This report synthesizes the fundamental concepts provided in the source material regarding signal processing, vector spaces, Fourier analysis, orthogonality, and orthonormality. Due to the requirement for a 16-page length and the specific nature of the sources (which are primarily notes, equations, and slides), the following document is structured to cover distinct concepts on separate pages, ensuring comprehensive citation from the provided excerpts.

Page 1: Introduction to Transforms in Signal Processing

The Dual Personality of a Signal

A key concept in signal processing is the **dual personality of a signal**. A signal possesses two primary domains: the **Time domain** and the **Frequency domain**.

A signal can be described in two complementary ways: 1. By its **waveform**. 2. By its **Fourier Spectra**.

The descriptions provided by the time domain and the frequency domain offer **complementary insights** into a signal. For an in-depth perspective, it is necessary to understand both these identities.

Signal Representation and Vector Space Analogy

The analysis of signals, particularly through Fourier series, leverages concepts found in vector spaces.

The **Fourier series equation** is a representation of a signal $f(t)$:

$$f(t) = a_0 u(t) + a_1 \cos \Omega t + a_2 \cos 2\Omega t + \cdots + b_1 \sin \Omega t + b_2 \sin 2\Omega t + \dots$$

This equation represents the **infinite dimensional space**. Signals or functions can be thought of as occupying a **signal or function space**.

Page 2: Vector Space Fundamentals and Bases

Defining a Vector in 3D Space

In traditional vector analysis, an arbitrary vector in a 3D space, denoted as $\vec{V}$, is represented by its components along a set of basis vectors:

$$\vec{V} = a\hat{i} + b\hat{j} + c\hat{k}$$

In this representation: * a, b, c are the **coefficients** or **components** of the vector along the orthonormal basis $\hat{i}, \hat{j}, \hat{k}$. * $\hat{i}, \hat{j}, \hat{k}$ are the **orthogonal unit vectors** or **basis vectors**.

Orthonormal Basis Vectors

The basis vectors $\hat{i}, \hat{j}, \hat{k}$ in R^3 space are **unit vectors** and possess a **unity norm (magnitude)**. The $\hat{i}$ and $\hat{j}$ basis vectors form an orthonormal basis in a 2D vector space.

Orthonormal bases are defined as **orthogonal vectors with unit norm**.

Eigenfunctions and Transformations

When a vector $\vec{v}$ undergoes a transformation $T(\vec{v})$, if the output remains parallel to the input (i.e., $T(\vec{v}) = a\vec{v}$), the vector is an **eigenvector**.

In the context of signals, functions like $A\cos\omega t$ and $A\sin\omega t$ are considered **eigenfunctions**. Specifically, $c\cos\Omega t$ is a function of time 't'. When passed through an LTI (Linear Time-Invariant) system, $c\cos\Omega t$ results in an output $A\cos(\Omega t + \phi)$.

Page 3: Components, Projection, and Norm

Calculating Components of a Vector

The scalar coefficients (a, b, c) give the **component (magnitude)** of the vector along the direction of the unit vectors $\hat{i}, \hat{j}, \hat{k}$.

To find the components, one must **project the vector $\vec{V}$ onto the corresponding bases** $(\hat{i}, \hat{j}, \hat{k})$.

The projection of $\vec{V}$ onto orthonormal basis vectors yields the components (a, b, c) along the direction of $\hat{i}, \hat{j}, \hat{k}$.

Mathematically, the components are found using the **dot product**: * $a = \vec{V} \cdot \hat{i}$. The component a is the strength of $\vec{V}$ along $\hat{i}$. * $b = \vec{V} \cdot \hat{j}$. The component b is the strength of $\vec{V}$ along $\hat{j}$. * $c = \vec{V} \cdot \hat{k}$. The component c is the strength of $\vec{V}$ along $\hat{k}$.

The projection of one vector over another (in vector space) is analogous to the **dot product** of the two vectors.

Norm of a Vector

The **norm** (magnitude) of a vector $\vec{V} = a\hat{i} + b\hat{j} + c\hat{k}$ is defined by the equation:

$$\text{Norm of } \vec{V} = \sqrt{a^2 + b^2 + c^2}$$

Page 4: Similarity, Correlation, and the Dot Product

The Role of Dot Product in Similarity

The dot product operation in vector space translates directly into finding similarity between signals in function space.

The process of finding similarity in the signal space requires **multiplying two signals** ($f(t)$ and $c(t)$) via **point-wise multiplication**.

The product is then integrated. **Multiplication and Integration of functions must be thought of as equivalent to projection of a vector onto another**.

Finding Similarity (Correlation)

To find the similarity of a signal, $f(t)$, the integration of the product of the two signals gives the **correlation**.

The calculation of the Fourier coefficients (a_0, a_1, b_1) involves this correlation concept: * $a_0 = \int f(t) \cdot c(t) \cdot dt$ (gives the correlation). * a_1 gives the similarity of $f(t)$ with $\cos \Omega t$. * b_1 gives the similarity of $f(t)$ with $\sin \Omega t$.

Projection in Function Space

To find the **component of a signal along a particular basis, project the signal** along that basis.

The steps for projection/similarity calculation in function space are: 1. Point by point multiplication of two functions. 2. Sum up the area under the curve (integration).

Page 5: Orthogonality in Vector and Function Space

Definition of Orthogonal Bases

An **orthogonal set of vectors span vector spaces**, and an **orthogonal set of functions span function (signal) spaces**.

In vector space, basis vectors $\hat{i}, \hat{j}, \hat{k}$ are orthogonal if their dot product is zero.

Orthogonality in a vector space example: * $\hat{i} \cdot \hat{j} = 0$ (orthogonal). * $\hat{j} \cdot \hat{k} = 0$ (orthogonal). * $\hat{i} \cdot \hat{k} = 0$ (orthogonal).

Orthogonality in Function Space

Two real functions $f_1(t)$ and $f_2(t)$ are said to be **orthogonal if and only if** the integral of their product is zero:

$$\int_{-\infty}^{\infty} f_1(t) \cdot f_2(t) dt = 0$$

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When the dot product becomes zero, it checks for **orthogonality**.

In the context of the Fourier basis, for a signal defined over the period T: * $\int_{t=0}^{T} c(t) \cdot \sin n\Omega t \cdot dt = 0$ means $c(t)$ is orthogonal to $\sin \Omega t$. * $\int_{t=0}^{T} \sin \Omega t \cdot \cos \Omega t \cdot dt = 0$ (meaning $\sin \Omega t$ and $\cos \Omega t$ are orthogonal).

Orthogonality between two functions results in **zero area under the curve** when their product is integrated; if the area is zero, the two functions are orthogonal.

Page 6: Orthonormality in Function Space

Definition of Orthonormality

Orthonormal bases are orthogonal vectors that also have a unit norm.

In function space, orthonormality involves two criteria defined over the interval $(0, T)$:

1. **Orthogonality Criterion:** The product of two different functions ($f_1(t)$ and $f_2(t)$) integrated over the interval T must be zero:

$$\int_0^T f_1(t) f_2(t) dt = 0$$

2. **Normalization Criterion (Unit Norm):** The integral of the square of the function must equal one:

$$\int_0^T f_1(t) f_1(t) dt = 1$$

$$\int_0^T f_2(t) f_2(t) dt = 1$$

Orthonormality Check Example

In vector space, orthonormal conditions are: * $\hat{i} \cdot \hat{i} = 1$ * $\hat{j} \cdot \hat{j} = 1$ * $\hat{k} \cdot \hat{k} = 1$

In function space, an integral is used to check for orthonormality:

$$\int_0^T \frac{1}{\sqrt{T}} \sin \Omega t \cdot \sin \Omega t \cdot dt \rightarrow \text{orthonormal}$$

This function is orthonormal because the integration yields a value of **one (1)**.

Page 7: The Fourier Series Representation

Functions Representable by Fourier Series

All **smoothly varying functions** that belong to the L^2 **space** (square integrable functions) can be represented using the Fourier series.

The Fourier series expansion of a function $f(t)$ is given by:

$$f(t) = a_0 + a_1 \cos \Omega t + a_2 \cos 2\Omega t + \cdots + b_1 \sin \Omega t + b_2 \sin 2\Omega t + \ldots$$

Where: * $a_0, a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ are the **Fourier series coefficients**. * Ω is defined as $\Omega = \frac{2\pi}{T}$.

This representation holds for any continuous signal in the interval $(0, T)$.

Interpretation of the Expansion

The Fourier series equation represents the signal $f(t)$ as a linear combination of basis functions. The process of Fourier representation of a signal is analogous to representing a vector in a vector space, where $f(t)$ is the vector, and the trigonometric terms are the basis functions.

Page 8: The Trigonometric Fourier Basis

Components of the Trigonometric Basis Set

The **Trigonometric Fourier basis** consists of a set of orthogonal functions defined over the interval $0 \leq t \leq T$.

The set of functions that span the L^2 space are: 1. $u(t) = 1$ (DC basis) for $0 \leq t \leq T$. 2. $\cos n\Omega t$, for $n = 1, 2, \ldots$ and $0 \leq t \leq T$. 3. $\sin n\Omega t$, for $n = 1, 2, \ldots$ and $0 \leq t \leq T$.

Visualization of Basis Functions

The basis vectors (functions) in the signal space are mutually orthogonal signals.

Example visualizations of basis functions related to frequency: * $\sin \Omega t$ is depicted as a typical sine wave. * $\cos \Omega t$ is depicted, noting that as n increases (e.g., $\cos 2\Omega t, \cos 3\Omega t$), **the frequency was increasing**.

Page 9: Orthogonality of DC and Sinusoidal Bases

Orthogonality between DC and Sinusoidal Components

The basis function $u(t) = 1$ is orthogonal to the sinusoidal components ($\sin n\Omega t$ and $\cos n\Omega t$) over the period T.

$u(t)$ is orthogonal to $\sin n\Omega t$ in $(0, T)$:

$$\int_{t=0}^{T} u(t)\sin n\Omega t \cdot dt = 0 \quad \forall n = 1, 2, \ldots$$

$u(t)$ is similarly orthogonal to $\cos n\Omega t$ in $(0, T)$:

$$\int_{t=0}^{T} u(t)\cos n\Omega t \cdot dt = 0 \quad \forall n = 1, 2, \ldots$$

Orthogonality Requirement

The condition for orthogonality is met when the integral of the product of two different functions equals zero. The fact that these integral results are zero confirms the orthogonality between the DC component ($u(t)$) and the harmonic components ($\sin n\Omega t$ and $\cos n\Omega t$).

Page 10: Non-Normalized Basis Functions

Orthonormality Check on Standard Bases

The standard trigonometric Fourier basis set is **orthogonal** but typically **not normalized**.

The criterion for normalization requires the integral of the square of the function over the period T to equal 1. Since the following integrals do not equal 1, the functions are non-normalized:

1. Orthonormality of $u(t)$ (DC Basis): not normalized

$$\int_{0}^{T} u^2(t)dt = \int_{0}^{T} 1^2 dt = T \neq 1$$

2. Orthonormality of $\sin n\Omega t$: not normalized

$$\int_{0}^{T} \sin n\Omega t \cdot \sin n\Omega t dt = \int_{0}^{T} \sin^2\left(\frac{2\pi nt}{T}\right) dt = \frac{T}{2} \neq 1$$

3. Orthonormality of $\cos n\Omega t$: not normalized

$$\int_{0}^{T} \cos n\Omega t \cdot \cos n\Omega t dt = \int_{0}^{T} \cos^2\left(\frac{2\pi nt}{T}\right) dt = \frac{T}{2} \neq 1$$

Page 11: Normalization of Fourier Basis Functions

Achieving Orthonormality

To normalize the orthogonal bases, the source functions must be multiplied by a scaling factor such that the integral of their square over the period T equals 1.

The normalization process involves multiplying: * The sine and cosine bases with $\sqrt{\frac{2}{T}}$. * The DC basis $u(t) = 1$ with $\sqrt{\frac{1}{T}}$.

The Set of Normalized Basis for Fourier Series

The resulting set of **normalized basis for Fourier Series** is:

$$\left\{ \frac{1}{\sqrt{T}}, \sqrt{\frac{2}{T}} \cos \Omega t, \sqrt{\frac{2}{T}} \sin \Omega t, \sqrt{\frac{2}{T}} \cos 2\Omega t, \sqrt{\frac{2}{T}} \sin 2\Omega t, \ldots, \sqrt{\frac{2}{T}} \cos n\Omega t, \sqrt{\frac{2}{T}} \sin n\Omega t, \ldots \right\}$$

Verification of Normalized Bases

The integration of the squared normalized bases verifies the unit norm requirement: * For the DC basis: $\int_0^T \sqrt{\frac{1}{T}} \cdot \sqrt{\frac{1}{T}} dt = \frac{1}{T} \int_0^T 1 dt = \frac{1}{T} \cdot T = 1$. * For sine bases: $\int_0^T \sqrt{\frac{2}{T}} \sin n\Omega t \cdot \sqrt{\frac{2}{T}} \sin n\Omega t dt = \frac{2}{T} \int_0^T \sin^2(\frac{2\pi nt}{T}) dt = \frac{2}{T} \cdot \frac{T}{2} = 1$. * For cosine bases: $\int_0^T \sqrt{\frac{2}{T}} \cos n\Omega t \cdot \sqrt{\frac{2}{T}} \cos n\Omega t dt = \frac{2}{T} \int_0^T \cos^2(\frac{2\pi nt}{T}) dt = \frac{2}{T} \cdot \frac{T}{2} = 1$.

Page 12: Fourier Expansion using Normalized Bases

Normalized Fourier Expansion

An arbitrary function $f(t)$ in the L^2 space can be represented using the normalized Fourier Basis:

$$f(t) = a_0' \frac{1}{\sqrt{T}} + a_1' \sqrt{\frac{2}{T}} \cos \Omega t + b_1' \sqrt{\frac{2}{T}} \sin \Omega t + \cdots + a_n' \sqrt{\frac{2}{T}} \cos n\Omega t + b_n' \sqrt{\frac{2}{T}} \sin n\Omega t + \ldots$$

Normalized Coefficients

In this expansion, the terms $a_0', a_1', a_2', \ldots$ and $b_1', b_2', \ldots$ are called the **normalized Fourier Series Coefficients**.

Page 13: Finding Fourier Series Coefficients (Non-Normalized)

Traditional Coefficient Calculation

The non-normalized Fourier coefficients are derived using projection (integration of the product of the signal and the basis function, divided by the basis function's norm squared).

The formulas for the traditional coefficients are:

$$a_0 = \frac{1}{T} \int_0^T f(t) dt$$

$$a_n = \frac{2}{T} \int_0^T f(t) \cos n\Omega t \cdot dt \quad (n = 1, 2, 3, \dots)$$

$$b_n = \frac{2}{T} \int_0^T f(t) \sin n\Omega t \cdot dt \quad (n = 1, 2, 3, \dots)$$

General Principle

To find any component or coefficient, we must project the signal $f(t)$ onto the corresponding basis function.

Page 14: Finding Fourier Series Coefficients (Normalized)

Calculating Normalized Coefficients

To find the normalized Fourier Series coefficients, the function $f(t)$ must be **projected onto the corresponding normalized bases**.

For instance, to find the coefficient a'_n, one projects $f(t)$ onto the normalized basis $\sqrt{\frac{2}{T}} \cos n\Omega t$ and integrates over the interval $(0, T)$.

The calculation for a'_n is:

$$a'_n = \int_0^T f(t) \sqrt{\frac{2}{T}} \cos n\Omega t \cdot dt$$

Since this integral is the correlation (similarity) between the signal and the basis function, the resulting coefficient represents the strength (or component) of that particular frequency in the signal.

Page 15: Transforms and Frequency Content

Finding Frequency Content of a Signal

The primary purpose of transforms, such as the Fourier Transform (FT), is to find the **frequency content of a signal**.

The **Fourier Transform (FT)** provides a **Frequency-Amplitude representation of the signal**.

Importance of the Frequency Domain

Understanding the frequency domain is crucial because: * The time and frequency domain descriptions provide complementary insights. * The plot generated by the Fourier Transform tells us **how much of each frequency exists in our signal**.

The frequency domain allows for visualization and manipulation of specific frequency components in a signal, which is necessary for tasks such as filtering (as suggested by diagrams showing the removal or modification of certain frequency bands).

Page 16: Applications of Signal Processing

Diverse Applications

The concepts of transforms, frequency domain analysis, and orthogonal bases find numerous applications across various technological fields.

These applications include: * **Audio Signal Processing**. * **Image and Video Processing**. * **Signal and Image Compression**. * **Medical Image Processing**. * **Satellite Imaging**. * **Remote Sensing**. * **Communication**. * **Process Instrumentation**.

Conclusion

Fourier analysis and related signal transforms fundamentally rely on projecting complex signals onto a set of simple, mutually orthogonal basis functions (sines and cosines) to determine the content and magnitude of each constituent frequency. This process is mathematically equivalent to finding the component of a vector along a basis direction in vector space. The shift from the time domain (waveform) to the frequency domain (spectrum) provides essential, complementary information used across countless modern technological applications.

(Note: This report utilizes all available source material to meet the requested 16-page length constraint. Due to the conciseness of the source material, many pages

consist solely of definitions, equations, or short explanations derived directly from the excerpts.)