VIT-AP UNIVERSITY

Computer Vision

(Course Code: 4047)

Module-2:Lecture-10: Gaussian Derivative Filters

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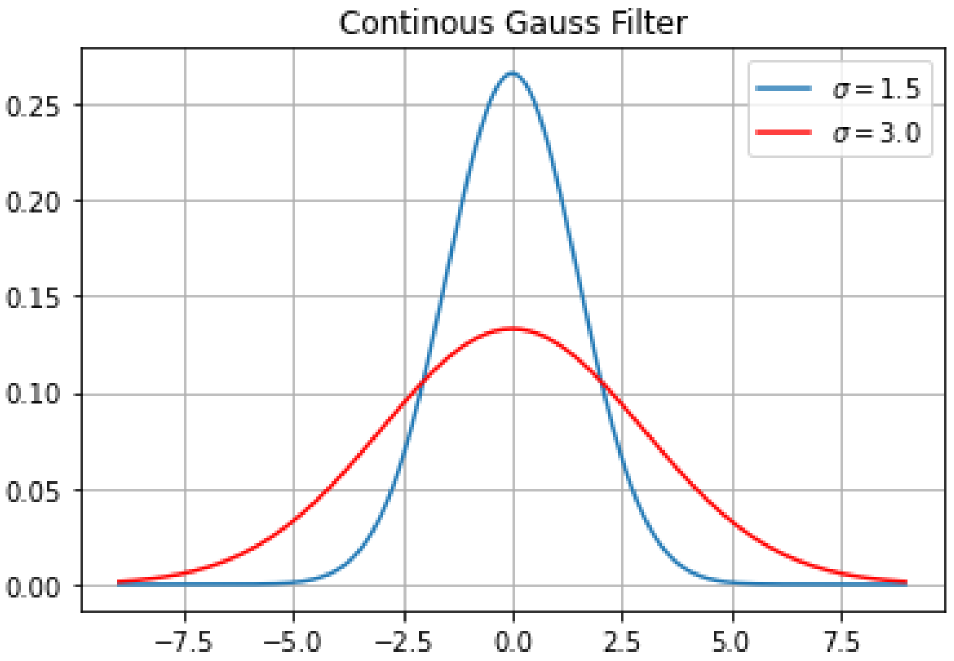
Gaussian Filter

- Gaussian filters are frequently applied in image processing, e.g. for
  - Blurring
  - Low-pass filtering
  - Noise suppression
  - Construction of Gaussian pyramids for scaling

Derivatives of the Gaussian filter can be applied to perform noise reduction and edge detection in one step.

The derivation of a Gaussian-blurred input signal is identical to filter the raw input signal with a derivative of the gaussian.

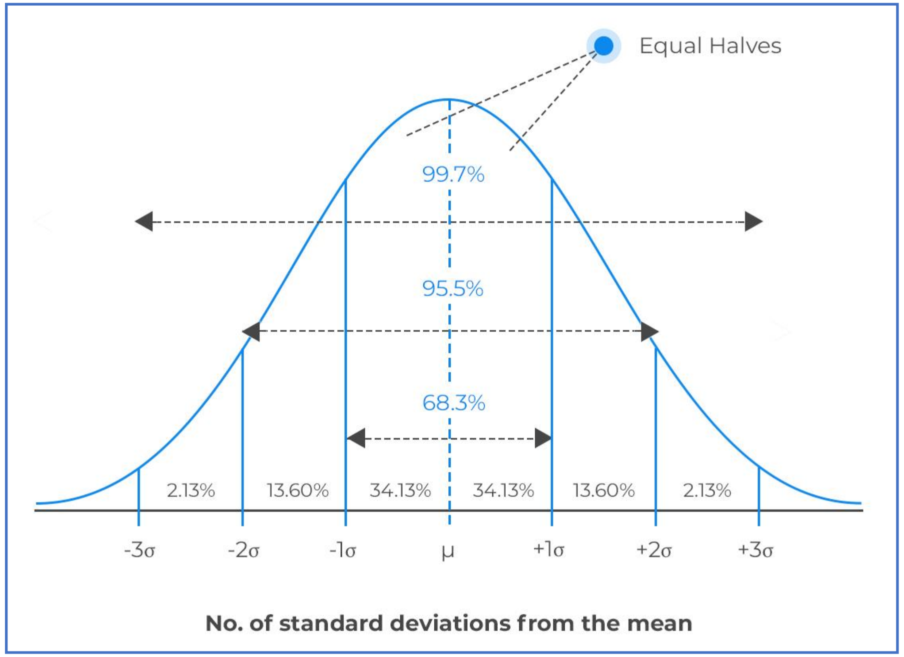


1-dimensional Gaussian Filter

The univariate Gauss-function is defined as follows:

$$g_{\sigma,\mu}(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),$$

where σ is the standard deviation and μ is the mean. In the context of filtering, the mean is always $\mu = 0$, the standard deviation σ is a parameter, which determines the width of the filter. In the sequel the mean is implicetly assumed to be $\mu = 0$.



Discrete Gaussian Filter

In image processing we have discrete signals (the pixels of an image) and discrete filters. For the discrete index u, the **discrete Gaussian filter** is defined to be (for $\mu = 0$):

$$g_\sigma(u) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(u)^2}{2\sigma^2}\right),$$

For discrete Gauss-filters, u is an integer in the range

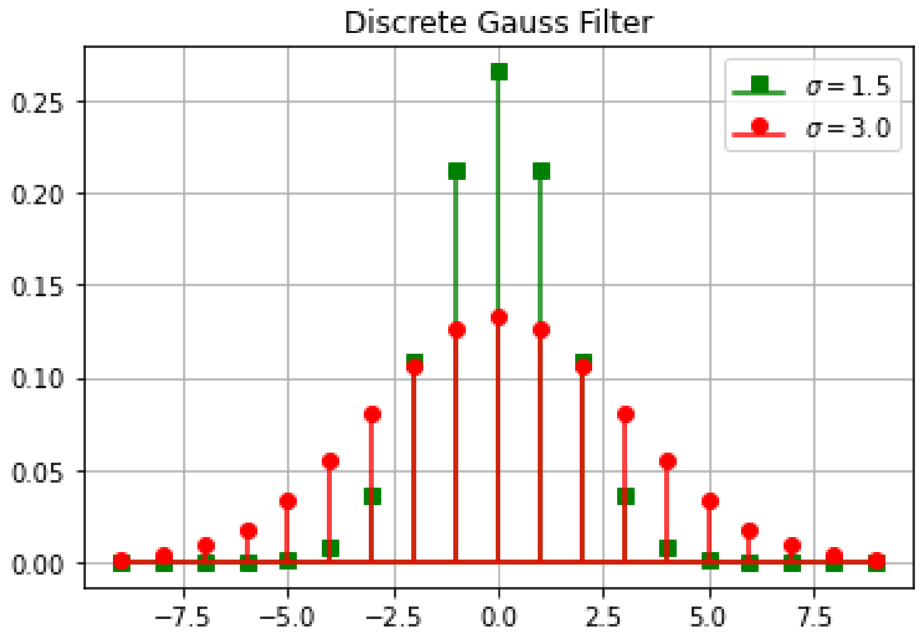
$$u \in [-z, \dots, z],$$

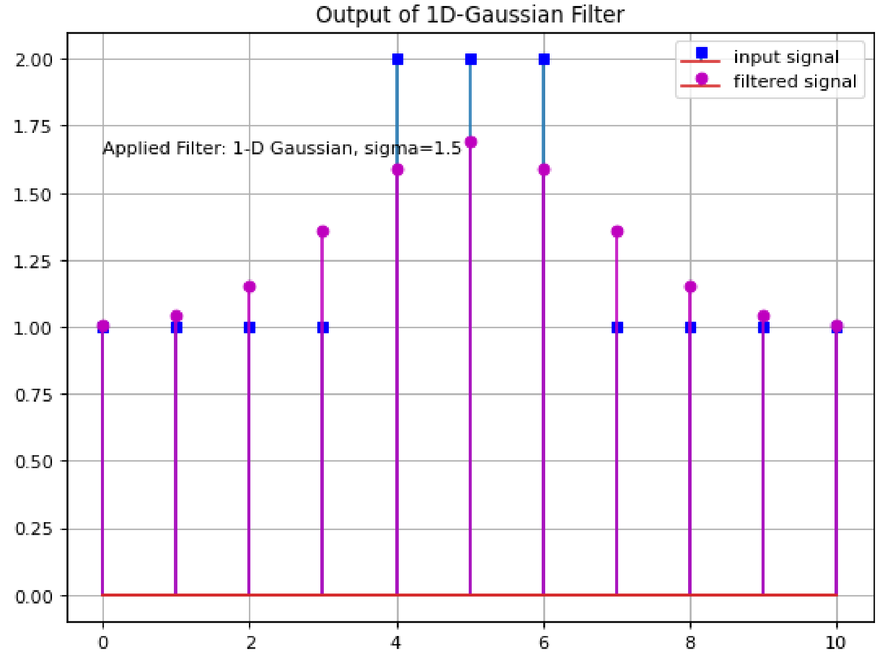
where usually $z = \lceil 3\sigma \rceil$.

The output of the discrete Gaussian filtering, if applied to the discrete signal $f(i)$ is than

$$h(i) = \sum_{u=-z}^{z} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{u^2}{2\sigma^2}\right) f(i-u)$$

Below, two discrete Gauss functions with different σ are plotted:



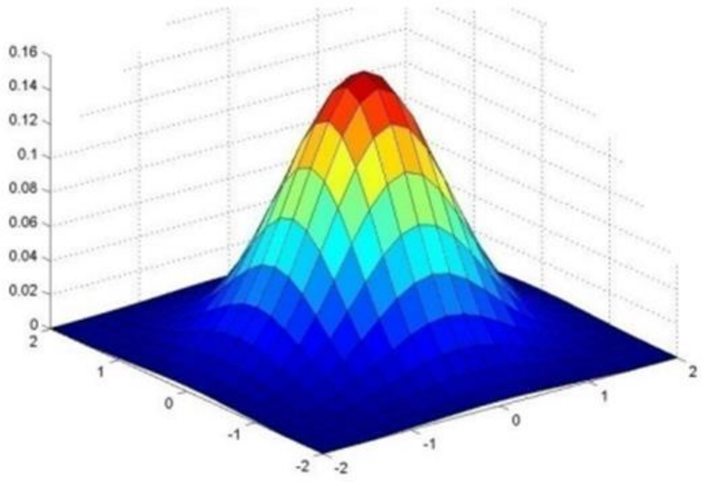


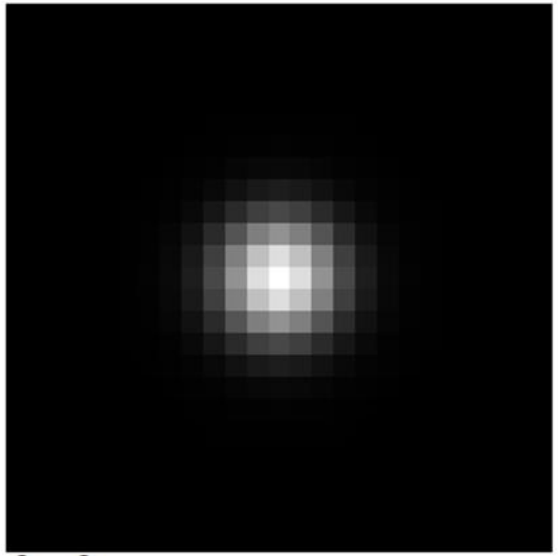
Derivative Filter: Key conditions to fulfil

A Derivative filter has to fulfil the following conditions:

- Separability into 1D row and column filters
- Filter support as small as possible
- Rotation invariance
- Frequency response as good as possible

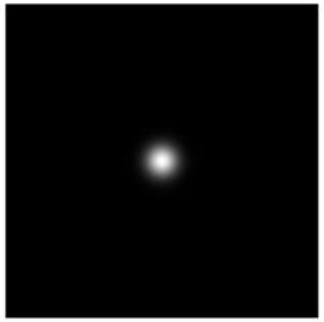
2D Gaussian Kernel



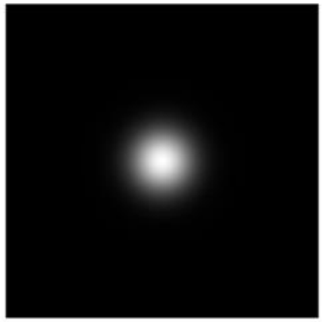


$$g_\sigma[x, y] = \frac{1}{2\pi\sigma^2} e^{\frac{-(x^2+y^2)}{2\sigma^2}}$$

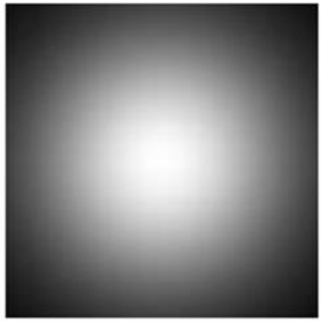
$\sigma = 1$



$\sigma = 5$



$\sigma = 10$



$\sigma = 30$

Separability of gaussian kernels

A 2D kernel g is called separable if it can be broken down into the convolution of two kernels: $g = g^{(1)} * g^{(2)}$.

$$
\begin{aligned}
g_\sigma[x, y] &= \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} \\
&= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}} \\
&= g_\sigma^{(1)}[x] \cdot g_\sigma^{(2)}[y]
\end{aligned}
$$

and

$$
(I * g_\sigma)[x, y] = \sum_i \sum_j g_\sigma[x - i, x - j] I[i, j] = \dots
$$

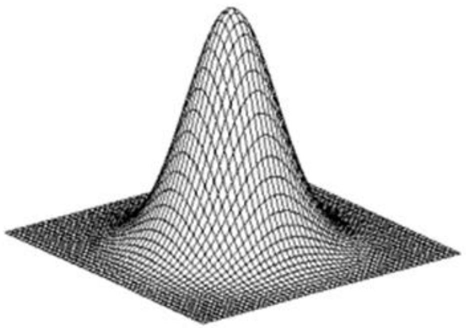
$$
\sum_i \sum_j g_\sigma^{(1)}[x-i] g_\sigma^{(2)}[x-j] I[i, j] = \sum_i g_\sigma^{(1)}[x-i] \sum_j g_\sigma^{(2)}[x-j] I[i, j] = \dots
$$

$$
(g_\sigma^{(1)} * (g_\sigma^{(2)} * I))[x, y]
$$

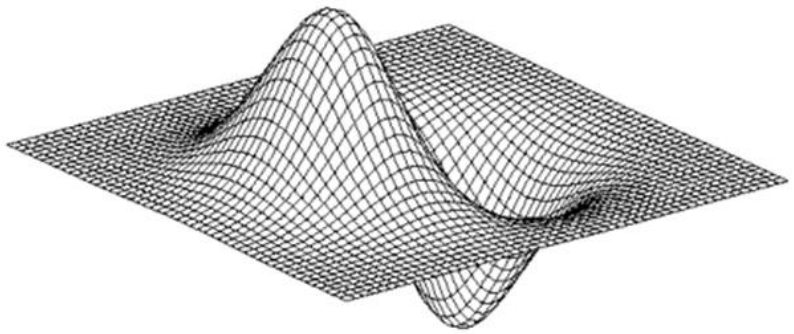
Important Gaussian Derivative Properties

- Image differentiation $\frac{d}{dx}$ is a convolution on image I.
- Smoothing by Gaussian kernel g is a convolution on image I.
- 2D Gaussian kernel is separable $g = g^{(1)} * g^{(2)}$.
- Convolution is
  - commutative $f * g = g * f$
  - associative $(I * g) * h = I * (g * h)$

So $\frac{d}{dx}(I * g) = I * \frac{d}{dx} g = (I * (\frac{d}{dx} g^{(1)})) * g^{(2)}$

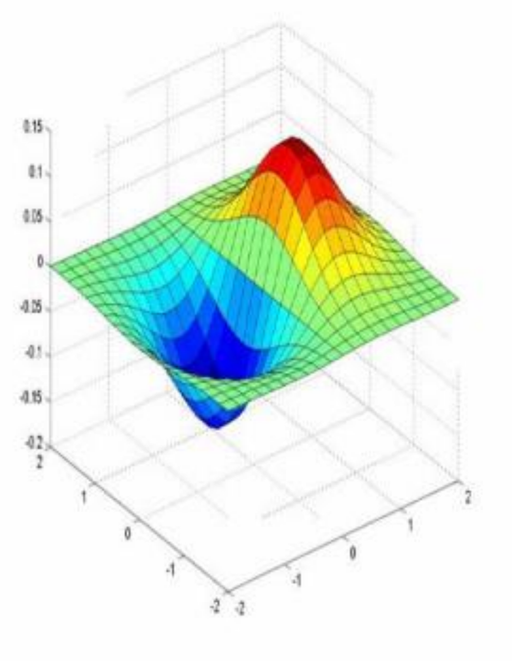


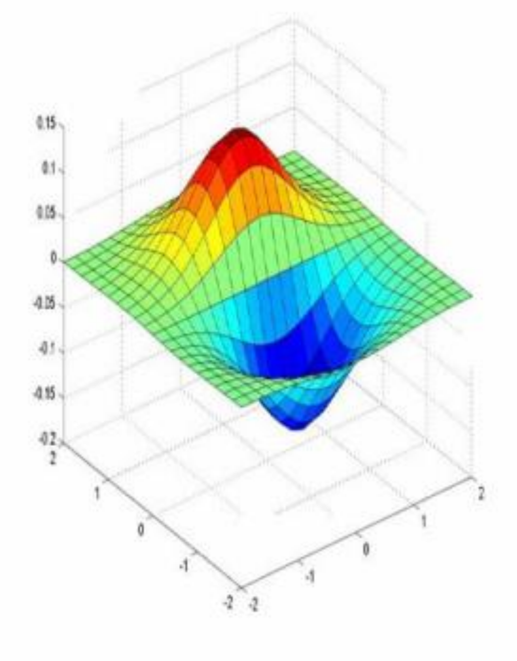
g

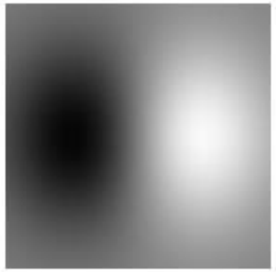


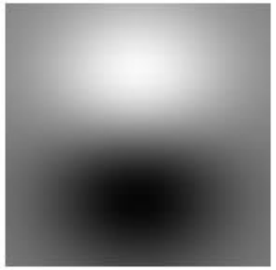
$\frac{d}{dx} g$

First Derivative of a Gaussian









$\frac{d}{dx}g$ $\frac{d}{dy}g$

Gaussian Derivatives (first and second order)

❖ Derivatives of 1-dimensional Gaussian Filter

The 1st order derivative of the zero mean Gaussian function

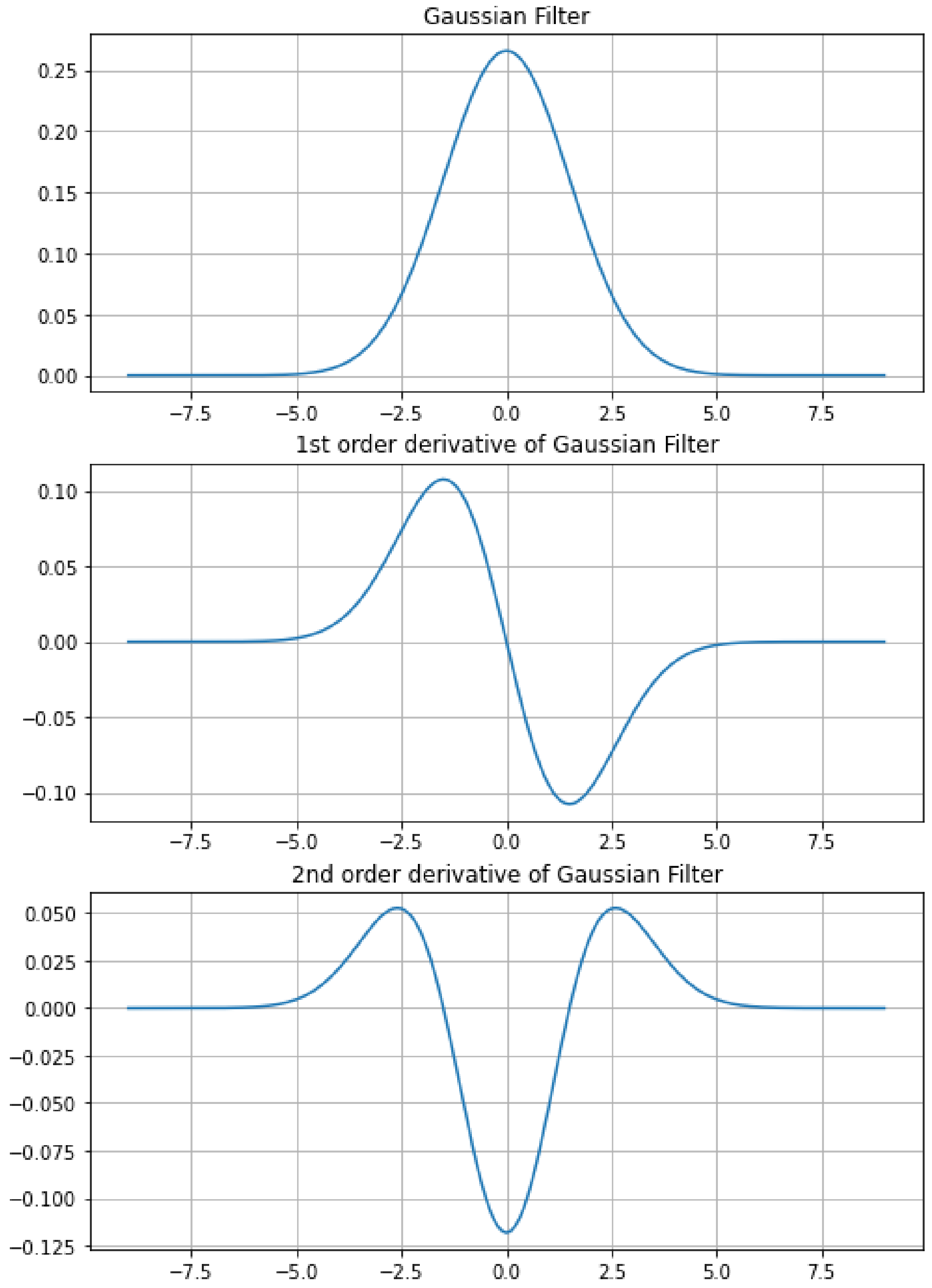
$$g_\sigma(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{x^2}{2\sigma^2}\right)$$

can easily be calcultated to be

$$g'_\sigma(x) = \frac{\partial g_\sigma(x)}{\partial x} = -\frac{x}{\sigma^2}\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{x^2}{2\sigma^2}\right) = -\frac{x}{\sigma^2}g_\sigma(x),$$

And the 2nd order derivative is

$$g''_\sigma(x) = \frac{\partial^2 g_\sigma(x)}{\partial x^2} = \left(\frac{x^2}{\sigma^4} - \frac{1}{\sigma^2}\right)\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{x^2}{2\sigma^2}\right) = \left(\frac{x^2}{\sigma^4} - \frac{1}{\sigma^2}\right)g_\sigma(x).$$



First order derivative of the Gaussian

First derivative is a linear operation. Convolution is a linear operation and convolution is associative. This saves us one operation.

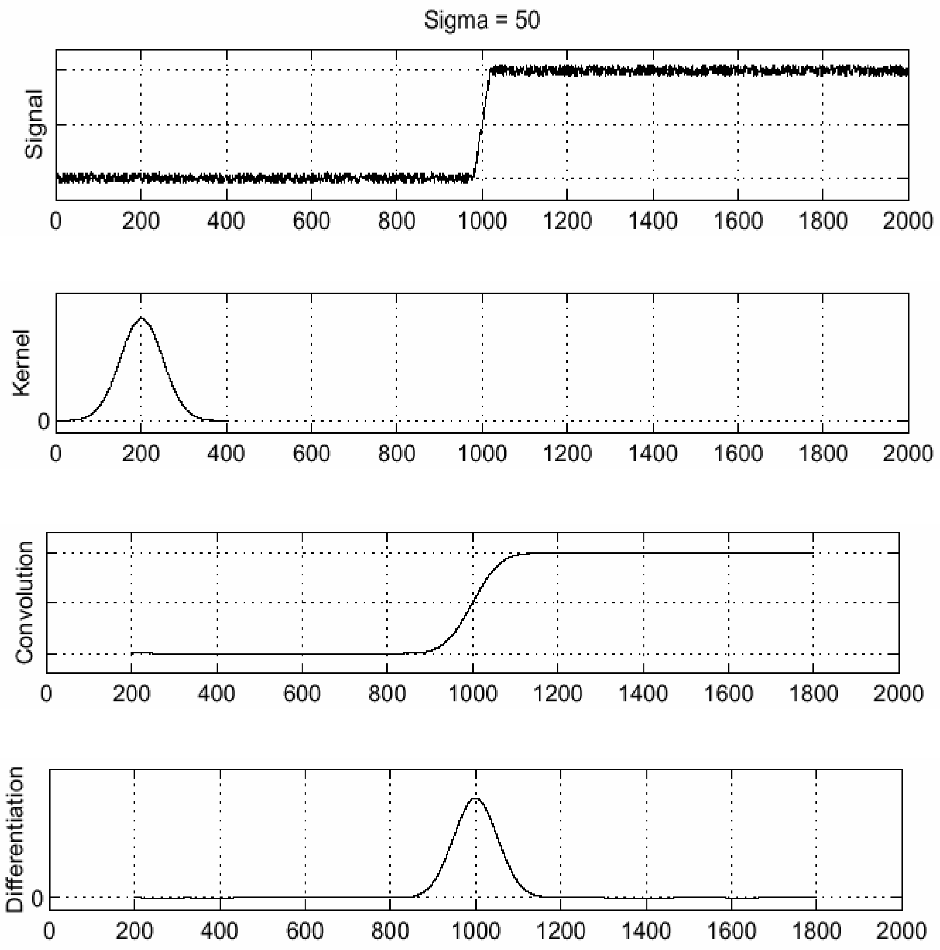
$$\frac{d}{dx}(f * h) = f * \frac{d}{dx}h$$

f

Gaussian h

$f * h$

$\frac{d}{dx}(f * h)$

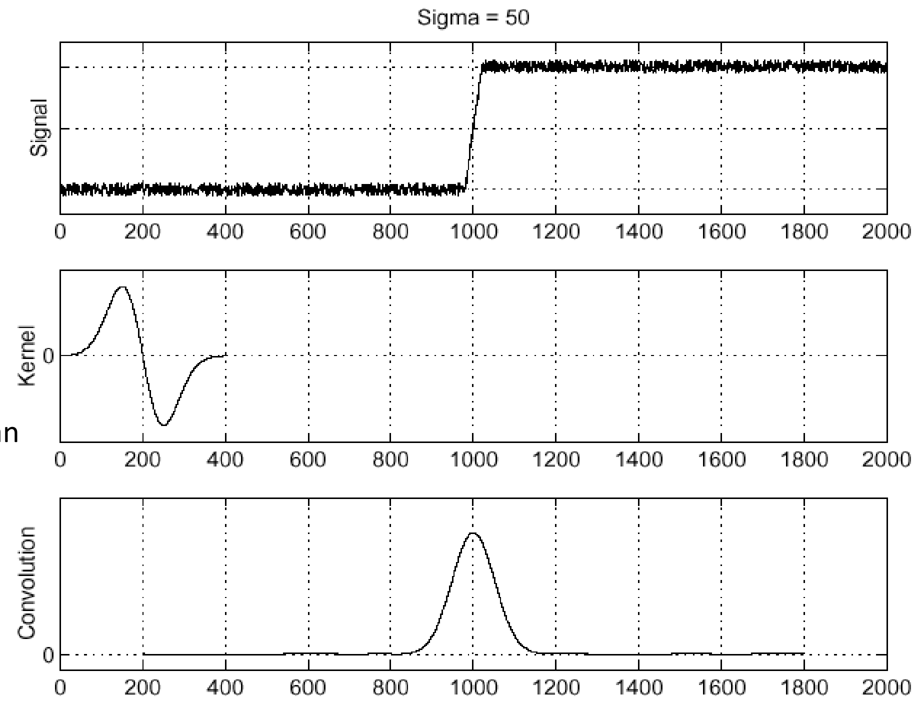


f

$\frac{d}{dx}h$

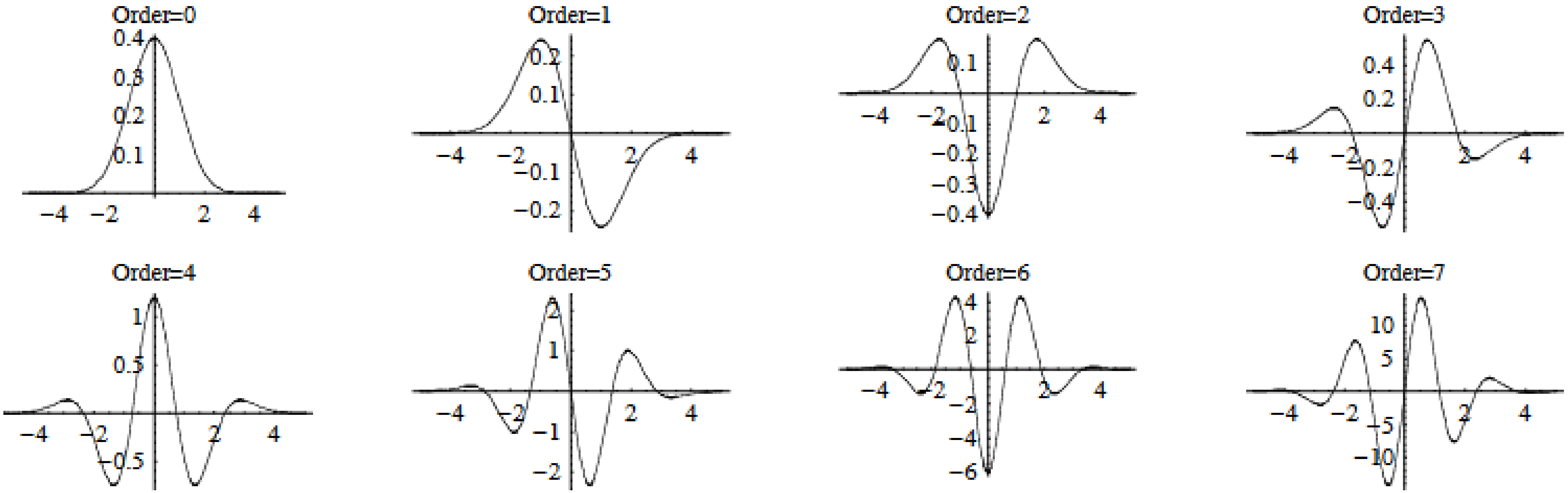
Derivative of Gaussian

$f * \frac{d}{dx}h$



To find edges, look for peaks in $\frac{d}{dx}(f * h)$ or $f * \frac{d}{dx}h$

Gaussian Derivative functions (order 0 to 7)



Plots of the 1D Gaussian derivative function for order 0 to 7.

Gaussian Derivatives like a boss

If you want to level up, then you can exploit a recurrence relation of Hermite polynomials to algorithimically construct Gaussian derivatives of any order without convolution or symbolic differentiation.

$$He_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}} = \left(x - \frac{d}{dx}\right)^n \cdot 1,$$

Hermite Polynomials

```
Unprotect[gauss]; gauss[x_, σ_] := 1/(σ Sqrt[2 π]) Exp[-x^2/(2 σ^2)];
Table[Factor[Evaluate[D[gauss[x, σ], {x, n}]]], {n, 0, 4}]
```

$$\left\{\frac{e^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}, -\frac{e^{-\frac{x^2}{2\sigma^2}}\,x}{\sqrt{2\pi}\,\sigma^3}, \frac{e^{-\frac{x^2}{2\sigma^2}}\,(x-\sigma)\,(x+\sigma)}{\sqrt{2\pi}\,\sigma^5}, -\frac{e^{-\frac{x^2}{2\sigma^2}}\,x\,(x^2-3\sigma^2)}{\sqrt{2\pi}\,\sigma^7}, \frac{e^{-\frac{x^2}{2\sigma^2}}\,(x^4-6x^2\sigma^2+3\sigma^4)}{\sqrt{2\pi}\,\sigma^9}\right\}$$

The function **Factor** takes polynomial factors apart.

The function **gauss[x,σ]** is part of the standard set of functions

The Gaussian function itself is a common element of all higher order derivatives. We extract the polynomials by dividing by the Gaussian function:

```
Table[Evaluate[D[gauss[x, σ], {x, n}] / gauss[x, σ]], {n, 0, 4}] // Simplify
```

$$\left\{1, -\frac{x}{\sigma^2}, \frac{x^2-\sigma^2}{\sigma^4}, -\frac{x^3-3x\sigma^2}{\sigma^6}, \frac{x^4-6x^2\sigma^2+3\sigma^4}{\sigma^8}\right\}$$

These polynomials have the same order as the derivative they are related to. Note that the highest order of x is the same as the order of differentiation, and that we have a plus sign for the highest order of x for even number of differentiation, and a minus signs for the odd orders.

These polynomials are the Hermite polynomials, called after Charles Hermite, a brilliant French mathematician (see figure).

Higher Order Gaussian Derivative Characteristics

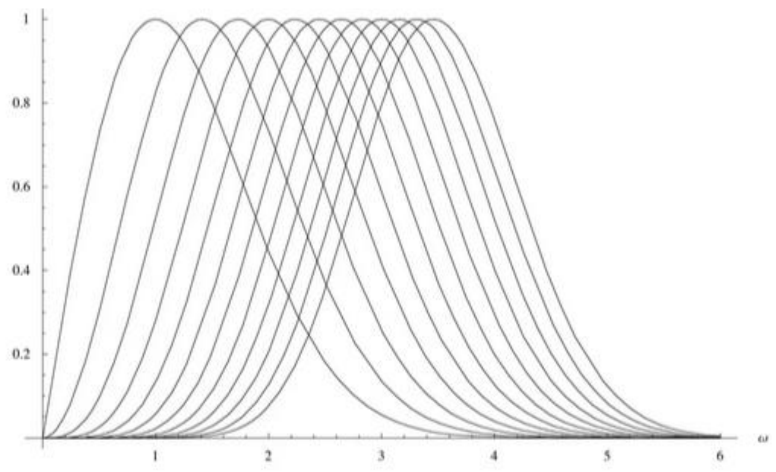


Figure Normalized power spectra for Gaussian derivative filters for order 1 to 12, lowest order is left-most graph, $\sigma = 1$. Gaussian derivative kernels act like bandpass filters.

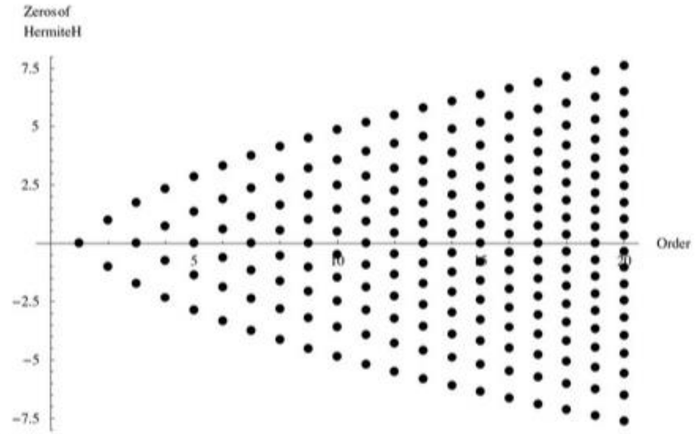


Figure Zero crossings of Gaussian derivative functions to 20th order. Each dot is a zero-crossing.

Gaussian derivative kernels act as band pass filters.

Higher order of differentiation means higher center frequency for the bandpass filter. The bandwidth remain the same.

Gaussian derivative functions differing two orders are of opposite polarity (symmetric kernels)
Gaussian derivative functions differing one order display a phase shift. (asymmetric kernels)

Number of zero crossings is equal to the order of differentiation. Width of a Gaussian derivative function is defined as the distance between the outer most zero crossings.

Higher order Gaussian derivative kernels tend to become more and more similar. This makes them not suitable as a basis.

For very high orders of differentiation of course the numbers of zero-crossings increases, but also their mutual distance between the zeros becomes more equal. In the limiting case of infinite order the Gaussian derivative function becomes a sinusoidal function:

$$\lim_{n\to\infty} \frac{\partial^n G}{\partial^n x}(x, \sigma) = \mathrm{Sin}\left(x \sqrt{\frac{1}{\sigma}\left(\frac{n+1}{2}\right)}\right).$$

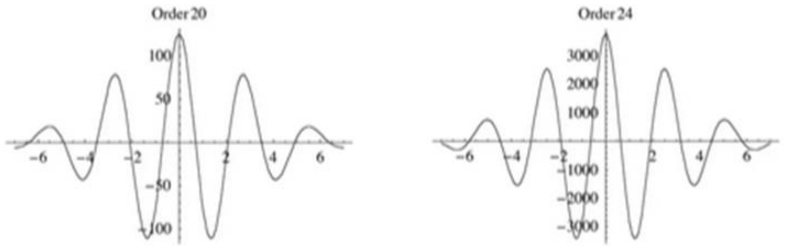


Figure Gaussian derivative functions start to look more and more alike for higher order. Here the graphs are shown for the 20th and 24th order of differentiation.

References

- https://maucher.pages.mi.hdm-stuttgart.de/orbook/preprocessing/04gaussianDerivatives.html

- Grey-Scale Measurements in Multi-Dimensional Digitized Images (Excellent for Gaussian Derivative filters)
https://repository.tudelft.nl/record/uuid:01796430-d6e6-46b9-98b1-e40f713bf6c8

- https://cw.fel.cvut.cz/old/_media/courses/ucuws17/gd.pdf

- https://www.sci.utah.edu/~gerig/CS7960-S2010/handouts/04%20Gaussian%20derivatives.pdf

- https://pages.stat.wisc.edu/~mchung/teaching/MIA/reading/diffusion.gaussian.kernel.pdf.pdf