Samir Khadka
Assignment 4
Linear Algebra

Question 1:

Calculation for Car Distribution:
Given:
Initial car distribution on Monday:

$$C_0 = \begin{bmatrix} 295 \\ 55 \\ 150 \end{bmatrix}$$

Return rate matrix R:

$$R = \begin{bmatrix} 0.97 & 0.05 & 0.10 \\ 0.00 & 0.90 & 0.05 \\ 0.03 & 0.05 & 0.85 \end{bmatrix}$$

Distribution on Tuesday:
$C_1 = R \times C_0$

$$C_1 = \begin{bmatrix} 0.97 & 0.05 & 0.10 \\ 0.00 & 0.90 & 0.05 \\ 0.03 & 0.05 & 0.85 \end{bmatrix} \begin{bmatrix} 295 \\ 55 \\ 150 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} (0.97 \times 295) + (0.05 \times 55) + (0.10 \times 150) \\ (0.00 \times 295) + (0.90 \times 55) + (0.05 \times 150) \\ (0.03 \times 295) + (0.05 \times 55) + (0.85 \times 150) \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 286.15 + 2.75 + 15 \\ 0 + 49.5 + 7.5 \\ 8.85 + 2.75 + 127.5 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 303.90 \\ 57.00 \\ 139.10 \end{bmatrix}$$

Distribution on Wednesday

$$C_2 = R \times C_1$$

$$C_2 = \begin{bmatrix} 0.97 & 0.05 & 0.10 \\ 0.00 & 0.90 & 0.05 \\ 0.03 & 0.05 & 0.85 \end{bmatrix} \begin{bmatrix} 303.90 \\ 57.00 \\ 139.10 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} (0.97 \times 303.90) + (0.05 \times 57) + (0.10 \times 139. \\ (0.00 \times 303.90) + (0.90 \times 57) + (0.05 \times 139) \\ (0.03 \times 303.90) + (0.05 \times 57) + (0.85 \times 139) \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 294.78 + 2.85 + 13.91 \\ 0 + 51.30 + 6.955 \\ 9.117 + 2.85 + 118.235 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 311.54 \\ 58.255 \\ 130.202 \end{bmatrix}$$

Thus, the approximate distribution of cars on wednesday is

$$\begin{bmatrix} 312 \\ 58 \\ 130 \end{bmatrix}$$

Question 2:

False.
The product AB is computed by multiplying matrix A with each column of B separately. If $B = [b_1\ b_2\ b_3]$ where $b_1,\ b_2,\ b_3$ are the columns of B, then $AB = [Ab_1\ Ab_2\ Ab_3]$, not $AB = [Ab_1 + Ab_2 + Ab_3]$. The addition of the products does not represent the multiplication of two matrices.

b. True.
The second row of AB is indeed the result of the second row of A multiplied on the right by matrix B. This is a direct application of the definition of matrix multiplication.

c. False.
Matrix multiplication is associative, so $(AB)C = A(BC)$, but it is generally not true that $(AB)C = (AC)B$. The order of multiplication cannot be changed because matrix multiplication is not commutative.
d. False.

The transpose of a product of two matrices is equal to the product of the transposes of the matrices in reverse order. That is, $(AB)^T = B^T A^T$, not $A^T B^T$.

e. True.
The transpose of a sum of matrices is equal to the sum of the transposes of those matrices. This means that $(A + B)^T = B^T + A^T$.

Question 3:

Let us consider two matrices: P and Q which have an order m x n and n x p respectively. Therefore, their product has order m x p according to matrix multiplication.
Let $q_1, q_2,........., q_n$ be the columns of matrix Q.
The product of P and Q is:
$PQ = P(\ * \ q_1, q_2,........., q_n)$
$= (* \ P q_1, P q_2,........., P q_n)$
Given: Product of AB
Let A be a matrix of order m x n.
Their product is:
$AB = A(* \ \ 0 \ \ * \ \ .... *)$
$= (* \ A(0) \ \ * \ \ .... *)$
$= (* \ \ 0 \ \ * \ \ .... *)$
Here, if the elements of matrix A multiplies the second=d column of matrix B which are all zero, the second column in final result of the product is also zero as shown above.

Question 4:

a. B is invertible.
   Since A and X are invertible, and it is given that $A - AX$ is invertible, we can write:
   $(A - AX)^{-1} = X^{-1}B$
   Multiplying both sides by $A - AX$ and then by X, we get:
   $I = (A - AX)X^{-1}B$
   $X = (A - AX)B$
   Since A and A - AX are invertible, their product with B is also invertible (the product of invertible matrices is invertible). Therefore, X is invertible. This also implies that B is invertible since the multiplication of an invertible matrix X^{-1} with B yields an invertible matrix .
b.
   To solve for X, we can use:
   $(A - AX)^{-1} = X^{-1}B$
   Multiplying both sides by X and then $(A - AX)$, we get

$X(A - AX)^{-1} = (A - AX)X^{-1}BX = A - AX$
Now, adding AX to both sides:
$X(A - AX)^{-1} + AX = A$
$X((A - AX)^{-1} + A) = A$
Since A is invertible, we can multiply both sides by A^{-1}:
$X((A - AX)^{-1}A + A^{-1}A) = A^{-1}A$
$X((A - AX)^{-1}A + I) = I$
$X(A^{-1} + I) = I$
Multiplying both sides by $(A^{-1} + I)^{-1}$, we get
$X = (A^{-1} + I)^{-1}$
The matrix $(A^{-1} + I)$ is invertible since it is the sum of an invertible matrix A^{-1} and the identity matrix I, and the sum of an invertible matrix and the identity matrix is always invertible.

Question 5:

Let $u, v \in R^n$, and let $x = S(u), y = S(v)$. We know that $T(x) = u$ and $T(y) = v$. For invertibility condition, $TS = I$, where I is the identity transformation.
For additivity,
To show: $S(u + v) = S(u) + S(v)$. Let
$T(x) + T(y) = T(x + y)$
Since T is a linear transformation,
Applying S to both sides:
$S(T(x) + T(y)) = S(T(x + y))$
$S(u + v) = S(T(S(u) + S(v)))$
$S(u + v) = S(T(x + y))$
Since $TS = I$, we have:
$S(u + v) = x + y$
$S(u + v) = S(u) + S(v)$
For scalar multiplication, we want to show that $S(cu) = cS(u)$ for any scalar c. Since $T(cx) = cT(x)$ (because T is linear), we can write:
$T(cx) = cT(x)$
Applying S to both sides
$S(T(cx)) = S(cT(x))$
$S(cu) = cS(T(x))$
$S(cu) = cS(u)$
Thus, S satisfies both additivity and scalar multiplication, so it is a linear transformation.

Question 6:

a. Answer

Given:

$$A = \begin{bmatrix} 2 & -4 & 4 & -2 \\ 6 & -9 & 7 & -3 \\ -1 & -4 & 8 & 0 \end{bmatrix}$$

Considering L to be an Identity matrix:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Considering ‘U’ to be matrix A:

Applying

$R_2 = R_2 - 3R_1$

$$U = \begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ -1 & -4 & 8 & 0 \end{bmatrix}$$

Coefficient that helps conduct the calculation will be updated in ‘L’.

We get:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying

$R_3 = R_3 + 0.5R_1$

$U =$ $\begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & -6 & 10 & -1 \end{bmatrix}$

$R_3 = R_3 + 2R_2$

$U =$ $\begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

Where, 'L' becomes:

$L =$ $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -0.5 & -2 & 1 \end{bmatrix}$

Thus, the matrices L and U where A = LU are:

$L =$ $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -0.5 & -2 & 1 \end{bmatrix}$

and

$U =$ $\begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

b.

Given:

$$U = \begin{bmatrix} 2 & -6 & 6 \\ -4 & 5 & -7 \\ 3 & 5 & -1 \\ -6 & 4 & -8 \\ 8 & -3 & 9 \end{bmatrix}$$

Considering 'L' to be an identity matrix:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Conducting LU decomposition:
Applying $R_2 = R_2 + 2R_1$ on U:

$$U = \begin{bmatrix} 2 & -6 & 6 \\ 0 & -7 & 5 \\ 3 & 5 & -1 \\ -6 & 4 & -8 \\ 8 & -3 & 9 \end{bmatrix}$$

Coefficient that helps conduct the calculation will be updated in 'L'.
We get:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $R_3 = R_3 - 3/2\, R_1$ on U:

$$U = \begin{bmatrix} 2 & -6 & 6 \\ 0 & -7 & 5 \\ 0 & 14 & -10 \\ -6 & 4 & -8 \\ 8 & -3 & 9 \end{bmatrix}$$

Here, the coefficient is 3/2.
Applying $R_4 = R_4 + 3\, R_1$ on U:

$$U = \begin{bmatrix} 2 & -6 & 6 \\ 0 & -7 & 5 \\ 0 & 14 & -10 \\ 0 & -14 & 10 \\ 8 & -3 & 9 \end{bmatrix}$$

Here, the coefficient is -3.
Applying $R_5 = R_5 - 4\, R_1$ on U:

$$U = \begin{bmatrix} 2 & -6 & 6 \\ 0 & -7 & 5 \\ 0 & 14 & -10 \\ 0 & -14 & 10 \\ 0 & 21 & -15 \end{bmatrix}$$

Here, the coefficient is 4.

Applying $R_3 = R_3 + 2\,R_2$ on U:

$$U = \begin{bmatrix} 2 & -6 & 6 \\ 0 & -7 & 5 \\ 0 & 0 & 0 \\ 0 & -14 & 10 \\ 0 & 21 & -15 \end{bmatrix}$$

Here, the coefficient is -2.
Applying $R_4 = R_4 - 2\,R_2$ on U:

$$U = \begin{bmatrix} 2 & -6 & 6 \\ 0 & -7 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 21 & -15 \end{bmatrix}$$

Here, the coefficient is 2.
Applying $R_5 = R_5 + 3\,R_2$ on U:

$$U = \begin{bmatrix} 2 & -6 & 6 \\ 0 & -7 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Here, the coefficient is -3.
Thus, ‘L’ becomes:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 3/2 & -2 & 1 & 0 & 0 \\ -3 & 2 & 0 & 1 & 0 \\ 4 & -3 & 0 & 0 & 1 \end{bmatrix}$$

Thus, the matrices L and U where A = LU are:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 3/2 & -2 & 1 & 0 & 0 \\ -3 & 2 & 0 & 1 & 0 \\ 4 & -3 & 0 & 0 & 1 \end{bmatrix}$$

and

$$U = \begin{bmatrix} 2 & -6 & 6 \\ 0 & -7 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$