Question 1:

a. Answer

Here, There are 3 sectors of economy:

Let us consider Manufacturing, Agriculture and Services be M, A and S respectively.

Given:

Let the consumption matrix be C.

$$\begin{bmatrix} 0.10 & 0.60 & 0.60 \\ 0.30 & 0.20 & 0 \\ 0.30 & 0.10 & 0.10 \end{bmatrix}$$

Let x be the vector for planned output when agriculture plans to produce 100 units.

$$x = \begin{bmatrix} 0 \\ 100 \\ 0 \end{bmatrix}$$

When ‘C’ is multiplied by ‘x’, we can obtain intermediate demands.

Here,

Intermediate demands = $C \times x$ =

$$= \begin{bmatrix} 0.10 & 0.60 & 0.60 \\ 0.30 & 0.20 & 0 \\ 0.30 & 0.10 & 0.10 \end{bmatrix} \times \begin{bmatrix} 0 \\ 100 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 60 \\ 20 \\ 10 \end{bmatrix}$$

Hence, the intermediate demands for
Manufacturing = 60
Agriculture = 20
Services = 10

b. Answer
Let the amount to be formed by sectors: Manufacturing, Agriculture and services be M, A and S respectively.

$$x = \begin{bmatrix} M \\ A \\ S \end{bmatrix}$$

$$\text{Let the final demand be } d = \begin{bmatrix} 0 \\ 18 \\ 0 \end{bmatrix}$$

Question 2:

To show that the composition of the two transformations A_1 and A_2 is a rotation in R^2 , we need to compute the matrix product A_1A_2 and compare it to the standard rotation matrix in R^2, which has the form:

$$R_\theta = \begin{bmatrix} cos\theta & -\ sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

Finding the product of A_1 and A_2

$$= \begin{bmatrix} sec\varphi & -\ tan\varphi & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ sin\varphi & cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} sec\varphi \cdot 1 + (-\ tan\varphi) \cdot sin\varphi & sec\varphi \cdot 0 + (-\ tan\varphi) \cdot cos\varphi & 0 \\ 0 \cdot 1 + 1 \cdot sin\varphi & 0 \cdot 0 + 1 \cdot cos\varphi & 0 \\ 0 \cdot 0 + 0 \cdot sin\varphi & 0 \cdot 0 + 0 \cdot cos\varphi & 1 \end{bmatrix}$$

$$= \begin{bmatrix} sec\varphi - tan\varphi \cdot sin\varphi & -\ tan\varphi \cdot cos\varphi & 0 \\ sin\varphi & cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now we need to simplify the entries in the matrix:
For element in first row and first column:

$sec\varphi - tan\varphi \cdot sin\varphi = sec\varphi - sin^2\varphi \cdot sec\varphi = sec\varphi(1 - sin^2\varphi) = sec\varphi \cdot cos^2\varphi = cos\varphi$

For element in first row and second column:
$-\ cos\varphi \cdot tan\varphi =-\ sin\varphi$
Therefore:

$$A_1A_2 = \begin{bmatrix} cos\varphi & -\ sin\varphi & 0 \\ sin\varphi & cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The resulting matrix A is a 2D rotation matrix with a rotation angle of φ. Thus, the composition of the two transformation $A_1\ and\ A_2$ results in a rotation matrix in R^2.

$$R = \begin{bmatrix} cos\varphi & -\ sin\varphi \\ sin\varphi & cos\varphi \end{bmatrix}$$

Therefore, the composition of the two transformations is indeed a rotation in R^2.

Question 3:

Answer:
Given matrix:
A =

$$\begin{bmatrix} 3 & -1 & 7 & 3 & 9 \\ -2 & 2 & -2 & -2 & 5 \\ -5 & 9 & 3 & 3 & 4 \\ -2 & 6 & 6 & 3 & 7 \end{bmatrix}$$

Obtaining reduced row echelon form of A:

A = $\begin{bmatrix} 3 & -1 & 7 & 3 & 9 \\ -2 & 2 & -2 & -2 & 5 \\ -5 & 9 & 3 & 3 & 4 \\ -2 & 6 & 6 & 3 & 7 \end{bmatrix}$

$R_1 \dashrightarrow \frac{1}{3}R_1$

$$\begin{bmatrix} 1 & \frac{-1}{3} & \frac{7}{3} & 1 & 3 \\ -2 & 2 & -2 & -2 & 5 \\ -5 & 9 & 3 & 3 & 4 \\ -2 & 6 & 6 & 3 & 7 \end{bmatrix}$$

$R_4 --> R_4 + 2R_1$ and $R_3 --> R_3 + 5R_1$ and $R_2 --> R_2 + 2R_1$

$$\begin{bmatrix} 1 & \frac{-1}{3} & \frac{7}{3} & 1 & 3 \\ 0 & 4/3 & 8/3 & 9 & 11 \\ 0 & \frac{8}{3} & \frac{44}{3} & 8 & 19 \\ 0 & \frac{16}{3} & \frac{46}{3} & 5 & 13 \end{bmatrix}$$

$R_2 --> \frac{3}{4}R_2$

$$\begin{bmatrix} 1 & \frac{-1}{3} & \frac{7}{3} & 1 & 3 \\ 0 & 1 & 2 & \frac{27}{4} & 33/4 \\ 0 & \frac{22}{3} & \frac{44}{3} & 8 & 19 \\ 0 & \frac{16}{3} & \frac{32}{3} & 5 & 13 \end{bmatrix}$$

$R_1 --> R_1 + \frac{1}{3}R_2$ and $R_3 --> R_3 - \frac{22}{3}R_2$ and $R_4 --> R_4 - \frac{16}{3}R_2$

$$\begin{bmatrix} 1 & \frac{-1}{3} & \frac{7}{3} & 1 & 3 \\ 0 & 1 & 2 & \frac{27}{4} & 33/4 \\ 0 & 0 & 0 & \frac{-83}{2} & \frac{-83}{2} \\ 0 & 0 & 0 & -31 & -31 \end{bmatrix}$$

Since, the element at row3 and column3 which are pivot elements, equals 0, we need to swap the rows. Finding the first nonzero element in column3 under the pivot entry. As can be seen there are no such entries, so we move to the next column.

$R_3 --> \frac{-2}{83} R_3$ then $R_1 --> R_1 - \frac{13}{4} R_3$ and $R_2 --> R_2 - \frac{-27}{4} R_2$ and $R_4 --> R_4 + 31R_3$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & \frac{5}{2} \\ 0 & 1 & 2 & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since, the element at row4 and column5(pivot element) equals 0, we need to swap the rows. Finding the first nonzero element in column5 under the pivot entry. As can be seen there are no such entries.

Hence,
The required matrix is

$$\begin{bmatrix} 1 & 0 & 3 & 0 & \frac{5}{2} \\ 0 & 1 & 2 & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We know that the pivot columns are 1, 2 and 4. Thus, the basis for the column space of A is formed by the original columns 1, 2 and 4 of matrix A:

Basis for Col A = $\left\{ \begin{bmatrix} 3 \\ -2 \\ -5 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 9 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 3 \\ 3 \end{bmatrix} \right\}$

The dimension of the column space is the number of pivot columns:
Dim (Col A) = 3

Null space (Nul A)
Solving Ax = 0

$x_1 + 3x_3 + \frac{5}{3}x_5 = 0$

$x_2 + 2x_3 + \frac{3}{2}x_5 = 0$

$x_4 + x_5 = 0$

Let x_3 *and* x_5 be the free variables.

$x_1 =- 3x_3 - \frac{5}{2}x_5$

$x_2 =- 2x_3 - \frac{3}{2}x_5$

$x_4 =- x_5$

Here the general solution to Ax = 0 is:

$$x = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} \frac{-5}{2} \\ \frac{-3}{2} \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Thus, the basis for the null space is:

Basis for Nul A= $\left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{5}{2} \\ -\frac{3}{2} \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

The dimension of the null space is the number of free variables:
dim(Nul A) = 2

R3= -2R3/83

$$\begin{bmatrix} 1 & 0 & 3 & 13/4 & 23/4 \\ 0 & 1 & 2 & 27/4 & 33/4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -31 & -31 \end{bmatrix}$$

R1= R1 – 13R3/4

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 5/2 \\ 0 & 1 & 2 & 27/4 & 33/4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -31 & -31 \end{bmatrix}$$

R2= R2 – 27R3/4 and R4=R4 + 31R3

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 5/2 \\ 0 & 1 & 2 & 0 & 3/2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since, the element at row4 and column5(pivot element) equals 0, we need to swap the rows. Finding the first nonzero element in column5 under the pivot entry. As can be seen there are no such entries.

Therefore,

Reduced Row echelon form(RREF) = $\begin{bmatrix} 1 & 0 & 3 & 0 & 5/2 \\ 0 & 1 & 2 & 0 & 3/2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Finding the Bases and Dimensions of Col A and Nul A

Column Space (Col A)

The pivot columns in the RREF are columns 1, 2, and 4. Thus, the basis for the column space of A is formed by the original columns 1, 2, and 4 of matrix A:

$$\text{Basis for Col } A = \left\{ \begin{bmatrix} 3 \\ -2 \\ -5 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 9 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 3 \\ 3 \end{bmatrix} \right\}$$

The dimension of the column space is the number of pivot columns:

$$\dim(\text{Col } A) = 3$$

R2=3R2/4

$$\begin{bmatrix} 1 & -1/3 & 7/3 & 1 & 3 \\ 0 & 1 & 2 & 27/4 & 33/4 \\ 0 & 22/3 & 44/3 & 8 & 19 \\ 0 & 16/3 & 32/3 & 5 & 13 \end{bmatrix}$$

R1=R1 + R2/3 and R3= R3 – 22R2/3

$$\begin{bmatrix} 1 & 0 & 3 & 13/4 & 23/4 \\ 0 & 1 & 2 & 27/4 & 33/4 \\ 0 & 0 & 0 & -83/2 & -83/2 \\ 0 & 16/3 & 32/3 & 5 & 13 \end{bmatrix}$$

R4= R4 – 16R2/3

$$\begin{bmatrix} 1 & 0 & 3 & 13/4 & 23/4 \\ 0 & 1 & 2 & 27/4 & 33/4 \\ 0 & 0 & 0 & -83/2 & -83/2 \\ 0 & 0 & 0 & -31 & -31 \end{bmatrix}$$

Since, the element at row3 and column3 which are pivot elements, equals 0, we need to swap the rows. Finding the first nonzero element in column3 under the pivot entry. As can be seen there are no such entries, we move to the next column.