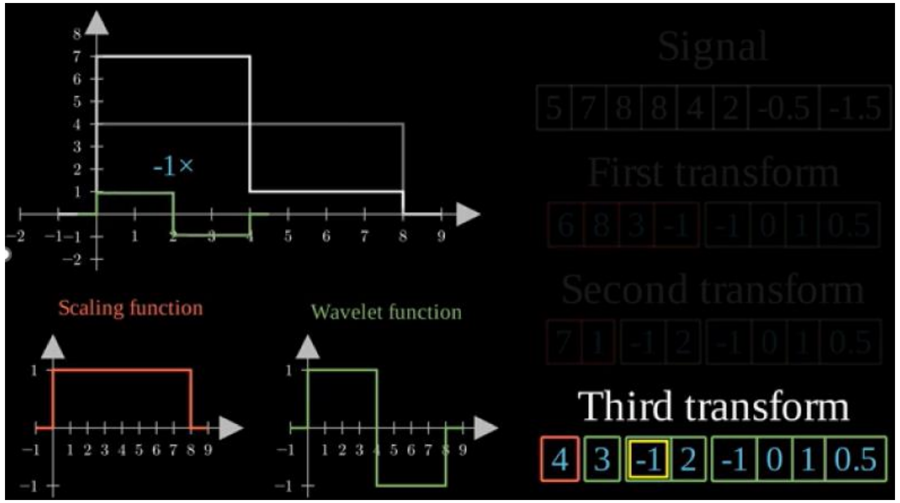


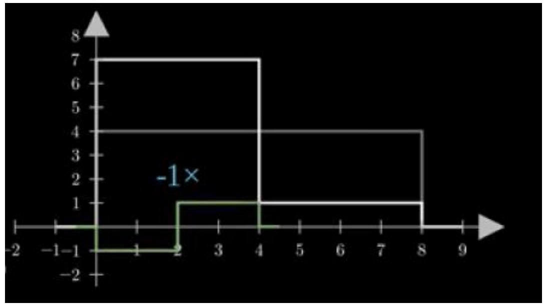


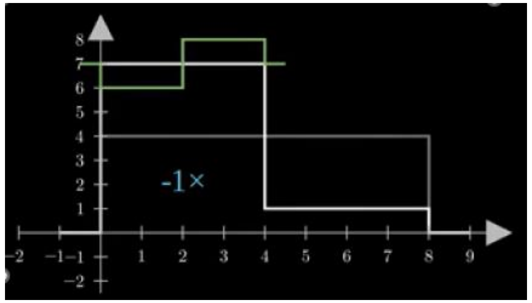


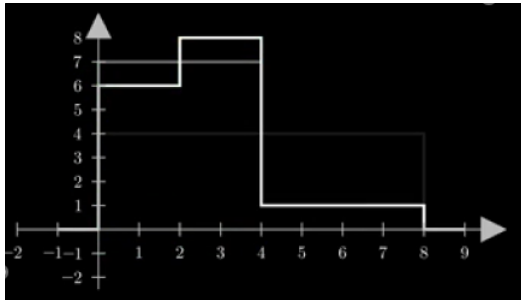


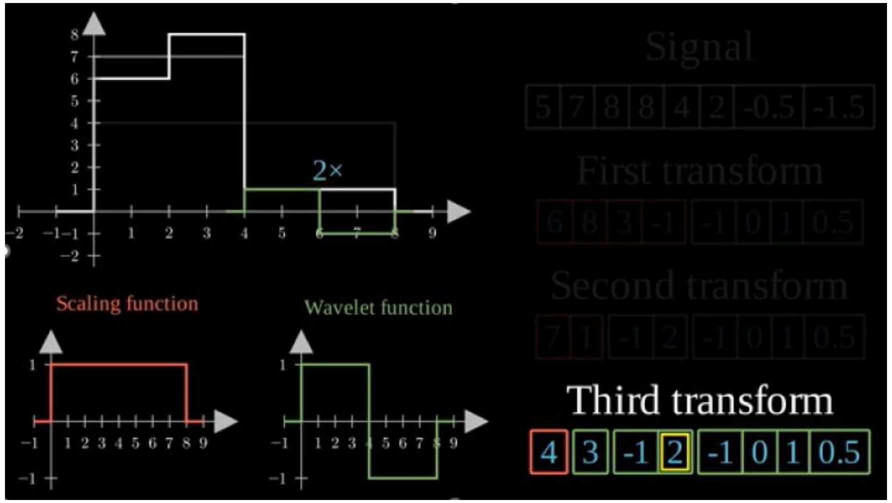
Haar Example: Signal Reconstruction (InverseTransform) 2/3

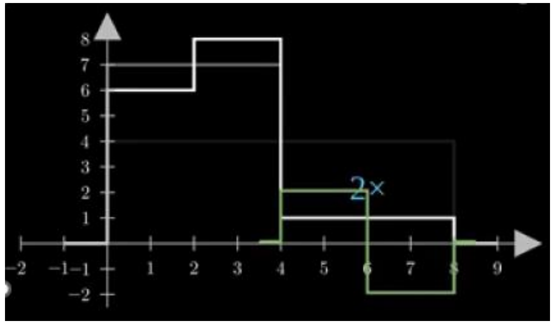


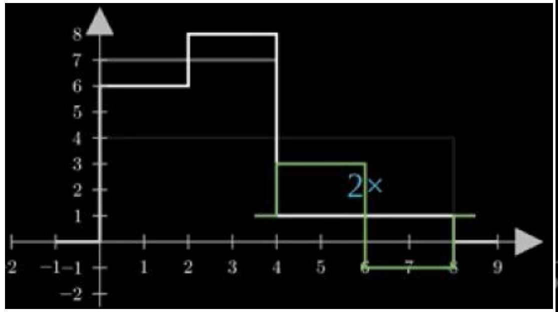


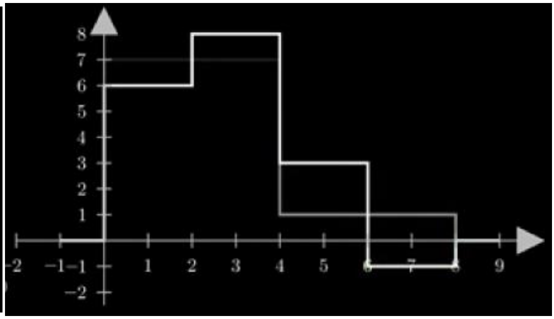




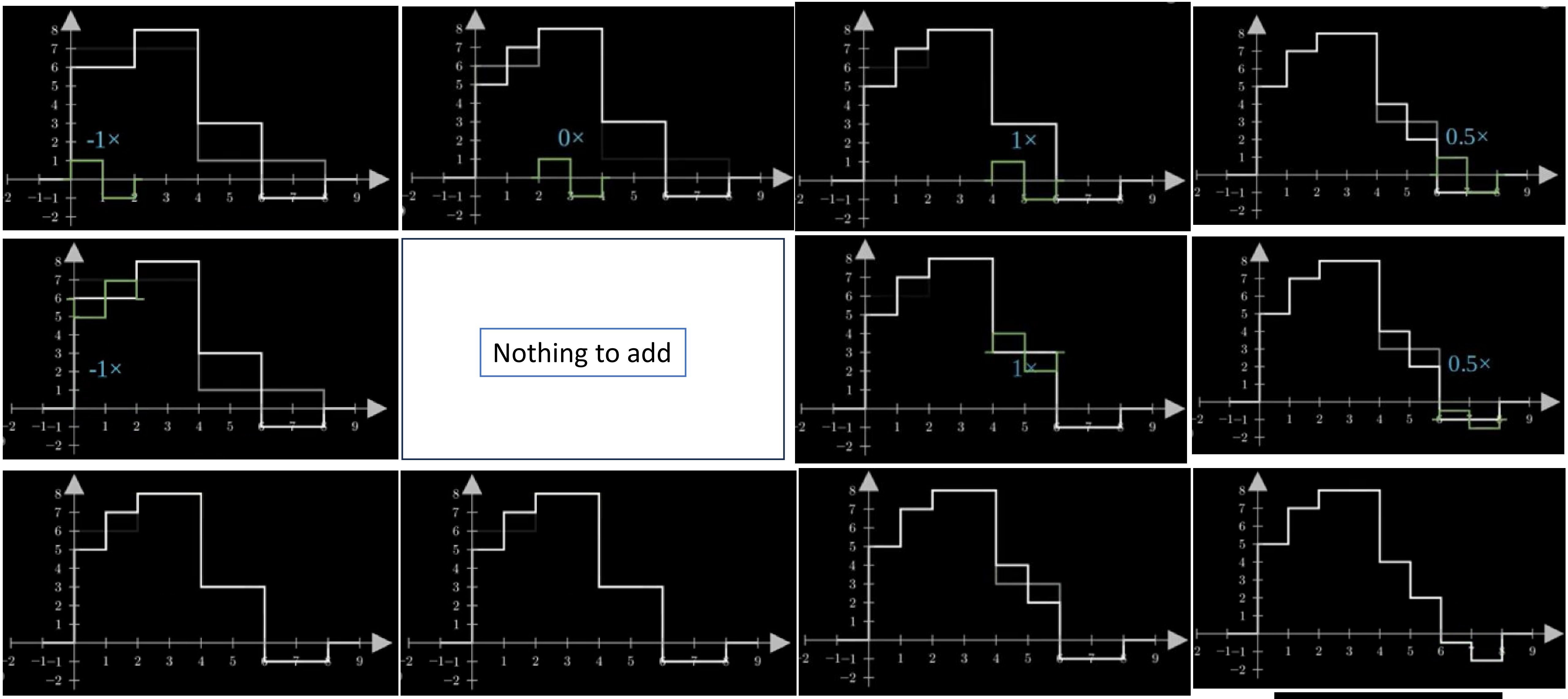








Haar Example: Signal Reconstruction (InverseTransform) 3/3



Third transform

4	3	-1	2	-1	0	1	0.5

Signal

5	7	8	8	4	2	-0.5	-1.5

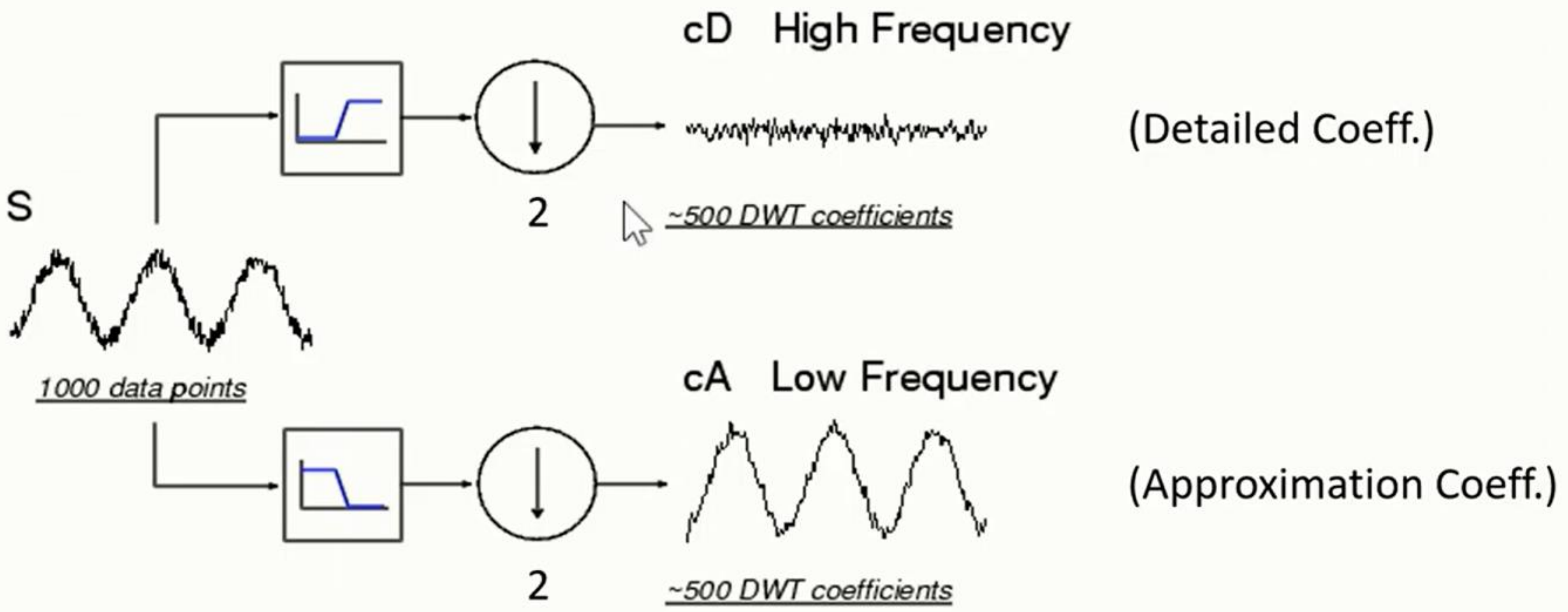
Discrete Wavelet Transform: Filter Banks

The wavelet decomposition of a signal $s(t)$ based on the multi resolution theory given by **S. Mallat** and **Meyer** can done using digital FIR filters as shown in figure.

Finite Impulse Response (*FIR*) : This type of *filter* gives a finite number of nonzero outputs (response) to an impulse function input.

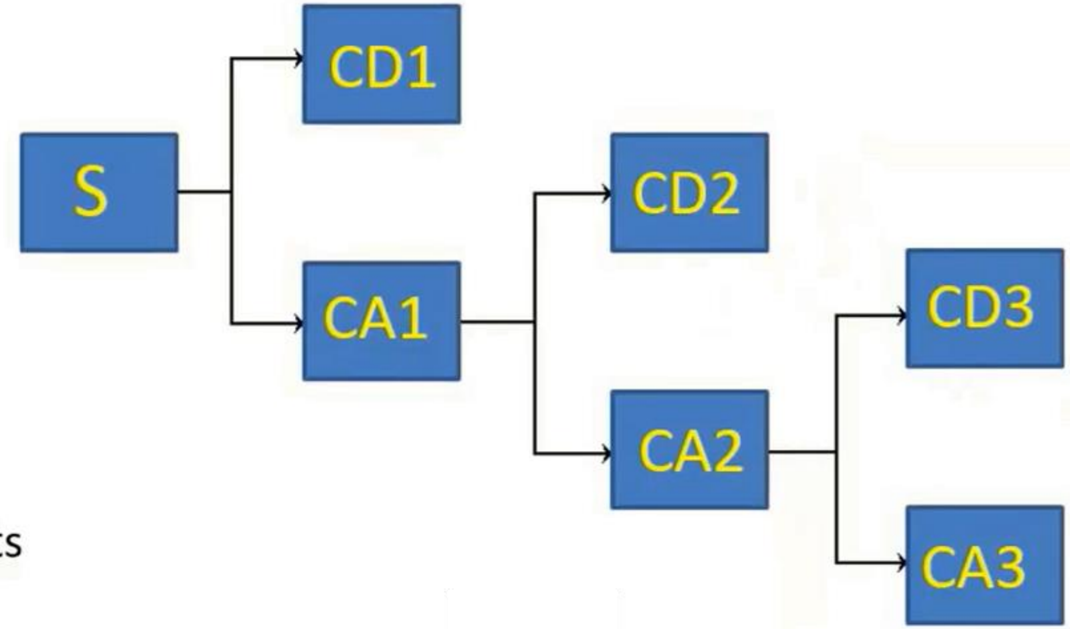
One level Decomposition

Signal Analysis



Discrete Wavelet transform (Filter Banks)

Multi Level Decomposition

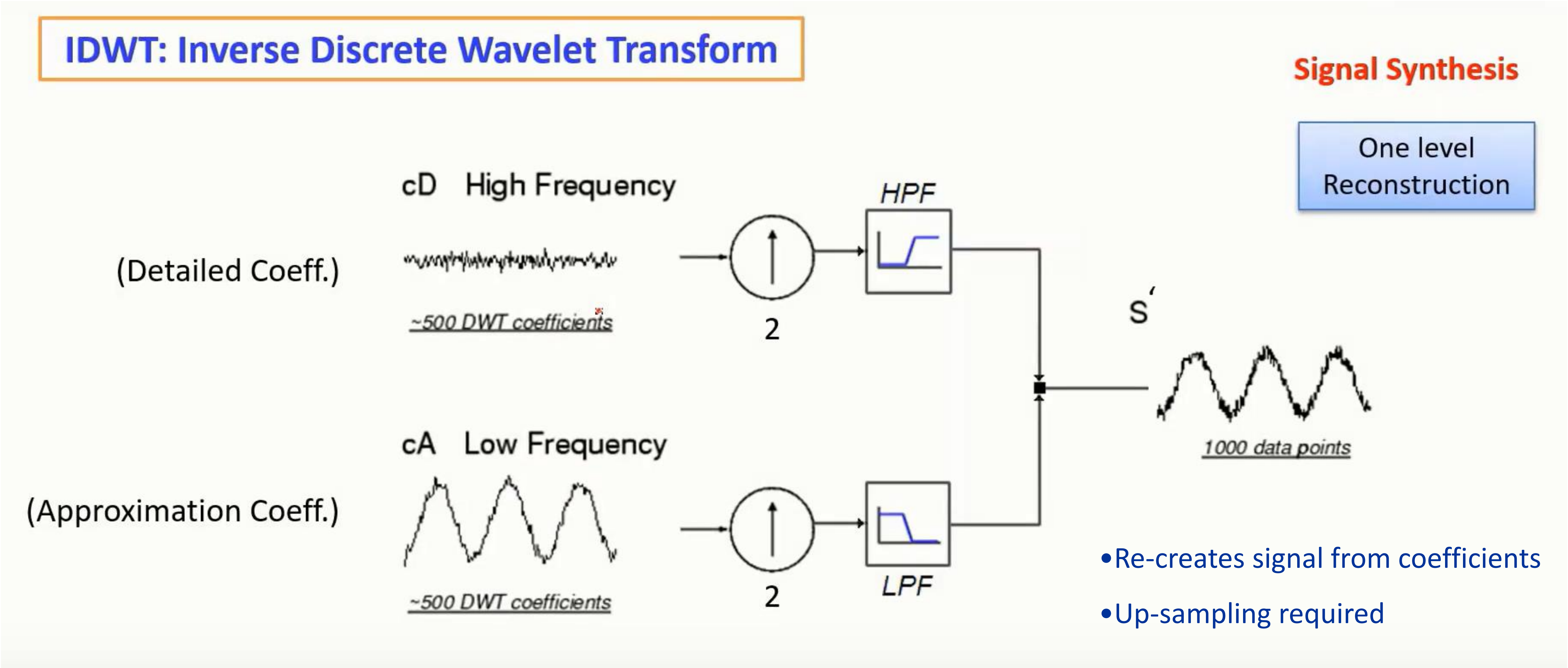


Here,
S: Signal,
CD: Detailed Coefficients
CA: Approximation Coefficients

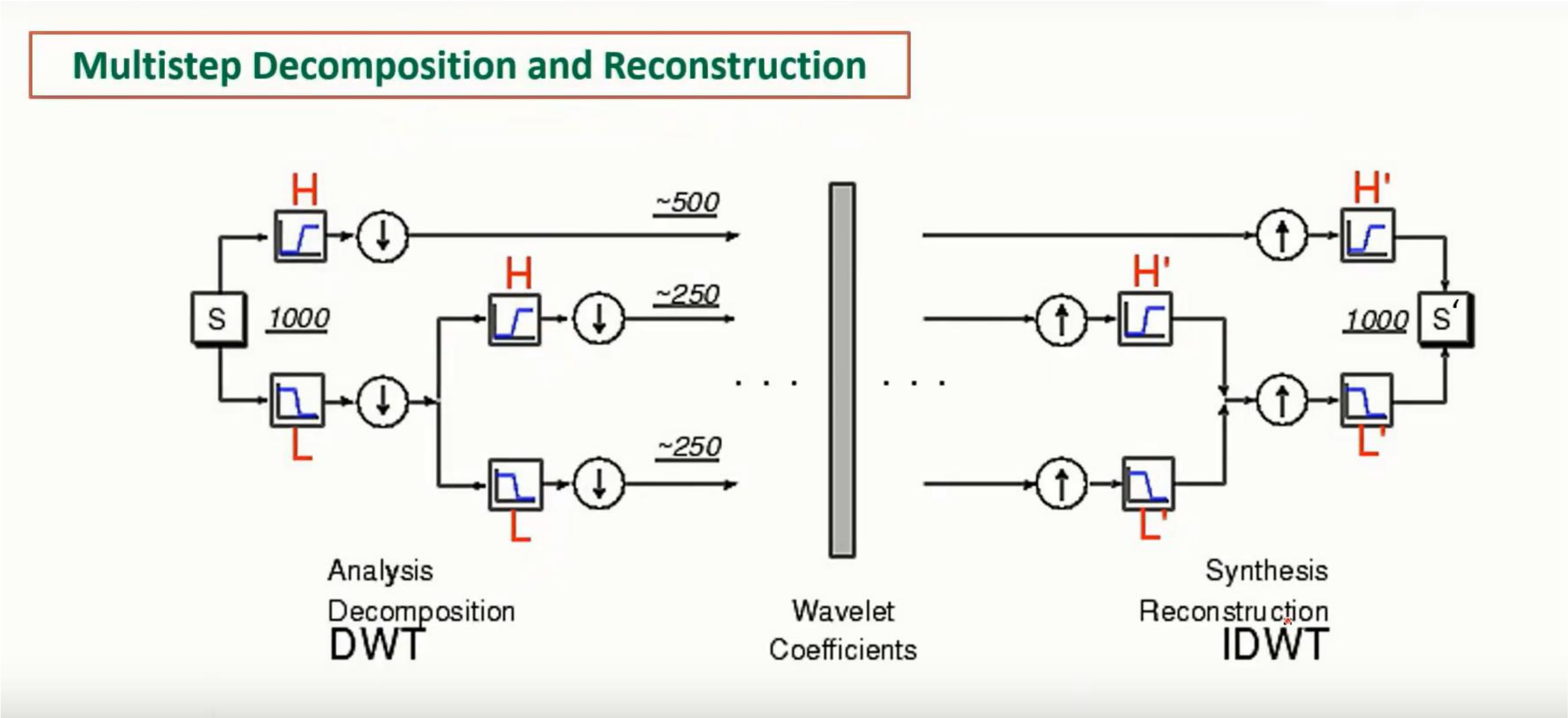
This is called Wavelet decomposition tree.

Max. Number of decomposition levels: $\log_2 N$

Inverse Discrete Wavelet Transform: Filter Banks

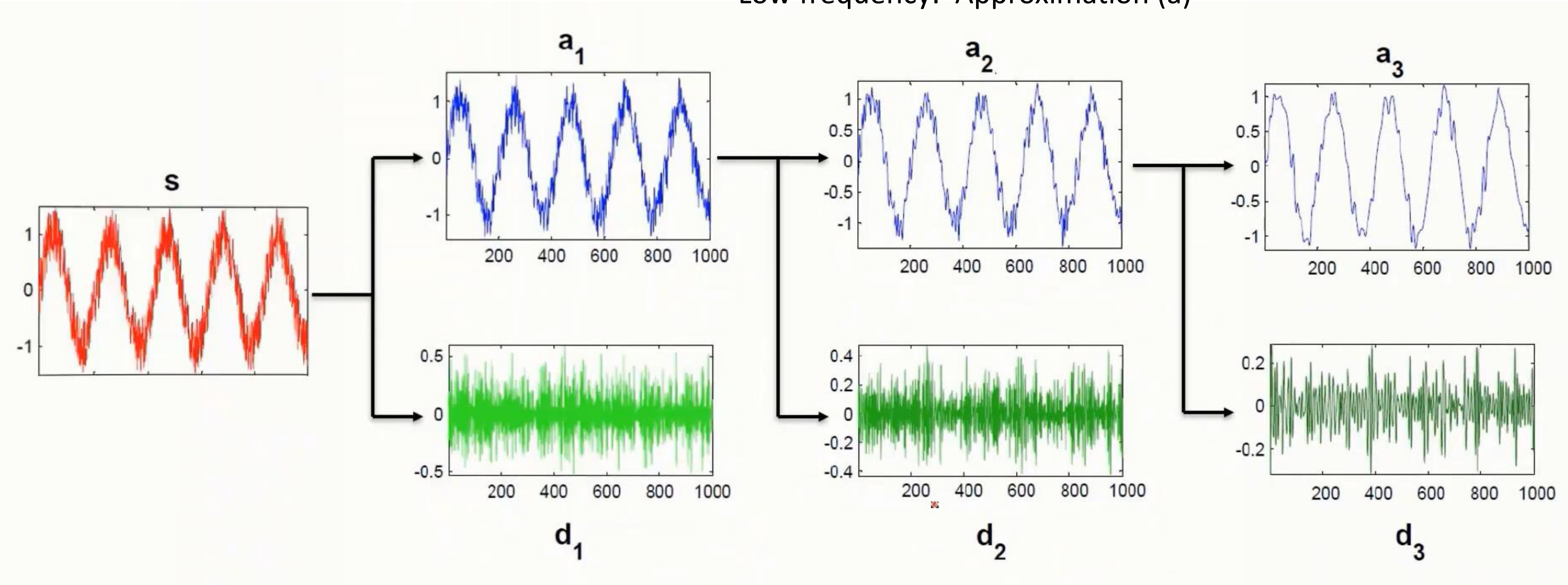


Multi-level Wavelet Analysis



DWT: Wavelet Decomposition of a Signal

Low frequency: Approximation (a)

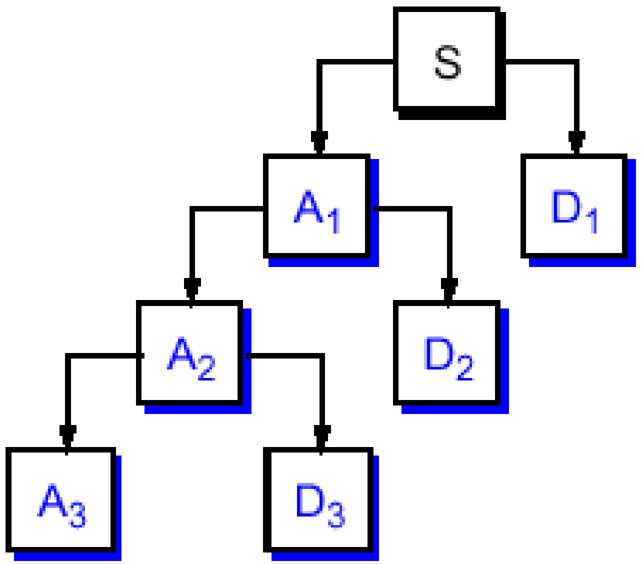


High frequency Details (d)

“Decomposition” can be performed iteratively

Multi-level Wavelet Analysis

Multi-level wavelet decomposition tree



Reassembling original signal

$$S = A_1 + D_1$$

$$= A_2 + D_2 + D_1$$

$$= A_3 + D_3 + D_2 + D_1$$

Mult-resolution image processing and wavelets

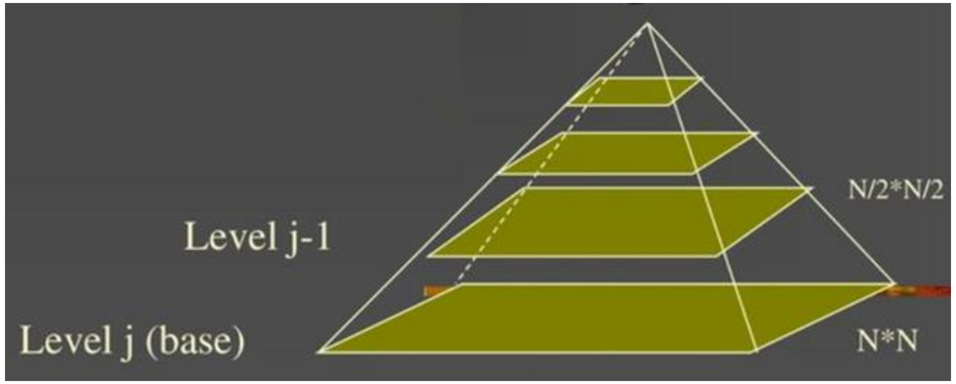
- ❖ Multi-resolution Analysis (MRA): An image cane be structured by a series of images with different resolutions
- ❖ Image Pyramids : Collection of decreasing resolution images arranged in the shape of a pyramid

- Coarse to fine analysis

- Low –resolution levels in a pyramid structure can be used to analyze the overall image context or large structures
- High-resolution levels can be used for analyzing individual object characteristics

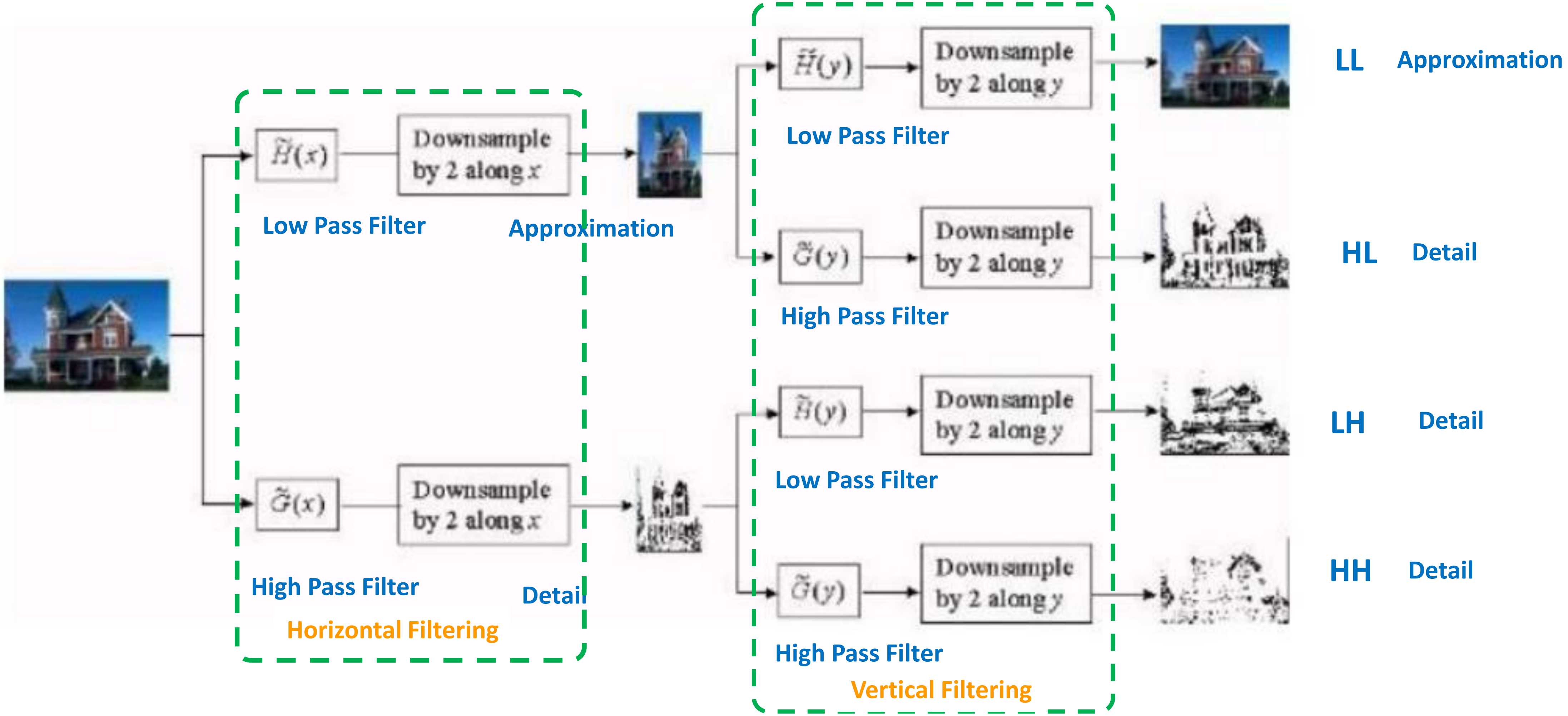
- Approximate pyramid (AP) & prediction residual pyramids (PRP)

- The approximation images in multiple levels construct an “AP”
- Prediction residual image is the difference of neighbor levels of images
- Encoding the PRP is more efficient than encoding the AP.



2D Discrete Wavelet Transform (One Level Decomposition)

The approximation shows an overall trend of pixel values and the details as the horizontal, vertical and diagonal components.



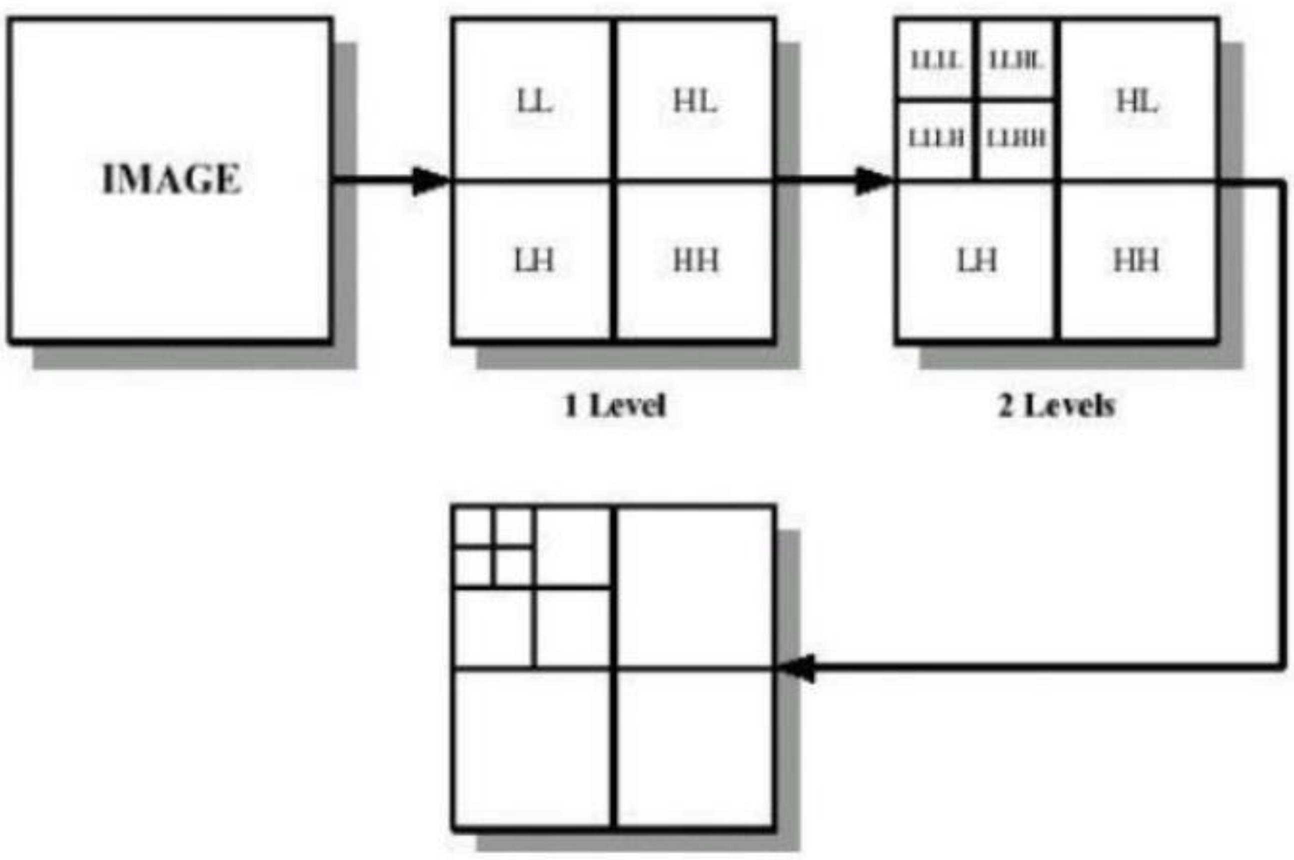
HPF (High Pass Filter)- to extract edges
LH — Passing through LPH and then HPF — Horizontal features (HPF along rows) - detail
HL — Passing through HPF and then LPF — Vertical features(HPF along columns.) - detail
HH — Passing through two simultaneous HPFs — Diagonal features(HPF distributed equally along both rows and columns)-detail

LPF (Low Pass Filter)-for approximation
LL — Passing through two simultaneous LPFs — gives an approximation

2-D DWT Multi-level Decomposition for an Image

A signal is passed through two filters, high pass and low pass filters. The image is then decomposed into high frequency (details) and low frequency components (approximation). At every level, we get 4 sub-signals.

The approximation shows an overall trend of pixel values and the details as the horizontal, vertical and diagonal components. If these details are insignificant, they can be valued as zero without significant impact on the image, thereby achieving filtering and compression.



Different Types of Wavelets

Wavelets	Abbreviations
Haar Wavelet	*Haar*
Daubechies Wavelet	*Db*
Symlets	*Sym*
Coiflets	*Coif*
Bi-Orthogonal Wavelet	*Bior*
Meyer Wavelet	*Meyr*
Discrete Meyer Wavelet	*Dmey*
Battle and Lemarié Wavelets	*Btlm*
Gaussian Wavelet	*Gaus*
Mexican Hat Wavelets	*Mexh*
Morlet Wavelet	*Morl*
Complex Gaussian Wavelets	*Cgau*
Complex Shannon Wavelets	*Shan*
Complex B-spline frequency Wavelets	*Fbsp*
Complex Morlet Wavelets	*Cmor*

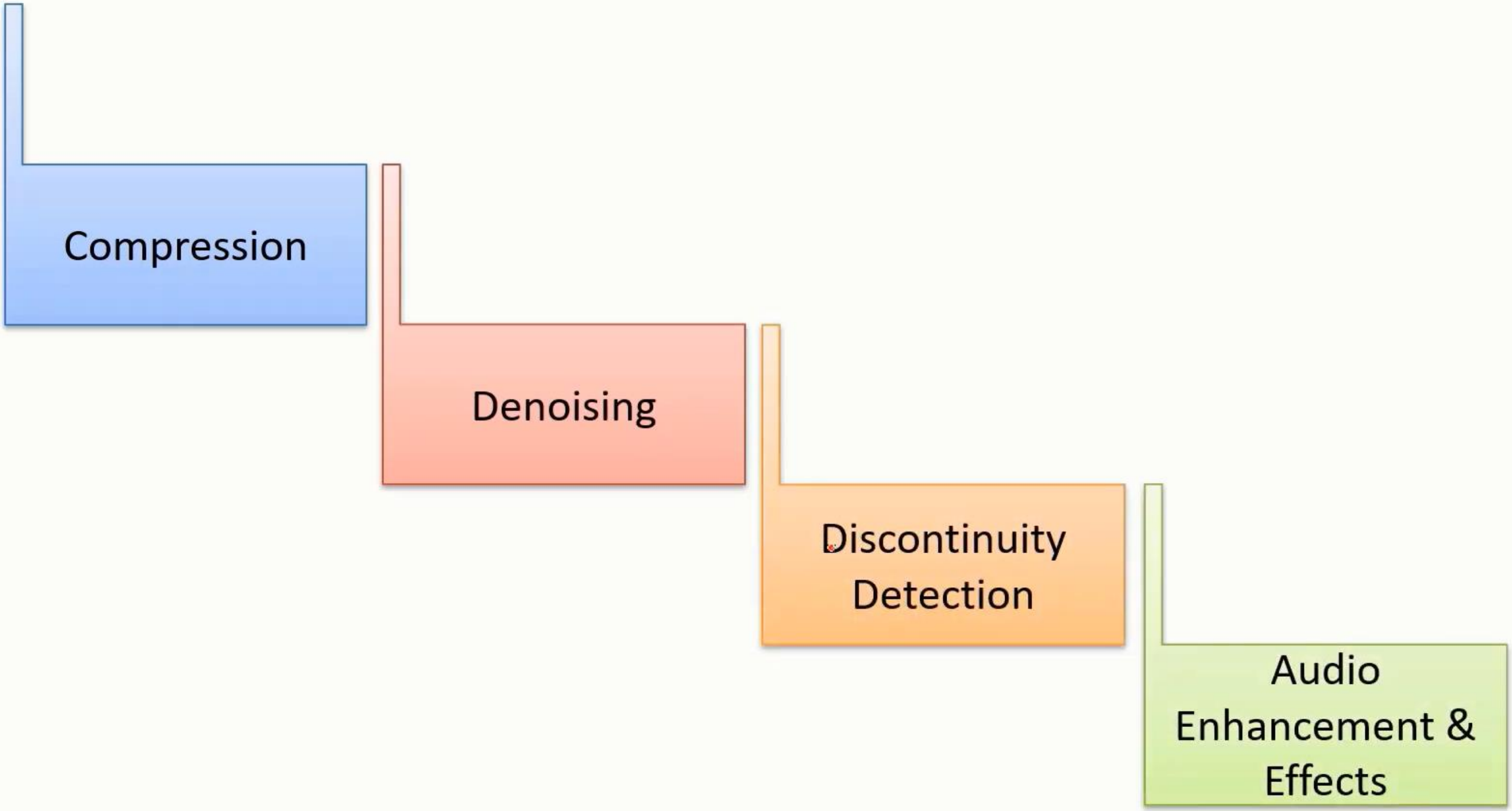
Different Wavelet Families based on their properties

Wavelets with filters			Wavelets without filters	
With compact support		With non-compact support	Real	Complex
Orthogonal	Biorthogonal	Orthogonal		
Db, haar, sym, coif	*bior*	*meyr, dmey, btlm*	*gaus, mexh, morl*	*cgau, shan, fbsp, cmor*

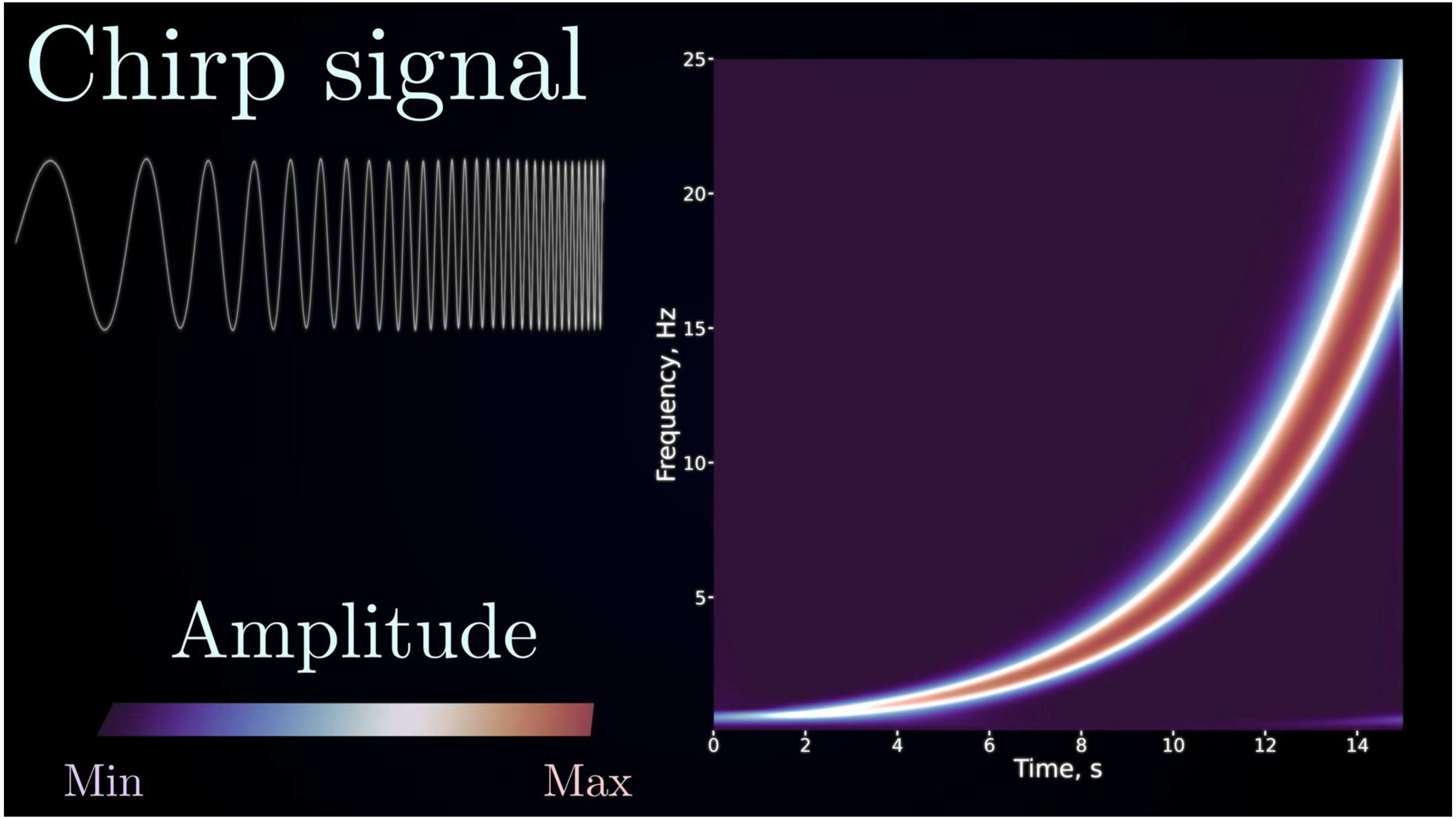
Compact Support: THE WAVELET IS ZERO OUTSIDE an INTERVAL [a,b] (FINITE & CLOSED). It means that the wavelet has a finite non-zero length which will result in increasing the probability of capturing events in short time instances.

Orthogonal wavelets come from one orthogonal basis set, the biorthogonal wavelets project from different basis sets. Orthogonal wavelet filter banks generate a single scaling function and wavelet, whereas biorthogonal wavelet filters generate one scaling function and wavelet for decomposition, and another pair for reconstruction.

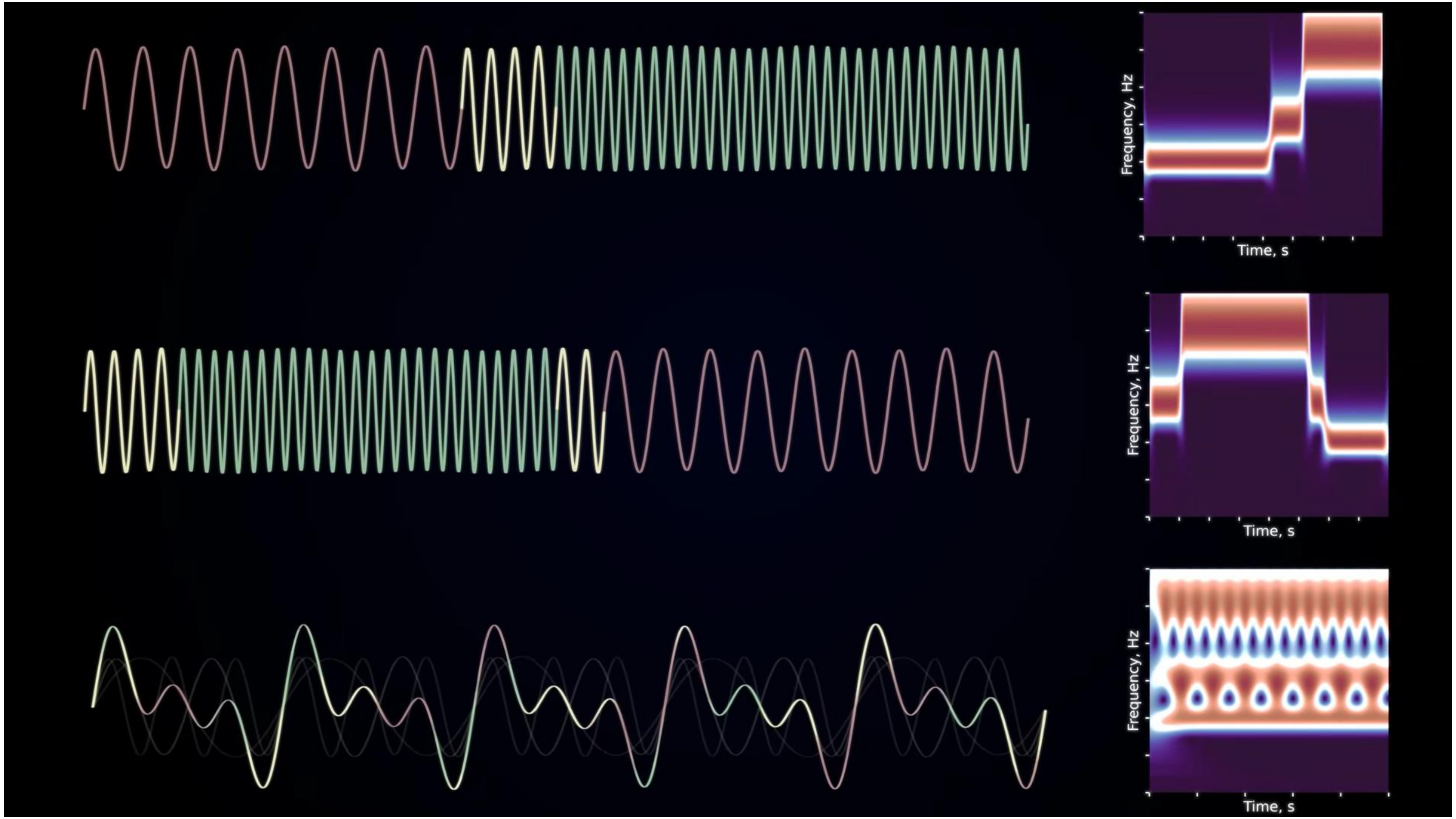
Wavelet Applications



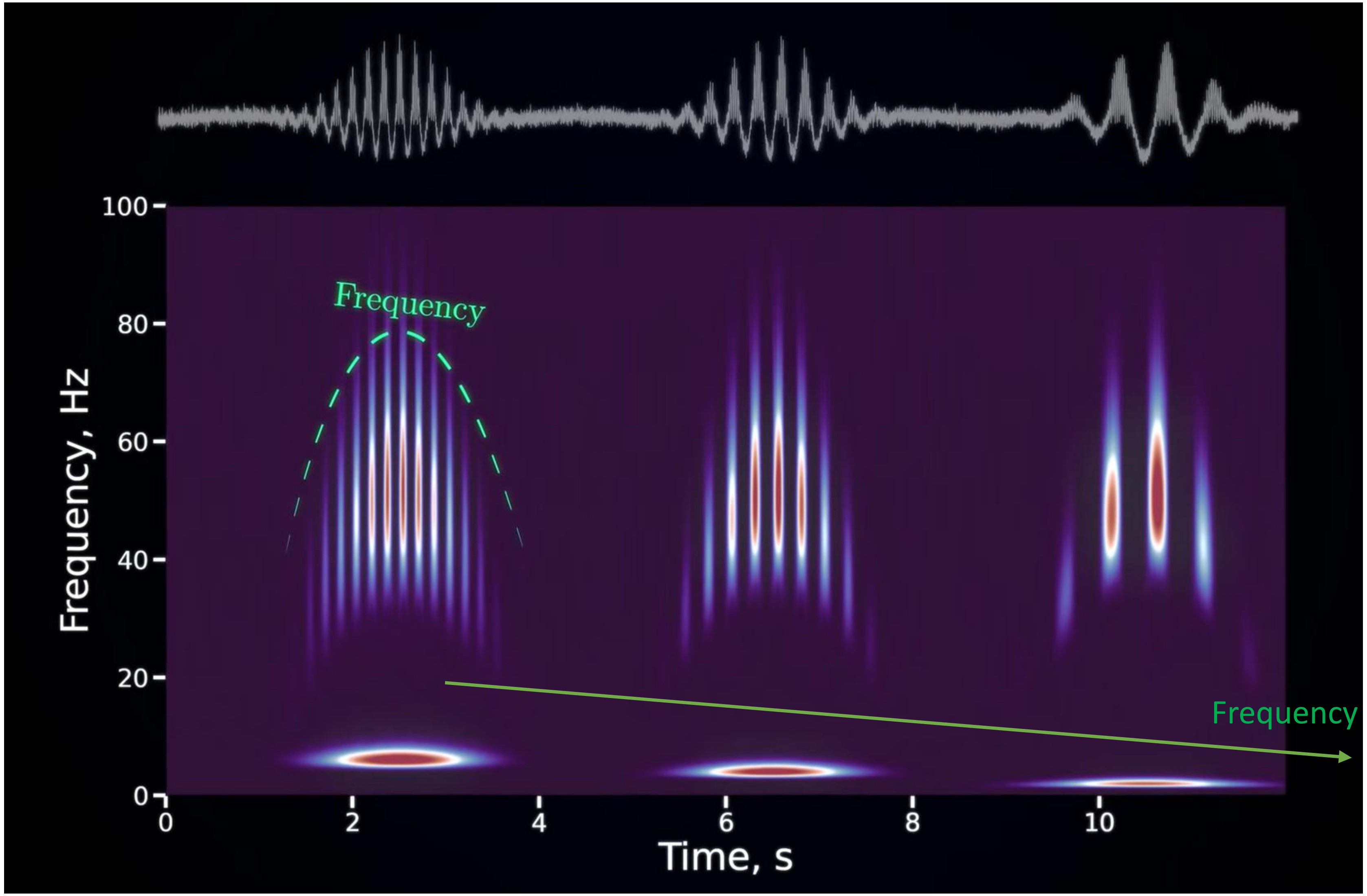
Scalogram for the Chirp signal



Traffic Signal Abnormality Detection



Brain Signal Analysis

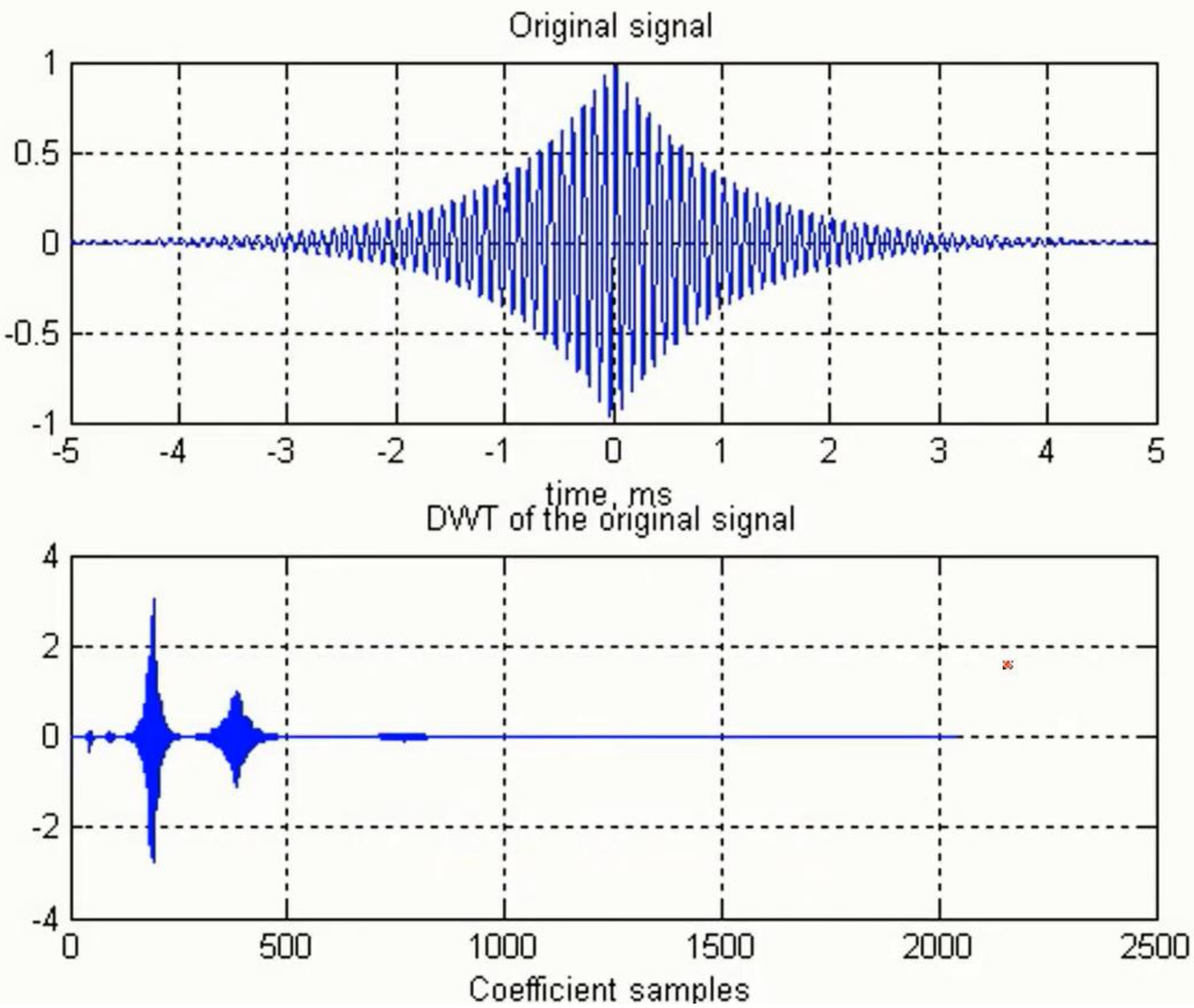


- ❖ Notice 3 distinct bouts of low frequency rhythm with a gradual decrease in frequency.
- ❖ We can easily quantify the duration and frequency of these patterns.

- ❖ Also, each of them is associated with a high frequency rhythm and their frequency follows the bell shape.

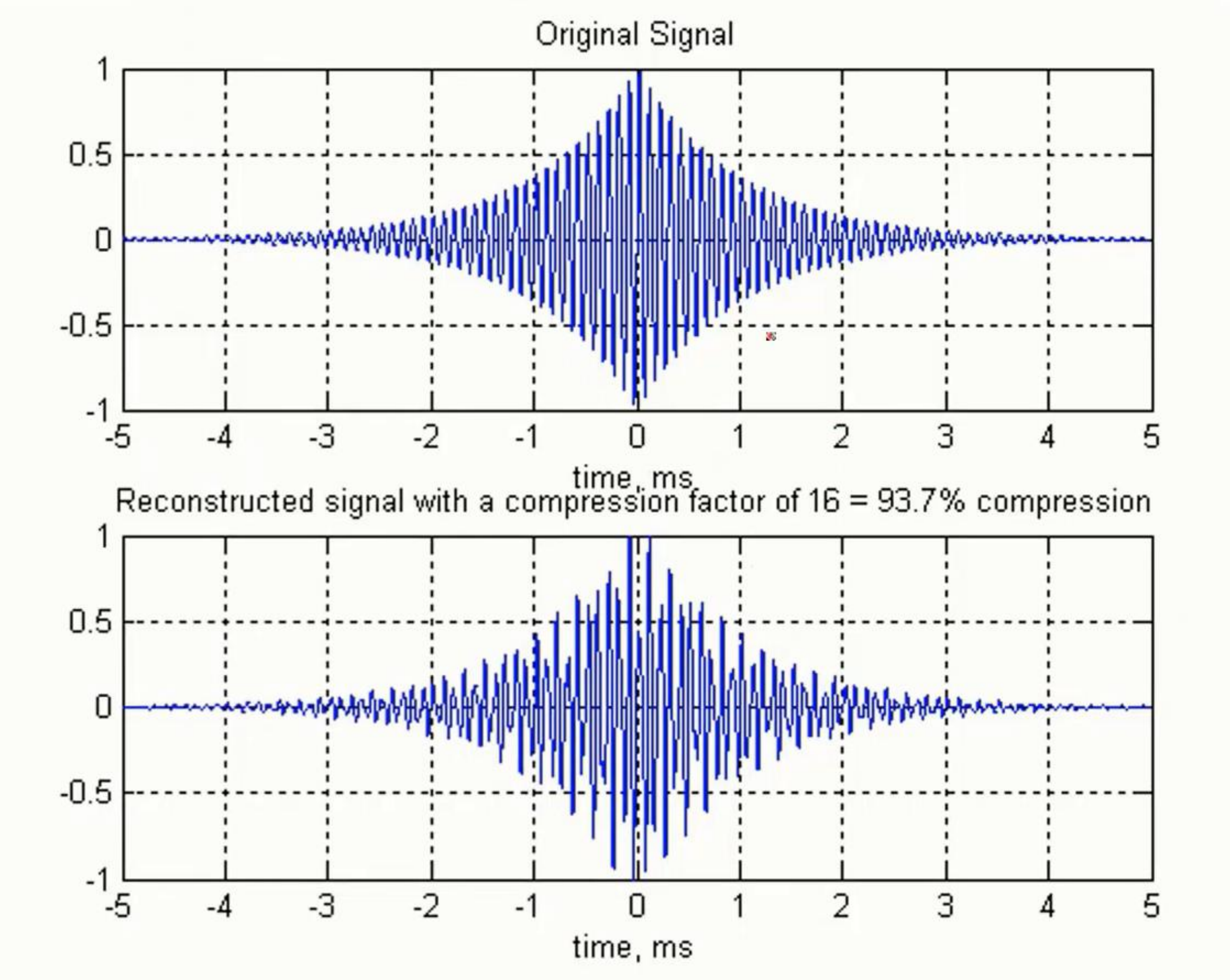
- ❖ We can quantify multiple parameters: Frequency values, durations, rise and decay of the frequency modulation and so on.

Signal Compression

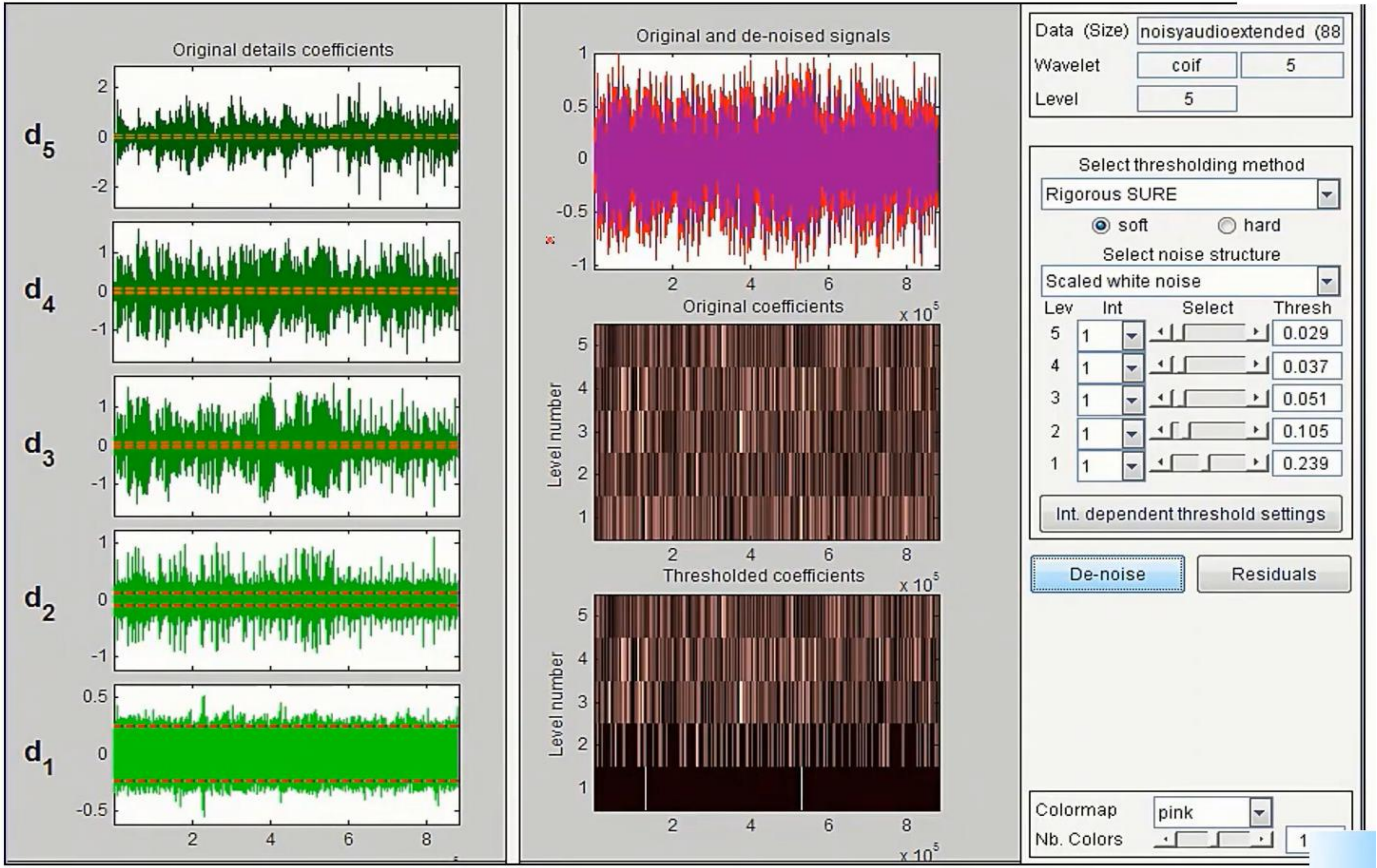


DWT is commonly used for compression, since most DWT coefficients are very small, hence can be zeroed-out!

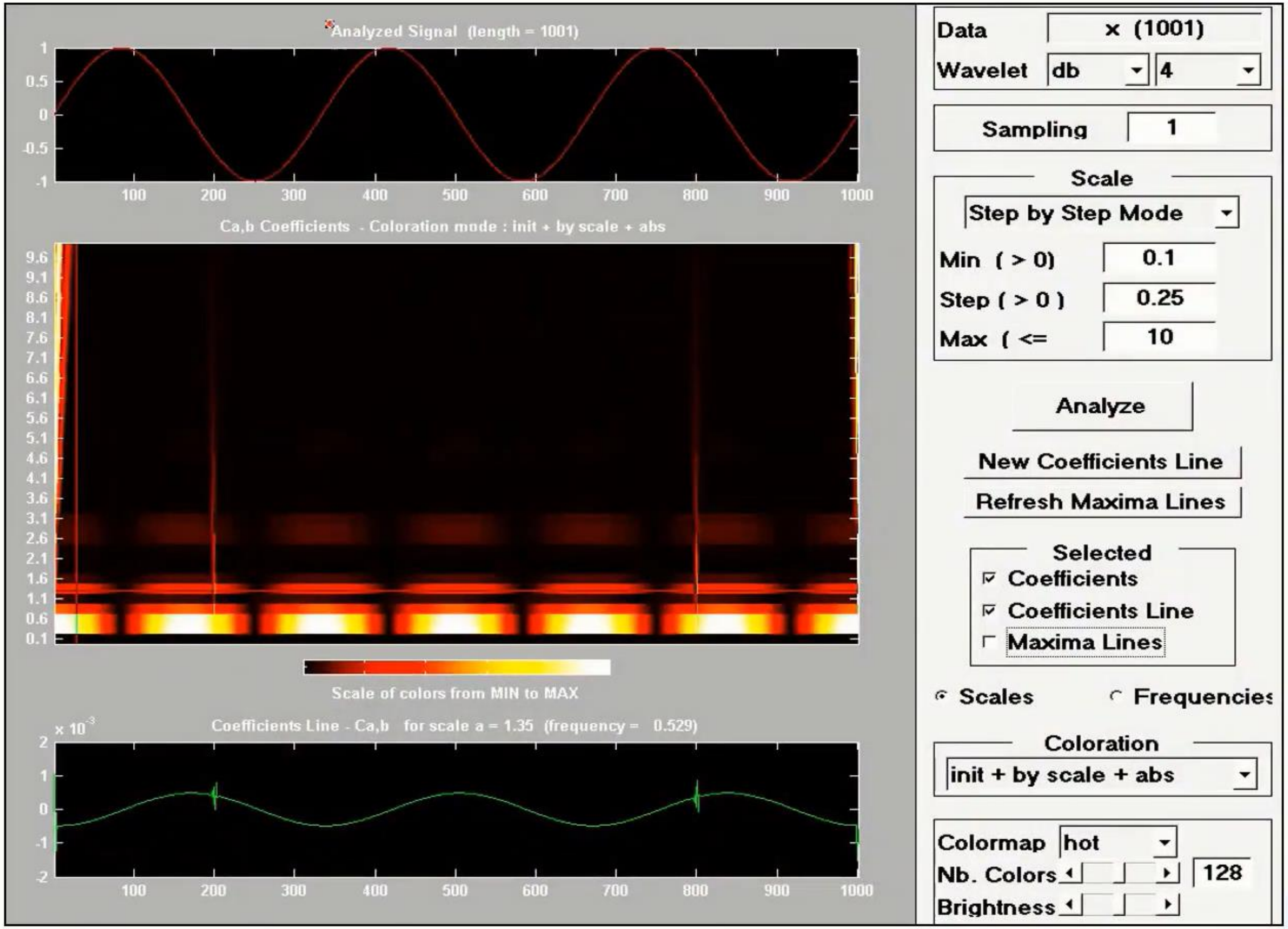
Signal Compression



Signal Denoising



Discontinuity Detection



Wavelet applications in Image Processing

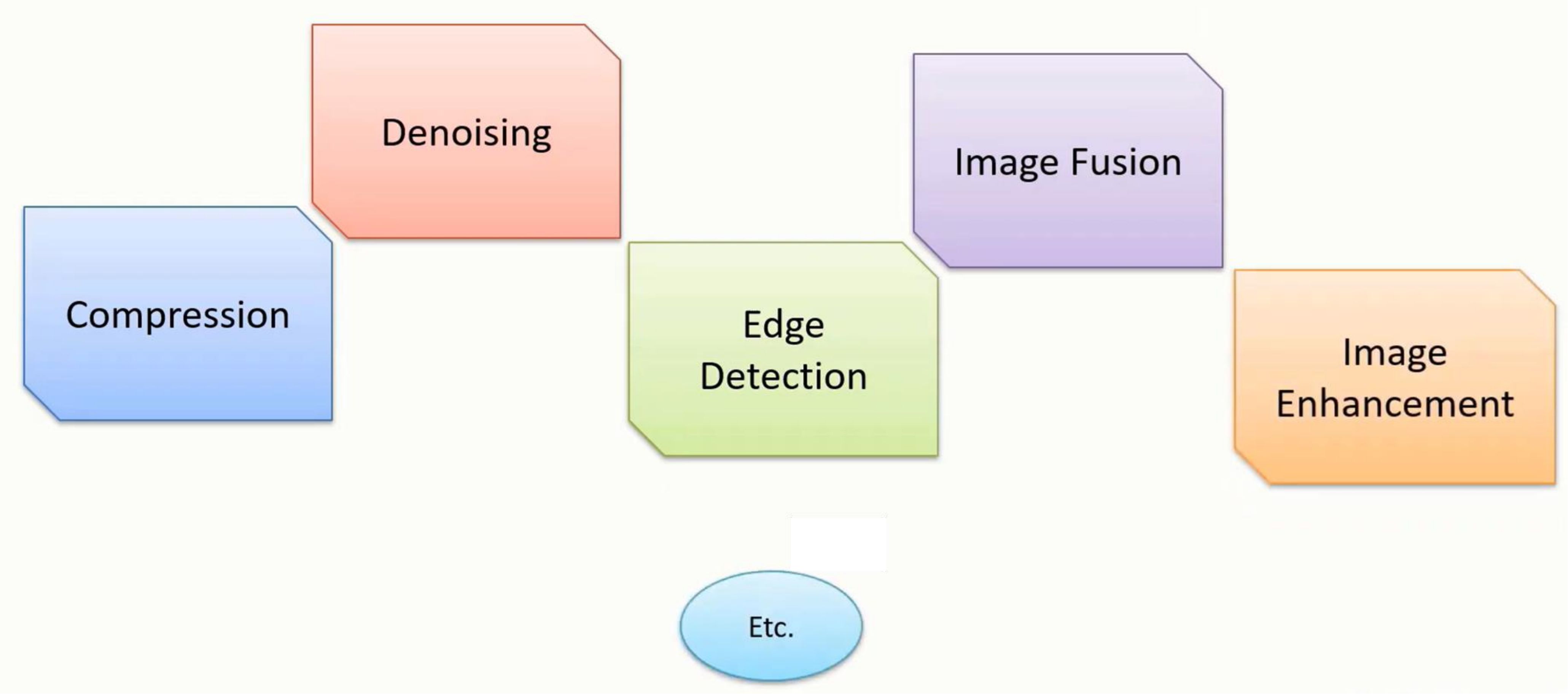


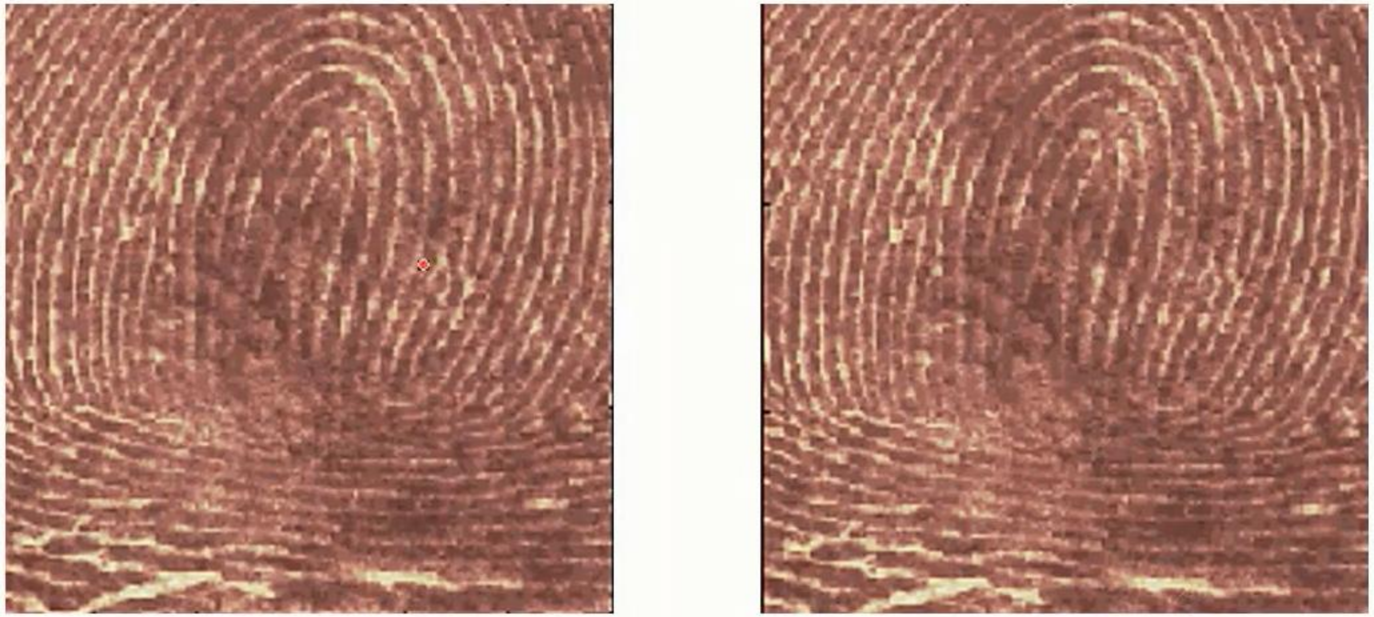
Image Compression

Image Compression
(Jpeg 2000)

Image Compression – Fingerprint images

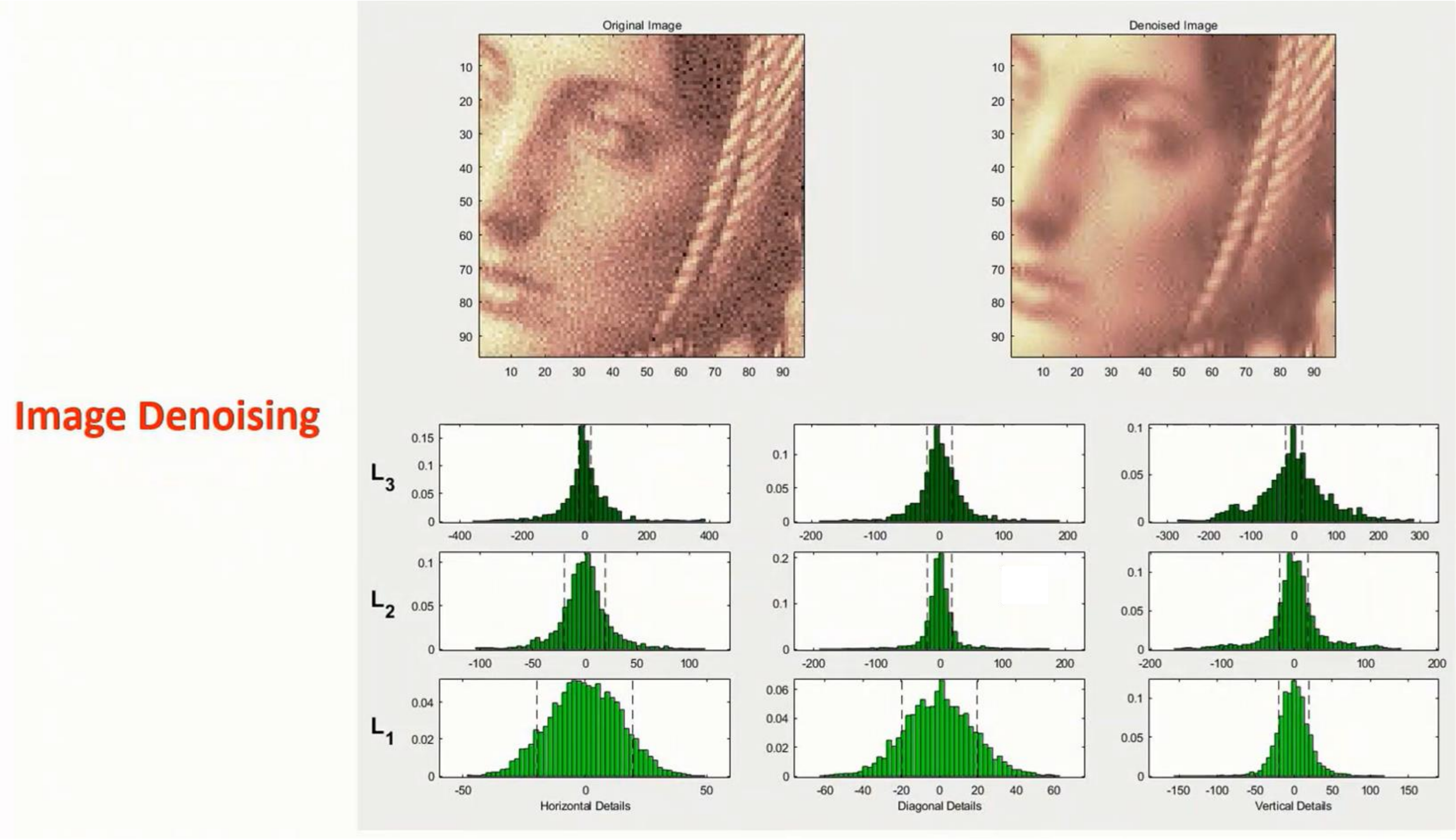
Image Compression

Original Image | Compressed Image

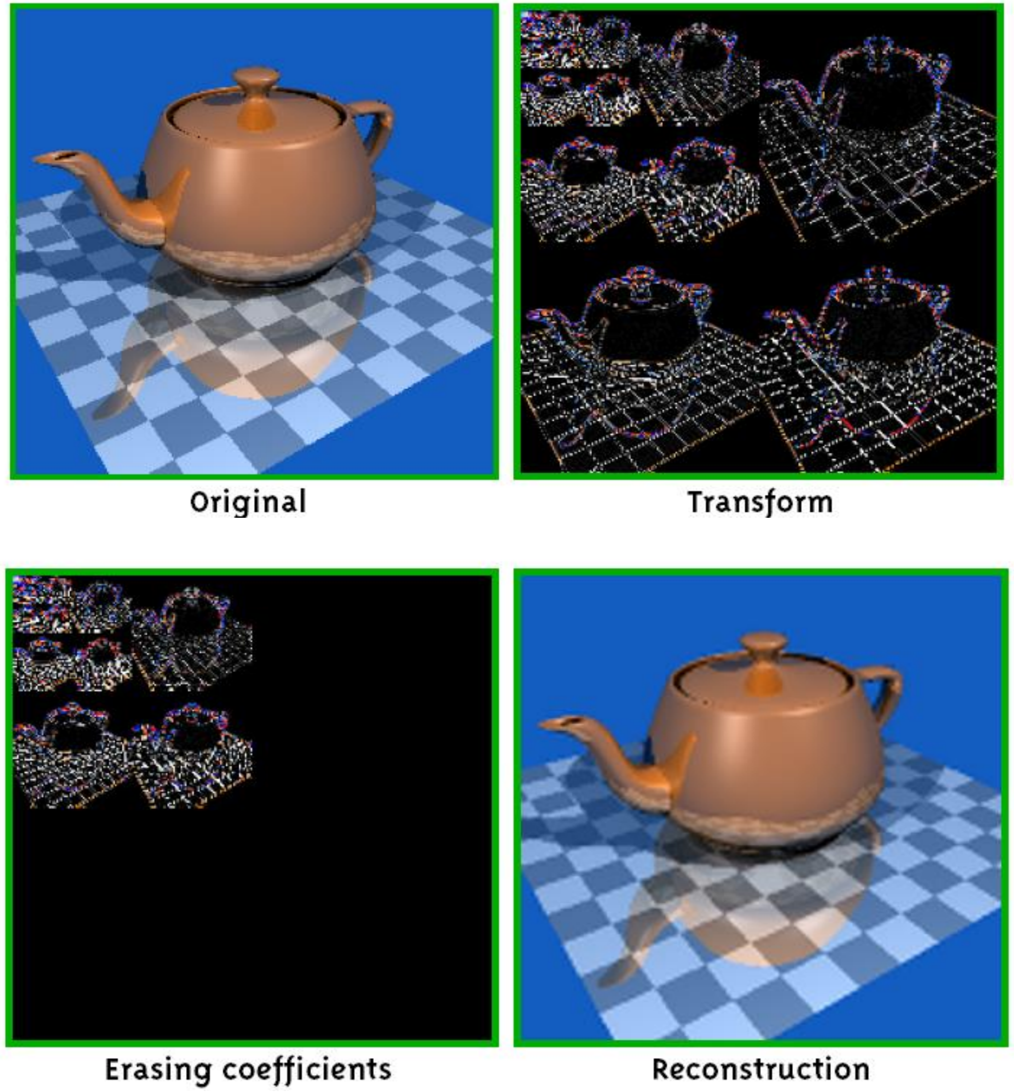


Threshold: 3.5, Zeros: 42%, Retained energy: 99.95%

Image Denoising



Wavelet Reconstruction



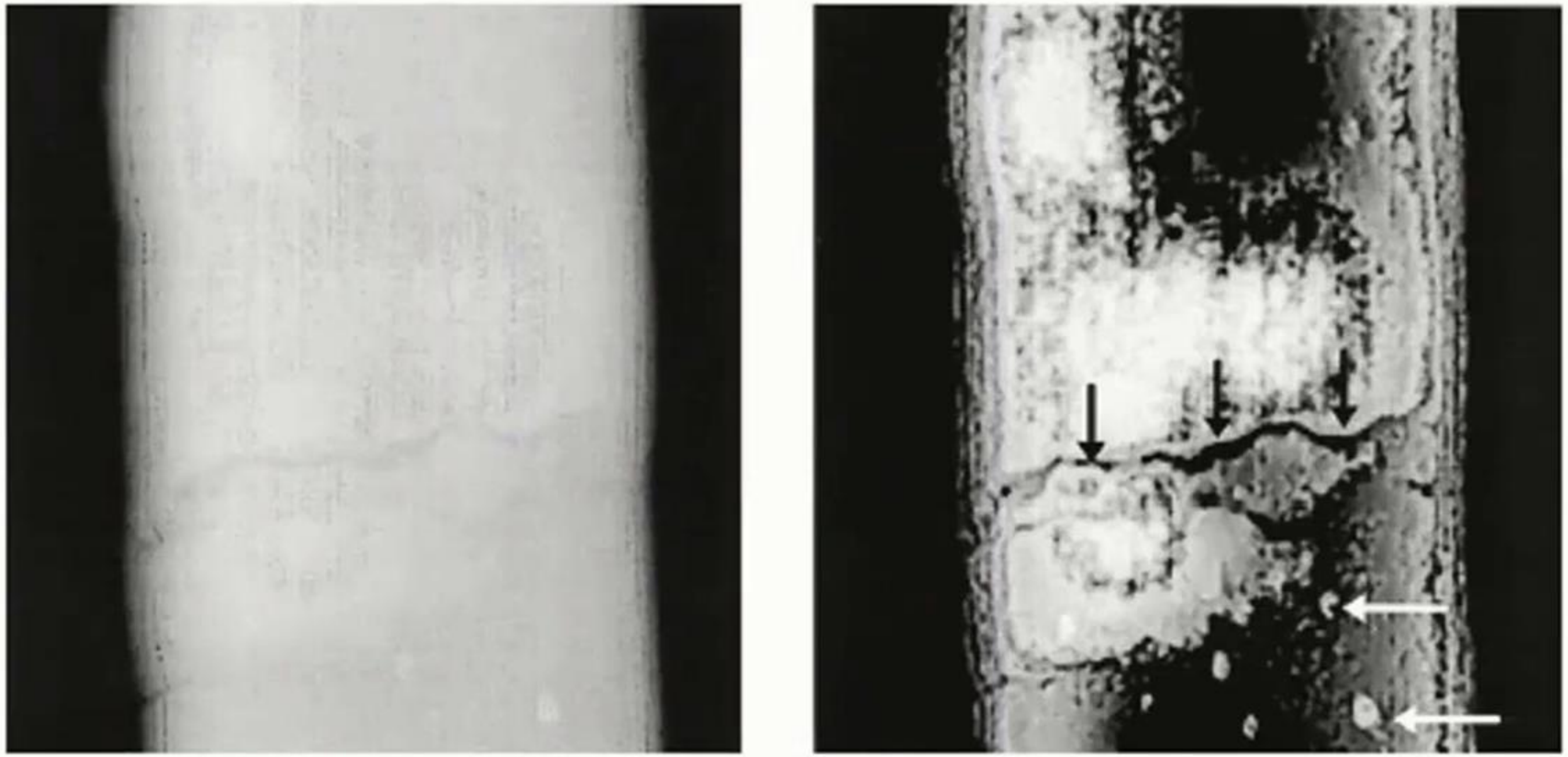
SIGGRAPH 96 Course Notes: Wavelets in Computer Graphics

Edge Detection

Edge Detection

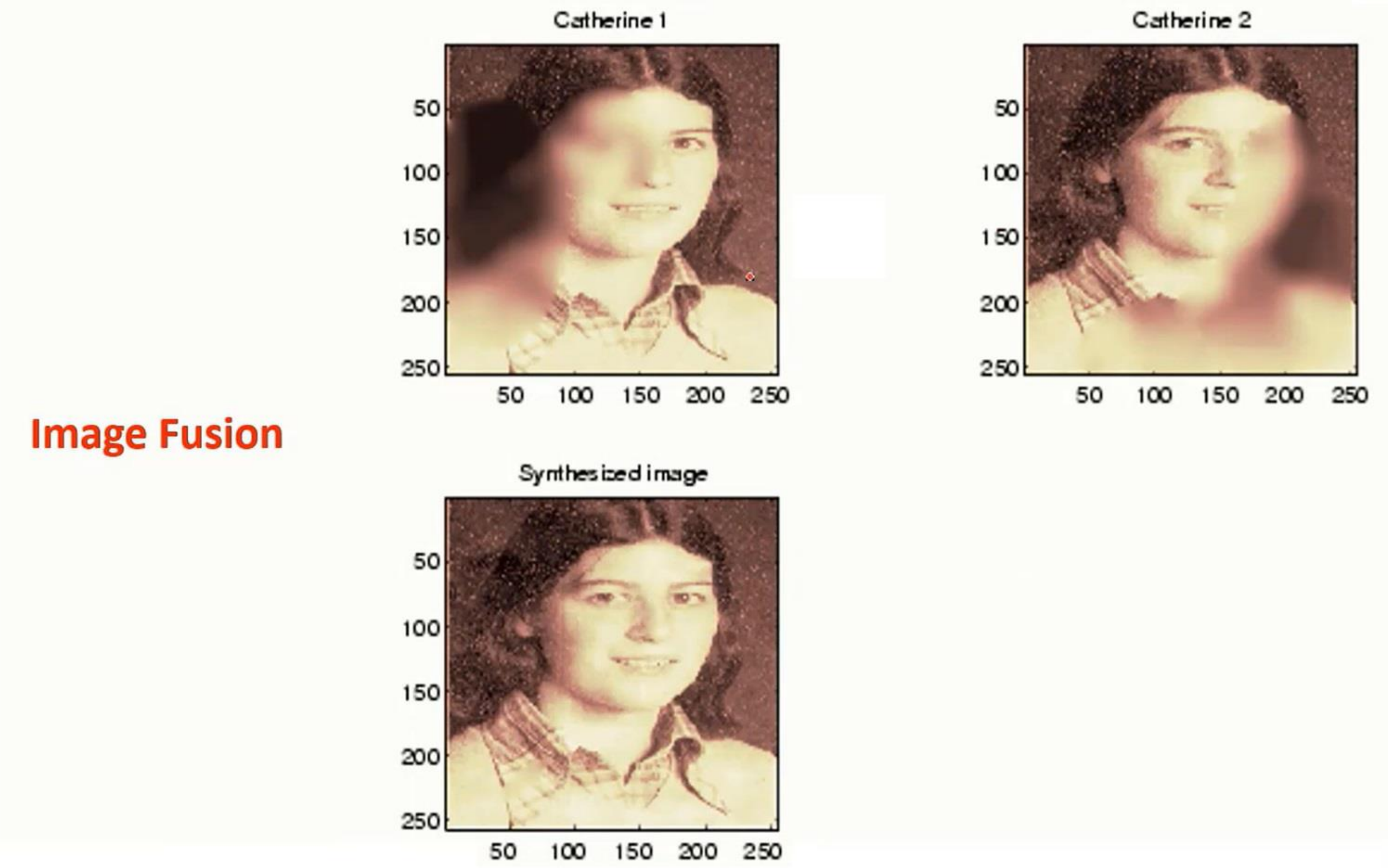
Image Enhancement

Image Enhancement



From: “Wavelet-based image enhancement in x-ray imaging and tomography”, by Bronnikov and Duifhuis

Image Fusion



Other application of wavelets

Audio, Image and Video Watermarking

Feature Extraction (Image Classification)

Face Recognition

Optical Character Recognition

Image Retrieval

Data Analysis and Prediction

Voice Recognition

Numerical Analysis

And Many More........

Wavelets for 2D Images

- Each color plane in image is signal
- Coefficients will represent visual features in the image
- Store as many coefficients as needed
  - Image compression (see next slide)
- c.f. Statistical shape descriptors

Why Wavelet Transform?

- Use hat-shape filters
  - Emphasize region where points cluster
  - Suppress weaker information in their boundaries
- Effective removal of outliers
  - Insensitive to noise, insensitive to input order
- Multi-resolution
  - Detect arbitrary shaped clusters at different scales
- Efficient
  - Complexity O(N)
- Only applicable to low dimensional data

References

- Wavelets: a mathematical microscope
- Understanding Wavelets by MATLAB
- An introduction to the wavelet transform (and how to draw with them!)
- An Introduction to Wavelets by Amara Graps
- Introduction to wavelets (IIT Bombay)
- https://in.mathworks.com/help/wavelet/ug/haar-transforms-for-time-series-data-and-images.html
- https://builtin.com/data-science/wavelet-transform
- https://www.robots.ox.ac.uk/~gari/teaching/cdt/A3/readings/ECG/Addison_0967-3334_26_5_R01.pdf

Fourier versus Wavelets

- Fourier
  - Loses time (location) coordinate completely
  - Analyses the ***whole*** signal
  - Short pieces lose “frequency” meaning

- Wavelets
  - Localized time-frequency analysis
  - Short signal pieces also have significance
  - *Scale = Frequency band*