Make optimal decisions

Operations Research

Make decisions

Maximize revenue/profit subject to a set of constraints

Domains

Linear Programming

Maximize $2X_1 + 3X_2$

Subject to

$$X_1 + X_2 \leq 120$$
$$2X_1 + 3X_2 \leq 320$$
$$X_1, X_2 \geq 0$$

- Deterministic
- Decision

Programming

Deterministic or stochastic?

Non-linear Programming

Decision or Game?

Linear Programming

Imagine you running an automobile firm which sells cars in three different segments – Hatchback, Sedan and SUV at prices ₹5,00,000, ₹10,00,000 and ₹25,00,000 respectively.

Suppose that the manufacturing of cars primarily requires the following raw materials A and B. The firm has 1,20,000 units of resource A and 1,40,000 units of resource B available. The resource requirements for the manufacturing of each car variant is given below.

Requirements	Resource A	Resource B
Hatchback	15	20
Sedan	20	50
SUV	60	100

How many cars of each type should be produced to maximize revenue?

Decision variables

X_1 - Number of Hatchback cars to be produced

X_2 - Number of Sedan cars to be produced

X_3 - Number of SUV cars to be produced

Objective function

Maximize $500000X_1 + 1000000X_2 + 2500000X_3$

Constraints

Requirements	Resource A	Resource B
Hatchback	15	20
Sedan	20	50
SUV	60	100

- Resource A constraint

$$15X_1 + 20X_2 + 60X_3 \leq 120000$$

- Resource B constraint

$$20X_1 + 50X_2 + 100X_3 \leq 140000$$

- Non-negativity restrictions

$$X_1, X_2, X_3 \geq 0$$

Linear program

Maximize $500000X_1 + 1000000X_2 + 2500000X_3$

Subject to

$$15X_1 + 20X_2 + 60X_3 \leq 120000$$

$$20X_1 + 50X_2 + 100X_3 \leq 140000$$

$$X_1, X_2, X_3 \geq 0$$

Dual of the linear program

Automobile firm

- Possesses resources A and B
- Manufactures and sells cars
- Aim: Maximize revenue

Dual variables: Shadow price or Marginal price of the resource at the optimum

Primal

Maximize $500000X_1 + 1000000X_2 + 2500000X_3$

Subject to

$$15X_1 + 20X_2 + 60X_3 \leq 120000$$
$$20X_1 + 50X_2 + 100X_3 \leq 140000$$
$$X_1, X_2, X_3 \geq 0$$

Dual

Minimize $120000Y_1 + 140000Y_2$

Subject to

$$15Y_1 + 20Y_2 \geq 500000$$
$$20Y_1 + 50Y_2 \geq 1000000$$
$$60Y_1 + 100Y_2 \geq 2500000$$
$$Y_1, Y_2 \geq 0$$

Let Y_1, Y_2 be the costs of resource A and resource B respectively

Buyer

- Purchase resources A and B
- Aim: Minimize total cost

Primal – Dual relationship

Primal

Maximize $500000X_1 + 1000000X_2 + 2500000X_3$

Subject to

$$15X_1 + 20X_2 + 60X_3 \leq 120000$$
$$20X_1 + 50X_2 + 100X_3 \leq 140000$$
$$X_1, X_2, X_3 \geq 0$$

Dual

Minimize $120000Y_1 + 140000Y_2$

Subject to

$$15Y_1 + 20Y_2 \geq 500000$$
$$20Y_1 + 50Y_2 \geq 1000000$$
$$60Y_1 + 100Y_2 \geq 2500000$$
$$Y_1, Y_2 \geq 0$$

Primal	Dual
Maximization	Minimization
Number of constraints	Number of variables
Number of variables	Number of constraints
Objective function coefficient	Right hand side in constraints
Right hand side in constraints	Objective function coefficient

How to construct a dual?

Primal

Maximize $500000X_1 + 1000000X_2 + 2500000X_3$

Subject to

$$15X_1 + 20X_2 + 60X_3 \leq 120000$$
$$20X_1 + 50X_2 + 100X_3 \leq 140000$$
$$X_1, X_2, X_3 \geq 0$$

Dual

Let Y_1, Y_2 be the dual variables corresponding to the two constraints

Minimize $120000Y_1 + 140000Y_2$

Subject to

$$15Y_1 + 20Y_2 \geq 500000$$
$$20Y_1 + 50Y_2 \geq 1000000$$
$$60Y_1 + 100Y_2 \geq 2500000$$
$$Y_1, Y_2 \geq 0$$

Dual of the dual is the primal!

Dual

Minimize $120000Y_1 + 140000Y_2$

Subject to

$$15Y_1 + 20Y_2 \geq 500000$$
$$20Y_1 + 50Y_2 \geq 1000000$$
$$60Y_1 + 100Y_2 \geq 2500000$$
$$Y_1, Y_2 \geq 0$$

Primal

Maximize $500000X_1 + 1000000X_2 + 2500000X_3$

Subject to

$$15X_1 + 20X_2 + 60X_3 \leq 120000$$
$$20X_1 + 50X_2 + 100X_3 \leq 140000$$
$$X_1, X_2, X_3 \geq 0$$

Standard form

Maximize $-120000Y_1 - 140000Y_2$

Subject to

$$-15Y_1 - 20Y_2 \leq -500000$$
$$-20Y_1 - 50Y_2 \leq -1000000$$
$$-60Y_1 - 100Y_2 \leq -2500000$$
$$Y_1, Y_2 \geq 0$$

Finding the dual

Minimize $-500000X_1 - 1000000X_2 - 2500000X_3$

Subject to

$$-15X_1 - 20X_2 - 60X_3 \geq -120000$$
$$-20X_1 - 50X_2 - 100X_3 \geq -140000$$
$$X_1, X_2, X_3 \geq 0$$

How to construct a dual?

Primal

Minimize $500X_1 + 100X_2 + 200X_3$

Subject to

$$15X_1 + 20X_2 + 60X_3 \geq 1200$$
$$20X_1 + 50X_2 + 100X_3 \leq 1400$$
$$X_1 \geq 0, X_2 \leq 0, X_3\ unrestricted$$

Dual

Minimize $-1200Y_1 + 1400Y_2$

Subject to

$$-15Y_1 + 20Y_2 \geq -500$$
$$20Y_1 - 50Y_2 \geq 100$$
$$-60Y_1 + 100Y_2 \geq -200$$
$$60Y_1 - 100Y_2 \geq 200$$
$$Y_1, Y_2 \geq 0$$

Convert to standard form

Define new variables $X_4, X_5, X_6 \geq 0$. Let $X_3 = X_4 - X_5$ and $X_6 = -X_2$.

Minimize $500X_1 - 100X_6 + 200(X_4 - X_5)$

Subject to

$$15X_1 - 20X_6 + 60(X_4 - X_5) \geq 1200$$
$$20X_1 - 50X_6 + 100(X_4 - X_5) \leq 1400$$
$$X_1, X_4, X_5, X_6 \geq 0$$

Convert objective function to maximization

Convert $\geq$ constraint to $\leq$ constraint

Maximize $-500X_1 + 100X_6 - 200X_4 + 200X_5$

Subject to

$$-15X_1 + 20X_6 - 60X_4 + 60X_5 \leq -1200$$
$$20X_1 - 50X_6 + 100X_4 - 100X_5 \leq 1400$$
$$X_1, X_4, X_5, X_6 \geq 0$$