WEEK 3: REVISION | FINAL EXAM

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1. FOUR FUNDAMENTAL SUBSPACES

Suppose **A** is a $m \times n$ matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

linearly independent
columns
rows

1. The column space is C(**A**), a subspace of $\mathbb{R}^m$.
2. The row space is C($\mathbf{A}^T$), a subspace of $\mathbb{R}^n$.
3. The nullspace is N(**A**), a subspace of $\mathbb{R}^n$.
4. The left nullspace is N($\mathbf{A}^T$), a subspace of $\mathbb{R}^m$.

$Ax = 0$

$A^T y = 0$

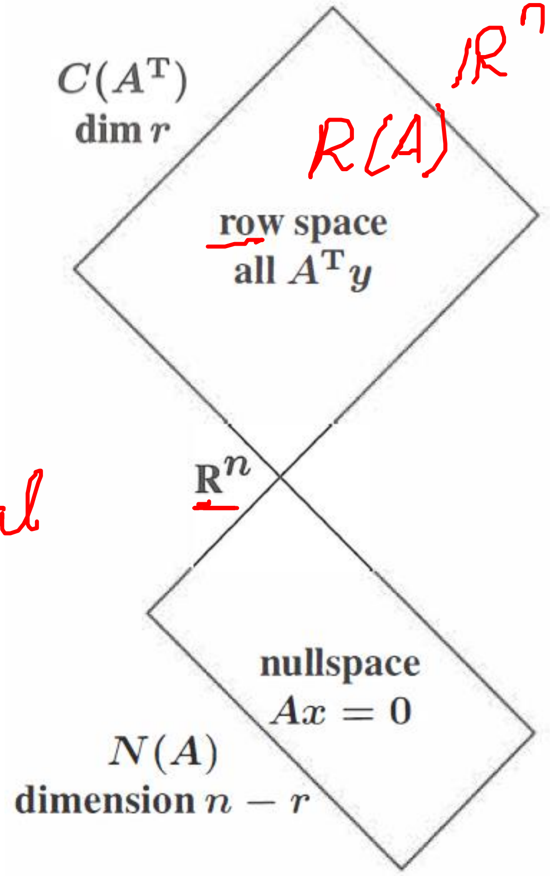
2. ORTHOGONAL VECTORS AND SUBSPACES

$\perp$

Two real vectors x and y are orthogonal if

$$x^T y = 0 \text{ or } y^T x = 0$$

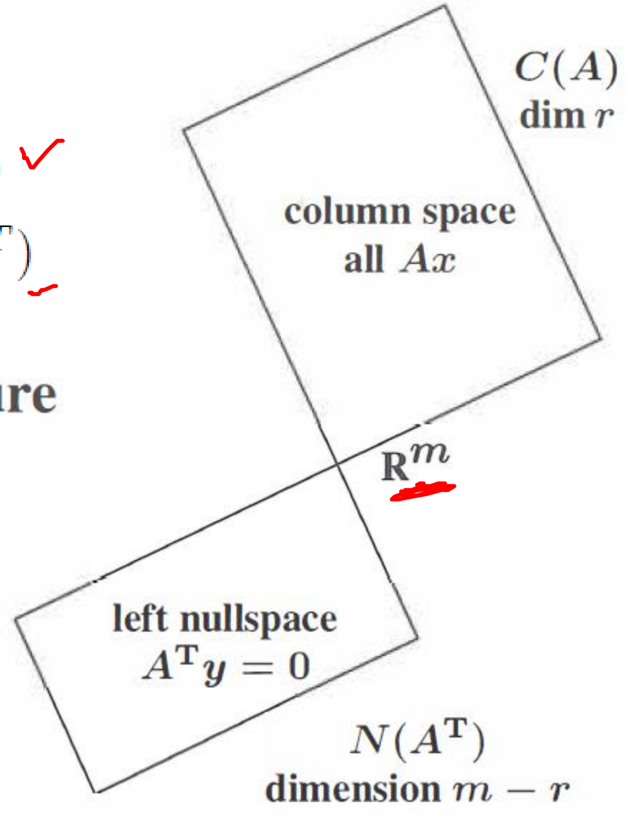
$$x.y = 0 \text{ or } y.x = 0$$



$R(A) \perp N(A)$ ✓

$C(A) \perp N(A^T)$

The big picture



orthogonal complement

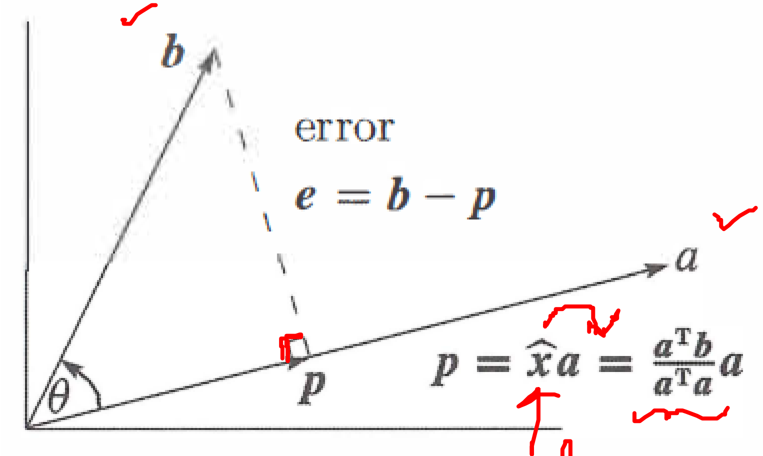
Rank → number of independent columns → rows

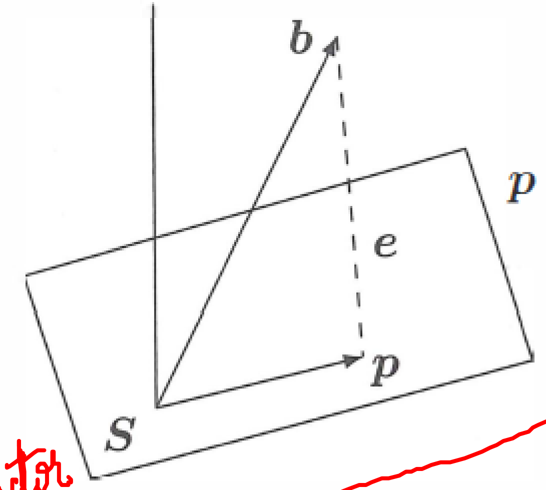
rank(A) + nullity(A) = n

$\dim(C(A^T)) + \dim(N(A^T)) = m$

3. PROJECTIONS

The projection (p) of b onto a line and onto S (column space of A).





$$p = A\hat{x}$$
$$= A(A^TA)^{-1}A^Tb$$
$$= Pb$$

matrix

$A\frac{a^Tb}{A^TA}$

$\frac{aa^T}{a^Ta}$

$a\hat{x} = \left(a \cdot \frac{a^Tb}{a^Ta}\right)$

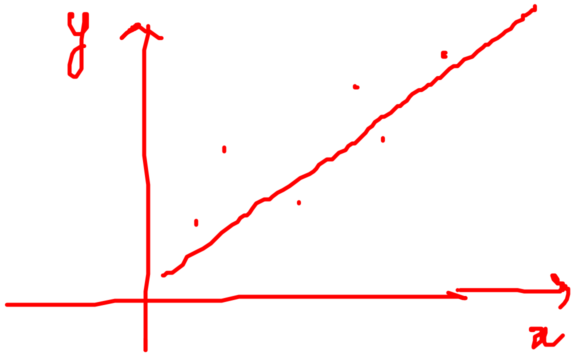
vector

Projection matrix of vector a, $\boxed{\mathbb{P} = \frac{aa^T}{a^Ta}}$

$$\boxed{\begin{array}{l} P^2 = P \\ P \text{ is symmetric} \end{array}}$$

4. LEAST SQUARES APPROXIMATIONS

$y = mx + c$



- It often happens that $Ax = \boldsymbol{b}$ has no solution.
- The usual reason is: *too many equations*.
  - The matrix A has more rows than columns.
  - There are more equations than unknowns (m is greater than n).
  - Then columns span a small part of m-dimensional space.

$\hat{\theta}'$ $\hat{\theta}''$

- We cannot always get the error $e = b - Ax$ down to zero.
- Least Squares method: Solve $A^T A\hat{x} = A^T b$ to get $\hat{x}$.

slope
intercept

$$\boxed{\widehat{x} \text{ is the "least-squares solution": } \|\boldsymbol{b} - A\widehat{x}\|^2 = \text{minimum.}}$$

SOME SOLVED PROBLEMS

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Find the fundamental spaces of A= $\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix}$

pivot free

$$\begin{bmatrix} 1 & 4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 4 & 8 & -4 \\ 2 & 4 & -2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Column space is C(**A**)

$$span \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Row space is C($\mathbf{A}^T$)

$$span \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$

$C(A^T)$

RREF ⇒ unique basis

Nullspace is N(**A**)

$$span \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Left nullspace is N($\mathbf{A}^T$)

$$span \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Rank = 1

SOLUTION TO $Ax = b$

$$A = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix}$$

Find the condition on (b_1, b_2, b_3) for $Ax = b$ to be solvable.

$$[\, A \;\; \boldsymbol{b} \,] = \left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & \boldsymbol{b}_1 \\ 0 & 0 & 1 & 4 & \boldsymbol{b}_2 \\ 1 & 3 & 1 & 6 & \boldsymbol{b}_3 \end{array}\right] \longrightarrow \begin{bmatrix} 1 & 3 & 0 & 2 & \boldsymbol{b}_1 \\ 0 & 0 & 1 & 4 & \boldsymbol{b}_2 \\ 0 & 0 & 0 & 0 & \underbrace{\boldsymbol{b}_3 - \boldsymbol{b}_1 - \boldsymbol{b}_2} \end{bmatrix}$$

Now we get $0 = 0$ in the third equation only if $b_3 - b_1 - b_2 = 0$.

$\boxed{b_1 + b_2 = b_3}$ This condition puts $\boldsymbol{b}$ in the column space of A.

How to find the projection matrix for $a = \begin{bmatrix} -1 \\ 3 \\ -2 \\ 1 \end{bmatrix}$?

$$\mathbb{P} = \frac{aa^T}{a^T a}$$

$$aa^T = \begin{bmatrix} -1 & 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 & -1 \\ -3 & 9 & -6 & 3 \\ 2 & -6 & 4 & -2 \\ -1 & 3 & -2 & 1 \end{bmatrix}$$

$$a^T a = \begin{bmatrix} -1 \\ 3 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -2 & 1 \end{bmatrix} = 15$$

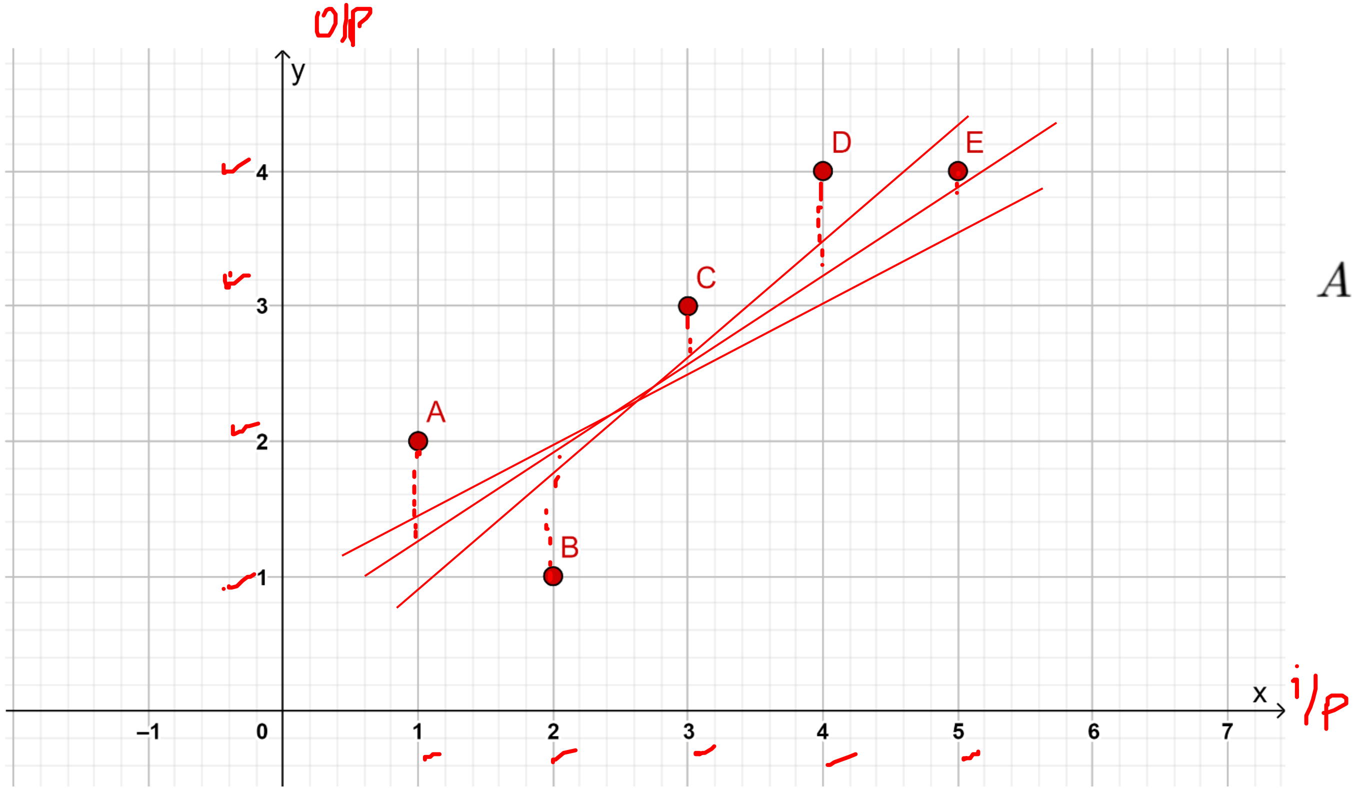
$$\mathbb{P} = \frac{1}{15} \begin{bmatrix} 1 & -3 & 2 & -1 \\ -3 & 9 & -6 & 3 \\ 2 & -6 & 4 & -2 \\ -1 & 3 & -2 & 1 \end{bmatrix}$$

Projection of $b = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ onto a:

$$p = \mathbb{P} * b$$

$$p = \frac{1}{15} \begin{bmatrix} 1 & -3 & 2 & -1 \\ -3 & 9 & -6 & 3 \\ 2 & -6 & 4 & -2 \\ -1 & 3 & -2 & 1 \end{bmatrix} * \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{15} \begin{bmatrix} 2 \\ -6 \\ 4 \\ -2 \end{bmatrix}$$

LEAST SQUARES METHOD



$y = mx + c$

Solve: $A^T A\hat{x} = A^T \mathbf{b}$

$\hat{x} = \begin{bmatrix} \hat{\theta}' \\ \hat{\theta}'' \end{bmatrix}$ → slope, → intercept

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \\ 4 \end{bmatrix}$$

x y

$$\underbrace{\begin{bmatrix} 55 & 15 \\ 15 & 5 \end{bmatrix}}_{A^TA} \begin{bmatrix} \hat{\theta}' \\ \hat{\theta}'' \end{bmatrix} = \underbrace{\begin{bmatrix} 49 \\ 14 \end{bmatrix}}_{A^Tb}$$

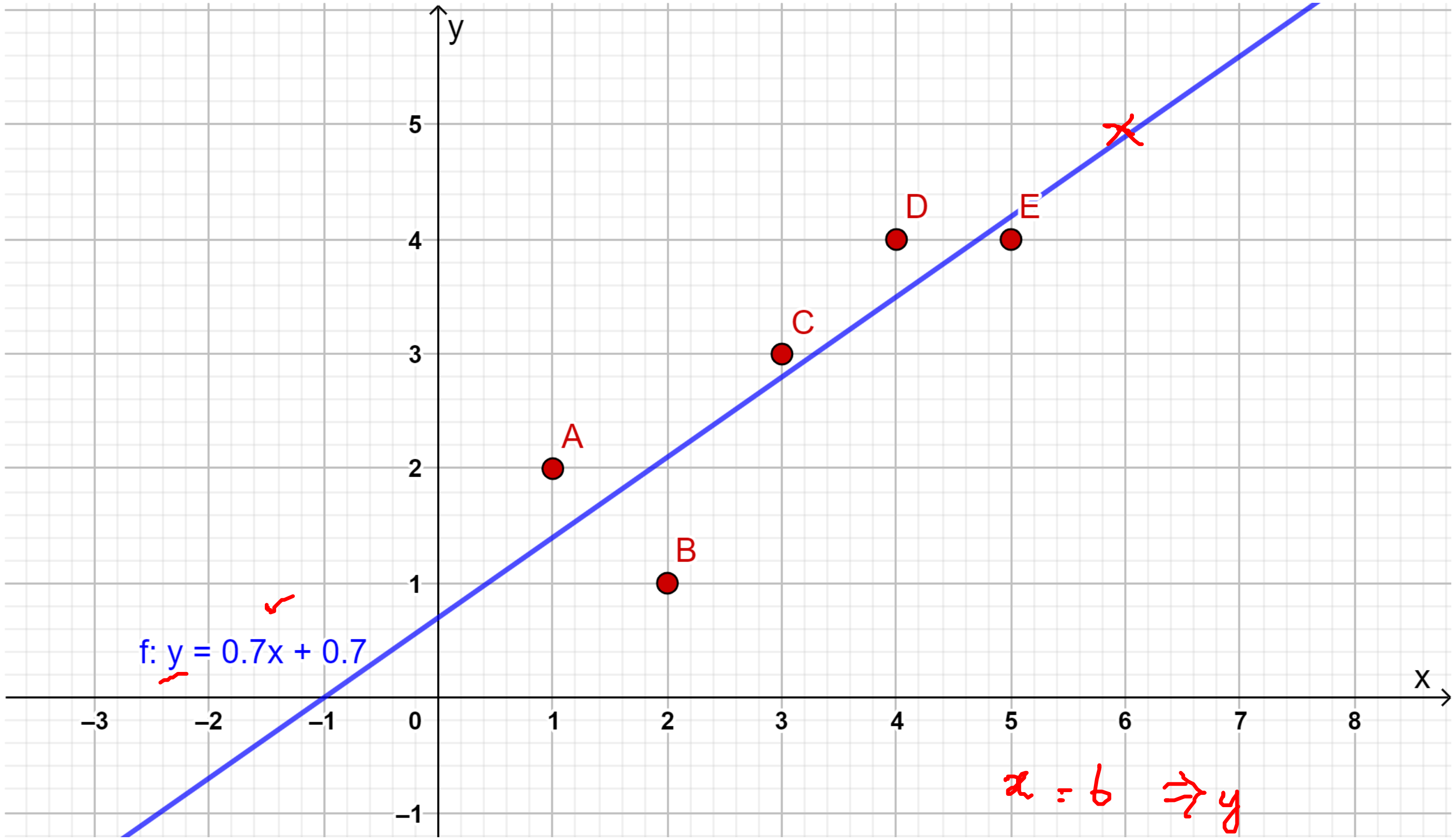
Solving this we get, $\hat{x} = \begin{bmatrix} 0.7 \\ 0.7 \end{bmatrix}$

Best fit line: $y = 0.7x + 0.7$

Distance of points from line = *residuals*

***Least squares regression line*: Minimizes sum of square residuals**

Best fit line: $y = 0.7x + 0.7$



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