Week 2

ANALYSIS OF ALGORITHMS

Measuring performance:

- Example of validating SIM cards against Aadhaar data.
    - Naive approach takes thousands of years
    - Smarter solution takes a few minutes.
- Two main resources of interest
    - Running time - how long the algorithm takes
    - Space - memory management
- Time depends on processing power
    - Impossible to change for given hardware
    - Enhancing hardware has only a limited impact at a practical level
- Storage is limited by available memory
    - Easier to configure, augment
- Typically, we focus on time rather than space.

Analysis of time dependence:

1. Input Size:
    - Running time depends on input size.
        - Larger arrays will take longer to sort.
    - Measure time efficiency as function of input size
        - Input size *n*
        - Running time `t(n)`
    - Different input of size *n* may take different amounts of time

Example 1: SIM cards v/s Aadhaar Cards

- $n \approx 10^9$ - number of cards
- Naive algorithm: `t(n)` $\approx n^2$
- Clever algorithm: `t(n)` $\approx n\log_2 n$
    - $\log_2 n$- number of times you need to divide *n* by 2 to reach 1

- $\log_2(n) = k \Rightarrow n = 2^k$

Example 2: Video game

- Several objects on screen
- Basic step: find closest pair of objects.
- n objects - naive algorithm is n^2
  - For each pair of objects, compute their distance
  - Report minimum distance across all pairs.
- There is a clever algorithm that takes time $n\log_2 n$
- High resolution gaming console may have 4000x2000 pixels.
  - 8 x 10^6 points - 8 million
- Suppose we have 100,000 = 1 x 10^5 objects.
- Naive algorithm takes 10^{10} steps
  - 1000 seconds, or 16.7 minutes in Python.
  - Unacceptable response time!
- $\log_2 100{,}000$ is under 20, so $n\log_2 n$ takes a fraction of a second.

Orders of magnitude

- When comparing `t(n)`, focus on orders of magnitude
  - Ignore constant factors
- `f(n)` = n^3 eventually grows faster than `g(n)` = 5000n^2
  - For small values of n, `f(n)` < `g(n)`
  - After n = 5000, `f(n)` overtakes `g(n)`
- Asymptotic complexity
  - What happens in the limit, as n becomes large
- Typical growth functions
  - Is `t(n)` proportional to $\log n$, ..., n^2, n^3, ..., 2^n?
    - Note: $\log n$ means $\log_2 n$ by default
  - Logarithmic, polynomial, exponential...

The red colour line is called the "**feasibility line**"

Orders of magnitude

Input size	Values of $t(n)$						
	$\log n$	n	$n \log n$	n^2	n^3	2^n	$n!$
10	3.3	10	33	100	1000	1000	10^6
100	6.6	100	66	10^4	10^6	10^{30}	10^{157}
1000	10	1000	10^4	10^6	10^9		
10^4	13	10^4	10^5	10^8	10^{12}		
10^5	17	10^5	10^6	10^{10}			
10^6	20	10^6	10^7	10^{12}			
10^7	23	10^7	10^8				
10^8	27	10^8	10^9				
10^9	30	10^9	10^{10}				
10^{10}	33	10^{10}	10^{11}				

Measuring running time

- Analysis should be independent of the underlying hardware
  - Don't use actual time
  - Measure in terms of basic operations
- Typical basic operations
  - Compare two values
  - Assign a value to a variable
- Exchange a pair of values?

  `(x,y) = (y,x)`

  ```
  t = x
  x = y
  y = t
  ```

  - If we ignore constants, focus on orders of magnitude, both are within a factor of 3.
  - Need not be very precise about defining basic operations.

What is the input size?

- Typically a natural parameter

    - Size of a list/array that we want to search or sort
    - Number of objects we want to rearrange
    - Number of vertices and number of edges in a graph
        - Separate parameters
- Numeric problems? Is *n* a prime?
    - Magnitude of *n* is not the correct measure.
    - Arithmetic operations are performed digit by digit
        - Addition with carry, subtraction with borrow, multiplication, long division ...
    - Number of digits is a natural measure of input size
        - Same as $\log_b n$, when we write *n* in base *b*

Which inputs should we consider?

- Performance varies across input instances
    - By luck, the value we are searching for is the first element we examine in an array
- Ideally, want the "average" behavior
    - Difficult to compute
    - Average over what? Are all inputs equally likely?
    - Need a probability distribution over inputs
- Instead, worst case input
    - Input that forces algorithm to take longest possible time
        - Search for a value that is not present in an unsorted list
        - Must scan all elements
    - Pessimistic - worst case may be rare
    - Upper bound for worst case guarantees good performance

Summary:

- Two important parameters when measuring algorithm performance
    - Running time, memory requirement (space)
    - Mainly focus on time
- Running time `t(n)` is a function of input size *n*
    - Interested in orders of magnitude
    - Asymptotic complexity, as *n* becomes large
- From running time, we can estimate feasible input sizes
- We focus on worst case inputs

- Pessimistic, but easier to calculate than average case
- Upper bound on worst case gives us an overall guarantee on performance

COMPARING ORDERS OF MAGNITUDE

Upper Bounds

- `f(x)` is said to be `O(g(x))` if we can find constants *c* and x_0 such that `c.g(x)` is an upper bound for `f(x)` for *x* beyond x_0
- `f(x)` $\leq$ `cg(x)` for every $x \geq x_0$

Examples:

1. $100n + 5$ is $O(n^2)$

- $100n + 5 \leq 100n + n = 101n$, for $n \geq 5$
- $101n \leq 101n^2$
- Choose $n_0 = 5$, c = 101

  $\Rightarrow \forall\ n \geq n_0$ (i.e 5) $100n \leq 101n^2$

  Alternatively,
- $100n + 5 \leq 100n + 5n = 105n$ for $n \geq 1$
- $105n \leq 105n^2$
- Choose $n_0 = 1$, c = 105

  $\Rightarrow \forall\ n \geq n_0$ (i.e 1) $100n \leq 105g(x)$

  Choice of n_0, c not unique.

2. $100n^2 + 20n + 5$ is $O(n^2)$

- $100n^2 + 20n + 5 \leq 100n^2 + 20n^2 + 5n^2$, for $n \geq 1$
- $100n^2 + 20n + 5 \leq 125n^2$, for $n \geq 1$
- Choose $n_0 = 1$, $c = 125$
- What matters is the highest term
  - $20n + 5$ is dominated by $100n^2$
- n^3 is not $O(n^2)$
  - No matter what c we choose, cn^2 will be dominated by n^3 for $n \geq c$

Useful properties:

- If $f_1(n)$ is $O(g_1(n))$ and $f_2(n)$ is $O(g_2(n))$, then $f_1(n) + f_2(n)$ is $O(\max(g_1(n), g_2(n)))$
- Proof
  - $f_1(n) \leq c_1 g_1(n)$ for $n > n_1$
  - $f_2(n) \leq c_2 g_2(n)$ for $n > n_2$
  - Let $c_3 = \max(c_1, c_2) = \max(n_1, n_2)$
  - For $n \geq n_3$, $f_1(n) + f_2(n) \leq c_1 g_1(n) + c_2 g_2(n) \leq c_3(g_1(n) + g_2(n)) \leq 2c_3(\max(g_1(n), g_2(n)))$
- Algorithm has two phases:
  - Phase A takes time $O(g_A(n))$
  - Phase B takes time $O(g_B(n))$
- Algorithm as a whole takes time $\max(O(g_A(n), g_B(n)))$

- Least efficient phase is the upper bound for the whole algorithm

Lower Bounds

- f(x) is said to be Ω(g(x)) if we can find constants *c* and x_0 such that *c*g(x) is a lower bound for f(x) for *x* beyond x_0
    - f(x) $\geq$ cg(x) for every x $\geq$ x_0
- n^3 is $\Omega(n^2)$
    - n^3 > n^2 for all *n*, so n_0 = 1, *c* = 1
- Typically we establish lower bounds for a problem rather than an individual algorithm
    - If we sort a list by comparing elements and swapping them, we require Ω(*n*logn) comparisons.
    - This is **independent** of the algorithm we use for sorting.

Tight bounds

- F(x) is said to be Θ(g(x))if it is both O(g(x)) and Ω(g(x))
    - Find constants c_1, c_2, x_0 such that c_1g(x) $\leq$ f(x) $\leq$ c_2g(x) for every *x* $\geq$ x_0
- n(n - 1)/2 is $\Theta(n^2)$
    - Upper bound:
        - n(n-1)/2 =n^2/2 - n/2 $\leq$ n^2/2 for all n $\geq$ 0; c_1 = $1/2$
    - Lower bound:
        - n(n-1)/2 = n^2/2 - n/2 $\geq$ n^2/2 - (n/2 x n/2) $\geq$ n^2/4 for all n $\geq$ 2; c_2 = $1/4$
    - Choose n_0 = 2, c_1 = 1/4, c_2 = 1/2

Summary

- `f(n)` is `O(g(n))` means `g(n)` is an upper bound for `f(n)`
    - Useful to describe asymptotic worst case running time
- `f(n)` is Ω(g(n)) means `g(n)` is a lower bound for `f(n)`
    - Typically used for a problem as a whole, rather than an individual algorithm
- `f(n)` is Θ(g(n)): matching upper and lower bounds
    - We have found an optimal algorithm for a problem

CALCULATING COMPEXITY - EXAMPLES

Calculating complexity

- Iterative programs
- Recursive programs

Iterative programs

EXAMPLE 1:

Find the maximum element in a list

- Input size is legth of the list
- Single loop scans all elements
- Always takes *n* steps
- Overall time is O(n)

```
def maxElement(L):
  maxval = L[0]
  for i in range(len(L)):
    if L[i] > maxval:
      maxval = L[i]
    return maxval
```

EXAMPLE 2:

Check whether a list contains duplicates

- Input size is length of the list
- Nested loop scans all pairs of elements
- A duplicate may be found in the very first iteration
- Worst case - no duplicates, both loops ran fully
- Time is: `(n - 1) + (n - 2) + ... + 1` = `n(n - 1)/2`
- Overall time is O(n^2)

```
def noDuplicates(L):
  for i in range(len(L)):
    for j in range(i + 1,len(L)):
      if L[i] == L[j]:
        return False
  return True
```

EXAMPLE 3:

Matrix multiplication

- Matrix is represented as list of lists
  - $$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$
  - `[[1,2,3],[4,5,6]]`
- Input matrices have size *m* x *n*, *n* x *p*

- Output matrix is *m* x *p*
- Three nested loops
- Overall time is O(*mnp*) - O(n^3) if both are *n* x *n*

```
def matrixMultiply(A,B):
  (m,n,p) = (len(A), len(B), len(B[0]))
  c = [[0 for i in range(p) ]
          for j in range(m) ]
  for i in range(m):
    for j in range(p):
      for k in range(n):
        c[i][j] = c[i][j] + A[i][k] * B[k][j]
  return c
```

EXAMPLE 4

Number of bits in binary representation of *n*

- log *n* steps for *n* to reach 1
- For number theoretic problems, input size is number of digits
- This algorithm is linear in input size

```
def numberOfBits(n):
  count = 1
  while n > 1:
    count = count + 1
    n = n // 2
  return count
```

EXAMPLE 5

Towers of Hanoi

- Three pegs A, B, C
- Move *n* disks from A to B, use C as transit peg
- Never put a larger disk on a snaller one

  Recursive Solution:
- Move *n* - 1 disks from A to C, use B as transit peg
- Move larger disk from A to B
- Move *n* - 1 disks from C to B, use A as transit peg

  Recurrence:
- `M(n)` - number of moves to transfer *n* disks
- `M(1)` = 1
- `M(n)` = `M(n - 1) + 1 + M(n - 1)` = `2M(n - 1) + 1`

  Unwind and solve:
- M(n) = 2M(n - 1) + 1
  = 2(2M(n - 2) + 1) + 1 = 2^2M(n - 2) + (2 + 1)
  = 2^2(2M(n - 3) + 1) + (2 + 1) = 2^3M(n - 3) + (4 + 2 + 1)

...
= 2^kM(n - k) + (2^k - 1))
...
= 2^{n-1}M(1) + (2^{n-1} - 1)
= 2^{n-1} + 2^{n-1} - 1 = 2^n - 1

Summary

- Iterative programs
    - Focus on loops
- Recursive programs
    - Write and solve a recurrence
- Need to be clear about accounting for "basic" operations

SEARCHING IN A LIST

Search problem

- Is value `v` present in list `l` ?
- Naive solution scans the list
- Input size *n*, the length of the list
- Worst case is when `v` is not present in `l`
- Worst case complexity is O(*n*)

```
def naivesearch(v,l):
  for x in l:
    if v == x:
      return True
  return False
```

Searching a sorted list

- What if `l` is sorted in ascending order?
- Compare `v` with the midpoint of `l`
    - If midpoint is `v`, the value is found
    - If `v` less than midpoint, search the first half
    - If `v` greater than midpoint, search the second half
    - Stop when the interval to search becomes empty

```
def binarysearch(v,l):
  if l == []:
```

```
    return False
  m = len(l)//2
  if v == l[m]:
    return True
  if v < l[m]:
    return binarysearch(v, l[:m])
  else:
    return binarysearch(v, l[m+1:])
```

Binary search

- How long does this take?
  - Each call halves the interval to search
  - Stop when the interval becomes empty
- log *n* - number of times to divide *n* by 2 to reach 1
  - `1 // 2 = 0`, so next call reaches empty interval
- O(log *n*) steps

Alternative calculation

- T(*n*) : the time to search a list of length *n*
  - If *n* = 0, we exit, so T(*n*) = 1
  - If *n* > 0, T(*n*) = T(*n*//2) + 1
- **Recurrence** for T(*n*)
  - T(0) = 1
  - T(*n*) = T(*n*//2) + 1, *n* > 0
- Solve by "**unwinding**"
  - T(*n*) = T(*n*//2) + 1
    = (T(*n*//4) + 1) + 1 = T(*n*//2^2) + 1 + 1
    ...
    = T(*n*//2^k) + 1 + ... + 1
    = T(1) + k, for k = log*n*
    = (T(0) + 1) + log *n* = 2 + log *n*

Summary

- Search in an unsorted list takes time O(*n*)
  - Need to scan the entire list
  - Worst case is when the value is not present in the list
- For a sorted list, binary search takes time O(log *n*)
  - Halve the interval to search each time

- In a sorted list, we can determine that `v` is absent by examing just `log n` values!

SELECTION SORT

Sorting a list

- Sorting a list makes many other computations easier
  - Binary search
  - Finding the median
  - Checking for duplicates
  - Building a frequency table of values
- How do we sort a list?
- Eg: You are the TA for a course
  - Instructor has a pile of evaluated exam papers
  - Papers in random order of marks
  - Your task is to arrange the papers in descending order of marks
- *Strategy 1*
  - Scan the entire pile and find the paper
  - Move this paper to a new pile
  - Repeat with the remaining papers
    - Add the paper with the next minimum marks to the second pile each time
  - Eventually, the new pile is sorted in descending order

Selection sort

- Select the next element in sorted order
- Append it to the final sorted list
- Avoid using a second list
  - Swap the minimum element into the first position
  - Swap the second minimum element into the second position
  - ...
- Eventually the list is rearranged in place in ascending order

```
def Selectionsort(L):
  n = len(L)
  if n < 1:
    return L
  for i in range(n):
    # Assume L[:i] is sorted - called invarient
    mpos = i
    # mpos: position of minimum in L[i:]
    for j in range(i+1, n):
```

```
    if L[j] == L[mpos]:
      mpos = j
  # L[mpos]: smallest value in L[i:]
  # Exchange L[mpos] and L[i]
  (L[i], L[mpos]) = (L[mpos], L[i])
  # Now L[:i+1] is sorted
return L
```

Analysis of selection sort

- Correctness follows from the invariant
- Efficiency
  - Outer loop iterates *n* times
  - Inner loop: *n* - *i* steps to find minimum in L[i:]
  - T(*n*) = *n* + (*n* - 1) + ... + 1
  - T(*n*) = *n*(*n* + 1)/2
- T(*n*) is O(n^2)

Summary

- Selection sort is an intuitive algorithm to sort a list
- Repeatedly find the minimum (or maximum) and append to sorted list
- Worst case complexity is O(n^2)
  - Every input takes this much time
  - No advantage even if list is arranged carefully before sorting

INSERTION SORT

Sorting a list

- Consider same example of TA as before
  Strategy 2
- Move the first paper to a new pile
- Second paper
  - Lower marks than first paper? Place below first paper in new pile
  - Higher marks than first paper? Place above the first paper in new pile
- Third paper
  - Insert into correct position with respect to first two
- Do this for the remaining papers
  - Insert each one into correct position in the second pile

Insertion sort

- Start building a new sorted list
- Pick next element and insert it into the sorted list
- An iterative formulation
    - Assume `L[:i]` is sorted
    - Insert `L[i]` in `L[:i]`

```
def InsertionSort(L):
  n = len(L)
  if n < 1:
    return L
  for i in range(n):
    # Assume L[:i] is sorted
    # Move L[i] to correct position is L
    j = i
    while(j > 0 and L[j] < L[j-1]):
      (L[j], L[j-1]) = (L[j-1], L[j])
      j = j-1
      # Now L[:i+1] is sorted
  return L
```

- A recursive formulation
    - Inductively sort `L[:i]`
    - Insert `L[i]` in `L[:i]`

```
def Insert(L,v):
  n = len(L)
  if n == 0:
    return [v]
  if v >= L[-1]:
    return L + [v]
  else:
    return Insert(L[:-1],v) + L[-1:]

def ISort(L):
  n = len(L)
  if n < 1:
    return L
  L = Insert(ISort(L[:-1]), L[-1])
  return L
```

In python, there are 2 types of sorts:

- `l.sort()` which is an in-place sort (iterative)
- `l2 = sorted(l)` which creates a new sorted list l1 (recursive)

Analysis of iterative insertion sort

- Correctness follows from the invarient
- Efficiency
    - Outer loop iterates *n* times
    - Inner loop: *i* steps to insert `L[i]` in `L[:i]`
    - T(*n*) = 0 + 1 + ... + (*n* - 1)
    - T(*n*) = *n*(*n* - 1)/2
- T(*n*) is O(n^2)

Analysis of recursive insertion

- For input of size *n*, let
    - TI(*n*) be the time taken by `Insert`
    - TS(*n*) be the time taken by `Isort`
- First calculate TI(*n*) for `Insert`
    - TI(0) = 1
    - TI(*n*) = TI(*n* - 1) + 1
    - Unwind to get TI(*n*) = *n*
- Set up a recurrence for TS(*n*)
    - TS(0) = 1
    - TS(*n*) = TS(*n*-1) + TI(*n*-1)
- Unwind to get 1 + 2 + ... + *n* - 1

Summary

- Insertion sort is another intuitive algorithm to sort a list
- Create a new sorted list
- Repeatedly insert elements into the sorted list
- Worst case complexity is O(n^2)
    - Unlike selection sort, not all cases take time n^2
    - If list is already sorted, `Insert` stops in 1 step
    - Overall time can be close to O(*n*)

MERGE SORT

Beating the O(n^2) barrier

- Both selection and insertion sort take time O(n^2)
- This is infeasible for n > 10000
- How can we bring the complexity below O(n^2)?
  Strategy 3

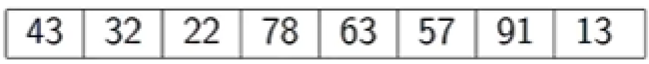
- Divide the list into two halves
- Separately sort the left and right half
- Combine the two sorted halves to get a fully sorted list

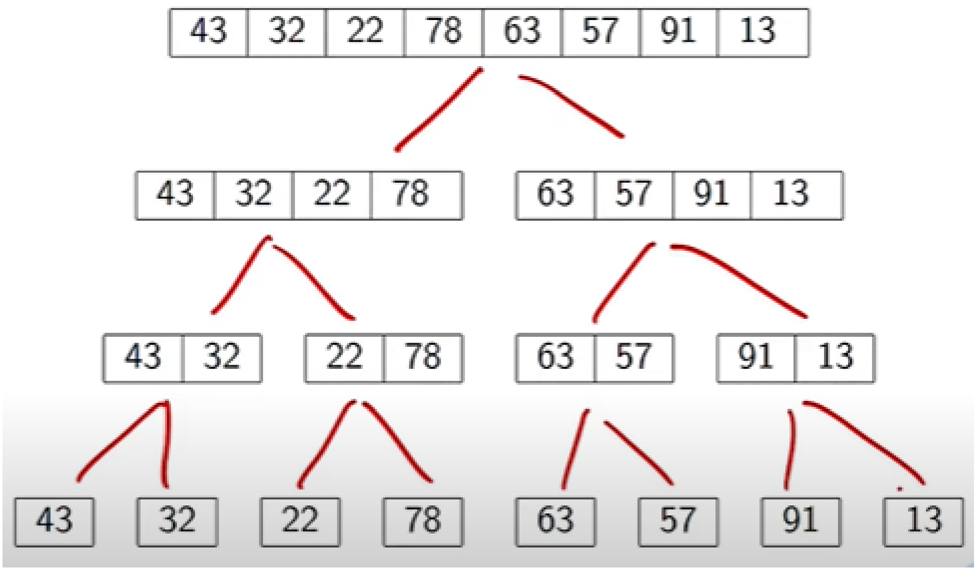
Combining 2 sorted lists

- Combine two sorted lists `A` and `B` into a single sorted list `C`
    - Compare first elements of `A` and `B`
    - Move the smaller of the two to `C`
    - Repeat till you exhaust `A` and `B`
- Merging `A` and `B`

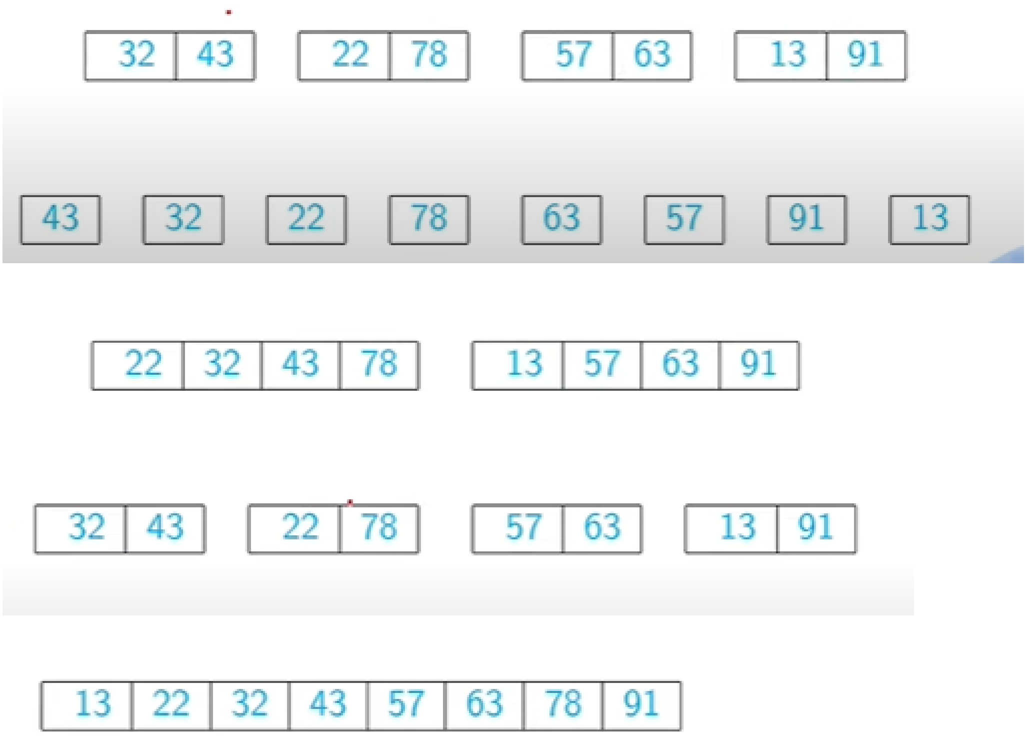
Merge Sort

- Let `n` be the length of `L`
- Sort `A[:n//2]`
- Sort `A[n//2:]`
- Merge the sorted halves into B
- How do we sort `A[:n//2]` and `A[n//2:]`?
    - Recursively, same strategy!





After sorting



Divide and Conquer

- Break up the problem into disjoint parts
- Solve each part separately
- Combine the solutions efficiently

Merging sorted lists

- Combine 2 sorted lists `A` and `B` into `C`
  - If `A` is empty, copy `B` into `C`
  - If `B` is empty, copy `A` into `C`
  - Otherwise, compare first elements of `A` and `B`
    - Move the smaller of the two to `C`
  - Repeat till all elements of `A` and `B` have been moved

```
def merge(A,B):
  (m,n) = (len(A), len(B))
  (C,i,j,k) = ([],0,0,0)
  while k < m+n:
    if i == m:
      C.extend(B[j:])
      k = k + (n-j)
    elif j == n:
      C.extend(A[i:])
      k = k + (n-1)
```

```
    elif A[i] < B[j]:
      C.append(A[i])
      (i,k) = (i+1,k+1)
    else:
      C.append(B[j])
      (j,k) = (j+1,k+1)
  return C
```

Merge Sort

- To sort `A` into `B`, both of length *n*
- If *n* $\leq$ 1, nothing to be done
- Otherwise
    - Sort `A[:n//2]` into `L`
    - Sort `A[n//2:]` into `R`
    - Merge `L` and `R` into `B

```
def mergesort(A):
  n = len(A)
  if n <= 1:
    return A
  L = mergesort(A[:n//2])
  R = mergesort(A[n//2:])
  B = merge(L,R)
  return B
```

Summary

- Merge sort using divide and conquer to sort a list
- Divide the list into two halves
- Sort each half
- Merge the sorted halves
- Next, we have to check that the complexity is less than O(n^2)

ANALYSIS OF MERGE SORT

Analysing merge

- Merge `A` of length *m*, `B` of length *n*
- Output list `C` has length *m* + *n*
- In each iteration we add (at least) one element to `C`
- Hence merge takes time O(*m* + *n*)
- Recall that *m* + *n* $\leq$ 2(max(m,n))
- If *m* $\approx$ *n*, `merge` take time O(*n*)

Analysing mergesort

- Let T(*n*) be the time taken for input of size *n*
  - For simplicity, assume n = 2^k for some *k*
- Recurrence
  - T(0) = T(1) = 1
  - T(*n*) = 2T(*n*/2) + *n*
    - Solve two subproblems of size *n*/2
    - Merge the soultions in time *n*/2 + *n*/2 = *n*
  - T(*n*) = 2[2T(*n*/4) + *n*/2] + n
    = 2^2T(*n*/2^2) + 2*n* = 2^3T(n/2^3) + 3*n*
    .
    .
    .
    = 2^kT(n/2^k) + *kn*
- When *k* = log *n*, T(n/2^k) = T(1) = 1
- T(n) = $2^{log n}$ T(1) + (log *n*)*n* = *n* + *n*log *n*
- T(n) = O(*n*log*n*)

Summary

- Merge sort takes time O(*n*log *n*) so can be used to effectively on large inputs
- Variations on merge are possible
  - Union of two sorted lists - discard duplicates, if `A[i] == B[j]` move just one copy to `C` and increment both `i` and `j`
  - Intersection of two sorted lists - when `A[i] == B[j]`, move one copy to `C`, otherwise discard the smaller of `A[i]`, `B[j]`
  - List difference - elements in `A` but not in `B`
- Merge needs to create a new list to hold the merged elements
  - No obvious way to effectively merge two two lists in place
  - Extra storage can be costly
- Inherently recursive
  - Recursive calls and returns are expensive