Penrose Superradiance in Nonlinear optics

Group 6

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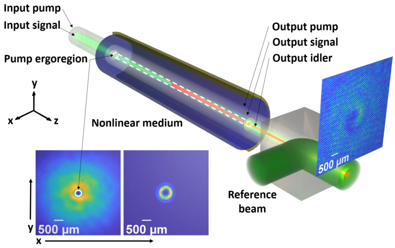
Abstract

Penrose Superradiance is a process which describes how scattered waves get amplified by extracting the rotational energy of a rotating Kerr black hole. In this paper, an analogous system in nonlinear optics is proposed for which similar effects to Penrose superradiance is shown. In this system, we consider a strong vortex pump beam in a nonlinear defocusing medium. A signal, which acts as a perturbation to the system experiences gain or amplification as it reflects-off the pump vortex. Amplification occurs only due to generation and trapping of negative energy modes in the core of the pump vortex.



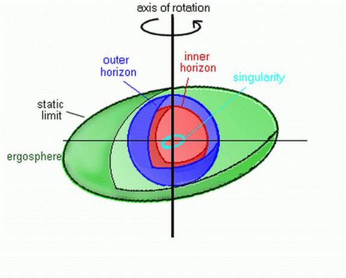


Figure: (a) Nonlinear optics setup (b) Ergoregion of a Kerr black hole

The paraxial wave equation is given by

$$
\begin{aligned}
2ik\frac{\partial E}{\partial z} + \nabla_\perp^2 E &= -\frac{\omega^2}{\epsilon_0 c^2} P_{NL} \\
&= -3\frac{\omega^2}{c^2}\chi^{(3)}|E|^2 E \\
&= -2k_0^2 n_0 n_2 |E|^2 E
\end{aligned}
$$

where n_0 is the linear refractive index, $k = 2\pi n_0/\lambda = k_0 n_0$, $\nabla_\perp^2$ is the transverse Laplacian which accounts for optical diffraction, and we consider a defocusing medium with nonlinear coefficient $n_2 < 0$. Hence we get the NLSE to be

$$
i\frac{\partial E}{\partial z} + \frac{1}{2k}\nabla_\perp^2 E + k_0 n_2 |E|^2 E = 0
$$

The 3D-NLSE describes the propagation of an optical beam through bulk nonlinear media in the presence of both nonlinearity and diffraction in two transverse dimensions.

Derivation of NLSE and pump vortex solution

The vortex pump solution to the NLSE is of the form

$$E(r,\theta,z) = \sqrt{I_\ell}\, u_\ell(r) e^{i(\beta_\ell z+\ell\theta)}$$

where I_l is the background intensity of the vortex of orbital angular momentum (OAM) l, $u_l(r) = \tanh^{|\ell|}(r/W)$ is the corresponding vortex profile which has a core size denoted r_l, and $\beta_l = k_0 n_2 I_l < 0$, where W is the vortex width.

Figure: An optical beam with OAM = +1

Signal and Idler beam differential equations

In the presence of a strong pump $\mathcal{E}_0$ with OAM ℓ and a weak externally applied signal field $\mathcal{E}_s$ of OAM n ($I_\ell >> I_n, I_q$), the total field can be written as

$$E(r,\theta,z) = \left[\mathcal{E}_0(r) + \mathcal{E}_s(r,z)e^{i(n-\ell)\theta} + \mathcal{E}_i(r,z)e^{-i(n-\ell)\theta}\right] e^{i(\beta_\ell z+\ell\theta)} \tag{1}$$

We now substitute the expression for total field in the NLSE upto first order in E_i, E_s and separate terms based on their differing OAM to obtain the differential equations for the signal and idler fields

$$\frac{\partial \mathcal{E}_s}{\partial z} = \frac{i}{2k}\nabla_n^2\mathcal{E}_s + i\beta_\ell u_\ell^2(r)\left[2\mathcal{E}_s + \mathcal{E}_i^*\right] - i\beta_\ell\mathcal{E}_s \tag{2}$$

$$\frac{\partial \mathcal{E}_i}{\partial z} = \frac{i}{2k}\nabla_q^2\mathcal{E}_i + i\beta_\ell u_\ell^2(r)\left[2\mathcal{E}_i + \mathcal{E}_s^*\right] - i\beta_\ell\mathcal{E}_i \tag{3}$$

Photon fluid

The propagation of a monochromatic laser beam in a self-defocusing medium can be described at steady state by the NLSE

$$\frac{\partial E}{\partial z} = i\frac{1}{2k}\nabla_\perp^2 E + i\frac{kn_2}{n_0}|E|^2 E$$

Taking the direction of propagation z as time coordinate $t = zn_0/c$, where c is the speed of light in vacuum and writing field $E = E_{bg}e^{i\phi}$ where $|E_{bg}|^2$ is the optical background intensity in the NLSE we get

$$\frac{n_0}{c}\partial_t E_{bg}e^{i\phi} = \frac{i}{2k}\nabla_\perp^2 E_{bg}e^{i\phi} + i\frac{kn_2}{n_0}E_{bg}^3 e^{i\phi}$$

Separating the real and imaginary terms, we get

$$\frac{n_0}{c}\partial_t E_{bg} = -\frac{1}{2k}\nabla_\perp(E_{bg}\nabla_\perp\phi) - \frac{1}{2k}\nabla_\perp E_{bg}\nabla_\perp\phi$$

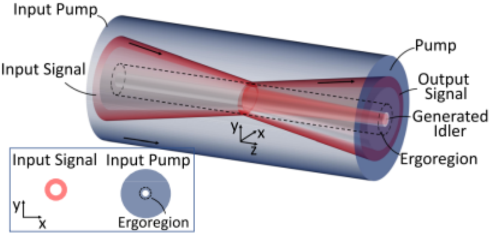
$$\frac{n_0}{c}E_{bg}\partial_t\phi = \frac{1}{2k}\nabla_\perp^2 E_{bg} - \frac{1}{2k}E_{bg}(\nabla_\perp\phi)^2 + \frac{kn_2}{n_0}E_{bg}^3$$

Now writing $|E_{bg}|^2 = \rho$ and $\mathbf{v} = (c/kn_0)\nabla\phi = \nabla\psi$ where $\psi = (c/kn_0)\phi$ in the above pair of equations we get

$$\partial_t \rho + \nabla(\rho v) = 0$$
$$\partial_t \psi + \frac{1}{2}v^2 - \frac{c^2 n_2}{n_0^3}\rho - \frac{c^2}{2k^2 n_0^2}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} = 0$$

As these equations resemble the hydrodynamical equations they effectively describe the laser beam as a 2+1-D quantum fluid of light or a photon fluid. Therefore in this mapping the optical intensity $|E_{bg}|^2$ is equivalent to a fluid density ρ, $\mathbf{v} = (c/kn_0)\nabla\phi$ to a fluid velocity, and $v_s^2 = c^2|n_2|\rho/n_0^3 = c^2|n_2||E_{bg}|^2/n_0^3 = c^2|\Delta n|/n_0^3$ is the squared sound speed.

Ergoregion



The ergoregion is defined as the region in the transverse (x, y) plane where $v > v_s$. In the case where we have only the pump present $(\phi = \ell\theta)$ the fluid velocity $v = (c/kn_0)\nabla\phi = (c/kn_0)\nabla(\ell\theta) = (c/kn_0)(\ell/r)$. Now if we introduce a weak signal beam which we shall treat as a perturbation the effective OAM of the pump becomes $|n - \ell|$ with respect to the signal. Hence, $v = (c/kn_0)(|n - \ell|/r)$. The ergoradius r_e is defined as the radius at which $v = v_s$ giving

$$r_e = (|n - \ell|/k)\sqrt{n_0/|\Delta n|}$$

Positive and negative energy modes

Penrose superradiance is based on the concept of positive and negative energy modes. As negative energy modes remain trapped within the ergoregion, allowing the positive energy modes to escape with additional energy. The idler and signal beams have associated with them a conserved quantity $N(z)$ which arises from the Noether current $j^0 = |E_s|^2 - |E_i|^2$, such that

$$N(z) = \int_0^{\infty} \left(|E_s|^2 - |E_i|^2\right) rdr = \text{ const.}$$

Using this we can now define reflection and transmission coefficients as $R(z) = \int_{r_e}^{\infty} \left(|E_s|^2 - |E_i|^2\right) rdr$ and $T(z) = \int_0^{r_e} \left(|E_s|^2 - |E_i|^2\right) rdr$.
Inside the ergoregion we find that the idler wave gets trapped and the the signal beam gets scattered outwards. This means that the intensity of idler beam will be greater than that of the signal beam and thus $j^0 < 0$ inside the ergoregion (or scattering region). In the case of superradiance $R(z) > 1$ implying that the scattered signal beam is amplified in this region. This balances the effect of the negative current as $j^0 > 0$ outside the ergoregion $r > r_e$.

Laguerre Gauss modes of signal beam

The signal wave is given by using Laguerre Gauss modes. The total wave equation is given by

$$\mathcal{E}_s(r,z) \approx c_s V_n(r,z) e^{-i(1+|n|)\phi_G(z)} e^{2i\beta_\ell \Gamma_n(z) z - i\beta_\ell z}$$

where $V_n(r,z)$ is the normalized z-dependent LG mode profile with radial mode index $p = 0$. $\phi_G(z) = \tan^{-1}(z/z_0)$ is the Gouy phase-shift at the focus with Rayleigh range defined as $z_0 = kw_0^2/2$ and $\Gamma_n(z) = \int_0^\infty 2\pi r dr \, |V_n(r,z)|^2 \, u_\ell^2(r)$ is the signal phase variation induced by the pump core on the signal. Considering the signal wave vector (nonlinear) shift as

$$\Delta K_s \approx \Delta K_s(0) = 2\beta_\ell \Gamma_n(0) - \beta_\ell,$$

since most of the nonlinear interaction will occur within a Rayleigh range around the beam focus at $z = 0$.
We derive radius at peak intensity as $r_n = w_0\sqrt{|n|/2}$. We require $r_n \approx r_e$ in order for the signal LG ring beam to glance off the ergosphere at it goes through its focus.

Signal beam plots

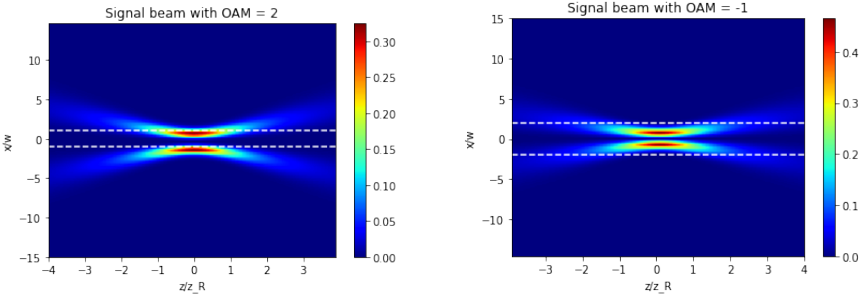


Figure: Signal beams with different OAMs at y=0

The idler beam

We consider the following ansatz to solve the differential equation

$$\mathcal{E}_i(r,z) = c_i U_{pq}(r) e^{i\left(\beta_\ell + \Lambda_{pq}\right)z}$$

We see that the wave vector shift of the idler wave can be defined as $\Delta K_i = \beta_\ell + \Lambda_{pq}$. Substituting the ansatz, we get,

$$\left[\frac{i}{2k}\nabla_q^2 + 2i\beta_l\left(u_l^2 - 1\right)\right] U_{pq} = \Lambda_{pq} U_{pq}$$

We arrive at an eigenproblem with eigenvalues Λ_{pq} and eigenfunctions U_{pq} with radial mode index p. We consider the lowest radial mode index $p = 0$. We use the LG mode solution of signal beam and the solution of U_q as described above to arrive at the following differential equation of c_i

$$\frac{dc_i}{dz} = ic_s^* \beta_\ell F(z) e^{-i[2\Delta K z - (1+|n|)\phi_G(z)]}$$

Here,

$$2\Delta K = 2\beta_\ell \Gamma(0) + \Lambda_{pq} = (2\beta_\ell \Gamma(0) - \beta_\ell) + (\beta_\ell + \Lambda_{pq}) = \Delta K_s + \Delta K_i$$

$$F(z) = \int_0^\infty 2\pi r dr V_n^*(r,z) u_\ell^2(r) U_q^*(r)$$

ΔK is the average wave vector shift of perturbations due to signal and idler waves

Zel'dovich Misner condition

If we consider the Guoy phase-shift term to be zero, we get the phase matching condition to be $\Delta K = 0$. Otherwise, at the vicinity of the origin (and positive z) we know that $\phi_G(z)$ will be positive and hence, the phase matching condition requires $\Delta K > 0$.

Taking the longitudinal wave vector shifts for the signal (s) and idler (i) fields as $\Delta K^{s,i} = k^{s,i} - \beta_\ell$ and the corresponding effective frequency shifts as $\Delta\omega_{s,i} = \omega_{s,i}\omega_p = c\Delta K_z^{s,i}/n_0$ with respect to the pump frequency $\omega_p = -c\beta_\ell/n_0$ (analogous to phonons in a 2D fluid) and we define $\Delta\omega = [(\Delta\omega_s + \Delta\omega_i)/2] = -(c/n_0)\,\Delta$ which is negative for $\Delta K > 0$. We note that $\Delta\omega = (\omega - \omega_p) = (\omega - m\Omega)$, with $m = (n - \ell)$.

So the condition $\Delta K > 0$ can also be written as $(\omega - m\Omega) < 0$ which is analogous to Zel'dovich-Misner condition for Penrose superradiance.

Zel'dovich Misner condition

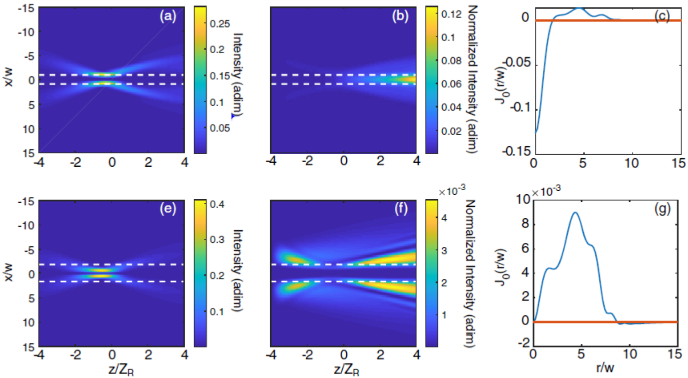


Figure: Idler, signal beams, Noether current (Images taken from paper)

Split Step method
Also commonly known as the beam propagation method in optics, this is a pseudo spectral method to solve nonlinear PDEs such as the Schrodinger equation, where linear and nonlinear parts are treated separately. Typically, Fourier transform is used as the linear step is made in the frequency domain while the nonlinear step is made in the time domain.
For the transverse transverse Laplacian in cylindrical coordinates (linear term of the NLSE), we use Hankel transforms to implement this algorithm.

We first propagate only the pump through the NLSE and obtain a solution. Then, we propagate both the pump and signal beams together and subtract the original solution. After performing an LG decomposition of this, we obtain idler and signal waves separately.

Conclusion

By making a physical connection between the conditions dictated by Penrose for rotating black holes and the optical case, we were able to identify the interaction conditions that transform the interaction from lossy or no amplification to one with amplification up to 50%. The results imply a new amplification regime in nonlinear optics that is tightly connected to the trapping and spatial separation of the idler beam that leads to a transient gain in the signal beam. These results pave the way toward future experiments on superradiant amplification in nonlinear optics and a deeper understanding of the fundamental physics and transient dynamics of Penrose superradiance.

The End

Questions? Comments?

Rewriting the NLSE with the above mapping we get,

$$\mathrm{i}\partial_t E + \alpha\nabla^2 E + \beta|E|^2 E = 0$$

where $\alpha = c/2n_0 k$ and $\beta = ckn_2/n_0^2$. Introducing small perturbations ψ on top of the background beam E_{bg} of the form $E = E_{bg}(1+\psi)$ and substituting back into the above NLSE we get the perturbation must satisfy we get the dispersion relation

$$(\omega - \mathbf{k}\cdot\mathbf{v})^2 - \alpha^2\mathbf{k}^4 - 2\alpha\beta\rho\mathbf{k}^2 = 0$$

Observe that for long wavelength modes we can ignore fourth order terms to get $\omega^2 = 2\alpha\beta\rho\mathbf{k}^2$, hence $\omega/k = \sqrt{2\alpha\beta\rho} = c/n\sqrt{|\Delta n|/n_0} = v_s$ i.e. these modes correspond to the constant sound speed in the photon fluid. (Here small oscillations on the transverse beam profile are analogously referred to as effective sound modes in the photon fluid)