Question 2:

Since $\{v_{1,} v_2, v_3\}$ is a linearly dependent set in R^n, there exist scalars $a_{1,} a_2, a_3$, not all zero, such that

$a_1v_1 + a_2v_2 + a_3v_3 = 0$

Applying the linear transformation T to both sides of the equation, we get

$T(a_1v_1 + a_2v_2 + a_3v_3) = T(0)$

Since T is linear, it preserves scalar multiplication and vector addition, so we have

$a_1T(v_1) + a_2T(v_2) + a_3T(v_3) = T(0)$

The image of the zero vector under any linear transformation is the zero vector in the codomain. Thus, T(0)=0 in R^m. Therefore, we have

$a_1T(v_1) + a_2T(v_2) + a_3T(v_3) = 0$

Since at least one of $a_{1,} a_2, a_3$ is non-zero, the set $\{T(v_1), T(v_2), T(v_3)\}$ is linearly dependent in R^m.

Question 3:

To show: $T(x_1, x_2) = (4x_1 - 2x_2, 3|x_2|)$ is not linear.

Let the vector $u = (x_1, x_2)$ where $x_2 < 0$ (i.e. $u = (0, -1)$) and a scalar $c =- 1$.

We know, $T(cu) = cT(u)$, according to the homogeneity property.

For $T(cu)$:

We have $cu = (-x_1, -x_2)$:

$T(-x_1, -x_2) = (4(-x_1) - 2(-x_2), 3|-x_3|) = (-4x_1 + 2x_2, 3x_2)$

Since $x_2 < 0, |x_2| =- x_2\, and\, |-x_2| = x_2$

Now,

For $cT(u)$:

We have, $u = (x_1, x_2)$:

$cT(x_1, x_2) =- 1 * (4x_1 - 2x_2, 3|x_3|) = (-4x_1 + 2x_2, -3|x_2|)$

Comparing $T(cu)\, and\, cT(u)$:

$(-4x_1 + 2x_2, 3x_2) \neq (-4x_1 + 2x_2, -3|x_2|)$

This inequality demonstrates that T does not satisfy the homogeneity property for all vectors in its domain and scalars c , thus proving that T is not a linear transformation.

Question 4:

We have, Ax = 0, for all x such that T(x) = 0

We need null space of matrix A.
Given,
A =

$$\begin{bmatrix} 4 & -2 & 5 & -5 \\ -9 & 7 & -8 & 0 \\ -6 & 4 & 5 & 3 \\ 5 & -3 & 8 & -4 \end{bmatrix}$$

Converting into row reduction echelon form:
R1=R1/4

$$\begin{bmatrix} 1 & -1/2 & 5/4 & -5/4 \\ -9 & 7 & -8 & 0 \\ -6 & 4 & 5 & 3 \\ 5 & -3 & 8 & -4 \end{bmatrix}$$

R2 = R2 + 9R1, R3 = R3 + 6R1, R4 = R4 - 5R1

$$\begin{bmatrix} 1 & -1/2 & 5/4 & -5/4 \\ 0 & 5/2 & 13/4 & -45/4 \\ 0 & 1 & 25/2 & -9/2 \\ 0 & -1/2 & 7/4 & 9/4 \end{bmatrix}$$

R2 = 2 * R2/5

$$\begin{bmatrix} 1 & -1/2 & 5/4 & -5/4 \\ 0 & 1 & 13/10 & -9/2 \\ 0 & 1 & 25/2 & -9/2 \\ 0 & -1/2 & 7/4 & 9/4 \end{bmatrix}$$

R1 = R1 + R2/2, R3 = R3 - R2, R4 = R4 + R2/2

$$\begin{bmatrix} 1 & 0 & 19/10 & -7/2 \\ 0 & 1 & 13/10 & -9/2 \\ 0 & 0 & 56/5 & 0 \\ 0 & 0 & 12/5 & 0 \end{bmatrix}$$

R3 = 5R3/56

$$\begin{bmatrix} 1 & 0 & 19/10 & -7/2 \\ 0 & 1 & 13/10 & -9/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 12/5 & 0 \end{bmatrix}$$

R1 = R1 - 19 R3/10, R2 = R2 - 13 R3/10, R4 = R4 - 12 R3/5

$$\begin{bmatrix} 1 & 0 & 0 & -7/2 \\ 0 & 1 & 0 & -9/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From above, we know that,
x_4 can be a free variable so, $x_4 = x_4$
$x_3 = 0$
Then,
$x_1 - \frac{7}{2}x_4 = 0$
$or,\ x_1 = \frac{7}{2}x_4$
Again,
$x_2 - \frac{9}{2}x_4 = 0$
$or,\ x_2 = \frac{9}{2}x_4$

We can define the variables in terms of x_4. Let x_4 be t.

Now,

$x_1 = \frac{7}{2}t,\ x_2 = \frac{9}{2}t$

Thus, x =

$$\begin{bmatrix} \frac{7t}{2} \\ \frac{9t}{2} \\ 0 \\ t \end{bmatrix}$$

$$or,\ x = \begin{bmatrix} \frac{7}{2} \\ \frac{9}{2} \\ 0 \\ 1 \end{bmatrix} \cdot t$$

Which is the null space of matrix A.