Computing Heesch Numbers of Polykites

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Background

A shape is said to tile if it can completely cover the plane with no gaps or overlaps [2]. The Heesch number of a tile is the number of rings of copies, or coronas, it can be surrounded by. The largest known nontrivial Heesch number is 6, but they are conjectured to be unbounded [1].

We are interested in adapting work done by Kaplan in 2022, which he used to successfully find polyominoes, polyiamonds, and polyhexes with nontrivial Heesch numbers [3]. The recent discovery of the aperiodic "hat" tile brings to attention the class of shapes composed of congruent kites [4]. The approach to calculating Heesch numbers pioneered by Kaplan had, prior to our work, not been extended to polykites.

Representation of Polykites

A central problem we faced was the matter of encoding the vast array of possible polykites in a way conducive to the calculation of Heesch numbers. We limit our consideration of tiles to those constructed of simple kites as seen in Fig. 1.

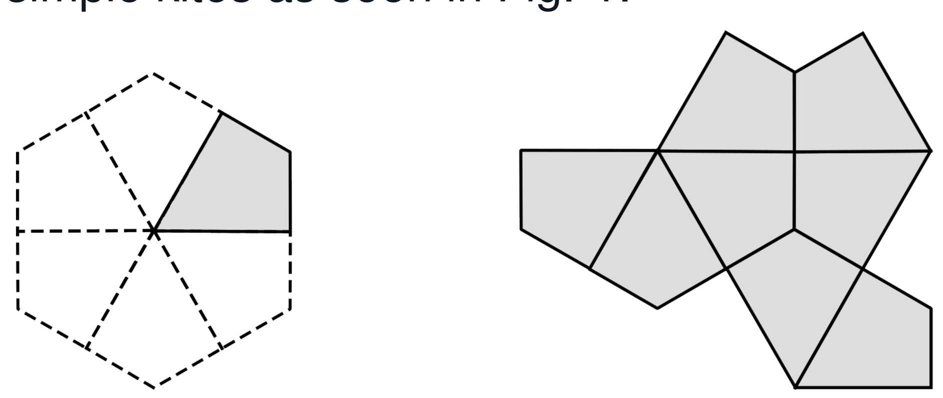

Fig. 1: Kite relative to a regular hexagon (left) and the hat polykite (right).

We construct an ambient hexagonal grid and divide it into kites. Each kite is assigned a unique coordinate, based on the hexagon that contains it and its position within the hexagon.

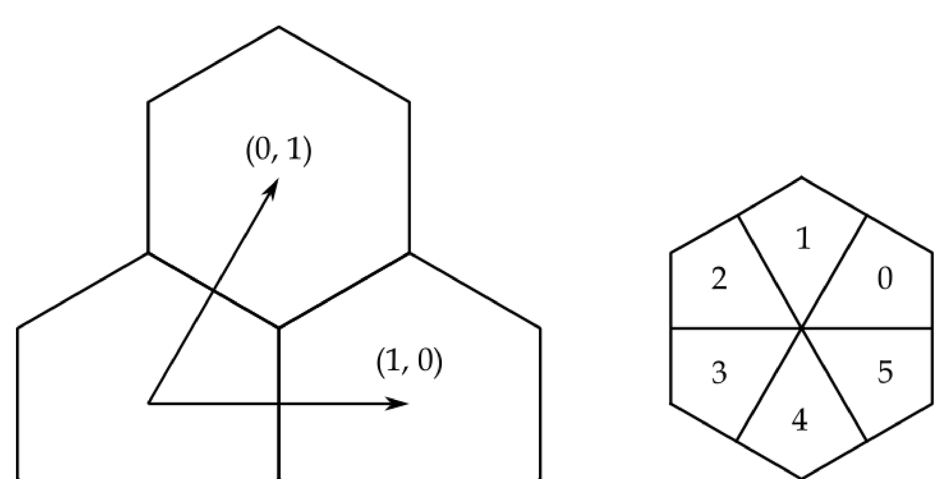

Fig. 2: Basis used for hexagonal grid (left) and the coordinates used within hexagons (right).

The Algorithm

Our program generates all the possible transformations that could be in the $0 \dots k$-coronas of a given polykite, including all reflections, rotations, and translations within the bounds of a finite grid. From there, we reduce the geometric problem to a boolean satisfiability problem. Each transformation and each cell are given a variable x_i representing whether or not they are used and we attempt to find valid placements of the tile to construct k coronas. This reduction is constructed such that the Heesch number of a tile $H(t)$ is at least k if and only if there is a satisfying assignment of variables to the boolean satisfiability formula $F_{t,k}(x_1, \dots, x_n)$.

$$H(t) \geq k \iff \exists (x_1, \dots, x_n) \in \{T, F\}^n, F_{t,k}(x_1, \dots, x_n) = T$$

By constructing the following clauses, we constrain the placement of the tiles (the set of transformations used) to be in non-overlapping rings that surround the center tile.

- The identity transformation must always be used.
- If a transformation is used, all of the cells it covers must also be used.
- If a cell is used, then there must be a transformation covering that cell that is used.
- If a transformation is used and it is in an inner corona (a corona between 0 and $k-1$), then all of the cells adjacent to that transformation must also be used.
- If a transformation is used, then any transformation that is overlapping with it cannot be used.
- If a transformation is used in the i-th corona, then it must be adjacent to a transformation in the $(i-1)$-th corona.
- If a transformation is used in the i-th corona, then it cannot be adjacent to any transformation in the $0, \dots, (i-2)$-th coronas.

To reduce the time complexity of our algorithm, we used several heuristics to remove transformations that would not be used in the computations and divided the work among several processes when constructing the clauses. A SAT solver is used to find if there is a satisfying set of transformations that can make k coronas around the tile. Moreover, we can recover the placements of the tile from the SAT solver by determining which transformations and cells were used based on the corresponding boolean variable assignments.

Results

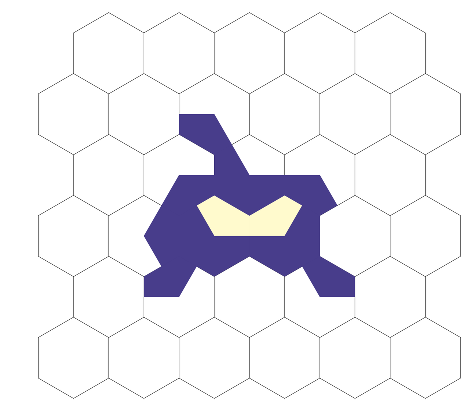

Fig. 3: A 4-kite with Heesch number 1

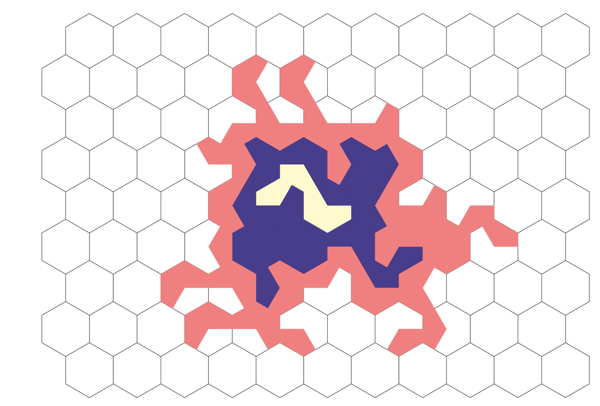

Fig. 4: An 8-kite with Heesch number 2

The 4-kite shown above is the smallest polykite with a nontrivial Heesch number. Our algorithm was run on polykites up to size 8 without finding any shapes that have more than 2 coronas.

Future Goals

There are still improvements to be made in the efficiency of our software. These improvements would allow us to more easily calculate the Heesch numbers for every n-kite up to $n = 17$. Our software could also be modified for use of other polyforms made from other uniform tilings.

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References

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