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Linear Algebra
Assignment 2

Question 1:

Given equation: $x_1q_1x_2q_2x_3q_3 = v$

Here, scalar coefficients can x_1, x_2, $x_3$be written into a column vector x i.e.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Then, vectors q_1, q_2, q_3 can also be written into the column of matrix Q i.e.

Q =$[q_1\, q_2\, q_3]$, WHERE Q is 5x1 matrix

Which is in the form of $Qx = v$

Question 2:

Let A be a 3 × 4 matrix, let $\boldsymbol{y_1}$ and $\boldsymbol{y_2}$ be vectors in $\mathbb{R}^3$, and let $\boldsymbol{w} = \boldsymbol{y_1} + \boldsymbol{y_2}$. Suppose $\boldsymbol{y_1} = A\boldsymbol{x_1}$ and $\boldsymbol{y_2} = A\boldsymbol{x_2}$ for some vectors $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ in $\mathbb{R}^4$. What fact allows you to conclude that the system $A\boldsymbol{x} = \boldsymbol{w}$ is consistent?

Solution:

The fact that allows us to conclude that the system $Ax = w$ is consistent is the linearity of matrix multiplication, which includes the property of additivity. This property states that for any matrix A and vectors u and v , the following holds:

$A(u + v) = Au + Av$

Given that $y_1 = Ax_1$ and $y_2 = Ax_2$ and $w = y_1 + y_2$, additivity property of matrix multiplication can be used for deducing:

$w = y_1 + y_2 = Ax_1 + Ax_2 = A(x_1 + x_2)$ ---- (1)

Let $x = x_1 + x_2$, then

$Ax = A(x_1 + x_2) = Ax_1 + Ax_2 = y_1 + y_2 = w$, we know from equation 1.

Hence, there exists a vector x in R^4 such that Ax = w . This proves that the system Ax = w is consistent. The existence of $x_1$and x_2 in R^4 satisfying the initial equations and the linearity of matrix multiplication guarantee that the vector x that satisfies Ax= w can be found by simply adding x_1 and x_2.

Question 3:

Mark each statement True or False. Justify each answer.

a. If $\boldsymbol{x}$ is a nontrivial solution of $A\boldsymbol{x} = \mathbf{0}$, then every entry in $\boldsymbol{x}$ is nonzero.
b. The equation $\boldsymbol{x} = x_2\boldsymbol{u} + x_3\boldsymbol{v}$, with x_2 and x_3 free (and neither $\boldsymbol{u}$ nor $\boldsymbol{v}$ a multiple of the other), describes a plane through the origin.
c. The equation $A\boldsymbol{x} = b$ is homogeneous if the zero vector is a solution.
d. The effect of adding **p** to a vector is to move the vector in a direction parallel to **p**.
e. The solution set of $A\boldsymbol{x} = b$ is obtained by translating the solution set of $A\boldsymbol{x} = 0$.

Solution:

a. False. A nontrivial solution $Ax = 0$ of means that at least one entry in x is nonzero, but not necessarily every entry.
b. True. The equation $x = x_2u + x_3v$, with x_2 and x_3 as free variables and u and v not multiples of each other, describes all linear combinations of u and v. This is a plane through the origin in three-dimensional space.
c. True. An equation of the form $Ax = b$ is homogeneous if b is the zero vector. If the zero vector is a solution, then by definition, the equation is homogeneous.
d. True. Adding a vector p to another vector moves the latter in a direction parallel to p, by the definition of vector addition.
e. True. The solution set of $Ax = b$ can be obtained by translating the solution set of the homogeneous system Ax = 0 by a particular solution to Ax = b.
f.

Question 4:

$$\text{Given } A = \begin{bmatrix} -2 & -6 \\ 7 & 21 \\ -3 & -9 \end{bmatrix} \text{ find solutions of } A\boldsymbol{x} = 0$$

Solution:

Given matrix:

$$\begin{bmatrix} -2 & -6 \\ 7 & 21 \\ -3 & -9 \end{bmatrix}$$

To find the solution of $Ax = 0$, we need to reduce the given matrix in row echelon form.

$$\left[\begin{array}{cc|c} -2 & -6 & 0 \\ 7 & 21 & 0 \\ -3 & -9 & 0 \end{array}\right]$$

Performing R2→ R2 + 7/2 R1 and R3 → R3 + 3/2 R1, we get

$$\left[\begin{array}{cc|c} -2 & -6 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The corresponding system is:

$-2x_1 - 6x_2 = 0$

$0 = 0$

$0 = 0$

Solving x_1 in terms of x_2:

$x_1 = -3x_2$

The general solution could be:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

where x_2 is a free variable.

Question 5:

Suppose an economy has four sectors, Agriculture (A), Energy (E), Manufacturing (M), and Transportation (T). Sector A sells 10% of its output to E and 25% to M and retains the rest. Sector E sells 30% of its output to A, 35% to M, and 25% to T and retains the rest. Sector M sells 30% of its output to A, 15% to E, and 40% to T and retains the rest. Sector T sells 20% of its output to A, 10% to E, and 30% to M and retains the rest.

a. Construct the exchange table for this economy.

b. Find a set of equilibrium prices for the economy.

Solution:

a. The exchange table for this economy can be constructed based on the given percentages, where the rows represent the sectors selling the output and the columns represent the sectors buying the output.

	A	E	M	T
A	0.65	0.30	0.35	0.20
E	0.10	0.10	0.15	0.10
M	0.25	0.35	0.15	0.30
T	0	0.25	0.40	0.40

b.

Equilibrium prices:

Let $p_A = 1$ as a reference price.

The system of equations:

$0.65p_A + 0.30p_E + 0.35p_M + 0.20p_T = p_A$

$0.10p_A + 0.10p_E + 0.15p_M + 0.10p_T = p_E$

$0.25p_A + 0.35p_E + 0.15p_M + 0.30p_T = p_M$

$0p_A + 0.25p_E + 0.40p_M + 0.40p_T = p_T$

Rearranging the equations:

$-0.35p_A + 0.30p_E + 0.35p_M + 0.20p_T = 0$

$0.10p_A - 0.90p_E + 0.15p_M + 0.10p_T = 0$

$0.25p_A + 0.35p_E - 0.85p_M + 0.30p_T = 0$

$0p_A + 0.25p_E + 0.40p_M - 0.60p_T = 0$

Rewriting the equations in a matrix form:

$$\left[\begin{array}{cccc|c} -0.35 & 0.30 & 0.35 & 0.20 & 0 \\ 0.10 & -0.90 & 0.15 & 0.10 & 0 \\ 0.25 & 0.35 & -0.85 & 0.30 & 0 \\ 0 & 0.25 & 0.40 & -0.60 & 0 \end{array}\right]$$

Question 6:

We know from the question, flow in is equal to flow out. So, we can write,
For node A, x1=100+x2
For node B, x2=x3-50
For node C, x3=120+x4
For node D, x4=x5-150
For node E, x5=80+x6
For node F, x6=x1-100

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & -1 & 0 & -150 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ -1 & 0 & 0 & 0 & 0 & 1 & -100 \end{bmatrix}$$

Solving this using row echelon method,
R6=R6+R1.

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & -1 & 0 & -150 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

R1=R1+R2

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 50 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & -1 & 0 & -150 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

R6=R6+R2, R1=R1+R3

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 170 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & -1 & 0 & -150 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & -1 & 0 & 0 & 1 & -50 \end{bmatrix}$$

R2=R2+R3, R6=R6+R3, R1=R1+R4

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 20 \\ 0 & 1 & 0 & -1 & 0 & 0 & 70 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & -1 & 0 & -150 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & -1 & 0 & 1 & 70 \end{bmatrix}$$

R2=R2+R4, R3=R3+R4, R6=R6+R4, R1=R1+R5

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 100 \\ 0 & 1 & 0 & 0 & -1 & 0 & -80 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 & -1 & 0 & -150 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 0 & -1 & 1 & -80 \end{bmatrix}$$

R2=R2+R5, R3=R3+R5, R4=R4+R5, R6=R6+R5

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 100 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 50 \\ 0 & 0 & 0 & 1 & 0 & -1 & -70 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution of the network flow in terms of x6 we get from above is: $x1 - x6 = 100$
$x2 - x6 = 0$
$x3 – x6 = 50$
$x4 - x6 = -70$
$x5 – x6 = 80$
$x6 = x6$.

This is due to the fact that x6 can equal anything since it is a free variable and the value the dependent variables will assume based on the given values of the independent variables shall determine this.
To minimize x6 we can consider the fact that traffic flows cannot be negative, that is ≥ 0 is an inherent constraint.

Since, $x4 = x6 - 70$, the smallest value for x6 that will not give a negative value of x4 is when $x6 - 70 = 0$.

Thus: $x6 = 70$ Again, it means that all traffic flows should remain non-negative and the smallest value for x6 to ensure this would be equal to 70.