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Linear Algebra

Assignment 5

Question 1:

a. Answer
Here, There are 3 sectors of economy:
Let us consider Manufacturing, Agriculture and Services be M, A and S respectively.
Given:
Let the consumption matrix be C.

$$\begin{bmatrix} 0.10 & 0.60 & 0.60 \\ 0.30 & 0.20 & 0 \\ 0.30 & 0.10 & 0.10 \end{bmatrix}$$

Let x be the vector for planned output when agriculture plans to produce 100 units.

$$x = \begin{bmatrix} 0 \\ 100 \\ 0 \end{bmatrix}$$

When 'C' is multiplied by 'x', we can obtain intermediate demands.
Here,
Intermediate demands = $C \times x$ =

$$= \begin{bmatrix} 0.10 & 0.60 & 0.60 \\ 0.30 & 0.20 & 0 \\ 0.30 & 0.10 & 0.10 \end{bmatrix} \begin{bmatrix} 0 \\ 100 \\ 0 \end{bmatrix}$$

$\times$

$$= \begin{bmatrix} 60 \\ 20 \\ 10 \end{bmatrix}$$

Hence, the intermediate demands for
Manufacturing = 60
Agriculture = 20
Services = 10

b. Answer
Let the amount to be formed by sectors: Manufacturing, Agriculture and services be M, A and S respectively.

$$x = \begin{bmatrix} M \\ A \\ S \end{bmatrix}$$

Let the final demand be d = $\begin{bmatrix} 0 \\ 18 \\ 0 \end{bmatrix}$

We know, C is the consumption matrix.
Then,
$x = Cx + d$
$d = x - cx$
$d = x(1 - C)$ ------> $eq\,1$
Again,

$$d=\left\{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}-\begin{bmatrix}0.10 & 0.60 & 0.60\\ 0.30 & 0.20 & 0.0\\ 0.30 & 0.10 & 0.10\end{bmatrix}\right\}\begin{bmatrix}M\\ A\\ S\end{bmatrix}$$

$$\begin{bmatrix}0\\ 18\\ 0\end{bmatrix}=\begin{bmatrix}1-0.10 & 0-0.60 & 0-0.60\\ 0-0.30 & 1-0.20 & 0-0.0\\ 0-0.30 & 0-0.10 & 1-0.10\end{bmatrix}\begin{bmatrix}M\\ A\\ S\end{bmatrix}$$

$$\begin{bmatrix}0\\ 18\\ 0\end{bmatrix}=\begin{bmatrix}0.90 & -0.60 & -0.60\\ -0.30 & 0.80 & 0\\ -0.30 & -0.10 & 0.90\end{bmatrix}\begin{bmatrix}M\\ A\\ S\end{bmatrix}$$

Rewriting the matrix into augmented matrix:

0.90	-0.60	-0.60	| 0
-0.30	0.80	0	| 18
-0.30	-0.10	0.90	| 0

$R_1 \text{ --> } \frac{1}{3}R_1$

0.30	-0.20	-0.20	| 0
-0.30	0.80	0	| 18
-0.30	-0.10	0.90	| 0

$R_2 \text{ --> } R_1 + R_2 \; and \; R_3 \text{ --> } R_3 + R_1$

0.30	-0.20	-0.20	| 0
0	0.60	-0.20	| 18
0	-0.30	0.70	| 0

$R_2 \text{ --> } \frac{1}{2}R_2$

$$\left[\begin{array}{ccc|c} 0.30 & -0.20 & -0.20 & 0 \\ 0 & 0.30 & -0.10 & 9 \\ -0.30 & -0.10 & 0.90 & 0 \end{array}\right]$$

$R_2 --> R_3 + R_2$

$$\left[\begin{array}{ccc|c} 0.30 & -0.20 & -0.20 & 0 \\ 0 & 0.30 & -0.10 & 9 \\ 0 & 0 & 0.60 & 9 \end{array}\right]$$

We can rewrite them in equations as follows:

$0.30M - 0.20A - 0.20S = 0 ---> eq2$

$0.30A - 0.10S = 9 ---> eq3$

$0.60S = 9$

$or,\ S = \frac{9}{0.60} = 15$

Here, the value of S is 15.

Substituting the value of S in eq 3, we get

$0.30A - 0.10S = 9 ---> eq3$

$or,\ 0.30A - 0.10 \times 15 = 9$

$or,\ 0.30A = 9 + 1.5$

$or,\ A = \frac{10.5}{0.30} = 35$

Again, substituting the value of A and S in eq 3, we get

$0.30M - 0.20A - 0.20S = 0$

$or,\ 0.30M - 0.20 \times 35 - 0.20 \times 15 = 0$

$or,\ 0.30M - 7 - 3 = 0$

$or,\ 0.30M = 10$

$or,\ M = 10/0.30 \approx 33.33$

Thus, the values of Manufacturing, Agriculture and Service are 33.33 approx., 35 and 15 respectively.

c. Answer:

Let the amount to be formed by sectors: Manufacturing, Agriculture and services be M, A and S respectively.

$$x = \begin{bmatrix} M \\ A \\ S \end{bmatrix}$$

Let the final demand be d = $\begin{bmatrix} 18 \\ 0 \\ 0 \end{bmatrix}$

We know, C is the consumption matrix.
Then,
$x = Cx + d$
$d = x - cx$
$d = x(1 - C) ------> eq\ 1$

$$\begin{bmatrix} 18 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.90 & -0.60 & -0.60 \\ -0.30 & 0.80 & 0 \\ -0.30 & -0.10 & 0.90 \end{bmatrix} \begin{bmatrix} M \\ A \\ S \end{bmatrix}$$

SImilar calculation to b
Rewriting the matrix into augmented form:

$$\left[\begin{array}{ccc|c} 0.90 & -0.60 & -0.60 & 18 \\ -0.30 & 0.80 & 0 & 0 \\ -0.30 & -0.10 & 0.90 & 0 \end{array}\right]$$

$R_1 --> \frac{1}{3}R_1$
$R_2 --> R_1 + R_2\ and\ R_3 --> R_3 + R_1$
$R_2 --> \frac{1}{2}R_2$
$R_2 --> R_3 + R_2$

$$\left[\begin{array}{ccc|c} 0.30 & -0.20 & -0.20 & 6 \\ 0 & 0.30 & -0.10 & 3 \\ 0 & 0 & 0.60 & 9 \end{array}\right]$$

The matrix can be written into system of equations:
$0.30M - 0.20A - 0.20S = 6 ---> eq\ 2$

$0.30A - 0.10S = 3 ---> eq\,3$

$0.60S = 9$

$or,\ S = \frac{9}{0.60} = 15$

Substituting the value of S into eq 3, we get

$0.30A - 0.10S = 3$

$or,\ 0.30A - 0.10 \times 15 = 3$

$or,\ 0.30A = 3 + 1.5$

$or,\ A = \frac{4.5}{0.30} = 15$

Substituting the value of A and S into eq 2, we get

$0.30M - 0.20 \times 15 - 0.20 \times 15 = 6$

$or,\ 0.30M = 6 + 3 + 3$

$or,\ M = \frac{12}{0.30} = 40$

Thus, the values of Manufacturing, Agriculture and Service are 40, 15 and 15 respectively.

d. Answer:
Let the amount to be formed by sectors: Manufacturing, Agriculture and services be M, A and S respectively.

$$x = \begin{bmatrix} M \\ A \\ S \end{bmatrix}$$

Let the final demand be $d = \begin{bmatrix} 18 \\ 0 \\ 0 \end{bmatrix}$

We know, C is the consumption matrix.
Then,

$x = Cx + d$

$d = x - cx$

$d = x(1 - C) -----> eq\,1$

The calculation is similar to that of part b, so

$$\begin{bmatrix} 18 \\ 18 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.90 & -0.60 & -0.60 \\ -0.30 & 0.80 & 0 \\ -0.30 & -0.10 & 0.90 \end{bmatrix} \begin{bmatrix} M \\ A \\ S \end{bmatrix}$$

Rewriting the matrix into augmented form:

$$\left[\begin{array}{ccc|c} 0.90 & -0.60 & -0.60 & 18 \\ -0.30 & 0.80 & 0 & 0 \\ -0.30 & -0.10 & 0.90 & 0 \end{array}\right]$$

$R_1 --> \frac{1}{3}R_1$
$R_2 --> R_1 + R_2 \; and \; R_3 --> R_3 + R_1$
$R_2 --> \frac{1}{2}R_2$
$R_2 --> R_3 + R_2$

$$\left[\begin{array}{ccc|c} 0.30 & -0.20 & -0.20 & 6 \\ 0 & 0.30 & -0.10 & 12 \\ 0 & 0 & 0.60 & 18 \end{array}\right]$$

The matrix can be written into system of equations:
$0.30M - 0.20A - 0.20S = 6 ----> eq\,2$
$0.30A - 0.10S = 12 ----> eq\,3$
$0.60S = 18$
$or,\ S = \frac{18}{0.60} = 30$
Substituting the value of S in eq 3, we get
$0.30A - 0.10 \times 30 = 12$
$or,\ 0.30A = 12 + 3$
$or,\ A = \frac{15}{0.30} = 50$
Substituting the value of A and S in eq 2, we get
$0.30M - 0.20 \times 50 - 0.20 \times 30 = 6$
$or,\ 0.30M = 6 + 10 + 6$
$or,\ M = 22/0.30 \approx 73.33$
Thus, the values of Manufacturing, Agriculture and Service are 73.33 approx., 50 and 30 respectively.

Question 2:

To show that the composition of the two transformations A_1 and A_2 is a rotation in R^2 , we need to compute the matrix product $A_1 A_2$ and compare it to the standard rotation matrix in R^2, which has the form:

$$R_\theta = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

Finding the product of A_1 and A_2

$$= \begin{bmatrix} sec\varphi & -tan\varphi & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ sin\varphi & cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} sec\varphi \cdot 1 + (-tan\varphi) \cdot sin\varphi & sec\varphi \cdot 0 + (-tan\varphi) \cdot cos\varphi & 0 \\ 0 \cdot 1 + 1 \cdot sin\varphi & 0 \cdot 0 + 1 \cdot cos\varphi & 0 \\ 0 \cdot 0 + 0 \cdot sin\varphi & 0 \cdot 0 + 0 \cdot cos\varphi & 1 \end{bmatrix}$$

$$= \begin{bmatrix} sec\varphi - tan\varphi \cdot sin\varphi & -tan\varphi \cdot cos\varphi & 0 \\ sin\varphi & cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now we need to simplify the entries in the matrix:
For element in first row and first column:
$sec\varphi - tan\varphi \cdot sin\varphi = sec\varphi - sin^2\varphi \cdot sec\varphi = sec\varphi(1 - sin^2\varphi) = sec\varphi \cdot cos^2\varphi = cos\varphi$

For element in first row and second column:
$-cos\varphi \cdot tan\varphi =- sin\varphi$
Therefore:

$$A_1 A_2 = \begin{bmatrix} cos\varphi & -\, sin\varphi & 0 \\ sin\varphi & cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The resulting matrix A is a 2D rotation matrix with a rotation angle of φ. Thus, the composition of the two transformation $A_1\ and\ A_2$ results in a rotation matrix in R^2.

$$R = \begin{bmatrix} cos\varphi & -\, sin\varphi \\ sin\varphi & cos\varphi \end{bmatrix}$$

Therefore, the composition of the two transformations is indeed a rotation in R^2.

Question 3:

Answer:
Given matrix:

A =

$$\begin{bmatrix} 3 & -1 & 7 & 3 & 9 \\ -2 & 2 & -2 & -2 & 5 \\ -5 & 9 & 3 & 3 & 4 \\ -2 & 6 & 6 & 3 & 7 \end{bmatrix}$$

Obtaining reduced row echelon form of A:

$$A = \begin{bmatrix} 3 & -1 & 7 & 3 & 9 \\ -2 & 2 & -2 & -2 & 5 \\ -5 & 9 & 3 & 3 & 4 \\ -2 & 6 & 6 & 3 & 7 \end{bmatrix}$$

$R_1 \text{ --> } \frac{1}{3}R_1$

$$\begin{bmatrix} 1 & \frac{-1}{3} & \frac{7}{3} & 1 & 3 \\ -2 & 2 & -2 & -2 & 5 \\ -5 & 9 & 3 & 3 & 4 \\ -2 & 6 & 6 & 3 & 7 \end{bmatrix}$$

$R_4 \text{ --> } R_4 + 2R_1$ and $R_3 \text{ --> } R_3 + 5R_1$ and $R_2 \text{ --> } R_2 + 2R_1$

$$\begin{bmatrix} 1 & \frac{-1}{3} & \frac{7}{3} & 1 & 3 \\ 0 & 4/3 & 8/3 & 9 & 11 \\ 0 & \frac{8}{3} & \frac{44}{3} & 8 & 19 \\ 0 & \frac{16}{3} & \frac{46}{3} & 5 & 13 \end{bmatrix}$$

$R_2 \text{ --> } \frac{3}{4}R_2$

$$\begin{bmatrix} 1 & \frac{-1}{3} & \frac{7}{3} & 1 & 3 \\ 0 & 1 & 2 & \frac{27}{4} & 33/4 \\ 0 & \frac{22}{3} & \frac{44}{3} & 8 & 19 \\ 0 & \frac{16}{3} & \frac{32}{3} & 5 & 13 \end{bmatrix}$$

$R_1 --> R_1 + \frac{1}{3}R_2$ and $R_3 --> R_3 - \frac{22}{3}R_2$ and $R_4 --> R_4 - \frac{16}{3}R_2$

$$\begin{bmatrix} 1 & \frac{-1}{3} & \frac{7}{3} & 1 & 3 \\ 0 & 1 & 2 & \frac{27}{4} & 33/4 \\ 0 & 0 & 0 & \frac{-83}{2} & \frac{-83}{2} \\ 0 & 0 & 0 & -31 & -31 \end{bmatrix}$$

Since the pivot items in rows three and column three equal zero, the rows must be switched. locating the column 3 pivot entry's first nonzero element. Since there are no such entries, as can be observed, we go on to the following column.

$R_3 --> \frac{-2}{83}R_3$ then $R_1 --> R_1 - \frac{13}{4}R_3$ and $R_2 --> R_2 - \frac{-27}{4}R_2$ and $R_4 --> R_4 + 31R_3$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & \frac{5}{2} \\ 0 & 1 & 2 & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The rows must be switched because the pivot element at row 4 and column 5 equals 0. identifying the column 5 pivot entry's first nonzero element. There are not any such entries, as can be observed.

Hence,
The required matrix is

$$\begin{bmatrix} 1 & 0 & 3 & 0 & \frac{5}{2} \\ 0 & 1 & 2 & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We know that the pivot columns are 1, 2 and 4. Thus, the basis for the column space of A is formed by the original columns 1, 2 and 4 of matrix A:

Basis for Col A = $\left\{ \begin{bmatrix} 3 \\ -2 \\ -5 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 9 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 3 \\ 3 \end{bmatrix} \right\}$

The dimension of the column space is the number of pivot columns:
dim (Col A) = 3

Null space (Nul A)
Solving Ax = 0

$x_1 + 3x_3 + \frac{5}{3}x_5 = 0$

$x_2 + 2x_3 + \frac{3}{2}x_5 = 0$

$x_4 + x_5 = 0$

Let x_3 and x_5 be the free variables.

$x_1 =- 3x_3 - \frac{5}{2}x_5$

$x_2 =- 2x_3 - \frac{3}{2}x_5$

$x_4 =- x_5$

Here the general solution to Ax = 0 is:

$$x = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} \frac{-5}{2} \\ \frac{-3}{2} \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Thus, the basis for the null space is:

Basis for Nul A= $\left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{5}{2} \\ -\frac{3}{2} \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

The dimension of the null space is the number of free variables:
dim(Nul A) = 2