5.7 Operations Research: Linear Programming and Duality in Spreadsheets and Python

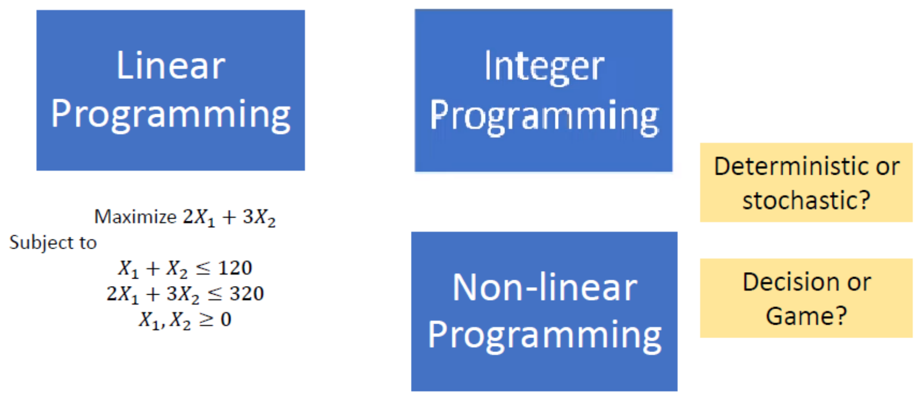
Monday, 24 October 2022 17:07

Summary

- Linear Programming

- What is Linear Programming (LP)?
- What is Integer Programming?
- What is Non-linear Programming?
- What assumptions about the variables do we make in LP?

Domains



- In **Linear programming,** you have **linear objective function and a linear set of constraints.**
  - Here, the **variables can take continuous values.**
- In **Integer programming, variables can take only integer values.**
  - Integer programming problems are **much more difficult to solve than linear programming problems.**
  - If you're solving a linear programming problem, and you obtain integer solutions, then that is also going to be integer programming optimal solution.
    - However, if you're obtaining continuous values, then you need to be careful about rounding off. Because rounded off values may not generate an optimal integer programming solution.
- In **non-linear programming, the objective function or the constraints are no longer linear.**

- An assumption in linear programming is that the parameters or variables are **deterministic**.
- It can sometimes happen that a level of uncertainty kicks in, in that case, you require stochastic programming to solve such problems.

- What we're looking here is a particular firm's problem, for a single person's problem.
- However, if there exists a scenario where the firm is trying to maximize its profit, given that a competitor firm has decided to produce a certain number of products. In that case, it's no longer a decision, and it becomes a game-theoretic problem.
- Game theory is a subfield of economics, wherein different agents interact, and each agent tries to maximize his/her profit in relation to other agents.

- Here, we're gonna deal with a linear programming problem, deterministic parameters and variables, and we're going to focus on decision.

Problem Statement:

Linear Programming

Imagine you running an automobile firm which sells cars in three different segments – Hatchback, Sedan and SUV at prices ₹5,00,000, ₹10,00,000 and ₹25,00,000 respectively.

Suppose that the manufacturing of cars primarily requires the following raw materials A and B. The firm has 1,20,000 units of resource A and 1,40,000 units of resource B available. The resource requirements for the manufacturing of each car variant is given below.

Requirements	Resource A	Resource B
Hatchback	15	20
Sedan	20	50
SUV	60	100

How many cars of each type should be produced to maximize revenue?

Decision variables

X_1 - Number of Hatchback cars to be produced

X_2 - Number of Sedan cars to be produced

X_3 - Number of SUV cars to be produced

Objective function

Maximize $500000X_1 + 1000000X_2 + 2500000X_3$

Constraints

Requirements	Resource A	Resource B
Hatchback	15	20
Sedan	20	50
SUV	60	100

- Resource A constraint
$$15X_1 + 20X_2 + 60X_3 \leq 120000$$
- Resource B constraint
$$20X_1 + 50X_2 + 100X_3 \leq 140000$$
- Non-negativity restrictions
$$X_1, X_2, X_3 \geq 0$$

Linear program

Maximize $500000X_1 + 1000000X_2 + 2500000X_3$

Subject to

$$15X_1 + 20X_2 + 60X_3 \leq 120000$$
$$20X_1 + 50X_2 + 100X_3 \leq 140000$$
$$X_1, X_2, X_3 \geq 0$$

Dual of the linear program

Automobile firm

- Possesses resources A and B
- Manufactures and sells cars
- Aim: Maximize revenue

Dual variables: Shadow price or Marginal price of the resource at the optimum

Primal

Maximize $500000X_1 + 1000000X_2 + 2500000X_3$
Subject to
$$15X_1 + 20X_2 + 60X_3 \leq 120000$$
$$20X_1 + 50X_2 + 100X_3 \leq 140000$$
$$X_1, X_2, X_3 \geq 0$$

Dual

Minimize $120000Y_1 + 140000Y_2$
Subject to
$$15Y_1 + 20Y_2 \geq 500000$$
$$20Y_1 + 50Y_2 \geq 1000000$$
$$60Y_1 + 100Y_2 \geq 2500000$$
$$Y_1, Y_2 \geq 0$$

Let Y_1, Y_2 be the costs of resource A and resource B respectively

Buyer

- Purchase resources A and B
- Aim: Minimize total cost

Primal – Dual relationship

Primal

Maximize $500000X_1 + 1000000X_2 + 2500000X_3$
Subject to
$$15X_1 + 20X_2 + 60X_3 \leq 120000$$
$$20X_1 + 50X_2 + 100X_3 \leq 140000$$
$$X_1, X_2, X_3 \geq 0$$

Dual

Minimize $120000Y_1 + 140000Y_2$
Subject to
$$15Y_1 + 20Y_2 \geq 500000$$
$$20Y_1 + 50Y_2 \geq 1000000$$
$$60Y_1 + 100Y_2 \geq 2500000$$
$$Y_1, Y_2 \geq 0$$

Primal	Dual
Maximization	Minimization
Number of constraints	Number of variables
Number of variables	Number of constraints
Objective function coefficient	Right hand side in constraints
Right hand side in constraints	Objective function coefficient

How to construct a dual?

Primal

Maximize $500000X_1 + 1000000X_2 + 2500000X_3$
Subject to

$$15X_1 + 20X_2 + 60X_3 \le 120000$$
$$20X_1 + 50X_2 + 100X_3 \le 140000$$
$$X_1, X_2, X_3 \ge 0$$

Dual

Let Y_1, Y_2 be the dual variables corresponding to the two constraints

Minimize $120000Y_1 + 140000Y_2$

Subject to

$$15Y_1 + 20Y_2 \ge 500000$$
$$20Y_1 + 50Y_2 \ge 1000000$$
$$60Y_1 + 100Y_2 \ge 2500000$$
$$Y_1, Y_2 \ge 0$$

Dual of the dual is the primal!

Dual

Minimize $120000Y_1 + 140000Y_2$
Subject to

$$15Y_1 + 20Y_2 \ge 500000$$
$$20Y_1 + 50Y_2 \ge 1000000$$
$$60Y_1 + 100Y_2 \ge 2500000$$
$$Y_1, Y_2 \ge 0$$

Standard form

Maximize $-120000Y_1 - 140000Y_2$
Subject to

$$-15Y_1 - 20Y_2 \le -500000$$
$$-20Y_1 - 50Y_2 \le -1000000$$
$$-60Y_1 - 100Y_2 \le -2500000$$
$$Y_1, Y_2 \ge 0$$

Primal

Maximize $500000X_1 + 1000000X_2 + 2500000X_3$
Subject to

$$15X_1 + 20X_2 + 60X_3 \le 120000$$
$$20X_1 + 50X_2 + 100X_3 \le 140000$$
$$X_1, X_2, X_3 \ge 0$$

Finding the dual

Minimize $-500000X_1 - 1000000X_2 - 2500000X_3$
Subject to

$$-15X_1 - 20X_2 - 60X_3 \ge -120000$$
$$-20X_1 - 50X_2 - 100X_3 \ge -140000$$
$$X_1, X_2, X_3 \ge 0$$

- Solved on the next page

How to construct a dual?

Primal

Minimize $500X_1 + 100X_2 + 200X_3$

Subject to

$$15X_1 + 20X_2 + 60X_3 \ge 1200$$
$$20X_1 + 50X_2 + 100X_3 \le 1400$$
$$X_1 \ge 0, X_2 \le 0, X_3 \; unrestricted$$

Convert to standard form

Define new variables $X_4, X_5, X_6 \ge 0$. Let $X_3 = X_4 - X_5$ and $X_6 = -X_2$.

Minimize $500X_1 - 100X_6 + 200(X_4 - X_5)$
Subject to

$$15X_1 - 20X_6 + 60(X_4 - X_5) \ge 1200$$
$$20X_1 - 50X_6 + 100(X_4 - X_5) \le 1400$$
$$X_1, X_4, X_5, X_6 \ge 0$$

Dual

Minimize $-1200Y_1 + 1400Y_2$

Subject to

$$-15Y_1 + 20Y_2 \ge -500$$
$$20Y_1 - 50Y_2 \ge 100$$
$$-60Y_1 + 100Y_2 \ge -200$$
$$60Y_1 - 100Y_2 \ge 200$$
$$Y_1, Y_2 \ge 0$$

Convert objective function to maximization
Convert $\ge$ constraint to $\le$ constraint

Maximize $-500X_1 + 100X_6 - 200X_4 + 200X_5$
Subject to

$$-15X_1 + 20X_6 - 60X_4 + 60X_5 \le -1200$$
$$20X_1 - 50X_6 + 100X_4 - 100X_5 \le 1400$$
$$X_1, X_4, X_5, X_6 \ge 0$$

- We require a maximization objective.
- And, we require less than or equal to constraints.
- And, we require all the variables to be greater than or equal to 0.

- X_4 and X_5 will handle the unrestricted sign of X_3, as X_3 can take any value.
- X_6 will handle the negative sign of X_2.

- We substitute these new variables in the primal.

- Solved on next page.

<u>Dealing with an equal to constraint:</u>

$$2X_1 + X_2 = 400 \,.$$

We want to convert this constraint to a less than or equal to type.

- First, write it as two constraints

$2X_1 + X_2 \le 400$
$2X_1 + X_2 \ge 400$

- Now, convert these two into a less than or equal to constraint.

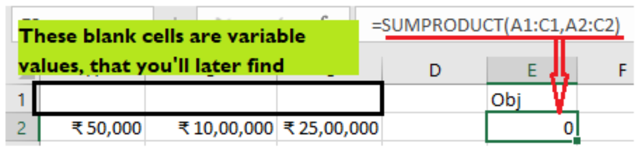
$2X_1 + X_2 \le 400$
$-2X_1 - X_2 \le -400$

- That's how you do it.

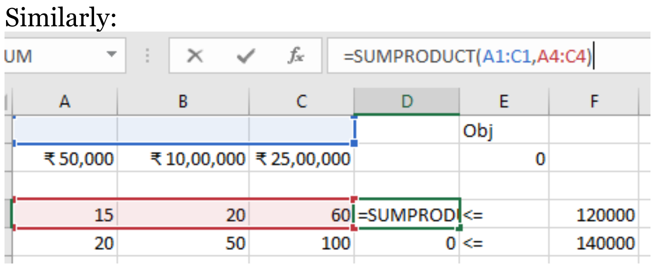
Excel Working

- To solve in Excel, you'll require the Solver add-in.
    - File >> Options >> Add-ins >> Go >> Tick Solver Add-in >> Ok

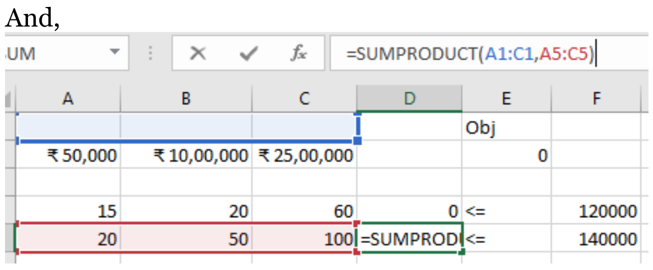
				Obj	
₹ 50,000	₹ 10,00,000	₹ 25,00,000		0	
15	20	60	0	<=	120000
20	50	100	0	<=	140000



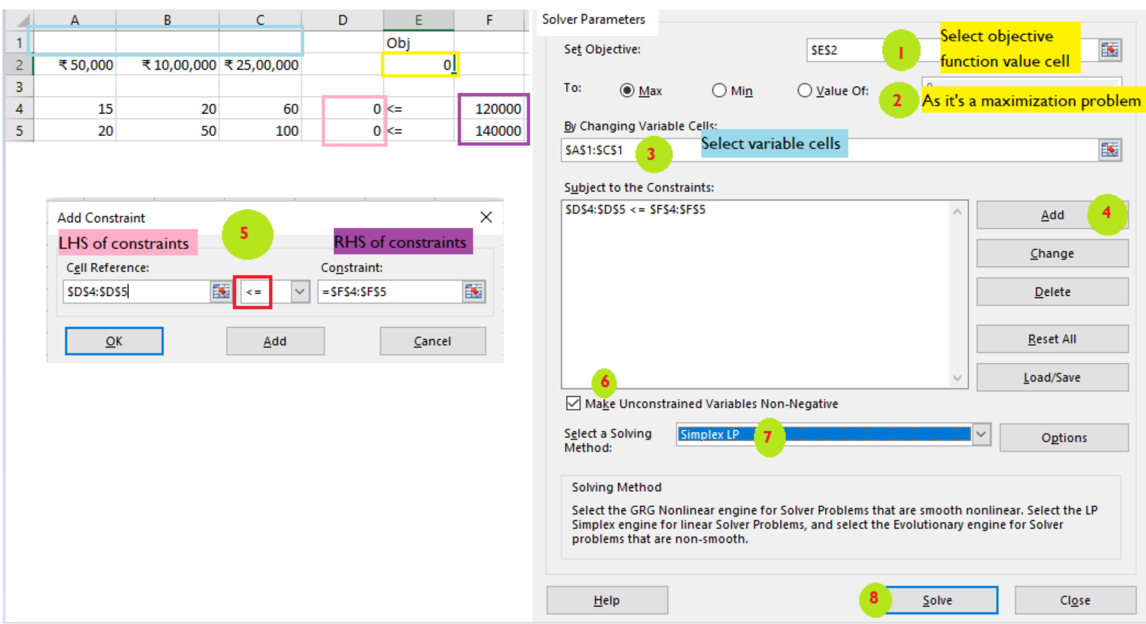
Similarly:

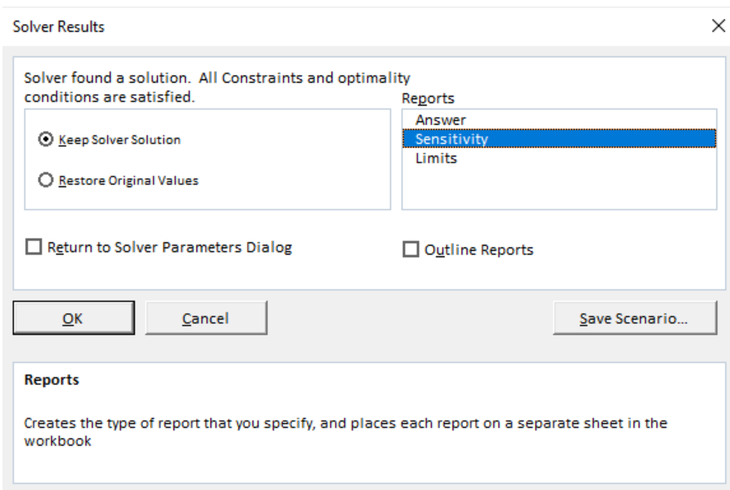


And,

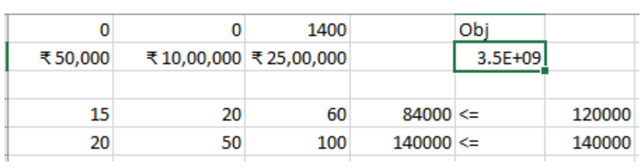


Go to Data >> Solver





This is the solution you get:



Sensitivity report on Primal problem:

Microsoft Excel 16.0 Sensitivity Report
Worksheet: [Book1]Sheet1
Report Created: Tue, 25-Oct-2022 12:58:11

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
A1		0	-450000	50000	450000	1E+30
B1		0	-250000	1000000	250000	1E+30
C1		1400	0	2500000	1E+30	500000

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
D4		84000	0	120000	1E+30	36000
D5		140000	25000	140000	60000	140000

- Allowable increase and decrease tells us in what range the solution will hold.
- Dual variable are also called the Shadow price.

- So, the sensitivity report directly gives you the value of dual the variables (in the final value column of constraints table).

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
D4	A	84000	0	120000	1E+30	36000
D5	B	140000	25000	140000	60000	140000

Resource B is exhausted

Whereas only 84,00 units of resource A are used
i.e., we have 1,20,000-84,000 = 36,000 units of resource A left

- Resource A is not that much valuable to us.
- If we have a tight constraint, meaning LHS = RHS, that resource is that much valuable.
- If it's not a tight constraint, that resource may not be of that much value to us.
- That's what we observe in the Shadow Price column.
- Let's verify this for solving for dual.

Dual problem

			Obj	
1,20,000	1,40,000		0	
15	20	0	>=	5,00,000
20	50	0	>=	10,00,000
60	100	0	>=	25,00,000

Follow the steps mentioned above and the solution you get:

0	25,000		Obj	
1,20,000	1,40,000		3.5E+09	
15	20	5,00,000	>=	5,00,000
20	50	12,50,000	>=	10,00,000
60	100	25,00,000	>=	25,00,000

And, the sensitivity report:

Microsoft Excel 16.0 Sensitivity Report
Worksheet: [Primal and Dual.xlsx]Dual
Report Created: Tue, 25-Oct-2022 13:28:58

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
A1		0	15000	120000	1E+30	15000
B1		25000	0	140000	20000	140000
C1		0	0	0	1E+30	0

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
C4		500000	7000	500000	1E+30	5.45697E-11
C5		1250000	0	1000000	250000	1E+30
C6		2500000	0	2500000	2.72848E-10	1E+30

- You can observe that the objective function value of the primal and the dual are the same.