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On Design of Suboptimal Tracking Controller for a Class of Nonlinear Systems

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Abstract—**In this paper, a new technique for solving the suboptimal tracking problem for a class of nonlinear dynamical systems based on the pseudo linearization is presented. Towards this end, an optimal tracking problem using a discounted cost function is defined and a control law with a feedback feedforward structure is designed. A state-dependent Riccati equation (SDRE) is solved in order to minimize the cost function in a suboptimal way. Due to the significant properties of the SDRE technique, the proposed method can handle the presence of input saturation, state constraint, time delay, and chaotic behavior. Two numerical examples are provided to illustrate the effectiveness and capabilities of the proposed design methodology.**

Keywords: Optimal tracking control, Nonlinear systems, Pseudo linearization, State-dependent Riccati equation (SDRE).

I. Introduction

Optimal control is a class of methods deals with the problem of finding a control law in order to achieve the best possible behavior with respect to a predefined criterion. These methods are usually constructed based on Pontryagin's minimum principle or Bellman's principle of optimality [1]. The optimal quadratic regulation problem for linear systems was solved in the 1960s [1]. The obtained results were also developed for the optimal tracking problems for linear systems [1]–[3]. Nevertheless, in many practical engineering problems, the system to be controlled is nonlinear. Although Jacobian linearization can be used to employ the linear optimal controllers for nonlinear systems, the performance of the designed system is degraded for a wide range of system operation. Therefore, it is so important to consider the nonlinearities of the system in the design procedure to obtain the optimal performance. As a result, development of the optimal controller (regulator and tracking) for nonlinear systems has been an area of extensive research over the last decades [4].

For a nonlinear optimal regulation problem several different methods have been proposed [5]–[9]. However, due to the complexity of the arising Hamilton-Jacobi-Bellman (HJB) equation, which is too difficult or even impossible to be solved, various methods are developed to find *approximate* solutions of the problem. For instance, Taylor series expansion is used to approximately solve the infinite-time horizon nonlinear optimal regulation (ITHNOR) problem [5]. Using adaptive dynamic programing technique, two algorithms are proposed in [6] and [7] to solve ITHNOR problem with known and unknown dynamics, respectively. State-dependent Riccati equation (SDRE) techniques have been also developed to approximately solve the nonlinear optimal regulation problem with both finite and infinite horizons [8]–[10]. Although some methods are proposed to solve the optimal tracking problem for nonlinear systems [11]–[12], it can be said that much less attention has been paid to this problem. This is mainly due to the fact that for most reference signals the cost function corresponding to the tracking problem becomes unbounded and also there are additional computational difficulties due to computing the feedforward control term which is not presented in the optimal regulator problem.

The SDRE technique was originally proposed by Person in 1962 to approximately solve the optimal regulation problem for nonlinear systems [13]. Representing the nonlinear system dynamics as a state-dependent linear system, called the pseudo linearization or extended linearization [4], is the main idea of the SDRE technique. Since then several methods have been developed based on the idea of the pseudo linearization to solve different problems such as robust H_∞ filter design [14], suboptimal sliding mode control design for delayed systems [15], observer design for nonlinear delayed systems [16], and so on. These methods are effectively applied in a wide variety of applications, such as turning process planning optimization [17] and drug administration in cancer treatment [18]. The main advantages of the SDRE technique are as follows: (i) the pseudo linearization technique maintains the nonlinear characteristics of the system since it does not neglect any nonlinear terms; (ii) the SDRE design procedure is simple and flexible; (iii) it is possible to make a tradeoff between the control effort and the closed-loop system performance by tuning two weighting matrices; and (iv) it can handle the presence of input saturation, state constraint, and time-delay. Due to these advantages, the SDRE based methods have been theoretically developed and some remarkable results are obtained recently [19]–[21]. Two complete surveys of the SDRE techniques and the related theories can be found in [4] and [13].

Based on the above discussion, the pseudo linearization techniques have been successfully developed for a wide variety of control engineering problems. For the set-point tracking problem, the SDRE technique is developed based on the integral action idea [13]. Nevertheless, to the best of

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authors' knowledge, the optimal tracking control problem for nonlinear systems, which is practically very important, has not been solved using the SDRE technique. There are two main reasons for this shortage. The first one is that the optimal tracking problem with the traditional quadratic cost function is only valid for the cases with desired trajectories generated by an asymptotically stable system. However, many of desired trajectories such as steps and sinusoidal signals are not generated by asymptotically stable systems and therefore, it is impossible to define an optimal tracking problem using the traditional quadratic cost function. The second reason is the complexity of the obtained HJB equation which is too difficult or even impossible to be analytically solved. Even if the trajectories are limited to be generated by stable systems, the complexity of the related HJB equation is a challenging issue. This motivates us to develop the SDRE control design procedure for the nonlinear tracking problem.

In this paper, a discounted cost function is used to tackle the first above mentioned problem and also to define the optimal tracking problem for more general desired trajectories. To develop the proposed tracking controller scheme, two steps must be taken. At first, the optimal nonlinear tracking problem is converted to an optimal nonlinear regulation problem. Then, the SDRE technique is used to find a suboptimal solution for the obtained optimal regulation problem or equivalently a solution for the original optimal tracking problem. It should be mentioned that using the proposed method, it is only required to solve a state-dependent Riccati equation and there is no need to solve any differential equations. It is shown that the proposed method yields to a control law with feedforward-feedback structure where both the feedback and the feedforward gains are found simultaneously. The proposed method inherits almost all of the interesting properties of the SDRE technique such as additional degrees of freedom, ability to consider input saturation, and so on.

The remainder of the paper is organized as follows. In Section II, the statement of the optimal tracking problem is presented. Section III explains the proposed SDRE tracking controller in detail. In Section IV, two numerical examples are provided to illustrate the design procedure and show the flexibility of the proposed method. Finally, Section V concludes the paper.

II. System Description and Problem Statement

Consider the following nonlinear dynamical system described by

$$\begin{aligned} \dot{x}(t) &= f\big(x(t)\big) + B\big(x(t)\big)u(t), \\ y(t) &= h\big(x(t)\big), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the control input, and $y \in \mathbb{R}^p$ is the system output. f, B, and h are assumed to be smooth functions and $f(0) = h(0) = 0$ and $B(x) \neq 0$ for all $x \in \mathbb{R}^n$.

As mentioned, the optimal tracking problem with traditional quadratic cost function is only valid for the cases where the desired trajectories are generated by an asymptotically stable system. However, there are many practically important trajectories which are not generated by such a system. In this paper, to tackle this problem, a discounted cost function is considered and a technique to solve the following optimal tracking problem is proposed.

Discounted infinite-time horizon nonlinear optimal tracking (DITHNOT) problem: Find the control input $u(t), t > 0$ to force the system output $y(t), t > 0$ to track the desired trajectory $y_d(t), t > 0$ such that the following discounted cost function is minimized:

$$\begin{aligned} J = \int_0^\infty e^{-2\gamma t} \Big(& \big(y(t) - y_d(t)\big)^{\mathrm{T}} Q \big(y(t) - y_d(t)\big) \\ & + u^{\mathrm{T}}(t) R u(t) \Big) dt, \end{aligned} \tag{2}$$

where $\gamma > 0$ is the discount factor. It is assumed that Q and R are respectively positive-semidefinite and positive-definite symmetric matrix-valued functions of the system state with appropriate dimensions. Assume further that the desired trajectory has the following nonlinear dynamics:

$$\begin{aligned} \dot{x}_d(t) &= \mathscr{f}\big(x_d(t)\big), \\ y_d(t) &= \mathscr{h}\big(x_d(t)\big), \end{aligned} \tag{3}$$

where $x_d \in \mathbb{R}^{n_d}$ and $y_d \in \mathbb{R}^p$ are the state and output of the desired trajectory system (3) and functions $\mathscr{f}$ and $\mathscr{h}$ are assumed to be smooth and $\mathscr{f}(0) = \mathscr{h}(0) = 0$.

It should be mentioned that many commonly used trajectories such as steps and sinusoidal signals can be generated by (3). In addition, the nonlinear dynamics (3) covers a broad class of interesting trajectories such as trajectories generated by Lorenz chaotic system which can be used for example in the chaos synchronization problem.

Applying the Bellman's principle of optimality to the above DITHNOT problem leads to an HJB equation which is too difficult or impossible to be analytically solved. Therefore, finding approximate solutions of the problem is considered as an alternative way in order to avoid encountering the complicated HJB equation. It the following section, an approximate method is proposed to solve the problem.

III. Main Result

In this section, based on the pseudo linearization idea, a technique to find a suboptimal solution of the DITHNOT problem is proposed. Since the nonlinear functions f, h, $\mathscr{f}$, and $\mathscr{h}$ are assumed to be smooth and $f(0) = h(0) = \mathscr{f}(0) = \mathscr{h}(0) = 0$, they can be rewritten in their pseudo linearized forms (also called state-dependent coefficient (SDC)) as follows [13]:

$$\begin{aligned} f(x) &= F(x)x, h(x) = H(x)x, \\ \mathscr{f}(x_d) &= \mathcal{F}(x_d)x_d, \mathscr{h}(x_d) = \mathcal{H}(x_d)x_d, \end{aligned} \tag{4}$$

where $F \in \mathbb{R}^{n\times n}, H \in \mathbb{R}^{p\times n}, \mathcal{F} \in \mathbb{R}^{n_d\times n_d}$, and $\mathcal{H} \in \mathbb{R}^{p\times n_d}$ are four matrix-valued functions. It should be mentioned that there are infinite ways to pseudo linearize non-scalar systems. This property of the pseudo linearization technique provides additional degrees of freedom which can enhance the design procedure of SDRE based methods [13]. Defining $\mathcal{X}(t) = e^{-\gamma t}\big[x^{\mathrm{T}}(t), x_d^{\mathrm{T}}(t)\big]^{\mathrm{T}}$ and $\mathcal{U}(t) = e^{-\gamma t}u(t)$ and substituting them in the cost function (2) in the DITHNOT problem leads to

$$J = \int_0^\infty \left(\mathcal{X}^{\mathrm{T}}(t)\mathcal{Q}(e^{\gamma t}\mathcal{X})\mathcal{X}(t) + \mathcal{U}^{\mathrm{T}}(t)R\mathcal{U}(t)\right)dt, \tag{5}$$

where

$$\mathcal{Q}\left(e^{\gamma t}\mathcal{X}(t)\right) = [H(x) - \mathcal{H}(x_d)]^{\mathrm{T}}Q[H(x) - \mathcal{H}(x_d)].$$

The nonlinear dynamics of the new state variable $\mathcal{X}(t)$ is given as:

$$\dot{\mathcal{X}}(t) = -\gamma\mathcal{X}(t) + e^{-\gamma t}\left[\dot{x}^{\mathrm{T}}(t), \dot{x}_d^{\mathrm{T}}(t)\right]^{\mathrm{T}}.$$

Now, by substituting $\dot{x}(t)$ and $\dot{x}_d(t)$ from (2) and (3), respectively, and using (4), we have the following augmented pseudo linearized dynamics:

$$\dot{\mathcal{X}}(t) = \left(-\gamma I + \begin{pmatrix} F\left(x(t)\right) & 0 \\ 0 & \mathcal{F}\left(x_d(t)\right)\end{pmatrix}\right)\mathcal{X}(t) + \begin{pmatrix} B\left(x(t)\right) \\ 0 \end{pmatrix}\mathcal{U}(t) = \mathcal{A}\left(e^{\gamma t}\mathcal{X}(t)\right)\mathcal{X}(t) + \mathcal{B}\left(e^{\gamma t}\mathcal{X}(t)\right)\mathcal{U}(t), \tag{6}$$

where I and 0 denote the identity and zero matrices with appropriate dimensions, respectively. Therefore, an infinite-time horizon nonlinear optimal regulation (ITHNOR) problem, described by (5) and (6), should be solved instead of the DITHNOT problem. The optimal solution of the ITHNOR problem is

$$\mathcal{U}(t) = -R^{-1}\mathcal{B}^{\mathrm{T}}\left(e^{\gamma t}\mathcal{X}(t)\right)\frac{\partial V(t,\mathcal{X})}{\partial \mathcal{X}},$$

where $V(t,\mathcal{X})$ is the solution of the following HJB equation which arises from the Bellman's principle of optimality [4]:

$$-\frac{\partial V}{\partial t} = \inf_{\mathcal{U}}\left(\left(\frac{\partial V}{\partial \mathcal{X}}\right)^{\mathrm{T}}\dot{\mathcal{X}} + \mathcal{X}^{\mathrm{T}}\mathcal{Q}(e^{\gamma t}\mathcal{X})\mathcal{X} + \mathcal{U}^{\mathrm{T}}R\mathcal{U}\right). \tag{7}$$

However, solving the above HJB equation is not generally easier than the HJB equation arising from the original DITHNOT problem. Nevertheless, there are some well-known approximate methods to solve ITHNOR problems [5]–[9] and among them, the SDRE is one of the most popular methods which yield to suboptimal performance [13]. In the following, the SDRE technique is used to find a suboptimal control law for the ITHNOR problem or equivalently the DITHNOT problem. Towards this end, some necessary definitions which are needed in the rest of the paper are presented.

Definition 1: The SDC representation (6) is point-wise stabilizable in the bounded open set $\Omega \subseteq \mathbb{R}^{n+n_d}$ if the pair $\left\{\mathcal{A}\left(e^{\gamma t}\mathcal{X}(t)\right), \mathcal{B}\left(e^{\gamma t}\mathcal{X}(t)\right)\right\}$ is stabilizable in the linear sense for all $\mathcal{X} \in \Omega$ and $t > 0$.

Definition 2: The SDC representation (6) is point-wise observable in the bounded open set $\Omega \subseteq \mathbb{R}^{n+n_d}$ if the pair $\left\{\mathcal{A}\left(e^{\gamma t}\mathcal{X}(t)\right), \mathcal{Q}^{1/2}\left(e^{\gamma t}\mathcal{X}(t)\right)\right\}$ is detectable in the linear sense for all $\mathcal{X} \in \Omega$ and $t > 0$.

In order to find a suboptimal solution for the above ITHNOR problem using the SDRE technique, two steps must be taken [22]. At the first step, the following state-dependent algebraic Riccati equation should be solved to find the matrix $\mathcal{P}\left(e^{\gamma t}\mathcal{X}(t)\right)$,

$$\mathcal{A}^{\mathrm{T}}\left(e^{\gamma t}\mathcal{X}(t)\right)\mathcal{P}\left(e^{\gamma t}\mathcal{X}(t)\right) + \mathcal{P}\left(e^{\gamma t}\mathcal{X}(t)\right)\mathcal{A}\left(e^{\gamma t}\mathcal{X}(t)\right) - \mathcal{P}\left(e^{\gamma t}\mathcal{X}(t)\right)\mathcal{B}\left(e^{\gamma t}\mathcal{X}(t)\right)R^{-1}\mathcal{B}^{\mathrm{T}}\left(e^{\gamma t}\mathcal{X}(t)\right)\mathcal{P}\left(e^{\gamma t}\mathcal{X}(t)\right) + \mathcal{Q}\left(e^{\gamma t}\mathcal{X}(t)\right) = 0. \tag{8}$$

The SDRE (8) has a unique symmetric positive-semidefinite solution $\mathcal{P}\left(e^{\gamma t}\mathcal{X}(t)\right)$ if the triple $\left(\mathcal{A}\left(e^{\gamma t}\mathcal{X}(t)\right), \mathcal{B}\left(e^{\gamma t}\mathcal{X}(t)\right), \mathcal{Q}^{1/2}\left(e^{\gamma t}\mathcal{X}(t)\right)\right)$ is point-wise stabilizable and detectable [22]. While this equation can be solved analytically for simple problems, there are some numerical methods to find its solution for complex systems [13]. The second step of the SDRE design procedure is to compute the control law $\mathcal{U}$ as:

$$\mathcal{U}(t) = -R^{-1}\mathcal{B}^{\mathrm{T}}\left(e^{\gamma t}\mathcal{X}(t)\right)\mathcal{P}\left(e^{\gamma t}\mathcal{X}(t)\right)\mathcal{X}(t).$$

It can be seen that the above technique uses the solution of the SDRE (8) instead of solving the HJB equation (7). Although it has been shown that there is always an SDC representation which yields to the optimal solution [18], finding such an SDC form is not straightforward. However, using any SDC representation leads to have a suboptimal control law [18].

Now, using the obtained control law $\mathcal{U}$, we can find the following control law for the original DITHNOT problem:

$$u(x, x_d) = -R^{-1}\mathcal{B}^{\mathrm{T}}(x)\mathcal{P}(x, x_d)\left[x^{\mathrm{T}}, x_d^{\mathrm{T}}\right]^{\mathrm{T}}.$$

The above control law can be rewritten as follows:

$$u(x, x_d) = -K_f(x, x_d)x - K_{ff}(x, x_d)x_d,$$

where K_f and K_{ff} are respectively the state-dependent feedback and feedforward gains which both are calculated from solving the SDRE (8).

The proposed SDRE algorithm for solving the DITHNOT problem can be summarized as follows.

SDRE tracking algorithm
1: Select SDC representations for $f, h, \mathcal{f}$, and $\mathcal{h}$.
2: Check point-wise stabilizability and detectability of the triple $\left(F(x), B(x), Q^{1/2}(x)\right)$.
3: Find $\mathcal{P}$ from the SDRE (8).
4: Apply $u(x, x_d) = R^{-1}\mathcal{B}^{\mathrm{T}}(x)\mathcal{P}(x, x_d)\left[x^{\mathrm{T}}, x_d^{\mathrm{T}}\right]^{\mathrm{T}}$ to the nonlinear system (1).

Remark: The proposed SDRE tracking controller can be used for some classes of nonlinear delayed systems based on the extensions of the SDRE regulator in [23].

IV. Simulation Results

This section presents the results of applying the proposed method on two practical engineering problems (Vander Pol's oscillator and Lorenz chaotic system). The obtained simulation results show the effectiveness of the proposed method to solve the DITHNOT problem.

A. Example 1 (Vander Pol's Oscillator)

Consider the Vander Pol's oscillator system [24]

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -x_1 + \alpha(1 - x_1^2)x_2 + u,$$
$$y = x_1.$$

Assume that the output y is supposed to track the desired trajectory $y_d(t) = \sin(t)$ in such a way that the following discounted cost function be minimized:

$$J = \int_0^\infty e^{-2\gamma t}\left(Q\big(y(t) - y_d(t)\big)^2 + R\big(u(t)\big)^2\right)dt.$$

To apply the proposed method, the following SDC representation is considered:

$$F(x) = \begin{bmatrix} 0 & 1 \\ -1 & \alpha(1 - x_1^2) \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathcal{F} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Therefore, the following SDC form for the augmented system is obtained:

$$\dot{\mathcal{X}} = \begin{bmatrix} -\gamma & 1 & 0 & 0 \\ -1 & \alpha(1 - e^{2\gamma t}\mathcal{X}_1^2) - \gamma & 0 & 0 \\ 0 & 0 & -\gamma & 1 \\ 0 & 0 & -1 & -\gamma \end{bmatrix}\mathcal{X} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}u. \quad (10)$$

In order to use the proposed method, the point-wise stabilizability and detectability of the above SDC representation must be check in advanced. The state-dependent controllability matrix is as follows:

$$\Phi_c = \begin{bmatrix} 0 & 1 & \star & \star \\ 1 & \alpha(1 - e^{2\gamma t}\mathcal{X}_1^2) - \gamma & \star & \star \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $\star$ is used to show the uncalculated elements. Obviously the first two state variables, i.e. x_1 and x_2, are controllable while the states of the trajectory are not. Nevertheless, for any $\gamma > 0$ these uncontrollable states are stabilizable and the SDC representation (10) is point-wise stabilizable. The state-dependent observability matrix is as follows:

$$\Phi_o = \begin{bmatrix} \sqrt{Q} & 0 & -\sqrt{Q} & 0 \\ \star & \sqrt{Q} & \star & 0 \\ \star & \star & \star & 0 \\ \star & \star & \star & 0 \end{bmatrix}.$$

One can see that only the last state variable $\mathcal{X}_4$ is not observable. Since this state variable is detectable, we can conclude that the SDC representation (10) is point-wise detectable for any positive constants γ and Q.

In the following simulations, we select $Q = 10, R = 1, x_0 = [2, 1]^T, \alpha = 0.9$, and $\gamma = 0.1$. The system output y and the control input u are shown in Figs. 1 and 2. From these figures, it can be concluded that the results are satisfying and tracking goal is successfully achieved.

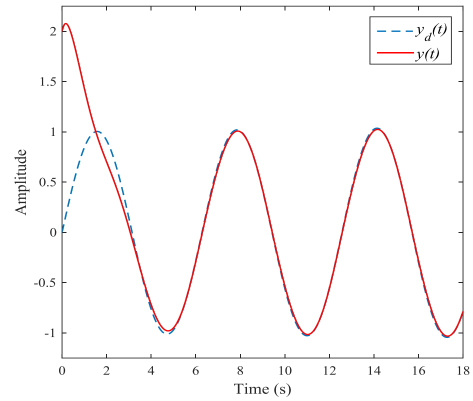

Figure 1. The system output $y(t)$ and the desired output $y_d(t)$ corresponding to Example 1.

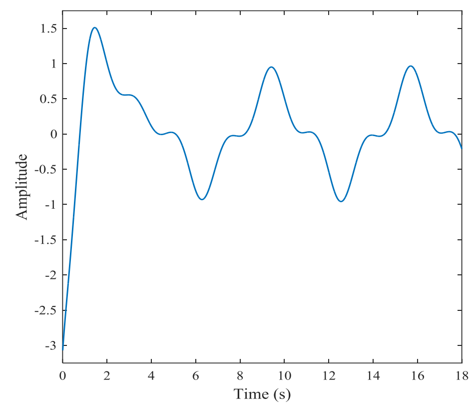

Figure 2. The control input $u(t)$ corresponding to Example 1.

B. Example 2 (Lorenz chaotic system)

In this example, the proposed method is used to solve the problem of chaos synchronization of the Lorenz system. The system is a set of three differential equations given by [26]:

$$\dot{x}_1(t) = a\big(x_2(t) - x_1(t)\big) + u_1(t),$$
$$\dot{x}_2(t) = dx_1(t) - x_1(t)x_3(t) + cx_2(t) + u_2(t),$$
$$\dot{x}_3(t) = x_1(t)x_2(t) - bx_3(t) + u_3(t),$$

where $a, b, c,$ and d are four parameters. We assume that $a = 5,\ b = 0.5,\ c = -1$, and $d = 10$. Our objective is to design a controller based on the proposed method in order to synchronize the system with another Lorenz system with different initial conditions. An SDC representation of the system is as follows:

$$\dot{x}(t) = A\big(x(t)\big)x(t) + Bu(t),$$
$$A\big(x(t)\big) = \begin{bmatrix} -5 & 5 & 0 \\ 10 & -1 & -x_1(t) \\ 0 & x_1(t) & -0.5 \end{bmatrix}, B = I_3.$$

One can see that the triple $(A(x(t), B, Q)$ is point-wise controllable and detectable for $Q > 0$. Therefore, it can be concluded that the proposed method can solve the above synchronization problem. The obtained results are depicted in Figs. 3 and 4. From Fig. 3, it can be seen that the synchronization is achieved in a short period of time after the start of the controller at $t = 10\ s$.

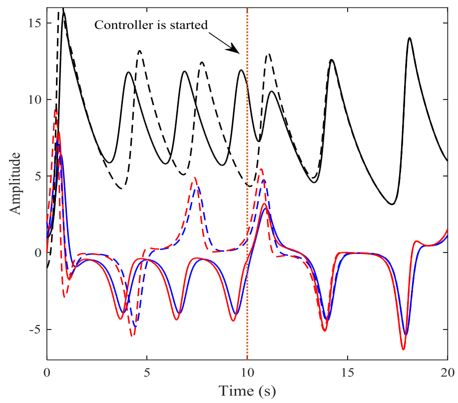

Figure 3. The system states (solid lines) and the desired trajectories (dashed lines) corresponding to Example 2.

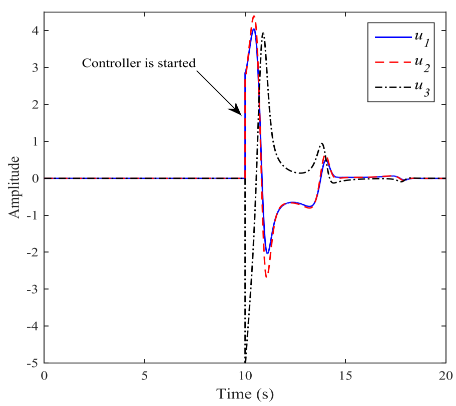

Figure 4. The control inputs corresponding to Example 2.

V. Conclusion

Using a discounted cost function, a general optimal tracking problem has been considered for a broad class of nonlinear systems. The tracking problem has been converted to an optimal regulation problem without any discount factor by defining some new state variables and control input. In order to avoid encountering any HJB equations, the SDRE technique has been used to find a suboptimal solution of the obtained regulation problem. It has been shown that this control law has actually feedback-feedforward structure for the original tracking problem, where both the feedback and feedforward gains are calculated by solving a state-dependent algebraic Riccati equation. Satisfactory results of applying the proposed method to two practical problems demonstrate its capabilities to solve the nonlinear tracking problem.

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