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Green's function approach to Chern-Simons extended electrodynamics: An effective theory describing topological insulators

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Boundary effects produced by a Chern-Simons (CS) extension to electrodynamics are analyzed exploiting the Green's function (GF) method. We consider the electromagnetic field coupled to a θ term in a way that has been proposed to provide the correct low-energy effective action for topological insulators (TI). We take the θ term to be piecewise constant in different regions of space separated by a common interface Σ, which will be called the θ boundary. Features arising due to the presence of the boundary, such as magnetoelectric effects, are already known in CS extended electrodynamics, and solutions for some experimental setups have been found, each with its specific configuration of sources. In this work we illustrate a method to construct the GF that allows us to solve the CS modified field equations for a given θ boundary with otherwise arbitrary configuration of sources. The method is illustrated by solving the case of a planar θ boundary but can also be applied for cylindrical and spherical geometries for which the θ boundary can be characterized by a surface where a given coordinate remains constant. The static fields of a pointlike charge interacting with a planar TI, as described by a planar discontinuity in θ, are calculated and successfully compared with previously reported results. We also compute the force between the charge and the θ boundary by two different methods, using the energy-momentum tensor approach and the interaction energy calculated via the GF. The infinitely straight current-carrying wire is also analyzed.

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I. INTRODUCTION

The relevance of Chern-Simons (CS) forms [1] in several branches of theoretical physics is well accounted for. In quantum field theory in regards to anomalies [2], they played a key role, and in particle physics they proved important as well [3–5]. In general relativity they also enjoy a prominent position, as clearly reviewed in [6]. Further studies involve its uses in topological quantum field theory [7], topological string theory [8] and as a quantum gravity candidate [9].

In general, CS forms are amenable for capturing topological features of the physical system they describe, which is why in the last decade their importance has also become apparent in the field of condensed matter physics for describing what came to be known as topological phases. The discovery made by von Klitzing of the astonishing precision with which the Hall conductance of a sample is quantized [10], despite the varying irregularities and geometry of the sample, turned out to have a topological origin. The reason for this lies in the band structure of the sample, but ultimately, the Hall conductance can be expressed as an invariant integral over the frequency in momentum space. More precisely, it can be expressed as an integral of the Berry curvature over the Brillouin zone [11], inasmuch as the genus of a manifold can be expressed in terms of an invariant integral of the local curvature over the surface enclosing it. This quantity plays the role of a topological order parameter uniquely determining the nature of the quantum state, as the order parameter in Landau-Ginzburg effective field theory determines the usual phases of quantum matter.

In this work we are concerned with a simple case of CS theories, to which we will refer as θ electrodynamics or simply θ ED, and it amounts to extending Maxwell electrodynamics by a parity-violating term of the form

$$\Delta\mathcal{L}_\theta = \theta(\alpha/4\pi^2)\mathbf{E}\cdot\mathbf{B} = -\frac{\theta}{4}(\alpha/4\pi^2)F_{\mu\nu}\tilde{F}^{\mu\nu}, \qquad (1)$$

where $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ and $\epsilon^{\mu\nu\alpha\beta}$ is the Levi-Cività symbol. In general, θ can be a dynamical field; however, we take it as a constant scalar, making Eq. (1) a pseudoscalar. Note that this extension is a total derivative, producing no contribution to the field equations when usual boundary conditions are met. If θ is not globally constant in the manifold where the theory is defined, then the CS term fails to be a topological invariant; therefore, the corresponding modifications to the field equations must be taken into consideration.

Here we study Maxwell theory extended by Eq. (1) defined on a manifold in which there are two domains

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defined by their different constant values of θ that are separated by a common interface or boundary Σ. The constant θ can be thought of as an effective parameter characterizing properties of a novel electromagnetic vacuum possibly arising from a more fundamental theory or, as applied to material media, as an effective macroscopic parameter to describe novel quantum degrees of freedom of matter apart from the usual permittivity ε and permeability μ. The former approach has been taken in the context of classical θ ED [12] and in the quantum vacuum framework [13]. For related analyses, in several contexts, see Ref. [14]. The latter approach has been used to describe topological insulators (TIs). Concretely, the low-energy limit of the electrodynamics of TIs can be described by extending Maxwell electrodynamics by Eq. (1), originally formulated in $4+1$ dimensions but appropriately adapted to lower dimensions by dimensional reduction [15]. Thus, θ ED as a topological field theory (TFT) serves as a model for many theoretical [16] and experimental realizations for studying detailed properties of topological states of quantum matter [17,18].

The formulation of θ ED pursued in this work can be considered as a particularly simple version of the so-called Janus field theories [19–24]. Generally speaking, such theories are characterized by having spacetime-dependent coupling constants, such as θ in our model. They have been actively explored in the context of the AdS/CFT correspondence. Nevertheless, as we have already mentioned, in the case of θ ED this idea is applied to a simpler but more realistic system that constitutes an effective low-energy theory that allows us to compute the response of topological insulators to arbitrary external sources and currents in a planar geometry, with direct extensions to cylindrical and spherical geometries. Janus field theories were motivated, from the gravitational sector of the AdS/CFT correspondence, by an exact and nonsingular solution for the dilatonic field in type II-B supergravity, which was found in a simple deformation of the $\mathrm{AdS}_5 \times S^5$ geometry [25]. Even though the solution breaks all the original supersymmetries, it proves to be stable under a large class of perturbations [25–27]. The dilaton acquires a constant value at the boundary, where AdS_5 is recovered but adopts different values at each half of the boundary. On the other hand, the AdS/CFT correspondence requires the existence of a dual gauge theory on the boundary for every nonsingular solution of type II-B supergravity in the bulk, which in this case is a four-dimensional $\mathcal{N}=4$ super Yang-Mills (SYM) theory living in the boundary [25]. In other words, a running dilaton induces spacetime-dependent coupling constants in the gauge theories in the dual sector, which defines the Janus field theory. In our case the four-dimensional $\mathcal{N}=4$ SYM theory is replaced by the CS modified ED, where we take the electromagnetic coupling to be globally constant, while the topological coupling to the Pontryagin invariant has different constant values at each side of a planar interface and suffers a jump across such a boundary. In relation to θ ED, it is interesting to recall that the authors of Ref. [19] proposed a model for the dual theory arising from the Janus solution, where the $\mathcal{N}=4$ SYM coupling $g(z)$ affects only the kinetic term of the non-Abelian gauge field, together with the interaction terms in the original Lagrangian for the standard $\mathcal{N}=4$ SYM theory. The model completely breaks the 16 original supersymmetries of the $\mathcal{N}=4$ SYM theory. Moreover, $g(z)$ is taken as constant on each side of a planar interface ($z=0$), with a sharp jump across it. In this way, the gauge field part of the action is the non-Abelian generalization of the Maxwell action in an inhomogeneous medium with permittivity ϵ and permeability μ related by $\epsilon(z)=1/\mu(z)=1/g^2(z)$. The YM fields satisfy boundary conditions at the interface, which are derived by integrating the equations of motion over the standard infinitesimal pill-shaped regions across the boundary, in a way similar to standard electrodynamics. The YM Green's function (GF) is also obtained by using image methods. Nevertheless, let us emphasize that this model does not include a coupling to the YM Pontryagin invariant, in such a way that its Abelian limit does not reproduce θ ED. The inclusion of the topological coupling $\theta(z)$ in addition to the YM coupling $g(z)$ is developed in Refs. [22,24], where $1/2$ BPS vacuum configurations are studied, in particular. As shown in Ref. [22] half of the original supersymmetries can be maintained provided such couplings are constrained by the relations $1/g^2(z)=D\sin 2\psi(z)$ and $\theta(z)=\theta_0+8\pi^2 D\cos 2\psi(z)$. The case of a sharp interface respecting the above constraints is also considered in Ref. [24], and it is studied in the Abelian Coulomb phase, by setting two different constant values ψ_1 and ψ_2 at each side of a planar boundary. However, in the case of θ ED, supersymmetry does not enter, and we are choosing the electromagnetic coupling to be globally constant, i.e., $g_1=g_2=e$, while only the topological coupling $\theta(z)$ becomes discontinuous at the sharp boundary, with constant values $\theta_1\neq\theta_2$ in each side. As can be seen already, these two systems are not equivalent, and later we will discuss this issue in more detail.

The paper is organized as follows. In Sec. II we review the basics of Chern-Simons electrodynamics defined on a four-dimensional spacetime in which the θ value is piecewise constant in different regions of space separated by a common boundary Σ. In Sec. III we restrict ourselves to the static case, and we construct the GF matrix for the planar geometry corresponding to a θ boundary located at $z=a$. Section IV is devoted to different applications, e.g., the problems of a pointlike charge and a current-carrying wire near a planar θ boundary. The interaction energy (and forces) between a charge-current distribution and a θ interface is briefly discussed. Contact between the results obtained with our method and others in the existing literature is made. A concluding summary of our results comprises Sec. V. Throughout the paper, Lorentz-Heaviside units are assumed ($\hbar = c = 1$), the metric signature will be taken as $(+,-,-,-)$, and the convention $\epsilon^{0123} = +1$ is adopted.

II. θ ELECTRODYNAMICS IN A BOUNDED REGION

Our model is based on Maxwell electrodynamics coupled to a gauge-invariant θ term as described by the following action:

$$\mathcal{S} = \int_{\mathcal{M}} d^4x \left[-\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} \theta \frac{\alpha}{4\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} - j^{\mu} A_{\mu} \right], \tag{2}$$

where $\alpha = e^2/\hbar c$ is the fine-structure constant and j^{μ} is a conserved external current. The coupling constant for the θ term, $\alpha/4\pi^2$, is chosen in such a way that the total electric charge $q_e = \frac{1}{4\pi} \int d\mathbf{S} \cdot \mathbf{D}$ has to be an integer multiple of the electron charge e, whereas the magnetic charge $q_m = \frac{1}{4\pi} \int d\mathbf{S} \cdot \mathbf{B}$ should be an integer multiple of $g = e/2\alpha$ by the Dirac quantization condition [5]. Later we recall the reasoning which shows that, quantum mechanically, the allowed values of θ are 0 or π (mod 2π).

The $(3+1)$-dimensional spacetime is $\mathcal{M} = \mathcal{U} \times \mathbb{R}$, where $\mathcal{U}$ is a three-dimensional manifold and $\mathbb{R}$ corresponds to the temporal axis. We make a partition of space in two regions: $\mathcal{U}_1$ and $\mathcal{U}_2$, in such a way that manifolds $\mathcal{U}_1$ and $\mathcal{U}_2$ intersect along a common two-dimensional boundary Σ, called the θ boundary, so that $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$ and $\Sigma = \mathcal{U}_1 \cap \mathcal{U}_2$, as shown in Fig. 1. We also assume that the field θ is piecewise constant in such a way that it takes the constant value $\theta = \theta_1$ in region $\mathcal{U}_1$ and the constant value $\theta = \theta_2$ in region $\mathcal{U}_2$. This situation is expressed in the characteristic function

$$\theta(\mathbf{x}) = \begin{cases} \theta_1, & \mathbf{x} \in \mathcal{U}_1 \\ \theta_2, & \mathbf{x} \in \mathcal{U}_2. \end{cases} \tag{3}$$

In this scenario the θ term in the action fails to be a global topological invariant because it is defined over a region with the boundary Σ. Varying the action gives rise to a set of Maxwell equations with an effective additional current density with support at the boundary

$$\partial_{\mu} F^{\mu\nu} = \tilde{\theta} \delta(\Sigma) n_{\mu} \tilde{F}^{\mu\nu} + 4\pi j^{\nu}, \tag{4}$$

where n_{μ} is the outward normal to Σ, and $\tilde{\theta} = \alpha(\theta_1 - \theta_2)/\pi$, which enforces the invariance of the classical dynamics under the shifts of θ by any constant, $\theta \to \theta + C$. Current conservation can be verified directly by taking the divergence at both sides of Eq. (4) and using symmetry properties. The set of equations (4) for θ ED can be written as

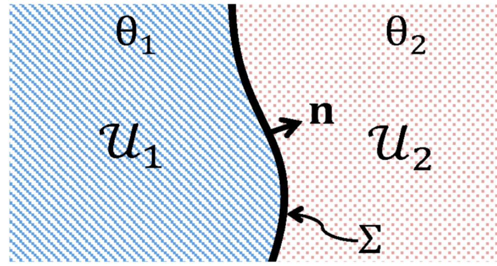

FIG. 1 (color online). Region over which the electromagnetic field theory is defined.

$$\nabla \cdot \mathbf{E} = \tilde{\theta} \delta(\Sigma) \mathbf{B} \cdot \mathbf{n} + 4\pi\rho, \tag{5}$$

$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \tilde{\theta} \delta(\Sigma) \mathbf{E} \times \mathbf{n} + 4\pi \mathbf{J}, \tag{6}$$

while the homogeneous equations are included in the Bianchi identity $\partial_{\mu} \tilde{F}^{\mu\nu} = 0$. Here $\mathbf{n}$ is the unit normal to Σ shown in Fig. 1. In this work we consider a simple geometry corresponding to a surface Σ taken as the plane $z = a$.

As we see from Eqs. (5) and (6) the behavior of θ ED in the bulk regions $\mathcal{U}_1$ and $\mathcal{U}_2$ is the same as in standard electrodynamics. The θ term modifies Maxwell equations only at the surface Σ. Here $F^{i0} = E^i$, $F^{ij} = -\varepsilon^{ijk} B^k$ and $\tilde{F}^{i0} = B^i$, $\tilde{F}^{ij} = \varepsilon^{ijk} E^k$. Equations (5) and (6) also suggest that the electromagnetic response of a system in the presence of a θ term can be described in terms of Maxwell equations in matter,

$$\nabla \cdot \mathbf{D} = 4\pi\rho, \qquad \nabla \times \mathbf{H} = 4\pi \mathbf{J}, \tag{7}$$

with constitutive relations

$$\mathbf{D} = \mathbf{E} + \frac{\alpha}{\pi} \theta(z) \mathbf{B}, \qquad \mathbf{H} = \mathbf{B} - \frac{\alpha}{\pi} \theta(z) \mathbf{E}, \tag{8}$$

where $\theta(z)$ is given in Eq. (3). If $\theta(z)$ is globally constant in $\mathcal{M}$, there is no contribution to Maxwell equations from the θ term in the action, even though θ is still present in the constitutive relations. In fact, the additional contributions of a globally constant θ to each of the modified Maxwell equations (5) and (6) cancel due to the homogeneous equations.

Now we return to the problem of the allowed values of θ to describe topological insulators. $U(1)$ gauge theories with nonzero θ (θ ED) exhibit an $SL(2,\mathbb{Z})$ duality group which strongly constrains the quantum physics [28,29]. This group is obtained by repeated applications of the S and T generators of electric-magnetic duality. The S generator is associated with the invariance of classical Maxwell equations in matter (7) (supplemented with magnetic charge and current densities) under duality rotations. Only the special case of a duality transformation by $\pi/2$ is consistent with the requirement that the electric charge and the magnetic charge are quantized.

The aforementioned rescaling symmetry $\theta \to \theta + C$ would allow us to set θ to zero at the classical level. Quantum mechanically, however, given that for properly quantized electric and magnetic fluxes $S_\theta/\hbar$ is an integer multiple of θ, only $C = 2\pi n$ for integer n is an allowed symmetry; otherwise, nontrivial contributions to the path integral would result. Furthermore, since $\mathbf{E} \cdot \mathbf{B}$ is odd under $t \to -t$, only $\theta = 0$ and $\theta = \pi$ give a time-reversal symmetric theory. Thus, time reversal takes θ into $-\theta$, so $\theta = 0$ is time-reversal invariant *per se*, whereas $\theta = \pi$ is invariant after the shift $\theta \to \theta + 2\pi$. This is typically referred as the T generator of the electric-magnetic duality. The two transformations S and T generate the $SL(2,\mathbb{Z})$ symmetry group acting on the fields [29].

Next we study the effects of a θ interface in the propagation of the fields. Assuming that the time derivatives of the fields are finite in the vicinity of the surface Σ, the field equations imply that the normal component of $\mathbf{E}$ and the tangential components of $\mathbf{B}$ acquire discontinuities additional to those produced by superficial free charges and currents, while the normal component of $\mathbf{B}$ and the tangential components of $\mathbf{E}$ are continuous. For vanishing free sources on the surface, the boundary conditions read

$$\mathbf{E}_z|_{z=a^-}^{z=a^+} = \tilde{\theta}\mathbf{B}_z|_{z=a}, \qquad \mathbf{B}_\parallel|_{z=a^-}^{z=a^+} = -\tilde{\theta}\mathbf{E}_\parallel|_{z=a}, \quad (9)$$

$$\mathbf{B}_z|_{z=a^-}^{z=a^+} = 0, \qquad \mathbf{E}_\parallel|_{z=a^-}^{z=a^+} = 0. \quad (10)$$

The notation is $\mathbf{V}_i|_{z=a^-}^{z=a^+} = \mathbf{V}_i(z)|_{z=a^-}^{z=a^+} = \lim_{\varepsilon\to 0}[\mathbf{V}_i(z = a+\varepsilon) - \mathbf{V}_i(z = a-\varepsilon)], \varepsilon > 0$ and $\mathbf{V}_i|_{z=a} = \mathbf{V}_i(z = a)$, for any vector $\mathbf{V}$. The continuity conditions (10) imply that the right-hand sides of Eqs. (9) are well defined and they represent self-induced surface charge and current densities, respectively. An immediate consequence of the boundary conditions (9) and (10) is that the presence of a magnetic field crossing the surface Σ is sufficient to generate an electric field, even in the absence of free electric charges. Many interesting magnetoelectric effects due to a θ boundary have been highlighted using different approaches. For example, electric charges close to the interface Σ induce magnetic mirror monopoles (and vice versa) [23,24,30]. Also, the propagation of electromagnetic waves across a θ boundary have been studied, finding that a nontrivial Faraday rotation of the polarizations appears [12,23,24,31]. It is worth mentioning that with the modified boundary conditions, several properties of conductors still hold for static fields as long as the conductor does not lie in the Σ boundary; in particular, conductors are equipotential surfaces, and the electric field just outside the conductor is normal to its surface.

III. GF METHOD

In this section we use the GF method to solve static boundary-value problems in θ ED in terms of the electromagnetic potential A^μ. Certainly, one could solve for the electric and magnetic fields from the modified Maxwell equations together with the boundary conditions (9) and (10); however, just as in ordinary electrodynamics, there might be occasions where information about the sources is unknown, and rather we are provided with information about the 4-potential at the given boundaries. In these cases the GF method provides the general solution to a given boundary-value problem (Dirichlet or Neumann) for arbitrary sources. Nevertheless, in what follows we restrict ourselves to contributions of free sources outside the θ boundary and without boundary conditions imposed on additional surfaces, except for the standard boundary conditions at infinity.

Since the homogeneous Maxwell equations that express the relationship between potentials and fields are not modified, the electrostatic and magnetostatic fields can be written in terms of the 4-potential $A^\mu = (\phi, \mathbf{A})$ according to $\mathbf{E} = -\nabla\phi$ and $\mathbf{B} = \nabla \times \mathbf{A}$ as usual. In the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$, the 4-potential satisfies the equation of motion

$$[-\eta^\mu{}_\nu \nabla^2 - \tilde{\theta}\delta(z-a)\epsilon^{3\mu\alpha}{}_\nu \partial_\alpha]A^\nu = 4\pi j^\mu, \quad (11)$$

together with the boundary conditions

$$A^\mu|_{z=a^-}^{z=a^+} = 0, \qquad (\partial_z A^\mu)|_{z=a^-}^{z=a^+} = -\tilde{\theta}\epsilon^{3\mu\alpha}{}_\nu \partial_\alpha A^\nu|_{z=a}. \quad (12)$$

One can further check that these boundary conditions for the 4-potential correspond to the ones obtained in Eqs. (9) and (10).

To obtain a general solution for the potentials ϕ and $\mathbf{A}$ in the presence of arbitrary external sources $j^\mu(\mathbf{x})$, we introduce the GF $G^\nu{}_\sigma(\mathbf{x},\mathbf{x}')$ solving Eq. (11) for a pointlike source,

$$[-\eta^\mu{}_\nu \nabla^2 - \tilde{\theta}\delta(z-a)\epsilon^{3\mu\alpha}{}_\nu \partial_\alpha]G^\nu{}_\sigma(\mathbf{x},\mathbf{x}') = 4\pi\eta^\mu{}_\sigma \delta^3(\mathbf{x}-\mathbf{x}'), \quad (13)$$

together with the boundary conditions (12), in such a way that the general solution for the 4-potential in the Coulomb gauge is

$$A^\mu(\mathbf{x}) = \int d^3\mathbf{x}' G^\mu{}_\nu(\mathbf{x},\mathbf{x}') j^\nu(\mathbf{x}'). \quad (14)$$

According to Eq. (13) the diagonal entries of the GF matrix are related to the electric and magnetic fields arising from the charge and current density sources, respectively, although they acquire θ dependence. However, the nondiagonal terms encode the magnetoelectric effect, i.e., the charge (current) density contributing to the magnetic (electric) field.

In the following we concentrate on constructing the solution to Eq. (13). The GF we consider has translational invariance in the directions parallel to Σ, that is, in the transverse x and y directions, while this invariance is broken in the z direction. Exploiting this symmetry we further introduce the Fourier transform in the direction parallel to the plane Σ, taking the coordinate dependence to be $(\mathbf{x}-\mathbf{x}')_{\parallel} = (x-x', y-y')$, and define

$$G^{\mu}{}_{\nu}(\mathbf{x},\mathbf{x}') = 4\pi \int \frac{d^2\mathbf{p}}{(2\pi)^2} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')_{\parallel}} g^{\mu}{}_{\nu}(z,z'), \quad (15)$$

where $\mathbf{p} = (p_x, p_y)$ is the momentum parallel to the plane Σ [32]. In Eq. (15) we have suppressed the dependence of the reduced GF g^{μ}_{ν} on $\mathbf{p}$.

Due to the antisymmetry of the Levi-Cività symbol, the partial derivative appearing in the second term of the GF Eq. (13) does not introduce derivatives with respect to z but only in the transverse coordinates. This allows us to write the full reduced GF equation as

$$[\partial^2 \eta^{\mu}{}_{\nu} + i\tilde{\theta}\delta(z-a)\epsilon^{3\mu\alpha}{}_{\nu} p_{\alpha}] g^{\nu}{}_{\sigma}(z,z') = \eta^{\mu}{}_{\sigma}\delta(z-z'), \quad (16)$$

where $\partial^2 = \mathbf{p}^2 - \partial_z^2$, $p^{\alpha} = (0,\mathbf{p})$ and $\mathbf{p}^2 = -p^{\alpha}p_{\alpha}$.

The solution to Eq. (16) is simple but not straightforward. To solve it we employ a method similar to that used for obtaining the GF for the one-dimensional δ-function potential in quantum mechanics, where the free GF is used for integrating the GF equation with δ interaction. To this end we consider a reduced free GF having the form $\mathcal{G}^{\mu}{}_{\nu}(z,z') = \mathfrak{g}(z,z')\eta^{\mu}{}_{\nu}$, associated with the operator ∂^2 previously defined, that solves

$$\partial^2 \mathcal{G}^{\mu}{}_{\nu}(z,z') = \eta^{\mu}{}_{\nu}\delta(z-z'), \quad (17)$$

satisfying the standard boundary conditions at infinity, where

$$\mathfrak{g}(z,z') = \frac{1}{2p} e^{-p|z-z'|} \quad (18)$$

and $p = |\mathbf{p}|$. Note that Eq. (17) demands the derivative of $\mathfrak{g}$ to be discontinuous at $z = z'$, i.e., $\partial_z \mathfrak{g}(z,z')|_{z=z'^-}^{z=z'^+} = -1$, and then the continuity of $\mathfrak{g}$ at $z = z'$ follows [32].

Now we observe that Eq. (16) can be directly integrated by using the free GF Eq. (17) together with the properties of the Dirac delta function, thus reducing the problem to a set of coupled algebraic equations,

$$g^{\mu}{}_{\sigma}(z,z') = \eta^{\mu}{}_{\sigma}\mathfrak{g}(z,z') - i\tilde{\theta}\epsilon^{3\mu\alpha}{}_{\nu} p_{\alpha}\mathfrak{g}(z,a) g^{\nu}{}_{\sigma}(a,z'). \quad (19)$$

Note that the continuity of $\mathfrak{g}$ at $z = z'$ implies the continuity of $g^{\mu}{}_{\sigma}$, but the discontinuity of $\partial_z \mathfrak{g}$ at the same point yields

$$\begin{aligned}\partial_z g^{\mu}{}_{\sigma}(z,z')|_{z=a^-}^{z=a^+} &= -i\tilde{\theta}\epsilon^{3\mu\alpha}{}_{\nu} p_{\alpha}\partial_z \mathfrak{g}(z,a)|_{z=a^-}^{z=a^+} g^{\nu}{}_{\sigma}(a,z') \\ &= i\tilde{\theta}\epsilon^{3\mu\alpha}{}_{\nu} p_{\alpha} g^{\nu}{}_{\sigma}(a,z'),\end{aligned} \quad (20)$$

from which the boundary conditions for the 4-potential in Eq. (12) are recovered. In this way the solution (19) guarantees that the boundary conditions at the θ interface are satisfied.

Now we have to solve for the various components $g^{\mu}{}_{\sigma}$. To this end we split Eq. (19) into $\mu = 0$ and $\mu = j = 1, 2, 3$ components:

$$g^{0}{}_{\sigma}(z,z') = \eta^{0}{}_{\sigma}\mathfrak{g}(z,z') - i\tilde{\theta}\epsilon^{30i}{}_{j} p_i \mathfrak{g}(z,a) g^{j}{}_{\sigma}(a,z'), \quad (21)$$

$$g^{j}{}_{\sigma}(z,z') = \eta^{j}{}_{\sigma}\mathfrak{g}(z,z') - i\tilde{\theta}\epsilon^{3ji}{}_{0} p_i \mathfrak{g}(z,a) g^{0}{}_{\sigma}(a,z'). \quad (22)$$

Now we set $z = a$ in Eq. (22) and then substitute into Eq. (21), yielding

$$\begin{aligned}g^{0}{}_{\sigma}(z,z') &= \eta^{0}{}_{\sigma}\mathfrak{g}(z,z') - i\tilde{\theta}\epsilon^{30i}{}_{j} p_i \eta^{j}{}_{\sigma}\mathfrak{g}(z,a)\mathfrak{g}(a,z') \\ &\quad - \tilde{\theta}^2 p^2 \mathfrak{g}(z,a)\mathfrak{g}(a,a) g^{0}{}_{\sigma}(a,z'),\end{aligned} \quad (23)$$

where we use the result $\epsilon^{30i}{}_{j}\epsilon^{3jk}{}_{0} p_k p_i = p^2$. Solving for $g^{0}{}_{\sigma}(a,z')$ by setting $z = a$ in Eq. (23) and inserting the result back into that equation, we obtain

$$\begin{aligned}g^{0}{}_{\sigma}(z,z') &= \eta^{0}{}_{\sigma}[\mathfrak{g}(z,z') + \tilde{\theta} p^2 \mathfrak{g}(a,a) A(z,z')] \\ &\quad + i\epsilon^{30i}{}_{\sigma} p_i A(z,z'),\end{aligned} \quad (24)$$

where

$$A(z,z') = -\tilde{\theta}\frac{\mathfrak{g}(z,a)\mathfrak{g}(a,z')}{1 + p^2\tilde{\theta}^2\mathfrak{g}^2(a,a)}. \quad (25)$$

The remaining components can be obtained by substituting $g^{0}{}_{\sigma}(a,z')$ in Eq. (22). The result is

$$\begin{aligned}g^{j}{}_{\sigma}(z,z') &= \eta^{j}{}_{\sigma}\mathfrak{g}(z,z') + i\epsilon^{3jk}{}_{0} p_k [\eta^{0}{}_{\sigma} - i\tilde{\theta}\epsilon^{30i}{}_{\sigma} p_i \mathfrak{g}(a,a)] \\ &\quad \times A(z,z').\end{aligned} \quad (26)$$

Equations (24) and (26) allow us to write the general solution as

$$\begin{aligned}g^{\mu}{}_{\nu}(z,z') &= \eta^{\mu}{}_{\nu}\mathfrak{g}(z,z') + A(z,z')\{\tilde{\theta}\mathfrak{g}(a,a)[p^{\mu}p_{\nu} \\ &\quad + (\eta^{\mu}{}_{\nu} + n^{\mu}n_{\nu})p^2] + i\epsilon^{\mu}{}_{\nu}{}^{\alpha 3} p_{\alpha}\},\end{aligned} \quad (27)$$

where $n_{\mu} = (0,0,0,1)$ is the normal to Σ.

The reciprocity between the position of the unit charge and the position at which the GF is evaluated $G_{\mu\nu}(\mathbf{x},\mathbf{x}') = G_{\nu\mu}(\mathbf{x}',\mathbf{x})$ is one of its most remarkable properties. From Eq. (15) this condition demands

$$g_{\mu\nu}(z,z',\mathbf{p}) = g_{\nu\mu}(z',z,-\mathbf{p}), \quad (28)$$

which we verify directly from Eq. (27). The symmetry $g_{\mu\nu}(z,z') = g^{*}_{\nu\mu}(z,z') = g^{\dagger}_{\mu\nu}(z,z')$ is also manifest.

The various components of the static GF matrix in the coordinate representation are obtained by computing the

Fourier transform defined in Eq. (15), with the reduced GF given by Eq. (18). The details are presented in Appendix. The final results are

$$G^0_{\ 0}(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{\tilde{\theta}^2}{4+\tilde{\theta}^2}\frac{1}{\sqrt{R^2+Z^2}}, \tag{29}$$

$$G^0_{\ i}(\mathbf{x},\mathbf{x}') = -\frac{2\tilde{\theta}}{4+\tilde{\theta}^2}\frac{\epsilon_{0ij3}R^j}{R^2}\left(1-\frac{Z}{\sqrt{R^2+Z^2}}\right), \tag{30}$$

$$G^i_{\ j}(\mathbf{x},\mathbf{x}') = \eta^i_{\ j}G^0_{\ 0}(\mathbf{x},\mathbf{x}') - \frac{i}{2}\frac{\tilde{\theta}^2}{4+\tilde{\theta}^2}\partial^i K_j(\mathbf{x},\mathbf{x}'), \tag{31}$$

where $Z=|z-a|+|z'-a|$, $R^j=(\mathbf{x}-\mathbf{x}')^j_\parallel=(x-x', y-y')$, $R=|(\mathbf{x}-\mathbf{x}')_\parallel|$ and

$$K^j(\mathbf{x},\mathbf{x}') = 2i\frac{\sqrt{R^2+Z^2}-Z}{R^2}R^j. \tag{32}$$

Finally, we observe that Eqs. (29)–(31) contain all the required elements of the GF matrix, according to the choices of z and z' in the function Z.

IV. APPLICATIONS

A. Pointlike charge near a planar θ boundary

Let us consider a pointlike electric charge q located at a distance $b>0$ from the $z=0$ plane, where we have chosen $a=0$. Also, the region $z<0$ is filled with a topologically nontrivial insulator, whereas the region $z>0$ is the vacuum ($\theta_2=0$). For simplicity we choose the coordinates such that $x'=y'=0$. Therefore, the current density is $j^\mu(\mathbf{x}')=q\eta^\mu_{\ 0}\delta(x')\delta(y')\delta(z'-b)$. According to Eq. (14), the solution for this problem is

$$A^\mu(\mathbf{x}) = qG^\mu_{\ 0}(\mathbf{x},\mathbf{r}), \tag{33}$$

where $\mathbf{r}=b\hat{\mathbf{e}}_z$. We first study the electrostatic potential. From Eq. (29),

$$z>0\colon\ G^0_{\ 0}(\mathbf{x},\mathbf{r}) = \frac{1}{|\mathbf{x}-\mathbf{r}|} - \frac{\tilde{\theta}^2}{4+\tilde{\theta}^2}\frac{1}{|\mathbf{x}+\mathbf{r}|}, \tag{34}$$

$$z<0\colon\ G^0_{\ 0}(\mathbf{x},\mathbf{r}) = \frac{4}{4+\tilde{\theta}^2}\frac{1}{|\mathbf{x}-\mathbf{r}|}. \tag{35}$$

For $z>0$ the GF yields the electric potential $A^0(\mathbf{x})=qG^0_{\ 0}(\mathbf{x},\mathbf{r})$ which can be interpreted as due to two pointlike electric charges, one of strength q at $\mathbf{r}$, and the other, the image charge, of strength $-q\tilde{\theta}^2/(4+\tilde{\theta}^2)$, at the point $-\mathbf{r}$. For $z<0$ only one pointlike electric charge appears, of strength $4q/(4+\tilde{\theta}^2)$ located at $\mathbf{r}$.

From Eq. (33) we see that two components of the magnetic vector potential are nonzero, $A^1(\mathbf{x})=qG^1_{\ 0}(\mathbf{x},\mathbf{r})$ and $A^2(\mathbf{x})=qG^2_0(\mathbf{x},\mathbf{r})$. The corresponding GF components for each region are given by

$$G^1_{\ 0}(\mathbf{x},\mathbf{r}) = -\frac{2\tilde{\theta}}{4+\tilde{\theta}^2}\frac{y}{R^2}\begin{cases}1-\frac{z+b}{|\mathbf{x}+\mathbf{r}|}, & z>0\\ 1+\frac{z-b}{|\mathbf{x}+\mathbf{r}|}, & z<0,\end{cases} \tag{36}$$

$$G^2_0(\mathbf{x},\mathbf{r}) = +\frac{2\tilde{\theta}}{4+\tilde{\theta}^2}\frac{x}{R^2}\begin{cases}1-\frac{z+b}{|\mathbf{x}+\mathbf{r}|}, & z>0\\ 1+\frac{z-b}{|\mathbf{x}+\mathbf{r}|}, & z<0,\end{cases} \tag{37}$$

according to Eq. (30). It is difficult to interpret the components of the vector potential directly; however, the magnetic field $\mathbf{B}=\nabla\times\mathbf{A}$ is illuminating. In fact

$$z>0\colon\ \mathbf{B}(\mathbf{x}) = \frac{2q\tilde{\theta}}{4+\tilde{\theta}^2}\frac{\mathbf{x}+\mathbf{r}}{|\mathbf{x}+\mathbf{r}|^3}, \tag{38}$$

$$z<0\colon\ \mathbf{B}(\mathbf{x}) = \frac{-2q\tilde{\theta}}{4+\tilde{\theta}^2}\frac{\mathbf{x}-\mathbf{r}}{|\mathbf{x}-\mathbf{r}|^3}. \tag{39}$$

Thus, we observe that the magnetic field for $z>0$ is that due to a magnetic monopole of strength $2q\tilde{\theta}/(4+\tilde{\theta}^2)$ located at $-\mathbf{r}$. For $z<0$ we have a magnetic monopole of strength $-2q\tilde{\theta}/(4+\tilde{\theta}^2)$ located at $\mathbf{r}$.

The solution shows that, for an electric charge near the planar surface of a topological insulator, both an image magnetic charge and an image electric charge will be induced. The appearance of magnetic monopoles in this solution seems to violate the Maxwell law $\nabla\cdot\mathbf{B}=0$, which remained unaltered in the case of θ ED. Nevertheless, recalling that $(\mathbf{x}\pm\mathbf{r})/|\mathbf{x}\pm\mathbf{r}|^3\sim\nabla_x(1/|\mathbf{x}\pm\mathbf{r}|)$, we have $\nabla\cdot\mathbf{B}\sim\nabla^2_x(1/|\mathbf{x}\pm\mathbf{r}|)\sim\delta(\mathbf{x}\pm\mathbf{r})$ in a region where $\mathbf{x}\neq\pm\mathbf{r}$. Physically, the magnetic field is induced by a surface current density

$$\mathbf{J} = \tilde{\theta}\delta(z)\mathbf{E}\times\mathbf{n} = \frac{4q\tilde{\theta}}{4+\tilde{\theta}^2}\frac{R}{(R^2+b^2)^{3/2}}\delta(z)\hat{\varphi}, \tag{40}$$

which is circulating around the origin. However, such an induced field has the correct magnetic field dependence expected from a magnetic monopole. This current is nothing but the Hall current [30].

It is worth mentioning that these results were also obtained using different approaches. On the one hand, the authors in Ref. [30] used the image method to show that an electric charge near a topological surface state induces an image magnetic monopole due to the magnetoelectric effect and emphasized the possible experimental verification via a gas of quantum particles carrying fractional statistics, consisting of the bound states of the electric charge and the image monopole charge.

At this stage we clarify the differences between the θ-ED approach we are following and the $1/2$ BPS construction in the sharp interface discussed in Ref. [24]. As we mentioned in the Introduction the eight remaining supersymmetries in the latter case are enforced by demanding the couplings to be related in the following way:

$$\frac{1}{e^2} = D \sin 2\psi(z), \qquad \theta = \theta_0 + 8\pi^2 D \cos 2\psi, \tag{41}$$

where one chooses the constant values ψ_1 and ψ_2 for $z > 0$ and $z < 0$, respectively. The constraint (41) does not allow us to simultaneously set $e_1 = e_2$ and $\theta_1 \neq \theta_2$, which corresponds to the case of θ ED, where supersymmetry is irrelevant. In other words, the limits $g = 0$ in the electric field and in the magnetic field of the single dyon at $z = a$ (Eqs. (5.10) of Ref. [24]), which were calculated using the method of images, do not reproduce the corresponding fields obtained from our Eqs. (34), (36) and (37). Also, the transmitted and reflected fields of massless waves propagating across the interface reported in Ref. [24] do not correspond to those calculated for θ ED in Refs. [12,31]. It is worth recalling that these couplings enter through the complexified parameter $\tau = \theta/2\pi + 4\pi i/g^2$, which is familiar in the study of the action of the group $SL(2,\mathbb{Z})$ on a topological insulator with nontrivial permittivity, permeability and θ angle [29].

B. Force between a charge and a planar θ boundary

In this section we formulate the interaction energy and the forces arising between external sources and a TI as represented by the θ boundary with a planar symmetry. We use both the GF matrix and the stress-energy tensor.

The interaction energy between a charge-current distribution and a topological insulator is

$$E_{\text{int}} = \frac{1}{2}\int d\mathbf{x} \int d\mathbf{x}' j^\mu(\mathbf{x})[G_{\mu\nu}(\mathbf{x},\mathbf{x}') - \eta_{\mu\nu}\mathcal{G}(\mathbf{x},\mathbf{x}')]j^\nu(\mathbf{x}'), \tag{42}$$

where $\mathcal{G}(\mathbf{x},\mathbf{x}') = 1/|\mathbf{x}-\mathbf{x}'|$ is the GF in vacuum. The first contribution represents the total energy of a charge-current distribution in the presence of the θ boundary, including mutual interactions. We evaluate this energy for the case considered in the previous subsection of a pointlike electric charge at position $\mathbf{r} = b\hat{\mathbf{e}}_z$. Making use of Eq. (34), the interaction energy is

$$E_{\text{int}} = \frac{q^2}{2}[G_{00}(\mathbf{r},\mathbf{r}) - \mathcal{G}(\mathbf{r},\mathbf{r})] = -\frac{q^2}{2}\frac{\tilde{\theta}^2}{4+\tilde{\theta}^2}\frac{1}{2b}. \tag{43}$$

Our result implies that the force on the charge exerted by the θ boundary is

$$\mathbf{F} = -\frac{\partial E_{\text{int}}}{\partial b}\hat{\mathbf{e}}_z = -\frac{q^2}{(2b)^2}\frac{\tilde{\theta}^2}{4+\tilde{\theta}^2}\hat{\mathbf{e}}_z, \tag{44}$$

noting that it is always attractive. This can be interpreted as the force between the charge q and the image charge $-q\tilde{\theta}^2/(4+\tilde{\theta}^2)$ according to Coulomb law.

The field point of view provides an alternative derivation of this result. To compute the force on the charge we calculate the normal component of the flow of momentum into the θ boundary. In terms of the stress-energy tensor this force is

$$\mathbf{F} = -\hat{\mathbf{e}}_z \int_{\Sigma^+} dS T_{zz}(\Sigma^+), \tag{45}$$

where the integration is over the surface Σ^+, just outside the θ interface, at $z = 0^+$.

The identification of the stress tensor in the case of θ electrodynamics proceeds along the standard lines of electrodynamics in a medium (see, for example, Ref. [32]), where we read the rate at which the electric field does work on the free charges,

$$\mathbf{J}\cdot\mathbf{E} = -\nabla\cdot\left(\frac{1}{4\pi}\mathbf{E}\times\mathbf{H}\right) - \frac{1}{4\pi}\left(\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} + \mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}\right), \tag{46}$$

and the rate at which momentum is transferred to the charges,

$$\begin{aligned}(\rho\mathbf{E} + \mathbf{J}\times\mathbf{B})_k = &-\frac{\partial}{\partial t}\left(\frac{1}{4\pi}\mathbf{D}\times\mathbf{B}\right)_k \\ &-\frac{1}{4\pi}[D_i\partial_k E_i - \partial_i(D_i E_k)] \\ &-\frac{1}{4\pi}[B_i\partial_k H_i - \partial_i(B_i H_k)]. \end{aligned}\tag{47}$$

Using the constitutive relations in Eq. (8), we recognize from Eq. (46) the energy flux $\mathbf{S}$ and the energy density U as

$$\mathbf{S} = \frac{1}{4\pi}\mathbf{E}\times\mathbf{B}, \qquad U = \frac{1}{8\pi}(\mathbf{E}^2 + \mathbf{B}^2), \tag{48}$$

while from Eq. (47) we obtain the momentum density $\mathbf{G}$ and we identify the stress tensor T_{ij} as

$$\begin{aligned}\mathbf{G} &= \frac{1}{4\pi}\mathbf{E}\times\mathbf{B}, \\ T_{ij} &= \frac{1}{8\pi}(\mathbf{E}^2 + \mathbf{B}^2)\delta_{ij} - \frac{1}{4\pi}(E_iE_j + B_iB_j).\end{aligned}\tag{49}$$

Outside the free sources, the conservation equations read

$$\nabla\cdot\mathbf{S} + \frac{\partial U}{\partial t} = 0, \qquad \frac{\partial G_k}{\partial t} + \partial_i T_{ik} = \frac{\alpha}{\pi}(E_iB_i)\partial_k\theta(z). \tag{50}$$

In other words, the stress tensor has the same form as in vacuum, but, as expected, it is not conserved on the θ

boundary because of the self-induced charge and current densities arising there.

Thus, the required expression for $T_{zz}(\Sigma^+)$ in Eq. (45) is the standard one,

$$T_{zz} = \frac{1}{8\pi}[E_\parallel^2 - E_z^2 + B_\parallel^2 - B_z^2], \tag{51}$$

where E_z (B_z) denotes the electric (magnetic) field component normal to the surface and $E_\parallel$ ($B_\parallel$) is the component of the electric (magnetic) field parallel to the surface. According to our results in the previous section, the electric and magnetic fields for $z > 0$ are

$$\mathbf{E}(\mathbf{x}) = q\frac{\mathbf{x}-\mathbf{r}}{|\mathbf{x}-\mathbf{r}|^3} - q\frac{\tilde{\theta}^2}{4+\tilde{\theta}^2}\frac{\mathbf{x}+\mathbf{r}}{|\mathbf{x}+\mathbf{r}|^3}, \tag{52}$$

$$\mathbf{B}(\mathbf{x}) = \frac{2q\tilde{\theta}}{4+\tilde{\theta}^2}\frac{\mathbf{x}+\mathbf{r}}{|\mathbf{x}+\mathbf{r}|^3}. \tag{53}$$

Thus, we find

$$\begin{aligned}\mathbf{F} &= \frac{1}{4}\frac{q^2}{(4+\tilde{\theta}^2)^2}\hat{\mathbf{e}}_z\int_0^\infty dR\frac{R}{(R^2+b^2)^3} \\ &\quad\times[16R^2 - (4+2\tilde{\theta}^2)^2 b^2 + 4\tilde{\theta}^2(R^2-b^2)] \\ &= -\frac{q^2}{(2b)^2}\frac{\tilde{\theta}^2}{4+\tilde{\theta}^2}\hat{\mathbf{e}}_z,\end{aligned}$$

in agreement with Eq. (44).

C. Infinitely straight current-carrying wire near a planar θ boundary

Let us now consider an infinitely straight wire parallel to the x axis and carrying a current I in the $+x$ direction. The wire is located in vacuum ($\theta_2 = 0$) at a distance b from an infinite topological medium with $\theta_1 \neq 0$ in the region $z < 0$. For simplicity we choose the coordinates such that $y' = 0$. Therefore, the current density is $j^\mu(\mathbf{x}') = I\eta^\mu{}_1\delta(y')\delta(z'-b)$.

The solution for this problem is given by

$$A^\mu(\mathbf{x}) = I\int_{-\infty}^{+\infty} G^\mu{}_1(\mathbf{x},\mathbf{r})dx', \tag{54}$$

where $\mathbf{x}-\mathbf{r} = (x-x')\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + (|z|+b)\hat{\mathbf{e}}_z$. Clearly the nonzero component $A^0(\mathbf{x})$ arising from the GF implies that an electric field is induced. The required component of the GF, $G^0{}_1$, defined in Eq. (30) is given by

$$G^0{}_1(\mathbf{x},\mathbf{r}) = -\frac{2\tilde{\theta}}{4+\tilde{\theta}^2}\frac{y}{R^2}\left[1 - \frac{|z|+b}{\sqrt{R^2+(|z|+b)^2}}\right]. \tag{55}$$

Substituting Eq. (55) into Eq. (54) yields the electric potential, which lacks an immediate interpretation. We can directly compute the electric field as $\mathbf{E}(\mathbf{x}) = -\nabla A^0(\mathbf{x})$, with the result

$$\mathbf{E}(\mathbf{x}) = \frac{4\tilde{\theta}I}{4+\tilde{\theta}^2}\left[\frac{|z|+b}{y^2+(|z|+b)^2}\hat{\mathbf{e}}_y - \frac{y\,\mathrm{sign}(z)}{y^2+(|z|+b)^2}\hat{\mathbf{e}}_z\right]. \tag{56}$$

We observe that the electric field for $z > 0$ is that due to a magnetic current located at $z' = -b$, $\mathbf{j}_{m,>} = -4\tilde{\theta}I/(4+\tilde{\theta}^2)\hat{\mathbf{e}}_x$. For $z < 0$ we have a magnetic current located at $z' = b$ of the same strength $\mathbf{j}_{m,<} = -\mathbf{j}_{m,>}$. Note that $\mathbf{j}_{m,>}$ is antiparallel to the current of the wire, while $\mathbf{j}_{m,<}$ is parallel.

Similarly, we compute the magnetic field. This is

$$\mathbf{B}(\mathbf{x}) = \nabla\times\left[I\hat{\mathbf{e}}_i\int_{-\infty}^{+\infty}G^i{}_1(\mathbf{x},\mathbf{r})dx'\right], \tag{57}$$

with $i = 1, 2$, where the corresponding GF are given by Eqs. (31). The result is

$$\begin{aligned}\mathbf{B}(\mathbf{x}) &= 2I\,\mathrm{sign}(z)\left[-\frac{|b-z|}{y^2+(b-z)^2}\right. \\ &\quad\left.+\frac{\tilde{\theta}^2}{4+\tilde{\theta}^2}\frac{(|z|+b)}{y^2+(|z|+b)^2}\right]\hat{\mathbf{e}}_y \\ &\quad+2Iy\left[\frac{1}{y^2+(b-z)^2} - \frac{\tilde{\theta}^2}{4+\tilde{\theta}^2}\frac{1}{y^2+(|z|+b)^2}\right]\hat{\mathbf{e}}_z.\end{aligned} \tag{58}$$

For $z > 0$ the magnetic field corresponds to the one produced by an image electric current located at $z' = -b$, flowing in the opposite direction to the current of the wire, $\mathbf{j}_{e,>} = -2\tilde{\theta}I/(4+\tilde{\theta}^2)\hat{\mathbf{e}}_x$. For $z < 0$ we have an electric current located at $z' = b$ of the same strength and flowing in the same direction of the current in the wire.

V. SUMMARY AND OUTLOOK

Classical electrodynamics is a fascinating field theory on which a plethora of technological devices rely. Advances in our theoretical understanding ignite new technological developments, and sometimes new discoveries demand extending the limits of theories that lead to them. Chern-Simons forms and topologically ordered materials are a good example of the above. In this work we study a particular kind of Chern-Simons extension to electrodynamics that consists of Maxwell Lagrangian supplemented by a parity-violating Pontryagin invariant coupled to a scalar field θ, restricted to the case where θ is piecewise constant in different regions of space separated by a common interface Σ.

It is well known that in this scenario the field equations in the bulk remain the standard Maxwell equations, but the discontinuity of θ alters the behavior of the fields at the interface Σ, giving rise to effects such as: (i) Induced effective charge and currents at Σ that are determined by the fields at the interface, (ii) Electric charges near a planar θ boundary induce magnetic mirror monopoles (and vice versa) and (iii) Nontrivial additional Faraday- and Kerr-like rotation of the plane of polarization of electromagnetic waves traversing the interface Σ.

Here we focus on the GF method applied for the static case in θ electrodynamics. The method is illustrated by the case of a planar θ interface, where the corresponding GF is calculated. The integral equation which defines the GF becomes an algebraic equation due to the delta interaction arising in the θ boundary plus the symmetries present in the parallel directions. We show how to compute the electromagnetic fields, on either side of the interface from the GF. Next we compute the force between a pointlike charge and a topological insulator. To this end we use the GF to compute the interaction energy between a charge-current distribution and a θ boundary that mimics the topological insulator, with vacuum energy removed. It can be shown that the above leads to the same interaction force as that computed by momentum flux perpendicular to the interface, for which the energy-momentum tensor and ensuing conservation laws of θ electrodynamics were analyzed. Finally, we use the GF to obtain the electromagnetic fields for an infinitely straight current-carrying wire parallel to the interface.

For the case of the pointlike charge in front of the θ interface, our results allow us to interpret the fields as those produced by the charge, its image, an induced magnetic monopole, and a circulating current density at the interface, in agreement with previously existing results. Similarly, the fields produced by the infinitely straight current-carrying wire and the θ boundary can be interpreted in terms of electric and magnetic current densities.

Let us emphasize that for a given θ boundary, the fields produced by arbitrary external sources can be calculated once the GF is known. Our method can be applied to a broader kind of geometries determined by the θ boundary. In fact, we can provide the GF for the spherical and the cylindrical cases [33]. Given that our results depend on $\tilde{\theta} = \alpha(\theta_1 - \theta_2)/\pi$, it is worth mentioning that they satisfy the quantum-mechanical periodicity condition $\theta \to \theta + 2\pi n$, with $\theta = 0, \pi$.

The GF method should also be useful for the extension to the dynamic case. In this respect, to our knowledge, little effort has been made in the context of topological insulators. Furthermore, GFs are also relevant for the computation of other effects, such as the Casimir effect. Therefore, we expect our method and results will be of considerable relevance and that they may constitute the basis for numerous other studies.

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APPENDIX: GF FOR PLANAR CONFIGURATION IN COORDINATE REPRESENTATION

Here we derive Eqs. (29)–(31) by computing explicitly the Fourier transform of the reduced GF, whose formula we take from (27). In the standard case ($\tilde{\theta} = 0$), the reduced vacuum GF is [32]

$$\mathfrak{g}(z, z') = \frac{1}{2p} e^{-p|z-z'|}. \tag{A1}$$

In the coordinate representation, the corresponding GF is obtained by Fourier transforming (A1) as defined in Eq. (15),

$$\mathcal{G}(\mathbf{x}, \mathbf{x}') = 4\pi \int \frac{d^2\mathbf{p}}{(2\pi)^2} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')_\parallel} \frac{1}{2p} e^{-p|z-z'|}. \tag{A2}$$

This double integral becomes easier to perform if we express the area element in polar coordinates, $d^2\mathbf{p} = p dp d\varphi$ (instead of the Cartesian ones), and choose the p_x axis in the direction of the vector $\mathbf{R} = (\mathbf{x} - \mathbf{x}')_\parallel$, as shown in Fig. 2. Noting that $\mathbf{p} \cdot \mathbf{R} = pR\cos\varphi$, we can write

$$\mathcal{G}(\mathbf{x}, \mathbf{x}') = \int_0^\infty dp e^{-p|z-z'|} \left\{ \frac{1}{2\pi} \int_0^{2\pi} e^{ipR\cos\varphi} d\varphi \right\}, \tag{A3}$$

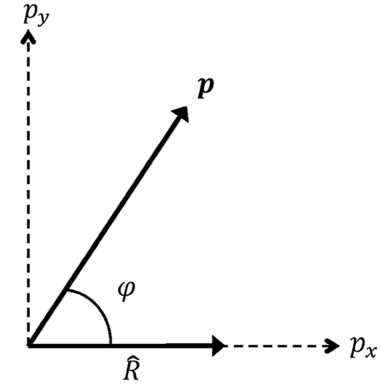

FIG. 2. The $\mathbf{p}$ plane.

where $R = |(\mathbf{x} - \mathbf{x}')_{\parallel}|$. The braces in this equation enclose an integral representation of the Bessel function $J_0(pR)$. The resulting integral,

$$\mathcal{G}(\mathbf{x}, \mathbf{x}') = \int_0^\infty J_0(pR) e^{-p|z-z'|} dp, \quad (A4)$$

is well known; see, for example, Ref. [34]. The final result is

$$\mathcal{G}(\mathbf{x}, \mathbf{x}') = \frac{1}{\sqrt{R^2 + |z - z'|^2}} = \frac{1}{|\mathbf{x} - \mathbf{x}'|}, \quad (A5)$$

which is the vacuum GF in the coordinate representation [32]. In the following we use a similar procedure to compute the required integrals for establishing Eqs. (29) and (30). We first consider the component G^0_0. From Eq. (27) we find

$$g^0_{\ 0}(z, z') = \mathfrak{g}(z, z') + A(z, z') p^2 \tilde{\theta} \mathfrak{g}(a, a), \quad (A6)$$

where the function $A(z, z')$ is

$$A(z, z') = -\frac{\tilde{\theta}}{4 + \tilde{\theta}^2} p^{-2} e^{-pZ}, \quad (A7)$$

with the notation $Z = |z - a| + |z' - a|$. In this way, the component $G^0_{\ 0}$ is given by

$$G^0_{\ 0}(\mathbf{x}, \mathbf{x}') = \mathcal{G}(\mathbf{x}, \mathbf{x}') - \frac{\tilde{\theta}^2}{4 + \tilde{\theta}^2} \int_0^\infty J_0(pR) e^{-pZ} dp, \quad (A8)$$

in the coordinate representation. As before, we use the integral representation of the Bessel function $J_0(pR)$ to perform the angular integration. The resulting integral is the same as in (A4); thus, we obtain

$$G^0_{\ 0}(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} - \frac{\tilde{\theta}^2}{4 + \tilde{\theta}^2} \frac{1}{\sqrt{R^2 + Z^2}}. \quad (A9)$$

Now we evaluate the components $G^0_{\ 1}$ and $G^0_{\ 2}$. The corresponding reduced GF are

$$g^0_{\ i}(z, z') = -i \epsilon_{0ij3} p^j A(z, z'), \quad (A10)$$

with $A(z, z')$ given by (A7). For convenience we define the vector

$$\mathbf{I}(\mathbf{x}, \mathbf{x}') = (I^1, I^2) = 4\pi \int \frac{d^2\mathbf{p}}{(2\pi)^2} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')_{\parallel}} \mathbf{p} p^{-2} e^{-pZ}, \quad (A11)$$

with $\mathbf{p} = (p_x, p_y)$, in terms of which we have

$$G^0_{\ i}(\mathbf{x}, \mathbf{x}') = i \frac{\tilde{\theta}}{4 + \tilde{\theta}^2} \epsilon_{0ij3} I^j(\mathbf{x}, \mathbf{x}'). \quad (A12)$$

We calculate the integral (A11) in the same coordinate system as before (see Fig. 2), and then we rewrite the result in a vector form. The integral can be written as

$$\mathbf{I}_p(\mathbf{x}, \mathbf{x}') = 2 \int_0^\infty dp e^{-pZ} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix} e^{ipR\cos\varphi} d\varphi \right\}, \quad (A13)$$

where the subscript p indicates that the vector $\mathbf{p}$ is written in the particular coordinate system of Fig. 2. Both the required angular and radial integrals are well known, and the result is

$$\begin{aligned} \mathbf{I}_p(\mathbf{x}, \mathbf{x}') &= 2i\hat{\mathbf{R}} \int_0^\infty J_1(pR) e^{-pZ} dp \\ &= \frac{2i}{R} \left(1 - \frac{Z}{\sqrt{R^2 + Z^2}} \right) \hat{\mathbf{R}}. \end{aligned} \quad (A14)$$

As a consequence of the chosen coordinate system, we find that $I_2 = 0$, in such a way that the vector $\mathbf{I}_p$ becomes parallel to $\hat{\mathbf{R}}$. However, this can be generalized in a direct way to an arbitrary coordinate system as

$$\mathbf{I}(\mathbf{x}, \mathbf{x}') = 2i \frac{\mathbf{R}}{R^2} \left(1 - \frac{Z}{\sqrt{R^2 + Z^2}} \right). \quad (A15)$$

Thus, we find

$$G^0_{\ i}(\mathbf{x}, \mathbf{x}') = -\frac{2\tilde{\theta}}{4 + \tilde{\theta}^2} \frac{\epsilon_{0ij3} R^j}{R^2} \left(1 - \frac{Z}{\sqrt{R^2 + Z^2}} \right). \quad (A16)$$

In order to evaluate the components $G^i_{\ j}$, we first observe that the corresponding reduced GF can be written as

$$g^i_{\ j}(z, z') = \eta^i_{\ j} g^0_{\ 0}(z, z') + \tilde{\theta} \mathfrak{g}(a, a) A(z, z') p^i p_j, \quad (A17)$$

where $g^0_{\ 0}$ is given by Eq. (A6). Now we need to compute the Fourier transformation of Eq. (A17) as defined in Eq. (15). However, the first term was studied before, and the result is given by Eq. (A9), thus leading us to study only the last term. To this end we introduce the vector

$$\mathbf{K}(\mathbf{x}, \mathbf{x}') = (K^1, K^2) = 4\pi \int \frac{d^2\mathbf{p}}{(2\pi)^2} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')_{\parallel}} \frac{\mathbf{p}}{p} p^{-2} e^{-pZ}, \quad (A18)$$

from which the required integral will be calculated by taking the spatial derivative. The integral (A18) can be computed again in the particular coordinate system of Fig. 2. In the polar coordinates defined in the $\mathbf{p}$ plane, the integral reads

$$\mathbf{K}_p(\mathbf{x},\mathbf{x}') = 2\int_0^\infty \frac{dp}{p} e^{-pZ}\left\{\frac{1}{2\pi}\int_0^{2\pi}\begin{bmatrix}\cos\varphi\\ \sin\varphi\end{bmatrix} e^{ipR\cos\varphi}d\varphi\right\}. \tag{A19}$$

Note that the braces in this equation enclose an integral representation of the Bessel function $J_1(pR)$. The resulting integral is well known, and the final result is

$$\mathbf{K}_p(\mathbf{x},\mathbf{x}') = 2i\int_0^\infty \frac{dp}{p} J_1(pR)e^{-pZ}\hat{\mathbf{R}} = 2i\frac{\sqrt{R^2+Z^2}-Z}{R}\hat{\mathbf{R}}, \tag{A20}$$

where $\hat{\mathbf{R}}$ is the unit vector shown in Fig. 2. The generalization to an arbitrary coordinate system is then

$$\mathbf{K}(\mathbf{x},\mathbf{x}') = 2i\frac{\sqrt{R^2+Z^2}-Z}{R^2}\mathbf{R}. \tag{A21}$$

Note that the required integral involves the term $p^i p_j$, which can be generated from (A18) as follows:

$$i\partial_j K^i(\mathbf{x},\mathbf{x}') = 4\pi\int \frac{d^2\mathbf{p}}{(2\pi)^2} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')_\parallel}\frac{p^i p_j}{p} p^{-2}e^{-pZ}. \tag{A22}$$

By using the final form of $\mathbf{K}(\mathbf{x},\mathbf{x}')$, given by Eq. (A21), one can further check the consistency condition $\partial_1 K^2(\mathbf{x},\mathbf{x}') = \partial_2 K^1(\mathbf{x},\mathbf{x}')$ required by the cross terms involving $p^1p_2 = p^2p_1 = -p_xp_y$. From the previous results, the G^i_j components of the GF matrix in the coordinate representation can be written as

$$G^i{}_j(\mathbf{x},\mathbf{x}') = \eta^i{}_j G^0{}_0(\mathbf{x},\mathbf{x}') - \frac{i}{2}\frac{\tilde{\theta}^2}{4+\tilde{\theta}^2}\partial_j K^i(\mathbf{x},\mathbf{x}'). \tag{A23}$$

These results establish Eq. (31).

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