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Linear Algebra
Assignment 3

Question 1:

A = $\begin{bmatrix} 8 & -3 & 0 & -7 & 2 \\ -9 & 4 & 5 & 11 & -7 \\ 6 & -2 & 2 & -4 & 4 \\ 5 & -1 & 7 & 0 & 10 \end{bmatrix}$

Reducing to row echelon form:
R1 → R1/8:

$$\begin{bmatrix} 1 & -3/8 & 0 & -7/8 & 1/4 \\ -9 & 4 & 5 & 11 & -7 \\ 6 & -2 & 2 & -4 & 4 \\ 5 & -1 & 7 & 0 & 10 \end{bmatrix}$$

R2 → R2 + 9R1, R3 → R3 - 6R1, R4 → R4 - 5R1

$$\begin{bmatrix} 1 & -3/8 & 0 & -7/8 & 1/4 \\ 0 & 5/8 & 5 & 25/8 & -19/4 \\ 0 & 1/4 & 2 & 5/4 & 5/2 \\ 0 & 7/8 & 7 & 35/8 & 35/4 \end{bmatrix}$$

R2 → 8R2/5:

$$\begin{bmatrix} 1 & -3/8 & 0 & -7/8 & 1/4 \\ 0 & 1 & 8 & 5 & -38/5 \\ 0 & 1/4 & 2 & 5/4 & 5/2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 7/8 & 7 & 35/8 & 35/4 \end{bmatrix}$$

R1 → R1 + 3 R2/8, R3 → R3 - R2/4, R4 → R4 - 7 R2/8

$$\begin{bmatrix} 1 & -3/8 & 0 & -7/8 & 1/4 \\ 0 & 1 & 8 & 5 & -38/5 \\ 0 & 0 & 0 & 0 & 22/5 \\ 0 & 0 & 0 & 0 & 77/5 \end{bmatrix}$$

Swapping the rows as row 3 and column 4 i.e. the pivot elements are zero.

R3 → 5R3/22

$$\begin{bmatrix} 1 & -3/8 & 0 & -7/8 & 1/4 \\ 0 & 1 & 8 & 5 & -38/5 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 77/5 \end{bmatrix}$$

R1 → R1 + 13R3/5, R2 → R2 + 38R3/5, R4 → R4 - 77/5 R3:

$$\begin{bmatrix} 1 & -3/8 & 0 & -7/8 & 1/4 \\ 0 & 1 & 8 & 5 & -38/5 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot columns are 1st, 2nd and 5th. Thus, matrix B becomes:

$$\begin{bmatrix} 8 & -3 & 2 \\ -9 & 4 & -7 \\ 6 & -2 & 4 \\ 5 & -1 & 10 \end{bmatrix}$$

Given: $Bx = 0$:

$$\begin{bmatrix} 8 & -3 & 2 \\ -9 & 4 & -7 \\ 6 & -2 & 4 \\ 5 & -1 & 10 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Reducing to row echelon form:
R1 → R1/8

$$\begin{bmatrix} 1 & -3/8 & 1/4 \\ -9 & 4 & -7 \\ 6 & -2 & 4 \\ 5 & -1 & 10 \end{bmatrix}$$

R2 → R2 + 9 R1, R3 → R3 - 6 R1, R4 → R4 - 5 R1:

$$\begin{bmatrix} 1 & -3/8 & 1/4 \\ 0 & 5/8 & -19/4 \\ 0 & 1/4 & 5/2 \\ 0 & 7/8 & 35/4 \end{bmatrix}$$

R2→ 8 R2/ 5

$$\begin{bmatrix} 1 & -3/8 & 1/4 \\ 0 & 1 & -38/5 \\ 0 & 1/4 & 5/2 \\ 0 & 7/8 & 35/4 \end{bmatrix}$$

R1 → R1 + 3 R2/ 8, R3 → R3 - R2/4, R4 → R4 - 7 R2/8

$$\begin{bmatrix} 1 & 0 & -13/5 \\ 0 & 1 & -38/5 \\ 0 & 0 & 22/5 \\ 0 & 0 & 77/5 \end{bmatrix}$$

R3 → 5 R3/22

$$\begin{bmatrix} 1 & 0 & -13/5 \\ 0 & 1 & -38/5 \\ 0 & 0 & 1 \\ 0 & 0 & 77/5 \end{bmatrix}$$

R1 → R1 + 13 R3/5, R2 → R2 + 38 R3/5, R4 → R4 - 77 R3/5

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The solution becomes:
$x_1 = 0,\ x_2 = 0,\ x_3 = 0$
Thus, $Bx = 0$ has a trivial solution.

b.

A= $$\begin{bmatrix} 12 & 10 & -6 & -3 & 7 & 10 \\ -7 & -6 & 4 & 7 & -9 & 5 \\ 9 & 9 & -9 & -5 & 5 & -1 \\ -4 & -3 & 1 & 6 & -8 & 9 \\ 8 & 7 & -5 & -9 & 11 & -8 \end{bmatrix}$$

Performing reduced row echelon method:
R1 → R1/12

$$\begin{bmatrix} 1 & 5/6 & -1/2 & -1/3 & 7/12 & 5/6 \\ -7 & -6 & 4 & 7 & -9 & 5 \\ 9 & 9 & -9 & -5 & 5 & -1 \\ -4 & -3 & 1 & 6 & -8 & 9 \\ 8 & 7 & -5 & -9 & 11 & -8 \end{bmatrix}$$

R2 → R2 +7 R1, R3 → R3 - 9 R1, R4 → R4 + 4 R1, R5 → R5 - 8 R1

$$\begin{bmatrix} 1 & 5/6 & -1/2 & -1/3 & 7/12 & 5/6 \\ 0 & -1/6 & 1/2 & 21/4 & -59/12 & 65/6 \\ 0 & 3/2 & -9/2 & -11/4 & -1/4 & -17/2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1/3 & -1 & 5 & -17/3 & 37/3 \\ 0 & 1/3 & -1 & -7 & 19/3 & -44/3 \end{bmatrix}$$

$R2 \to -6\ R2$

$$\begin{bmatrix} 1 & 5/6 & -1/2 & -1/3 & 7/12 & 5/6 \\ 0 & 1 & -3 & -63/2 & 59/2 & -65 \\ 0 & 3/2 & -9/2 & -11/4 & -1/4 & -17/2 \\ 0 & 1/3 & -1 & 5 & -17/3 & 37/3 \\ 0 & 1/3 & -1 & -7 & 19/3 & -44/3 \end{bmatrix}$$

$R1 \to R1 - 5R2/6$, $R3 \to R3 - 3\ R2/2$, $R4 \to R4 - R2/3$, $R5 \to R5 - R2/3$

$$\begin{bmatrix} 1 & 5/6 & -1/2 & -1/3 & 7/12 & 5/6 \\ 0 & 1 & -3 & -63/2 & 59/2 & -65 \\ 0 & 0 & 0 & 89/2 & -89/2 & 89 \\ 0 & 0 & 0 & 31/2 & -31/2 & 34 \\ 0 & 0 & 0 & 7/2 & -7/2 & 7 \end{bmatrix}$$

Swapping the rows as row 3 and column 3 i.e. the pivot elements are zero.
$R3 \to 2\ R3/89$

$$\begin{bmatrix} 1 & 0 & 2 & 26 & -24 & 55 \\ 0 & 1 & -3 & -63/2 & 59/2 & -65 \\ 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 31/2 & -31/2 & 34 \\ 0 & 0 & 0 & 7/2 & -7/2 & 7 \end{bmatrix}$$

R1 → R1 - 26R3, R2 → R2 + 63 R3/2, R4 → R4 - 31 R3/2, R5 → R5 - 7 R3/2

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 2 & 3 \\ 0 & 1 & -3 & 0 & -2 & -2 \\ 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Swapping the rows as row 4 and column 5 i.e. the pivot elements are zero.
R4 → R4/3

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 2 & 3 \\ 0 & 1 & -3 & 0 & -2 & -2 \\ 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

R1 → R1 - 3 R4, R2 → R2 + 2 R4, R3 → R3 - 2 R4:

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 2 & 3 \\ 0 & 1 & -3 & 0 & -2 & -2 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot columns are 1, 2, 4 and 5,
Here, matrix B is

$$\begin{bmatrix} 12 & 10 & -3 & 10 \\ -7 & -6 & 7 & 5 \\ 9 & 9 & -5 & -1 \\ -4 & -3 & 6 & 9 \\ 8 & 7 & -9 & -8 \end{bmatrix}$$

R1 → R1 / 12

$$\begin{bmatrix} 1 & 5/6 & -1/4 & 5/6 \\ -7 & -6 & 7 & 5 \\ 9 & 9 & -5 & -1 \\ -4 & -3 & 6 & 9 \\ 8 & 7 & -9 & -8 \end{bmatrix}$$

R2 → R2 + 7R1, R3 → R3 - 9 R1, R4 → R4 + 4R1, R5 → R5 - 8R1

$$\begin{bmatrix} 1 & 5/6 & -1/4 & 5/6 \\ 0 & -1/6 & 21/4 & 65/6 \\ 0 & 3/2 & -11/4 & -17/2 \\ 0 & 1/3 & 5 & 37/3 \\ 0 & 1/3 & -7 & -44/3 \end{bmatrix}$$

R2 → -6R2

$$\begin{bmatrix} 1 & 5/6 & -1/4 & 5/6 \\ 0 & 1 & -63/2 & -65 \\ 0 & 3/2 & -11/4 & -17/2 \\ 0 & 1/3 & 5 & 37/3 \\ 0 & 1/3 & -7 & -44/3 \end{bmatrix}$$

R1 → R1 - 5R2/6, R3 → R3 - 3R2/2, R4 → R4 - R2/3, R5 → R5 - R2/3

$$\begin{bmatrix} 1 & 0 & 26 & 55 \\ 0 & 1 & -63/2 & -65 \\ 0 & 0 & 89/2 & 89 \\ 0 & 0 & 31/2 & 34 \\ 0 & 0 & 7/2 & 7 \end{bmatrix}$$

R3 → 2 R3/89

$$\begin{bmatrix} 1 & 0 & 26 & 55 \\ 0 & 1 & -63/2 & -65 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 31/2 & 34 \\ 0 & 0 & 7/2 & 7 \end{bmatrix}$$

R1 → R1 - 26 R3, R2 → R2 + 63 R3/2, R4 → R4 - 31 R3/2, R5 → R5 - 7 R3 /2

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

R4 → R4 / 3

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

R1 → R1 - 3 R4, R2 → R2 + 2 R4, R3 → R3 - 2 R4

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, the solution is $x_1 = 0,\ x_2 = 0,\ x_3 = 0$

Thus, $Bx = 0$ has only the trivial solution.

Question 2:

Since $\{v_{1,}\ v_2,\ v_3\}$ is a linearly dependent set in R^n, there exist scalars $a_{1,}\ a_2,\ a_3$, not all zero, such that

$a_1v_1 + a_2v_2 + a_3v_3 = 0$

Applying the linear transformation T to both sides of the equation, we get

$T(a_1v_1 + a_2v_2 + a_3v_3) = T(0)$

Since T is linear, it preserves scalar multiplication and vector addition, so we have

$a_1T(v_1) + a_2T(v_2) + a_3T(v_3) = T(0)$

The image of the zero vector under any linear transformation is the zero vector in the codomain.

Thus, T(0)=0 in R^m. Therefore, we have

$a_1T(v_1) + a_2T(v_2) + a_3T(v_3) = 0$

Since at least one of $a_{1,}\ a_2,\ a_3$ is non-zero, the set $\{T(v_1),\ T(v_2),\ T(v_3)\}$ is linearly dependent in R^m.

Question 3:

To show: $T(x_1,\ x_2) = (4x_1 - 2x_2,\ 3|x_2|)$ is not linear.

Let the vector $u = (x_1,\ x_2)$ where $x_2 < 0$ (i.e. $u = (0,\ -1)$) and a scalar $c =- 1$.

We know, $T(cu) = cT(u)$, according to the homogeneity property.

For $T(cu)$:

We have $cu = (-x_1, -x_2)$:

$T(-x_1, -x_2) = (4(-x_1) - 2(-x_2), 3|-x_3|) = (-4x_1 + 2x_2, 3x_2)$

Since $x_2 < 0, |x_2| =- x_2\ and\ |-x_2| = x_2$

Now,

For $cT(u)$:

We have, $u\ = (x_1, x_2)$:

$cT(x_1, x_2) =- 1\ *\ (4x_1 - 2x_2, 3|x_3|) = (-4x_1 + 2x_2, - 3|x_2|)$

Comparing $T(cu)\ and\ cT(u)$:

$(-4x_1 + 2x_2, 3x_2)\ \neq (-4x_1 + 2x_2, - 3|x_2|)$

This inequality demonstrates that T does not satisfy the homogeneity property for all vectors in its domain and scalars c , thus proving that T is not a linear transformation.

Question 4:

We have, Ax = 0, for all x such that T(x) = 0

We need null space of matrix A.

Given,

A =

$$\begin{bmatrix} 4 & -2 & 5 & -5 \\ -9 & 7 & -8 & 0 \\ -6 & 4 & 5 & 3 \\ 5 & -3 & 8 & -4 \end{bmatrix}$$

Converting into row reduction echelon form:

R1=R1/4

$$\begin{bmatrix} 1 & -1/2 & 5/4 & -5/4 \\ -9 & 7 & -8 & 0 \\ -6 & 4 & 5 & 3 \\ 5 & -3 & 8 & -4 \end{bmatrix}$$

R2 = R2 + 9R1, R3 = R3 + 6R1, R4 = R4 - 5R1

$$\begin{bmatrix} 1 & -1/2 & 5/4 & -5/4 \\ 0 & 5/2 & 13/4 & -45/4 \\ 0 & 1 & 25/2 & -9/2 \\ 0 & -1/2 & 7/4 & 9/4 \end{bmatrix}$$

R2 = 2 * R2/5

$$\begin{bmatrix} 1 & -1/2 & 5/4 & -5/4 \\ 0 & 1 & 13/10 & -9/2 \\ 0 & 1 & 25/2 & -9/2 \\ 0 & -1/2 & 7/4 & 9/4 \end{bmatrix}$$

R1 = R1 + R2/2, R3 = R3 - R2, R4 = R4 + R2/2

$$\begin{bmatrix} 1 & 0 & 19/10 & -7/2 \\ 0 & 1 & 13/10 & -9/2 \\ 0 & 0 & 56/5 & 0 \\ 0 & 0 & 12/5 & 0 \end{bmatrix}$$

R3 = 5R3/56

$$\begin{bmatrix} 1 & 0 & 19/10 & -7/2 \\ 0 & 1 & 13/10 & -9/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 12/5 & 0 \end{bmatrix}$$

R1 = R1 - 19 R3/10, R2 = R2 - 13 R3/10, R4 = R4 - 12 R3/5

$$\begin{bmatrix} 1 & 0 & 0 & -7/2 \\ 0 & 1 & 0 & -9/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From above, we know that,
x_4 can be a free variable so, $x_4 = x_4$
$x_3 = 0$
Then,
$x_1 - \frac{7}{2}x_4 = 0$
$or,\ x_1 = \frac{7}{2}x_4$
Again,
$x_2 - \frac{9}{2}x_4 = 0$
$or,\ x_2 = \frac{9}{2}x_4$
We can define the variables in terms of x_4. Let x_4 be t.
Now,
$x_1 = \frac{7}{2}t, x_2 = \frac{9}{2}t$
Thus, x =

$$\begin{bmatrix} \frac{7t}{2} \\ \frac{9t}{2} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} t \end{bmatrix}$$

$$or,\ x = \begin{bmatrix} \frac{7}{2} \\ \frac{9}{2} \\ 0 \\ 1 \end{bmatrix} \cdot t$$

Which is the null space of matrix A.

Question 5:
Given:

$$T(e_1) = \begin{bmatrix} 3 \\ 1 \\ 3 \\ 1 \end{bmatrix}$$

$$T(e_2) = \begin{bmatrix} -5 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

For the standard matrix A for the linear transformation $T: R^2 -> R^4$, Taking $T(e_1)$ and $T(e_2)$ as its columns:

$$A = [T(e_1)\ T(e_2)] = \begin{bmatrix} 3 & -5 \\ 1 & 2 \\ 3 & 0 \\ 1 & 0 \end{bmatrix}$$

Therefore, the standard matrix A of the linear transformation T is:

$$A = \begin{bmatrix} 3 & -5 \\ 1 & 2 \\ 3 & 0 \\ 1 & 0 \end{bmatrix}$$

This matrix represents the linear transformation T from R^2 to R^4.

Question 6:

To find:
The standard matrix representation of T which reflects through
$x_1 - axis$ and through the line $x_2 = x_1$
Let R_{x1} be the reflection through the $x_1 - axis$ and $R_{x2=x1}$ be the reflection through the line
$x_2 = x_1$.
The standard matrix for R_{x1} is:

$$R_{x1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Through $x_1 - axis$,
$(x,\ y)$ −−−−−> $(x,\ -y)$

The standard matrix for $R_{x1=x2}$ is:

$$R_{x1=x2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Through the line $x_1 = x_2$,
$(x,\ y)\ ----->\ (y,\ x)$
For standard matrix of T,

$$T = R_{x1=x2} \cdot R_{x1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

For angle of rotation θ,
We need the standard form of a 2D rotation matrix $R(\theta)$:

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\ sin(\theta \\ sin(\theta) & cos(\theta) \end{bmatrix}$$

Comparing T with $R(\theta)$,
$cos(\theta) = 0\ and\ sin(\theta) = 1$
Which corresponds to
$\theta = 90^{\circ}$ i.e. counterclockwise rotation about the origin.

Question 7:
Given that:
$A = [a_1,\ a_2]$
Where a_1 and a_3 are given in the figure.

Let a_1 and a_3 be the vectors such that

$$a_1 = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}, \ a_2 = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$$

To find the image of the vector $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$,

$$T\begin{bmatrix} -1 \\ 3 \end{bmatrix} = A\begin{bmatrix} -1 \\ 3 \end{bmatrix} = [a_1, \ a_2]\begin{bmatrix} -1 \\ 3 \end{bmatrix} = - a_1 + 3a_2 \rightarrow (1)$$

From the figure,

$$a_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ a_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

From eq (1),

$$- a_1 + 3a_2 = - 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Plotting the vectors in a graph,