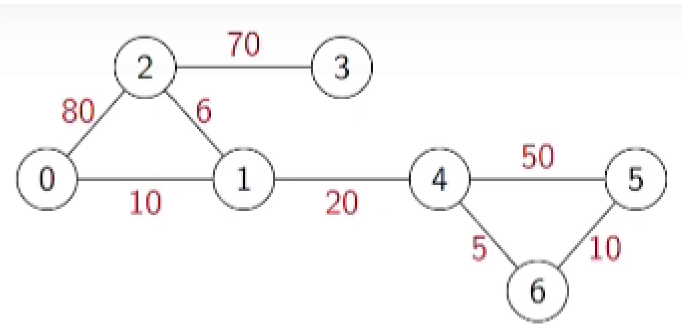
Week 5

SHORTEST PATH IN WEIGHTED GRAPHS

Weighted graphs

- BFS explores a graph level by level
- BFS compputes shortest path,in terms of number of edges, to every reachable vertex
- May assign values to edges
    - Cost, time, distance,...
    - Weighted graphs
- $G = (V, E), W : E \to \mathbb{R}$



- Adjacency matrix: Record the weights along with edge information - weight is always 0 if no edge

	0	1	2	3	4	5	6
0	(0,0)	(1,10)	(1,80)	(0,0)	(0,0)	(0,0)	(0,0)
1	(1,10)	(0,0)	(1,6)	(0,0)	(1,20)	(0,0)	(0,0)
2	(1,80)	(1,6)	(0,0)	(1,70)	(0,0)	(0,0)	(0,0)
3	(0,0)	(0,0)	(1,70)	(0,0)	(0,0)	(0,0)	(0,0)
4	(0,0)	(1,20)	(0,0)	(0,0)	(0,0)	(1,50)	(1,5)
5	(0,0)	(0,0)	(0,0)	(0,0)	(1,50)	(0,0)	(1,10)
6	(0,0)	(0,0)	(0,0)	(0,0)	(1,5)	(1,10)	(0.0)

- Adjacency list: Record weights along with the edge information

0	[(1,10),(2,80)]
1	[(0,10),(2,6),(4,20)]
2	[(0,80),(1,6),(3,70)]
3	[(2,70)]
4	[(1,20),(5,50),(6,5)]
5	[(4,50),(6,10)]
6	[(4,5),(5,10)]

Shortest paths in weighted graphs

- BFS computes shortest path, in terms of number of edges, to every reachable vertex
- In a weighted graph, add up the weights along a path
- Weighted shortest path need not have minimum number of edges
    - Shortest path from 0 to 2 is via 1 (weight = 16)

Shortest path problems

Single source shortest paths

- Find shortest paths from a fixed vertex to every other vertex
- Transport finished product from factory (single source) to all retail outlets
- Courier company delivers items from distribution centre (single source) to addresses

All pairs shortest path

- Find shortest paths between every pair of vertices i and j
- Optimal airline, railway, road routes between cities

Negative edges weights

- Can negative edge weights be meaningful?
- Taxi driver trying to head home at the end of the day
    - Roads with few customer, drive empty (positive weight)
    - Roads with many customers, make profit (negative weight)
    - Find route towards home that minimizes cost

Negative cycles

- A negative cycle is one whose weight is negative
    - Sum of the weights of edges that make up the cycle
- By repeatedly traversing a negative cycle, total cost keeps decreasing
- If a graph has a negative cycle, total cost keeps decreasing

- Without negative cycles, we can compute shortest paths even if some weights are

Summary

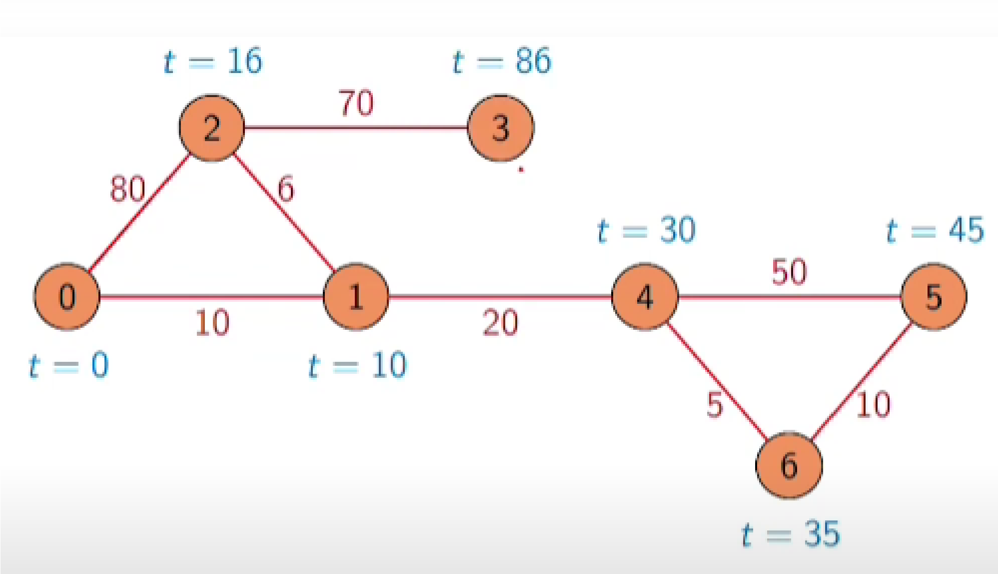
- In a weighted graph, each edge has a cost
  - Entries in adjacency matrix capture edge weights
- Length of a path is the sum of the weights
  - Shortest path in a weighted graph need not be minimum in terms of number of edges
- Different shortest path problems
  - Single source - from one designated vertex to all others
  - All-pairs - Between every pair of vertices
- Negative edge weights
  - Should not have negative cycles
  - Without negative cycles, shortest paths still well defined

SINGLE SOURCE SHORTEST PATHS

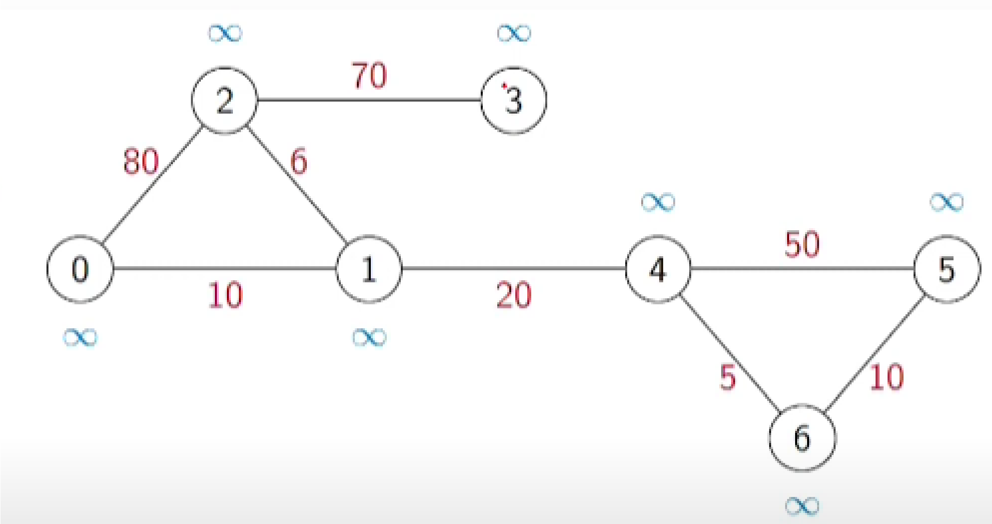
Single source shortest paths

- Weighted graph:
  - G - (V, E)
  - $W : E \rightarrow \mathbb{R}$
- Single source shortest paths
  - Find shortest paths from a fixed vertex to every other vertex
- Assume, for now, that edge weights are all non-negative

- Compute shortest path from 0 to all other vertices
- Imagine vertices are oil depots, edges are pipelines
- Set fire to oil depot at vertex 0
- Fire travels at uniform speed along each pipeline
- First oil depot to catch fire after 0 is nearest vertex
- Next oil depot is second nearest vertex
- ...



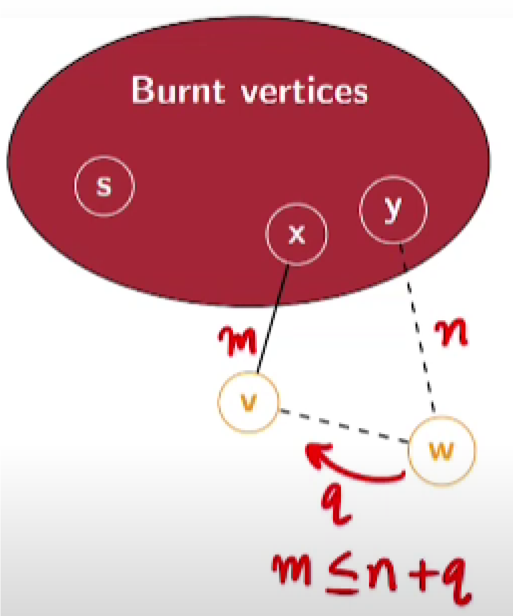
- Compute expected burn time for each vertex
- Each time a new vertex burns, update the expected burn times of its neighbours
- Algorithm due to Edsger W Dikjstra



Dikjstra's algorithm: Proof of correctness

- Each new shortest path we discover extends an earlier one (Greedy method)
- By induction, assume we have found shortest paths to all vertices already burnt

- Next vertex to burn is v, via x
- Can't find a shorter path later from y to v via w
  - Burn time of $w \geq$ burn time of v
  - Edge from w to v has weight ≥ 0
- This argument breaks down if edge (w, v) can have negative weight
  - Can't use Dikjstra's algorithm with negative edge weights



Implementation

- Maintain 2 dictionaries with vertices as keys
  - `visited` initially `False` for all v (burnt vertices)
  - `distance` initially `infinity` for all v (expected burn time)
- Set `distance[s]` to 0
- Repeat, until all reachable vertices are visited
  - Find unvisited vertex `nextv` with minimum distance
  - Set `visited[nextv]` to `True`
  - Recompute `distance[v]` for every neighbour `v` of `nextv`

```
def dijkstra(WMat, s):
  # s is the source vertex; WMat is the weighted adj matrix
  (rows,cols,x) = WMat.shape # x is the edge/weight info
  # x[0] edge info; x[1] weight info
  infinity = np.max(WMat) * rows + 1
  # max value in the matrix multipied by rows + 1 is larger than all
  (visited, distance) = ({},{})
  for v in range(rows):
    (visted[v], distance[v]) = (False, infinity)
  distance[s] = 0
  for u in range(rows):
    nextd = min([distance[v] for v in range(rows) if not visted[v]])
    nextvlist = [v for v in range(rows) if(not visited[v]) and distance[v] == nextd]
    if nextvlist == []:
      break
    nextv = min(nextvlist)
    visited[nextv] = True
    for v in range(cols):
      if WMat[nextv,v,0] == 1 and (not visited[v]):
```

```
      distance[v] = min(distance[v],distance[nextv] + WMat[nextv,v,1])
  return(distance)
```

Complexity

- Setting `infinity` takes $O(n^2)$ time
- Main loop runs n times
  - Each iteration visits one more vertex
  - $O(n)$ to find next vertex to visit
  - $O(n)$ to update `distance[v]` for neighbours
- Overall $O(n^2)$
- If we use an adjacency list
  - Setting `infinity` and updating distances both $O(m)$, amortised
  - $O(n)$ bottleneck remains to find next vertex to visit
  - Better data structure? Later

```
def dijkstralist(WList, s):
  infinity = 1 + len(WList.keys()) * max([d for u in WList.keys() for (v,d) in WList[u]])
  (visited,distance) = ({},{})
  for v in WList.keys():
    (visited[v], distance[v]) = (False,infinity)
  distance[s] = 0
  for u in WList.keys():
    nextd = min([distance[v] for v in WList.keys() if not visited[v]])
    nextvlist = [v for v in WList.keys() if (not visited[v]) and distance[v] == nextd]
    if nextvlist == []:
      break
    nextv = min(nextvlist)
    visited[nextv] = True
    for (v,d) in WList[nextv]:
      if not visited[v]:
        distance[v] = min(distance[v], distance[nextv])
  return(distance)
```

Summary

- Dijkstra's algorithm computes single source shortest paths
- Use a greedy strategy to identify vertices to visit
  - Next vertex to visit is based on shortest distance computed so far
  - Need to prove that such a strategy is correct
  - Correctness requires edge weights to be non-negative
- Complexity is $O(n^2)$
  - Even with adjacency lists
  - Bottleneck is identifying unvisited vertex with minimum distance
  - Need a better data structure to identify and remove minimum (or maximum) from a collection

SINGLE SOURCE SHORTEST PATHS WITH NEGATIVE WEIGHTS

Dijkstra's Algorithm

Burning pipe analogy

- We keep track of the following
  - The vertices that have been burnt
  - The expected burn time of vertices
- Initially
  - No vertex is burnt
  - Expected burn time of vehicles
  - Expected burn time of rest is ∞
- While there are vertices yet to burn
  - Pick unburnt vertex with minimum expected burn time, mark it s burnt
  - Update the expected burn time of its neighbours

Initialization (assume source vertex 0)

- $B(i)$ = `False`, for $0 \leq i < n$
  - UB = {k | $B(k)$ = `False`}
- $EBT(i) = \begin{cases} 0, & \text{if i} = 0 \\ \infty & \text{otherwise.} \end{cases}$

Update, if $UB \neq \emptyset$

- Let $j \in UB$ such that $EBT(j) \leq EBT(k)$ for all $k \in UB$
- Update $B(j)$ = `True`,$UB = UB \setminus \{ j \}$
- For each $(j, k) \in E$ such that $k \in UB$, $EBT(k) = min(EBT(k), EBT(j) + W(j, k))$

Extending to negative edge weights

- The difficulty with negative edge weights is that we stop updating the burn time once a vertex is burnt
- What if we allow updates evenafter a vertex is burnt?
- Negative edge weights are allowed, but no negative cycles
- Going around a cycle can only add to the length
- Shortest route to every vertex is a path, no loops

- Suppose minimum weight path from 0 to k is

$$0 \xrightarrow{w_1} j_1 \xrightarrow{w_2} j_2 \xrightarrow{w_3} \cdots \xrightarrow{w_{\ell-1}} j_{\ell-1} \xrightarrow{w_\ell} k$$

- Need not be minimum in terms of number of edges
- Every prefix of this path must be a minimum weight path

- $0 \xrightarrow{w_1} j_1$
- $0 \xrightarrow{w_1} j_1 \xrightarrow{w_2} j_2$
- $\ldots$
- $0 \xrightarrow{w_1} j_1 \xrightarrow{w_2} j_2 \xrightarrow{w_3} \cdots \xrightarrow{w_{\ell-1}} j_{\ell-1}$

- Once we discover shortest path to j_{l-1}, next update will fix shortest path to k
- Repeatedly update shortest distance to each vertex based on shortest distance to its neighbours
  - Update cannot push this distance below actual shortest distance
- After l updates,all shortest paths using $\geq l$ edges have stabilized
  - Minimum weight path to any node has at most $n-1$ edges
  - After $n-1$ update, all shortest paths have stabilized

Bellman-Ford Algorithm
Initialization (source vertex 0)

- $D(j)$: minimum distance known so far to vertex j
- $D(j) = \begin{cases} 0, & \text{if j} = 0 \\ \infty & \text{otherwise.} \end{cases}$

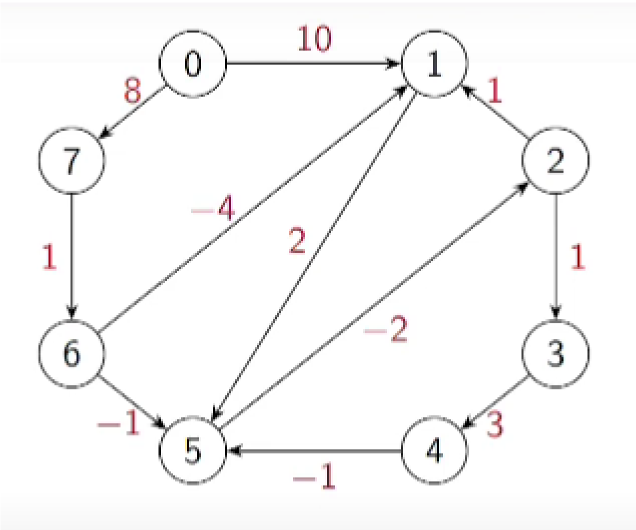
Repeat $n-1$ times

- For each vertex $j \in \{0,1,\ldots,n-1\}$, for each edge $(j,k) \in E$,
  $D(k) = min(D(k), D(j) + W(j,k))$

  Works for directed and undirected graphs

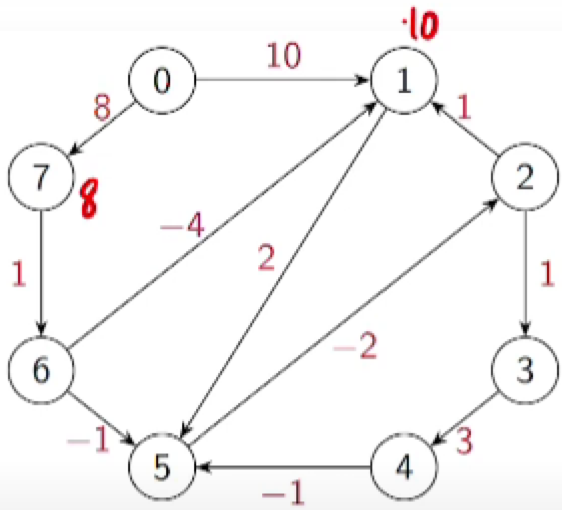
```
def bellmanford(WMat,s):
  (rows,cols,x) = WMat.shape
  infinity = np.max(WMat)*rows + 1
  distance = {}
  for v in range(rows):
    distance[v] = infinity
  distance[s] = 0
  for i in range(rows):
```

```
    for u in range(rows):
      for v in range(cols):
        if WMat[u,v,0] == 1:
          distance[v] = min(distance[v], distance[u] + WMat[u,v,1])
  return(distance)
```

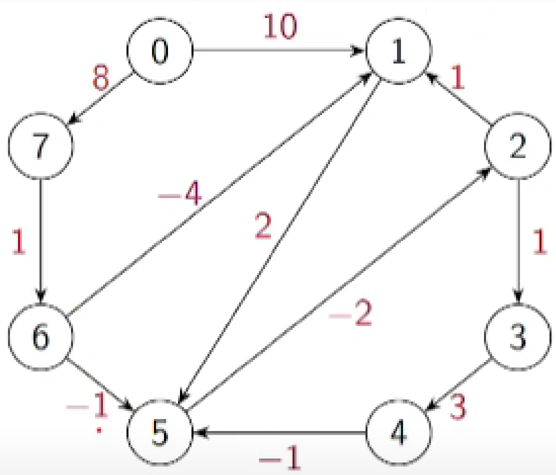


v	$D(v)$							
0	0							
1	∞							
2	∞							
3	∞							
4	∞							
5	∞							
6	∞							
7	∞							

- Initialize $D(0) = 0$

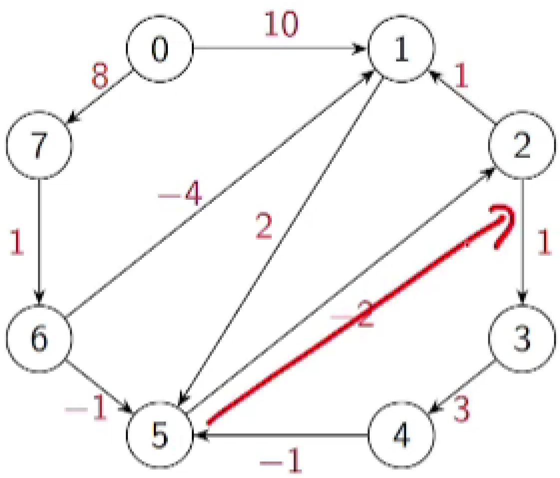


v	D(v)							
0	0	0	0					
1	∞	10	10					
2	∞	∞	∞					
3	∞	∞	∞					
4	∞	∞	∞					
5	∞	∞	12					
6	∞	∞	9					
7	∞	8	8					



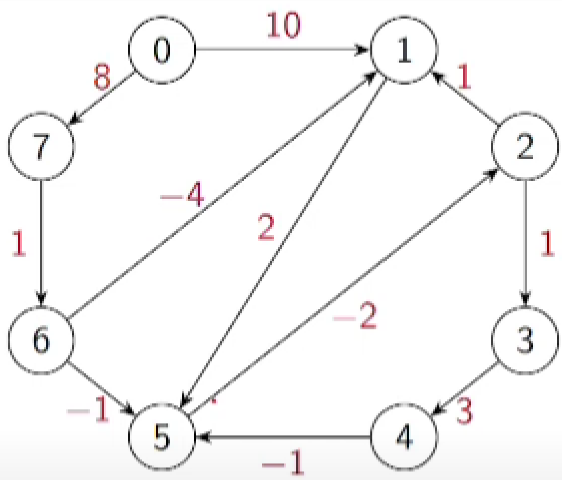
- Initialize $D(0) = 0$
- For each $(j,k) \in E$, update
  $D(k) = \min(\ D(k),$
  $\qquad D(j) + W(j,k)\ )$

v	D(v)							
0	0	0	0	0				
1	∞	10	10	5				
2	∞	∞	∞	10				
3	∞	∞	∞	∞				
4	∞	∞	∞	∞				
5	∞	∞	12	8				
6	∞	∞	9	9				
7	∞	8	8	8				



- Initialize $D(0) = 0$
- For each $(j,k) \in E$, update
  $D(k) = \min(\ D(k),$
  $\qquad D(j) + W(j,k)\ )$

v	$D(v)$							
0	0	0	0	0	0	0	0	0
1	∞	10	10	5	5	5	5	5
2	∞	∞	∞	10	6	5	5	5
3	∞	∞	∞	∞	11	7	6	6
4	∞	∞	∞	∞	∞	14	10	9
5	∞	∞	12	8	7	7	7	7
6	∞	∞	9	9	9	9	9	9
7	∞	8	8	8	8	8	8	8



- Initialize $D(0) = 0$
- For each $(j, k) \in E$, update
  $D(k) = \min(\ D(k),$
  $D(j) + W(j, k)\ )$

- What if there was a negative cycle? Distance would continue to decrease
- Check if update n reduces any $D(v)$

Complexity

- Initialising `infinity` takes $O(n^2)$ time
- The outer update loop runs $O(n)$ times
- In each iteration, we have to examine every edge in the graph
  - This take $O(n^2)$ for an adjacency matrix
- Overall, $O(n^3)$

- If we shift to adjacency lists
  - Initializing `infinity` is $O(m)$
  - Scanning all edges in each update iteration is $O(m)$

```
def bellmanfordlist(WList,s):
  infinity =  1 + len(WList.keys())*max([d for u in WList.keys() for (v,d) in WList[u]])
  distance = {}
  for v in WList.keys():
    distance[v] = infinity

  distance[s] = 0

  for i in WList.keys():
    for u in WList.keys():
      for (v,d) in WList[u]:
```

```
        distance[v] = min(distance[v], distance[u] + d)
  return(distance)
```

Summary

- Dijkstra's algorithm assumes non-negative edge weights
    - Final distance is frozen each time a vertex "burns"
- Without negative cycles, every shortest route is a path
- Every prefix of a shortest path is also a shortest path
- Iteratively find shortest paths of length $1, 2, \ldots, n-1$
- Update distance to each vertex with every iteration - *Bellman-Ford algorithm*
- $O(n^3)$ time with adjacency matrix, $O(mn)$ time with adjacency list
- If Bellman-Ford algorithm does not converge after $n-1$ iterations, there is a negative cycle

ALL PAIRS SHORTEST PATHS (FLOYD-WARSHALL ALGORITHM)

Shortest paths in weighted graphs

Two types of shortest path problems of interest

Single source shortest paths

- Find shortest paths from a fixed vertex to every other vertex
- Transport finished product from factory (single source) to all retail outlets
- Courier company delivers items from distribution centre (single source) to addresses

All pairs shortest paths

- Find shortest paths between every pair of vertices i and j
- Optimal airline, railway, road routes between cities
- Run Dijkstra or Bellman-Ford from each vertex
- Is there is another way?

Transitive closure

- Adjacency matrix A represents paths of length 1
- Matrix multiplication, A^2 = A x A
    - $A^2[i,j] = 1$ if there is a path of length 2 from i to j
    - For some k, $A[i,k] = 1$, $A[k,j] = 1$
- $A^+ = A + A^2 + \ldots + A^{n-1}$

An alternative approach

- $B^k[i,j] = 1$ if there is path from i to j via vertices $\{0, 1, \ldots, k-1\}$

    - Constraint applies only to intermediate vertices between i and j
    - $B^0[i,j] = 1$ if there is a direct edge
    - $B^0 = A$
- $B^{k+1}[i,j] = 1$ if
    - $B^k[i,j] = 1$ - can already reach j from i via $\{0,1,\ldots,k-1\}$
    - $B^k[i,k] = 1$ and $B^k[k,j] = 1$ - use $\{0,1,\ldots,k-1\}$ to go from i to k and then from k to j

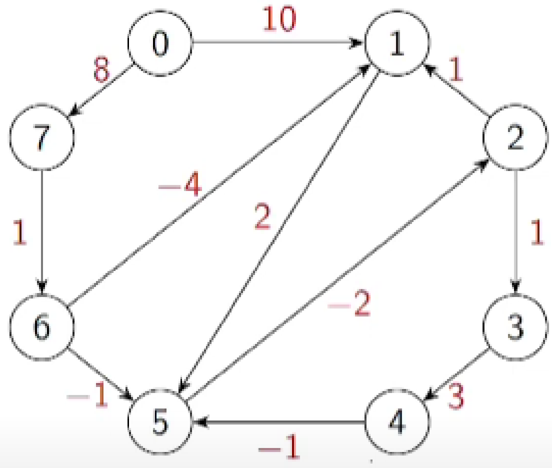
Warshall's Algorithm

- $B^k[i,j] = 1$ if there is path from i to j via vertices $\{0,1,\ldots,k-1\}$
- $B^0[i,j] = A[i,j]$
    - Direct edges, no intermediate vertices
- $B^{k+1}[i,j] = 1$ if
    - $B^k[i,j] = 1$, or
    - $B^k[i,k] = 1$ and $B^k[k,j] = 1$
- This algorithm also computes transitive closure - Warshall's algorithm
- $B^n[i,j] = 1$ if there is some path from i to j with intermediate vertices in $\{0,1,\ldots,n-1\}$
- $B^n = A^+$
- We adapt Warshall's algorithm to compute all-pairs shortest paths

Floyd-Warshall Algorithm

- Let $SP^k[i,j]$ be the length of the shortest path i to j via vertices $\{0,1,\ldots,k-1\}$
- $SP^0[i,j] = W[i,j]$
    - No intermediate vertices, shortest path is weight of direct edge
    - Assume $W[i,j] = \infty$ if $(i,j) \notin E$
- $SP^{k+1}[i,j]$ is the minimum of
    - $SP^k[i,j]$
      Shortest path using only $\{0,1,\ldots,k-1\}$
    - $SP^k[i,k] + SP^k[k,j]$
      Combine shortest path from i to k and k to j

SP^0	0	1	2	3	4	5	6	7
0	∞	10	∞	∞	∞	∞	∞	8
1	∞	∞	∞	∞	∞	2	∞	∞
2	∞	1	∞	1	∞	∞	∞	∞
3	∞	∞	∞	∞	3	∞	∞	∞
4	∞	∞	∞	∞	∞	-1	∞	∞
5	∞	∞	-2	∞	∞	∞	∞	∞
6	∞	-4	∞	∞	∞	-1	∞	∞
7	∞	∞	∞	∞	∞	∞	1	∞



SP^1	0	1	2	3	4	5	6	7
0	∞	10	∞	∞	∞	∞	∞	8
1	∞	∞	∞	∞	∞	2	∞	∞
2	∞	1	∞	1	∞	∞	∞	∞
3	∞	∞	∞	∞	3	∞	∞	∞
4	∞	∞	∞	∞	∞	-1	∞	∞
5	∞	∞	-2	∞	∞	∞	∞	∞
6	∞	-4	∞	∞	∞	-1	∞	∞
7	∞	∞	∞	∞	∞	∞	1	∞

SP^2	0	1	2	3	4	5	6	7
0	∞	10	∞	∞	∞	12	∞	8
1	∞	∞	∞	∞	∞	2	∞	∞
2	∞	1	∞	1	∞	3	∞	∞
3	∞	∞	∞	∞	3	∞	∞	∞
4	∞	∞	∞	∞	∞	-1	∞	∞
5	∞	∞	-2	∞	∞	∞	∞	∞
6	∞	-4	∞	∞	∞	-2	∞	∞
7	∞	∞	∞	∞	∞	∞	1	∞

SP^3	0	1	2	3	4	5	6	7
0	∞	10	∞	∞	∞	12	∞	8
1	∞	∞	∞	∞	∞	2	∞	∞
2	∞	1	∞	1	∞	3	∞	∞
3	∞	∞	∞	∞	3	∞	∞	∞
4	∞	∞	∞	∞	∞	-1	∞	∞
5	∞	-1	-2	-1	∞	1	∞	∞
6	∞	-4	∞	∞	∞	-2	∞	∞
7	∞	∞	∞	∞	∞	∞	1	∞

SP^7	0	1	2	3	4	5	6	7
0	∞	10	10	11	14	12	∞	8
1	∞	1	0	1	4	2	∞	∞
2	∞	1	1	1	4	3	∞	∞
3	∞	1	0	1	3	2	∞	∞
4	∞	-2	-3	-2	1	-1	∞	∞
5	∞	-1	-2	-1	2	1	∞	∞
6	∞	-4	-4	-3	0	-2	∞	∞
7	∞	-3	-3	-2	1	-1	1	∞

SP^8	0	1	2	3	4	5	6	7
0	∞	5	5	6	9	7	9	8
1	∞	1	0	1	4	2	∞	∞
2	∞	1	1	1	4	3	∞	∞
3	∞	1	0	1	3	2	∞	∞
4	∞	-2	-3	-2	1	-1	∞	∞
5	∞	-1	-2	-1	2	1	∞	∞
6	∞	-4	-4	-3	0	-2	∞	∞
7	∞	-3	-3	-2	1	-1	1	∞

Implementation

- Shortest path matrix SP is n x n x $(n+1)$
- Initialize $SP[i,j,0]$ to edge weight $W(i,j)$, or ∞ if no edge
- Update $SP[i,j,k]$ from $SP[i,j,k-1]$ using the Floyd-Warshall update rule
- Time complexity is $O(n^3)$
- We only need $SP[i,j,k-1]$ to compute $SP[i,j,k]$
- Maintain two "slices" $SP[i,j]$, $SP'[i,j]$, compute SP' from SP, copy SP' to SP, save space

```
def floydwarshall(WMat):
  (rows, cols, x) = WMat.shape
  infinity = np.max(WMat) * rows * rows + 1
  SP = np.zeros(shape=(rows,cols,cols+1))
  for i in range(rows):
    for j in range(cols):
      SP[i,j,0] = infinity
  for i in range(rows):
    for j in range(cols):
      if WMat[i,j,0] == 1:
        SP[i,j,0] = WMat[i,j,1]

  for k in range(1,cols+1):
    for i in range(rows):
      for j in range(cols):
        SP[i,j,k] = min(SP[i,j,k-1], SP[i,k-1,k-1]+SP[k-1,j,k-1])
  return(SP[:,:,cols])
```

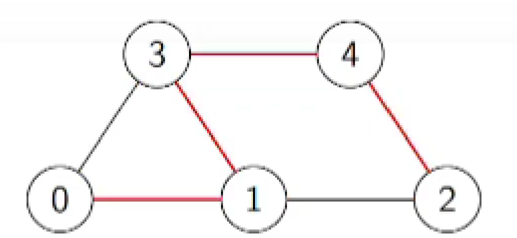
Summary

- Warshall's algorithm is an alternative way to compute transitive closure
    - $B^k[i,j] = 1$ if we can reach j from i using vertices in $\{0, 1, \ldots, k-1\}$
- Adapt Warshall's algorithm to compute all pairs shortest paths
    - $SP^k[i,j]$ is the length of the shortest path from i to j using vertices in $\{0, 1, \ldots, k-1\}$
    - $SP^n[i,j]$ is the length of the overall shortest path
    - Floyd-Warshall algorithm
- Works with negative edge weights, assuming no negative cycles
- Simple nested loop implementation, time $O(n^3)$
- Space can be limited to $O(n^2)$by reusing two "slices" SP and SP'

MINIMUM COST SPANNING TREES

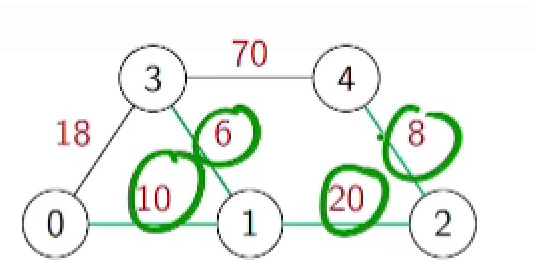
Spanning trees

- Retain a minimal set of edges so that graph remains connected
- A minimally connected graph is a tree
    - Adding an edge to a tree creates a loop
    - Removing an edge disconnects the graph
- Want a tree tht connects all the vertices - spanning tree



Spanning trees with costs

- Restoring a road or laying a fibre optic cable has a cost
- Minimum cost spanning tree
    - Add the cost of all edges in the tree
    - Among the different spanning trees, choose one with minimum cost
- Example
    - Spanning tree, Cost is 114 - not minimum cost spanning tree
    - Another spanning tree, Cost is 44 - minimum cost spanning tree



Some facts about trees

Defn: A tree is a connected acyclic graph

Fact 1: A tree on n vertices has exactly $n-1$ edges

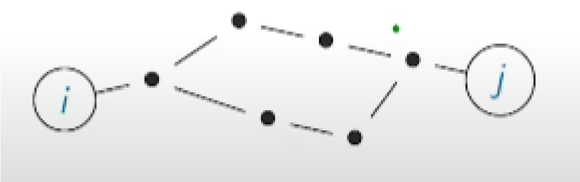
- Initially, one single component
- Deleting edge (i,j) msut split component
  - Otherwise, there is still a path from i to j, combine with (i,j) to form cycle
- Each edge deletion creates one more component
- Deleting $n-1$ edges creates n components, each an isolated vertex

Fact 2: Adding an edge to a tree must create a cycle

- Suppose we add an edge (i,j)
- Tree is connected, so there is already a path from i to j forms a cycle

Fact 3: In a tree, every pair of vertices is connected by a unique path

- If there are two paths from i to j, there must be a cycle



Observation: Any two of the following facts about a graph G implies the third

- G is connected
- G is acyclic
- G has $n-1$ edges

Summary

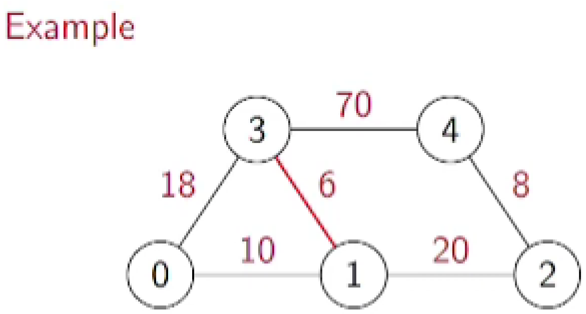
- We will use these facts about trees to build minimum cost spanning trees
- Two natural strategies
- Start with the smallest edge and "grow" a tree

    - Prim's algorithm
- Scan the edges in ascending order of weight to connect components without forming cycles
    - Kruskal's algorithm

MINIMUM COST SPANNING TREES - PRIM'S ALGORITHM

Minimum Cost Spanning tree (MCST)

- Weighted undirected graph, $G = (V, E)$, $W : E \to \mathbb{R}$
    - G assumed to be connected
- Find a minimum cost spanning tree
    - Tree connecting all vertices in V
- *Strategy*
    - Incrementally grow the minimum cost spanning tree
    - Start with a smallest weight edge overall
    - Extend the current tree by adding the smallest edge from the tree to a vertex not yet in the tree

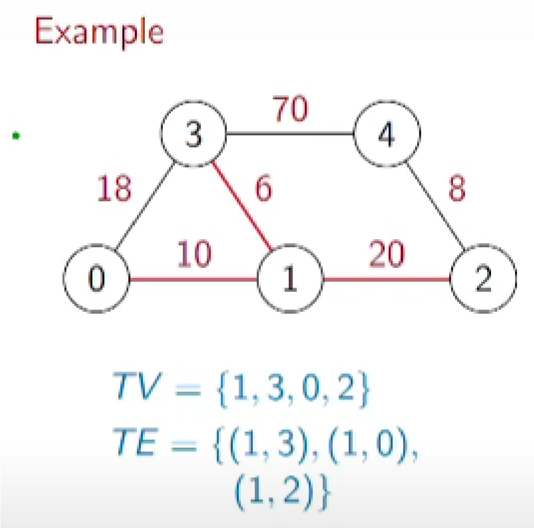


- Start with smallest edge $(1, 3)$
- Extend the tree with $(1, 0)$
- Can't add $(0, 3)$, forms a cycle
- Instead, extend the tree with $(1, 2)$
- Extend the tree with $(2, 4)$

Prim's algorithm

- $G = (V, E)$, $W : E \to \mathbb{R}$
- Incrementally build an MCST
    - $TV \subseteq V$: tree vertices, already added to MCST
    - $TE \subseteq E$: tree edges, already added to MCST

- Initially, $TV = TE = \emptyset$
- Choose minimum weight edge $e = (i, j)$
  - Set TV = $\{i, j\}$, TE = $\{e\}$ MCST
- Repeat $n - 2$ times
  - Choose minimum weight edge $f = (u, v)$ such that $u \in TV, v \notin TV$
  - Add v to TV, f to TE

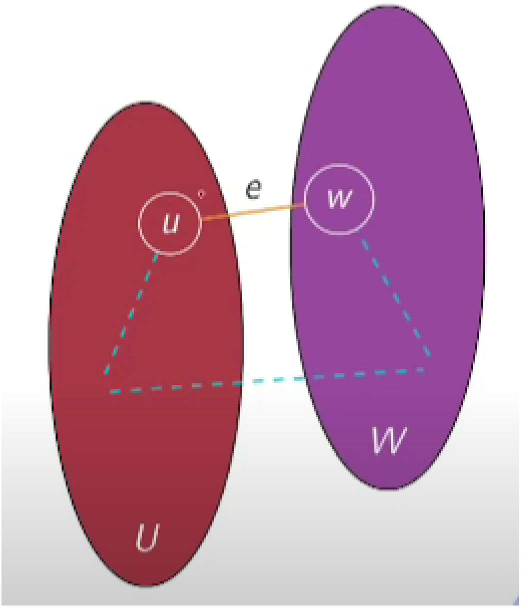


Correctness of Prim's Algorithm

Minimum Separator Lemma

- Let V be partitioned into two non-empty sets U and $W = V \setminus U$
- Let $e = (u, w)$ be the minimum cost edge with $u \in U, w \in W$
- Every MCST must include e

- Assume for now, all edge weights distinct
- Let T be an MCST, $e \notin T$
- T contains a path p from u to
  - p starts in U, ends in W
  - Let $f = (u', w')$ be the first edge on p crossing from U to W
  - Drop f, add e to get a cheaper spanning tree
- What if two edges have same weights?
- Assign each edge a unique index from 0 to $m - 1$
- Define $(e, i) < (f, j)$ if $W(e) < W(j)$ or $W(e) = W(j)$ and $i < j$



- In Prim's algorithm, TV and $W - V \setminus TV$ partition V
- Algorithm picks smallest edge connecting TV and W, which must belong to every MCST
- In fact, for any $v \in V$, $\{v\}$ and $V \setminus \{v\}$ form a partition
- The smallest weight edge leaving any vertex must belong to every MCST
- We started with overall minimum cost edge
- Instead, can start at any vertex v, with TV = $\{v\}$ and $TE = \emptyset$

Implementation

1)

- Keep track of
  - `visited[v]` - is v in the spanning tree?
  - `distance[v]` - shortest distance from v to the tree
  - `TreeEdges` - edges in the current spanning tree

```
def primlist(WList):
  infinity = 1 + max([d for u in WList.keys() for (v,d) in WList[u]])
  (visited, distance, TreeEdges) = ({},{},[])
  for v in WList.keys():
    (visited[v], distance[v]) = (False, infinity)
  visited[0] = True
  for (v,d) in WList[0]:
    distance[v] = d
  for i in WList.keys():
    (mindist, nextv) = (infinity, None)
    for u in WList.keys():
      for (v,d) in WList[u]:
        if visited[u] and (not visited[v]) and d < mindist:
          (mindist, nextv, nexte) = (d,v,(u,v))
    if nextv is None:
      break
```

```
    visited[nextv] = True
    TreeEdges.append(nexte)
    for (v,d) in WList[nextv]:
      if not visited[v]:
        distance[v] = min(distance[v], d)
  return(TreeEdges)
```

- Initialize `visited[v]` to `False`, `distance[v]` to `infinity`
- First add vertex 0 to tree
- Find edge (u, v) leaving the tree where `distance[v]` is minimu, add it to the tree, update `distance[w]` of neighbours

Complexity

- Initialization takes $O(n)$
- Loop to add nodes to the tree runs $O(n)$ times
- Each iteration takes $O(m)$ time to find a node to add
- Overall time is $O(mn)$, which could be $O(n^3)$!

2)

- For each v, keep track of its nearest neighbour in the tree
  - `visited[v]` - is v in the spanning tree?
  - `distance[v]` - shortest distance from v to the tree
  - `nbr[v]` - nearest neighbour of v in tree
- Scan all non-tree vertices to find `nextv` with minimum distance
- Very similar to Dijkstra's algorithm,except for the update rule for distance
- Like Dijksta's algorithm, this is still $O(n^2)$ even for adjacency lists
- With a more clever data structure to extract the minimum, we can do better

```
def primlist2(WList):
  infinity = 1 + max([d for u in WList.keys() for (v,d) in WList[u]])
  (visited, distance, nbr) = ({},{},{})
  for v in WList.keys():
    (visited[v],distance[v], nbr[v]) = (False, infinity, -1)
  visited[0] = True
  for (v,d) in WList[0]:
    (distance[v], nbr[v]) = (d,0)
  for i in range(1, len(WList.keys())):
    nextd = min([distance[v] for v in WList.keys() if not visited[v]])
    nextvlist =[v for v in WList.keys() if (not visted[v]) and distance[v] == nextd]
    if nextvlist == []:
      break
    nextv = min(nextvlist)
    visited[nextv] = True
    for (v,d) in WList[nextv]:
```

```
    if not visited[v]:
      (distance[v],nbr[v]) = (min(distance[v], d), nextv)
return(nbr)
```

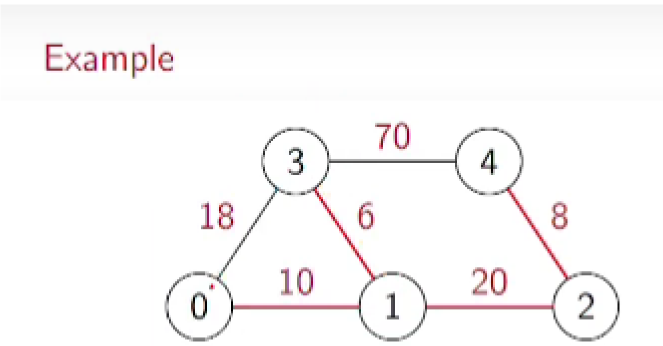
Summary

- Prim's algorithm grows an MCST starting with any vertex
- At each step, connect one more vertex to the tree using minimum cost edge from inside the tree to outside the tree
- Correctness follows from Minimum Seperator Lemma
- Implementation similar to Dijksta's algorithms
  - Update rule for distance is different
- Complexity is $O(n^2)$
  - Even with adjacency lists
  - Bottleneck is identifying unvisited vertex with minimum distance
  - Need a better data structure to identify and remove minimum (or maximum) from a collection

MINIMUM COST SPANNING TREES - KRUSAL'S ALGORITHM

Minimum Cost Spanning tree (MCST)
Strategy 2

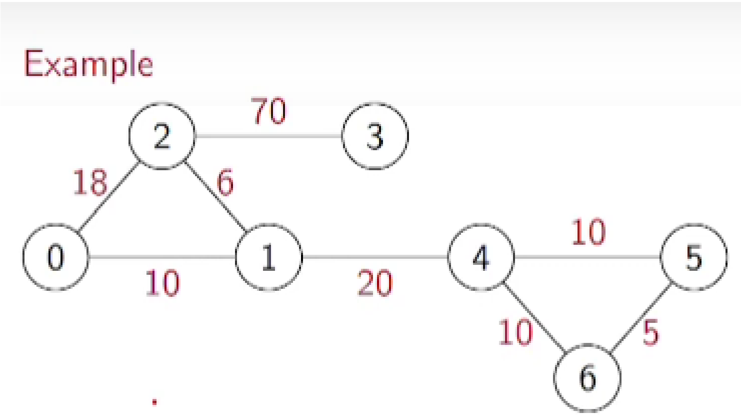
- Start with n components, each a single vertex
- Process edges in ascending order of cost
- Include edge if it does not create a cycle



- Start with smallest edge, $(1, 3)$
- Add next smallest edge, $(2, 4)$
- Add next smallest edge, $(0, 1)$
- Can't add $(0, 3)$, forms a cycle
- Add next smallest edge, $(1, 2)$

Kruskal's Algorithm

- $G = (V, E), W : E \to \mathbb{R}$
- Let $E = \{e_0, e_1, \ldots e_{m-1}\}$ be edges sorted in asceding order by weight
- Let $TE \subseteq E$ be the set of tree edges already added to MCST
- Initially, $TS = \emptyset$
- Scan E from e_0 to e_{m-1}
  - If adding e_i to TE creates a loop, skip it
  - Otherwise, add e_i to TE



- Sort E as $\{(5,6),(1,2),(0,1),(4,5),(0,2),(1,4),(2,3)\}$
- Set $TE = \emptyset$
- Kepe adding each into TE after checking
- $TE = \{(5,6),(1,2),(0,1),(4,5),(1,4),(2,3)\}$

Correctness of Kruskal's Algorithm

From *Minimum Separator Lemma*

- Edges in TE partition vertices into connected components
  - Initially each vertex is a separate component
- Adding $e = (u, w)$ merges components of u and w
  - If u and w are in the same component, e forms a cycle and is discarded
- Let U be component of u, W be $V \setminus U$
  - U, W form a partition of V with $u \in U$ and $w \in W$
  - Since we are scanning edges in ascending order of cost, e is minimum cost edge connecting U and W,soit must be part of any MCST

Implementaion

- Collect edges in a list as (d,u,v)

    - Weight as first component for easy sorting
- Main challenge is to keep track of connected components
    - Dictionary to record component of each vertex
    - Initially each vertex is an isolated component
    - When we add an edge (u, v), merge the components of u and v

```
def kruskal(WList):
  (edges, component, TE) = ([],{},[])
  for u in WList.keys():
    # Weight as first component to sort easily
    edges.extend([(d,u,v)] for (v,d) in WList[u])
    component[u] = u
  edges.sort()
  print(edges)

  for (d,u,v) in edges:
    if component[u] != component[v]:
      TE.append((u,v))
      c = component[u]
      for w in WList.keys():
        if component[w] == c:
          component[w] = component[v]
  return(TE)
```

Analysis

- Sorting the edges is $O(mlogm)$
    - Since m is atmost n^2, equivalently $O(mlogn)$
- Outer loop runs m times
    - Each time we add a tree edge, we have to merge components - $O(n)$ scan
    - $n-1$ tree edges, so this is done $O(n)$ times
- Overall $O(n^2)$
- Bottleneck is naive strategy to label and merge components
- Components partition vertices
    - Collection of disjoint sets
- Data structure to maintain collectio of disjoint sets
    - `find(v)` - return set containing v
    - `union(u,v)` - merge sets of u, v
- Efficient union-find brings complexity down to $O(mlogn)$

Summary

- Kruskal's algorithm builds an MCST bottom up

- Start with n components, each an isolated vertex
  - Scan edges in ascending order of cost
  - Whenever an edge merges disjoint components, add it to the MCST

- Correctness follows from Minimum Separator Lemma
- Complexity is $O(n^2)$ due to naive handling components
  - Will see he to improve to $O(mlogn)$
- If edge weights repeat, MCST is not unique
- "Choose minimum cost edge" will allow choices
  - Consider a triangle on 3 vertices with all edges equal
- Different choices lead to different spanning trees
- In general, there may be a very number of minimum cost spanning trees