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Day-3

Energy & Power signal.

Energy of a signal

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

Energy of aperiodic signal

Average power of aperiodic signal over fundamental period

$$P_{avg} = \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$

Average power of aperiodic signal.

$$P_{avg} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$

For discrete time sequence $x[n]$.

Energy of Sequence $x[n]$

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

Average power of aperiodic sequence $x[n]$

$$P_{avg} = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

Average Power of Periodic Sequence over one fundamental period

$$P_{avg} = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2$$

* A Signal / sequence is said to be $\boxed{\text{power type}}$ if its average power is finite & energy is infinite.

i.e. $0 < P_{avg} < \infty$.

$E \to \infty$.

eg. Most of periodic signals.

⇒ A signal is said to be $\boxed{\text{energy type}}$ if its total energy is finite and average power is 0.

i.e $0 < E < \infty$.

and

$P_{avg} = 0$.

eg. Most of aperiodic signals

Check energy & Power of Signal

1) unit step signal

$x[n] = u[n]$

$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases}$$

Solution:-

Energy of $x[n]$.

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$= \sum_{n=-\infty}^{\infty} [u[n])^2$$

$$= \sum_{n=0}^{\infty} 1.$$

$$=$$

$$= \infty$$

Average power of $x[n]$

$$P_{avg} = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |u[n]|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} 1.$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \times (N+1)$$

$$= \lim_{N\to\infty} \frac{1+1/N}{2+1/N}$$

$$= 1/2 \,//$$

Avg power finite & energy infinite.

So

This is power type signal.

ii) $x[n] = \delta[n]$ where $\delta[n] = \begin{cases} 1, & \text{for } n=0 \\ 0 & \text{for } n\neq 0 \end{cases}$

Now;

Energy of $x[n]$

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$= 1.$$

Average power,

$$P_{avg} = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} 1.$$

$$= 1/\infty$$

$$= 0$$

Since energy is finite & average power is 0.

The signal is Energy Type.

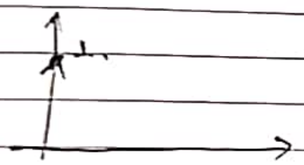
Basic Standard Signals:

will that work properly when & supply to customer or Not ?

i) Unit sample or delta function

For continuous Time.

$$\delta(t) = \begin{cases} 1 & \text{for } t = 0 \\ 0 & \text{for } t \neq 0 \end{cases}$$



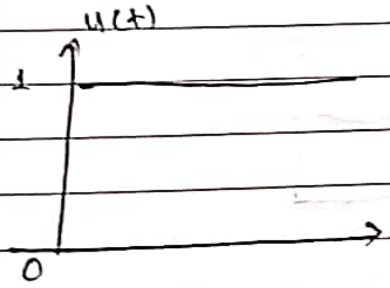
For discrete time

$$\delta[n] = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$

) Unit Step function.

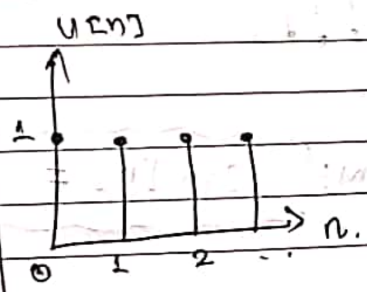
Continuous

$$u(t) = \begin{cases} 1, & \text{for } t \geq 0 \\ 0, & \text{for } t < 0 \end{cases}$$



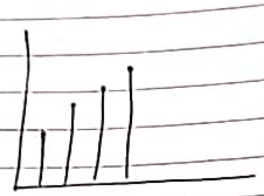
exactly $t=0$ ना
Continuous है |

$$u[n] = \begin{cases} 1, & \text{for } n \geq 0. \\ 0 & \text{for } n < 0 \end{cases}$$



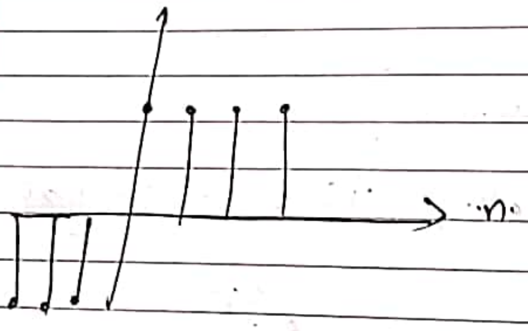
iii) Unit Ramp signal

sgn [n] = n for n >, 0
0 for n < 0



Signum function

Sgn [n] = 1 for n >, 0
-1 for n < 0.



KIM:- $\boxed{\text{Sgn}[n] = 2u[n] - 1}$

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SYSTEM

input $x[n]$ → Discrete time system → output $y[n]$.

$$y[n] = T\{x[n]\}.$$

Classification of system

1) Linear & Non linear system.

A Relaxed time System is said to be linear, if

~~input~~

If input $x_1[n] \rightarrow$ output $y_1[n] = T[x_1[n]]$

if input $x_2[n] \rightarrow$ output $y_2[n] = T\{x_2[n]\}$

then,

input $a_1 x_1[n] + a_2 x_2[n]$.

$$T[a_1 x_1[n] + a_2 x_2[n]\} = a_1 y_1[n] + a_2 y_2[n]$$

System using Superposition symmetry is called linear.

System which doesnt satisfy above condition is called non linear

1) Check linearity

$y[n] = x.[n^2]$.

$x[n]$ —— $\boxed{\text{System}}$ —— $y[n]$.

$y_1[n] = T[x_1[n]] = x_1[n^2]$.

$y_2[n] = T[x_2[n]] = x_2[n^2]$.

$y_3[n] = T[a_1x_1[n] + a_2x_2[n]]$.

$= a_1x_1[n^2] + a_2x_2[n^2]$.

$y_4[n] = a_1y_1[n] + a_2y_2[n]$.

$= a_1x_1[n^2] + a_2x_2[n]$.

Since, $y_3[n] = y_4[n]$,

Given System is linear

iii] $y[n] = x^2[n]$.

$x[n] \longrightarrow \boxed{\text{System}} \longrightarrow y[n]$.

$y_1[n] = T[x_1[n]] = x_1^2[n]$

$y_2[n] = T[x_2[n]] = x_2^2[n]$

$\cancel{y_3[n] = T[a_1x_1^2[n] + a_2x_2^2[n]]}$

$y_3[n] = T[a_1x_1[n] + a_2x_2[n]]$.

$= a_1^2x_1^2[n] + a_2^2x_2^2[n] + 2a_1a_2x_1[n]x_2[n]$.

Now

$y_4n = a_1y_1[n] + a_2y_2[n]$

$= a_1x_1^2[n] + a_2x_2^2[n]$.

Since

$y_3[n] \neq y_4[n]$ the given system is Non linear.

iii) $y[n] = Ax[n] + B$.

Soln

$y_1[n] = T[x_1[n]] = Ax_1[n] + B$.

$y_2[n] = T[x_2[n]] = Ax_2[n] + B$.

Now

$y_3[n] = T[a_1x_1[n] + a_2[x_2[n]]]$

$= A[a_1x_1[n] + a_2x_2[n]] + B$.

$= Aa_1x_1[n] + Aa_2x_2[n] + B$

$y_4[n] = a_1y_1[n] + a_2y_2[n]$

$= a_1[Ax_1[n] + B] + a_2[Ax_2[n] + B]$

$= a_1Ax_1[n] + Aa_2x_2[n] + B(a_1 + a_2)$.

Since.

$y_3[n] \neq y_4[n]$.

The given system is Non linear.

iii) $y[n] = Ax[n] + B$.

$x[n] \longrightarrow$ system $\longrightarrow y[n]$.

$y_1[n] = T\{x_1[n]\} = Ax_1[n] + B$.

$y_2[n] = T\{x_2[n]\} = Ax_2[n] + B$.

$y_3[n] = T[a_1x_1[n] + a_2x_2[n]]$

$= A[a_1x_1[n] + a_2x_2[n]] + B$.

$y_4[n] = a_1y_1[n] + a_2y_2[n]$

$= a_1Ax_1[n] + a_1B + a_2Ax_2[n] + a_2B$.

$y_3[n] \neq y_4[n]$.

Though system doesn't satisfy condition of linearity the system can't be said Non linear. because

for $x[n] = 0$, $y[n] = B$.

i.e system is Not Relaxed But this system can be said linear if $B = 0$.

ii) Memory and Memoryless System.

System is said to be memory less if its output $y(n)$ at $n = no$ only depends upon input $x[n]$ at $n = no$.

$$\text{eg} \quad y[n] = a x[n]$$

A system is said to be memory type if its output $y[n]$ at $n = no$ also depends upon the input $x[n]$ at $n \neq no$.

$$y[n] = y[n] + x[n-1]$$

iii) Casual & Non Casual signal.

System is said to be casual if its output $y[n]$ at $n = no$ only depends upon the input $x[n]$ at $n \leq no$.

A System is said to be Non Casual if its output $y[n]$ at $n = no$ also depends upon the input $x[n]$ at $n > no$.

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Check whether System are casual or Non casual, memory or Non memory Type.

i) $y[n] = a x[n]$

ii) $y[n] = x[n] + 2x[n+1]$ → memory, Non casual

iii) $y[n] = x[n] + x[n-1]$ → memory, casual

iv) $y[n] = \sum_{k=-\infty}^{n} x[k]$. → memory, casual

v) $y[n] = x[-n]$. → Memory. Non casual.

Sol^n i) $y[n] = a x[n]$ → memory less, casual.

ii) $y[n]$

KIM :

A memory less system are casual, but vice versa is Not True

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* Stable & unstable system :

System is said ~~to~~ to be BIBO stable
if bounded input produce output.
i.e if input $|x[n]| < K_1 < \infty$ produce $y[n]$
then
$|y[n]| < K_2 < \infty$.

A system ~~is~~ which doesnt satisfy above condition is called unstable

eg. $y[n] = x[n] + x[n-1]$.

let

$x[n] = u[n]$.
$x[0] = u[0] = 1 < \infty$ finite

then
$y[n] = u[n] + u[n-1]$.
$y[0] = u[0] + u[0-1]$

$= 1 + 1 = 2 < \infty$ finite

Determine system is BIBO stable or Not.

$y[n] = 1/3 \, [x[n] + x[n-1] + x[n-2]]$

Solution

Assume that,

$|x[n]| < M_x < \infty$ for all n.

Then

$y[n] = 1/3 \, [x[n] + x[n-1] + x[n-2]]$.

$\leq 1/3 \, [|x[n]| + x[n-1] + |x[n-2]|]$.

$\leq 1/3 \, [M_x + M_x + M_x]$.

$\leq M_x$ ✓

v) Time variant & Time invarient System

Conceptually System is said to be time invarient if its Characteristics & behaviour is fixed over time.

Mathematically,

System is said to be time invarient if input $x[n]$ produces output $y[n] = T\{x[n]\}$

input $x[n-k]$ produces output $y[n,k] = T\{x[n-k]\}$

then

$y[n,k] = y[n-k]$.

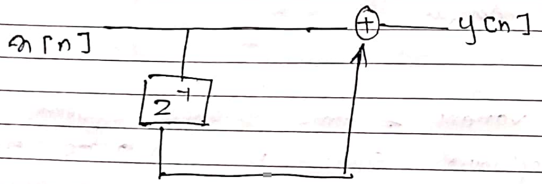
$x[n-k]$ —— System → $y[n,k]$

$x[n]$ $\qquad\qquad\qquad$ $y[n]$

System is called time varient if it doesn't satisfy above condition.

Example.

i) $y[n] = x[n] + x[n-1]$.



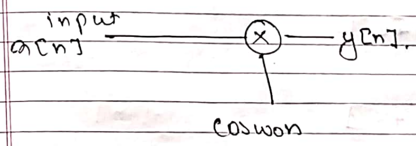
$y[n,k] = T[x[n-k]]$.

$= x[n-k] + x[n-k-1]$

$y[n-k] = x[n-k] + x[n-1-k]$.

Here, since $y[n,k] = y[n-k]$, given system is Time invarient.

ii) $y[n] = x[n] \cos w_0 n$.



$y[n,k] = T\{x[n-k]\}$

$= \cancel{x[n-k] \cdot \cos \cos w_0 [n-k]}$.

$= x[n-k] \cos w_0 n$

$y[n-k] = x[n-k] \cos w_0 (n-k)$

Since $y[n,k] \neq y[n-k]$, given system is Time varient

no time invarient.

$y[n] = x[-n]$.

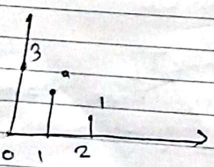
$x[n] \longrightarrow \boxed{\text{System}} \longrightarrow y[n] = x[-n]$.

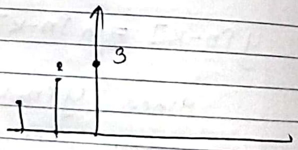
$$y[n,k] = T[x[n-k]]$$
$$= x[-[n-k]]$$
$$= x[-n+k].$$

$$y[n-k] = x[-n+k]$$

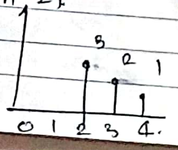
Since,
$y[n,k] \neq y[n-k]$.
The given system is ~~time~~ not time invariant

Practically [illegible]





$x[n-2]$



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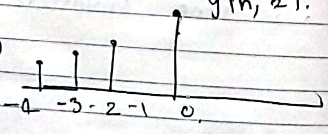
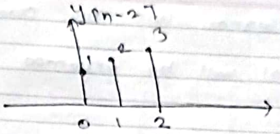


$y[n-2]$

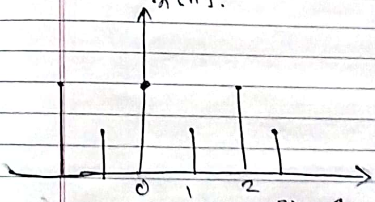


Operation on Signals:

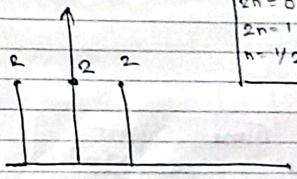
Operation on independent Parameter.

1) Time Scaling.
given
$x[n]$.
then $y[n] = x[k.n]$.
where k is integer multiple of n.

$x[n]$.

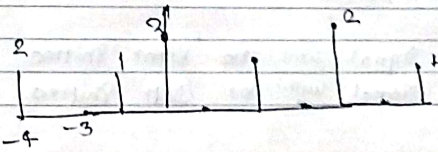


$x[2n]$



$2n = 0$
$2n = 1$
$n = 1/2$

$x[1/2\, n]$



for $K < 1$, signal will be compressed

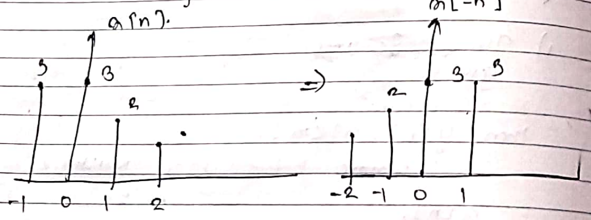
for $K > 1$, signal will be expanded

Time Reflection.

$x[n]$.

then $y[n] = x[-n]$.

Mirroring, folding



Time shifting.

Given $x[n]$.

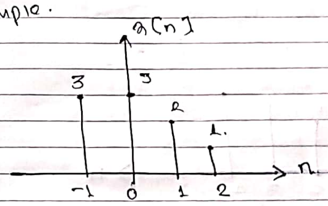
then.

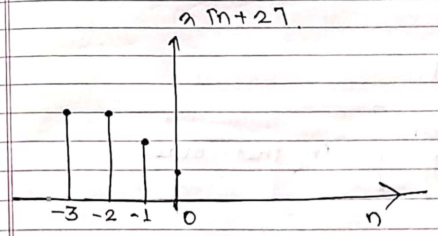
$y[n] = x[n - n_0]$.

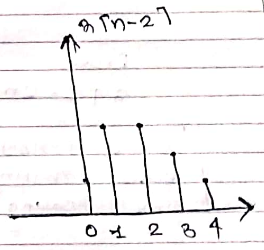
if $n_0 > 0$, signal will be Right shifted

if $n_0 < 0$, signal will be left shifted.

Example.



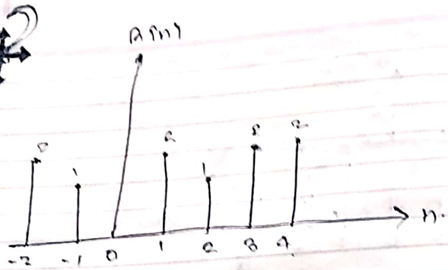




n	$n+2$
0	-2
1	-1
2	0
-1	-3

$n + 2 = 0$

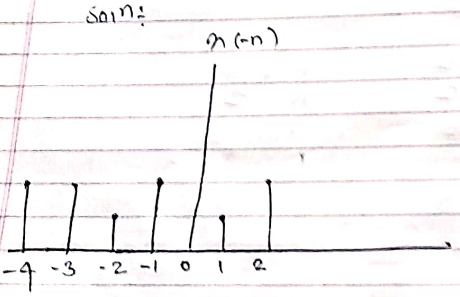
$n = -2$



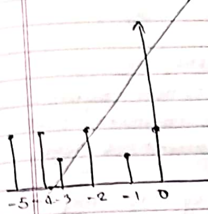
Draw
$x[2-2n]$.

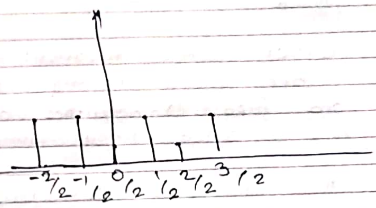
[Time reflection
Time shifting
Time scaling] in that order.

Soln:

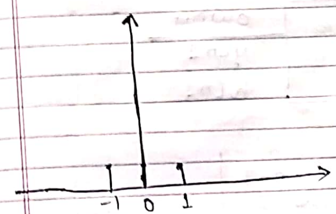


$x[2-n]$ येसलाई right shift हुन्छ





$x[2-2n]$



compression.

$2-2n=0$
$n=1$

$2-2n=-1$
$-2n=-3/2$

$2-2n=-2$
$-2n=-4/2$
$=2$

$2-2n=-4$
$=-3$

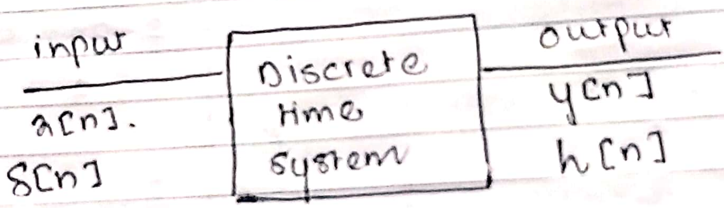
KTM

either graphically गर्ने OR सबै पुरै equation मा compare गरेर गर्ने।

Clossification of system Done.:-
Contd

:- Linear Time invarient system:
The system Having Superposition symmetry and fixed Characteristics and behaviour over time is called linear time invarient system.

it is completely characterized by its impulse Response:



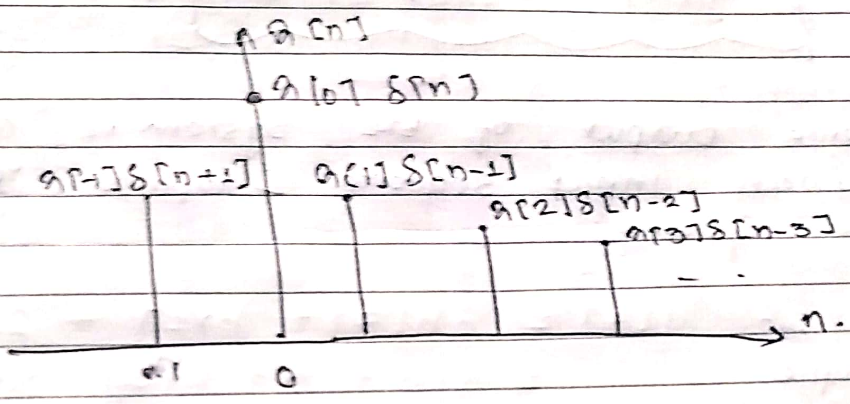
impulse response $h[n] = y[n]. \Big|_{x[n] = \delta[n]}$

Convolution sum of Descrete time LTI system.

let $x[n]$ be the discrete time input sequence to the system and can be Represented by superposition principle as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\, \delta[n-k]$$

where $x[k]$ is coefficient.
$\delta[n-k]$ is impulse occuring at $n=k$.



output of any discrete time system due to input.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\ h[n,k].$$

where $h[n,k]$ is response of system due to input $\delta[n-k]$.

Since $h[n]$ is the response due to input $\delta[n]$ and given system is time invarient,

$$h[n,k] = h[n-k].$$

So, output will be

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\ h[n-k]$$

$$y[n] = x[n] * h[n]$$

Hence output of LTI system is convolution between input sequence & impulse response of the system.

Example:
find output of LTI system having input sequence

$x[n] = \{3, 1, 2, 5\}$ and impulse response

$h[n] = \{1, 3, \underset{\uparrow}{2}, 4, 6\}$

Solution ⇒
Output of LTI system is given by

$$y[n] = x[n] * h[n].$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \ldots + x[-1]\, h[n+1] + x[0].\, h[n] + x[1]\, h[n-1] + x[2]\, h[n-2] + x[3]\, h[n-3]. +$$

$= 3h[n] + h[n-1] + 2h[n-2] + 5h[n-3]$

for $n=0$, $y[0] = 3h[0] + h[-1] + 2h[-2]$
$+ 5h[-3]$

$= 3\times2 + 3 + 2\times1 + 0$

$= 6+3+2$

$= 11.$

for $n=1$, $y[1] = 3h[1] + h[0] + 2h[-1] + 5h[-2].$

$= 3\times4 + 2 + 2\times3 + 5\times1$

$= 25$

for $n=2$, $y(2) = 3h[2] + h[1] + 0 + 5h[-1]$

$= 3\times6 + 1 + 0 + 5\times3$

$= 18 + 1 + 15$

$= 19 + 15$

$= 34$

for $n=3$, $y(3) = 3h[3] + h[2] + 2h[1] + 5$

$= 3\times0 + 2 + 2\times4$

$= 2+8$

$= 10.$

for $n=4$, $y(4) = 3h[4] + h[3] + 2h[2] +$

for $n=2$, $y[2] = 3h[2] + h[1] + 2h[0] + 5h[-1]$

$= 3\times 6 + 4 + 2\times 2 + 5\times 3$

$= 18 + 4 + 4 + 15$

$= 18 + 8 + 15 \quad 41$

$= \cancel{18+19}$

$= \cancel{37}.$

for $n=3$, $y[3] = 3h[3] + h[2] + 2h[1] + 5h[0]$

$= 3\times 0 + 6 + 2\times 4 + 5\times 2$

$= 16 + 8$

$= 24.$

for $n=4$, $y[4] = 3\times h[4] + h[3] + 2h[2] + 5h[1]$

$= 3\times 0 + 0 + 2\times 6 + 5\times 4$

$= 12 + 20$

$= 32$

for $n=-1$, $y[-1] = 3\times h[-1] + h[-2] + 2h[-3]$
$+ 5h[-4].$

$= 3\times 3 + 1 + 0 + 0$

$= 10.$

for $n = -2$, $y[-2] = 3 \times h[-2] + h[-3] + 2 \times h[-4]$
$+ 2 \times h[-5]$
$= 3 \times 1$
$= 3$

for $n = 5$, $y[5] = 3h[5] + h[4] + 2h[3] + 5h[2]$
$= 3 \times 0 + 0 + 2 \times 0 + 5 \times 6 = 30.$

$\therefore y[n] = [3, 10, \underset{\uparrow}{11}, 25, 41, 24, 32, 30]$

To Check the Answer.

		h[-2]	h[-1]	h[0]	h[1]	h[2]
x[n]		1	3	2	4	6
x[0]	3	3	9	6	12	18
x[1]	1	1	3	2	4	6
x[2]	2	2	6	4	8	12
x[3]	5	5	15	10	20	30

Diagonally Sum डालें 1

$y[n] = [3, 10, \underset{\uparrow}{11}, 25, 41, 24, 32, 30]$

H.W:

Find the output of LTI system.

i) $x[n] = \{1, 3, 5, 2\}$ and $h[n] = [2, 3, 4, 6]$.
$\uparrow$

ii) $x[n] = (1/2)^n u[n]$ and $h[n] = 2u[n] - 2u[n-4]$.

o/p of LTI system
$y[n] = \sum_{k=-\infty}$

i) Folding: get value of $h[-k]$.
ii) Shifting: shift $h[-k]$ by n sample to get $h[n-k]$
iii) multiplication: multiply $h[n-k]$ with $x[k]$ to get value
iv) Summation: Sum all value of $x[k]h[n-k]$ to get value of $y[n]$ for each n.

find o/p of LTI system having

$x[n] = (1/3)^n u[n]$ and $h[n] = u[n+1] - u[n-4]$.

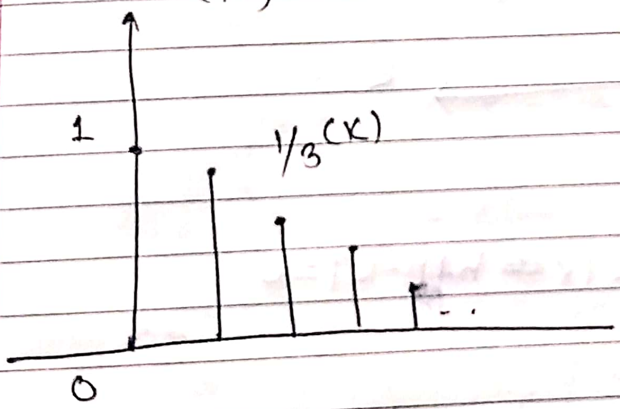
Soln:

o/p of LTI system.

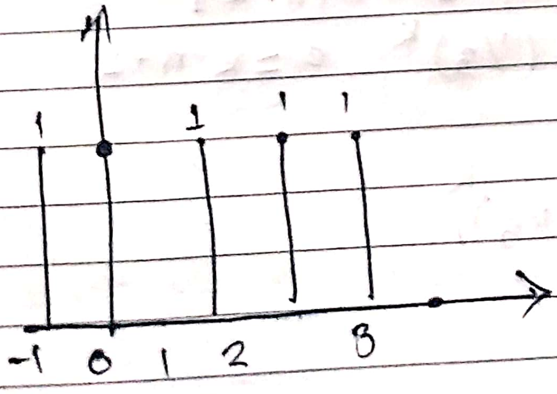
$$y[n] = x[n] * h[n]$$

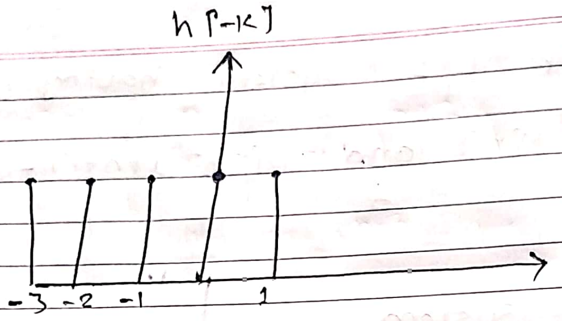
$$= \sum_{k=\infty}^{\infty} x[k]\, h[n-k]$$

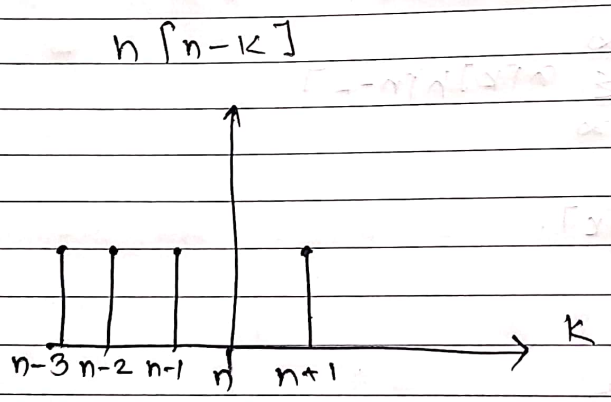
$x[k] = (1/3)^k \cdot u[k]$.



$h[k] = u[k+1] - u[k-4]$.







for $n < -1$, $x[k] \times h[n-k] = 0$.

$y[n] = 0$.

for $-1 \leq n < 3$, $x[k] \times h[n-k]$

$= (1/3)^k \quad 0 \leq k \; n+1$

$$y[n] = \sum_{k=0}^{n+1} (1/3)^k$$

$$= \frac{1\left(1-(1/3)^{n+2}\right)}{1-1/3}$$

$$= \frac{3}{2}\left[1-(1/3)^{n+2}\right]$$

for $n \geq 3$

$$x[k] * h[n-k] = (1/3)^k, \quad n-3 \leq k \leq n+1$$

$$y[n] = \sum_{k=n-3}^{n+1} (1/3)^k$$

$$= \frac{(1/3)^{n-3}\left(1-(1/3)^5\right)}{1-1/3}$$

$$= \frac{3}{2}\left(\frac{1}{3}\right)^{n-3} \times \left(\frac{242}{243}\right)$$

$$= \frac{121}{81}\left(1/3\right)^{n-3}$$

$$y[n] = \begin{cases} 0 \\ \frac{3}{2}\left[1-\left(\frac{1}{3}\right)^{n+2}\right], & \text{for } -1 \le n < 3 \\ \frac{121}{21}\left(\frac{1}{3}\right)^{n-3}, & \text{for } n \geq 3 \end{cases}$$

June - 3 - 2021
Thursday.

Due date for Assignment - Tuesday

Last week Qn.

Chapter 1 Completed

Given,

$x_a(t) = 10 \cos 3000\pi t + 2 \sin 7000\pi t + 12 \cos 14000\pi t$

Find.

i) Nyquist sampling rate

ii) if $f_s = 8000$ samples/sec, then find $x[n]$.

iii) if signal is constructed from $x[n]$ obtained in ii from what will be $x_r(t)$.

A. Given.

$x_a(t) = 10 \cos(2\pi 1500 t)$

$f_1 = 1500 Hz$ $f_2 = 3500$ $f_3 = 7000$.

i) $f_N = 2 f_{max} = 1400 Hz = 14 kHz$

ii) If $f_s = 8 kHz$.

then $x[n] = x_a(t)/t = n/f_s = \frac{n}{8000}$

$x[n] = 10 \cos 3000\pi \frac{n}{8000} + 2 \sin 7000\pi \frac{n}{8000}$

$+ 12 \cos 14000 \frac{\pi n}{8000}$

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$$= 10\cos\frac{8\pi}{8}n + 2\sin\frac{7\pi}{8}n + 12\cos\frac{14\pi}{8}n$$

$$= 10\cos\frac{8\pi}{8}n + 2\sin\frac{7\pi}{8}n + 12\cos\left(\left(2\pi - \frac{2\pi}{8}\right)n\right)$$

$$= 10\cos\frac{8\pi}{8}n + 2\sin\frac{7\pi}{8}n + 12\cos\frac{2\pi}{8}n$$

iii) $$x[n] = 10\cos\left(2\pi\frac{3}{16}n\right) + 2\sin\left(2\pi\frac{7}{16}n\right)$$

$$+ 12\cos\left(2\pi\frac{2}{16}n\right)$$

$f_1' = 3/16 \qquad f_2' = 7/16 \qquad f_3' = 2/16$

$f_s = 8$ KHz.

$f_{r1}' = f_1' f_s = 8 \times \frac{3}{16}$ KHz $= 1500$ Hz

$f_{r2}' = f_2' f_s = 8 \times \frac{7}{16}$ KHz $= 3500$ Hz

$f_{r3}' = f_3' f_s = 8 \times \frac{2}{16}$ KHz $= 1000$ Hz.

$x_r(t)$

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$$x_r(t) = 10\cos 2\pi f_{r}' t + 2\sin 2\pi f_{r2}' t + 12\cos 2\pi f_{r3}' t.$$

$$= 10\cos 800\pi t + 2\sin 700\pi t + 12\cos 2000\pi t,$$

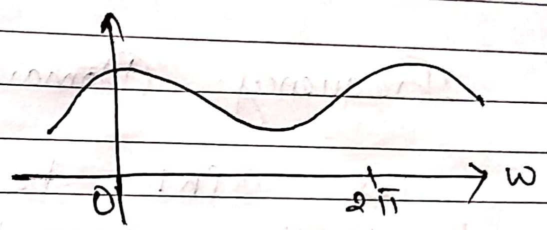
CHAPTER - 7

Discrete fourier Transform.

why we need DFT ?

discrete/ aperiodic $\qquad$ continuous/ periodic

$x[n] \xrightarrow{FT} X(e^{jw})$



why we n

In many digital signal processors frequency analysis of discrete time sequence are often performed.

Since fourier transform $X(e^{jw})$ of sequence $x[n]$ ~~x(e^{jw})~~ is continuous function of frequency (w) from $w=0$ to 2π.

It is not the computationally convinient representation of sequence n.

So this requires it is necessory to represent by samples of $X(e^{jw})$ in frequency domain.

Such a frequency domain representation is called discrete fourier transform DFT.

DFT is a very important and powerful computational tool for performing frequency analysis during digital signal processing.

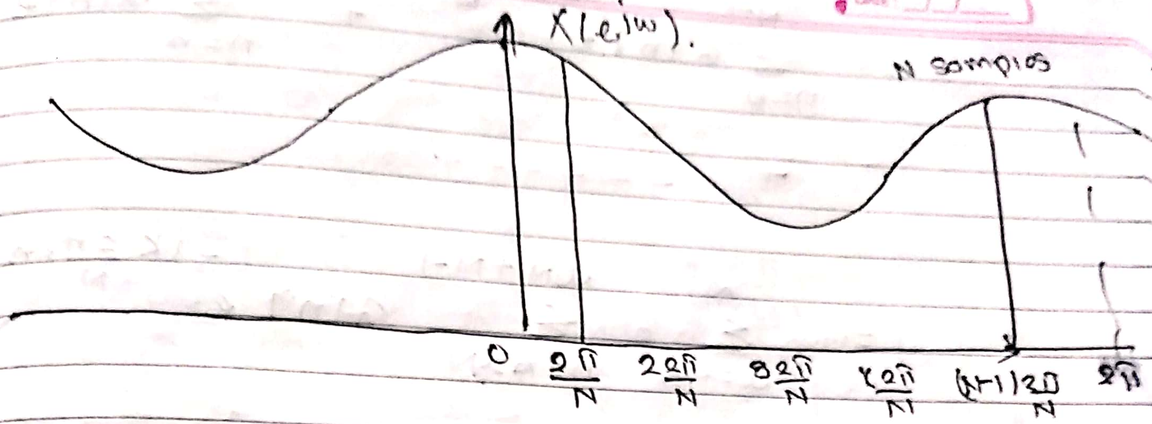
Frequency Domain Sampling.

let $x[n]$ be the finite energy in discrete time aperiodic sequence and its frequency spectrum can be represented as $X(e^{jw})$

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n] e^{-jw} \quad \text{---} (i)$$

let us take n frequency sample in frequency spectrum $X(e^{jw})$ with Spacing of $\delta w = \frac{2\pi}{N}$ between successive samples.

Since frequency spectrum $X(e^{jw})$ is a



periodic of 2π is only necessary to take samples in fundamental way.

for simplicity we take N equidistant frequency samples.

Put $w = k\frac{2\pi}{N}$ for $n = 0, 1, 2 \ldots k-1$ in

eqn (i).

$$X[e^{jk\frac{2\pi}{N}}] = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jk\frac{2\pi}{N}n}$$

$$X(k) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jk\frac{2\pi}{N}n}$$

$$= \cdots + \sum_{n=-N}^{-1} x[n]\, e^{-jk\frac{2\pi}{N}n} +$$

$$\sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi}{N} n} + \sum_{n=N}^{2N-1} x[n] e^{-jk\frac{2\pi}{N} n}$$

$$+ \; - - - - - -$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x[n] e^{-jk\frac{2\pi}{N} n}$$

$$x(k) = \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x[n] e^{-jk\frac{2\pi}{N} n}$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n-lN=0}^{N-1} x[n] e^{-jk\frac{2\pi}{N} n}$$

By changing index in inner summation from $n-lN$ to N and interchanging the order of summation, we obtain the result

$$X(k) = \sum_{n=0}^{N-1} \sum_{l=-\infty}^{\infty} x[n-lN] e^{-jk\frac{2\pi}{N}(n-lN)}$$

$$= \sum_{n=0}^{N-1} \sum_{l=-\infty}^{\infty} x[n-lN] e^{-jk\frac{2\pi}{N} n} \; e^{jkl2\pi}$$

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$\therefore e^{jkl2\pi} = \cos kl2\pi + j\sin kl2\pi$
$= 1$

$$X(k) = \sum_{n=0}^{N-1} \underbrace{\sum_{l=-\infty}^{\infty} x[n-lN]}\, e^{-jk\frac{2\pi}{N}n}$$

The periodic sequence

$x_p(n) = \sum_{l=-\infty}^{\infty} x[n-lN]$ can be obtained

from periodic ~~req~~ repetition of sequence $x[n]$ for every N samples.

$$x(K) = \sum_{n=0}^{N-1} x_p(n)\, e^{-jk\frac{2\pi}{N}n} \quad \text{--- (ii)}$$

Periodic sequence can be expressed by fourier series x_p.

$$x_p[n] = \sum_{k=\langle N\rangle} C_k\, e^{jk\frac{2\pi}{N}n} \quad \text{--- (iii)}$$

$$C_k = 1/N \sum_{n=\langle N\rangle} x_p[n]\, e^{-jk\frac{2\pi}{N}n} \quad \text{--- (iv)}$$

for $k = 0, 1 \text{ --- } N-1$

from eqn (ii) & (iv)

$$C_k = \frac{1}{N} x(k) \quad \text{for } k = 0, 1, \dots N-1$$

Put the value of C_k in eqn (iii), we get

$$x_p[n] = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{ik\frac{2\pi}{N}n} \quad \text{--- (v)}$$

for $n = 0, 1, \dots N-1$

Discrete time/aperiodic

$$\begin{tikzcd} x[n] \arrow[r, "FT"] & X(e^{i\omega}) \arrow[r, "FDS"] & \underset{X(k)}{DFT} \arrow[ll, bend left, "x_p[n]"] \end{tikzcd}$$

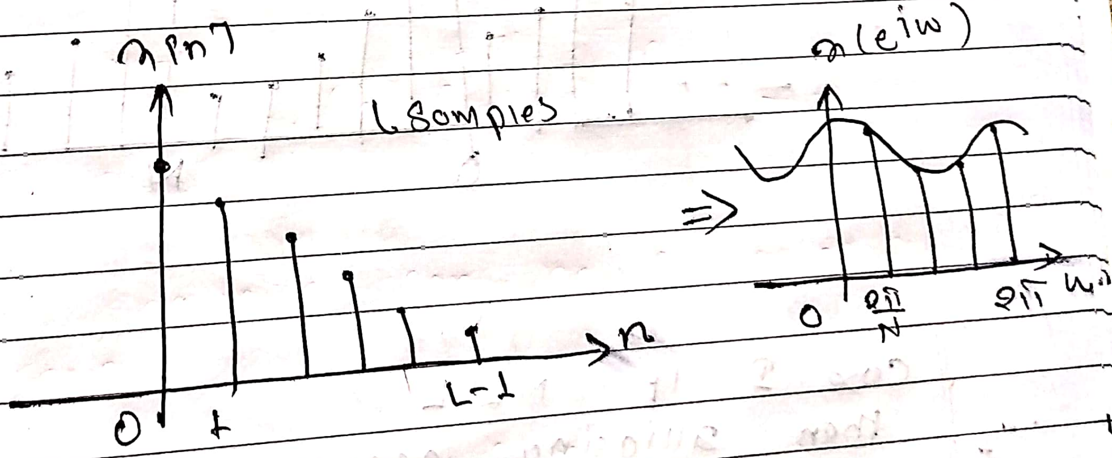
This eqn (5) implies that periodic sequence $x_p[n]$ can be recovered from the frequency samples of sequence $x[n]$ but this doesn't imply that

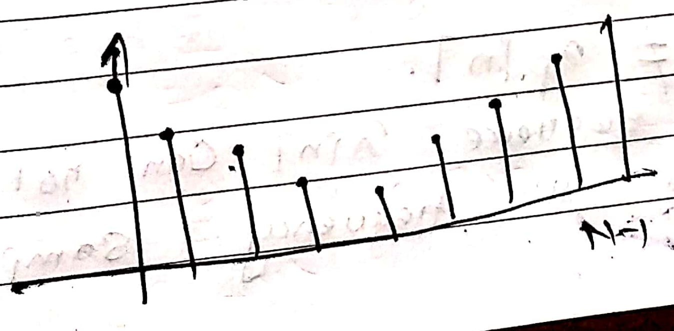
$x[n]$ or $X(e^{jw})$ can be recovered from its frequency samples.

since;

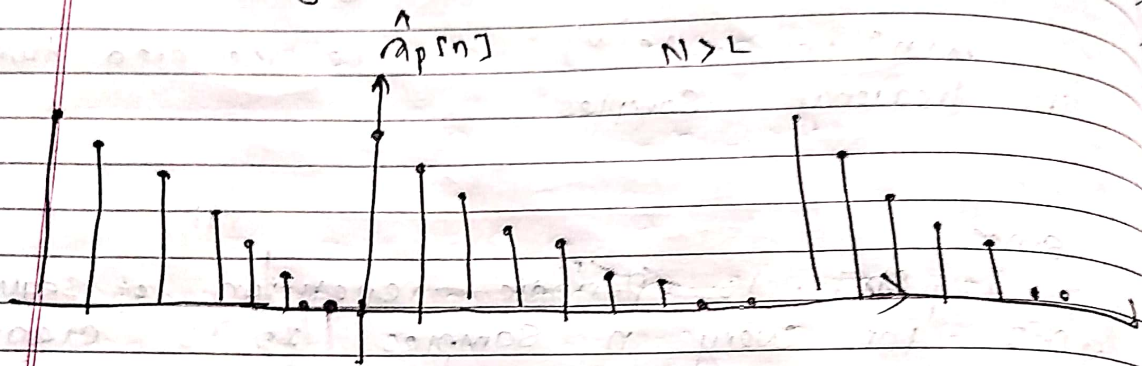
$x_p[n]$ is periodic repetition of sequence $x[n]$ for every n samples it is clear that $x[n]$ can be recovered from $x_p[n]$ for one fundamental period if there is no time domain aliasing.

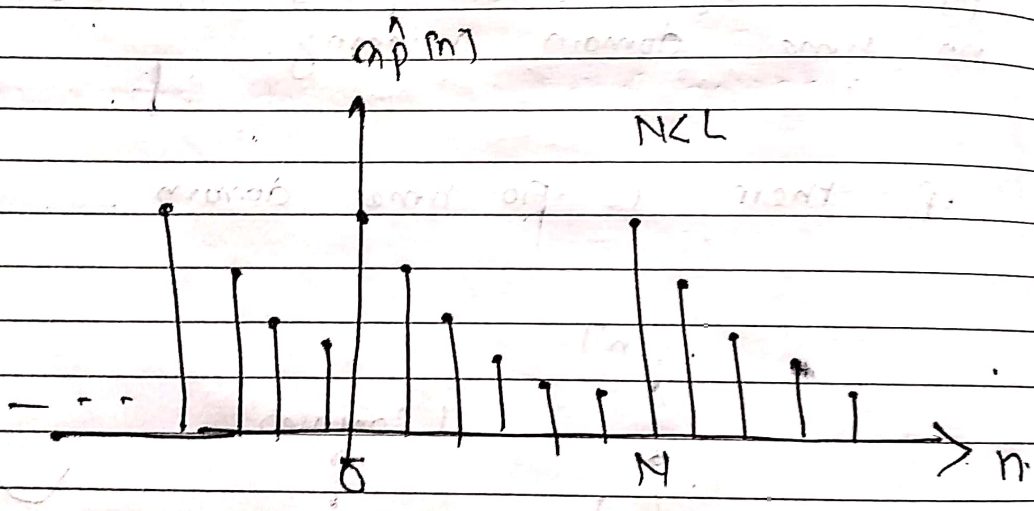
if their is no time domain





Possibility of $x_p[n]$.





Case I if $N < L$
then alliasing occurs.

$$x[n] \neq x_p[n].$$

original sequence $x[n]$ can not be recovered from it's frequency samples.

case II if $N \geq L$.
then
No alliasing occur.

origiginal signal can be recovered from its frequency samples by taking $x_p[n]$ only for one fundamental period.

$x[n] = x_p[n], \quad 0 \leq n \leq N-1$

$$\boxed{x[n] = \frac{1}{N} \sum_{K=0}^{N-1} x(K) e^{-jK\frac{2\pi}{N}n}}$$

for $k = 0, 1, \ldots, N-1$.

called inverse Discrete fourier transform (IDFT).

so
N-point DFT of $x[n]$ can be written as.

$$\boxed{x[k] = \sum_{n=0}^{N-1} x[n] e^{-jK\frac{2\pi}{N}n}}$$

for $k = 0, 1, \ldots N-1$

In another way

$$x_p[n] = \begin{cases} x[n], & 0 \le n \le L-1 \\ 0, & L \le n \le N-1 \end{cases}$$

Padding the sequence $x[n]$ with $N-L$ zeros and computing N-point DFT $X(k)$ doesn't provide any additional information but DFT display will be better.

Example:
Find 4-point DFT of sequence
$x(n) = \{2, 3, 1, 4\}$

Soln:
N point DFT of $x(n)$

$$x(k) = \sum_{n=0}^{N-1} x(n)\, e^{-jk\frac{2\pi}{N}n}$$

for $k = 0, 1, 2, \ldots, N-1$.

$$= \sum_{n=0}^{3} x[n]\, e^{-jk\frac{2\pi n}{4}}$$

$$= x[0] + x[1]\, e^{-jk\pi/2} + x[2]e^{-jk\pi}$$

$$+ x[3]\, e^{-jk\,3\pi/2}.$$

$$= 2 + 3e^{-jk\pi/2} + e^{-jk\pi} + 4e^{-jk\,3\pi/2}.$$

for $k = 0$, $x(0) = 2 + 3 + 1 + 4$

$= 10$

for $k = (1)$, $x(1) = 3 + 1e^{-jk\pi/2} + 4e^{-jk\pi}$

$+ 0$

$= 2+3+ \quad 1 e^{-j\pi/2} + 4 e^{-j\pi}$

$\cancel{x(2),\ k(2)} = 0 + 2 + 1.e^{-j2\pi} + x[3] e^{-j3\pi}$

for

$(X(1), k(1) = 2 + 3e^{-j\pi/2} + e^{-j\pi} + 4e^{-j3\pi/2}$

$= 2 + 3 \left[\cos\pi/2 - j\sin\pi/2\right] + (\cos\pi - j\sin\pi)$

$+ 4(\cos 3\pi/2 - j\sin 3\pi/2)$

$= 2 - 3j - 1 + 4j = 1 + j.$

for(k=2) $X(2), k(2) = 2 + 3e^{-j\pi} + e^{-j2\pi} + 4e^{-j3\pi}$

$= 2 + 3(\cos\pi - j\sin\pi) + (\cos 2\pi - j\sin 2\pi)$

$+ 4(\cos 3\pi - j\sin 3\pi).$

$= 2 - 3 + 1 - 4 = -4.$

for k=3 $X(3), \cancel{k(3)} = 2 + 3e^{-j3\pi/2} + e^{-j3\pi} + 4e^{-j9\pi/2}$

$= 2 + 3j - 1 - 4j$

$= 1 - j.$

$X(k) = \{10, 1+j, -4, 1-j\}$

H/W.

find 4-point DFT of
$x[n] = \{3, 1, -1, 2\}$.

DFT as a linear Transformation.

N-point DFT of sequence $x[n]$.

$$X(K) = \sum_{n=0}^{N-1} x[n] e^{-jK \frac{2\pi}{N} n}$$

$$= \sum_{n=0}^{N-1} x[n] W_N^{Kn}$$

where, $W_N = e^{-j2\pi/N}$

$\rightarrow$ N^{th} root of unity.

→ x^t Matrix to formula.

$$(x_N) = \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ \vdots \\ x[N-1] \end{bmatrix}, \quad (X_N) = \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-1) \end{bmatrix}$$

$$(W_N) = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{(N-1)} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & & & & \vdots \\ 1 & W_N^{(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

we represent DFT $x[n]$ in matrix form

$$(X_N) = (W_N)(x)_N.$$

Similarly IDFT of $X(k)$ can be written as.

$$(x_N) = (W_N)^{-1}(X_N).$$

$$= \frac{1}{N}(W_N^*)(X_N)$$

Example of in DFT Matrix form.

Find the 4-point DFT of sequence,

$x[n] = \{2, 3, 1, 4\}$

Soln:

$$(x_4) = \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 4 \end{bmatrix}$$

$$(X_4) = \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix}$$

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$W_4 = e^{-j2\pi/4} = e^{-j\pi/2} = \cos\pi/2 - j\sin\pi/2$$
$$= -j$$

$(X_1) = (W_4)(x_4)$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -J & -1 & J \\ 1 & -1 & 1 & -1 \\ 1 & J & -1 & -J \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2+3+1+4 \\ 2-3J-1+4J \\ 2-3+1-4 \\ 2+3J-1-4J \end{bmatrix}$$

$$= \begin{bmatrix} 10 \\ 1+J \\ -4 \\ 1-J \end{bmatrix}$$

Hence,

4 point DFT will be

$$X(K) = \{10,\ 1+J,\ -4,\ 1-J\}$$

Properties of Discrete Fourier Transform (DFT)

N-point DFT of sequence $x[n]$.

$$X(k) = \sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi}{N}n}$$

for $k = 0, 1, \ldots, N-1$.

and,
N-point IDFT of $X(k)$.

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{jk\frac{2\pi}{N}n}$$

for $n = 0, 1, \ldots, N-1$

Notation:

$$x[n] \xleftrightarrow[N]{DFT} X(k)$$

① Linearity

If $x_1[n] \xleftrightarrow[N]{DFT} X_1(k)$.

$$x_2[n] \xleftrightarrow[N]{DFT} X_2(k)$$

then.

$$a_1 x_1[n] + a_2 x_2[n] \xleftrightarrow[N]{DFT} a_1 X_1(K) + a_2 X_2(K)$$

where a_1 and a_2 are arbitrary constants

ii) Periodicity

If $x[n] \xleftrightarrow[N]{DFT} X(K)$

then

$X(K+N) = X(K)$ for all value of K

also

$x[n+N] = x[n]$ for all value of n.

iii) Time Reversal (Sequence Reversal). (Circular Time Reversal)

If $x[n] \xleftrightarrow[N]{DFT} X(K)$

then

$$x[-n]_N = x[N-n]. \xleftrightarrow[N]{DFT} X(-K)_N$$

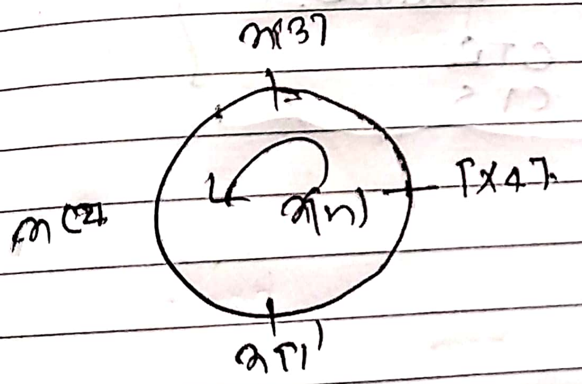
$$= X[N-K]$$

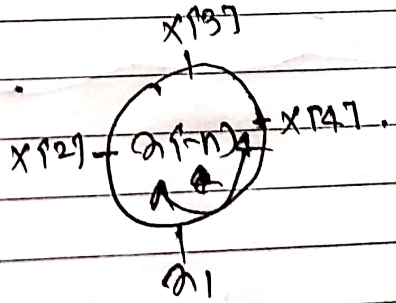
This property states that reversing a sequence $x[n]$ in time domain is equivalent to reversing its DFT values.

That is, magnitude of its DFT values remain unchanged but phase of DFT value will be reversed.

$X[N]$.

Circular Time Reversal Periodicity represent ওর example.





iv). Circular Time Shift of Sequence.

If $x[n] \xleftrightarrow[N]{DFT} X(K)$

then $x[n-n_0]_N \xleftrightarrow[N]{DFT} e^{-jK\frac{2\pi}{N}n_0} X(K)$

PTO

This property states that if sequence $x[n]$ is shifted by n_0 samples in time domain then magnitude of it's DFT value remains unchanged. but phase of its DFT values will be shifted by an amount of $-K \frac{2\pi}{N} n_0$.

$$\text{DFT}\{x(N-n)\} = \sum_{n=0}^{N-1} x(N-n$$

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Circular frequency shift

If $x[n] \xleftrightarrow[N]{DFT} X(K)$

then

$$e^{jl\frac{2\pi n}{N}} x[n] \xleftrightarrow[N]{DFT} X(K-l)_N$$

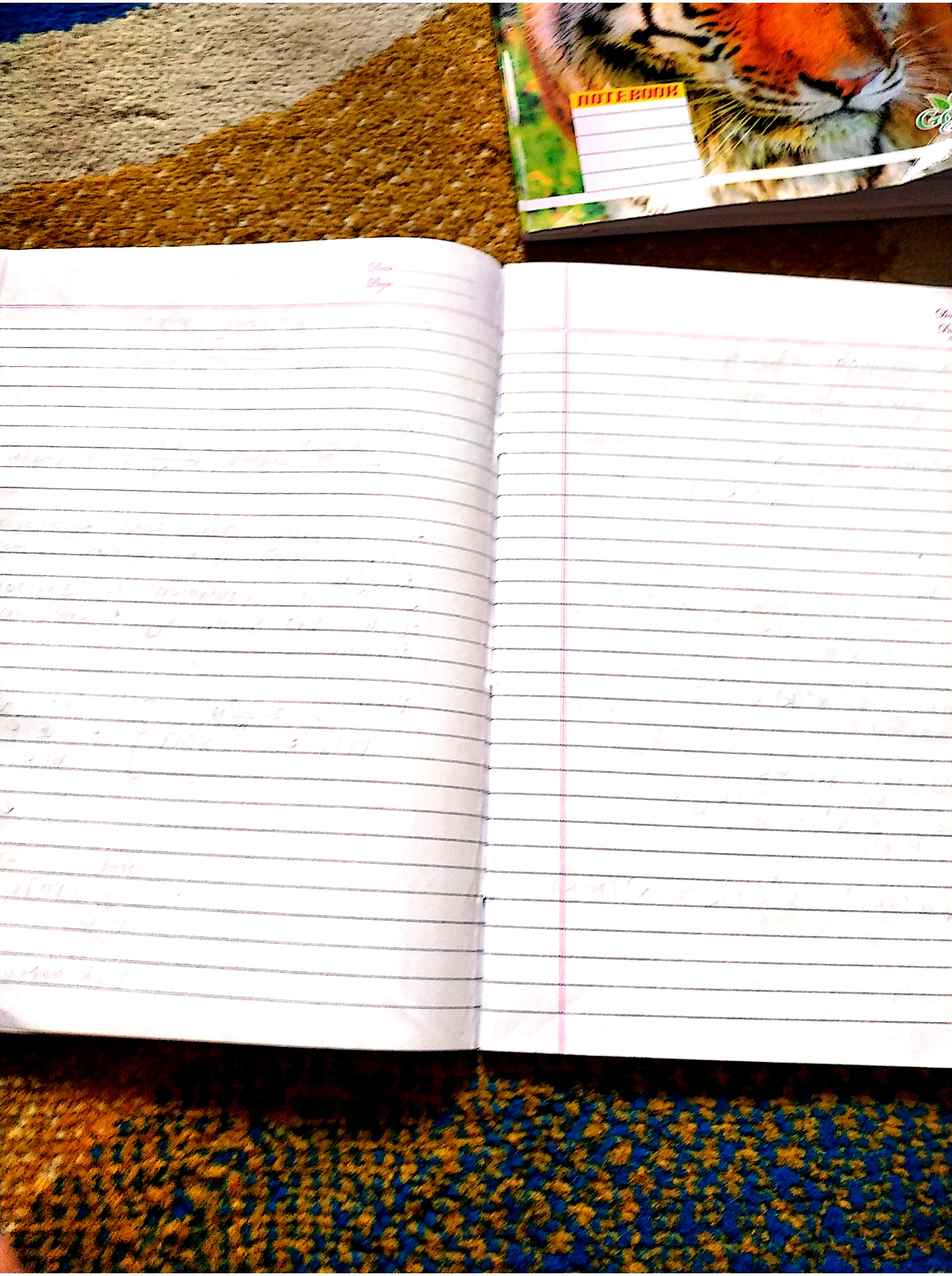
This property states that multiplication of sequence $x[n]$ with complex exponential $e^{jl\frac{2\pi n}{N}}$ is equivalent to circular shift of its DFT values by l units in frequency domain.

proof:

$$DFT\left\{ e^{jl\frac{2\pi n}{N}} x[n] \right\} = \sum_{n=0}^{N-1} e^{jl\frac{2\pi}{N}n} \cdot x[n] \cdot e^{-jK\frac{2\pi}{N}n}$$

$$= \sum_{n=0}^{N-1} x[n] e^{-j(K-l)\frac{2\pi}{N}n}$$

$$= X(K-l)_N$$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vii) Complex Conjugate Property

If $x[n] \xleftrightarrow[N]{DFT} X(K)$

then

$$x^*[n] \xleftrightarrow{DFT} X^*(-K)$$

$$= X^*(N-K)$$

Proof:

we have

$$x[n] = \frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{jK\frac{2\pi}{N}n}$$

$$x^*[n] = \frac{1}{N} \sum_{K=0}^{N-1} X^*(K) e^{-jK\frac{2\pi}{N}n}$$

This is also True for $-K$

$$x^*[n] = \frac{1}{N} \sum_{K=0}^{N-1} X^*(-K) e^{jK\frac{2\pi}{N}n}$$

so,

$$DFT\{x^*[n]\} = X^*(-K)_N = X^*(N-K)$$

vii) Correlation Property. (circular correlation)

If $x[n] \xleftrightarrow[N]{DFT} X(K)$

$y[n] \xleftrightarrow[N]{DFT} Y(K)$

where,

$x[n]$ and $y[n]$ are complex valued sequence

then,

a) Cross Correlation

$$r_{xy}(l) \xleftrightarrow[N]{DFT} R_{xy}(K) = X(K)\,Y^*(K)$$

where $r_{xy}(l) = \sum_{n=0}^{N-1} x[n]\,y^*[n-l]_N$

b) Auto Correlation

$$r_{xx}(l) \xleftrightarrow[N]{DFT} R_{xx}(K) = X(K)\,X^*(K)$$
$$= |X(K)|^2$$

where.

$$r_{xx}(l) = \sum_{n=0}^{N-1} x[n]\,x^*[n-l]_N$$

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V. imp.

viii) Circular convolution property

If $x_1[n] \xleftrightarrow[N]{DFT} X_1(K)$.

$x_2[n] \xleftrightarrow[N]{DFT} X_2(K)$

Then,

$x_1[n] \, Ⓝ \, x_2[n] \xleftrightarrow[N]{DFT} X_1(K)X_2(K)$

where

$$x_1[n] \, Ⓝ \, x_2[n] = \sum_{m=0}^{N-1} x_1[m]\, x_2[n-m]_N$$

Proof:

let $X_3(K) = X_1(K)X_2(K)$

IDFT of $X_3(K)$.

$$x_3[n] = 1/N \sum_{K=0}^{N-1} X_3(K)\, e^{jK\frac{2\pi}{N}n}$$

for $n = 0, 1 \ldots N-1$.

$$x_3[n] = 1/N \sum_{K=0}^{N-1} X_1(K)\, X_2(K)\, e^{jK\frac{2\pi}{N}n}\, e^{-jK\frac{2\pi}{N}m}$$

we have

$$X_1(K) = \sum_{m=0}^{N-1} x_1(m)\, e^{-jK\frac{2\pi}{N}m}$$

for $K = 0, 1, \ldots N-1$

Contd:

Defn: This property states that if two sequences are circularly convoluted in time domain then, its DFT will be multiplication of between DFT of individual sequences.

Proo

Contd:

$$x_3[n] = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} x_1(m)\, e^{-jk\frac{2\pi}{N}m} \; X_2(k)\, e^{jk\frac{2\pi n}{N}}$$

$$= \sum_{m=0}^{N-1} x_1(m) \frac{1}{N} \sum_{k=0}^{N-1} X_2(k)\, e^{jk\frac{2\pi}{N}(n-m)}$$

Using circular frequency shift property of DFT, we get

$$X_3[n] = \sum_{m=0}^{N-1} x_1[m]\, x_2[n-m]_N$$

$$= x_1[n] \, (N) \, x_2[n]. //$$

Imp

ix) Multiplication of two sequences

If $x_1[n] \xleftrightarrow[N]{DFT} X_1(K)$

$x_2[n] \xleftrightarrow[N]{DFT} X_2(K)$

then

$$x_1[n]\,x_2[n] \xleftrightarrow[N]{DFT} \frac{1}{N}[X_1(K) \; (N) \; X_2(K)]$$

$$= \frac{1}{N}\sum_{l=0}^{N-1} X_1(l)\, X_2(K-l)_N$$

This property states that if two sequences are multiplied in time Domain then its DFT will be circular convolution between DFT of individual sequence

Proof:

Let, $x_3[n] = x_1[n]\,x_2[n]$ then

DFT of $x_3[n]$.

$$X_3(K) = \sum_{n=0}^{N-1} x_3[n]\, e^{-jK\frac{2\pi}{N}n}$$

for $K = 0, 1, \ldots N-1$.

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$$= \sum_{n=0}^{N-1} x_1[n]\, x_2[n]\, e^{-jk\frac{2\pi}{N}n}$$

we have

$$x_1[n] = 1/N \sum_{l=0}^{N-1} X_1(l)\, e^{jl\frac{2\pi}{N}n}$$

$$X_3(k) = \sum_{n=0}^{N-1} 1/N \sum_{l=0}^{N-1} X_1(l)\, e^{jl\frac{2\pi}{N}n}\, x_2[n]\, e^{-jk\frac{2\pi}{N}n}$$

$$= \frac{1}{N} \sum_{l=0}^{N-1} X_1(l) \sum_{n=0}^{N-1} x_2[n]\, e^{-j(k-l)\frac{2\pi}{N}n}$$

Using ~~frequency~~ circular frequency shift property of DFT,

we get $$X_3(k) = 1/N \sum_{l=0}^{N-1} X_1(l)\, X_2(k-l)$$

$$= 1/N \left[X_1(k) \; (N) \; X_2(k) \right]$$

x) Parsevals Relation

If $x_1[n] \xleftrightarrow[N]{DFT} X_1(K)$

$x_2[n] \xleftrightarrow[N]{DFT} X_2(K)$

then

$$\sum_{n=0}^{N-1} x_1[n]\, x_2^*[n] = 1/N \sum_{K=0}^{N-1} X_1(K)\, X_2^*(K)$$

Proof:

we have

$$x_2[n] = 1/N \sum_{K=0}^{N-1} X_2(K)\, e^{jK\frac{2\pi}{N}n}$$

$$x_2^*[n] = 1/N \sum_{K=0}^{N-1} X_2^*(K)\, e^{-jK\frac{2\pi}{N}n}$$

$$LHS = \sum_{n=0}^{N-1} x_1[n]\, x_2^*[n].$$

$$= \sum_{n=0}^{N-1} x_1[n]\, 1/N \sum_{K=0}^{N-1} X_2^*(K)\, e^{-jK\frac{2\pi}{N}n}$$

$$= 1/N \sum_{K=0}^{N-1} X_2^*(K)\, X_1(K) = RHS$$

Proved

If $x_1[n] = x_2[n] = x[n]$.

Then energy of signal $x[n]$

$$E = \sum_{n=0}^{N-1} |x[n]|^2 = \sum_{n=0}^{N-1} x[n]\, x^*[n]$$

$$= \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, X^*(K).$$

$$E = \frac{1}{N} \sum_{K=0}^{N-1} |X(K)|^2 \quad //$$

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Q) Find circular convolution between two sequences

$x_1[n] = \{1, 2, 3, 4\}$

$x_2[n] = \{3, 5, -2, 1\}$

or,

Find IDFT of $X_3(k)$ if $X_3(k) = X_1(k)X_2(k)$ where $X_1(k)$ and $X_2(k)$ are 4-point DFT of sequence $x_1[n] = \{1, 2, 3, 4\}$ and $x_2[n] = \{3, 5, -2, 1\}$ respectively.

Solution:-

4-point DFT of $x_1[n]$

Solving it

$x_1[n] = \{1, 2, 3, 4\}$

$$x_4 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$w_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}.$$

$$X_4 = (w_4)\, x_4.$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

$$= \begin{bmatrix} 1+2+3+4 \\ 1-2j-3+4j \\ 1-2+3-4 \\ 1+2j-3-4j \end{bmatrix}$$

$$= \begin{bmatrix} 10 \\ 2j-2 \\ -2 \\ -2j-2 \end{bmatrix}.$$

Hence,

4 point DFT will be

$X_1(k) = \{10,\ \cancel{2j-2}\ -2+2j,\ -2,\ -2-2j\}$

Similary 4-point DFT of $x_2[n]$

$$X_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3+5+-2+1 \\ 3-5j+2+j \\ 3-5-2-1 \\ 3+5j+2-j \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ 5-4j \\ -5 \\ 5+4j \end{bmatrix}.$$

Hence 4 point DFT will be,

$x_2(k) = \{7, 5-4j, -5, 5+4j\}$

multiplying

$$X_3(k) = X_1(k)\ X_2(k).$$

In matrix form

$$(X_3)_4 = (X_1)_4\ I_{1\times 4}\ (X_2)_4$$

$$= \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & -2+2j & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2-j2 \end{bmatrix} \begin{bmatrix} 7 \\ 5-4j \\ -5 \\ 5+4j \end{bmatrix}$$

$$= \begin{bmatrix} 70 \\ (-2+2j)(5-4j) \\ 10 \\ (-2-j2)(5+4j) \end{bmatrix}$$

$$= \begin{bmatrix} 70 \\ -2+j18 \\ 10 \\ -2-i18 \end{bmatrix}$$

Using computing IDFT of $X_3(k)$.

$$(x_3)_4 = \frac{1}{4}(W_4^*)(X_3)_4$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 70 \\ -2+18j \\ 10 \\ -2+18j \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 70-2+18j+10-2+18j \\ 70-2j-18-10+2j-18 \\ 70+2-18j+10+2+18j \\ 70+2j+18-10-2j+18 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 76 \\ 24 \\ 84 \\ 96 \end{bmatrix} = \begin{bmatrix} 19 \\ 6 \\ 21 \\ 24 \end{bmatrix}.$$

Hence

Required Sequence

$x_3[n] = \{19, 6, 21, 24\}.$

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श्री 9n की अकों Method.

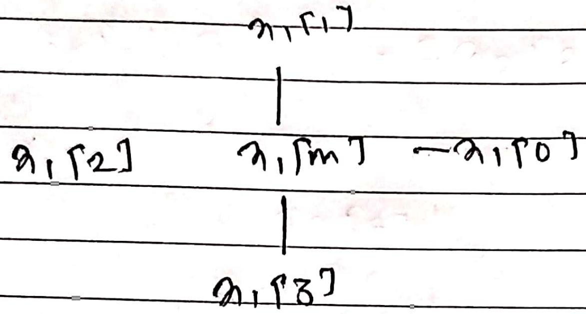
P.T.O

using circular convolution

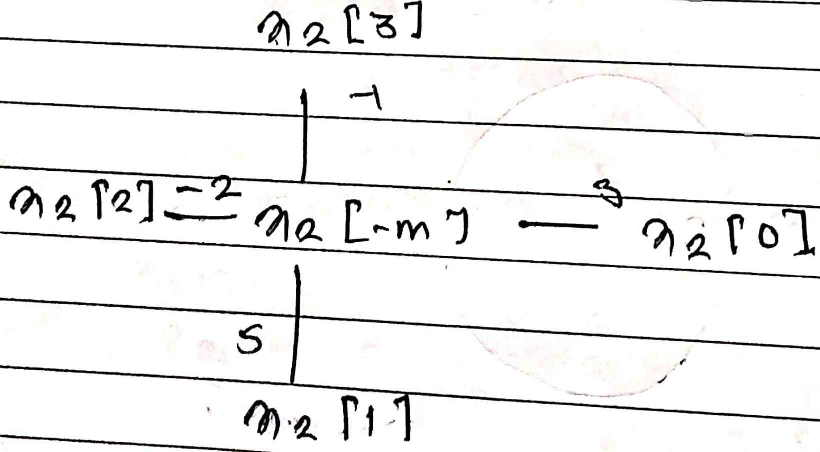
we know $x_3(n) = x_1[n] \circledast_N x_2[n]$

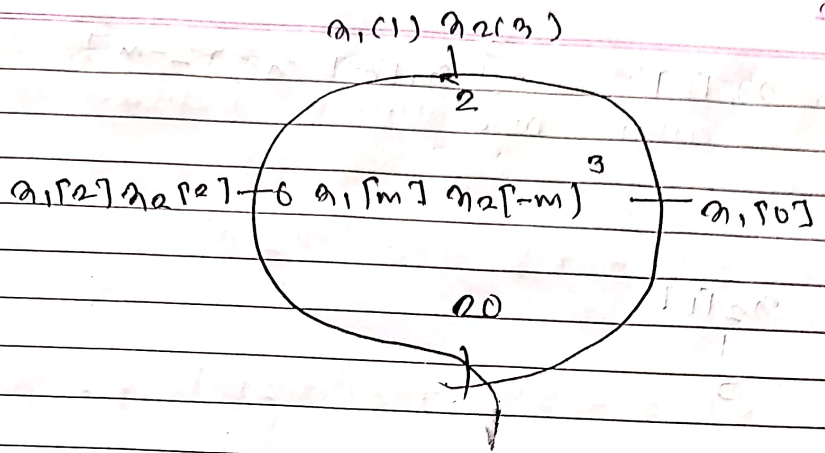
$$= \sum_{m=0}^{N-1} x_1[m]\, x_2[n-m]_N$$

$$= \sum_{m=0}^{3} x_1[m]\, x_2[n-m]_4.$$



for $n=0$, $x_3[0] = \sum_{m=0}^{3} x_1[m]\, x_2[-m]$





10 days Complete

$x_3[1] = 5 + 6 + 3 - 8 = 6$

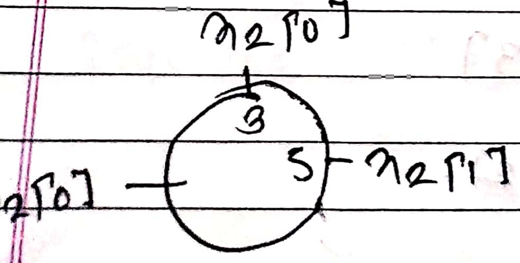
for $x_3[2]$.

$x_3[2]$

5 3 x.

-2

for $n=1$, $x_3[1] = \sum_{m=0}^{3} x_1[m]\, x_2[1-m]$.



for $n=2$, $x_3[2] = \sum_{m=0}^{3} x_1[m]\, x_2[2-m]$.

$x_2[1]$

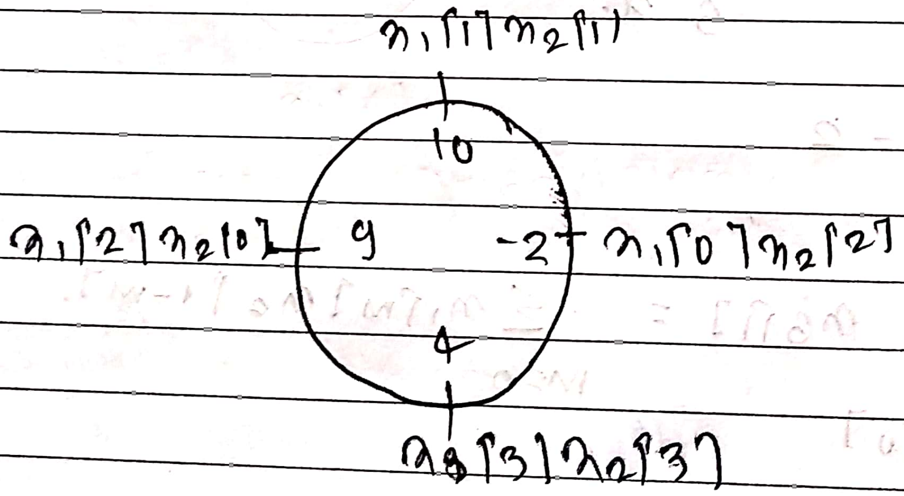
5

$x_2[0]$. -3 $\quad$ -2 $x_2[2]$

1

$x_2[3]$

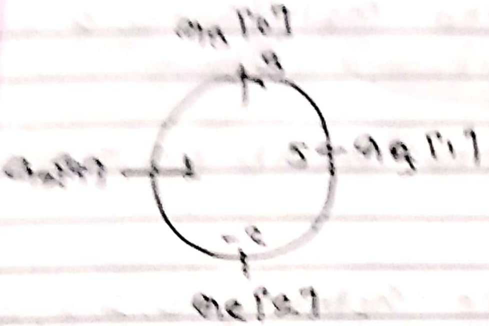
Now,



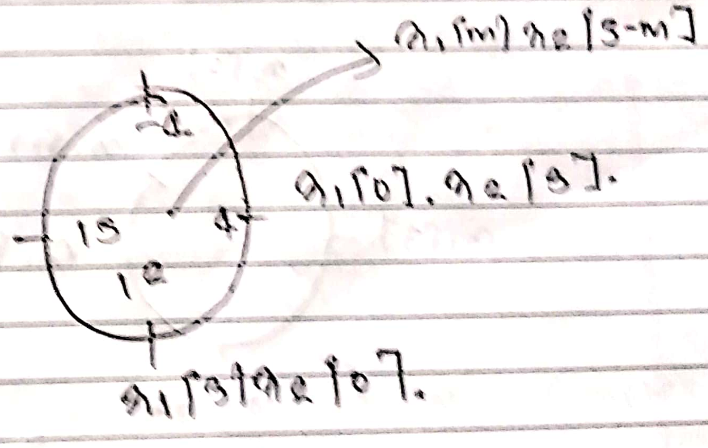
$x_3[2] = -2 + 10 + 9 + 4 = 21$.

for $n=3$, $x_3[3] = \sum_{m=0}^{3} x_1[m]\, x_2[3-m]$.

Now



Multiplying.



$x_3[3] = 1 - 4 + 15 + 12 = 24$.

Required sequence $x_3[n] = \{19, 6, 21, 24\}$

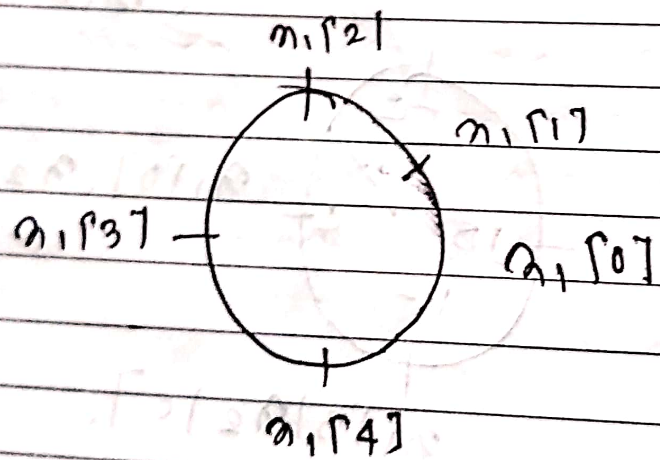
H.W //

Find circular convolution between these sequence

i) $x_1[n] = \{3, 5, 4, 7\}$ and $x_2[n] = [-5, 2, 1, 6]$

ii) $x_1[n] = \{1, 3, 4, 5\}$ and $x_2[n] = \{2, 5, 6\}$

iii) $x_1[n] = \{1, 2, 3, 4, 5\}$ and $x_2[n] = [-5, 2, 1, 4, 6\}$



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Q7: Circular convolution find

Computational complexicity of DFT calculation

N-point DFT of sequence $x[n]$.

$$X(k) = \sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi}{N} n} \quad \text{for } k = 0, 1, \ldots N-1.$$

$$= x[0] e^{1.0} + x[1] e^{-jk\frac{2\pi}{N}} + \ldots x[N-1] e^{-jk\frac{2\pi}{N}(N-1)}$$

i) Number of multiplication

for N-point DFT, it requires N^2 complex multiplication.

ii) No of Addition

for N-point DFT, it requires $N(N-1)$ complex addition.

iii) $\delta\omega = \frac{2\pi}{N}$ should be very small.

iv) for smaller $\delta\omega$, N seems larger & larger and increases number of complex multiplication and addition.

Fast fourier Transform (FFT.)

Divide and conquer Approach.

Radix - 2 FFT Algorithm.

→ most popular Algorithm

→ developed in Middle of 1980's.

FFT ~~b~~ computes N-point DFT in such a efficient manner that it reduces computational time by decreasing the number ~~ee~~ of complex multiplication and addition.

Most of FFT algorithms use divide & conquer approach.

i) Decimation In Time fast fourier Transform (DITFFT)

ii) Decimation in frequency " " " (DIFFFT).

Decimation in Time Fast Fourier Transform (DITFFT)

Let $x[n]$ be the discrete time sequence of length N. for $n = 0, 1, 2 \ldots N-1$ and it can be seperated for odd and even value of n. as below.

$$f_1[n] = x[2n] \quad \text{for } n = 0, 1, \ldots N/2 - 1.$$

even

$$f_2[n] = x[2n+1] \quad \text{for } n = 0, 1 \ldots N/2 - 1.$$

odd.

$f_1[n]$ and $f_2[n]$ are obtained by decimating the sequence $x[n]$ by a factor of 2. so this algorithm is called Radix-2 decimation in time fast fourier transform algorithm. (DITFFT)

N-point DFT of sequence $x[n]$ is given by

$$X(K) = \sum_{n=0}^{N-1} x[n] W_N^{Kn}$$

$$= \sum_{n=0}^{N-1} x[n] e^{-jK\frac{2\pi}{N}n} \quad \text{for } K = 0, 1 \ldots N-1$$

where

$$W_N = e^{-j\frac{2\pi}{N}}$$

$$W_N^2 = e^{-j\frac{2\pi}{N}2}$$

$$= e^{-j\frac{2\pi}{N/2}} = W_{N/2}$$

$$X(k) = \sum_{m=0}^{\frac{N}{2}-1} x[2m]\, W_N^{2mk} + \sum_{m=0}^{N/2-1} x[2m+1]\, W_N^{(2m+1)k}$$

$$= \sum_{m=0}^{N/2-1} f_1[m]\, W_{N/2}^{km} + W_N^{k} \sum_{m=0}^{N/2-1} f_2[m]\, W_{N/2}^{km}$$

N-point DFT of $x[n]$

$$X(k) = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

$$X(k) = F_1(k) + W_N^{k} F_2(k) \quad \text{for } k = 0, 1 \;--\; N/2 - 1.$$

where $F_1(k)$ and $F_2(k)$ are $N/2$-point DFT of sequence $f_1[n]$ and $f_2[n]$ respectively.

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Since DFT values are periodic in Nature,

$f_1(K+N/2) = f_1(K)$

$f_2(K+N/2) = f_2(K)$.

Now,

$X(K+N/2) = f_1(K+N/2) + W_N^{(K+N/2)} f_2(K+N/2)$

$= f_1(K) + W_N^K \cdot W_N^{N/2} f_2(K) \rightarrow (1)$

$= f_1(K) - W_N^K F_2(K)$.

for $K = 0, 1, 2, \ldots N/2 - 1$.

$\because W_N^{N/2}$

$= e^{-J \frac{2\pi}{N} \times N/2}$

$= e^{-J\pi}$

$= \cos\pi - J\sin\pi$

$= -1$.

This value replaced value of $W_N^{N/2}$ in eqn (1).

At this stage Number of multiplication

$$= \left(\frac{N}{2}\right)^2 + \left(N/2\right)^2 + N/2$$

$$= \left(\frac{N}{2}\right)^2 + \frac{N}{2}$$

Number of multiplication reduces from N^2 to $\frac{N^2}{2} + \frac{N}{2}$

Similarly again processing for $f_1[n]$ and $f_2[n]$

$$\begin{tikzcd}
 & & v_{11}[n] \\
 & f_1[n] \arrow[ur, "\text{even}"] \arrow[r, "\text{odd}"'] & v_{12}[n] \\
x[n] \arrow[ur, "\text{even}"] \arrow[dr, "\text{odd}"'] & & \\
 & f_2[n] \arrow[r, "\text{even}"] \arrow[dr, "\text{odd}"'] & v_{21}[n] \\
 & & v_{22}[n]
\end{tikzcd}$$

We get four $\frac{N}{4}$ - point DFTs.

$$f_1(k) = v_{11}(k) + W_{N/2}^{k} v_{12}(k)$$

$$f_1(k + N/4) = v_{11}(k) - W_{N/2}^{k} v_{12}(k)$$

for $k = 0, 1 \ldots \frac{N}{4} - 1$

$$f_2(k) = v_{21}(k) + W_{N/2}^{k} v_{22}(k).$$

$$f_2(k + N/4) = v_{21}(k) - W_{N/2}^{k} v_{22}(k).$$

for $k = 0, 1 \ldots N/4 - 1$

$$X(k) = f_1(k) + W_N^{k} f_2(k)$$

At this stage,

No of multiplication $= 4\left(\frac{N}{4}\right)^2 + 4\,N/4$

$= \frac{N^2}{4} + N.$

No of multiplication at this stage reduces from N^2 to $\frac{N^2}{4} + N$.

This process continues till 2-point DFT.

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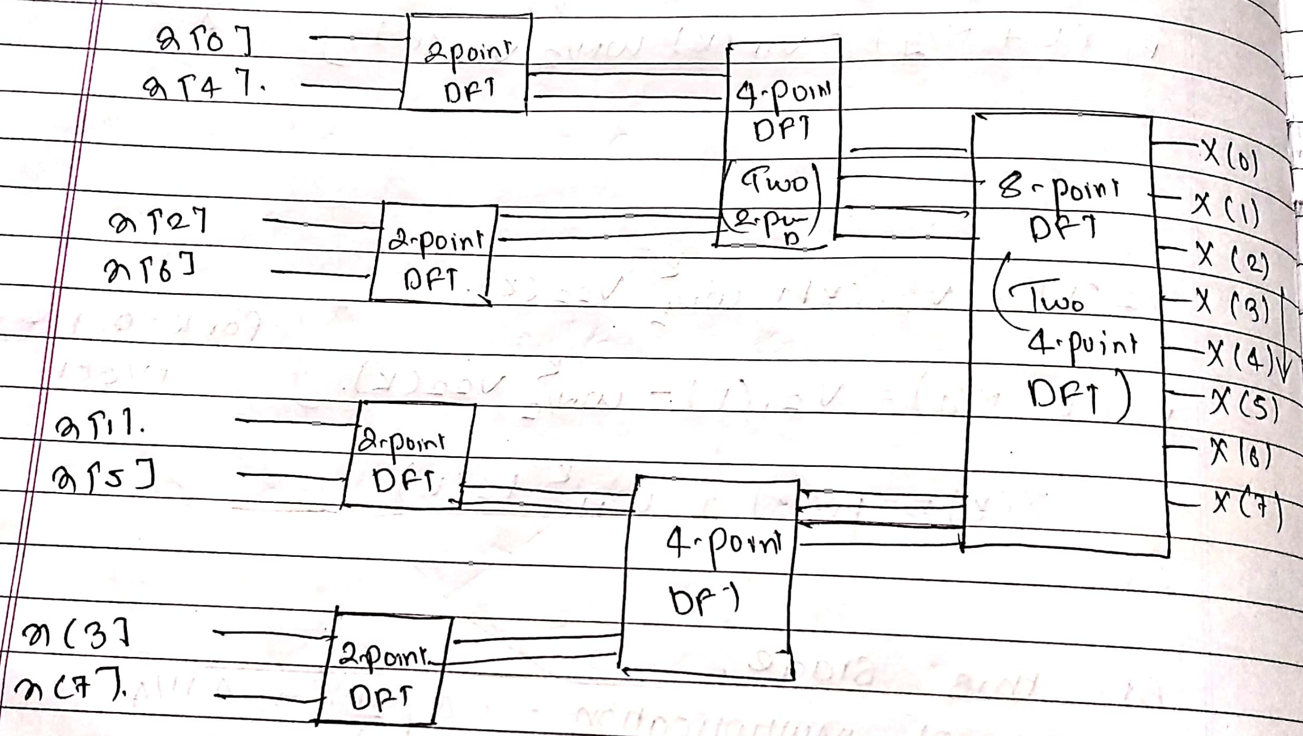
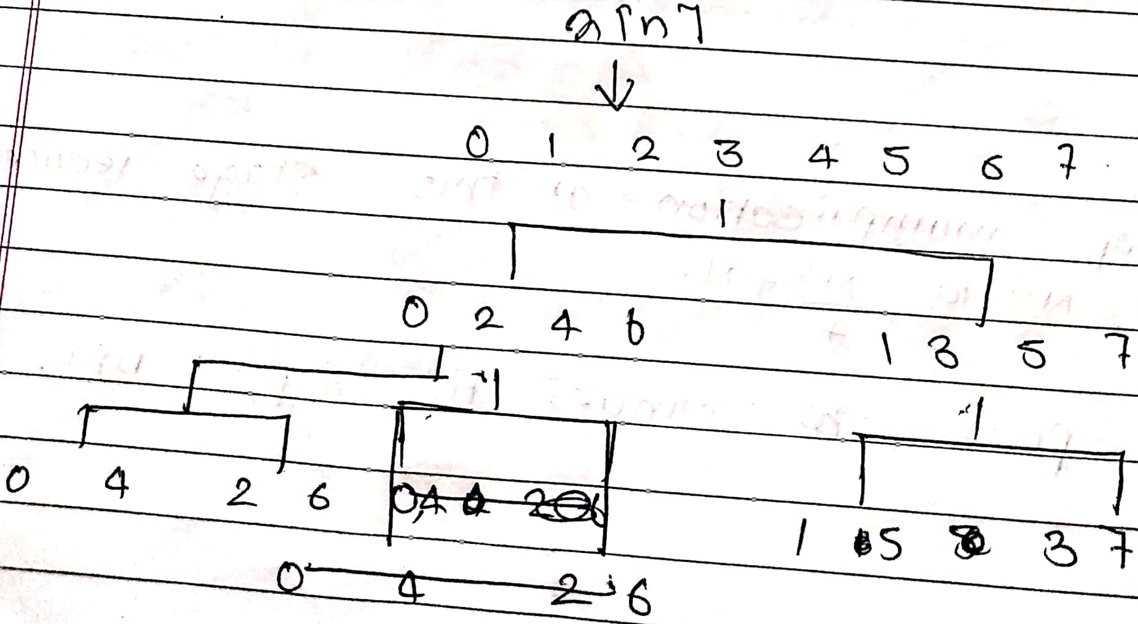
For N=8, DITFFT.





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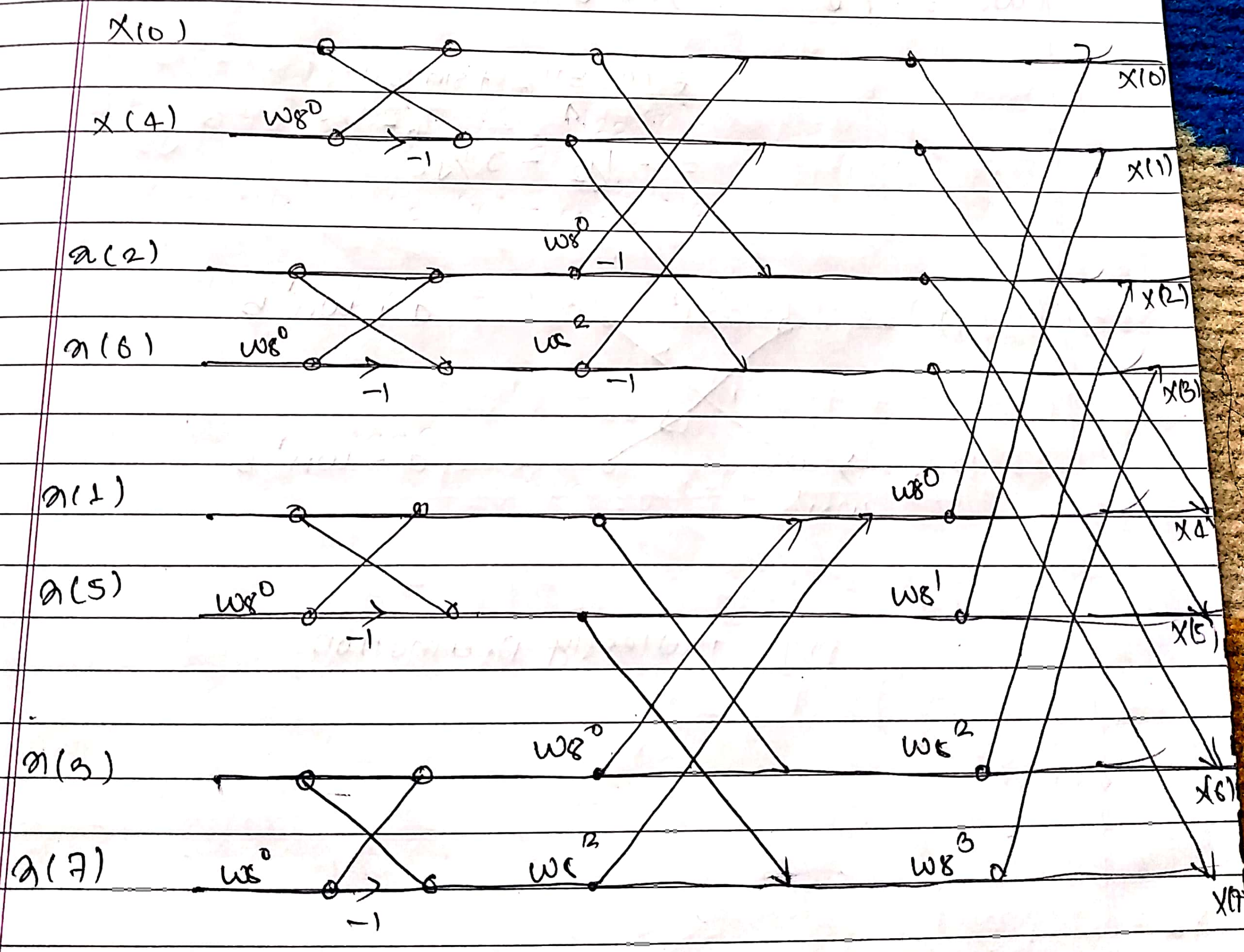
For N=8, DITFFT $W_8 = e^{-j2\pi/8} = e^{-j\pi/4}$

$= \cos\pi/4 - i\sin\pi/4$

$= \frac{1}{\sqrt{2}} - i\,{}^{1}/_{\sqrt{2}}.$



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$$W_8^2 = (e^{-j2\pi/8})^2 = e^{-j\pi/2}$$

$$= \cos\pi/2 - j\sin\pi/2$$

$$= -j.$$

$$W_8^3 = (e^{-j2\pi/8})^3 = e^{-j3\pi/4}$$

$$= \cos\frac{3\pi}{4} - j\sin 3\pi/4$$

$$= -1/\sqrt{2} - j\,1/\sqrt{2}$$

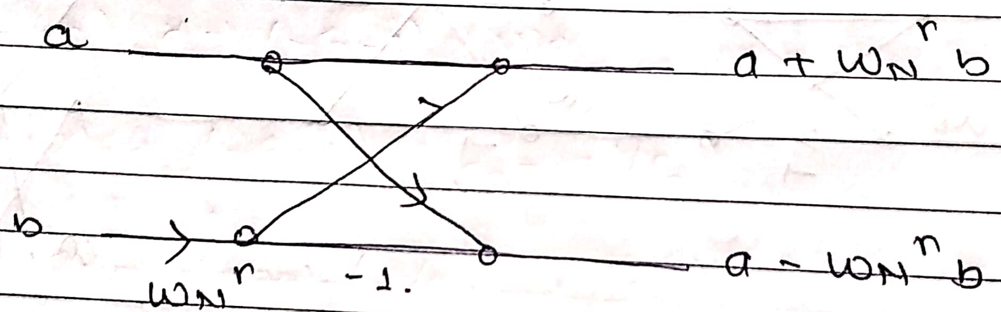

fig. Butterfly Decimation.

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Example:

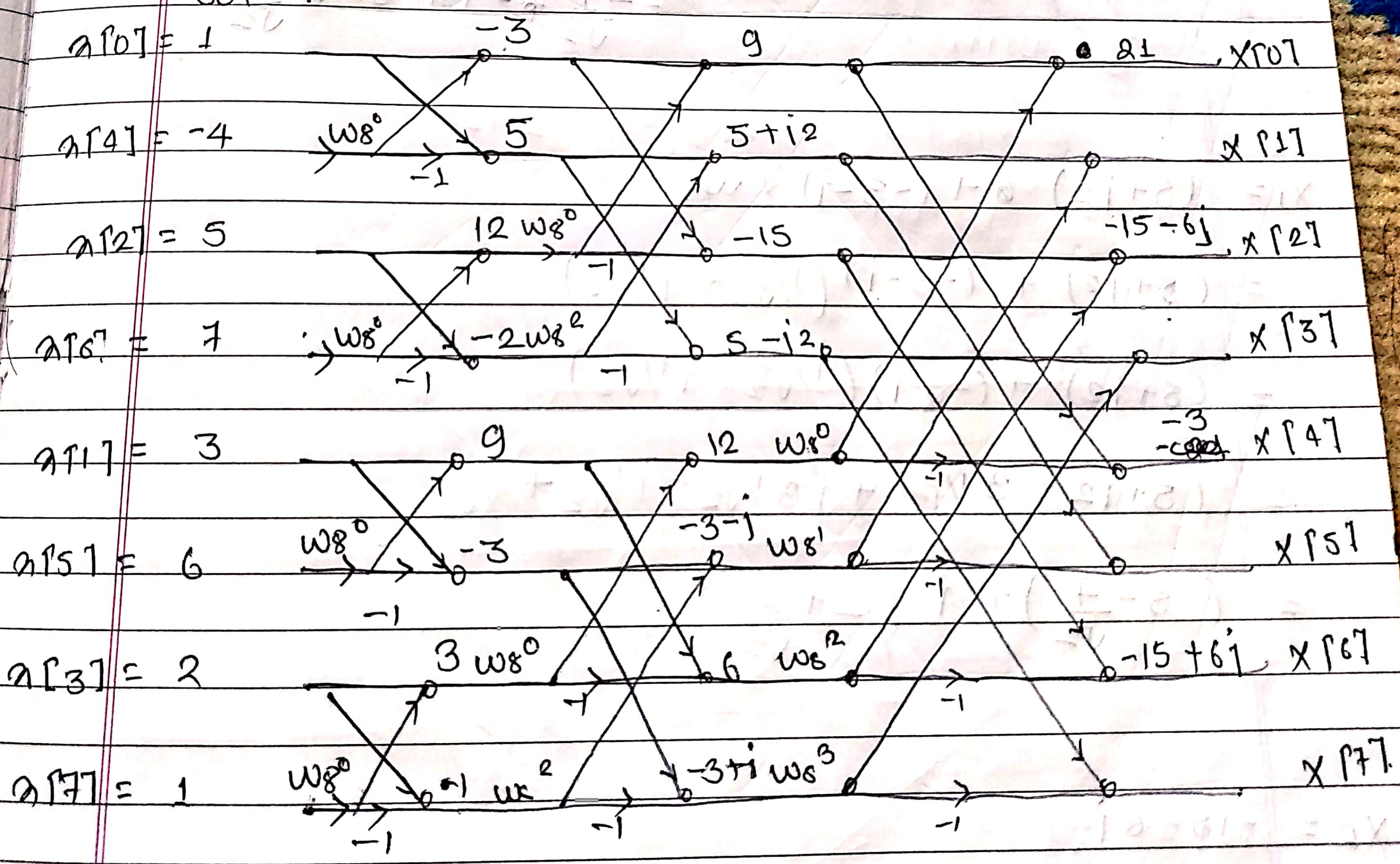
find 8-point DFT of sequence

$x(n) = \{ \overset{0}{1}, \overset{1}{3}, \overset{2}{5}, \overset{3}{2}, \overset{4}{-4}, \overset{5}{6}, \overset{6}{7}, \overset{7}{1} \}$

using DITFFT algorithm.

(Solve using formula).

Sol^n:



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$$w_8 = e^{-i2\pi/8} = e^{-j\pi/4} = \cos\pi/4 - i\sin\pi/4$$

$$= \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$w_8^2 = e^{-i\pi/2} = \cos\pi/2 - i\sin\pi/2 = -i$$

$$w_8^3 = e^{-i6\pi/8} = e^{-3\pi/4} = \cos 3\pi/4 - i\sin 3\pi/4$$

$$= -\frac{1}{\sqrt{2}} \cancel{p} - \cancel{i\sin 3\pi/4} \frac{1}{\sqrt{2}}$$

$$X_1 = (5+i2) \cancel{x} + (-3-i) \times w_8$$

$$= (5+i2) + (-3-i)(1/\sqrt{2} - i1/\sqrt{2})$$

$$= (5+i2) + (-3-i)(1/\sqrt{2} - i1/\sqrt{2})$$

$$= (5+i2 - 3 1/\sqrt{2} + i3 1/\sqrt{2} - i1/\sqrt{2} + 1/\sqrt{2}$$

$$= \left(5 - \frac{4}{\sqrt{2}}\right) + i\left(2 + \frac{2}{\sqrt{2}}\right)$$

$$X_2 = -15 - 6i$$

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$X_3 = (5 - j2) + (-3 + j)\, W_8^3$

$= (5 - j2) + (-3 + j)\left(-\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)$

$= 5 - j2 + \frac{3}{\sqrt{2}} + j\frac{3}{\sqrt{2}} - j\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$

$= \left(5 + \frac{4}{\sqrt{2}}\right) - j\left(2 - \frac{2}{\sqrt{2}}\right)$

$X(5) = (5 + j2) - (-3 - j)\, W_8^1$

$= (5 + j2) - (-3 - j)\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)$

$= (5 + j2) - \left[\frac{-3}{\sqrt{2}} + \frac{j3}{\sqrt{2}} - \frac{j}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right]$

$= (5 + j2) - \left[\frac{-4}{\sqrt{2}} + \frac{j2}{\sqrt{2}}\right]$

$= \left(5 + \frac{4}{\sqrt{2}}\right) + j\left(2 - \frac{2}{\sqrt{2}}\right)$

$X(7) = (5 - j2) - (-3 + j)\, W_8^{3}$

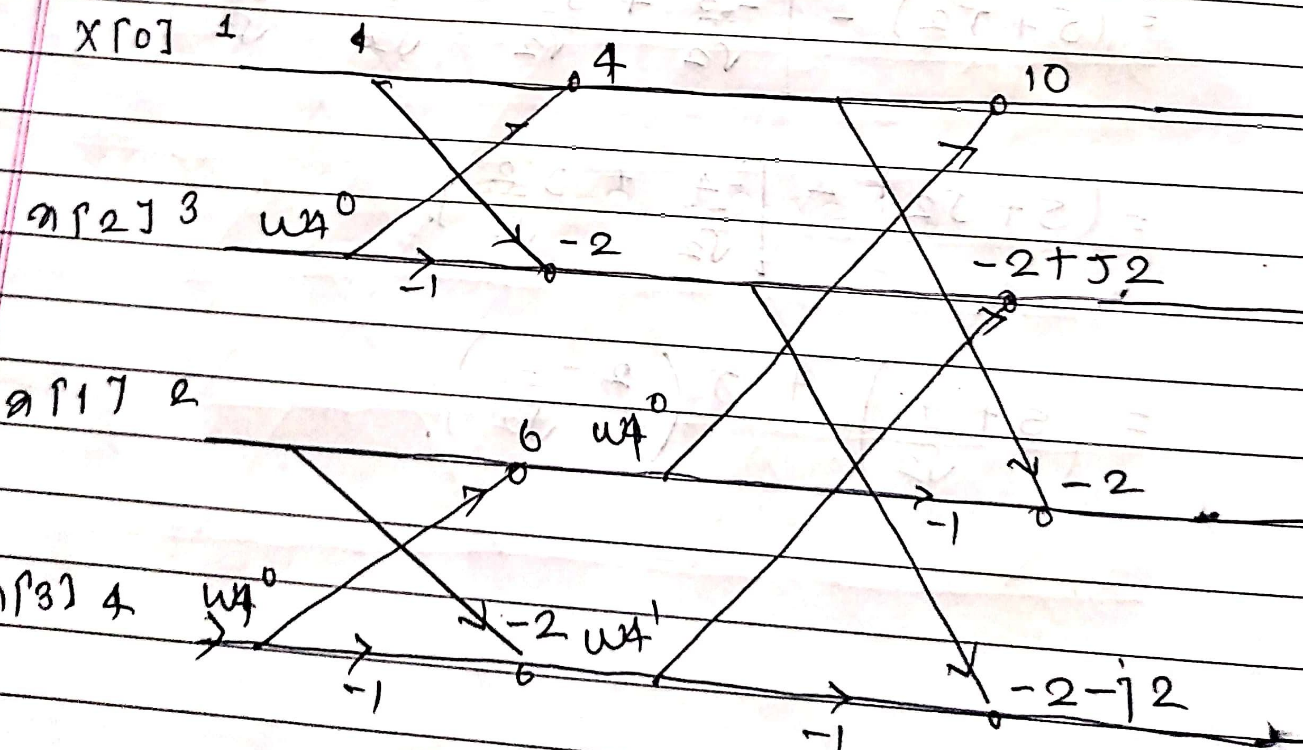
$= \left(5 - \frac{4}{\sqrt{2}}\right) - j\left(2 + \frac{2}{\sqrt{2}}\right)$

Q. H.W.

Find 4-point DFT of sequence $x[n] = \{1, 2, 3, 4\}$ using DITFFT.

Soln: $X(k) = \{10, -2+j2, -2, -2-j2\}$

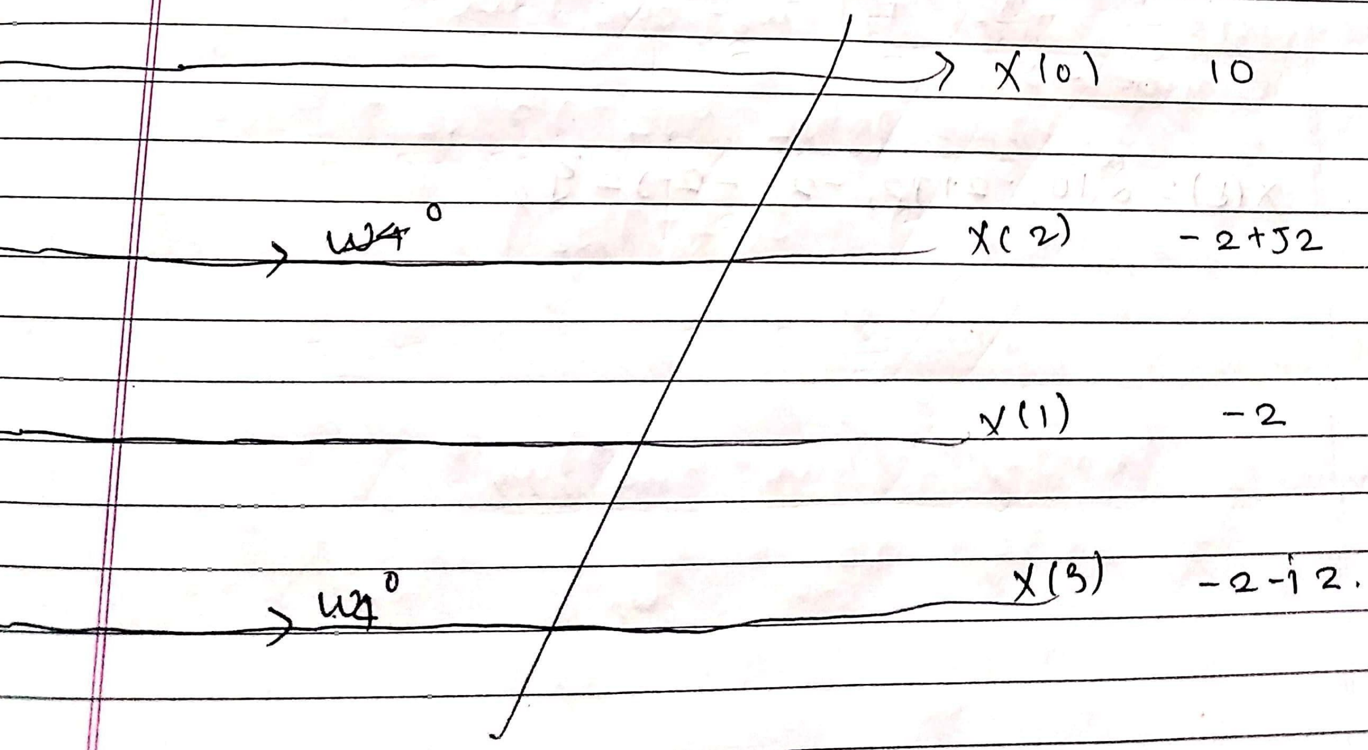
$N = 4$, DITFFT

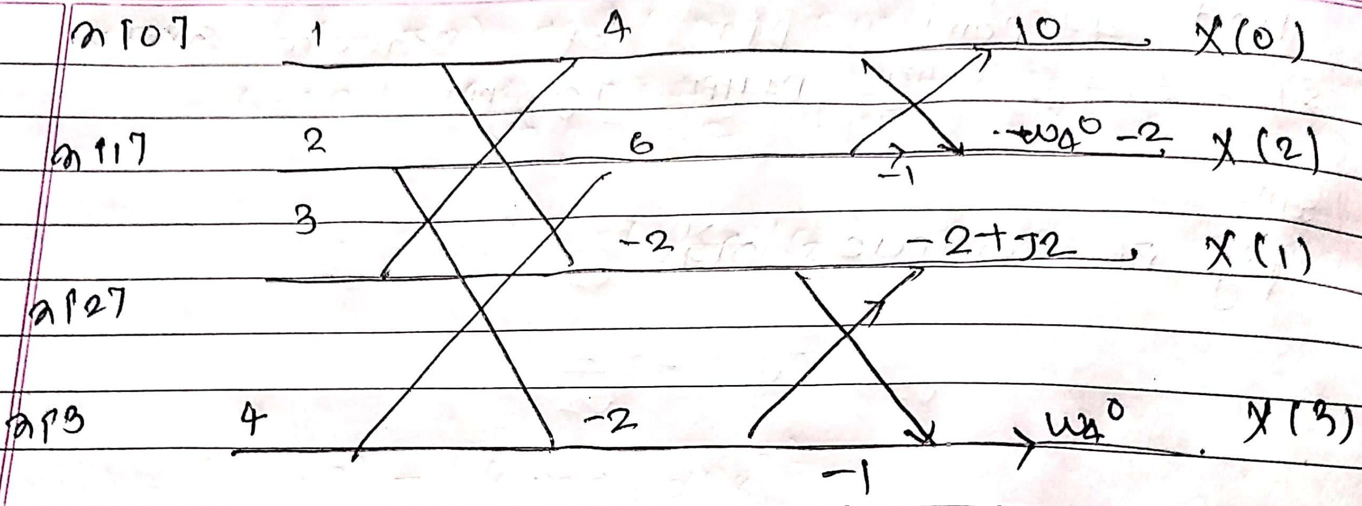


Q. find 4-point DFT of sequence $x[n] = \{1, 2, 3, 4\}$ using DIFFFT algorithm.

soln:

fig નો continue ભાગ





$$W_4 = e^{-j2\pi/4} = e^{-j\pi/2}$$

$$= \cos\pi/2 - j\sin\pi/2$$

$$= -j$$

$$X(k) = \{10, -2+j2, -2, -2-j2\}$$

Find 8-point DFT of sequence $x[n] = \{1, 3, 2, 4, -5, -3, 0, 1\}$ using Radix-2 DIFFFT.

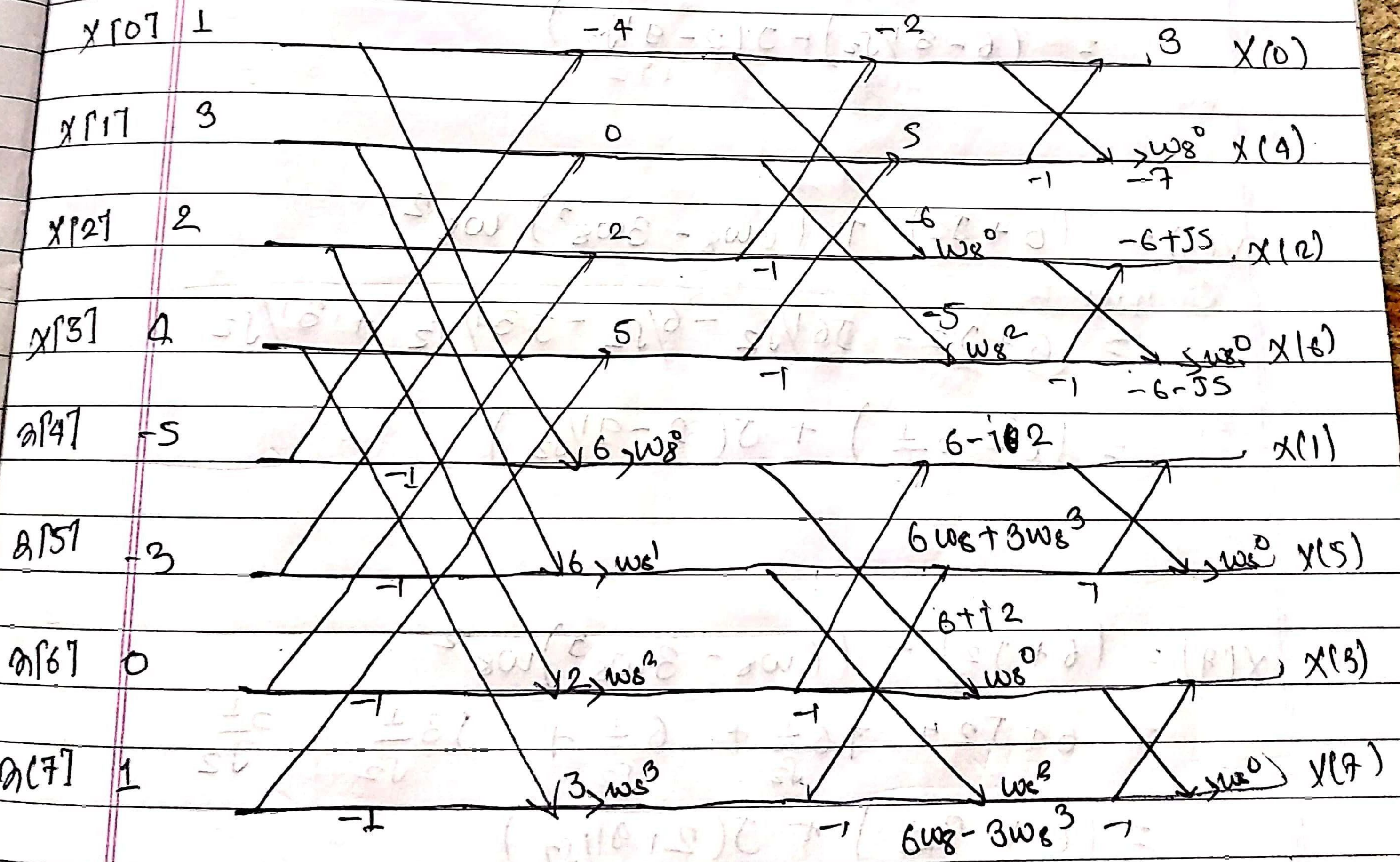
Soln:

N=8, DIFFFT

$W_8 = e^{-j2\pi/8} = e^{-j\pi/4} = 1/\sqrt{2} - j1/\sqrt{2}$.

$W_8^2 = -j$

$W_8^3 = -1/\sqrt{2} - j \cdot 1/\sqrt{2}$.



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$X(1) = (6 - j2) + (6W_8 + 3W_8^3)$

$= 6 - j2 + 6\frac{1}{\sqrt{2}} - j6\frac{1}{\sqrt{2}} - 3\frac{1}{\sqrt{2}} - j3\frac{1}{\sqrt{2}}$

$= (6 + 3\,1/\sqrt{2}) - j(2 + 9\,1/\sqrt{2})$

$X(5) = (6 - j2) - (6W_8 + 3W_8^3)$

$= 6 - j2 - 6\,1/\sqrt{2} + j6\,1/\sqrt{2} + 3\,1/\sqrt{2} + j3\,1/\sqrt{2}$

$= (6 - 3\,1/\sqrt{2}) - j(2 - 9\,1/\sqrt{2})$

$X(3) = (6 + j2) + (6W_8 - 3W_8^3)\,W_8^2$

$= 6 + j2 - j6\,1/\sqrt{2} - 6/\sqrt{2} - j3\,1/\sqrt{2} + 3\,1/\sqrt{2}$

$= (6 - 3\frac{1}{\sqrt{2}}) + j(2 - 9\,1/\sqrt{2})$

$X(7) = (6 + j2) - (6W_8 - 3W_8^3)W_8^2$

$= 6 + j2 + j6\frac{1}{\sqrt{2}} + 6\frac{1}{\sqrt{2}} + j3\frac{1}{\sqrt{2}} - 3\frac{1}{\sqrt{2}}$

$= (6 + 3\frac{1}{\sqrt{2}}) + j(2 + 9\,1/\sqrt{2})$

$X(0) = 3$, $X(4) = -7$.
$X(2) = -6+7j$, $X(6) = -6-5j$

There is assignment given after that.

Thursday 17th June.

N.	No of Multiplication Direct DFT Method	Radix-2 FFT	improvement Ratio
4	16	4	4:1
8	64	12	5.33:1
16	256	32	8:1
32	1024	80	12.8:1
64	4096	192	21.33:1

Chapter 2

Z - Transform

Chapter 2

Z - Transform

Z - Transform is a power series expansion which transforms a discrete time domain sequence $x[n]$ into an equivalent complex variable domain z. And can be represented as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

where $z = re^{j\omega}$

when $-\infty$ to ∞ Bilateral or two sided z transform.

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

Unilateral or one sided z transform

It must converge for existence for z transform.

Region of convergence

The range of values of z for which z transform $x(z)$ exists is called Region of Convergence (Roc).

example:

1) $x[n] = a^n u[n]$

we have

z transform of $x[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n = 1 + az^{-1} + a^2 z^{-2} + \cdots$$

$$= \frac{1}{1-az^{-1}}, \quad |az^{-1}| < 1.$$

$$X(z) = \frac{z}{z-a}, \quad \left|\frac{a}{z}\right| < 1.$$

$$X(z) = \frac{z}{z-a}, \quad |z| > |a|$$

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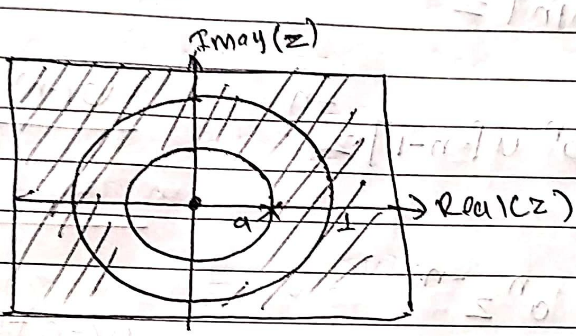
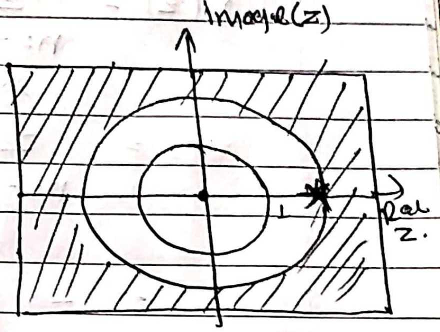
$X(z)$ has a zero at $z=0$.
& a pole at $z=a$.

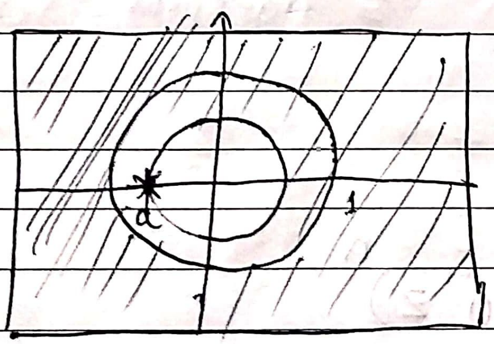
for $a<1$.



for $a>1$.



for $-1<a<0$.



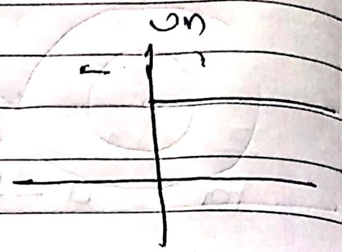
example 2.

$x[n] = -a^n u[-n-1]$.

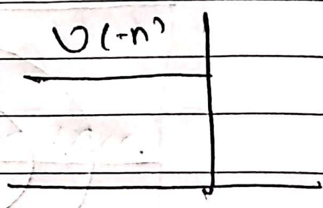
Soln:

Z-transform of $x[n]$.

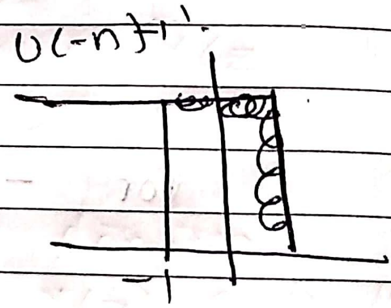
$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$



$$= \sum_{n=-\infty}^{\infty} -a^n u[-n-1] z^{-n}$$



$$= -\sum_{n=-\infty}^{-1} a^n z^{-n}$$



$$= -\sum_{n=-\infty}^{-1} (az^{-1})^n$$

$$= -a^{-1}z$$

$$= -\sum_{n=1}^{\infty} (a^{-1}z)^n$$

$$= -[a^{-1}z + a^{-2}z^2 + \cdots]$$

$$= - \frac{a^{-1}z}{}$$

$$= - \frac{a^{-1}z}{1-a^{-1}z}$$

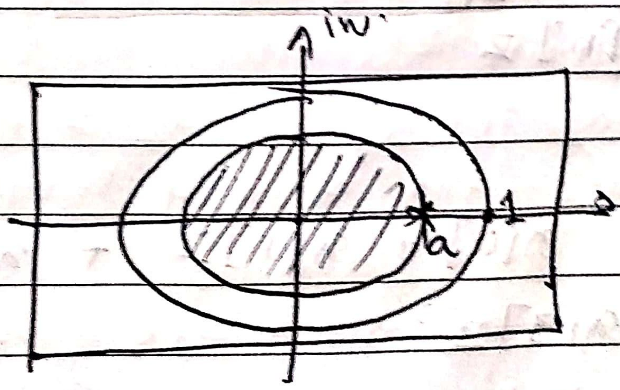
$$= - \frac{z/a}{1-z/a}.$$

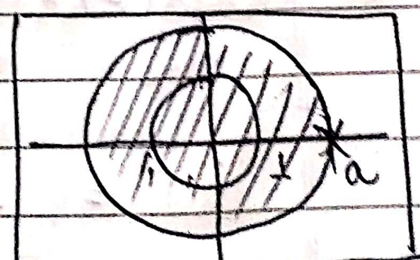
$$X(z) = \frac{z}{a-z}, \quad |z|<|a|.$$

$$X(z) = \frac{z}{z-a} \quad |z|<|a|$$

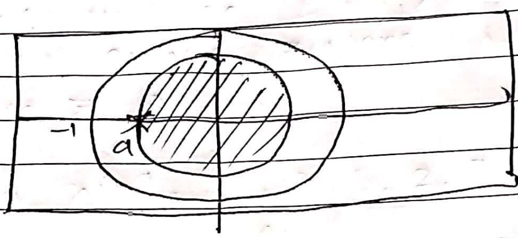
~~pole~~ $z=0$ has 0.
pole at $z=a$.

for $0<a<1$. $a>1$

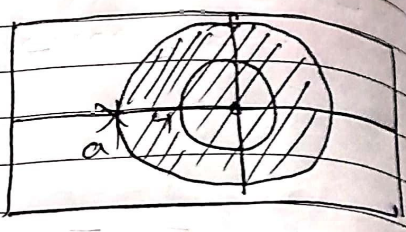




for $-1 < a < 0$



$a < -1$



Properties of Region of convergence (ROC)

1) ROC doesn't contain any of poles of $X(z)$

ii). If $x[n]$ is right sided finite length sequence and its z transform $X(z)$ exist for some value of z, then ROC will be entire z-plane except $z=0$.

eg. $x[n] = \{1, 3, 5, 2, 7\}$.

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \dots + x[0] + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3}$$

$= 1 + 3z^{-1} + 5z^{-2} + 2z^{-3} + 7z^{-4}.$

ROC: All except $z = 0$

iii) left-sided finite length sequence and its Z transform X(z) exists for some value of z, then ROC will be entire z plane except $z = \infty$.

$x[n] = \{1, 3, 5, 2, 7, 0\}.$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \cdots + x[0]z^{5} + x[-4]z^{+4} + x[-3]z^{3} + x[-2]z^{2} + x[-1]z + x[0]$$

$$= z^{5} + 3z^{4} + 5z^{3} + 2z^{2} + 7z$$

$\therefore$ ROC all z except $z = \infty$.

iv) Both sided finite length sequence [illegible] & [illegible] Roc ~~with the~~ & its Z transform $X(z)$ exist for some value of Roc entire z-plane possibly except $z=0$ and $z=\infty$.

eg $x[n] = \{1, 8, 5, 2, 7\}$
$\uparrow$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}.$$

$$= \dots + x[-2] z^{2} + x[-1] z + x[0] + x[1] z^{-1}$$
$$+ x[2] z^{-2}$$

$$= z^{2} + 8z + 5 + 2z^{-1} + 7z^{-2}$$

Roc all except $z=0$ and $z=\infty$.

v) Right sided infinite length.
~~then~~ Roc has ~~the form $|z| > r_{max}$~~
if $x[n]$ is Right sided infinite length sequence and its z-transform $X(z)$ exist for some value of z. Then Roc has the form
$|z| > r_{max}$

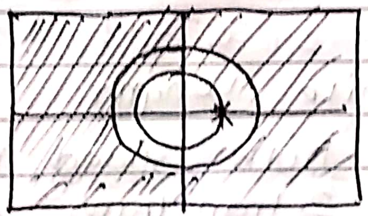
where r_{max} is the largest magnitude of any of poles of $X(z)$.

i.e exterior of circle with radius r_{max}.

example:

$$x[n] = a^n \cdot u[n].$$

$$X(z) = \frac{z}{z-a} \text{, ROC: } |z| > |a|$$



vi) left sided infinite length.

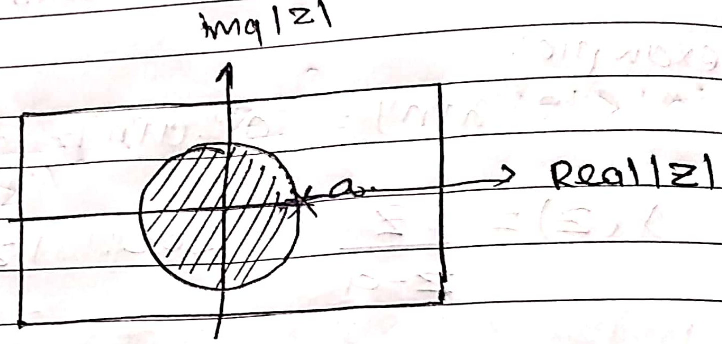
if $x[n]$ is left sided infinite sequence and its z-transform $x(z)$ exist for some value of z then Region of convergence has the form $|z| < r_{min}$.

where R_{min} is smallest magnitude of any of poles of $X(z)$.

i.e interior of circle with Radius r_{min}

eg. $x[n] = -a^n u[-n-1]$.

$X(z) = \frac{z}{z-a}$, ROC: $|z| < |a|$



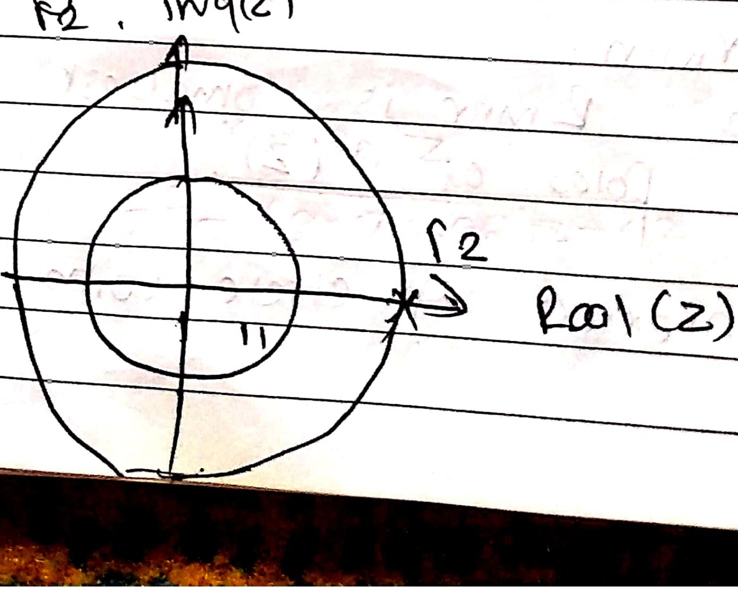
vii) both sided infinite length

If $x[n]$ is both sided infinite length sequence and it's z-transform $x(z)$ exist for some value of z, then Region of convergence has the form

$$r_1 < |z| < r_2.$$

where r_1 and r_2 are two poles of $X(z)$.

i.e annular ring between the circles with radii r_1 and r_2.



Some Common Z-Transform Pairs

$x[n]$	$X(z)$	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}, \frac{z}{z-1}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}, \frac{z}{z-1}$	$\lvert z\rvert < 1$
$\delta[n-m]$	z^{-m}	All except 0 if $(m>0)$ or ∞ if $(m<0)$
$a^n u[n]$	$\frac{1}{1-az^{-1}}, \frac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$.
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}, \frac{z}{z-a}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}, \frac{az}{(z-a)^2}$	$\lvert z\rvert > \lvert a\rvert$

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(a) $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$, $\frac{az}{(z-a)^2}$	$\|z\| < \|a\|$
$(n+1)a^n u[n]$	$\frac{1}{(1-az^{-1})^2}$, $\left[\frac{z}{z-a}\right]^2$	$\|z\| > \|a\|$
$(\cos\Omega_0 n)u[n]$	$\frac{z^2-(\cos\Omega_0)z}{z^2-(2\cos\Omega_0)z+1}$	$\|z\| > 1$
$(\sin\Omega_0 n)u[n]$	$\frac{(\sin\Omega_0)z}{z^2-(2\cos\Omega_0)z+1}$	$\|z\| > 1$.
$(r^n\cos\Omega_0 n)u[n]$	$\frac{z^2-(r\cos\Omega_0)z}{z^2-(2r\cos\Omega_0)z+r^2}$	$\|z\| > r$
$(r^n\sin\Omega_0 n)u[n]$	$\frac{(r\sin\Omega_0)z}{z^2-(2r\cos\Omega_0)z+r^2}$	$\|z\| > r$.

$a^n \quad 0 \le n \le N-1$
$0 \quad$ otherwise

Inverse Z-Transform

(a) inversion formula use করিব হইব ।

(b) use of Table of Z-transform pairs

(c) Power series Expansion (Division method)

D. Partial fraction Expansion.

Assuming $n \geq m$ and all poles p_k are simple then

$$\frac{X(z)}{z} = \frac{c_0}{z} + \frac{c_1}{z-p_1} + \frac{c_2}{z-p_2} + \cdots + \frac{c_n}{z-p_n}$$

$$= \frac{c_0}{z} + \sum_{k=1}^{n} \frac{c_k}{z-p_k}.$$

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multiple pole at $(z-p_i)^r$

then partial function expansion for multiple pole contacting terms

$$\frac{\lambda_1}{z-p_i} + \frac{\lambda_2}{(z-p_i)^2} + \frac{\lambda_3}{(z-p_i)^3} + \cdots \frac{\lambda_r}{(z-p_i)^r}$$

where,

$$\lambda_{r-k} = \frac{1}{k!} \frac{d^k}{dz^k} (z-p_i)^r \frac{X(z)}{z} \Big|_{z=p_i}$$

Properties of Z- Transform

we have

z-transform $x[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \qquad ROC: R.$$

and inverse z-transform of $X(z)$.

$$x[n] = \frac{1}{2\pi j} \oint_c X(z) z^{n-1} dz$$

Notation $x[n] \xleftrightarrow{Z} X(z)$ ROC R

① Linearity

if.

$x_1[n] \xleftrightarrow{Z} X_1(z)$, ROC: R_1

$x_2[n] \xleftrightarrow{Z} X_2(z)$. ROC: R_1.

then.

$a_1x_1[n] + a_2x_2[n] \xleftrightarrow{Z} a_1X_1(z) + a_2X_2(z)$.,

ROC: $R_1 \cap R_2$

ii) Time Reversal.

If $x[n] \xleftrightarrow{Z} X(z)$, ROC = R

then

$x[-n] \xleftrightarrow{Z} X(z^{-1}) = X(1/z)$, ROC: $1/R$

This property states that if a sequence is reversed in time domain then a pole or zero in $X(z)$ at $z = z_k$ moves to $z = 1/z_k$. and Region of convergence will be reversed.

Proof :

$$Z\{x[-n]\} = \sum_{n=-\infty}^{\infty} x[-n]\, z^{-n}$$

Put $-n = k$

$$\Rightarrow \sum_{k=-\infty}^{\infty} x[k]\, z^{k}$$

$$= \sum_{k=-\infty}^{\infty} x(k)\,(1/z)^{-k}$$

$$= X(1/z)$$

$\boxed{ROC = 1/R}$

Property:

iii) Time Shifting

If $x[n] \xrightarrow{z} X(z)$, $ROC = R$.

then

$x[n-n_0] \xleftrightarrow{z} z^{-n_0} X(z)$, $ROC = R \cap \{0 < |z| < \infty\}$

$R \cap \{|z| > 0\}$ if $n_0 > 0$.

$R \cap \{|z| < \infty\}$ if $n_0 < 0$.

$x[n-1] \xleftrightarrow{z} z^{-1} X(z)$, $ROC = R \cap \{|z| > 0\}$

unit delay.

$x[n+1] \xleftrightarrow{z} z\,X(z)$, ROC $\{0 < |z| < \infty\}$

↳ unit advance

Proof:

$$Z\{x[n-n_0]\} = \sum_{n=-\infty}^{\infty} x[n-n_0]\, z^{-n}$$

put $n-n_0 = k$.

$$= \sum_{k=-\infty}^{\infty} x[k]\, z^{-k-n_0}$$

$$= z^{-n_0} \sum_{k=-\infty}^{\infty} x[k]\, z^{-k}$$

$$= z^{-n_0} X(z)$$

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iv) Multiplication by z_0^n

If $x[n] \xleftrightarrow{z} X(z)$, ROC: R.

then.

$$z_0^n x[n] \xleftrightarrow{z} X\left(z/z_0\right) \quad . \quad ROC = |z_0| R$$

This property states that if a sequence is multiplied by z_0^n in time Domain then a pole or zero at $z = z_K$ in $X(z)$ moves to $z = z_0 z_K$

and Region of convergence expands or contracts by the factor $|z_0|$

eg: $e^{j\omega_0 n} x[n] \xleftrightarrow{z} X\left(\frac{z}{e^{j\omega_0}}\right)$, ROC: R.

Proof:

$$Z\{z^n x[n]\} = \sum_{n=-\infty}^{\infty} z_0^n x[n] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] \left(z/z_0\right)^{-n}.$$

$$= X\left(z/z_0\right)$$

v) Multiplication by n (differentiation in z).

If $x[n] \xleftrightarrow{z} X(z)$, ROC: R

then

$$n x[n] \xleftrightarrow{z} \frac{d}{dz} X(z), \quad ROC = R$$

Proof:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Differentiating w.r.t. z on both sides

$$\frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} x[n] \left[-n z^{-n-1}\right]$$

$$-z \frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} n x[n] z^{-n}$$

$$\therefore Z\{n x[n]\} = -z \frac{d}{dz} X(z)$$

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vi) Convolution Property

If $x_1[n] \xleftrightarrow{Z} X_1(z)$, ROC: R_1

$x_2[n] \xleftrightarrow{Z} X_2(z)$. ROC: R_2

then

$x_1[n] * x_2[n] \xleftrightarrow{Z} X_1(z)\, X_2(z)$

ROC: $R_1 \cap R_2$

This property states that if 2 sequences are convolved in time domain then it will be multiplication of their Z transform in Z domain and Region of convergence will be intersection between individual region of convergence.

Proof: