8.3 Optimization Method - Data Envelopment Analysis

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Summary	• Optimization Method - Data Envelopment Analysis
	## Optimization method Data Envelopment Analysis
• Full form of DEA. • DEA is used for? • In DEA, an EU is called as?	## Data Envelopment Analysis (DEA) • A non-parametric mathematical method to find the production frontier. • Can be used to calculate the productive efficiency of an economic unit. • In the DEA terminology, an economic unit is referred to as a "Decision Making Unit (DMU)". • What DEA measures is actually a relative efficiency – Calculate individual efficiencies for each DMU in a set of DMU's. • Formulates an optimization problem for each DMU.

• We solve optimization problem to find ?
• What are the constraints we apply on weights?
	## DEA logic
	• For multiple inputs and multiple outputs, define the weighted ratio.
	• Since various inputs can't be directly added, define weights for each input. Similar approach for output.
	• But we don't know the value of each weight?
	• Solution: Let each DMU choose the input and the output weights to its advantage.
	• Objective for each DMU: maximize its efficiency by choosing its weights carefully.
	• Constraints: Choose the weights such that using these weights, they shouldn't get an efficiency more than 1!
	
	• You invested 1L rupees in your company and you hired 10 number of workers. You can't directly add 1L and 10. That's why you define weights.
	• **We solve optimization problem to find optimal weights.**
	• We solve for each DMU independent of the others in a way that the efficiency for a given DMU is maximized.
	• So, each DMU will have different weights.
	• Constraint: Choose weights as such the efficiency does not get more than one.
	• Also, let's say that sales office 1 has solved the optimization problem and obtained weights s.t. the efficiency is within the constraint. Now, using the weights of sales office 1, the other sales offices should not report an efficiency more than 1.
	• **That is, your weights should**
	  ○ **not make your efficiency more than 1, and**
	  ○ **not make other's efficiencies more than 1 when they use your weights.**
	• Given weights, none of the DMU's should get an efficiency more than 1.

• Given weights, how do you decide efficient and inefficient DMUs?	## DEA logic • Using a DMU's weights, if it can't achieve an efficiency of 1, then it is truly inefficient.

- Using a DMU's weight, if an other DMU gets an efficiency of 1, then that other DMU is really good!
- Note that we are referring to only relative efficiency.

- Variables in DEA:
  - $K = ?$
  - $N = ?$
  - $M = ?$
  - $I_{ik} = ?$
  - $O_{jk} = ?$
  - $x_{ik} = ?$
  - $y_{jk} = ?$
  - $E_k = ?$

DEA – Mathematical formulation

- K = # of DMU's considered in the dataset.
- N = # of inputs considered for the DMU's
- M = # of outputs considered for the DMU's.
- I_{ik} = Recorded value of input i for the DMU k. ($i = 1, 2, \ldots N$, $k = 1, 2, \ldots, K$).
- O_{jk} = Recorded value of output j for the DMU k. ($j = 1, 2, \ldots M$, $k = 1, 2, \ldots, K$).
- x_{ik} = Weight assigned to input i by the DMU k. ($i = 1, 2, \ldots N$, $k = 1, 2, \ldots, K$).
- y_{jk} = Weight assigned to output j by the DMU k. ($j = 1, 2, \ldots M$, $k = 1, 2, \ldots, K$).
- E_k = Efficiency of the DMU k. ($k = 1, 2, \ldots, K$)

Example:

Inputs

Sales office	Budget (INR)	Team size
1	3,00,000	13
2	2,56,000	9
3	5,00,000	7
4	3,90,000	10
5	1,85,000	14

- Input #2 for sales office 3, $I_{23} = 7$
- Input #1 for sales office 4, $I_{14} = 3,90,000$

Outputs

Sales office	Sales (INR)	No of leads
1	11,10,000	15
2	17,50,000	10
3	34,50,000	12
4	12,24,000	23
5	24,00,000	20

- Output #1 for sales office 5, $O_{15} = 24,00,000$
- Output #2 for sales office 1, $O_{21} = 15$

Corresponding weights will be:
- x_{23}
- x_{14}

- y_{15}
- y_{21}

- Efficiency $= ?$
- For a particular DMU k,
  - $E_k = ?$

DEA – Mathematical formulation

- Efficiency is defined as a ratio of weighted outputs to the weighted inputs.

$$Efficiency = \frac{Weighted\ Output}{Weighted\ Input}$$

- Each DMU defines its own efficiency using the weights they want to assign to their inputs and outputs.
- For a particular DMU k, the efficiency is:

$$E_k = \frac{y_{1k}O_{1k}+y_{2k}O_{2k}+y_{3k}O_{3k}\ldots+y_{Mk}O_{Mk}}{x_{1k}I_{1k}+x_{2k}I_{2k}+x_{3k}I_{3k}\ldots+x_{Nk}I_{Nk}}$$

- Each DMU tries to maximize ?
- The optimization problem for each DMU k is ?
- Decision variables are ?

DEA – Optimization problem

- Each DMU tries to maximize their own efficiency by adjusting the weights assigned to the inputs and the outputs.
- Only constraint on this: using these weights, none of the DMU's should get an efficiency more than 1!
- For each DMU k, the optimization problem is:

$$Max\ E_k$$

$$subject\ to\ \ E_k \leq 1, \qquad k = 1, 2, \ldots K$$

$$Decision\ variables\colon\ x_{ik}, y_{jk} \geq 0, \forall i, \forall j.$$

• Issues with DEA Optimization problem given above.	**DEA – Optimization problem** Complexities • Remember that the efficiency was defined as a ratio of weighted inputs to weighted outputs. • Weights are the decision variables. • Hence the objective function and the constraints of this optimization problem are ratios of decision variables. • That is, they are NON LINEAR! • Is there a way to linearize the problem? • A linear optimization problem is much easier to solve, of course.
• How do you linearize DEA optimization problem?	**DEA – Optimization problem** • To linearize, we maximize the numerator of the efficiency equation for the DMU k. • And normalize the denominator to 1. • For the constraints, we rearrange the efficiency terms to make it linear. • So the revised formulation is: $Efficiency = \frac{Weighted\ output}{Weighted\ input} \leq 1$ To linearize: $Weighted\ output \leq Weighted\ input$
• Write DEA optimization problem after linearization.	**DEA – Optimization problem** • For a DMU k, $$Max\ y_{1k}O_{1k} + y_{2k}O_{2k} + y_{3k}O_{3k} \ldots + y_{Mk}O_{Mk}$$ $$subject\ to$$ $$x_{1k}I_{1k} + x_{2k}I_{2k} + x_{3k}I_{3k} \ldots + x_{Nk}I_{Nk} = 1$$ $$y_{1k}O_{11} + y_{2k}O_{21} \ldots + y_{Mk}O_{M1} \leq x_{1k}I_{11} + x_{2k}I_{21} \ldots + x_{Nk}I_{N1}$$ $$y_{1k}O_{12} + y_{2k}O_{22} \ldots + y_{Mk}O_{M2} \leq x_{1k}I_{12} + x_{2k}I_{22} \ldots + x_{Nk}I_{N2}$$ $$\vdots$$ $$y_{1k}O_{1k} + y_{2k}O_{2K} \ldots + y_{Mk}O_{MK} \leq x_{1k}I_{1K} + x_{2k}I_{2K} \ldots + x_{Nk}I_{NK}$$ $$Decision\ variables\text{:}\ x_{ik}, y_{jk} \geq 0, \forall i, \forall j.$$ • We want weights that keep efficiencies of all DMU ≤ 1 • Output weights for DMU k : $y_{1k}, y_{2k}, \ldots, y_{Mk}$ • Input weights for DMU k : $x_{1k}, x_{2k}, \ldots, x_{Nk}$ Objective: $Max\ Weighted\ Output\ of\ DMU\ k$ subject to $Weighted\ Input\ of\ DMU\ k = 1,$ and **using the weights of DMU k**: $DMU1\ weighted\ output \leq DMU1\ weighted\ input$ $DMU2\ weighted\ output \leq DMU2\ weighted\ input$ $\vdots$ $DMUK\ weighted\ output \leq DMUK\ weighted\ input$ Decision variables: $x_{ik}, y_{jk} \geq 0, \quad \forall i, \forall j$